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Geometric Properties of Runge-Kutta Discretizations for Nonautonomous Index 2 Differential Algebraic Systems

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Abstract

We analyze Runge-Kutta discretizations applied to nonautonomous index 2 differential algebraic equations (DAE's) in semi-explicit form. It is shown that for half-explicit and projected Runge-Kutta methods there is an attractive invariant manifold for the discrete system which is close to the invariant manifold of the DAE. The proof combines reduction techniques to autonomous index 2 differential algebraic equations with some invariant manifold results of Schropp [9]. The results support the favourable behavior of these Runge-Kutta methods applied to index 2 DAE's for $t \geq 0$.

1 Introduction

The treatment of higher index DAE's is a standard problem in numerical analysis. An important class of problems which serve as sources are multibody systems with constraints. Due to the structure of the forces in the model the resulting DAE is autonomous or nonautomous. Multibody systems originally appear on the position level as index 3 DAE's but there are several well known possibilities to reformulate the problem as an index 2 DAE. Higher index problems can easily be rewritten as index 2 system by differentiating the constraints, provided the derivatives are explicitely available. A disadvantage of this approach is that, in general, the constraints of the original problem are not satisfied any more. To avoid these drift problems, it is advantegous to remain the original constraints in the system and to enforce the problem by additional Lagrange-multipliers. This was introduced by Gear [2], Gear, Gupta & Leimkuhler [3] to preserve the qualitative properties of the original system.

In the present paper we analyze the behavior of some widely spreaded Runge-Kutta type discretizations applied to nonautonomous index 2 DAE's in semi-explicit form. To be more precise, we focus our interest onto the following aspect. It is well known that the solution flow of a nonautonomous DAE takes place in a submanifold of the state times time times control space. We characterize how that submanifold persists under discretization with projected as well as half-explicit Runge-Kutta methods and show that the discretized dynamics possesses an invariant submanifold too which is situated nearby the original one. We rewrite the original DAE as an autononous DAE with one additional space variable. Then, using the results of Schropp [9] and discrete invariant manifold techniques of Nipp, Stoffer [8] we mimic that approach for the projected and half-explicit Runge-Kutta dynamics. This underpins the use of projected or half explicit Runge-Kutta DAE methods when dealing with the longtime behavior of nonautonomous index 2 DAE's.

2 The main result

We are motivated to consider a nonautonomous, semi-explicit DAE

$$\begin{aligned} \Gamma(u,t)\dot{u} &= f(u,t,\lambda), \quad u(t_0)=u_0, \\ 0 &= g(u,t), \quad \lambda(t_0)=\lambda_0, \end{aligned} \tag{2.1}$$

$u \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}^l$ which includes in particular various formulations of multibody systems with constraints.
Let C_b^r denote the space of functions of class C^r with bounded derivatives up to order r. We make the following assumptions.

(A1) f, g and Γ are C_b^r-functions for r sufficiently big.

(A2) $\Gamma(u,t) \in \mathbb{R}^{N,N}$ is invertible for $(u,t) \in D_\tau := \{(u,t) \in \mathbb{R}^{N+1} \mid \| g(u,t) \|_2 < \tau\}$, $\tau > 0$ and $\| \Gamma(u,t)^{-1} \|$ is bounded.

(A3) There is a C_b^r-function ψ_0 satisfying $\frac{\partial g}{\partial u}(u,t)\Gamma(u,t)^{-1}f(u,t,\psi_0(u,t)) + \frac{\partial g}{\partial t}(u,t) = 0$ for $(u,t) \in D_\tau$.

(A4) $\frac{\partial g}{\partial u}(u,t)\Gamma(u,t)^{-1}\frac{\partial f}{\partial \lambda}(u,t,\psi_0(u,t))$ is invertible for $(u,t) \in D_\tau$ and the inverse has bounded norm.

Remark: In the case of constraint mechanical systems the matrix Γ plays the the role of the mass matrix which is symmetric and positive definite.

(A3) and (A4) ensure that problem (2.1) is of index 2. Consistent initial values (u_0, λ_0) must satisfy $g(u_0,t_0) = 0$ and $\frac{\partial g}{\partial u}(u_0,t_0)\Gamma(u_0,t_0)^{-1}f(u_0,t_0,\lambda_0) + \frac{\partial g}{\partial t}(u_0,t_0) = 0$. Condition (A4) says that $\frac{\partial g}{\partial u}(u,t)$ is of full rank so that the second equation of (2.1) for fixed t defines the submanifold $M^t := \{u \in \mathbb{R}^N \mid g(u,t) = 0\}$ of $\mathbb{R}^N$. Moreover, the total solution set of the second equation of (2.1) has the form $M := \{(u,t) \in \mathbb{R}^{N+1} \mid g(u,t) = 0\} = \cup_{t\in\mathbb{R}} M^t$. Then, the underlying index 0 ODE reads

$$\dot{u} \;= \Gamma(u,t)^{-1} f(u,t,\psi_0(u,t)),\; u \in M^t,\; t \in \mathbb{R}. \tag{2.2}$$

Here, the reader may notice, that the "underlying manifold" M^t changes with t. We denote the solution of (2.2) together with the initial condition

$$u(t_0) \;=\; u_0, \;\; (u_0,t_0) \in M \tag{2.3}$$

by $\bar{u}(t,t_0,u_0)$. In this situation, (A3) implies the solution flow $(\bar{u}(t,t_0,u_0), \bar{\lambda}(t,t_0,u_0))$, $\bar{\lambda}(t,t_0,u_0) = \psi_0(\bar{u}(t,t_0,u_0),t)$ for equation (2.1). This means that the manifold

$$M_0 \;=\; \{(u,t,\lambda) \in D_\tau \times \mathbb{R}^l \mid g(u,t) = 0,\; \lambda = \psi_0(u,t)\}$$

is invariant under the solution flow of (2.1).

We are interested in the qualitative, geometric features of s-stage Runge-Kutta type methods with Butcher tableau

$$\begin{array}{c|c} c & A \\ \hline & b^T, \end{array} \qquad A = (a_{ij})_{1\le i,j\le s} \in \mathbb{R}^{s,s}, \qquad b,c \in \mathbb{R}^s \tag{2.4}$$

and constant step size h when applied to (2.1). The Runge-Kutta method possesses stage order q, if

$$\sum_{j=1}^{s} a_{ij} c_j^{k-1} \;=\; \frac{c_i^k}{k},\; k = 1,\dots,q,\; i = 1,\dots,s.$$

To avoid drift problems in the discrete long time run we have to focus our interest to Runge-Kutta type methods which retain the first order constraint $g(u,t) = 0$. This leads us to the widely spreaded projected Runge-Kutta methods introduced by Ascher & Petzold [1] or to the half-explicit Runge-Kutta methods due to Hairer, Lubich and Roche [5]. For the Butcher tableau of the projected Runge-Kutta method we impose the conditions:

(B1) The Runge-Kutta matrix A is invertible.

(B2) $R(\infty) = 1 - b^T A^{-1}\mathbb{I}$, $\mathbb{I} = (1,\dots,1) \in \mathbb{R}^s$ satisfies $|R(\infty)| < 1$.

(B3) The method is of classical order p and possesses stage order q with $p \ge q \ge 1$.

For the half-explicit method, that is, $a_{i,j} = 0$ for $i \le j$ we assume

(B1') $a_{i+1,i} \neq 0$ for $i = 1,\dots s-1$ and $b_s \neq 0$.

(B2') The method is of order p and $c = A\mathbb{I}$ holds.

Let (u_n, λ_n) denote the Runge-Kutta approximations of (2.1) at the time $t_n = nh$. Applied to equation (2.1) the projected Runge-Kutta method has the following form. First we compute a classical Runge-Kutta step

$$\begin{aligned} \tilde{u}_{n+1} &= u_n + h(b^T \otimes I)\bar{\Gamma}(U^n, \mathbb{I}t_n + hc)^{-1}\bar{f}(U^n, \mathbb{I}t_n + hc, \Lambda^n), \\ \lambda_{n+1} &= (1 - b^T A^{-1}\mathbb{I})\lambda_n + (b^T A^{-1} \otimes I)\Lambda^n \end{aligned} \tag{2.5}$$

where $U^n = (U_1^n, \ldots, U_s^n) \in \mathbb{R}^{Ns}$, $\Lambda^n = (\Lambda_1^n, \ldots, \Lambda_s^n) \in \mathbb{R}^{ls}$ denote the solution of the algebraic system

$$\begin{aligned} U - (\mathbb{I} \otimes u_n) &= h(A \otimes I)\bar{\Gamma}(U, \mathbb{I}t_n + hc)^{-1}\bar{f}(U, \mathbb{I}t_n + hc, \Lambda), \\ 0 &= \bar{g}(U, \mathbb{I}t_n + hc). \end{aligned} \tag{2.6}$$

$\bar{f}$ stands for $\bar{f}(U^n, \mathbb{I}t_n + hc, \Lambda^n) = (f(U_1^n, t_n + hc_1, \Lambda_1^n), \ldots, f(U_s^n, t_n + hc_s, \Lambda_s^n))$, $\bar{g}$ is defined via $\bar{g}(U^n, \mathbb{I}t_n + hc) = (g(U_1^n, t_n + hc_1), \ldots, g(U_s^n, t_n + hc_s))$ and $\bar{\Gamma}$ denotes $\bar{\Gamma}(U^n, \mathbb{I}t_n + hc) = \operatorname{diag}(\Gamma(U_1^n, t_n + hc_1), \ldots, \Gamma(U_s^n, t_n + hc_s))$. Finally, the projection step

$$\begin{aligned} u_{n+1} &= \tilde{u}_{n+1} + \frac{\partial}{\partial \lambda}\Big(\Gamma(u_{n+1}, t_{n+1})^{-1} f(u_{n+1}, t_{n+1}, \lambda_{n+1})\Big)\gamma, \\ 0 &= g(u_{n+1}, t_{n+1}) \end{aligned} \tag{2.7}$$

determines u_{n+1}.

A Runge-Kutta method satisfying $a_{sj} = b_j$, $j = 1, \ldots, s$ is called stiffly accurate (see, e.g., Hairer, Wanner [7], p. 44). Stiffly accurate Runge-Kutta solutions satisfy the first order constraint $g(u, t) = 0$ and, hence, the projection step (2.7) is superfluous.
The application of a half-explicit Runge-Kutta method to (2.1) reads as follows. Solve (2.6) in the case $a_{i,j} = 0$ for $j \geq i$ and obtain U^n and Λ_i^n, $i = 1, \ldots, s-1$. Then Λ_s^n and u_{n+1} are computed by

$$\begin{aligned} u_{n+1} &= u_n + h(b^T \otimes I)\bar{\Gamma}(U^n, \mathbb{I}t_n + hc)^{-1}\bar{f}(U^n, \mathbb{I}t_n + hc, \Lambda^n), \\ 0 &= g(u_{n+1}, t_{n+1}). \end{aligned} \tag{2.8}$$

In order to compute the λ-component one has several possibilities. The most accurate is the computation of λ from the index 2 condition, that is, $\lambda_n = \psi_0(u_n, t_n)$. Here we follow the more efficient approach of Hairer, Lubich, Roche [6]. They propose to require $c_s = 1$ and take

$$\lambda_{n+1} = \Lambda_s^n. \tag{2.9}$$

Moreover, we assume

(B3') $\Lambda_s^n - \bar{\lambda}(t_n + h, t_n, u_n) = O(h^r)$, $(u_n, t_n) \in M$, $r \leq p$ (see, e.g., Hairer, Brasey [4] for sufficient conditions on A, b, c).

The qualitative properties of the discrete schemes are characterized in

Theorem 2.1 *Consider the DAE (2.1) and assume (A1)-(A4). Let* (u_n, λ_n) *denote the sequences generated with a projected [half-explicit] Runge-Kutta method satisfying (B1)-(B3) [(B1')-(B3')], when applied to (2.1) with consistent initial values* (u_0, λ_0).
Then for $0 < h < h_0$, $h_0 > 0$ *sufficiently small there is* $\gamma \in]0, \tau]$ *and a* C_b^r*-function* $\psi_{0,h} : M \to \mathbb{R}^l$, $M = \{(u,t) \in \mathbb{R}^{N+1} \mid g(u,t) = 0\}$ *such that the following assertions hold.*

i) The set $M_{0,h} = \{(u,t,\lambda) \in D_\gamma \times \mathbb{R}^l \mid g(u,t) = 0,\ \lambda = \psi_{0,h}(u,t)\}$ *is invariant for the projected [half-explicit] Runge-Kutta map (2.5)-(2.7) [(2.8)-(2.9)].*

ii) The manifold $M_{0,h}$ *is uniformly attractive with attractivity constant* $\chi_h = |R(\infty)| + O(h^{q+1})$ $[\chi_h = 0]$.

iii) For every initial value (u_0, λ_0) *at time* t_0 *with* $\| \lambda_0 - \psi_0(u_0, t_0) \|$ *sufficiently small there is* $(\tilde{u}_0, \tilde{\lambda}_0) \in M_{0,h}$ *and* $\alpha, \hat{\alpha} > 0$ *such that the corresponding evolutions* (u_n, λ_n) *and* $(\tilde{u}_n, \tilde{\lambda}_n)$ *satisfy*

$$\begin{aligned} \| (u_i, t_i) - (\tilde{u}_i, t_i) \| &\leq \alpha \chi_h^i \| \lambda_0 - \psi_0(u_0, t_0) \|, \quad i = 0, 1, 2, \ldots, \\ \| \lambda_i - \tilde{\lambda}_i \| &\leq \hat{\alpha} \chi_h^i \| \lambda_0 - \psi_0(u_0, t_0) \| \quad i = 0, 1, 2, \ldots. \end{aligned}$$

iv) $\| \psi_0(u,t) - \psi_{0,h}(u,t) \| \leq Ch^q$ $[Ch^r]$ *for* $(u,t) \in M$.

Remark: The invariant manifold $M_{0,h}$ for the projected Runge-Kutta iteration is highly attractive, if $R(\infty) = 0$. The manifold is infinite attractive, that is, $(u_1, \lambda_1) \in M_{0,h}$ for every (u_0, λ_0), if $\chi_h = 0$. This is valid for half-explicit and stiffly accurate Runge-Kutta methods. In addition, the closeness of M_0 and $M_{0,h}$ is governd by the stage order of the projected Runge-Kutta method or the consistency error in the control variable λ for the half-explicit Runge-Kutta scheme, respectively.

3 Proof of the main result

In this section we prove Theorem 2.1 by reducing the nonautonomous, semi-explicit equation (2.1) to an autonomous index 2 DAE in Hessenberg form. Then our goal is to apply the invariant manifold theorem of Schropp [9].
Using (A2) we rewrite the nonautonomous semi-explicit DAE

$$\begin{aligned} \Gamma(u,t)\dot{u} &= f(u,t,\lambda), \quad u(t_0) = u_0, \\ 0 &= g(u,t), \quad \lambda(t_0) = \lambda_0, \end{aligned} \tag{3.10}$$

$u \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}^l$ as an autonomous DAE in Hessenberg form by introducing the additional space variable $v \in \mathbb{R}$. This yields

$$\begin{aligned} \dot{u} &= \Gamma(u,v)^{-1} f(u,v,\lambda), \quad u(t_0) = u_0, \\ \dot{v} &= 1, \quad v(t_0) = t_0, \\ 0 &= g(u,v), \quad \lambda(t_0) = \lambda_0. \end{aligned} \tag{3.11}$$

If we denote the solution flow of (3.10) with $(\bar{u}(t,t_0,u_0), \bar{\lambda}(t,t_0,u_0))$, equation (3.11) possesses the flow $(\bar{u}(t,t_0,u_0), \bar{v}(t,t_0,u_0), \bar{\lambda}(t,t_0,u_0))$, $\bar{v}(t,t_0,u_0) = t$.
First we remark that with $\tilde{f}(u,v,\lambda) = (\Gamma(u,v)^{-1} f(u,v,\lambda), 1)$, $\tilde{g} = g$ and $\tilde{\psi}_0 = \psi_0$ problem (3.11) satisfies the assumptions (A1)-(A3) of Theorem 2.1 in Schropp [9].
Our next step is to discretize equation (3.11) with a projected or a half-explicit Runge-Kutta method with tableau (2.4) satisfying (B1)-(B3), (B1')-(B3') respectively. We denote the Runge-Kutta iterates of (3.11) at time $t_n = nh$ with $\hat{u}_n$, $\hat{v}_n$, $\hat{\lambda}_n$ and the corresponding internal stages with $\hat{U}^n$, $\hat{V}^n$, $\hat{\Lambda}^n$. For the projected Runge-Kutta method we obtain the discrete scheme

$$\begin{aligned} \bar{u}_{n+1} &= \hat{u}_n + h(b^T \otimes I)\bar{\Gamma}(\hat{U}^n, \hat{V}^n)^{-1} \bar{f}(\hat{U}^n, \hat{V}^n, \hat{\Lambda}^n), \\ \hat{v}_{n+1} &= \hat{v}_n + hb^T \mathbb{I} \\ \hat{\lambda}_{n+1} &= (1 - b^T A^{-1} \mathbb{I})\hat{\lambda}_n + (b^T A^{-1} \otimes I)\hat{\Lambda}^n \end{aligned} \tag{3.12}$$

where $\hat{U}^n \in \mathbb{R}^{Ns}$, $\hat{V}^n \in \mathbb{R}^s$, $\hat{\Lambda}^n \in \mathbb{R}^{ls}$ denote the solution of

$$\begin{aligned} \hat{U} - (\mathbb{I} \otimes \hat{u}_n) &= h(A \otimes I)\bar{\Gamma}(\hat{U}, \hat{V})^{-1} \bar{f}(\hat{U}, \hat{V}, \hat{\Lambda}), \\ \hat{V} - \mathbb{I}\hat{v}_n &= hA\mathbb{I}, \\ 0 &= \bar{g}(\hat{U}, \hat{V}). \end{aligned} \tag{3.13}$$

Then the projection step

$$\begin{aligned} \hat{u}_{n+1} &= \bar{u}_{n+1} + \frac{\partial}{\partial \lambda}\Big(\Gamma(\hat{u}_{n+1}, \hat{v}_{n+1})^{-1} f(\hat{u}_{n+1}, \hat{v}_{n+1}, \hat{\lambda}_{n+1})\Big)\gamma, \\ 0 &= g(\hat{u}_{n+1}, \hat{v}_{n+1}) \end{aligned} \tag{3.14}$$

determines $\hat{u}_{n+1}$.
The application of a half-explicit Runge-Kutta method to (3.11) reads as follows. Solve (3.13) in the case $a_{i,j} = 0$ for $j \geq i$ and obtain $\hat{U}^n$, $\hat{V}^n$ and $\hat{\Lambda}^n_i$, $i = 1, \ldots, s-1$. Then $\hat{\Lambda}^n_s$ and $\hat{u}_{n+1}$, $\hat{v}_{n+1}$ are computed by

$$\begin{aligned} \hat{u}_{n+1} &= \hat{u}_n + h(b^T \otimes I)\bar{\Gamma}(\hat{U}^n, \hat{V}^n)^{-1} \bar{f}(\hat{U}^n, \hat{V}^n, \hat{\Lambda}^n), \\ \hat{v}_{n+1} &= \hat{v}_n + hb^T \mathbb{I}, \\ 0 &= g(\hat{u}_{n+1}, \hat{v}_{n+1}). \end{aligned} \tag{3.15}$$

Let (u_n, λ_n) stand for the Runge-Kutta iterates of (3.10). Provided that $\hat{u}_n = u_n$, $\hat{v}_n = t_n$ and $\hat{\lambda}_n = \lambda_n$ hold, we can proceed as follows.

The second line of equation (3.13) and (B3) directly show $\hat{V}^n = v_n\mathbb{I} + hc$. Then a simple comparison of (2.6) and (3.13) shows $\hat{U}^n = U^n$ and $\hat{\Lambda}^n = \Lambda^n$. Moreover, the reader should keep in mind that the unique solubility of (3.13) and (2.6) for $0 < h < h_0$, $h_0 > 0$ sufficiently small follows from Lemma 3.4 in Schropp [9].
Next we can deduce the relation

$$\hat{v}_{n+1} \quad = \quad \hat{v}_n + h = t_{n+1} \tag{3.16}$$

from the second line in (3.12) and (3.15). Finally, the comparison of the step forward maps (2.5), (3.12) and (2.8), (3.15) gives

$$\hat{u}_{n+1} = u_{n+1}, \qquad \hat{\lambda}_{n+1} = \lambda_{n+1}, \tag{3.17}$$

that is, the $u-$ and the $\lambda-$component of the projected Runge-Kutta maps applied to (3.10) and (3.11) coincide. Analogously, we can show formula (3.17) for the half-explicit Runge-Kutta map.
Since the autonomous DAE (3.11) satisfies the assumptions (A1)-(A3) of Theorem 2.1 in Schropp [9] and the discrete schemes fulfil (B1)-(B3), (B1')-(B3') we obtain

Lemma 3.1 *Consider the DAE (3.11) and assume (A1)-(A4). Let* $(\hat{u}_n, \hat{v}_n, \hat{\lambda}_n)$ *denote the sequences generated with a projected [half-explicit] Runge-Kutta map satisfying (B1)-(B3) [(B1')-(B3')], when applied to (3.11) with consistent initial values* $(\hat{u}_0, \hat{v}_0, \hat{\lambda}_0)$.
Then for $0 < h < h_0$, $h_0 > 0$ *sufficiently small there is a* C_b^r*-function* $\psi_{0,h} : M \to \mathbb{R}^l$, $M = \{(u,v) \in \mathbb{R}^{N+1} \mid g(u,v) = 0\}$ *such that the following assertions hold.*

i) The set $M_{0,h} = \{(u,v,\lambda) \in D_\gamma \times \mathbb{R}^l \mid g(u,v) = 0,\ \lambda = \psi_{0,h}(u,v)\}$ *is invariant for the projected [half-explicit] Runge-Kutta map (3.12)-(3.14) [(3.15), (2.9)].*

ii) The manifold $M_{0,h}$ *is uniformly attractive with attractivity constant* $\chi_h = |R(\infty)| + O(h^{q+1})$ $[\chi_h = 0]$.

iii) For every initial value $(\hat{u}_0, \hat{v}_0, \hat{\lambda}_0)$ *with* $\| \hat{\lambda}_0 - \psi_0(\hat{u}_0, \hat{v}_0) \|$ *sufficiently small there is* $(\tilde{u}_0, \tilde{v}_0, \tilde{\lambda}_0) \in M_{0,h}$ *and* $\alpha, \hat{\alpha} > 0$ *such that the corresponding evolutions* $(\hat{u}_n, \hat{v}_n, \hat{\lambda}_n)$ *and* $(\tilde{u}_n, \tilde{v}_n, \tilde{\lambda}_n)$ *satisfy*

$$\begin{aligned} \| (\hat{u}_i, \hat{v}_i) - (\tilde{u}_i, \tilde{v}_i) \| \quad &\leq \quad \alpha \chi_h^i \| \hat{\lambda}_0 - \psi_0(\hat{u}_0, \hat{v}_0) \|, \quad i = 0,1,2,\ldots, \\ \| \hat{\lambda}_i - \tilde{\lambda}_i \| \quad &\leq \quad \hat{\alpha} \chi_h^i \| \lambda_0 - \psi_0(u_0, t_0) \| \quad i = 0,1,2,\ldots. \end{aligned}$$

iv) $\| \psi_0(u,v) - \psi_{0,h}(u,v) \| \leq Ch^q$ $[Ch^r]$ *for* $(u,v) \in M$.

Rewriting the assertions i)-iv) of Lemma 3.1 for the half-explicit and projected Runge-Kutta map applied to the nonautonomous DAE (3.10) and using $u_n = \hat{u}_n$, $t_n = \hat{v}_n$ and $\lambda_n = \hat{\lambda}_n$ immediately shows the claims i), ii) and iv) of Theorem 2.1. To complete the proof of iii) in Theorem 2.1 it remains to show that the in phase condition iii) in Lemma

3.1 holds with $\tilde{v}_n = \hat{v}_n = t_n$.
Since all norms in finite-dimensional vector spaces are equivalent we rewrite the first assertion of Lemma 3.1 iii) in the infinity norm with a possibly different constant $\tilde{\alpha}$ instead of α. This gives

$$\| (\hat{u}_i, \hat{v}_i) - (\tilde{u}_i, \tilde{v}_i) \|_\infty \quad \leq \quad \tilde{\alpha}\chi_h^i \| \hat{\lambda}_0 - \psi_0(\hat{u}_0, \hat{v}_0) \|_\infty, \;\; i = 0, 1, 2, \ldots . \tag{3.18}$$

Using $| \hat{v}_0 - \tilde{v}_0 | = | \hat{v}_i - \tilde{v}_i | \leq \| (\hat{u}_i, \hat{v}_i) - (\tilde{u}_i, \tilde{v}_i) \|_\infty$ (see equation (3.16) and formula (3.18)) we obtain

$$| \hat{v}_0 - \tilde{v}_0 | \quad \leq \quad \tilde{\alpha}\chi_h^i \| \hat{\lambda}_0 - \psi_0(\hat{u}_0, \hat{v}_0) \|_\infty, \;\; i = 0, 1, 2, \ldots . \tag{3.19}$$

Now, Lemma 3.4 in Schropp [9] ensures the existence of the discrete iterates for $i \in \mathbb{N}$. Letting $i \to \infty$ in (3.19) then shows $t_0 = \hat{v}_0 = \tilde{v}_0$ and the proof of Theorem 2.1 is complete.

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