First publ. in: Integral equations and operator theory 39 (2001), 1, pp. 15-40

ON ELLIPTIC OPERATOR PENCILS WITH GENERAL BOUNDARY CONDITIONS

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In this paper parameter-dependent partial differential operators are investigated which satisfy the condition of N-ellipticity with parameter, an ellipticity condition formulated with the use of the Newton polygon. For boundary value problems with general boundary operators we define N-ellipticity including an analogue of the Shapiro–Lopatinskii condition. It is shown that the boundary value problem is N-elliptic if and only if an a priori estimate with respect to certain parameter-dependent norms holds. These results are closely connected with singular perturbation theory and lead to uniform estimates for problems of Vishik–Lyusternik type containing a small parameter.

1. Introduction

Let us consider an operator pencil depending polynomially on the complex parameter λ and being of the form

$$A(x, D, \lambda) = A_{2m}(x, D) + \lambda A_{2m-1}(x, D) + \cdots + \lambda^{2m-2\mu} A_{2\mu}(x, D)\,, \qquad (1.1)$$

where m and μ are integer numbers with $m > \mu > 0$ and $A_j(x, D) = \sum_{|\alpha|\le j} a_{\alpha j}(x) D^\alpha$ is a partial differential operator with smooth coefficients. We assume that the pencil (1.1) acts on a smooth compact manifold M with smooth boundary ∂M. Here and in the following, we use the standard multi-index notation.

In the paper [5] the authors obtained basic results on N-elliptic pencils of the form (1.1) and the Dirichlet boundary value problem connected with this pencil. The present paper is a continuation of [5] and deals with general boundary conditions and corresponding a priori estimates. Moreover, we will prove the necessity of N-ellipticity for these estimates, construct a right parametrix and show the connection to problems with small parameter.

*Supported in part by the Deutsche Forschungsgemeinschaft and by Russian Foundation of Fundamental Research, Grant 00-01-00387.

Let the boundary operators B_j, for simplicity independent of the complex parameter λ, be of the form

$$B_j(x,D) = \sum_{|\beta|\le m_j} b_{\beta j}(x)D^\beta \quad (j=1,\dots,m)\,, \tag{1.2}$$

where the numbering is chosen such that for the orders of the operators B_j we have $m_1 \le m_2 \le \ldots \le m_m$. Additionally, we assume that

$$m_\mu < m_{\mu+1}\,. \tag{1.3}$$

The coefficients of B_j are supposed to be defined in $\overline{M}$ and to be infinitely smooth.

The principal symbol $A^{(0)}(x,\xi,\lambda)$ of (1.1) is defined as

$$A^{(0)}(x,\xi,\lambda) := A^{(0)}_{2m}(x,\xi) + \lambda A^{(0)}_{2m-1}(x,\xi) + \ldots + \lambda^{2m-2\mu}A^{(0)}_{2\mu}(x,\xi)\,, \tag{1.4}$$

where

$$A^{(0)}_j(x,\xi) := \sum_{|\alpha|=j} a_{\alpha j}(x)\xi^\alpha \quad (j=2\mu,\dots,2m) \tag{1.5}$$

stands for the principal symbol of A_j. The principal symbols (1.4) and (1.5) are invariant under change of coordinates and thus globally defined on the cotangent bundle $T^*M\backslash\{0\}$. The principal symbols $B^{(0)}_j$ of the boundary operators B_j are defined analogously.

In [5] the Newton polygon approach was used to formulate and prove an a priori estimate for the Dirichlet boundary value problem. This method (which was also applied to Douglis–Nirenberg systems in [4]) turns out to be suitable for general boundary conditions, too. The concept of the Newton polygon makes it possible to define the general notion of N-ellipticity with parameter which is a generalization of the classical definition of ellipticity with parameter given by Agmon [1] and Agranovich–Vishik [3]. For the connection to N-parabolic problems and Douglis–Nirenberg systems, the reader is referred to [5], Section 1.

The main ideas of the present paper are to use the language of function spaces connected with the Newton polygon and to find equivalent (Shapiro–Lopatinskii type) conditions for estimates in these spaces. These conditions, in particular the condition of regular degeneration (see below) which might seem surprising at the first moment, become clearer if we replace in (1.1) λ by ε^{-1}. We obtain a problem of singular perturbation theory as it was studied, for instance, by Vishik and Lyusternik [12]. The a priori estimate stated below in Section 4 corresponds to a uniform (with respect to ε) estimate in the Vishik–Lyusternik theory (see also [6], [9], [11]). We will come back to this close connection in the Appendix.

2. The Shapiro–Lopatinskii condition

As the manifold M is compact we may fix a finite number of coordinate systems. Locally in each of these coordinate systems the operator pencil $A(x,D,\lambda)$ is of the form (1.1) and acts in $\mathbb{R}^n$. We can suppose without loss of generality that the coefficients of $A(x,D,\lambda)$ are (in local coordinates) of the form

$$a_{\alpha j}(x) = a_{\alpha j} + a'_{\alpha j}(x), \quad a'_{\alpha j} \in \mathcal{D}(\mathbb{R}^n)\,. \tag{2.1}$$

Definition 2.1. Let $x^0 \in \overline{M}$ be fixed. The interior symbol $A(x^0, \xi, \lambda)$ is called N-elliptic with parameter in $[0, \infty)$ at x^0 (cf. [5]) if the estimate

$$|A^{(0)}(x^0, \xi, \lambda)| \geq C|\xi|^{2\mu} (\lambda + |\xi|)^{2m-2\mu} \quad \big(\xi \in \mathbb{R}^n, \lambda \in [0, \infty)\big) \tag{2.2}$$

holds with a constant C which does not depend on ξ or λ. If this is true for every $x^0 \in \overline{M}$, the symbol $A(x, \xi, \lambda)$ and the operator $A(x, D, \lambda)$ are called N-elliptic with parameter in $[0, \infty)$.

By continuity and compactness, for an N-elliptic operator the constant C in (2.2) can be chosen independently of x^0.

Now we shall define the analogue of the Shapiro–Lopatinskii condition for our problem. For this, we fix a point $x^0 \in \partial M$ and a coordinate system in the neighbourhood of x^0 such that in this system locally the boundary ∂M is given by the equation $x_n = 0$. We use in $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$ the coordinates $x = (x', x_n)$ and the dual coordinates $\xi = (\xi', \xi_n)$. If A is N-elliptic with parameter, it follows from (2.2) that for every $x^0 \in \overline{M}$ we have

$$A^{(0)}(x^0, \xi, \lambda) \neq 0 \quad \big(\xi \in \mathbb{R}^n \backslash \{0\},\ \lambda \in [0, \infty)\big)\,. \tag{2.3}$$

In the case $n > 2$ this implies that $A^{(0)}$, considered as a polynomial in ξ_n, has exactly m roots with positive imaginary part for every $\xi' \neq 0$. In the case $n = 2$ this is an additional condition which we assume to hold in the following. Similar considerations hold for $A^{(0)}_{2\mu}$.

Let A be N-elliptic with parameter in $[0, \infty)$, fix $x^0 \in \partial M$ and write A in local coordinates corresponding to x^0 as considered above. Then we define the polynomial in $\tau \in \mathbb{C}$

$$Q(x^0, \tau) = \tau^{-2\mu} A^{(0)}(x^0, 0, \tau, 1)\,. \tag{2.4}$$

Definition 2.2. The operator $A(x, D, \lambda)$ degenerates regularly at the boundary ∂M if for every $x^0 \in \partial M$ the polynomial (2.4) has exactly $m - \mu$ roots in the upper half-plane of the complex plane.

Remark 2.3. a) It is easily seen that if for a fixed $x^0 \in \partial M$ and a fixed coordinate system polynomial (2.4) has $m - \mu$ roots in the upper half-plane, then this polynomial has this property for arbitrary $x^0 \in \partial M$ and for an arbitrary coordinate system. This is due to the fact that $Q(x^0, \cdot)$ has no real roots (this follows from inequality (2.2)) and that its roots depend continuously on the coefficients.
b) The condition of regular degeneration has its direct counterpart in the theory of singular perturbations (see, e.g., [12], Section 6).
c) Some examples where the condition of regular degeneration (Definition 2.2) holds automatically can be found in [5], Remark 3.4.

If A is N-elliptic with parameter in $[0, \infty)$, then for any fixed $x^0 \in \overline{M}$ and $\xi' \in \mathbb{R}^{n-1} \backslash \{0\}$, we see from (2.3) that we can factorize the principal symbol $A^{(0)}(x^0, \xi, \lambda)$ in the form

$$A^{(0)}(x^0, \xi, \lambda) = A^{(0)}_+(x^0, \xi, \lambda)\, A^{(0)}_-(x^0, \xi, \lambda)\,.$$

Here

$$A_+^{(0)}(x^0,\xi',\tau,\lambda) := \prod_{j=1}^{m}(\tau-\tau_j(x^0,\xi',\lambda))\,, \tag{2.5}$$

where $\tau_1,\dots,\tau_m$ are the zeros of $A^{(0)}$ with positive imaginary part.

Now let $x^0\in\partial M$ and denote by $B_j'(x^0,\xi',\xi_n,\lambda)$ the remainder of $B_j^{(0)}(x^0,\xi)$ after division by $A_+^{(0)}(x^0,\xi,\lambda)$, where all polynomials are considered as polynomials in ξ_n. We write B_j' in the form

$$B_j'(x^0,\xi',\xi_n,\lambda) = \sum_{k=1}^{m} b_{jk}(x^0,\xi',\lambda)\xi_n^{k-1}\,. \tag{2.6}$$

and define the Lopatinskii determinant by

$$\mathrm{Lop}(x^0,\xi',\lambda) := \det\Big(b_{jk}(x^0,\xi',\lambda)\Big)_{j,k=1,\dots,m}\,. \tag{2.7}$$

Then the condition

$$\mathrm{Lop}(x^0,\xi',\lambda)\neq 0 \tag{2.8}$$

means that $B_j^{(0)}(x^0,\xi',\cdot)$ are linearly independent modulo $A_+^{(0)}(x^0,\xi',\cdot,\lambda)$. It is well-known that condition (2.8) is satisfied if and only if the ordinary differential equation on the half-line

$$\begin{aligned} A^{(0)}(x^0,\xi',D_t,\lambda)\,w(t) &= 0 \qquad (t>0)\,, && (2.9)\\ B_k^{(0)}(x^0,\xi',D_t)\,w(t)|_{t=0} &= h_k \quad (k=1,\dots,m)\,, && (2.10)\\ w(t) &\to 0 \qquad (t\to+\infty)\,, \end{aligned}$$

is uniquely solvable for every $(h_1,\dots,h_m)\in\mathbb{C}^m$. Here D_t stands for $-i\frac{\partial}{\partial t}$.

Definition 2.4. Let A satisfy the regular degeneration condition. Then the boundary problem $(A,B_1,\dots,B_m)$ is called N-elliptic with parameter $\lambda\in[0,\infty)$ if the following conditions hold:

a) The interior symbol $A(x,\xi,\lambda)$ is N-elliptic with parameter in $[0,\infty)$ in the sense of Definition 2.1.

b) For every fixed $x^0\in\partial M$, every $\xi'\neq 0$ and every $\lambda\in[0,\infty)$ the polynomials $(B_j^{(0)}(x^0,\xi',\cdot))_{j=1,\dots,m}$ are linearly independent modulo $A_+^{(0)}(x^0,\xi',\cdot,\lambda)$, i.e. (2.8) holds.

c) For every fixed $x^0\in\partial M$, the boundary problem

$$(A_{2\mu}^{(0)}(x^0,D),B_1(x^0,D),\dots,B_\mu(x^0,D))$$

fulfills the Shapiro–Lopatinskii condition, i.e. $(B_j^{(0)}(x^0,\xi))_{j=1,\dots,\mu}$ are linearly independent modulo $(A_{2\mu}^{(0)})_+(x^0,\xi)$. Here $(A_{2\mu}^{(0)})_+$ is defined in analogy to (2.5) with A replaced by $A_{2\mu}$.

d) Let $Q_+(x^0,\tau) := \prod_{j=\mu+1}^{m}(\tau-\tau_j^1(x^0))$ where $\tau_{\mu+1}^1,\dots,\tau_m^1$ denote the zeros of $Q(x^0,\tau)$ with positive imaginary part. Then $(B_j^{(0)}(x^0,0,\tau))_{j=\mu+1,\dots,m}$ are linearly independent modulo $Q_+(x^0,\tau)$ for every $x^0\in\partial M$.

Remark 2.5. a) Note that the degree of $B_j^{(0)}(x^0, 0, \cdot)$ is m_j which may be greater than $2m - 2\mu$.
b) Condition b) in Definition 2.4 differs from the Agmon–Agranovich–Vishik condition of ellipticity with parameter. If the symbols $A(x, \xi, \lambda)$ and $B_j(x, \xi)$ are homogeneous with respect to (ξ, λ), the Agmon–Agranovich–Vishik condition means that

$$\operatorname{Lop}(x^0, \xi', \lambda) \neq 0 \text{ for } |\xi'|^2 + \lambda^2 = 1,\ \lambda \geq 0\,. \tag{2.11}$$

In particular, in this case inequality (2.8) holds for $\lambda = 1$ and $\xi' = 0$. In the case of N-ellipticity, however, the Lopatinskii determinant is in general not defined for $\xi' = 0$ and may tend to zero as $\xi' \to 0$.
c) Taking in 2.4 b) $\lambda = 0$ and $|\xi'| = 1$, we obtain the standard Shapiro–Lopatinskii condition for the boundary value problem $(A_{2m}, B_1, \ldots, B_m)$.
d) Conditions 2.4 c) and d) correspond in some sense to the limit $\lambda \to \infty$ in 2.4 b). To explain this, we replace $\lambda = \varepsilon^{-1}$ and write (2.9) in the form

$$\Big[A_{2\mu}^{(0)}(x^0, \xi', D_t) + \cdots + \varepsilon^{2m-2\mu} A_{2m}^{(0)}(x^0, \xi', D_t)\Big] w_\varepsilon = 0\,.$$

For $\varepsilon = 0$ we supplement this equation with the first μ boundary conditions and obtain a problem for w_0 which is, due to condition 2.4 c), uniquely solvable.

For $\varepsilon > 0$ the solution $w_0(t)$ will be a good approximation of $w_\varepsilon(t)$ for $t \geq t_0$ for each $t_0 > 0$. However, w_0 does not satisfy the last $m - \mu$ boundary conditions and therefore will not be a good approximation in a neighbourhood of $t = 0$. To satisfy all boundary conditions, we have to add boundary layers which exist due to condition 2.4 d). These considerations are a basic part of the Vishik–Lyusternik theory of boundary value problems with small parameter (see [12], Section 6).
e) Condition 2.4 d) can be formulated as unique solvability of an ordinary differential equation system on the half-line, similarly to (2.9)–(2.10). If this condition holds, we have the strict inequalities

$$m_\mu < m_{\mu+1} < m_{\mu+2} < \cdots < m_m\,.$$

3. The basic ODE estimate

In a first step we consider the model problem in the half space. Let $(A, B_1, \ldots, B_m)$ be of the form (1.1), (1.2) and acting in $\mathbb{R}^n_+$. We suppose that A is homogeneous in (ξ, λ), i.e. has the form

$$A(\xi, \lambda) = A_{2m}(\xi) + \lambda A_{2m-1}(\xi) + \ldots + \lambda^{2m-2\mu} A_{2\mu}(\xi)\,, \tag{3.1}$$

where $A_j(\xi)$ is a homogeneous polynomial in ξ of degree j. Similarly we assume that B_j is given by

$$B_j(\xi) = \sum_{|\beta| = m_j} b_{\beta j} \xi^\beta \quad (j = 1, \ldots, m)\,. \tag{3.2}$$

For fixed $\lambda \geq 0$ and $\xi' \in \mathbb{R}^{n-1} \backslash \{0\}$ we investigate the boundary problem

$$A(\xi', D_t, \lambda)\, w_j(t) = 0 \qquad (t > 0)\,, \tag{3.3}$$
$$B_k(\xi', D_t)\, w_j(t)|_{t=0} = \delta_{jk} \quad (k = 1, \ldots, m)\,, \tag{3.4}$$
$$w_j(t) \to 0 \qquad (t \to +\infty)\,.$$

In [5], the following lemma on the roots of the polynomial $A(\xi', \cdot, \lambda)$ is proved.

Lemma 3.1. *Let the polynomial $A(\xi, \lambda)$ in* (3.1) *be N-elliptic with parameter in* $[0, \infty)$ *and assume that A degenerates regularly. Then, with a suitable numbering of the roots $\tau_j(\xi', \lambda)$ of $A(\xi', \tau, \lambda)$ with positive imaginary part, we have:*
(i) *Let $S(\xi') = \{\tau_1^0(\xi'), \ldots, \tau_\mu^0(\xi')\}$ be the set of all zeros of $A_{2\mu}(\xi', \tau)$ with positive imaginary part. Then for all $r > 0$ there exists a $\lambda_0 > 0$ such that the distance between the sets $\{\tau_1(\xi', \lambda), \ldots, \tau_\mu(\xi', \lambda)\}$ and $S(\xi')$ is less than r for all ξ' with $|\xi'| = 1$ and all $\lambda \geq \lambda_0$.*
(ii) *Let $\tau_{\mu+1}^1, \ldots, \tau_m^1$ be the roots of the polynomial $Q(\tau)$ (cf.* (2.4)) *with positive imaginary part. Then*

$$\tau_j(\xi', \lambda) = \lambda \tau_j^1 + \tilde{\tau}_j^1(\xi', \lambda) \quad (j = \mu + 1, \ldots, m)\,, \tag{3.5}$$

and there exist constants K_j and λ_1, independent of ξ' and λ, such that for $\lambda \geq \lambda_1$ the inequality

$$|\tilde{\tau}_j^1(\xi', \lambda)| \leq K_j |\xi'|^{\frac{1}{k_1}} \lambda^{1 - \frac{1}{k_1}} \quad (|\xi'| \leq \lambda) \tag{3.6}$$

holds, where k_1 is the maximal multiplicity of the roots of $Q(\tau)$.

Theorem 3.2. *Assume that the operator $(A, B_1, \ldots, B_m)$ is of the form* (3.1)–(3.2)*. Assume that condition* (1.3) *holds and that A degenerates regularly at the boundary* (*cf. Definition* 2.2) *and $(A, B_1, \ldots, B_m)$ is N-elliptic with parameter in $\mathbb{R}_+^n$ in the sense of Definition* 2.4*. Then for every $\xi' \in \mathbb{R}^{n-1} \backslash \{0\}$ and $\lambda \in [0, \infty)$ the ordinary differential equation* (3.3)–(3.4) *has a unique solution $w_j(t, \xi', \lambda)$, and the estimate*

$$\|D_t^l w_j(\cdot, \xi', \lambda)\|_{L_2(\mathbb{R}_+)} \leq$$
$$\leq C \begin{cases} |\xi'|^{l - m_j - \frac{1}{2}}, & j \leq \mu, \quad l \leq m_{\mu+1}, \\ |\xi'|^{m_{\mu+1} - m_j} (\lambda + |\xi'|)^{l - m_{\mu+1} - \frac{1}{2}}, & j \leq \mu, \quad l > m_{\mu+1}, \\ |\xi'|^{l - m_\mu - \frac{1}{2}} (\lambda + |\xi'|)^{m_\mu - m_j}, & j > \mu, \quad l \leq m_\mu, \\ (\lambda + |\xi'|)^{l - m_j - \frac{1}{2}}, & j > \mu, \quad l > m_\mu, \end{cases} \tag{3.7}$$

holds with a constant C not depending on ξ' and λ.

Proof. The existence and the uniqueness of the solution follows immediately from conditions a) and b) in Definition 2.4. From the homogeneity of the symbols and from the uniqueness of the solution we see that

$$w_j(t, \xi', \lambda) = r^{-m_j} w_j\Big(rt, \frac{\xi'}{r}, \frac{\lambda}{r}\Big) \tag{3.8}$$

holds for every $r > 0$. If we set $r = |\xi'|$ and $\omega' = \frac{\xi'}{|\xi'|}$ we obtain

$$\|D_t^l w_j(\cdot, \xi', \lambda)\|_{L_2(\mathbb{R}_+)} = |\xi'|^{l - m_j - \frac{1}{2}} \Big\| D_t^l w_j\Big(\cdot, \omega', \frac{\lambda}{|\xi'|}\Big) \Big\|_{L_2(\mathbb{R}_+)}. \tag{3.9}$$

The theorem will be proved if we show that for $|\omega'| = 1$ we have

$$\|(D_t^l w_j)(\cdot,\omega',\Lambda)\|_{L_2(\mathbb{R}_+)} \leq \begin{cases} C\,, & j \leq \mu\,,\ l \leq m_{\mu+1}\,, \\ C\,\Lambda^{l-m_{\mu+1}-\frac{1}{2}}, & j \leq \mu\,,\ l > m_{\mu+1}\,, \\ C\,\Lambda^{m_\mu - m_j}, & j > \mu\,,\ l \leq m_\mu\,, \\ C\,\Lambda^{l-m_j-\frac{1}{2}}, & j > \mu\,,\ l > m_\mu\,, \end{cases} \tag{3.10}$$

for $\Lambda \geq 1$ and that the left-hand side is bounded by a constant for $\Lambda \leq 1$.

The boundedness for $\Lambda \leq 1$ follows easily from conditions a) and b) of Definition 2.4. We have to consider the case of large Λ.

To find an estimate in this case, we represent the solution in a form suggested in a paper of Frank [6]. This representation is different from the (more explicit) representation which is possible for the Dirichlet boundary value problem and which was used in [5].

Due to Lemma 3.1, the roots of this polynomial consist of two groups, the first group, denoted by $\{\tau_1(\omega',\Lambda),\ldots,\tau_\mu(\omega',\Lambda)\}$, being bounded for $\Lambda \to \infty$, the other group, denoted by $\{\tau_{\mu+1}(\omega',\Lambda),\ldots,\tau_m(\omega',\Lambda)\}$, being of order Λ for $\Lambda \to \infty$.

We define

$$A_1(\omega',\tau,\Lambda) := \prod_{j=1}^{\mu}(\tau - \tau_j(\omega',\Lambda))\,. \tag{3.11}$$

Let $\gamma^{(1)}$ be a contour in the upper half of the complex plane enclosing the zeros $\tau_1,\ldots,\tau_\mu$. From Lemma 3.1 we see that $\gamma^{(1)}$ can be chosen independently of ω' and Λ for all $|\omega'| = 1$ and $\Lambda \geq \Lambda_0$.

From the same lemma we see that $A_1(\omega',\tau,\Lambda) \to (A_{2\mu})_+(\omega',\tau)$ as $\Lambda \to \infty$. Therefore we obtain from condition c) in Definition 2.4 that there exists Λ_1 such that for $\Lambda \geq \Lambda_1$ and for all $|\omega'| = 1$ the polynomials $\{B_j(\omega',\tau)\}_{j=1,\ldots,\mu}$ are independent modulo $A_1(\omega',\tau,\Lambda)$. Thus there exist polynomials (with respect to τ) $N_j(\omega',\tau,\Lambda)$, depending continuously on (ω',Λ), such that

$$\frac{1}{2\pi i}\int_{\gamma^{(1)}} \frac{B_k(\omega',\tau)N_j(\omega',\tau,\Lambda)}{A_1(\omega',\tau,\Lambda)}d\tau = \delta_{kj} \quad (k,j = 1,\ldots,\mu)\,. \tag{3.12}$$

From the construction of N_j (cf., e.g., [2], p. 634) it is clear that for $|\omega'| = 1$ the polynomial $N_j(\omega',\tau,\Lambda)$ tends to the corresponding polynomial connected with $A_{2\mu}$ for $\Lambda \to \infty$; in particular, $N_j(\omega',\tau,\Lambda)$ is bounded for $|\omega'| = 1$, $\tau \in \gamma^{(1)}$ and $\Lambda \geq \Lambda_1$.

Analogously, we define

$$A_2(\omega',\tau,\Lambda) := \prod_{j=\mu+1}^{m}(\tau - \tau_j(\omega',\Lambda))\,. \tag{3.13}$$

Let $\tilde{\gamma}^{(2)}(\omega',\Lambda)$ be a contour in the upper half of the complex plane enclosing the zeros $\tau_{\mu+1}(\omega',\Lambda),\ldots,\tau_m(\omega',\Lambda)$. From Lemma 3.1 we know that this contour is of order Λ for $\Lambda \to \infty$. Therefore we may fix a contour $\gamma^{(2)}$, independent of ω' and Λ such that $\gamma^{(2)}$ encloses all values τ_j/Λ with $j = \mu+1,\ldots,m$. We also remark that due to the regular degeneration we may choose $\gamma^{(2)}$ with a positive distance to the real axis (cf. also (3.5)).

From condition d) in 2.4 we know that $\{B_j(0,\tau)\}_{j=\mu+1,\dots,m}$ is linearly independent modulo $Q_+(\tau)$. From Lemma 3.1 b) we know that

$$A_2\Big(\frac{\omega'}{\Lambda},\tau,1\Big) \to Q_+(\tau) \quad (\Lambda\to\infty)\,.$$

Due to continuity, the polynomials $\{B_j(\frac{\omega'}{\Lambda},\tau,1)\}_{j=\mu+1,\dots,m}$ are for sufficiently large Λ linearly independent modulo $A_2(\frac{\omega'}{\Lambda},\tau,1)$. Therefore there exist polynomials (in τ) $N_j(\omega',\tau,\Lambda)$ for $j=\mu+1,\dots,m$, depending continuously on ω' and Λ, such that

$$\frac{1}{2\pi i}\int_{\gamma^{(2)}} \frac{B_k(\frac{\omega'}{\Lambda},\tau)N_j(\omega',\tau,\Lambda)}{A_2(\frac{\omega'}{\Lambda},\tau,1)}d\tau = \delta_{kj} \quad (k,j=\mu+1,\dots,m)\,. \tag{3.14}$$

Now we need a lemma which will be proved below.

Lemma 3.3. *The solution $w_j(t,\omega',\Lambda)$ of the problem* (3.3)–(3.4) *can be represented in the form*

$$w_j(t,\omega',\Lambda) = \frac{1}{2\pi i}\int_{\gamma^{(1)}} \frac{M_j^{(1)}(\omega',\tau,\Lambda)}{A_1(\omega',\tau,\Lambda)}e^{it\tau}d\tau + \frac{1}{2\pi i}\int_{\gamma^{(2)}} \frac{M_j^{(2)}(\omega',\tau,\Lambda)}{A_2(\frac{\omega'}{\Lambda},\tau,1)}e^{it\Lambda\tau}d\tau \tag{3.15}$$

where for $|\tau|=O(1)$ and $|\omega'|=1$ we have

$$M_j^{(1)}(\omega',\tau,\Lambda) \le \begin{cases} C\,, & j\le\mu\,,\\ C\,\Lambda^{m_\mu-m_j}\,, & j>\mu\,,\end{cases}$$

and

$$M_j^{(2)}(\omega',\tau,\Lambda) \le \begin{cases} C\,\Lambda^{-m_{\mu+1}}\,, & j\le\mu\,,\\ C\,\Lambda^{-m_j}\,, & j>\mu\,,\end{cases}$$

As a direct corollary of the lemma we obtain

$$\|(D_t^l w_j)(\cdot,\omega',\Lambda)\|_{L_2(\mathbb{R}_+)} \le \begin{cases} O(1)+O(\Lambda^{l-m_{\mu+1}-\frac12})\,, & j\le\mu\,,\\ O(\Lambda^{m_\mu-m_j})+O(\Lambda^{l-m_j-\frac12})\,, & j>\mu\,.\end{cases}$$

The estimate (3.10) trivially follows from these relations. □

Proof of Lemma 3.3. Let $w(t,\omega',\Lambda)$ be a solution of the problem (3.3)–(3.4) with δ_{jk} replaced by $\phi=(\phi_1,\dots,\phi_m)\in\mathbb{C}^m$. We seek the solution in the form

$$\begin{aligned} w(t,\omega',\Lambda) &= \sum_{k=1}^{\mu}\psi_k(\omega',\Lambda)\frac{1}{2\pi i}\int_{\gamma^{(1)}}\frac{N_k(\omega',\tau,\Lambda)}{A_1(\omega',\tau,\Lambda)}e^{it\tau}d\tau \\ &+ \sum_{k=\mu+1}^{m}\psi_k(\omega',\Lambda)\frac{1}{2\pi i}\int_{\gamma^{(2)}}\frac{N_k(\omega',\tau,\Lambda)}{A_2(\frac{\omega'}{\Lambda},\tau,1)}e^{it\Lambda\tau}d\tau \end{aligned} \tag{3.16}$$

where the functions ψ_k still have to be found.

Applying the boundary operator $B_l(\xi',D_t)$ to both sides of (3.16) and taking $t=0$ we obtain the following system for the unknown functions $\psi_k(\omega',\Lambda)$:

$$\psi_l(\omega',\Lambda)+\Lambda^{m_l}\sum_{k=\mu+1}^{m}\psi_k(\omega',\Lambda)h_{lk}(\omega',\Lambda) \;=\; \phi_l \quad (l=1,\dots,\mu)\,, \tag{3.17}$$

$$\sum_{k=1}^{\mu}\psi_k(\omega',\Lambda)h_{lk}(\omega',\Lambda)+\Lambda^{m_l}\psi_l(\omega',\Lambda) \;=\; \phi_l \quad (l=\mu+1,\dots,m)\,. \tag{3.18}$$

Here we have set

$$h_{lk}(\omega',\Lambda)=\frac{1}{2\pi i}\int_{\gamma^{(2)}}\frac{B_l(\frac{\omega'}{\Lambda},\tau)N_k(\omega',\Lambda,\tau)}{A_2(\frac{\omega'}{\Lambda},\tau,1)}\,d\tau \qquad (l=1,\dots,\mu;\ k=\mu+1,\dots,m)\,, \tag{3.19}$$

$$h_{lk}(\omega',\Lambda)=\frac{1}{2\pi i}\int_{\gamma^{(1)}}\frac{B_l(\omega',\tau)N_k(\omega',\Lambda,\tau)}{A_1(\omega',\tau,\Lambda)}\,d\tau \qquad (l=\mu+1,\dots,m;\ k=1,\dots,\mu)\,. \tag{3.20}$$

We remark that we have used $B_l(\omega',\Lambda\tau)=\Lambda^{m_l}B_l(\frac{\omega'}{\Lambda},\tau)$.

Now we write $\psi=(\psi',\psi'')$, where ψ' consists of the first μ components of the vector ψ, and ψ'' consists of the other $m-\mu$ components. In the same way we write $\phi=(\phi',\phi'')$. In these notations the system (3.17)–(3.18) can be rewritten in the form

$$\begin{aligned} \psi' \;+\; \Delta_1 H_{12}\psi'' &= \phi'\,,\\ H_{21}\psi' \;+\; \Delta_2\psi'' &= \phi''\,, \end{aligned}$$

where we use the notation

$$\Delta_1:=\begin{pmatrix}\Lambda^{m_1} & & \\ & \ddots & \\ & & \Lambda^{m_\mu}\end{pmatrix},\quad \Delta_2:=\begin{pmatrix}\Lambda^{m_{\mu+1}} & & \\ & \ddots & \\ & & \Lambda^{m_m}\end{pmatrix}$$

and

$$H_{12}:=\Big(h_{lk}\Big)_{\substack{l=1,\dots,\mu\\ k=\mu+1,\dots,m}}\,,\quad H_{21}:=\Big(h_{lk}\Big)_{\substack{l=\mu+1,\dots,m\\ k=1,\dots,\mu}}\,.$$

If me multiply the second equation by the matrix $\Delta_1 H_{12}\Delta_2^{-1}$ from the left and subtract it from the first equation we obtain

$$(I-\Delta_1 H_{12}\Delta_2^{-1}H_{12})\psi'=\phi'-\Delta_1 H_{12}\Delta_2^{-1}\phi''\,.$$

In a similar way we obtain

$$(I-\Delta_2^{-1}H_{21}\Delta_1 H_{12})\psi''=-\Delta_2^{-1}H_{21}\phi'+\Delta_2^{-1}\phi''\,.$$

The matrices in brackets in the left-hand sides of above relations differ from the identity by matrices whose elements can be estimated by a constant times $\Lambda^{m_\mu-m_{\mu+1}}$. According to

(1.3), their norms tend to zero as $\Lambda \to \infty$. From this it follows that the matrices in brackets for large Λ have inverses which we denote by G_1 and G_2, respectively. Then we obtain

$$\begin{aligned} \psi' &= G_1\phi' - G_1\Delta_1 H_{12}\Delta_2^{-1}\phi'' , \\ \psi'' &= -G_2\Delta_2^{-1}H_{21}\phi' + G_2\Delta_2^{-1}\phi'' . \end{aligned}$$

If we take $\phi = e_j \quad (1 \le j \le \mu)$, where e_j stands for the j-th unit vector, and denote by e'_j the first μ components of e_j, we obtain

$$\psi'_{(j)} = G_1 e'_j, \quad \psi''_{(j)} = -G_2\Delta_2^{-1}H_{21}e'_j .$$

In the same way if $j > \mu$ and e''_j denotes the components $\mu+1, \dots, m$ of e_j, we obtain

$$\psi'_{(j)} = -G_1\Delta_1 H_{12}\Lambda^{-m_j}e''_j, \quad \psi''_{(j)} = G_2\Lambda^{-m_j}e''_j .$$

The statement of the lemma directly follows from these relations. □

4. A priori estimate and parametrix construction

Theorem 3.2 is the key result for proving a priori estimates. The norms used in these estimates are based on the Newton polygon $N_{r,s}$ (cf. Fig. 1) defined for $r > s \ge 0$ as the convex hull of the set

$$\{(0,0)\,,\ (0,r-s)\,,\ (s,r-s)\,,\ (r,0)\,\} .$$

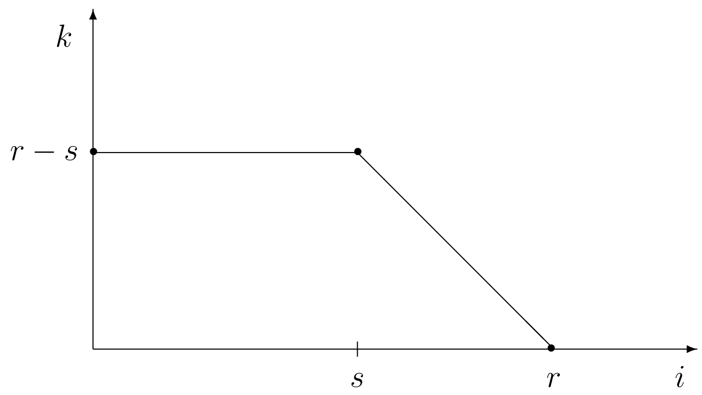

Fig. 1. The Newton polygon $N_{r,s}$.

The weight function $\Xi_{r,s}(\xi,\lambda)$ is defined by

$$\Xi_{r,s}(\xi,\lambda) := \sum_{(i,k)\in N_{r,s}\cap\mathbb{Z}^2} |\xi|^i\,|\lambda|^k . \tag{4.1}$$

For a discussion of general Newton polygons we refer the reader to [4], [5], [7].

It is easily seen (cf. [5], Section 2) that we have the equivalence

$$\Xi_{r,s}(\xi,\lambda) \approx (1+|\xi|)^s(\lambda+|\xi|)^{r-s} . \tag{4.2}$$

The sign $\approx$ means that the quotient of the left-hand and the right-hand side is bounded from below and from above by positive constants independent of ξ and λ. Taking the right-hand side of (4.2) as a definition, we may define $\Xi_{r,s}$ for every $r, s \in \mathbb{R}$. The Sobolev space $H^{(r,s)}(\mathbb{R}^n) := H^{\Xi_{r,s}}(\mathbb{R}^n)$ is defined as

$$\{u \in \mathcal{S}'(\mathbb{R}^n) : \Xi_{r,s}(\xi,\lambda)Fu(\xi) \in L_2(\mathbb{R}^n)\}$$

with the norm

$$\|u\|_{(r,s),\mathbb{R}^n} := \|u\|_{\Xi_{r,s},\mathbb{R}^n} := \|F^{-1}\Xi_{r,s}(\xi,\lambda)Fu(\xi)\|_{L_2(\mathbb{R}^n)} . \tag{4.3}$$

Here Fu stands for the Fourier transform of u and $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of all tempered distributions. The space $H^{\Xi_{r,s}}(\mathbb{R}^{n-1})$ is defined analogously with the weight function $\Xi_{r,s}(\xi',\lambda) := \Xi_{r,s}(\xi',0,\lambda)$. These spaces can be defined on the half-space $\mathbb{R}^n_+$ in accordance with the general theory of Sobolev spaces with weight functions as it can be found, e.g., in [13]. On the manifold M and the boundary ∂M, the spaces $H^{\Xi_{r,s}}(M)$ and $H^{\Xi_{r,s}}(\partial M)$, respectively, are defined in the usual way, using a partition of unity.

In [5], Section 2, Sobolev spaces connected with Newton polygons were investigated in detail. In particular, for $r \in \mathbb{N}$ it was shown that $(\frac{\partial}{\partial\nu})^j : u \mapsto (\frac{\partial}{\partial\nu})^j u|_{\partial M}$ acts continuously from $H^{\Xi_{r,s}}(M)$ to $H^{\Xi_{r,s}^{(-j-1/2)}}(\partial M)$ for $j = 0,\dots,r-1$. Here $\Xi_{r,s}^{(-j-1/2)}$ denotes the weight function corresponding to the Newton polygon which is constructed from $N_{r,s}$ by a shift of length $j+1/2$ to the left parallel to the abscissa. More precisely, the shifted polygon is the convex hull of the points

$$\begin{array}{ll} (0,0),\ (0,r-s),\ (s-j+1/2,r-s),\ (r-j+1/2,0) & \text{if } j-1/2 \le s\,,\\ (0,0),\ (0,r-j+1/2),\ (r-j+1/2,0) & \text{if } s < j-1/2 \le r\,. \end{array}$$

While in [5] the basic Sobolev space was $H^{\Xi_{m,\mu}}$, we now have to deal with more general spaces. For the remainder of this section, we fix integer numbers

$$r \ge m_m + 1 \quad\text{and}\quad m_\mu + 1 \le s \le m_{\mu+1} \tag{4.4}$$

and consider the Newton polygon $N_{r,s}$, its weight function $\Xi := \Xi_{r,s}$ and the corresponding Sobolev space. We remark that for the Dirichlet problem the values $r = m$ and $s = \mu$ used in [5] are included as an example.

Analogously to (4.1), we define the function $\Phi = \Phi_{r,s}$ by

$$\Phi(\xi,\lambda) := \sum_{i,k} |\xi|^i\lambda^k\,, \tag{4.5}$$

where the sum runs over all integer points (i,k) belonging to the side of $N_{r,s}$ which is not parallel to one of the coordinate lines. This means that we have

$$\Phi(\xi,\lambda) \approx |\xi|^s(\lambda+|\xi|)^{r-s}\,. \tag{4.6}$$

By $\Phi^{(-l)}$ we again denote the corresponding function for the shifted Newton polygon. From Theorem 3.2 we obtain the following estimate for the fundamental solution w_j defined in (3.3)–(3.4):

Lemma 4.1. *For the solution $w_j(t,\xi',\lambda)$ considered in Theorem* 3.2 *we have the estimate*

$$\|D_t^l w_j(\cdot,\xi',\lambda)\|_{L_2(\mathbb{R}_+)} \le C\,\frac{\Phi^{(-m_j-1/2)}(\xi',\lambda)}{\Phi^{(-l)}(\xi',\lambda)}\,. \tag{4.7}$$

Proof. To see this, we only have to remark that the right-hand side of (4.7) is equivalent to

$$\begin{cases} |\xi'|^{l-m_j-\frac{1}{2}}, & j\le\mu,\quad l\le s,\\ |\xi'|^{s-m_j-\frac{1}{2}}(\lambda+|\xi'|)^{l-s}, & j\le\mu,\quad l>s,\\ |\xi'|^{l-s}(\lambda+|\xi'|)^{s-m_j-\frac{1}{2}}, & j>\mu,\quad l\le s,\\ (\lambda+|\xi'|)^{l-m_j-\frac{1}{2}}, & j>\mu,\quad l>s. \end{cases}$$

The first and fourth lines above coincide with the corresponding lines in the right-hand side of (3.7). The ratio of the second line in (3.7) and the second line above is equal to

$$\Big(\frac{|\xi'|}{\lambda+|\xi'|}\Big)^{m_{\mu+1}-s+1/2}\,.$$

Respectively, the ratio of the third line in (3.7) and the third line above is equal to

$$\Big(\frac{|\xi'|}{\lambda+|\xi|}\Big)^{s-m_\mu-1/2}\,.$$

Now our statement follows from (4.4). □

In [5] it was shown how an inequality of the form (4.7) leads to the proof of an a priori estimate in terms of the parameter-dependent norms. Following the same steps as in the proof of Theorem 5.6 in [5], we obtain from Lemma 4.1:

Theorem 4.2. *Let $A(x,D,\lambda)$ be an operator pencil of the form* (1.1)*, acting on the manifold M with boundary ∂M. Let $B_j(x,D)$, $j=1,\dots,m$, be boundary operators of the form* (1.2)*. Assume that A degenerates regularly at the boundary and that $(A,B_1,\dots,B_m)$ is N-elliptic with parameter in the sense of Definition* 2.4*. Set $\Xi=\Xi_{r,s}$ with r and s satisfying* (4.4)*. For simplicity, assume that r and s are integers. Then for $\lambda\ge\lambda_0$ there exists a constant $C=C(\lambda_0)$, independent of u and λ, such that*

$$\begin{aligned} \|u\|_{\Xi,M} \;&\le\; C\Big(\|A(x,D,\lambda)u\|_{(r-2m,s-2\mu),M}\\ &\quad+\sum_{j=1}^m \|B_j(x,D)u\|_{\Xi^{(-m_j-1/2)},\partial M}+\lambda^{r-s}\|u\|_{L_2(M)}\Big)\,. \end{aligned} \tag{4.8}$$

Now we want to construct a right (rough) parametrix for the operator $(A,B)=(A,B_1,\dots,B_m)$. We restrict ourselves to the construction of local parametrices in $\mathbb{R}^n$ and $\mathbb{R}^n_+$; after this the definition of the parametrix on the manifold is standard.

Lemma 4.3. *Let $A(x, D, \lambda)$ in (1.1) be N-elliptic in $\mathbb{R}^n$ with coefficients of the form (2.1). Then there exists a bounded operator*

$$P_0 : H^{(r-2m,s-2\mu)}(\mathbb{R}^n) \to H^{(r,s)}(\mathbb{R}^n) \tag{4.9}$$

such that

$$AP_0 = I + T \tag{4.10}$$

where I denotes the identity operator in $H^{(r-2m,s-2\mu)}(\mathbb{R}^n)$ and

$$T : H^{(r-2m,s-2\mu)}(\mathbb{R}^n) \to H^{\Theta}(\mathbb{R}^n) \tag{4.11}$$

is bounded. Here we have set

$$\Theta(\xi, \lambda) := \Xi_{r-2m+1,s-2\mu+1}(\xi, \lambda) \Big[= (1 + |\xi|)\, \Xi_{r-2m,s-2\mu}(\xi, \lambda) \Big] . \tag{4.12}$$

Here and in the following, by a bounded operator we understand a continuous operator with norm bounded by a constant independent of λ.

Proof. We define P_0 as a classical pseudodifferential operator (ps.d.o.) with symbol

$$P_0(x, \xi, \lambda) := \frac{\psi(\xi)}{A^{(0)}(x, \xi, \lambda)}$$

where $\psi \in C^\infty(\mathbb{R}^n)$ is a cut-off function with $\psi \equiv 0$ for $|\xi| \le 1$ and $\psi \equiv 1$ for $|\xi| \ge 2$. The continuity of the operator (4.9) is equivalent to the statement that the L_2-L_2-norm of the operator

$$(1 + |D|^2)^{\frac{s}{2}} (\lambda^2 + |D|^2)^{\frac{r-s}{2}} P_0(x, D, \lambda)(1 + |D|^2)^{\frac{-s+2\mu}{2}} (\lambda^2 + |D|^2)^{\frac{-r+s-2\mu+2m}{2}}$$

can be estimated by a constant independent of λ.

Using standard results on the L_2-boundedness of ps.d.o. (cf. [10], Section 2.4) we have to show the inequalities

$$\psi(\xi) \Big| D_x^\alpha (A^{(0)}(x, \xi, \lambda)^{-1}) \Big| \le C_\alpha (1 + |\xi|)^{-2\mu} (\lambda + |\xi|)^{-2m+2\mu} .$$

For $|\alpha| = 0$ this inequality directly follows from N-ellipticity with parameter, the proof for arbitrary α can be made using the chain rule.

To prove (4.10)–(4.11) we write the operator T in the form

$$T = \tilde{T} + (A(x, D, \lambda) - A^{(0)}(x, D, \lambda))P_0$$

with $\tilde{T}u = A^{(0)}(x, D, \lambda)P_0 u - u$. Noting that

$$A(x, D, \lambda) - A^{(0)}(x, D, \lambda) : H^{(r,s)}(\mathbb{R}^n) \to H^{\Theta}(\mathbb{R}^n)$$

is continuous, it is sufficient to prove (4.11) with T replaced by $\tilde{T}$. As above, this is equivalent to the uniform L_2-L_2 boundedness of

$$(1 + |D|^2)^{-\frac{s-2\mu+1}{2}} (\lambda^2 + |D|^2)^{-\frac{r-s-2m+2\mu}{2}} \tilde{T}\, (1 + |D|^2)^{\frac{s-2\mu}{2}} (\lambda^2 + |D|^2)^{\frac{r-s-2m+2\mu}{2}} .$$

For this it is enough to show that the symbol $\tilde{T}(x,\xi,\lambda)$ of $\tilde{T}$ satisfies

$$(1+|\xi|)\Big|D_x^\beta \tilde{T}(x,\xi,\lambda)\Big| \le C_\beta\,. \tag{4.13}$$

The last inequality follows easily from the fact that for $|\xi|\ge 2$ we have

$$\tilde{T}(x,\xi,\lambda) = \sum_{0<|\alpha|\le 2m} \frac{1}{\alpha!}\partial_\xi^\alpha A^{(0)}(x,\xi,\lambda)\; D_x^\alpha \,\frac{1}{A^{(0)}(x,\xi,\lambda)} \tag{4.14}$$

and from the estimates

$$\Big|D_x^\beta\,\frac{1}{A^{(0)}(x,\xi,\lambda)}\Big| \le C\,(\Xi_{2m,2\mu}(\xi,\lambda))^{-1} \quad (|\xi|\ge 2) \tag{4.15}$$

and

$$|D_x^\gamma \partial_\xi^\alpha A^{(0)}(x,\xi,\lambda)| \le C\;\Xi_{2m,2\mu}^{(-|\alpha|)}(\xi,\lambda) \quad (0\le|\alpha|\le 2m)\,. \tag{4.16}$$

□

Now assume that (A,B) acts in the half space $\mathbb{R}^n_+$, the coefficients of (A,B) are of the form (2.1) and that (A,B) is N-elliptic in the sense of Definition 2.4. We will use the cut-off function $\psi'\in C^\infty(\mathbb{R}^{n-1})$ defined by $\psi'(\xi') := \psi(\xi',0)$ with ψ from the proof of Lemma 4.3.

To define a parametrix, we use a cut-off function $\psi'\in C^\infty(\mathbb{R}^{n-1})$ with

$$\psi'(\xi') = \begin{cases} 0\,, & |\xi'|\le 1\,,\\ 1\,, & |\xi'|\ge 2\,.\end{cases}$$

For $j=1,\dots,m$ we define the ps.d.o. P_j in $\mathbb{R}^{n-1}$ (with x_n as parameter) by

$$(P_j g)(x',x_n) := \psi'(D')\,w_j(x',x_n,D',\lambda)\,g\,, \tag{4.17}$$

where $w_j(x',x_n,\xi',\lambda)$ is the unique solution of (3.3)–(3.4) with

$$\begin{aligned} A(\xi',D_t,\lambda) &= A^{(0)}(x',0,\xi',D_t,\lambda)\,,\\ B_k(\xi',D_t) &= B_k^{(0)}(x',0,\xi',D_t)\,. \end{aligned} \tag{4.18}$$

Due to Lemma 3.3, for large λ the symbol of $w_j(x',x_n,D',\lambda)$ can be written in the form

$$\begin{aligned} w_j(x',x_n,\xi',\lambda) &= \frac{1}{2\pi i}\int_{\gamma^{(1)}} \frac{M_j^{(1)}(x',\xi',\tau,\lambda)}{A_1(x',\xi',\tau,\lambda)}\,e^{ix_n\tau}\,d\tau\\ &+ \frac{1}{2\pi i}\int_{\gamma^{(2)}} \frac{M_j^{(2)}(x',\xi',\tau,\lambda)}{A_2(x',\xi'/\lambda,\tau,1)}\,e^{ix_n\lambda\tau}\,d\tau\,. \end{aligned} \tag{4.19}$$

Lemma 4.4. *The operator P_j defined in* (4.17) *is continuous from $H^{\Xi^{(-m_j-1/2)}}(\mathbb{R}^{n-1})$ to $H^{\Xi}(\mathbb{R}^n_+)$.*

Proof. Let $g \in H^{\Xi^{(-m_j-1/2)}}(\mathbb{R}^{n-1})$ and set $u := P_j g$. Using the equivalent norm

$$\left[\sum_{l=0}^{r} \int_0^\infty \|D_n^l u(\cdot, x_n)\|^2_{\Xi^{(-l)},\mathbb{R}^{n-1}} \, dx_n \right]^{1/2} \tag{4.20}$$

in $H^{\Xi}(\mathbb{R}^n_+)$, we see that we have to show that

$$\left\| \Xi^{(-l)}(D', \lambda) D_n^l P_j \left[\Xi^{(-m_j-1/2)}(D', \lambda) \right]^{-1} \right\|_{L_2(\mathbb{R}^{n-1}) \to L_2(\mathbb{R}^{n-1})} \le C(x_n)$$

for some function $C = C(x_n)$ whose $L_2(\mathbb{R}_+)$-norm is bounded by a constant independent of λ. For this it is sufficient to show that for $|\xi'| \ge 1$ we have

$$\left(\int_0^\infty |D_{x'}^{\alpha'} D_n^l w_j(x', x_n, \xi', \lambda)|^2 \, dx_n \right)^{1/2} \le C \, \frac{\Xi^{(-m_j-1/2)}(\xi', \lambda)}{\Xi^{(-l)}(\xi', \lambda)} \, .$$

As we have for $|\xi'| \ge 1$ the equivalence $\Xi_{r,s}(\xi', \lambda) \approx \Phi_{r,s}(\xi', \lambda)$ for all $r, s \in \mathbb{R}$ (with $\Phi_{r,s}$ defined by the right-hand side of (4.6)), the case $\alpha' = 0$ is already covered by Lemma 4.1. Here we take into account that, due to condition (2.1), the constant C in Lemma 4.1 applied to the symbols (4.18) may be chosen independently of $x' \in \mathbb{R}^{n-1}$.

The case $\alpha' > 0$ follows after differentiation of (4.19) with respect to x' along the same lines as in the proof of Lemma 3.3. □

Lemma 4.5. *The operator*

$$C_j := A(x, D, \lambda) P_j : H^{\Xi^{(-m_j-1/2)}}(\mathbb{R}^{n-1}) \to H^{\Theta}(\mathbb{R}^n_+)$$

is bounded. Here $\Theta(\xi, \lambda)$ *is defined in* (4.12).

Proof. The symbol of the ps.d.o. C_j in $\mathbb{R}^{n-1}$ with parameter x_n is given by

$$\sum_{|\alpha'|=1}^{2m} \frac{1}{(\alpha')!} \partial_{\xi'}^{\alpha'} A(x, \xi', D_n, \lambda) \, D_{x'}^{\alpha'} P_j(x', x_n, \xi', \lambda)$$

with

$$P_j(x, \xi', \lambda) = \psi(\xi') \, w_j(x, \xi', \lambda) \, .$$

Consider the family $\mathcal{F} = \{A(x, \xi, \lambda) : x \in \mathbb{R}^n_+\}$ of polynomials in $(\xi, \lambda) \in \mathbb{R}^{n+1}$. As the degree of the polynomial $A(x, \cdot)$ is equal to $2m$ for all $x \in \mathbb{R}^n_+$, the family $\mathcal{F}$ is a subset of the finite-dimensional vector space of all polynomials in (ξ, λ) of degree not greater than $2m$. Therefore, there exists a finite set $x^{(1)}, \dots, x^{(K)} \in \mathbb{R}^n_+$ such that every $A \in \mathcal{F}$ may be represented in the form

$$A(x, \xi, \lambda) = \sum_{k=1}^{K} c_k(x) \, A(x^{(k)}, \xi, \lambda)$$

with smooth coefficients $c_k(x)$.

Taking into account that the operators of multiplication by $c_k(x)$ are bounded in $H^{\Theta}(\mathbb{R}^n_+)$, we reduce our problem to the proof of the boundedness of operators of the form

$$C_{\alpha',l} := a_{\alpha',l}(D',\lambda)\, D_n^l D_{x'}^{\alpha'} P_j(x,D',\lambda) : H^{\Xi^{(-m_j-1/2)}}(\mathbb{R}^{n-1}) \to H^{\Theta}(\mathbb{R}^n_+)\,,$$

where

$$|a_{\alpha',l}(\xi',\lambda)| \le C\; \Xi^{(-l-1)}_{2m,2\mu}(\xi',\lambda)\,. \tag{4.21}$$

Literally repeating the proof of Lemma 4.4 we establish the boundedness of the operator

$$D_{x'}^{\alpha'} P_j(x,D,\lambda) : H^{\Xi^{(-m_j-1/2)}}(\mathbb{R}^{n-1}) \to H^{\Xi}(\mathbb{R}^n_+)\,.$$

According to (4.21) the operator

$$a_{\alpha',l}(D',\lambda) : H^{\Xi}(\mathbb{R}^n_+) \to H^{\Theta}(\mathbb{R}^n_+)$$

is bounded. As $C_{\alpha',l}$ is the product of of the above operators this operator is also bounded. □

Theorem 4.6. *Consider in the half space $\mathbb{R}^n_+$ the boundary value problem $(A,B) = (A,B_1,\ldots,B_m)$ of the form* (1.1), (1.2) *with coefficients of the form* (2.1). *Assume that A degenerates regularly at the boundary and that (A,B) is N-elliptic with parameter in $[0,\infty)$ in the sense of Definition* 2.4. *Then there exists a bounded operator*

$$P : H^{(r-2m,s-2\mu)}(\mathbb{R}^n_+) \times \prod_{j=1}^m H^{\Xi^{(-m_j-1/2)}}(\mathbb{R}^{n-1}) \to H^{\Xi}(\mathbb{R}^n_+)$$

such that

$$(A,B)P = I + T$$

where I stands for the identity operator in the space

$$H^{(r-2m,s-2\mu)}(\mathbb{R}^n_+) \times \prod_{j=1}^m H^{\Xi^{(-m_j-1/2)}}(\mathbb{R}^{n-1}) \tag{4.22}$$

and T is a continuous operator from the space (4.22) *to the space*

$$H^{\Theta}(\mathbb{R}^n_+) \times \prod_{j=1}^m H^{\Xi^{(-m_j+1/2)}}(\mathbb{R}^{n-1})$$

with $\Theta(\xi,\lambda)$ being defined in (4.12).

Proof. We define

$$P(f,g_1,\ldots,g_m) := RP_0Ef + \sum_{j=1}^m P_j(g_j - B_jP_0f)\,.$$

Here E is a fixed operator of extension from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, R denotes the operator of restriction onto $\mathbb{R}^n_+$, the operator P_0 is given in Lemma 4.3 and P_j $(j = 1, \dots, m)$ is given by (4.17). The continuity of P follows from Lemma 4.3 and Lemma 4.4. In order to see that the operator T is continuous with respect to the spaces given in the theorem, we denote the components of T by $T_0, T_1, \dots, T_m$. The operator T_0 is given by

$$T_0(f, g_1, \dots, g_m) = ARP_0Ef - f + \sum_{j=1}^{m} AP_j(g_j - B_jRP_0Ef)\,.$$

We see from Lemma 4.3 and Lemma 4.5 that T_0 maps the space (4.22) continuously into $H^{\Theta}(\mathbb{R}^n_+)$.

Turning to the other components $T_1, \dots, T_m$, we remark that for $j, k = 1, \dots, m$ the operator B_kP_j equals $\delta_{kj}\, I$ up to operators of lower order. More precisely, the operator

$$B_k(x, D)P_j - \delta_{kj}I$$

is a ps.d.o. in $\mathbb{R}^{n-1}$ which is continuous from

$$H^{\Xi^{(-m_j-1/2)}}(\mathbb{R}^{n-1}) \quad \text{to} \quad H^{\Xi^{(-m_k+1/2)}}(\mathbb{R}^{n-1})\,.$$

This is due to the fact that $w_j(x', x_n, \xi', \lambda)$ satisfies (3.3)–(3.4); the estimates for the lower order terms of the ps.d.o. B_kP_j can be found in the same way as it was done for AP_j in the proof of Lemma 4.5. From the continuity of $B_kP_j - \delta_{kj}I$ the continuity of T_k in the spaces given in the theorem immediately follows. □

Remark 4.7. The main feature of the parametrix constructed in the previous theorem is that the spaces defined in terms of Newton polygons appear. The existence of a parametrix for fixed λ is clear due to the ellipticity of the boundary value problem $(A_{2m}, B_1, \dots, B_m)$ (see Remark 2.5 c)). Similarly, on a compact manifold with boundary, the Fredholm property of (A, B) follows from the ellipticity of $(A_{2m}, B_1, \dots, B_m)$ as (for fixed λ) the norm in $H^{\Xi_{r,s}}$ is equivalent to the standard norm in H^r.

5. Proof of the necessity

The aim of this section is to prove the following theorem.

Theorem 5.1. *Let A degenerate regularly at the boundary ∂M and assume that inequality* (1.3) *holds. Let r and s be integers satisfying* (4.4) *and assume, in addition, that*

$$r \geq m \quad \textit{and} \quad \mu \leq s \leq r - m + \mu\,. \tag{5.1}$$

If the a priori estimate (4.8) *holds, then $(A, B_1, \dots, B_m)$ is N-elliptic with parameter in the sense of Definition* 2.4.

Note that if the orders m_j of the boundary operators B_j are all different (e.g., if the boundary operators are normal), then (1.3) is satisfied and (5.1) is a consequence of (4.4). The proof of this theorem is divided into several steps.

Necessity of condition 2.4 a). First of all note that applying estimate (4.8) to functions whose support does not intersect with the boundary, we obtain the estimate in $\mathbb{R}^n$

$$||u||_{r,s} \leq C\Big(||A(x, D, \lambda)u||_{r-2m,s-2\mu} + \lambda^{r-s}||u||_{L_2}\Big) \tag{5.2}$$

Proposition 5.2. *Suppose that* (5.2) *takes place and* x^0 *is an arbitrary point of* $\mathbb{R}^n$. *Denote* $A(D, \lambda) = A^0(x^0, D, \lambda)$. *Then the estimate*

$$\Big\||D|^s(|D| + \lambda)^{r-s}u\Big\|_{L_2} \leq C\Big\||D|^{s-2\mu}(|D| + \lambda)^{r-s-2m+2\mu}A(D, \lambda)u\Big\|_{L_2} \tag{5.3}$$

holds, where the constant C *does not depend on* x^0 *or* λ.

The necessity of a) easily follows from (5.3). Indeed, applying the Fourier transform, we can rewrite (5.3) in the form

$$\int_{\mathbb{R}^n} \Big[|\xi|^{2s}(|\xi| + \lambda)^{2(r-s)} - C^2|\xi|^{2(s-2\mu)}(|\xi| + \lambda)^{2(r-s-2m+2\mu)}|A(\xi, \lambda)|^2\Big]|(Fu)(\xi)|^2 d\xi \leq 0\,.$$

Since $u \in \mathcal{D}$ is arbitrary, the expression in the square brackets is non-positive. From this part a) follows.

To prove (5.3) we replace in (5.2) λ by $\rho\lambda$ with $\rho > 0$ and $u(x)$ by

$$u_\rho(x) = \rho^{-r+n/2}u(\rho(x - x^0)) \tag{5.4}$$

and tend ρ to $+\infty$. To carry out the calculations we need the following

Lemma 5.3. *Denote*

$$(S_{\rho,x^0}u)(x) = u(\rho(x - x^0)). \tag{5.5}$$

Then for an arbitrary ps.d.o. $a(x, D)$ *we have*

$$\Big[a(x, D)S_{\rho,x^0}u\Big](x) = \Big[S_{\rho,x^0}a(x^0 + \rho^{-1}x, \rho D)u\Big](x)\,. \tag{5.6}$$

Proof. Direct calculation shows that

$$(FS_{\rho,x^0}u)(\xi) = \rho^{-n}\exp(-ix^0\xi)(Fu)\Big(\frac{\xi}{\rho}\Big)\,.$$

If we substitute the last expression in the left-hand side of (5.6) and change ξ to $\rho\xi$ we obtain the right-hand side of (5.6). □

Proof of Proposition 5.2. Applying the a priori estimate (5.2) to the function u_ρ (cf. (5.4)), we obtain, according to the lemma,

$$((1 + |D|)^s(\rho\lambda + |D|)^{r-s}u_\rho)(x) = \rho^{n/2}S_{\rho,x^0}\Big[(\rho^{-1} + |D|)^s(\lambda + |D|)^{r-s}u\Big](x)\,.$$

The $L_2(\mathbb{R}^n)$ norm of this expression tends to the left-hand side of (5.3), as ρ tends to $+\infty$.

Now we turn to the right-hand side of (5.2). We have

$$\begin{aligned}&(1+|D|)^{s-2\mu}(\rho\lambda+|D|)^{r-s-2m+2\mu}A(x,D,\rho\lambda)u_\rho(x)\\ &\quad=\rho^{n/2}S_{\rho,x^0}\Big[(\rho^{-1}+|D|)^{s-2\mu}(\lambda+|D|)^{r-s-2m+2\mu}h_\rho\Big](x)\end{aligned}$$

where

$$h_\rho(x)=\rho^{-2m}A(x^0+\rho^{-1}x,\rho D,\rho\lambda)u(x)\quad\to\quad A(D,\lambda)u$$

as $\rho\to+\infty$.

It is easy to check that the limit of the second term of the right-hand side of (5.2) is equal to zero. □

To prove the necessity of 2.4 b), c) and d) we consider (4.8) for functions with supports belonging to a small neighbourhood of a point $x^0\in\partial M$. In this case the norms in (4.8) can be taken in $\mathbb{R}^n_+$ and $\mathbb{R}^{n-1}$, respectively. Now we use the fact that we have the norm equivalence (4.20) and the equivalence

$$\Xi^{(-l)}(\xi,\lambda)\approx\begin{cases}(1+|\xi|)^{s-l}\,(\lambda+|\xi|)^{r-s}\,, & l\le s\,,\\ (\lambda+|\xi|)^{r-l}\,, & l>s\,.\end{cases}$$

According to [5], Section 2, the norm

$$\|(iD_n+\sqrt{1+|D'|^2})^q\,(iD_n+\sqrt{\lambda^2+|D'|^2})^{p-q}u\|_{L_2(\mathbb{R}^n_+)}$$

is defined for any $p,q\in\mathbb{R}$ and is equivalent to $\|u\|_{(p,q),\mathbb{R}^n_+}$ (cf. (4.3)). Substituting these expressions into the a priori estimate (4.8) for the half space, we obtain in explicit form

$$\begin{aligned}&\sum_{l=0}^{s}\Big\|(1+|D'|^2)^{(s-l)/2}(\lambda^2+|D'|^2)^{(r-s)/2}D_n^l u\Big\|_{L_2(\mathbb{R}^n_+)}\\ &+\sum_{l=s+1}^{r}\Big\|(\lambda^2+|D'|^2)^{(r-l)/2}D_n^l u\Big\|_{L_2(\mathbb{R}^n_+)}\\ &\le\ C\Bigg(\|\sigma(D,\lambda)A(x,D,\lambda)u\|_{L_2(\mathbb{R}^n_+)}+\lambda^{r-s}||u||_{L_2(\mathbb{R}^n_+)}\\ &\quad+\sum_{j=1}^{\mu}\Big\|(1+|D'|^2)^{(s-m_j-1/2)/2}(\lambda^2+|D'|^2)^{(r-s)/2}B_j(x',D)u\Big\|_{L_2(\mathbb{R}^{n-1})}\\ &\quad+\sum_{j=\mu+1}^{m}\Big\|(\lambda^2+|D'|^2)^{(r-m_j-1/2)/2}B_j(x',D)u\Big\|_{L_2(\mathbb{R}^{n-1})}\Bigg)\,, \qquad (5.7)\end{aligned}$$

where we used the abbreviation

$$\sigma(D,\lambda):=(iD_n+\sqrt{1+|D'|^2})^{s-2\mu}(iD_n+\sqrt{\lambda^2+|D'|^2})^{r-s-2m+2\mu}\,.$$

Proposition 5.4. *Suppose estimate (5.7) holds. Let x^0 be an arbitrary point in $\mathbb{R}^{n-1}$ and set $A(D,\lambda) = A^{(0)}(x^0, D, \lambda)$, $\quad B_j(D,\lambda) = B_j^{(0)}(x^0, D, \lambda)$, $j = 1, \ldots, m$. Then the following estimate holds*

$$\begin{aligned}
&\sum_{l=0}^{s} \Big\| |D'|^{s-l} (\lambda^2 + |D'|^2)^{(r-s)/2} D_n^l u \Big\|_{L_2(\mathbb{R}^n_+)} \\
&+ \sum_{l=s+1}^{r} \Big\| (\lambda^2 + |D'|^2)^{(r-l)/2} D_n^l u \Big\|_{L_2(\mathbb{R}^n_+)} \\
&\le \; C \Bigg(\| \tilde{\sigma}(D,\lambda) A(D,\lambda) u \|_{L_2(\mathbb{R}^n_+)} \\
&+ \; \sum_{j=1}^{\mu} \Big\| |D'|^{s-m_j-1/2} (\lambda^2 + |D'|^2)^{(r-s)/2} B_j(D) u \Big\|_{L_2(\mathbb{R}^{n-1})} \\
&+ \; \sum_{j=\mu+1}^{m} \Big\| (\lambda^2 + |D'|^2)^{(r-m_j-1/2)/2} B_j(D) u \Big\|_{L_2(\mathbb{R}^{n-1})} \Bigg), \qquad (5.8)
\end{aligned}$$

where we have set

$$\tilde{\sigma}(D,\lambda) := (iD_n + |D'|)^{s-2\mu} (iD_n + \sqrt{\lambda^2 + |D'|^2})^{r-s-2m+2\mu}.$$

Proof. We apply (5.7) with λ replaced by $\rho\lambda$ to the function u_ρ defined in (5.4), noting that $S_{\rho,x^0} u$ is again defined in $\mathbb{R}^n_+$ because of $x^0 \in \mathbb{R}^{n-1}$ and $\rho > 0$. From Lemma 5.3 and the fact that for any function $v \in L_2(\mathbb{R}^n_+)$ we have

$$\rho^{n/2} \| S_{\rho,x^0} v \|_{L_2(\mathbb{R}^n_+)} = \| v \|_{L_2(\mathbb{R}^n_+)},$$

we see that the l-th term in the first sum in (5.7) is equal to

$$\Big\| \Big(\frac{1}{\rho^2} + |D'|^2 \Big)^{(s-l)/2} \Big(\lambda^2 + |D'|^2 \Big)^{(r-s)/2} D_n^l u \Big\|_{L_2(\mathbb{R}^n_+)}$$

which tends to the corresponding term in (5.8) for $\rho \to \infty$. The remaining expressions in (5.7) can be treated analogously; the term $(\rho\lambda)^{r-s} \| u_\rho \|_{L_2(\mathbb{R}^n_+)}$ tends to zero for $\rho \to \infty$.

For the terms involving the boundary operators we remark that $\gamma_0 S_{\rho,x^0} = S_{\rho,x^0} \gamma_0$ where $\gamma_0 : u \mapsto u(\cdot, 0)$ stands for the trace operator. Therefore we may apply Lemma 5.3 to the function $B_j(x', D) u_\rho$ defined in $\mathbb{R}^{n-1}$. □

If we apply (5.8) to a function of the form

$$u(x) = \phi(x') V(x_n), \quad \phi(x') \in \mathcal{D}(\mathbb{R}^{n-1})$$

we obtain an estimate on the half-line (cf. [8], Chapter 3, Proposition 2 in Subsection 2.3):

$$\begin{aligned}
&\sum_{l=0}^{s} |\xi'|^{s-l}(\lambda^2+|\xi'|^2)^{(r-s)/2}||D_n^l V||_{L_2(\mathbb{R}_+)} \\
&+ \sum_{l=s+1}^{r} (\lambda^2+|\xi'|^2)^{(r-l)/2}||D_n^l V||_{L_2(\mathbb{R}_+)} \\
&\le C\Bigg(\|\tilde\sigma(\xi',D_n,\lambda)A(\xi',D_n,\lambda)V\|_{L_2(\mathbb{R}_+)} \\
&+ \sum_{j=1}^{\mu} |\xi'|^{s-m_j-1/2}(\lambda^2+|\xi'|^2)^{(r-s)/2}|B_j(\xi',D_n)V(0)| \\
&+ \sum_{j=\mu+1}^{m} (\lambda^2+|\xi'|^2)^{(r-m_j-1/2)/2}|B_j(\xi',D_n)V(0)| \Bigg) . \qquad (5.9)
\end{aligned}$$

Necessity of condition 2.4 b). Suppose $V(x_n) \in L_2(\mathbb{R}_+)$ is a solution of the homogeneous equation

$$A(\xi',D_n,\lambda)V(x_n)=0, \quad x_n>0\,.$$

Then this function satisfies the equation

$$A_+(\xi',D_n,\lambda)V(x_n)=0, \quad x_n>0. \qquad (5.10)$$

Now from (5.9) we deduce the estimate

$$c(\xi',\lambda)\sum_{l=0}^{r}||D_n^l V||_{L_2(\mathbb{R}_+)} \le \sum_{j=1}^{m}|B_j'(\xi',\lambda,D_n)V(0)|\,. \qquad (5.11)$$

Here B_j' are remainders of B_j after the division by A_+ and $c(\xi,\lambda)>0$ for $\xi'\ne 0$ and $\lambda\ge 0$. From a standard trace result for Sobolev spaces on $\mathbb{R}_+$ we know that

$$\sum_{j=1}^{r}|D_n^{j-1}V(0)| \le C\sum_{j=1}^{r+1}\|D_n^{j-1}V\|_{L_2(\mathbb{R}_+)}\,. \qquad (5.12)$$

From this and (5.11) we obtain, using $r\ge m$ (see (5.1)),

$$\tilde c(\xi',\lambda)\sum_{j=1}^{m}|D_n^{j-1}V(0)| \le \sum_{j=1}^{m}\Big|\sum_{k=1}^{m} b_{jk}(\xi',\lambda)D_n^{k-1}V(0)\Big|, \qquad (5.13)$$

where

$$B_j'(\xi',\lambda,z)=\sum_{k=1}^{m} b_{jk}(\xi',\lambda)z^{k-1}\,. \qquad (5.14)$$

The constant $\tilde c(\xi',\lambda)$ in (5.13) is positive for $\xi'\ne 0$ and $\lambda\ge 0$. Note that the Cauchy problem

$$D_n^{k-1}V(0)=\zeta_k, \quad k=1,\dots,m$$

for ODE (5.10) has a unique solution for arbitrary $\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{C}^m$. This means that for an arbitrary complex vector ζ we have the estimate

$$\tilde{c}(\xi', \lambda)|\zeta| \le |\mathcal{B}(\xi', \lambda)\zeta|$$

where $\mathcal{B}(\xi', \lambda) := (b_{jk}(\xi', \lambda))_{j,k=1,\dots,m}$. The last inequality means that the matrix $\mathcal{B}(\xi', \lambda)$ is nonsingular as $|\xi'| \neq 0, \lambda \ge 0$, i.e. the necessity of b) is proved.

Proposition 5.5. *Suppose the estimate* (5.7) *holds. Let x^0 be an arbitrary point of $\mathbb{R}^{n-1}$. Then the inequality*

$$\begin{aligned} \sum_{l=0}^{s} \left\| |D'|^{s-l} D_n^l u \right\|_{L_2(\mathbb{R}^n_+)} \le\ & C\Bigg(\left\| (iD_n + |D'|)^{s-2\mu} A_{2\mu}(D) u \right\|_{L_2(\mathbb{R}^n_+)} \\ &+ \sum_{j=1}^{\mu} \left\| |D'|^{s-m_j-1/2} B_j(D) u \right\|_{L_2(\mathbb{R}^{n-1})} \Bigg) \end{aligned} \tag{5.15}$$

holds, where $A_{2\mu}(D) = A_{2\mu}^{(0)}(x^0, D),\ B_j(D) = B_j^{(0)}(x^0, D),\ j = 1, \dots, \mu.$

Proof. This can be seen in exactly the same way as Proposition 5.4, now applying the a priori estimate (5.7) with $\rho^t\lambda$ instead of λ to the function

$$u_\rho(x) := \rho^{-t(r-s)-s+n/2} u(\rho(x - x^0))$$

where $t > 1$ is fixed and $\rho > 0$ tends to infinity. □

Necessity of condition 2.4 c). From Proposition 5.5 the estimate on the half-line can be obtained

$$\begin{aligned} \sum_{l=0}^{s} |\xi'|^{s-l} ||D_n^l V||_{L_2(\mathbb{R}_+)} \le\ & C\Bigg(\left\| (iD_n + |\xi'|)^{s-2\mu} A_{2\mu}(\xi', D_n) V \right\|_{L_2(\mathbb{R}_+)} \\ &+ \sum_{j=1}^{\mu} |\xi'|^{s-m_j-1/2} |B_j(\xi', D_n) V(0)| \Bigg). \end{aligned}$$

As above we see that for solutions $V(x_n) \in L_2(\mathbb{R}_+)$ of

$$A_{2\mu}(\xi', D_n) V(x_n) = 0, \quad x_n > 0$$

we obtain the inequality

$$\sum_{l=1}^{s} |D_n^{l-1} V(0)| \le C \sum_{j=1}^{\mu} |B_j'(\xi', \lambda, D_n) V(0)| \tag{5.16}$$

with a constant C independent of ξ', $|\xi'| = 1$, and λ, where now B_j' denotes the remainder of B_j after division by $(A_{2\mu})_+$. Replacing in (5.16) the germ of V in 0 by an arbitrary vector $\zeta \in \mathbb{C}^\mu$ and using $s \ge \mu$ (see (5.1)), we obtain the necessity of c).

Proposition 5.6. *Suppose the estimate* (5.7) *holds and* x^0 *is an arbitrary point of* $\mathbb{R}^{n-1}$. *Then the estimate*

$$\sum_{l=s}^{r} ||D_n^l u||_{L_2(\mathbb{R}^n_+)} \leq C\Bigg(||(D_n - i)^{r-s-2m+2\mu} D_n^s Q(x^0, D_n)u||_{L_2(\mathbb{R}^n_+)} + \sum_{j=\mu+1}^{m} ||B_j^{(0)}(x^0, 0, D_n)u||_{L_2(\mathbb{R}^{n-1})}\Bigg) \tag{5.17}$$

holds.

Proof. We apply (5.7) with λ replaced by ρ to the function

$$u_\rho(x) := \rho^{1/2+\varepsilon(n-1)/2-r} u\big(\rho^\varepsilon(x' - x^0), \rho x_n\big)$$

with $0 < \varepsilon < 1$ fixed. Now we use

$$u\big(\rho^\varepsilon(x' - x^0), \rho x_n\big) = \Big[S^{(x')}_{\rho^\varepsilon, x^0}\, S^{(x_n)}_{\rho,0} u\Big](x)\,,$$

where $S^{(x')}_{\dots}$ indicates that the operator $S_{\dots}$ acts on the first $n-1$ variables (and analogously that $S^{(x_n)}_{\dots}$ acts on the last variable), and apply Lemma 5.3 twice. For the l-th term in the first sum of (5.7) we obtain the expression

$$\rho^{(1-\varepsilon)(l-s)} \Big\|(\rho^{-2\varepsilon} + |D'|^2)^{(s-l)/2}(1 + \rho^{2(\varepsilon-1)}|D'|^2)^{(r-s)/2} D_n^l u\Big\|_{L_2(\mathbb{R}^n_+)}.$$

For $l \leq s-1$ this expression tends to zero for $\rho \to \infty$, for $l = s$ its limit equals $\|D_n^s u\|_{L_2(\mathbb{R}^n_+)}$.

The remaining terms can be treated analogously; to finish the proof we use

$$\rho^{-2m} A\Big(x^0 + \frac{x'}{\rho^\varepsilon}, \frac{x_n}{\rho}, \rho^\varepsilon D', \rho D_n, \rho\Big) \;\to\; D_n^{2\mu} Q(x^0, D_n)$$

and

$$\rho^{-m_j} B_j\Big(x^0 + \frac{x'}{\rho^\varepsilon}, \rho^\varepsilon D', \rho D_n\Big) \;\to\; B_j^{(0)}(x^0, 0, D_n)$$

as $\rho \to \infty$. □

Necessity of condition 2.4 d). From Proposition 5.6 we obtain the estimate on the half-line

$$\sum_{l=s}^{r} ||D_n^l V||_{L_2(\mathbb{R}_+)} \leq C\Bigg(\Big\|(D_n - i)^{r-s-2m+2\mu} D_n^s Q(x^0, D_n)V\Big\|_{L_2(\mathbb{R}_+)} + \sum_{j=\mu+1}^{m} |B_j^{(0)}(x^0, 0, D_n)V(0)|\Bigg). \tag{5.18}$$

Since $m_j \geq s$ for $j \geq \mu + 1$, each $B_j^{(0)}(x^0, 0, \tau)$ contains the factor τ^s, and it is easily seen that condition d) follows from the analogous condition for

$$\tilde{B}_j^{(0)}(x^0, 0, \tau) := \tau^{-s} B_j^{(0)}(x^0, 0, \tau)\,.$$

Now we apply (5.18) to a solution $V \in L_2(\mathbb{R}_+)$ of

$$Q(x^0, D_n)V(x_n) = 0, \quad x_n > 0$$

and substitute $W(x_n) := D_n^s V(x_n)$. We obtain

$$\sum_{l=1}^{r-s+1} \|D_n^{l-1} W\|_{L_2(\mathbb{R}_+)} \le C \sum_{j=\mu+1}^{m} |B_j'(x^0, 0, D_n)W(0)|, \tag{5.19}$$

where now B_j' stands for the remainder of $\tilde{B}_j^{(0)}$ after division by Q_+. Using $r - s \ge m - \mu$ (cf. (5.1)) and the trace result (5.12), we obtain the linear independence of B_j' modulo Q_+ from (5.19) and therefore condition d).

Appendix. Singularly perturbed problems

One of the most important features of the Newton polygon approach is to provide an easy formulation and proof of a priori estimates in the theory of singularly perturbed problems. All results of the previous sections can be rewritten for boundary value problems with small parameter as treated by Vishik–Lyusternik [12], Nazarov [11], Frank [6] and others. As an example, we formulate an a priori estimate for such problems.

Consider for $\varepsilon > 0$ the operator

$$A_\varepsilon(x, D) := \varepsilon^{2m-2\mu} A_{2m}(x, D) + \varepsilon^{2m-2\mu-1} A_{2m-1}(x, D) + \ldots + A_{2\mu}(x, D)$$

with $A_j = \sum_{|\alpha| \le j} a_{\alpha j}(x) D^\alpha$. Let A_ε act on a smooth compact manifold M with boundary ∂M and assume that we have boundary conditions $B_1(x, D), \ldots, B_m(x, D)$ of the form (1.2) satisfying (1.3).

We fix integer numbers r and s satisfying (4.4) and consider the weight function

$$\Xi_\varepsilon(\xi) := \Xi_{\varepsilon,(r,s)}(\xi) := (1 + |\xi|)^s \, (1 + \varepsilon|\xi|)^{r-s} .$$

The norms corresponding to this weight function will be denoted by
$\| \cdot \|_{\Xi_\varepsilon, M} = \| \cdot \|_{\varepsilon,(r,s),M}$.

Definition A.1. a) The operator $A_\varepsilon(x, D)$ is called N-elliptic if

$$|A_\varepsilon^{(0)}(x, \xi)| \ge C|\xi|^{2\mu} \, (1 + \varepsilon|\xi|)^{2m-2\mu} \quad (\xi \in \mathbb{R}^n \,, \ \varepsilon > 0 \,, \ x \in \overline{M})$$

holds where C does not depend on x, ξ or ε.

b) The operator A_ε is said to degenerate regularly at the boundary if the polynomial

$$Q(x^0, \tau) := \tau^{-2\mu} A_1^{(0)}(x^0, 0, \tau)$$

has exactly $m - \mu$ roots in the upper half plane.

Definition A.2. The boundary problem $(A_\varepsilon, B_1, \dots, B_m)$ is called N-elliptic if the following conditions hold:

a) The operator $A_\varepsilon(x, D)$ is N-elliptic in the sense of Definition A.1.

b) For every fixed $x^0 \in \partial M$ the boundary problem

$$\Big(A_\varepsilon^{(0)}(x^0, \xi', D_n),\ B_1^{(0)}(x^0, \xi', D_n), \dots, B_m^{(0)}(x^0, \xi', D_n)\Big)$$

for each $\varepsilon > 0$ and $\xi' \neq 0$ is uniquely solvable on the half-line $x_n \geq 0$ in the space of functions tending to zero as $x_n \to \infty$. Moreover we suppose that the problem

$$\Big(A_{2m}^{(0)}(x^0, \xi', D_n),\ B_1^{(0)}(x^0, \xi', D_n), \dots, B_m^{(0)}(x^0, \xi', D_n)\Big)$$

(corresponding to $\varepsilon = \infty$) has the same property.

c) For every $x^0 \in \partial M$ the boundary problem

$$(A_{2\mu}(x^0, D), B_1(x^0, D), \dots, B_\mu(x^0, D))$$

fulfills the Shapiro–Lopatinskii condition.

d) For every $x^0 \in \partial M$ the polynomials $(B_j^{(0)}(x^0, 0, \tau))_{j=\mu+1,\dots,m}$ are linearly independent modulo $Q_+(x^0, \tau)$ with Q_+ defined in Definition 2.4 d).

If the conditions of Definition A.1 and A.2 hold, we can apply Theorem 4.2 to the operator

$$A(x, D, \lambda) := \lambda^{2m-2\mu} A_{1/\lambda}(x, D)\,.$$

The connection between $\Xi_\varepsilon(\xi)$ and $\Xi(\xi, \varepsilon^{-1})$ (defined in (4.2)) is given by

$$\Xi_\varepsilon(\xi) = \varepsilon^{r-s}\, \Xi(\xi, \varepsilon^{-1})$$

and

$$\Xi_\varepsilon^{(-m_j-1/2)}(\xi) = \begin{cases} \varepsilon^{r-s}\, \Xi^{(-m_j-1/2)}(\xi, \varepsilon^{-1}) & \text{if } j \leq \mu\,, \\ \varepsilon^{r-m_j-1/2}\, \Xi^{(-m_j-1/2)}(\xi, \varepsilon^{-1}) & \text{if } j > \mu\,. \end{cases}$$

Using these relations, we obtain from Theorem 4.2 the following result which can be found (without the notation of the Newton polygon) in [6]:

Theorem A.3. *Assume that A_ε degenerates regularly and that $(A_\varepsilon, B_1, \dots, B_m)$ is N-elliptic in the sense of Definition* A.2. *Then the following a priori estimate holds with a constant C independent of $\varepsilon > 0$:*

$$\begin{aligned} \|u\|_{\Xi_\varepsilon, M} \quad &\leq \quad C\Bigg(\|A_\varepsilon u\|_{\varepsilon,(r-2m,s-2\mu),M} + \sum_{j=1}^{\mu} \|B_j u\|_{\Xi_\varepsilon^{(-m_j-1/2)},\partial M} \\ &+ \quad \sum_{j=\mu+1}^{m} \varepsilon^{m_j+1/2-s}\, \|B_j u\|_{\Xi_\varepsilon^{(-m_j-1/2)},\partial M} + \|u\|_{L_2(M)} \Bigg)\,. \end{aligned}$$

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2000 Mathematics Subject Classification: 35J40, 46E35.