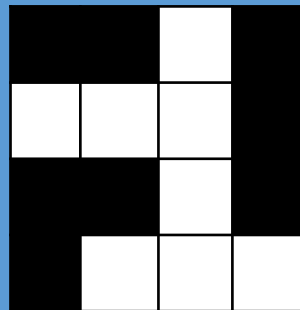
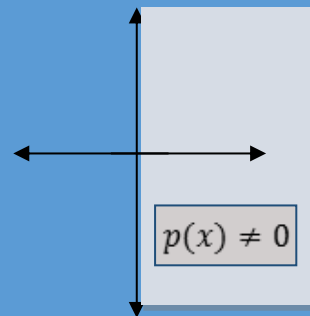


Perturbation and Intervals of Totally Nonnegative Matrices and Related Properties of Sign Regular Matrices

$A \pm tE_{ij}$

$A \leq^* Z \leq^* B$

$$\begin{pmatrix} a_{11} & a_{12} & ? \\ a_{21} & a_{22} & a_{23} \\ ? & a_{32} & a_{33} \end{pmatrix}$$



Mohammad Adm

Konstanz 2016

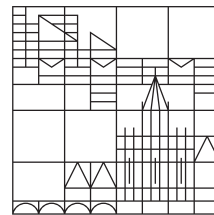
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Perturbation and Intervals of Totally Nonnegative Matrices and Related Properties of Sign Regular Matrices

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Mohammad Adm

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Referent: Prof. Dr. Jürgen Garloff, University of Konstanz

Referent: Prof. Dr. Shaun Fallat, University of Regina

Abstract

A real matrix is called *sign regular* if all of its minors of the same order have the same sign or are allowed to vanish and is said to be *totally nonnegative* and *totally positive* if all of its minors are nonnegative and positive, respectively. Such matrices arise in a remarkable variety of ways in mathematics and many areas of its applications such as differential and integral equations, function theory, approximation theory, matrix analysis, combinatorics, reductive Lie groups, numerical mathematics, statistics, computer aided geometric design, mathematical finance, and mechanics.

A great deal of papers is devoted to the problem of determining whether a given n -by- n matrix is totally nonnegative or totally positive. One could calculate each of its minors, but that would involve evaluating about $4^n/\sqrt{\pi n}$ subdeterminants for a given n -by- n matrix. The question arises whether there is a smaller collection of minors whose nonnegativity or positivity implies the nonnegativity or positivity, respectively, of all minors. In this thesis we derive a condensed form of the Cauchon Algorithm which provides an efficient criterion for total nonnegativity of a given matrix and give an optimal determinantal test for total nonnegativity. By this test we reduce the number of needed minors to only n^2 .

It has long been conjectured, by the supervisor of the thesis, that each matrix in a matrix interval with respect to the checkerboard partial ordering is nonsingular totally nonnegative if the two corner matrices are so, i.e., the so-called *interval property* holds. Invariance of the totally nonnegativity under element-wise perturbation lays at the core of this conjecture. The conjecture was affirmatively answered for the totally positive matrices and some subclasses of the totally nonnegative matrices. In this thesis we settle the conjecture and solve the perturbation problem for totally nonnegative matrices under perturbation of their single entries. The key in settling the conjecture is that the entries of the matrix that is obtained by the application of the Cauchon Algorithm to a given nonsingular totally nonnegative matrix can be represented as ratios of two minors formed from consecutive rows and columns of the given matrix.

Several analogous results hold for nonsingular *totally nonpositive* matrices, i.e., matrices having all of their minors nonpositive. By using the Cauchon Algorithm we derive a characterization of these matrices and an optimal determinantal test for their recognition. We also prove that these matrices and the nonsingular almost strictly sign regular matrices, a subclass of the sign regular matrices, possess the interval property. These and related results evoke the (open) question whether the interval property holds for general nonsingular sign regular matrices.

A *partial totally nonnegative (positive)* matrix has specified and unspecified entries, and all minors consisting of specified entries are nonnegative (positive). A *totally nonnegative (positive) completion problem* asks which partial totally nonnegative (positive) matrices allow a choice of the unspecified entries such that the resulting matrix is totally nonnegative

(positive). Much is known about the totally nonnegative completion problem which is easier to solve than the totally positive completion problem, while very little is known about the latter problem. In this thesis we define new patterns of partial totally positive matrices which are totally positive completable. These patterns settle partially two conjectures which are recently posed by Johnson and Wei.

Finally, a real polynomial is called *Hurwitz* or *stable* if all of its roots have negative real parts. The importance of the Hurwitz polynomials arises in many scientific fields, for example in control theory and dynamical system theory. A rational function is called an *R-function of negative type* if it maps the open upper complex half plane into the open lower half plane. With these kinds of polynomials and rational functions some special structured matrices are associated which enjoy important properties like total nonnegativity. Conversely, from the total nonnegativity of these matrices several properties of polynomials and rational functions associated with these matrices can be inferred. We use these connections to give new and simple proofs of some known facts, prove some properties of these structured matrices, and derive new results for interval polynomials and "interval" *R*-functions of negative type.

Zusammenfassung

Eine reelle Matrix wird *zeichenfest* genannt, wenn alle ihre Minoren gleicher Ordnung dasselbe Vorzeichen besitzen oder verschwinden; sie heißt *total nichtnegativ* bzw. *total positiv*, wenn ihre sämtlichen Minoren nichtnegativ bzw. positiv sind. Derartige Matrizen treten in einer bemerkenswerten Vielzahl von Gebieten der Mathematik und ihren Anwendungen auf, so z. B. in den Differential- und Integralgleichungen, Funktionen-, Approximations- und Matrixtheorie, Kombinatorik, reduktiven Lie-Gruppen, Numerischer Mathematik, Statistik, Computer Aided Geometric Design, Mathematischen Finanzökonomie und Mechanik.

Eine große Anzahl von Arbeiten beschäftigt sich mit dem Problem zu entscheiden, ob eine gegebene quadratische Matrix n -ter Ordnung total nichtnegativ (positiv) ist. Man könnte jeden Minor der Matrix berechnen, aber das würde die Auswertung von ungefähr $4^n/\sqrt{\pi n}$ Unterdeterminanten der gegebenen Matrix erfordern. Die Frage erhebt sich, ob es eine geringere Anzahl von Minoren gibt, von deren Nichtnegativität bzw. Positivität auf die Nichtnegativität bzw. Positivität sämtlicher Minoren der Matrix geschlossen werden kann. In der vorliegenden Arbeit geben wir eine komprimierte Form des Cauchon-Algorithmus an, die ein effektives Kriterium für die totale Nichtnegativität einer Matrix ermöglicht, und leiten einen optimalen Determinantentest für den Nachweis der totalen Nichtnegativität her. Mit Hilfe dieses Testes lässt sich die Anzahl der zu untersuchenden Minoren auf nur n^2 reduzieren.

Lange Zeit wurde von dem Betreuer dieser Arbeit vermutet, dass jede Matrix aus einem Matrixintervall bezüglich der Schachbrett-Halbordnung regulär und total nichtnegativ ist, wenn die beiden Eckmatrizen des Intervalls diese Eigenschaft besitzen, d. h. die sogenannte *Intervall-Eigenschaft* gilt. Die Frage der Erhaltung der totalen Nichtnegativität einer Matrix bei Störung ihrer einzelnen Koeffizienten ist hiermit eng verknüpft. Die Vermutung wurde bestätigt für die total positiven Matrizen und einige Unterklassen der total nichtnegativen Matrizen. In der vorliegenden Arbeit beweisen wir die Vermutung und lösen das Problem der Erhaltung der totalen Nichtnegativität bei Störung einzelner Matrixkoeffizienten. Der Schlüssel zum Beweis der Vermutung ist der Sachverhalt, dass die Koeffizienten der Matrix, die mittels Anwendung des Cauchon-Algorithmus auf eine reguläre und totalnichtnegative Matrix gewonnen wird, dargestellt werden können als Quotient zweier Minoren, die aus aufeinanderfolgenden Zeilen und Spalten der Ausgangsmatrix gebildet werden.

Verschiedene entsprechende Ergebnisse sind für die regulären *total nichtpositiven* Matrizen gültig, d. h. für Matrizen, deren sämtliche Minoren nichtpositiv sind. Mit Hilfe des Cauchon-Algorithmus leiten wir eine Charakterisierung dieser Matrizen und einen optimalen Determinantentest zu ihrer Erkennung her. Wir zeigen ferner, dass diese Matrizen und die regulären fast streng zeichenfesten¹⁾ (*almost strictly sign regular*) Matrizen, eine

¹⁾ Wenn uns ein entsprechender deutscher Begriff aus der Literatur nicht bekannt ist, verwenden wir die Übersetzung des englischen Begriffs und geben diesen in Klammern an.

Unterklasse der zeichenfesten Matrizen, die Intervall-Eigenschaft besitzen. Diese und verwandte Resultate legen die (offene) Frage nahe, ob die Intervall-Eigenschaft auch allgemein für die regulären zeichenfesten Matrizen gilt.

Eine total nichtnegative bzw. positive Teilmatrix (*partial matrix*) ist eine Matrix, die sowohl spezifizierte als auch unspezifizierte Koeffizienten besitzt, wobei die Minoren, die nur aus spezifizierten Koeffizienten gebildet werden, sämtlich nichtnegativ bzw. positiv sind. Ein *total nichtnegatives* bzw. *positives Vervollständigungsproblem* (*completion problem*) fragt, welche total nichtnegativen bzw. positiven Teilmatrizen eine Wahl sämtlicher ihrer unspezifizierten Koeffizienten erlauben derart, dass die jeweilige vervollständigte Matrix total nichtnegativ bzw. positiv ist. Über das total nichtnegative Vervollständigungsproblem sind viele Ergebnisse bekannt; dieses ist einfacher zu lösen als das total positive Problem, über welches erst sehr wenige Resultate vorliegen. In dieser Arbeit geben wir neue Muster von total positiven Teilmatrizen an, welche eine total positive Vervollständigung erlauben. Diese Muster bestätigen teilweise zwei Vermutungen aus einer kürzlich erschienenen Arbeit von Johnson und Wei.

Ein Polynom heißt (*Hurwitz*-)stabil, wenn sämtliche seiner Nullstellen negativen Realteil besitzen. Derartige Polynome treten in vielen Gebieten auf, so z. B. in der Regelungstheorie und bei dynamischen Systemen. Eine rationale Funktion wird eine *R-Funktion von negativem Typ* genannt, wenn sie die obere offene komplexe Halbebene auf die untere offene Halbebene abbildet. Mit den stabilen Polynomen und den *R-Funktionen vom negativen Typ* sind strukturierte Matrizen verknüpft, die wichtige Eigenschaften wie die der totalen Nichtnegativität besitzen. Umgekehrt können aus der totalen Nichtnegativität dieser Matrizen verschiedene Eigenschaften der zugrundeliegenden Polynome und rationalen Funktionen gefolgert werden. Wir verwenden diese Beziehungen, um neue und einfache Beweise für einige bekannte Sachverhalte anzugeben, Eigenschaften dieser strukturierten Matrizen zu beweisen und neue Ergebnisse für Intervallpolynome und "Intervall"-*R-Funktionen vom negativen Typ* herzuleiten.

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1. Introduction

Several types of matrices play an important role in various branches of mathematics and other sciences. A particular instance of these matrices are the sign regular matrices which appear in remarkably diverse areas of mathematics and its applications, including approximation theory, numerical mathematics, statistics, economics, computer aided geometric design, and other fields, cf. [GK60], [Kar68], [And87], [GM96], [P99], [Pin10], [FJ11], [HT12]. A real matrix is called *sign regular* and *strictly sign regular* if all its minors of the same order have the same sign or vanish and are nonzero and have the same sign, respectively. If the sign of all minors of any order is nonnegative (positive, nonpositive, and negative) then the matrix is called *totally nonnegative* (*totally positive*, *totally nonpositive*, and *totally negative*, respectively). In the existing literature, the terminologies of totally nonnegative and totally positive are not consistent with that we are using throughout the thesis. Elsewhere in the literature the terms totally nonnegative and totally positive correspond to totally positive and strictly totally positive, respectively, see, e.g., [Kar68], [And87], [GM96], [Pin10].

In this thesis we consider the following problems:

- The condensed form of the Cauchon Algorithm, characterizations of totally nonnegative matrices and nonsingular totally nonpositive matrices by using the Cauchon Algorithm, relationship between Neville elimination and the Cauchon Algorithm, representations of the entries of the matrix that is obtained by the application of the Cauchon Algorithm to a nonsingular totally nonnegative or a nonsingular totally nonpositive matrix, optimal determinantal criteria for total nonnegativity and nonsingular total nonpositivity, and characterization of several subclasses of totally nonnegative matrices by using the Cauchon Algorithm.
- Matrix intervals of nonsingular sign regular matrices with respect to the checkerboard ordering.
- Invariance of total nonnegativity under element-wise perturbation and total nonnegativity of the extended Perron complement.
- Totally nonnegative and totally positive completion problems.
- Total nonnegativity of special structured matrices, stability of given polynomials and location of zeros and poles of rational functions by using total nonnegativity of certain matrices, stability of interval polynomials, and interval problems for a subclass of the rational functions, viz. the R -functions.

1.1. Overview

This thesis is divided into eight chapters. In the following we give a brief overview of each chapter.

Chapter 1: Most of the definitions, notations, and determinantal identities and inequalities that will be used throughout the thesis are introduced.

Chapter 2: We present several criteria and properties of sign regular, strictly sign regular, almost strictly sign regular, totally nonpositive, totally negative, totally positive, and totally nonnegative matrices and some of their subclasses. The variation diminishing property, Neville elimination, and planar networks and their relations to totally nonnegative matrices are described. We also give some spectral properties of some subclasses of the sign regular matrices.

Chapter 3: We introduce the totally nonnegative cells and the Cauchon Algorithm. We study the totally nonnegative and totally nonpositive matrices through the Cauchon Algorithm. Hereby determinantal criteria for totally nonnegative matrices and nonsingular totally nonpositive matrices, representations of the entries of the matrix that is obtained by the application of the Cauchon Algorithm to a nonsingular totally nonnegative and a nonsingular totally nonpositive matrix, and characterizations of several subclasses of totally nonnegative matrices are derived.

Chapter 4: We present several known results of matrix intervals of sign regular matrices with respect to the checkerboard partial ordering and prove new results, e.g., Garloff's Conjecture and analogous results for the nonsingular totally nonpositive, nonsingular almost strictly sign regular, nonsingular tridiagonal sign regular, and nonsingular sign regular matrices of special signatures.

Chapter 5: We close the book on the perturbation problem of the single entries of totally nonnegative matrices and extend some known results on the total nonnegativity of the extended Perron complement.

Chapter 6: We review some totally nonnegative and totally positive completion problems and present several new results which lead us to settle partially two conjectures posed recently.

Chapter 7: We consider the problem of stability of polynomials and introduce several notions and structured matrices which play an important role in the study of polynomials and rational functions.

Chapter 8: We apply our results on totally nonnegative matrices to the matrices that are introduced in Chapter 7 to investigate the relationships between total nonnegativity of

these matrices and the stability of the associated polynomials and the location of zeros and poles of the associated rational functions. We also apply our results on intervals of totally nonnegative matrices to derive sufficient conditions for the stability of interval polynomials and special "interval" rational functions, viz. R -functions of negative type. These problems include invariance of exclusively positive poles and exclusively negative roots in the presence of variation of the coefficients of the polynomials within given intervals.

1.2. Definitions and Notation

In this section we introduce most of the definitions and notations that will be used throughout this thesis. All of these definitions are extended verbatim to infinite matrices.

The set of the n -by- m real matrices is denoted by $\mathbb{R}^{n,m}$. For integers κ, n we denote by $Q_{\kappa,n}$ the set of all strictly increasing sequences of κ integers chosen from $\{1, 2, \dots, n\}$. We use the set theoretic symbols \cup and \setminus to denote somewhat not precisely but intuitively the union and the difference, respectively, of two index sequences, where we consider the resulting sequence as strictly increasing ordered. For $\alpha \in Q_{\kappa,n}$ we define $\alpha^c := \{1, \dots, n\} \setminus \alpha$.

For $A \in \mathbb{R}^{n,m}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\kappa) \in Q_{\kappa,n}$, and $\beta = (\beta_1, \beta_2, \dots, \beta_\mu) \in Q_{\mu,m}$, we denote by $A[\alpha|\beta]$ the κ -by- μ submatrix of A lying in the rows indexed by $\alpha_1, \alpha_2, \dots, \alpha_\kappa$ and columns indexed by $\beta_1, \beta_2, \dots, \beta_\mu$. We suppress the brackets when we enumerate the indices explicitly. By $A(\alpha|\beta)$ we denote the $(n-\kappa)$ -by- $(m-\mu)$ submatrix $A[\alpha^c|\beta^c]$ of A . When $\alpha = \beta$, the *principal submatrix* $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$ and $\det A[\alpha]$ is called a *principal minor*, with the similar notation $A(\alpha)$ for the complementary principal submatrix. In the special case when $\alpha = (1, 2, \dots, \kappa)$, we refer to the principal submatrix $A[\alpha]$ as the *leading principal submatrix* (and to $\det A[\alpha]$ as the *leading principal minor*) of order κ . We denote by $|\alpha|$ the number of members of α . A measure of the gaps in an index sequence α is the *dispersion* of α , denoted by $d(\alpha)$, and is defined to be $d(\alpha) := \alpha_\kappa - \alpha_1 - \kappa + 1$. If $d(\alpha) = 0$ we call α *contiguous*, if $d(\alpha) = d(\beta) = 0$ we call the submatrix $A[\alpha|\beta]$ *contiguous* and in the case $\kappa = \mu$ we call the corresponding minor *contiguous*.

For $\alpha, \beta \in Q_{\kappa,n}$, we set $\alpha_{\hat{i}} := \alpha \setminus \{i\}$ for some $i \in \alpha$ and $\det A[\alpha|\beta] := 1$ if α or β is not strictly increasing or empty, in particular, we put $\det A[\alpha_1, \alpha_2] = 1$ if $\alpha_1 > \alpha_2$ (possibly $\alpha_2 = 0$).

We order the sequences from $Q_{\kappa,n}$ with respect to the lexicographical and colexicographical ordering. We denote by \leq and \leq_c the *lexicographical* and *colexicographical ordering*, respectively, i.e.,

$$\alpha = (\alpha_1, \dots, \alpha_\kappa) \leq \alpha^* = (\alpha_1^*, \dots, \alpha_\kappa^*)$$

if and only if $\alpha = \alpha^*$ or the first nonvanishing difference in the following sequence

$$\alpha_1^* - \alpha_1, \alpha_2^* - \alpha_2, \dots, \alpha_\kappa^* - \alpha_\kappa$$

is positive and

$$\alpha = (\alpha_1, \dots, \alpha_\kappa) \leq_c \alpha^* = (\alpha_1^*, \dots, \alpha_\kappa^*)$$

if and only if $\alpha = \alpha^*$ or the first nonvanishing difference in the following sequence

$$\alpha_\kappa^* - \alpha_\kappa, \alpha_{\kappa-1}^* - \alpha_{\kappa-1}, \dots, \alpha_1^* - \alpha_1$$

is positive.

In the remaining chapters we will consider the lexicographical and colexicographical ordering on pairs of indices instead on strictly increasing sequences.

A minor $\det A[\alpha|\beta]$ of A is called *row-initial* if $\alpha = (1, 2, \dots, \kappa)$ and $\beta \in Q_{\kappa, m}$ is contiguous, *column-initial* if $\alpha \in Q_{\kappa, n}$ is contiguous while $\beta = (1, 2, \dots, \kappa)$, *initial* if it is row-initial or column-initial, and *quasi-initial* if either $\alpha = (1, 2, \dots, \kappa)$ and $\beta \in Q_{\kappa, m}$ is arbitrary or $\alpha \in Q_{\kappa, n}$ is arbitrary, while $\beta = (1, 2, \dots, \kappa)$.

The identity matrix of order n is denoted by I_n . The n -by- n matrix whose only nonzero entry is a one in the $(i, j)^{th}$ position is denoted by E_{ij} . We reserve throughout the notation $T_n = (t_{ij})$ for the (anti-diagonal matrix) permutation matrix of order n with $t_{ij} = \delta_{i, n-j+1}$, $i, j = 1, \dots, n$, and set $A^\# := T_n A T_n$ for $A \in \mathbb{R}^{n, m}$. A matrix $A = (a_{ij}) \in \mathbb{R}^{n, n}$ is referred to as a *tridiagonal* (or *Jacobi*), *pentadiagonal*, *lower triangular*, and *upper triangular* matrix if $a_{ij} = 0$ whenever $|i - j| > 1$, $|i - j| > 2$, $j > i$, and $i > j$, respectively.

An n -by- n matrix A is termed *irreducible* if either $n = 1$ and $A \neq 0$ or $n \geq 2$ and there is no permutation matrix P such that

$$PAP^T = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where 0 is the $(n - r)$ -by- r zero matrix ($1 \leq r \leq n - 1$). Otherwise it is called *reducible*.

A sequence ϵ is termed a *signature sequence* if all its members are -1 or 1 . A matrix $A \in \mathbb{R}^{n, m}$ is called *strictly sign regular* abbreviated as *SSR* and *sign regular SR with signature* $\epsilon = (\epsilon_1, \dots, \epsilon_{n'})$ if $0 < \epsilon_\kappa \det A[\alpha|\beta]$ and $0 \leq \epsilon_\kappa \det A[\alpha|\beta]$, respectively, for all $\alpha \in Q_{\kappa, n}$, $\beta \in Q_{\kappa, m}$, $\kappa = 1, 2, \dots, n'$, where $n' := \min\{n, m\}$. If A is *SSR* (*SR*) with signature $\epsilon = (1, 1, \dots, 1)$, then A is termed *totally positive TP* (*totally nonnegative TN*). A is said to be *totally positive (totally nonnegative) of order $k \leq n'$ TP_k (TN_k)* if all its minors of order less than or equal to k are positive (nonnegative). If a square matrix A is *TN* and has a *TP* integral power then it is called *oscillatory*. If A is *SSR* (*SR*) with signature $\epsilon = (-1, -1, \dots, -1)$, then A is called *totally negative t.n. (totally nonpositive t.n.p.)*. If a square matrix A is in a certain class of *SR* matrices and in addition also nonsingular then we affix *Ns* to the name of the class, i.e., if A is *TN* and in addition nonsingular we write A is *NsTN*. If all the principal minors of A are positive then A is referred to as *P-matrix*. A real square matrix A is termed *positive definite* if it is symmetric and a *P-matrix*. In

passing we note that if A is TN then so are its transpose, denoted by A^T , and $A^\#$, see, e.g., [FJ11, Theorem 1.4.1].

A lower (upper) triangular matrix A is called *totally positive* (abbreviated ΔTP) if it is TN and has its minors positive unless they are identical zero because of the triangular structure of A . An n -by- n matrix A is said to have an LU (LDU) factorization if A can be written as $A = LU$ ($A = LDU$), where L , (D), and U are n -by- n lower triangular, (diagonal), and upper triangular matrices, respectively.

For an n -by- n matrix A with $A[\alpha]$ nonsingular for some $\alpha \in Q_{k,n}$, the *Schur complement of $A[\alpha]$ in A* , denoted by $A/A[\alpha]$, is defined as

$$A/A[\alpha] := A[\alpha^c] - A[\alpha^c|\alpha](A[\alpha])^{-1}A[\alpha|\alpha^c]. \quad (1.1)$$

Finally, we endow $\mathbb{R}^{n,n}$ with two partial orderings: Firstly, with the usual *entry-wise partial ordering* ($A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{n,n}$)

$$A \leq B \Leftrightarrow a_{ij} \leq b_{ij}, \quad i, j = 1, \dots, n,$$

and with the *checkerboard partial ordering* which is defined as follows. Let $S := \text{diag}(1, -1, \dots, (-1)^{n+1})$ and $A^* := SAS$. Then we define

$$A \leq^* B \Leftrightarrow A^* \leq B^*.$$

In particular, we say that a real matrix A is *nonnegative (positive)* if $0 \leq (<)A$.

1.3. Compound Matrices and Kronecker's Theorem

In this section we present compound matrices, some of their properties, their relation to the sign regular matrices, and Kronecker's Theorem. For more details, the interested reader is referred to [Kar68], [GK60], [Pin10].

Let $A = (a_{ij})$ be an n -by- m matrix. The sequences in $Q_{p,n}$ are considered to be ordered with respect to the lexicographical order. Hence each sequence will occupy a definite place comparable to the other sequences in $Q_{p,n}$ and consequently it will have a definite number s , which can run through the values $1, 2, \dots, \binom{n}{p}$. The same process is done for the sequences in $Q_{p,m}$.

For the minors of order p of the matrix A set

$$a_{st}^{(p)} := \det A[\alpha|\beta], \quad (1.2)$$

where s and t are the numbers of the sequences α in $Q_{p,n}$ and β in $Q_{p,m}$, respectively. The matrix

$$A^{(p)} = (a_{st}^{(p)})_{s=1, t=1}^{N, M}, \quad N = \binom{n}{p}, \quad M = \binom{m}{p}, \quad p = 1, \dots, \min\{n, m\},$$

is called the p -th compound matrix of A or the p -th associated matrix of A .

Compound matrices play a fundamental role in matrix theory and provide a method of constructing new matrices from given ones.

In order to find all the compound matrices of a given n -by- m matrix, one needs to calculate

$$\sum_{p=1}^{\min\{n,m\}} \binom{n}{p} \binom{m}{p} = \binom{n+m}{n} - 1 \quad (1.3)$$

minors. For $n = m$ this number is of order $4^n n^{-1/2}$ (by using Stirling's approximation to the factorial).

The following theorem gives some properties which are very useful in studying compound matrices.

Theorem 1.1. [Kar68], [Kus12] *Let $A, B \in \mathbb{R}^{n,n}$. Then the following hold:*

1. $A^{(j)} = 0$ if and only if $j \geq r$, where r is the rank of A .
2. $I_n^{(j)} = I_{\binom{n}{j}}$.
3. $(AB)^{(j)} = A^{(j)}B^{(j)}$.
4. For any positive integer m , $(A^{(j)})^m = (A^m)^{(j)}$.
5. If A is nonsingular, then $(A^{(j)})^{-1} = (A^{-1})^{(j)}$.

By using the compound matrix terminology we can restate the definition of SR matrices in terms of the compound matrices. A matrix $A \in \mathbb{R}^{n,m}$ is SR (SSR) with signature $(\epsilon_1, \dots, \epsilon_{n'})$ if each p -th compound matrix of A has entry-wise the sign ϵ_p , where 0 is permitted, (strict sign ϵ_p), and TN (TP) if all its p -th compound matrices are entry-wise nonnegative (positive), $p = 1, \dots, n'$.

The following theorem due to Kronecker provides the relationship between the eigenvalues of a given matrix and that of its p -th compound matrices.

Theorem 1.2. [Kar68], [GK02, Theorem 23, p. 65] *Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the complete system of eigenvalues of the matrix A . Then the complete system of eigenvalues of the p -th compound matrix $A^{(p)}$ consists of all possible products of the numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ taken p at a time.*

1.4. Determinantal Identities and Inequalities

In this section we list and briefly describe various determinantal identities which are now indispensable ingredients in matrix theory. Also we give some determinantal inequalities that hold for all totally nonnegative matrices.

We begin with the following fundamental formula in the theory of matrices which is attributed to Cauchy and Binet. This identity can be seen as a generalization of the multiplication formula of matrices and is in the square case identical with property 3. in Theorem 1.1.

Lemma 1.1. [FJ11, Cauchy-Binet Identity, Theorem 1.1.1] *Let A be an m -by- p matrix and B an p -by- n matrix, and let $\alpha \subseteq \{1, \dots, m\}$ and $\beta \subseteq \{1, \dots, n\}$ be k -element sets with $k \leq \min\{n, m, p\}$. Then*

$$\det AB[\alpha|\beta] = \sum_{\gamma} \det A[\alpha|\gamma] \det B[\gamma|\beta], \quad (1.4)$$

where γ ranges over $Q_{k,p}$.

The following determinantal identity due to Sylvester serves as a basis tool in some proofs of this thesis.

Lemma 1.2. [GK02, Theorem 1, p. 13], [Pin10, Sylvester's Determinant Identity] *Let A be an n -by- m matrix such that $\det A[\alpha|\beta] \neq 0$, where α, β are both in $Q_{k,n}$ and $B = (b_{ij})$ is the matrix obtained from A by*

$$b_{ij} := \frac{\det A[\alpha \cup \{i\} | \beta \cup \{j\}]}{\det A[\alpha|\beta]}, \quad \text{for all } (i, j) \in \alpha^c \times \beta^c, \quad (1.5)$$

then

$$\det B[\eta|\zeta] = \frac{\det A[\alpha \cup \eta | \beta \cup \zeta]}{\det A[\alpha|\beta]}, \quad \text{for all } \eta \subseteq \alpha^c, \zeta \subseteq \beta^c. \quad (1.6)$$

The submatrix $A[\alpha|\beta]$ in Lemma 1.2 is called the *pivot block*, see, e.g., [Kar68, p. 5], since this identity is proven by observing that B is the *Schur complement* of $A[\alpha|\beta]$ in A . A consequence of Sylvester's determinant identity is the next corollary.

Corollary 1.1. [dBP82, Corollary 1, p. 84] *Let A, B be given as in Lemma 1.2. Then*

$$\text{rank}(A[\alpha \cup \alpha' | \beta \cup \beta']) = |\alpha| + \text{rank}(B[\alpha' | \beta']),$$

where $\alpha \cap \alpha' = \beta \cap \beta' = \phi$.

In the sequel we will often make use of the following special case of Sylvester's determinant identity, see, e.g., [FJ11, pp. 29-30].

Lemma 1.3. *Partition $A \in \mathbb{R}^{n,n}$, $n \geq 3$, as follows:*

$$A = \begin{pmatrix} c & A_{12} & d \\ A_{21} & A_{22} & A_{23} \\ e & A_{32} & f \end{pmatrix},$$

where $A_{22} \in \mathbb{R}^{n-2,n-2}$ and c, d, e, f are scalars. Define the submatrices

$$C := \begin{pmatrix} c & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, D := \begin{pmatrix} A_{12} & d \\ A_{22} & A_{23} \end{pmatrix},$$

$$E := \begin{pmatrix} A_{21} & A_{22} \\ e & A_{32} \end{pmatrix}, F := \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & f \end{pmatrix}.$$

Then if $\det A_{22} \neq 0$, we have

$$\det A = \frac{\det C \det F - \det D \det E}{\det A_{22}}.$$

The following lemma relates the minors of a given nonsingular square matrix with the minors of its inverse. In particular, it plays a fundamental role in proving that for a given n -by- n *NsTN* matrix A , the matrix $SA^{-1}S$ is *NsTN*, where $S = \text{diag}(1, -1, \dots, (-1)^{n+1})$.

Lemma 1.4. [FJ11, Jacobi's Identity, pp. 28-29] *Let $A \in \mathbb{R}^{n,n}$ be nonsingular. Then for any nonempty subsets $\alpha, \beta \subseteq \{1, \dots, n\}$ with $|\alpha| = |\beta|$ the following equality holds:*

$$\det A^{-1}[\alpha|\beta] = (-1)^s \frac{\det A[\beta^c|\alpha^c]}{\det A}, \quad (1.7)$$

where $s := \sum_{\alpha_i \in \alpha} \alpha_i + \sum_{\beta_i \in \beta} \beta_i$.

The next lemma relates the determinant of a given square matrix to its minors.

Lemma 1.5. [Kar68, Laplace Expansion by Minors, p. 6] *Let $A \in \mathbb{R}^{n,n}$. Then for any $k = 1, \dots, n$ and a fixed $\alpha \in Q_{k,n}$ we have*

$$\det A = \sum_{\beta \in Q_{k,n}} (-1)^s \det A[\alpha|\beta] \det A[\alpha^c|\beta^c],$$

where s is defined for each $\alpha, \beta \in Q_{k,n}$ as in the above lemma.

The next lemma will be applied to recognize *SSR* matrices.

Lemma 1.6. [GK02, Lemma 1, p. 259] *For an arbitrary matrix $A = (a_{ij}) \in \mathbb{R}^{n+1,n}$, the following equality holds:*

$$\begin{aligned} & \det A[1, \dots, n|1, \dots, n] \det A[2, \dots, n-1, n+1|1, \dots, n-1] + \\ & \det A[n, 2, \dots, n-1, n+1|1, \dots, n] \det A[2, \dots, n-1, 1|1, \dots, n-1] + \\ & \det A[n+1, 2, \dots, n-1, 1|1, \dots, n] \det A[2, \dots, n-1, n|1, \dots, n-1] = 0. \end{aligned}$$

The following lemma provides a determinantal identity which can be viewed as a generalization of [And87, (1.39)], see also [GLL11b, (B. 1)].

Lemma 1.7. *Let $A \in \mathbb{R}^{n,m}$, $\alpha = (\alpha_1, \dots, \alpha_l) \in Q_{l,n}$, and $\beta = (\beta_1, \dots, \beta_{l-1}) \in Q_{l-1,m-1}$ with $0 < d(\beta)$. Then for all η such that $\beta_{l-1} < \eta \leq m$, $k \in \{1, \dots, l\}$, $s \in \{1, \dots, h\}$, and $\beta_h < t < \beta_{h+1}$ for some $h \in \{1, \dots, l-2\}$ or $\beta_{l-1} < t < \eta$ the following determinantal identity holds:*

$$\det A[\alpha_{\hat{\alpha}_k} | \beta_{\hat{\beta}_s} \cup \{t\}] \det A[\alpha | \beta \cup \{\eta\}] = \det A[\alpha_{\hat{\alpha}_k} | \beta_{\hat{\beta}_s} \cup \{\eta\}] \det A[\alpha | \beta \cup \{t\}] + \det A[\alpha_{\hat{\alpha}_k} | \beta] \det A[\alpha | \beta_{\hat{\beta}_s} \cup \{t, \eta\}]. \quad (1.8)$$

Proof. The proof proceeds by applying Lemma 1.3 to the following $(l+1)$ -by- $(l+1)$ matrix

$$B := \begin{bmatrix} a_{\alpha_1, \beta_1} & \cdots & a_{\alpha_1, \beta_h} & a_{\alpha_1, t} & a_{\alpha_1, \beta_{h+1}} & \cdots & a_{\alpha_1, \beta_{l-1}} & a_{\alpha_1, \eta} \\ a_{\alpha_2, \beta_1} & \cdots & a_{\alpha_2, \beta_h} & a_{\alpha_2, t} & a_{\alpha_2, \beta_{h+1}} & \cdots & a_{\alpha_2, \beta_{l-1}} & a_{\alpha_2, \eta} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \\ a_{\alpha_l, \beta_1} & \cdots & a_{\alpha_l, \beta_h} & a_{\alpha_l, t} & a_{\alpha_l, \beta_{h+1}} & \cdots & a_{\alpha_l, \beta_{l-1}} & a_{\alpha_l, \eta} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}. \quad (1.9)$$

□

The determinants of some special kind of matrices can be evaluated by using recursion relations, for instance, the determinant of an n -by- n tridiagonal matrix $A = (a_{ij})$ can be evaluated by using the following recursion equations:

$$\det A = a_{11} \det A[2, \dots, n] - a_{12} a_{21} \det A[3, \dots, n] \quad (1.10)$$

$$= a_{nn} \det A[1, \dots, n-1] - a_{n-1, n} a_{n, n-1} \det A[1, \dots, n-2]. \quad (1.11)$$

The next proposition extends the above two relations.

Proposition 1.1. [Pin10, Formula (4.1)] *For an n -by- n tridiagonal matrix $A = (a_{ij})$ the following relation holds:*

$$\det A = \det A[1, \dots, i-1] \det A[i, \dots, n] - a_{i-1, i} a_{i, i-1} \det A[1, \dots, i-2] \det A[i+1, \dots, n], \quad i = 2, \dots, n. \quad (1.12)$$

Now we turn to matrix inequalities that hold for TN matrices.

Theorem 1.3. [GK02, Theorem 16, p. 270] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be TN . Then*

$$\begin{aligned} a_{11} \det A(1) &- a_{12} \det A(1|2) + \dots - a_{1,2p} \det A(1|2p) \leq \det A \\ &\leq a_{11} \det A(1) - a_{12} \det A(1|2) + \dots + a_{1,2q-1} \det A(1|2q-1) \\ &\quad \left(p = 1, 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor; \quad q = 1, 2, \dots, \left\lfloor \frac{n+1}{2} \right\rfloor \right). \end{aligned}$$

Theorem 1.4. [FJ11, Corollary 6.2.4, Koteljanskii Inequality] *Let $A \in \mathbb{R}^{n,n}$ be TN. Then for any $\alpha \in Q_{k,n}$ and $\beta \in Q_{l,n}$, the following inequality holds:*

$$\det A[\alpha \cup \beta] \cdot \det A[\alpha \cap \beta] \leq \det A[\alpha] \cdot \det A[\beta]. \quad (1.13)$$

We conclude this section with a proposition which provides a useful determinantal inequality that plays a fundamental role in the solution of the perturbation problem of TP matrices.

Proposition 1.2. [Ska04, Theorem 4.2] *Let $\alpha, \alpha', \beta, \beta', \gamma, \gamma', \delta, \delta'$ be subsets of $\{1, 2, \dots, n\}$ with $\alpha \cup \gamma = \{1, 2, \dots, p\}$ and $\alpha' \cup \gamma' = \{1, 2, \dots, p'\}$, $q = |\alpha \cap \gamma|$, $q' = |\alpha' \cap \gamma'|$, and $r := \frac{1}{2}(p - q + p' - q')$. Let η be the unique order preserving map*

$$\eta : (\alpha \setminus \gamma) \cup (\gamma \setminus \alpha) \rightarrow \{1, 2, \dots, p - q\},$$

and let η' be the unique order reversing map

$$\eta' : (\alpha' \setminus \gamma') \cup (\gamma' \setminus \alpha') \rightarrow \{p - q + 1, \dots, 2r\}.$$

Define the subsets α'' and β'' of $\{1, 2, \dots, 2r\}$ by

$$\begin{aligned} \alpha'' &:= \eta(\alpha \setminus \gamma) \cup \eta'(\gamma' \setminus \alpha'), \\ \beta'' &:= \eta(\beta \setminus \delta) \cup \eta'(\delta' \setminus \beta'). \end{aligned}$$

Then the following two statements are equivalent:

1. For each square TN matrix A of order at least n the following relation holds:

$$\det A[\alpha|\alpha'] \det A[\gamma|\gamma'] \leq \det A[\beta|\beta'] \det A[\delta|\delta'].$$

2. The relations $\alpha \cup \gamma = \beta \cup \delta$ and $\alpha' \cup \gamma' = \beta' \cup \delta'$ are fulfilled and the sets α'', β'' satisfy the inequality

$$\max\{|\omega \cap \beta''|, |\omega \setminus \beta''|\} \leq \max\{|\omega \cap \alpha''|, |\omega \setminus \alpha''|\} \quad (1.14)$$

for each subset $\omega \subseteq \{1, 2, \dots, 2r\}$ of even cardinality.

2. Sign Regular, Totally Nonpositive, and Totally Nonnegative Matrices

In order to benefit from the properties of sign regular matrices or of any of its subclasses one needs first to check whether a given matrix is sign regular or not. In this chapter we present several methods in order to check the membership of a given matrix in a certain subclass of sign regular matrices. Furthermore, we present some spectral properties of these subclasses. These methods consist of determinantal criteria, the variation diminishing property, planar networks, and the Neville elimination. We start with determinantal criteria that are needed in order to check the membership of a given n -by- m matrix in the class of sign regular (totally nonpositive, totally nonnegative) matrices. In the case that we are given a square matrix of order n and want to employ the definition of sign regularity (totally nonpositivity, totally nonnegativity), we need to check about $4^n (\pi n)^{-1/2}$ minors which is a very large number of minors when n is relatively large. Thus it is obviously impractical to check whether a matrix is sign regular by naively checking all its minors. So it is very important to search for criteria by which this number of minors could be decreased. In the first three sections we are concerned with the recognition problem: can one decide, by checking a restricted number of minors, whether or not a real matrix is sign regular of any subclass? Many of the inequalities in the definition of a strictly sign regular matrix are superfluous as we see later.

The organization of this chapter is as follows. In Section 2.1, determinantal criteria for checking sign regularity, strict sign regularity, and almost strict sign regularity, and related results are given. In Section 2.2, we focus on a subclass of sign regular matrices which is the class of the totally nonpositive matrices and present some determinantal criteria and properties of this subclass. In Section 2.3, we turn to another important subclass of sign regular matrices which is the class of the totally nonnegative matrices and introduce some determinantal criteria and properties of it and its subclasses. In Section 2.4, we present the variation diminishing property and its connections to the sign regular matrices. In Section 2.5, the relationships between totally nonnegative matrices and planar networks are investigated. In Section 2.6, the Neville elimination is introduced and its usefulness in ascertaining the total nonnegativity of a given matrix is discussed. Finally, in Section 2.7, some spectral properties of totally positive, oscillatory, totally nonnegative, totally negative, and totally nonpositive matrices are given.

2.1. Sign Regular Matrices

In this section we present some determinantal conditions that are sufficient for a given matrix to be strictly sign regular or sign regular. We start with strictly sign regular matrices and introduce an efficient determinantal criterion for checking strictly sign regularity. The following theorem states that it is sufficient to check only contiguous minors for strictly sign regularity. We remind the reader that we have defined in Section 1.2 $n' = \min \{n, m\}$.

Theorem 2.1. [GK02, Corollary, p. 261], [And87, Theorem 2.5] *Let $A \in \mathbb{R}^{n,m}$. Then A is SSR with signature $\epsilon = (\epsilon_1, \dots, \epsilon_{n'})$ if*

$$0 < \epsilon_k \det A[\alpha|\beta] \text{ whenever } \alpha \in Q_{k,n}, \beta \in Q_{k,m}, \text{ and } d(\alpha) = d(\beta) = 0, k = 1, \dots, n'. \quad (2.1)$$

Theorem 2.1 in the case of *TP* matrices is proved by Fekete [Fek13]. The following theorem shows that the rank of a given matrix plays a significant role in reducing the number of minors that one needs to check for sign regularity. The case of *TN* matrices was proved in [Cry76].

Theorem 2.2. [And87, Theorem 2.1] *Let $A \in \mathbb{R}^{n,m}$ be of rank r and $\epsilon = (\epsilon_1, \dots, \epsilon_{n'})$ be a signature sequence. If*

$$0 \leq \epsilon_k \det A[\alpha|\beta] \text{ for } \alpha \in Q_{k,n}, \beta \in Q_{k,m}, k = 1, 2, \dots, r, \quad (2.2)$$

is valid whenever $d(\beta) \leq m - r$, then A is SR with signature ϵ .

As a consequence of Theorem 2.2, if a given square matrix is nonsingular then it is sufficient to check the minors corresponding to submatrices whose columns are consecutive.

Unfortunately, this is all what can be positively said concerning the minimal number of minors that is sufficient for sign regularity of arbitrary signature. The following theorem can be easily concluded by using the Cauchy-Binet Identity; it shows that the set of sign regular matrices is closed under matrix multiplication.

Theorem 2.3. [And87, Theorem 3.1] *Let $A, B \in \mathbb{R}^{n,n}$ be SR with signatures $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ and $\delta = (\delta_1, \dots, \delta_n)$, respectively. Then AB is SR with signature $(\epsilon_1\delta_1, \dots, \epsilon_n\delta_n)$.*

In the next subsection and two sections we present some subclasses of *SR* matrices which need a fewer number of minors to be checked than that of general *SR* matrices.

There are no efficient criteria for general *SR* matrices as for *SSR* matrices but each *SR* matrix can be approximated arbitrary closely by an *SSR* matrix. The next theorem, which was first showed by A. M. Whitney [Whi52] in the case of *TN* matrices, shows that we can approximate any square *SR* matrix by an *SSR* and for a given square *SR* matrix whose rank is less than its order, we can approximate it by a strictly one not only with the same signature but also with a possible different signature.

Theorem 2.4. [GK02, Theorem 17, p. 272], [And87, Theorem 2.7] *Every SR matrix $A \in \mathbb{R}^{n,n}$ can be approximated arbitrarily closely by SSR matrices with the same signature. Moreover, if A has rank r , $r < n$, then it can be approximated by a SSR matrix with arbitrary prescribed signs of the minors of order greater than r .*

Several decompositions of *SSR* matrices are studied in [CPn08]. In [BPn12] it was shown that a given tridiagonal *SR* matrix can have only one of the following special signature sequences:

$$\epsilon = (1, 1, \dots, 1, \epsilon_n), \text{ or} \quad (2.3)$$

$$\epsilon = (-1, 1, \dots, (-1)^{n-1}, \epsilon_n), \quad (2.4)$$

with $\epsilon_n = \pm 1$.

The next two theorems provide efficient criteria for sign regularity of a given nonnegative tridiagonal matrix.

Theorem 2.5. [HC10, Theorem 9] *Let $A \in \mathbb{R}^{n,n}$ be a nonnegative tridiagonal matrix. Then the following three statements are equivalent:*

(a) *A is *NsSR* with signature $\epsilon = (1, 1, \dots, 1, \epsilon_n)$.*

(b) *The following conditions are satisfied:*

$$0 < \epsilon_n \det A, \quad (2.5)$$

$$0 \leq \det A[2, \dots, n], \quad (2.6)$$

$$0 \leq \det A[1, \dots, n-1], \quad (2.7)$$

$$0 < \det A[1, \dots, k], \text{ for all } 1 \leq k \leq n-2. \quad (2.8)$$

(c) *The following conditions are satisfied:*

$$0 < \epsilon_n \det A, \quad (2.9)$$

$$0 \leq \det A[2, \dots, n], \quad (2.10)$$

$$0 \leq \det A[1, \dots, n-1], \quad (2.11)$$

$$0 < \det A[k+1, \dots, n], \text{ for all } 2 \leq k \leq n-1. \quad (2.12)$$

Theorem 2.6. [BPn12, Theorem 4.1] *Let $3 \leq n$, and $A \in \mathbb{R}^{n,n}$ be a nonsingular nonnegative tridiagonal matrix. Then A is *SR* if and only if $A[1, \dots, n-1]$ and $A[2, \dots, n]$ are *TN*, and $A[1, \dots, n-2]$ and $A[2, \dots, n-1]$ are nonsingular.*

We close this section with the following subsection which introduces an intermediate subclass between the *SR* and *SSR* matrices.

2.1.1. Almost Strictly Sign Regular Matrices

In this subsection we consider a subclass of sign regular matrices that is intermediate between the sign regular and the strictly sign regular matrices.

We start with the following lemma which characterizes the zero-nonzero pattern of a given *NsSR* matrix according to the value of the second component of its signature.

Lemma 2.1. [HLZ12, Lemma 7], see also [Pn02, Lemma 2.2] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be $NsSR$ with signature $\epsilon = (\epsilon_1, \dots, \epsilon_n)$. Then one of the following two statements holds:*

(i) *If $\epsilon_2 = 1$, then*

$$\begin{cases} a_{ii} \neq 0, & i = 1, \dots, n, \\ a_{ij} = 0, & j < i \Rightarrow a_{kl} = 0 \quad \forall l \leq j < i \leq k, \\ a_{ij} = 0, & i < j \Rightarrow a_{kl} = 0 \quad \forall k \leq i < j \leq l, \end{cases}$$

which is called a type-I staircase matrix.

(ii) *If $\epsilon_2 = -1$, then*

$$\begin{cases} a_{1n} \neq 0, a_{2,n-1} \neq 0, \dots, a_{n1} \neq 0, \\ a_{ij} = 0, & n - i + 1 < j \Rightarrow a_{kl} = 0 \quad \forall i \leq k, j \leq l, \\ a_{ij} = 0, & j < n - i + 1 \Rightarrow a_{kl} = 0 \quad \forall k \leq i, l \leq j, \end{cases}$$

which is called a type-II staircase matrix.

Following [HLZ12], we call a minor *trivial* if it vanishes and its zero value is determined already by the pattern of its zero-nonzero entries. We illustrate this definition by the following example. Let

$$A := \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix},$$

where the asterisk denotes a nonzero entry. Then $\det A[2, 3|1, 2]$ and $\det A[1, 2|1, 3]$ are trivial, whereas $\det A$ and $\det A[1, 2|2, 3]$ are nontrivial minors.

The following lemma identifies the trivial/nontrivial minors in a given staircase matrix.

Lemma 2.2. [HLZ12, p. 4183] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ and $\alpha, \beta \in Q_{\kappa,n}$. Then the staircase matrices possess the following properties:*

(i) *If A is a type-I staircase matrix, then*

$$\det A[\alpha|\beta] \text{ is a nontrivial minor} \Leftrightarrow a_{\alpha_1, \beta_1} \cdot a_{\alpha_2, \beta_2} \cdots a_{\alpha_\kappa, \beta_\kappa} \neq 0.$$

(ii) *If A is a type-II staircase matrix, then*

$$\det A[\alpha|\beta] \text{ is a nontrivial minor} \Leftrightarrow a_{\alpha_1, \beta_\kappa} \cdot a_{\alpha_2, \beta_{\kappa-1}} \cdots a_{\alpha_\kappa, \beta_1} \neq 0.$$

(iii) *A is a type-I staircase matrix if and only if $T_n A$ is a type-II staircase matrix, and*

$$\det A[\alpha|\beta] \text{ is a nontrivial minor} \Leftrightarrow \det (T_n A)[\alpha'|\beta] \text{ is a nontrivial minor,}$$

where $\alpha' := (n - \alpha_i + 1, i = 1, \dots, \kappa)$.

Now we present the definition of the almost strictly sign regular matrices and characterize them.

Definition 2.1. [HLZ12, Definition 8] *Let $A \in \mathbb{R}^{n,n}$ and $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ be a signature sequence. If for all the nontrivial minors the following inequalities*

$$0 < \epsilon_k \det A[\alpha|\beta], \quad \text{where } \alpha, \beta \in Q_{k,n}, \quad k = 1, \dots, n,$$

hold, then A is called almost strictly sign regular (abbreviated ASSR) with signature ϵ .

The next theorem provides an efficient criterion for nonsingular almost strictly sign regularity.

Theorem 2.7. [HLZ12, Theorem 10] *Let $A \in \mathbb{R}^{n,n}$ and $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ be a signature sequence. Then A is $NsASSR$ with signature ϵ if and only if A is a type-I or type-II staircase matrix, and for all the nontrivial minors the following inequalities hold:*

$$0 < \epsilon_k \det A[\alpha|\beta], \quad \text{where } \alpha, \beta \in Q_{k,n} \text{ such that } d(\alpha) = d(\beta) = 0, \quad k = 1, \dots, n.$$

One can predict some components of the signature sequence of a given type-I staircase matrix by using the positions of vanishing entries of a given $NsASSR$ matrix.

Lemma 2.3. [HLZ12, Lemma 9] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be a type-I staircase matrix and $NsASSR$ with signature $\epsilon = (\epsilon_1, \dots, \epsilon_n)$. Set*

$$r := \min \{ |j - i| \mid a_{ij} = 0, \text{ for some } i, j \in \{1, \dots, n\} \}.$$

Then the following relations hold

$$\epsilon_2 = \epsilon_1^2, \quad \epsilon_3 = \epsilon_1^3, \quad \dots, \quad \epsilon_{n-r+1} = \epsilon_1^{n-r+1}.$$

In [APnS15], $NsASSR$ matrices are studied and characterized through the Neville elimination, see Section 2.6.

2.2. Totally Nonpositive Matrices

In this section we present a subclass of sign regular matrices which is called totally nonpositive matrices, some of its properties, and determinantal criteria for checking the membership of a given matrix in this subclass.

Recall that a given matrix is called *totally nonpositive t.n.p.* (*totally negative t.n.*) if all of its minors are nonpositive (negative). The study of *t.n.* matrices was initiated in [FD00]. Some aspects of this subclass including existence, spectral properties, Schur complements, and factorizations were considered therein. The following propositions introduce methods to generate TN , TP , $t.n.p.$, and $t.n.$ matrices from given TP , $t.n.$, and $t.n.p.$ matrices.

Proposition 2.1. [FD00, Proposition 2.1] *Let $A \in \mathbb{R}^{n,m}$ be $t.n.$ ($t.n.p.$) and let $B, C \in \mathbb{R}^{m,p}$ be $t.n.$ and TP , respectively. Then $AB \in \mathbb{R}^{n,p}$ is TP (TN) and $AC \in \mathbb{R}^{n,p}$ is $t.n.$ ($t.n.p.$).*

Proposition 2.2. [FD00, Proposition 3.2] *Let $A \in \mathbb{R}^{n,n}$ be TP and define $A' := A - xE_{11}$, where $x := \frac{\det A}{\det A[2, \dots, n]}$. Then there exists a sufficiently small positive number t such that $B := A' - tE_{11}$ is nonsingular and $SB^{-1}S$ is t.n., where $S = \text{diag}(1, -1, \dots, (-1)^{n+1})$.*

The negativity of the entry at the position $(1, 1)$ (or (n, n)) plays an important role in characterizing the *Ns.t.n.p.* matrices.

Lemma 2.4. [CKRU08, Proposition 3.2] *If $A = (a_{ij}) \in \mathbb{R}^{n,n}$ is Ns.t.n.p. with $a_{11} < 0$, then $a_{ij} < 0$ for all $i, j = 1, \dots, n$ with $(i, j) \neq (n, n)$.*

If $A = (a_{ij}) \in \mathbb{R}^{n,n}$ is Ns.t.n.p. with $a_{nn} < 0$, then by applying the above lemma to $A^\#$ we conclude that all the entries of A are negative except possibly the a_{11} entry. The next theorem presents necessary and sufficient conditions for a given nonsingular matrix to be t.n.p.

Theorem 2.8. [HC10, Theorem 5] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ with $2 \leq n$ be nonsingular. Then the following three statements are equivalent:*

(a) *A is t.n.p.*

(b) *For any $k \in \{1, \dots, n-1\}$,*

$$a_{11} \leq 0, \quad a_{nn} \leq 0, \quad a_{n1} < 0, \quad a_{1n} < 0, \quad (2.13)$$

$$\det A[\alpha|k+1, \dots, n] \leq 0, \quad \text{for all } \alpha \in Q_{n-k,n}, \quad (2.14)$$

$$\det A[k+1, \dots, n|\beta] \leq 0, \quad \text{for all } \beta \in Q_{n-k,n}, \quad (2.15)$$

$$\det A[k, \dots, n] < 0. \quad (2.16)$$

(c) *For any $k \in \{1, \dots, n-1\}$,*

$$a_{11} \leq 0, \quad a_{nn} \leq 0, \quad a_{n1} < 0, \quad a_{1n} < 0, \quad (2.17)$$

$$\det A[\alpha|1, \dots, k] \leq 0, \quad \text{for all } \alpha \in Q_{k,n}, \quad (2.18)$$

$$\det A[1, \dots, k|\beta] \leq 0, \quad \text{for all } \beta \in Q_{k,n}, \quad (2.19)$$

$$\det A[1, \dots, k+1] < 0. \quad (2.20)$$

In the case of total negativity the following theorem can be used.

Theorem 2.9. [CRU09, Theorem 6] *Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$ with $a_{nm} < 0$. Then A is t.n. if and only if all its initial minors are negative.*

The *LDU* factorization plays a fundamental role in characterizing and generating t.n. and *Ns.t.n.p.* matrices.

Theorem 2.10. [FD00, Corollary 4.3] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ with $a_{11} < 0$. Then A is t.n. if and only if A can be written as $A = UDL$, where L (U) are unit diagonal lower (upper) triangular Δ TP matrices and D is a diagonal matrix with positive diagonal entries except for a negative (n, n) entry.*

Remark 2.1. [CKRU08, Remark 2.6] *By the fact that $A^\#$ is t.n. if and only if A is so and by application of the above theorem to $A^\#$ we can conclude that an n -by- n matrix $A = (a_{ij})$ with $a_{nn} < 0$ is t.n. if and only if $A = LDU$, where L (U) is a unit diagonal lower (upper) triangular ΔTP matrix, and D is a diagonal matrix with positive diagonal entries except for a negative $(1, 1)$ entry.*

Theorem 2.11. [CKRU08, Theorem 2.5] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be nonsingular with $a_{11} < 0$ and $a_{nn} \leq 0$. Then A is t.n.p. if and only if $A = LDU$, where L (U) is a unit diagonal lower (upper) triangular TN matrix with positive entries below (above) the diagonal, and D is a diagonal matrix with positive diagonal entries except for a negative $(1, 1)$ entry.*

Characterizations of t.n.p. matrices by minors and by their full rank factorization are given in [CRU09], [CRU10], [CRU13], and [CRU14]. The following two propositions provide important properties of *Ns.t.n.p.* matrices.

Proposition 2.3. [Pn03, Theorem 3.2], [CKRU08, Proposition 3.3] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be *Ns.t.n.p.* with $a_{11} < 0$ and $a_{nn} < 0$. Then $\det A[\alpha] < 0$ for all $\alpha \in Q_{k,n}$, $k = 1, 2, \dots, n$.*

Proposition 2.4. [CRU13, Proposition 9] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be *Ns.t.n.p.* with $a_{11} = 0$. Then $\det A[\alpha] < 0$ for all $\alpha \in Q_{k,n}$, $k = 1, \dots, n$, except for $k = 1$ and $\alpha = (1)$ or $\alpha = (n)$.*

For any contiguous submatrix $A[\alpha|\beta]$, we call the submatrix of A having $A[\alpha|\beta]$ in its upper right corner the *left shadow* of $A[\alpha|\beta]$, and, analogously, we call the submatrix having $A[\alpha|\beta]$ in its lower left corner the *right shadow* of $A[\alpha|\beta]$, see [dBP82, p. 84], [Pin10, p. 13]. The following proposition is analogous to Proposition 2.8 in the next section with a similar proof.

Proposition 2.5. *Let $A \in \mathbb{R}^{n,m}$ be t.n.p. and let $\alpha = (i+1, \dots, i+r)$, $\beta = (j+1, \dots, j+r)$ for some $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, and $2 \leq r \leq \min\{n, m\} - 1$. If $A[\alpha|\beta]$ has rank $r - 1$, then*

(i) *either the rows $i+1, \dots, i+r$ or the columns $j+1, \dots, j+r$ of A are linearly dependent,*
or

(ii) *the right or left shadow of $A[i+1, \dots, i+r|j+1, \dots, j+r]$ has rank $r - 1$.*

Proof. We follow the proof of [Pin10, Proposition 1.15]. Since $A[\alpha|\beta]$ has rank $r - 1$ and $2 \leq r$ then there exist $p, q \in \{1, \dots, r\}$ such that $\det A[\alpha_{\widehat{i+p}}|\beta_{\widehat{j+q}}] < 0$. Set $B := (b_{kl})$ be the $(n - r + 1) \times (m - r + 1)$ matrix with

$$b_{kl} := \frac{\det A[\alpha_{\widehat{i+p}} \cup \{k\} | \beta_{\widehat{j+q}} \cup \{l\}]}{\det A[\alpha_{\widehat{i+p}} | \beta_{\widehat{j+q}}]}, \quad k \in \{1, \dots, n\} \setminus \alpha_{\widehat{i+p}}, \quad l \in \{1, \dots, m\} \setminus \beta_{\widehat{j+q}}, \quad (2.21)$$

where the rows and columns are rearranged in increasing order. By Lemma 1.2 B is a TN matrix since A is t.n.p. and $B[i+1|j+1] = 0$ ($b_{i+p, j+q} = 0$ in the notation of (2.21)) since $A[\alpha|\beta]$ has rank $r - 1$. Hence by Proposition 2.8 applied to $B[i+1|j+1]$ as submatrix either

the row $i+1$ of B or the column $j+1$ of B is zero which implies by Corollary 1.1 that either the rows $i+1, \dots, i+r$ or the columns $j+1, \dots, j+r$ of A are linearly dependent, respectively, or $B[i+1, \dots, n-r+1|1, \dots, j+1]$ or $B[1, \dots, i+1|j+1, \dots, m-r+1]$ has rank 0 which implies by Corollary 1.1 that the right or left shadow of $A[i+1, \dots, i+r|j+1, \dots, j+r]$ has rank $r-1$. Therefore the claim follows. \square

The following lemma is a special case of the above proposition; we present it here for later reference.

Lemma 2.5. *Let $A \in \mathbb{R}^{n,n}$ be Ns.t.n.p. and let $\alpha = (i+1, \dots, i+r)$, $\beta = (j+1, \dots, j+r)$ for some $i, j \in \{1, \dots, n\}$ and $2 \leq r \leq n-1$. If $A[\alpha|\beta]$ has rank $r-1$, then either $A[i+1, \dots, n|1, \dots, j+r]$ has rank $r-1$ if $j < i$ or $A[1, \dots, i+r|j+1, \dots, n]$ has rank $r-1$ if $i < j$.*

The Schur complement of any nonsingular submatrix of a given SR matrix is investigated in [HL10]. Proposition 2.2 and the following theorem show that there is a strong relation between $t.n.$ and TP matrices.

Theorem 2.12. [FD00, Theorem 2.6] *Let $A \in \mathbb{R}^{n,n}$ be $t.n.$ and $\alpha \subset \{1, \dots, n\}$. Then the Schur complement $A/A[\alpha]$ is similar to a TP matrix via a diagonal matrix with diagonal entries ± 1 .*

We conclude this section with the following two propositions which are useful in getting our results on $Ns.t.n.p.$ matrix intervals.

Proposition 2.6. [CRU13, Proposition 6] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be Ns.t.n.p. with $a_{11} = 0$. Then there exists $0 < \epsilon_0$ such that for all $\epsilon \in [0, \epsilon_0]$ the matrix $A_\epsilon := A - \epsilon E_{11}$ is Ns.t.n.p.*

Proposition 2.7. [CRU13, Proposition 7] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be Ns.t.n.p. with $a_{11} < 0$. Then $A - a_{11}E_{11}$ is Ns.t.n.p.*

2.3. Totally Nonnegative Matrices

In this section we present some criteria for totally nonnegative matrices and recall some of their properties. As for the sign regular matrices there is no efficient determinantal criterion for a general totally nonnegative matrix. However, if the matrix is in addition nonsingular the number of minors which are needed to be checked can be reduced significantly. In Section 3.6 we give a procedure depending on the given matrix which provides an optimal determinantal criterion for total nonnegativity; it requires the determination of only $n \cdot m$ minors for an n -by- m matrix.

From a topological viewpoint, TN matrices form a closed set and the set of TP matrices is dense in the set of TN matrices as the following theorem states; which is a special case of Theorem 2.4.

Theorem 2.13. [Whi52, Theorem 1], see also [Pin10, Theorem 2.6] and [FJ11, p. 62] *Given a TN matrix $A = (a_{ij}) \in \mathbb{R}^{n,m}$ and a number $0 < \epsilon$, a TP matrix $B = (b_{ij}) \in \mathbb{R}^{n,m}$ can be found such that*

$$0 \leq b_{ij} - a_{ij} < \epsilon, \quad \text{for all } i = 1, \dots, n, \quad j = 1, \dots, m.$$

Zero entries and zero minors of TN matrices are not arbitrary in nature and have an effect on other entries and minors. The following theorem states that a zero entry or a zero contiguous minor of a TN matrix "throws a shadow" or portends a linear dependence. That is, under suitable linear independence assumptions all minors of the same order in the left shadow or the right shadow of the corresponding submatrix, are also zero, see [dBP82], [Pin10, p. 13].

Proposition 2.8. [dBP82], [Pin10, Proposition 1.15] *If $A \in \mathbb{R}^{n,m}$ is TN and rank $A[i+1, \dots, i+r | j+1, \dots, j+r] = r-1$, then*

- (i) *either the rows $i+1, \dots, i+r$ or the columns $j+1, \dots, j+r$ of A are linearly dependent, or*
- (ii) *the right or left shadow of $A[i+1, \dots, i+r | j+1, \dots, j+r]$ has rank $r-1$.*

The next proposition relates some special minors of a given TN matrix with its rank.

Proposition 2.9. [Pin08, Proposition 2.5] *Let $A \in \mathbb{R}^{n,m}$ be TN. Assume that $1 = i_1 < i_2 < \dots < i_{r+1} = n$ and $1 = j_1 < j_2 < \dots < j_{r+1} = m$. If*

$$\det A[i_1, \dots, i_{r+1} | j_1, \dots, j_{r+1}] = 0,$$

while

$$\det A[i_1, \dots, i_r | j_1, \dots, j_r], \det A[i_2, \dots, i_{r+1} | j_2, \dots, j_{r+1}] > 0,$$

then A is of rank r .

Principal minors of *NsTN* matrices enjoy surprising and useful properties.

Lemma 2.6. [Kar68, Corollary 9.1, p. 89], [Cry73, Theorem 5.4] *All principal minors of a NsTN matrix are positive.*

The following lemmata are special cases of Proposition 2.8; we present them here for later reference.

Lemma 2.7. [FJ11, Corollary 7.2.10] *Suppose that $A \in \mathbb{R}^{n,n}$ is a NsTN matrix and that $A[\alpha|\beta]$ is a p -by- p submatrix with rank $(A[\alpha|\beta]) < p$. If $A[\alpha|\beta]$ is contiguous, then either its left shadow or its right shadow has rank equal to that of $A[\alpha|\beta]$. In particular, it is the one to the side of the diagonal of A on which more entries of $A[\alpha|\beta]$ lie.*

By Lemma 2.6, *NsTN* matrices have positive diagonal entries, and so a zero entry in the lower part and upper part necessarily throws a left shadow and a right shadow, respectively.

Lemma 2.8. (Shadow property) [FJ11, Corollary 1.6.5] *Suppose that $A \in \mathbb{R}^{n,n}$ is $NsTN$. Then A is a type-I staircase matrix.*

For more results about the zero-nonzero entries and minors patterns of TN matrices see, e.g., [Pin08], [Pin10, Section 1.3], and [FJ11, Section 1.6, Chapter 7].

For nonsingular matrices, the set of minors that are needed for checking total nonnegativity can be reduced to only the set of quasi-initial minors.

Theorem 2.14. [GPn93, Theorem 3.1], [FJ11, Theorem 3.3.5] *Let $A \in \mathbb{R}^{n,n}$ be nonsingular. Then A is TN if and only if for each $1 \leq k \leq n$, A satisfies the following conditions:*

$$\begin{aligned} 0 &\leq \det A[\alpha|1, \dots, k] \quad \text{for all } \alpha \in Q_{k,n}, \\ 0 &\leq \det A[1, \dots, k|\beta] \quad \text{for all } \beta \in Q_{k,n}, \\ 0 &< \det A[1, \dots, k]. \end{aligned}$$

The LU factorization of TP and TN matrices are studied and investigated in [Cry73] and [Cry76]. As in the other subclasses of SR matrices lower and upper triangular matrices are very important in the characterization of TN matrices. For the nonsingular lower (upper) triangular matrices, the quasi-initial minors play a fundamental role in their characterizations. The following theorem represents a special case of Theorem 2.14.

Theorem 2.15. [Cry76, Theorem 1.4], [And87, Corollary 2.2] *Let $A \in \mathbb{R}^{n,n}$ be nonsingular and lower triangular. Then A is TN if $0 \leq \det A[\alpha|1, \dots, k]$ for every $k = 1, \dots, n$ and $\alpha \in Q_{k,n}$.*

Theorem 2.16. [Cry73, Theorem 5.3], [Pin10, Proposition 2.11] *Let $A \in \mathbb{R}^{n,n}$ be $NsTN$. Then A has a unique factorization of the form*

$$A = LDU, \tag{2.22}$$

where L (U) is a unit diagonal lower (upper) triangular TN matrix, and D is a diagonal matrix whose diagonal entries are positive.

By using Theorems 2.15, 2.16, and Lemma 2.6, one can easily prove Theorem 2.14.

Unfortunately, if A is singular then the above theorem is no longer true. But we still have a weaker result; the unit diagonal triangular nature of the L and U , and the uniqueness are lost. In [GL12] the uniqueness theorem for an LU factorization of TN matrices by modifying the Neville elimination was obtained.

Theorem 2.17. [Cry76, Theorem 1.1], [Pin10, Theorem 2.12] *Let $A \in \mathbb{R}^{n,n}$ be TN . Then A has a factorization of the form*

$$A = LU, \tag{2.23}$$

where L (U) is a lower (upper) triangular TN matrix.

Very easy criteria for total nonnegativity which apply to tridiagonal matrices are given by the next propositions.

Proposition 2.10. [And87, Theorem 2.3], [Pin10, Theorem 4.3 and p. 100] *Let A be a nonnegative tridiagonal matrix. Then*

- (a) *A is TN if and only if all its principal minors formed from consecutive rows and columns are nonnegative.*
- (b) *A is $NsTN$ if and only if all its leading principal minors are positive.*

Proposition 2.10 (a) can be improved as the following proposition states.

Proposition 2.11. [HC10, Lemma 6] *Let $A \in \mathbb{R}^{n,n}$ with $2 < n$ be nonnegative tridiagonal. Then A is TN if*

- (i) $\det A \geq 0$,
- (ii) $\det A[1, \dots, n-1] \geq 0$,
- (iii) $\det A[1, \dots, k] > 0$, $k = 1, \dots, n-2$.

The following theorem provides another factorization of a given TN matrix; it was first proved in the nonsingular case in [RH72] and in the general case in [Cry76].

Theorem 2.18. [Cry76, Theorem 1.2] *If $A \in \mathbb{R}^{n,n}$ is TN , then there exist a TN matrix S and a tridiagonal TN matrix T such that*

- (i) $TS = SA$,
- (ii) *the matrices A and T have the same eigenvalues.*

If A is nonsingular, then S is also nonsingular.

The above factorization is very useful for the investigation of the spectral problem of TN matrices.

We conclude this section with three subsections each of which investigates a subclass of the TN matrices.

2.3.1. Totally Positive Matrices

In this subsection we present efficient determinantal criteria that can be used to check a given matrix for total positivity, characterization of totally positive matrices through their LDU factorizations, and sufficient conditions for a given positive matrix to be totally positive.

By using the Cauchy-Binet Identity, it is easy to see that the product of two TP (TN) matrices is a TP (TN) matrix. The sum of two TP (TN) matrices is not necessarily a TP (TN) matrix. However, in [JO04] it was shown that any positive matrix is the sum of TP matrices.

The following lemma plays a fundamental role in reducing the number of minors that are needed to check total positivity of a given matrix.

Lemma 2.9. [Pin10, Lemma 2.1, (Fekete's Lemma)] *Let $A \in \mathbb{R}^{n,m}$ with $m \leq n$ such that all $(m-1)$ st order minors with columns $1, \dots, m-1$ are positive and the m^{th} order minors composed from consecutive rows are also positive. Then all m^{th} order minors of A are positive.*

The above lemma can be shown by using Lemma 1.6. By repeated application of Lemma 2.9 we obtain Theorem 2.1 in the case of TP matrices. So in order to check whether a given matrix is TP it is enough to consider only the contiguous minors. This fact is used to prove the following theorem which shows that it suffices to consider only the initial minors for total positivity.

Theorem 2.19. [GPn92b, Theorem 4.1], [GPn92a] *Let $A \in \mathbb{R}^{n,m}$. Then A is TP if and only if all of its initial minors are positive.*

Theorem 2.19 specifies exactly $n \cdot m$ minors to be checked since each entry of a given matrix is a lower right corner of exactly one initial minor. For $n = m$, we have n^2 minors which is significantly smaller than $4^n(\pi n)^{-1/2}$. The following natural questions arise [LL14]:

- (1) Can the number of needed minors be further reduced?
- (2) Is the set of minors given in Theorem 2.19 unique?

The answer to the first question is no, since this number is equal to the number of the entries of the given matrix, and so this determinantal criterion is optimal [Pn13]. The answer of the second question is also no since there are many other possible choices for a set of n^2 minors which will give an efficient test for total positivity, see [FZ00, Theorem 16].

Let a given square matrix of order n be TN . Then in order to check whether this matrix is TP we need only to check the positivity of $2n-1$ minors, which are called *corner minors* [FJ11], as the next theorem states.

Theorem 2.20. [SS95], [Gla98, Theorem 4.2], [FJ11, Theorem 3.1.10] *Let $A \in \mathbb{R}^{n,n}$ be TN . Then A is TP if and only if it satisfies, for each $1 \leq k \leq n$,*

$$\begin{aligned} 0 &< \det A[1, \dots, k | n-k+1, \dots, n], \\ 0 &< \det A[n-k+1, \dots, n | 1, \dots, k]. \end{aligned}$$

For the totally positive matrices of order k , TP_k , it suffices to check the total positivity of all contiguous submatrices of order k as the following theorem states.

Theorem 2.21. [FJ11, Corollary 3.1.6] *Let $A \in \mathbb{R}^{n,m}$ with the property that all its k -by- k contiguous submatrices are TP . Then A is TP_k .*

TP matrices are also studied and characterized in terms of their factorization. The next theorem gives necessary and sufficient conditions for a given lower or upper triangular matrix to be ΔTP .

Theorem 2.22. [Cry73, Theorem 3.1] *Let $A \in \mathbb{R}^{n,n}$ be a lower (upper) triangular matrix. Then A is ΔTP if and only if $0 < \det A[\alpha|1, \dots, k]$ ($0 < \det A[1, \dots, k|\alpha]$) for all $\alpha \in Q_{k,n}$, $1 \leq k \leq n$, such that $d(\alpha) = 0$.*

The following theorem characterizes the total positivity in terms of the LDU factorizations.

Theorem 2.23. [Pin10, Theorem 2.10], see also [Cry73, Theorem 5.1] *Let $A \in \mathbb{R}^{n,n}$ be TP . Then A has a unique factorization of the form*

$$A = LDU, \quad (2.24)$$

where L (U) is a unit diagonal lower (upper) triangular ΔTP , and D is a diagonal matrix whose diagonal entries are positive.

The next theorem provides a simple but only sufficient determinantal criterion for total positivity. It uses the sign of the entries of the given matrix and a measure of the magnitude of the contiguous 2×2 minors.

Theorem 2.24. [CC98, Theorem 2.2] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ with the property that*

- (a) $0 < A$ and
- (b) $c_0 \cdot a_{i,j+1} \cdot a_{i+1,j} \leq a_{ij} \cdot a_{i+1,j+1}$ ($1 \leq i, j \leq n-1$),

where $c_0 = 4.07959562349\dots$ is the unique real root of $x^3 - 5x^2 + 4x - 1$. Then A is TP .

Analogously to Theorem 2.24, by using the density property of TP matrices and Theorem 2.24 one obtains the following sufficient condition for total nonnegativity.

Theorem 2.25. [DPn05, Theorem 1] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be a nonnegative type-I staircase matrix. Assume that, for any $1 \leq i, j \leq n-1$, the following condition holds:*

$$\text{if } 0 < a_{ij} \cdot a_{i+1,j+1}, \text{ then } c_0 \cdot a_{i,j+1} \cdot a_{i+1,j} \leq a_{ij} \cdot a_{i+1,j+1}, \quad (2.25)$$

where c_0 is given in Theorem 2.24. Then A is TN . Moreover, if the second inequality in (2.25) is strict, then A is $NsATP$, see Subsection 2.3.3.

The question arises whether the constant c_0 can be decreased. This question is answered in [KV06]; where the following theorem was proved.

Theorem 2.26. [KV06, Theorem 2], see also [Pin10, Theorem 2.16] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ with the property that*

- (a) $0 < A$ and
- (b) $c_n \cdot a_{i,j+1} \cdot a_{i+1,j} < a_{ij} \cdot a_{i+1,j+1}$ ($1 \leq i, j \leq n-1$),

where $c_n := 4\cos^2\left(\frac{\pi}{n+1}\right)$. Then A is TP .

The next theorem shows that the best possible value for c_n is the one that given by the above theorem.

Theorem 2.27. [DPn05, Theorem 4] *Let $n \in \mathbb{N}$ and $2 \leq n$. Then for any $0 < \epsilon$ there exists an n -by- n positive matrix $A_{n,\epsilon} = (a_{ij})$ for which*

$$(1 - \epsilon)c_n \cdot a_{i,j+1} \cdot a_{i+1,j} \leq a_{ij} \cdot a_{i+1,j+1}, \quad 1 \leq i, j \leq n - 1, \quad (2.26)$$

but $A_{n,\epsilon}$ is not TP .

Letting n tend to infinity, we have that the bound on c_0 given in Theorem 2.24 cannot be reduced to less than 4 when we consider matrices of any order.

Corollary 2.1. [DPn05, Corollary 5] *For any $0 < \epsilon$, there exist $n \in \mathbb{N}$, $n \geq 2$, and an n -by- n positive matrix $A = (a_{ij})$ such that*

$$4(1 - \epsilon)a_{i,j+1} \cdot a_{i+1,j} \leq a_{ij} \cdot a_{i+1,j+1}, \quad 1 \leq i, j \leq n - 1, \quad (2.27)$$

and which is not TP .

2.3.2. Oscillatory Matrices

In this subsection we present a class of matrices that lies between the nonsingular totally nonnegative and the totally positive matrices; called oscillatory matrices. This class was introduced and intensively studied by Gantmacher and Krein, see, e.g., [GK35], [GK37], [GK60], and [GK02]. We recall from Section 1.2 the following definition.

Definition 2.2. *A square matrix A is said to be an oscillatory matrix if A is TN and some integral power of A is TP .*

This class of matrices enjoys the same spectral properties as the TP matrices, see Section 2.7. Simple conditions are needed in order to check whether a given TN matrix is oscillatory or not. The following lemma can be easily shown by using Proposition 2.8.

Lemma 2.10. [FJ11, Corollary 1.6.6] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be TN with no zero rows or columns. Then A is irreducible if and only if $a_{ij} > 0$ for all $|i - j| \leq 1$.*

The next theorem was first proved by Gantmacher and Krein, it provides simple necessary and sufficient conditions for a given TN matrix to be oscillatory.

Theorem 2.28. [GK02, Theorem 10, p. 100], [And87, Theorem 4.2] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be TN . Then A is oscillatory if and only if it is nonsingular and*

$$0 < a_{i,i+1}, a_{i+1,i}, \quad i = 1, \dots, n - 1. \quad (2.28)$$

By Lemma 2.10 and Theorem 2.28, a given TN matrix is oscillatory if and only if it is nonsingular and irreducible. The key for proving Theorem 2.28 is the following theorem.

Theorem 2.29. [And87, Theorem 4.4] Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be TN. If A is nonsingular and satisfies (2.28), then $0 < \det A[\alpha|\beta]$ for every pair $\alpha, \beta \in Q_{k,n}$ such that

$$|\alpha_i - \beta_i| \leq 1, \text{ and } \max(\alpha_i, \beta_i) < \min(\alpha_{i+1}, \beta_{i+1}), \quad i = 1, \dots, k, \quad (2.29)$$

where $\alpha_{k+1} = \beta_{k+1} := \infty$.

Remark 2.2. [GK02, p. 102], [Pin10, p. 130] The largest power of a given n -by- n TN matrix A that suffices to consider for checking that A is oscillatory is $n - 1$, i.e., if A is an oscillatory matrix, then A^{n-1} is TP.

2.3.3. Almost Totally Positive Matrices

In this subsection we consider another subclass of totally nonnegative matrices which lies between the totally nonnegative and totally positive matrices; it is called the almost totally positive matrices, denoted by *ATP*. This subclass of matrices was introduced in [GMPn92].

Definition 2.3. [GMPn92, Definition 3.1] Let $A \in \mathbb{R}^{n,m}$ be TN. Then it is said to be almost totally positive (*ATP*) if it satisfies the following two conditions:

- (i) Any contiguous minor of A is positive if and only if the diagonal entries of the corresponding submatrix are positive.
- (ii) In the case that A has a zero row or column, the subsequent rows or columns are also zero, respectively.

It was proved in [GMPn92, Theorem 3.1] that if A is *ATP* then (i) holds for any minor of A (with not necessarily consecutive rows and columns).

The *ATP* matrices form an important subclass of the *TN* matrices since they appear in many practical applications such as approximation theory and computer aided geometric design [GMPn92], [P99]. There are many important examples of *ATP* matrices under certain conditions such as the Hurwitz matrix and matrices of Hurwitz type [DPn05], [AGT16], and the *B*-splines collocation matrix [dB76], [GMPn92].

Definition 2.4. Given an n -by- n matrix A , let $B := A[\alpha|\beta]$ with $\alpha, \beta \in Q_{k,n}$ be a contiguous submatrix and $a_{\alpha_1, \beta_1} \cdot a_{\alpha_2, \beta_2} \cdots a_{\alpha_k, \beta_k} \neq 0$. Then B is called a column boundary submatrix if either $\beta_1 = 1$ or $1 < \beta_1$ and $A[\alpha|\beta_1 - 1] = 0$. Analogously, B is called a row boundary submatrix if either $\alpha_1 = 1$ or $1 < \alpha_1$ and $A[\alpha_1 - 1|\beta] = 0$.

Minors corresponding to column and row boundary submatrices are called *column* and *row boundary minors*, respectively.

Theorem 2.30. [GPn06, Theorem 2.4] Let $A \in \mathbb{R}^{n,n}$ be a nonnegative type-I staircase matrix. Then the following two statements are equivalent:

- (i) A is a *NsATP* matrix.
- (ii) All row and column boundary minors of A are positive.

In Subsection 3.8.4 we give a new characterization of *ATP* matrices by using the zero-nonzero pattern of a given matrix and that of a related matrix. We conclude this subsection with the remark that in [Gla04] a subclass of matrices which is called *inner totally positive* was introduced. It was pointed out that the set of *NsATP* matrices coincides with the set of inner totally positive matrices.

2.4. Variation Diminishing Property

In this section we present an important feature of the sign regular matrices which is called *variation diminishing property*. To describe the property that for a given n -by- m matrix A the number of sign changes in the sequence of the components of $A\mathbf{x}$ is less than or equal to the number of that of \mathbf{x} for any $\mathbf{x} \in \mathbb{R}^m$, see [Pin10]. It was introduced by I. J. Schoenberg in 1930. He discovered this connection between total nonnegativity and the variation diminishing property [Sch30]. This property provides an additional criterion for sign regularity.

Let us define two counts on the number of sign changes of the components of a vector $\mathbf{x} = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, see, e.g., [Pin10, Chapter 3].

$S^-(\mathbf{x})$ is the number of the sign changes in the sequence x_1, \dots, x_n , with zero terms are discarded (with the convention $S^-(\mathbf{0}) := 0$).

$S^+(\mathbf{x})$ is the maximum number of the sign changes in the sequence x_1, \dots, x_n , where zero terms are arbitrarily assigned values $+1$ or -1 (with the convention that $S^+(\mathbf{0}) := n$).

Example 2.1. Let $\mathbf{x} = (1, 0, 2, -3, 8, -1)^T$. Then $S^-(\mathbf{x}) = 3$ and $S^+(\mathbf{x}) = 5$.

Lemma 2.11. [Pin10, Lemma 3.1] For every $\mathbf{x} = (x_1, \dots, x_n)^T \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ we have

$$S^+(\mathbf{x}) + S^-(\hat{\mathbf{x}}) = n - 1,$$

where $\hat{\mathbf{x}} = (x_1, -x_2, \dots, (-1)^{n-1}x_n)^T$.

The next two definitions introduce important classes of systems of real vectors.

Definition 2.5. [GK02, Definition 3, Definition 4, pp. 246, 248] Let

$$\mathbf{u}^i = (u_{1i}, u_{2i}, \dots, u_{ni})^T \in \mathbb{R}^n, \quad i = 1, 2, \dots, m,$$

be a system of vectors. Set for $k = 1, \dots, m$

$$\begin{aligned} \mathbf{u}^k &:= c_1 \mathbf{u}^1 + c_2 \mathbf{u}^2 + \dots + c_k \mathbf{u}^k, \\ \mathbf{c}^k &:= (c_1, \dots, c_k) \in \mathbb{R}^k \setminus \{\mathbf{0}\}. \end{aligned} \tag{2.30}$$

Then we say that the system of vectors has property T^+ (T^-) if for every $\mathbf{c}^m \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ and (2.30) the following inequality holds:

$$S^+(\mathbf{u}^m) \leq m - 1 \quad (\text{respectively, } S^-(\mathbf{u}^m) \leq m - 1),$$

and has property D^+ (D^-) if for every $\mathbf{c}^k \in \mathbb{R}^k \setminus \{\mathbf{0}\}$ and (2.30) the following inequality holds:

$$S^+(\mathbf{u}^k) \leq S^-(\mathbf{c}^k) \quad (\text{respectively, } S^-(\mathbf{u}^k) \leq S^-(\mathbf{c}^k)), \quad \text{for all } k = 1, \dots, m.$$

Definition 2.6. [GK02, Definition 6, p. 258] *The vectors*

$$\mathbf{u}^1, \mathbf{u}^2, \dots, \mathbf{u}^m \in \mathbb{R}^n \quad (m \leq n)$$

form a Markov system if for every $k \leq m$ the vectors $\mathbf{u}^1, \mathbf{u}^2, \dots, \mathbf{u}^k$ have the property T^+ .

The following theorem provides necessary and sufficient conditions for a given system of vectors to have property D^+ or be a Markov system in terms of the matrix whose columns are these vectors.

Theorem 2.31. [GK02, Theorem 2, p. 249, Corollary, p. 259] *Let*

$$\mathbf{u}^1, \mathbf{u}^2, \dots, \mathbf{u}^m, \quad (\mathbf{u}^j = (u_{1j}, u_{2j}, \dots, u_{nj})^T \in \mathbb{R}^n; \quad j = 1, 2, \dots, m, \quad m < n)$$

be a system of vectors and $U := [\mathbf{u}^1 : \mathbf{u}^2 : \dots : \mathbf{u}^m]$ be the n -by- m matrix whose columns are the vectors $\mathbf{u}^1, \mathbf{u}^2, \dots, \mathbf{u}^m$. Then

(i) *in order for this system to have property D^+ it is necessary and sufficient that U is SSR.*

(ii) *in order for this system to be a Markov system, it is necessary and sufficient that for every $k \leq m$ all minors*

$$\det U[i_1, i_2, \dots, i_k | 1, 2, \dots, k], \quad (1 \leq i_1 < i_2 < \dots < i_k \leq n)$$

be different from zero and have the same sign ϵ_k , i.e.,

$$0 < \epsilon_k \det U[i_1, i_2, \dots, i_k | 1, 2, \dots, k], \quad (1 \leq i_1 < i_2 < \dots < i_k \leq n, \quad k = 1, 2, \dots, m).$$

Matrices that never increase the number of sign variations of a vector are of interest in a variety of applications, including approximation theory and shape preserving transformations [Goo96]. The next theorems represent necessary and sufficient conditions for a given real matrix to be SSR or SR.

Theorem 2.32. [GK02, Theorem 2, Theorem 2', pp. 249-250], [Kar68, Theorem 1.2, p. 2019] *Let $A \in \mathbb{R}^{n,m}$ with $m < n$. Then A is SSR if and only if $S^+(A\mathbf{x}) \leq S^-(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$.*

Theorem 2.33. [GK02, Theorem 4, Theorem 4', p. 253] *Let $A \in \mathbb{R}^{n,m}$ with $m \leq n$ and rank $A = m$. Then A is SR if and only if $S^-(A\mathbf{x}) \leq S^-(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m$.*

Theorem 2.34. [And87, Theorem 5.6] *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular matrix. Then the following conditions are equivalent:*

- (a) A is SR,
- (b) $S^+(A\mathbf{x}) \leq S^+(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$,
- (c) $S^-(A\mathbf{x}) \leq S^+(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$,
- (d) $S^-(A\mathbf{x}) \leq S^-(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.

The variation diminishing property gives an additional way to check whether a given matrix is TP or TN as the following two theorems state.

Theorem 2.35. [Pin10, Theorem 3.3] *Let $A \in \mathbb{R}^{n,m}$. Then A is TP if and only if the following two statements hold:*

- (a) for each $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$,

$$S^+(A\mathbf{x}) \leq S^-(\mathbf{x});$$

- (b) if $S^+(A\mathbf{x}) = S^-(\mathbf{x})$ and $A\mathbf{x} \neq \mathbf{0}$, then the sign of the first (and last) component of $A\mathbf{x}$ (if zero, the sign given in determining $S^+(A\mathbf{x})$) agrees with the sign of the first (and last) nonzero component of \mathbf{x} .

Theorem 2.36. [Pin10, Theorem 3.4], see also [GK02, Theorem 5, p. 254], [Kar68, Theorem 1.5, p. 223] *Let $A \in \mathbb{R}^{n,m}$. Then A is TN if and only if the following two statements hold:*

- (a) for each $\mathbf{x} \in \mathbb{R}^m$,

$$S^-(A\mathbf{x}) \leq S^-(\mathbf{x});$$

- (b) if $S^-(A\mathbf{x}) = S^-(\mathbf{x})$ and $A\mathbf{x} \neq \mathbf{0}$, then the sign of the first (and last) nonzero component of $A\mathbf{x}$ agrees with the sign of the first (and last) nonzero component of \mathbf{x} .

We conclude this section with the following corollary which represents a special case when the given matrix is $NsTN$.

Corollary 2.2. [Pin10, Corollary 3.5] *Let $A \in \mathbb{R}^{n,n}$ be $NsTN$. Then for each $\mathbf{x} \in \mathbb{R}^n$*

$$S^+(A\mathbf{x}) \leq S^+(\mathbf{x}).$$

2.5. Planar Networks

In this section we present planar networks and their relationships to totally nonnegative matrices. Our approach in this section relies on the combinatorial constructions. It associates a totally nonnegative matrix to a planar network with nonnegative weighted edges, and conversely. The role of totally nonnegative matrices in graph theory was discovered by Karlin-McGregor [KM59], and Lindström [Lin73]. Since then this approach was used by many mathematicians to show that many properties of totally nonnegative matrices can be applied to graph theory, and conversely, see, e.g., [Bre95], [BFZ96], [CIM98], [Pn98], [FZ00], [FGJ00], [Ska03], [Ska04], [FG05], [FL07], [FJ11], [CN12].

Definition 2.7. [CN12, Definition 1] *A weighted planar network (Γ, w) of order n is a finite directed acyclic planar graph containing exactly n sources and n sinks, denoted by s_1, \dots, s_n and t_1, \dots, t_n , respectively, which lie on the boundary, and each edge e is assigned a scalar weight $w(e)$. Furthermore, the sources and sinks are configured such that they may be labeled in counterclockwise order as $t_1, \dots, t_n, s_n, \dots, s_1$. It will be assumed that the network is drawn with the sources s_1, \dots, s_n on the left and the sinks t_1, \dots, t_n on the right, with no vertical edges, and with the edges directed from left to right.*

By a *path* π in Γ we mean a directed continuous curve in the network beginning at a source s_i and terminating at a sink t_j . A family of paths is called *vertex-disjoint* if no two paths from this family intersect. The *weight of a path* π , denoted $w(\pi)$, is the product of the weights of the edges of the path π .

Definition 2.8. [CN12, Definition 2] *The weight matrix $W := W(\Gamma, w)$ of a weighted planar network (Γ, w) of order n is the n -by- n matrix $W = (w_{ij})$, where*

$$w_{ij} := \sum_{\pi \in P_{i,j}} w(\pi), \quad (2.31)$$

and $P_{i,j}$ is the set of paths from source s_i to sink t_j , $i, j = 1, \dots, n$. By convention empty sums are 0.

The minor, $\det W[\alpha|\beta]$, of the weight matrix W equals to

$$\det W[\alpha|\beta] = \sum_{\sigma \in S_k} \text{sign}(\sigma) \prod_{l=1}^k w_{\alpha_l, \beta_{\sigma(l)}}, \quad (2.32)$$

where S_k is the group of permutations of the set $\{1, \dots, k\}$ and $\alpha = (\alpha_1, \dots, \alpha_k), \beta = (\beta_1, \dots, \beta_k) \in Q_{k,n}$.

The minors of the weight matrix of a planar network have an important combinatorial interpretation which can be attributed to Lindström [Lin73] and further to Karlin and McGregor [KM59]. The following lemma describes the exact relation between planar networks with nonnegative weights and TN matrices.

Lemma 2.12. [FZ00, Lemma 1, (Lindström’s Lemma), Corollary 2] *A minor $\det W[\alpha|\beta]$ of the weight matrix W of a weighted planar network (Γ, w) of order n is equal to the sum of weights of all collections of vertex-disjoint paths that connect the sources labeled by α with the sinks labeled by β . If the planar network has nonnegative weights, then its weight matrix is TN .*

The above lemma can be used to show the total nonnegativity of various combinatorial matrices, formed from, e.g., the Pascal triangle, the q -binomial coefficients, and the Stirling numbers of both kinds [FZ00].

This combinatorial approach can also be used to check the total positivity. Before we present this criterion we recall the following definition [FZ00]. A planar network Γ is called *totally connected* if for any two subsets $\alpha, \beta \subseteq \{1, \dots, n\}$ of the same cardinality, there exists a collection of vertex-disjoint paths in Γ connecting the sources labeled by α and the sinks labeled by β .

Corollary 2.3. [FZ00, Corollary 3] *If a totally connected planar network has positive weights, then its weight matrix is TP .*

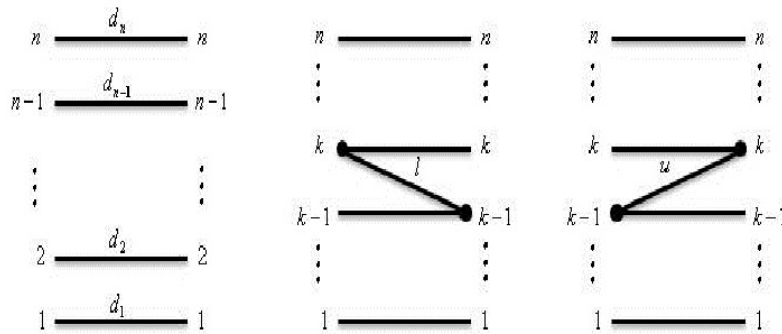


Figure 2.1.: Elementary diagrams for D (left), $L_k(l)$ (middle), and $U_j(u)$ (right), [FJ11, Figure 2.1]

In Figure 2.1, the slanted edges and the horizontal edges in the left diagram of the networks are called *essential edges*. A weighting w of a planar network Γ_0 given in Figure 2.2 is called *essential* if $w(e) \neq 0$ for any essential edge e and $w(e) = 1$ for all other edges.

Initial minors of TP matrices play an important role since for any set α of k consecutive indices of $\{1, \dots, n\}$ there is a unique collection of k vertex-disjoint paths connecting the sources labeled by $\{1, \dots, k\}$ (respectively by α) with the sinks labeled by α (respectively by

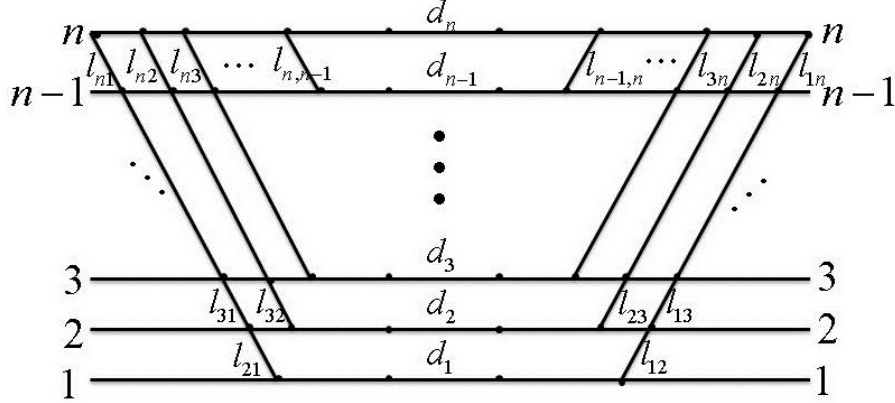


Figure 2.2.: Planar network Γ_0

$\{1, \dots, k\}$). Moreover a square matrix is uniquely determined by its initial minors, provided that all of them are nonzero [FZ00, Lemma 7].

Lemma 2.13. [FZ00, Lemma 6] *The n^2 weights of essential edges in an essential weighting w of Γ_0 are related to the n^2 initial minors of the weight matrix $W = W(\Gamma_0, w)$ by an invertible monomial transformation. Thus, an essential weighting w of Γ_0 is uniquely recovered from W .*

The following theorem describes a parametrization of the TP matrices by n^2 positive real numbers (initial minors). It can be proved by using the above discussion and Lemma 2.13.

Theorem 2.37. [FZ00, Theorem 5] *The map $w \mapsto W(\Gamma_0, w)$ restricts to a bijection between the set of all essential positive weightings of Γ_0 and the set of all n -by- n TP matrices.*

By Lindström's Lemma, for a given planar network with nonnegative weights (totally connected planar network with positive weights) the weight matrix is TN (TP). Conversely, if we are given a $NsTN$ (TP) matrix, A say, one can find a planar network such that A equals to the weight matrix. To show this we need first the following definitions and theorem.

Definition 2.9. [JOvdD99], [FJ11, Definition 2.1.1, (elementary bidiagonal matrices)] *For any positive integer n and complex numbers s, t , we let*

$$L_i(s) = I_n + sE_{i,i-1} \quad \text{and} \quad U_j(t) = I_n + tE_{j-1,j} \quad (2.33)$$

for $2 \leq i, j \leq n$. Matrices of the form $L_i(s)$ or $U_j(t)$ above are called elementary bidiagonal matrices, and the class of elementary bidiagonal matrices will be denoted by EB .

Definition 2.10. [FJ11, Definition 2.1.2 (generalized elementary bidiagonal matrices)] *Matrices of the form $D + sE_{i,i-1}$ or $D + tE_{j-1,j}$, where D is an n -by- n diagonal matrix and n is a positive integer, s, t are complex numbers, and $2 \leq i, j \leq n$, are called generalized elementary bidiagonal matrices, and this class will be denoted by *GEB*.*

It is easy to see that

$$(L_i(s))^{-1} = L_i(-s). \quad (2.34)$$

It is obvious that any GEB matrix is *TN* whenever D is nonnegative and $s, t \geq 0$, and any EB matrix is *TN* whenever $s, t \geq 0$.

One of the fundamental ways to better understand *TN* matrices is through the study of their generators. That is, we would like to find their bidiagonal factorizations which play an important role in *TN* matrices theory. For instance, bidiagonal factorization has an important combinatorial interpretation via planar networks, which have lead to an important connection between total nonnegativity and combinatorics, and plays a significant role in performing accurate computations with *TN* matrices, see e.g., [Koe05], [Koe07], and with *t.n.p.* matrices, see e.g., [Hua12]. The following theorem represents a fundamental theorem in the factorization of a given *NsTN* (*TP*) matrix. It appeared in [Loe55] for *NsTN* matrices, where it was attributed to A. M. Whitney and her work in [Whi52] although she did not make such a claim in her paper.

Theorem 2.38. [Loe55], [FZ00, Theorem 12 (Loewner-Whitney)], [Fal01], [FJ11, Theorem 2.2.2 and Corollary 2.2.3] *Any n -by- n *NsTN* (respectively, *TP*) matrix can be written as*

$$(L_n(l_k)L_{n-1}(l_{k-1}) \cdots L_2(l_{k-n+2}))(L_n(l_{k-n+1}) \cdots L_3(l_{k-2n+4})) \cdots (L_n(l_1)) \cdot D \cdot (U_n(u_1))(U_{n-1}(u_2)U_n(u_3)) \cdots (U_2(u_{k-n+2}) \cdots U_{n-1}(u_{k-1})U_n(u_k)), \quad (2.35)$$

where $k = \binom{n}{2}$; $l_i, u_j \geq 0$ (resp., $l_i, u_j > 0$) for all $i, j \in \{1, 2, \dots, k\}$; and $D = \text{diag}(d_1, \dots, d_n)$ is a diagonal matrix with all $d_i > 0$.

The diagonal matrices and elementary bidiagonal matrices can be represented by elementary diagrams as in Figure 2.1, where each horizontal edge of $L_k(l)$ and $U_j(u)$ has a weight of 1.

By Theorem 2.38 a *NsTN* matrix A can be written as a product of elementary bidiagonal matrices (2.35). Hence the weighted planar network (Γ, w) is then constructed by concatenating the "chips" corresponding to the factors $L_{n-i}(l_{k-i})$, D , and $U_{n-j}(u_{k-j})$, see, e.g., [FZ00, p. 26]. The concatenation of planar networks corresponds to multiplying their weight matrices. Hence the product of elementary bidiagonal matrices equals the weight matrix $W(\Gamma_0, w)$, see Figure 2.2.

Bidiagonal factorization also introduces an interesting criterion for a given *NsTN* matrix to be oscillatory.

Theorem 2.39. [Fal01], [FJ11, Theorem 2.6.4] *Suppose A is a $NsTN$ matrix with corresponding bidiagonal factorization given by (2.35). Then A is oscillatory if and only if at least one of the multipliers from each of L_n, L_{n-1}, \dots, L_2 and each of U_n, U_{n-1}, \dots, U_2 is positive.*

The following theorem shows that all TN matrices can be factored as products of non-negative generalized elementary bidiagonal matrices [Cry76].

Theorem 2.40. [Cry76, Theorem 1.1, Theorem 4.1] *Any n -by- n TN matrix A can be written as*

$$A = \prod_{i=1}^M L^{(i)} \prod_{j=1}^N U^{(j)}, \quad (2.36)$$

where the matrices $L^{(i)}$ and $U^{(j)}$ are, respectively, lower and upper bidiagonal TN matrices with at most one nonzero entry off the main diagonal.

The factors $L^{(i)}$ and $U^{(j)}$ are not required to be nonsingular, nor are they assumed to have constant main diagonal entries, as they are GEB matrices. There has been a significant amount of work done on the factorization of TN matrices into elementary bidiagonal matrices, see, e.g., [BFZ96], [Cry76], [GPn96], [Loe55]. Parallel to the method used in the case of $NsTN$ matrices, it is easy to show that any TN matrix equals to a weight matrix $W(\Gamma, w)$, [Ska03, Theorem 3].

2.6. Neville Elimination

In this section we present an elimination process which is called Neville elimination that gives an efficient method for checking total nonnegativity. This scheme proceeds by producing zeros in the columns of a matrix by adding to each row an appropriate multiple of the preceding one (instead of using a fixed row with a fixed pivot as in Gaussian elimination). Some authors called this elimination scheme Whitney elimination because A. M. Whitney was in their opinion the first who proved that the property of total nonnegativity is preserved under this row operation scheme (Theorem 2.41 below), see, e.g., [Loe55], [FZ00], [FJ11]. Neville elimination is also used to find the bidiagonal factorization of nonsingular totally nonnegative matrices which allows one to obtain algorithms with high relative accuracy for the computation of singular values, eigenvalues, and inverses of totally nonnegative matrices, cf. [Koe05], [Koe07]. Neville elimination is applied to several classes of matrices, see, e.g., [GT04], [GT08], [CPn97], [Hua13], [APnS15]. In [GPn94a] a matricial description of this elimination process is given.

The essence of Neville elimination dates back to the work of A. M. Whitney [Whi52], where she showed the following. Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$ be TN . Suppose that A has no zero rows and assume that

$$0 < a_{11}, \dots, 0 < a_{p1}, \quad a_{p+1,1} = \dots = a_{n1} = 0, \quad (1 \leq p \leq n).$$

Let C be the n -by- $(m-1)$ matrix that is obtained from A by dividing the elements of the i^{th} row by a_{i1} for $i = 1, \dots, p$ and then subtracting from the i^{th} row ($i = 2, \dots, p$) its preceding row and then deleting the first column.

Theorem 2.41. [Whi52, Theorem 2], [FJ11, Theorem 2.2.1, (Whitney's key reduction result)] *The matrix A is TN if and only if C is a TN matrix.*

Theorem 2.38 can be obtained from [Whi52]. In 1955, Theorem 2.38 was first appeared without proof in [Loe55], where it was referred to as Whitney's Theorem. In [FZ00] it is called the Loewner-Whitney Theorem.

We recall from [GPn92b] and [GPn96] the following description of Neville elimination. We formulate the procedure only for the square case because we will consider only the nonsingular case. Let $A \in \mathbb{R}^{n,n}$. The *Neville elimination* results in a sequence of matrices

$$A = \ddot{A}^{(1)} \rightarrow \dot{A}^{(1)} \rightarrow \ddot{A}^{(2)} \rightarrow \dot{A}^{(2)} \rightarrow \dots \rightarrow \ddot{A}^{(n)} = \dot{A}^{(n)} = U,$$

where U is an upper triangular matrix. For each k , $1 \leq k \leq n$, the matrix $\ddot{A}^{(k)} = (\ddot{a}_{ij}^{(k)})$ has zeros below its main diagonal in the first $k-1$ columns. $\dot{A}^{(k)}$ is obtained from the matrix $\ddot{A}^{(k)}$ by shifting to the bottom the rows with a zero entry in column k in such a way that the relative order among them is the same as in $\ddot{A}^{(k)}$. The matrix $\ddot{A}^{(k+1)}$ is obtained from $\dot{A}^{(k)}$ according to the following formula

$$\ddot{a}_{ij}^{(k+1)} := \begin{cases} \dot{a}_{ij}^{(k)} & \text{for } i = 1, \dots, k, j = 1, \dots, n, \\ \dot{a}_{ij}^{(k)} - \frac{\dot{a}_{ik}^{(k)}}{\dot{a}_{i-1,k}^{(k)}} \dot{a}_{i-1,j}^{(k)} & \text{if } \dot{a}_{i-1,k}^{(k)} \neq 0, \\ \dot{a}_{ij}^{(k)} & \text{if } \dot{a}_{i-1,k}^{(k)} = 0, \end{cases} \quad \text{for } i = k+1, \dots, n, \\ j = k, \dots, n.$$

The number $p_{ik} := \dot{a}_{ik}^{(k)}$, $1 \leq k \leq i \leq n$, is called the (i, k) *pivot* and the number

$$m_{ik} := \begin{cases} \frac{\dot{a}_{ik}^{(k)}}{\dot{a}_{i-1,k}^{(k)}} & \text{if } \dot{a}_{i-1,k}^{(k)} \neq 0, \\ 0 & \text{if } \dot{a}_{i-1,k}^{(k)} = 0, \end{cases} \quad (2.37)$$

is called the (i, k) *multiplier of Neville elimination*, where $1 \leq k < i \leq n$.

If all the pivots are nonzero then the entries of the intermediate matrices allow the following determinantal representation, see [GPn92b, formula (2.8)],

$$\dot{a}_{ij}^{(k)} = \frac{\det A[i-k+1, \dots, i|1, \dots, k-1, j]}{\det A[i-k+1, \dots, i-1|1, \dots, k-1]}, \quad i, j = k, \dots, n, k = 2, \dots, n. \quad (2.38)$$

Complete Neville elimination of the matrix A consists of two steps: First Neville elimination is performed to get the upper triangular matrix U and in the second step Neville elimination is applied to U^T . The (i, k) *pivot* (respectively, *multiplier*) of the *complete Neville elimination* of A is that of the Neville elimination if $k \leq i$ and the (k, i) pivot (respectively, multiplier) is that of the Neville elimination applied to U^T if $i \leq k$.

Complete Neville elimination allows an efficient test for a given matrix to be *NsTN* or *TP*.

Theorem 2.42. [GPn92b], [GPn93, Theorem 3.1] *Let $A \in \mathbb{R}^{n,n}$. Then the following three conditions are equivalent:*

- (i) *A is TP (NsTN).*
- (ii) *Complete Neville elimination applied to A can be performed without exchange of rows and columns, and all of the pivots are positive (nonnegative and the diagonal pivots are positive).*
- (iii) *All initial minors (leading principal minors) are positive (and all quasi-initial minors¹⁾ are nonnegative).*

Complete Neville elimination can also be used to check whether a given matrix is *TN* or not even if it is singular or rectangular. To formulate this result we need the following definition.

Definition 2.11. [GPn92b, Definition 5.1] *An n -by- m matrix A satisfies the condition *N* if, whenever we have carried some rows to the bottom in the performance of Neville elimination of A , those rows were zero rows, and the same condition is also satisfied in the Neville elimination of U^T .*

Theorem 2.43. [GPn92b, Theorem 5.4] *Let $A \in \mathbb{R}^{n,m}$. Then A is *TN* if and only if it satisfies the condition *N* and all the pivots are nonnegative.*

Complete Neville elimination can also be used in order to check for sign regularity in some special instances.

Theorem 2.44. [GPn94b, Theorem 3.4] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be nonsingular. Then A is *t.n.* if and only if $a_{nn} < 0$ and satisfies condition *N*, with positive multipliers, and with diagonal pivot p_{ii} satisfying*

$$p_{11} < 0, \quad 0 < p_{ii}, \quad i = 2, \dots, n. \quad (2.39)$$

In the following theorem we denote for a given $A \in \mathbb{R}^{n,n}$ by $A^{(k)}$ the principal submatrix $A[k, \dots, n]$ and $p_{11}^{(k)}, \dots, p_{n-k+1, n-k+1}^{(k)}$ the diagonal pivots of $A^{(k)}$ for $k = 1, \dots, n$.

Theorem 2.45. [GPn94b, Theorem 4.1] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be nonsingular. Then A is *SSR* if and only if the following conditions hold:*

- (i) *The complete Neville elimination of $A^{(1)} = A$ can be performed without row or column exchanges, with positive multipliers, and with nonzero diagonal pivots p_{11}, \dots, p_{nn} .*
- (ii) *$\text{sign}(a_{nn}) = \text{sign}(p_{11})$.*

¹⁾ For the definition see Section 1.2.

(iii) Let r be the integer such that $\text{sign}(p_{11}) = \dots = \text{sign}(p_{rr}) \neq \text{sign}(p_{r+1,r+1})$ ($r := n$ if $\text{sign}(p_{ii}) = \text{sign}(p_{11})$ for all $i = 2, \dots, n$). If $r = 1$, let s be the integer such that $\text{sign}(p_{11}) \neq \text{sign}(p_{22}) = \dots = \text{sign}(p_{s+1,s+1}) \neq \text{sign}(p_{s+2,s+2})$ ($s := n - 1$ if $\text{sign}(p_{ii}) = \text{sign}(p_{22})$ for all $i = 3, \dots, n$). If $1 < r$ ($r = 1$), then for each $2 \leq l \leq n - r + 1$ ($2 \leq l \leq n - s$) the complete Neville elimination of $A^{(l)}$ can be performed without row or column exchanges, with positive multipliers, and with nonzero diagonal pivots $p_{ii}^{(l)}$ ($1 \leq i \leq n - l + 1$) having signs

$$\text{sign}(p_{11}^{(l)}) = \text{sign}(p_{11}), \dots, \text{sign}(p_{n-l+1,n-l+1}^{(l)}) = \text{sign}(p_{n-l+1,n-l+1}). \quad (2.40)$$

In [CPn07], an elimination procedure which is called *two-determinant pivoting strategy* and which is associated to Neville elimination was presented. It was used to provide some properties of SR matrices. Neville elimination and so-called quasi-Neville elimination are employed to give a characterization and a bidiagonal factorization of rectangular TN matrices, see, e.g., [GT04] and [GT08], and for *t.n.p.* matrices, see, e.g., [Pn03], [CRU09], and [CRU14].

2.7. Some Spectral Properties of Totally Nonnegative and Totally Nonpositive Matrices

In this section we present some spectral properties of totally positive, totally nonnegative, oscillatory, totally negative, and totally nonpositive matrices. The theory of nonnegative and positive matrices which summarized under the heading Perron-Frobenius theory, see, e.g., [BP94] and [Gan59], forms an indispensable ingredient in the study of the sign regular matrices. The following theorem serves as a basis for the study of the eigenvalues and eigenvectors of sign regular matrices.

Theorem 2.46. [GK02, Theorem 5, (Perron), p. 83] *If a real square matrix is positive, then there exists a positive simple eigenvalue which exceeds the moduli of all other eigenvalues. To this "maximum" eigenvalue there corresponds a positive eigenvector.*

Let $\mathbf{x} = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, \dots, y_{n-1})^T \in \mathbb{R}^{n-1}$ be such that

$$x_n < x_{n-1} < \dots < x_2 < x_1, \quad y_{n-1} < y_{n-2} < \dots < y_2 < y_1.$$

Then we say that \mathbf{y} *strictly interlace* \mathbf{x} if the following inequalities hold:

$$x_n < y_{n-1} < x_{n-1} < y_{n-2} < x_{n-2} < \dots < x_2 < y_1 < x_1.$$

Gantmacher and Krein, see, e.g., [GK37] and [GK02] introduced the concept of total positivity and proved that the eigenvalues of every TP matrix are positive and distinct. The following theorem gives a complete characterization of the spectral properties of TP and oscillatory matrices.

Theorem 2.47. [Pn98], [GK02, Theorem 6, p. 87], [Pin10, Theorem 5.3, (Gantmacher-Krein)] *Let $A \in \mathbb{R}^{n,n}$ be oscillatory. Then A has n positive and simple eigenvalues. Let*

$$\lambda_n < \lambda_{n-1} < \dots < \lambda_2 < \lambda_1 \quad (2.41)$$

denote these eigenvalues and \mathbf{u}^k be a real eigenvector (unique up to multiplication by a nonzero constant) associated with the eigenvalue λ_k , $k = 1, \dots, n$. Then

$$q - 1 \leq S^- \left(\sum_{i=q}^p c_i \mathbf{u}^i \right) \leq S^+ \left(\sum_{i=q}^p c_i \mathbf{u}^i \right) \leq p - 1 \quad (2.42)$$

for each $1 \leq q \leq p \leq n$ (and c_i not all zero). In particular, $S^-(\mathbf{u}^j) = S^+(\mathbf{u}^j) = j - 1$ for all $j = 1, \dots, n$. In addition, the eigenvalues of the principal submatrices obtained by deleting the first (or last) row and column of A strictly interlace those of A .

Formula (2.42) holds for the eigenvectors of any square SR matrix whose some integral power is SSR . The eigenvalues of such a matrix are all real, simple, and distinct, see [GK02, Theorem 9, p. 263].

Conversely to the above theorem, one can check whether a given matrix or some of its powers are TP or not by using its eigenvalues and the left and right eigenvectors.

Theorem 2.48. [GK02, Theorem 19, p. 272], [And87, Theorem 6.4] *If a matrix $A \in \mathbb{R}^{n,n}$ has n nonzero real eigenvalues*

$$\lambda_1, \lambda_2, \dots, \lambda_n$$

whose absolute values are distinct:

$$|\lambda_n| < |\lambda_{n-1}| < \dots < |\lambda_1|, \quad (2.43)$$

and the eigenvectors

$$\mathbf{u}^j = (u_{1j}, u_{2j}, \dots, u_{nj})^T \text{ and } \mathbf{v}^j = (v_{1j}, v_{2j}, \dots, v_{nj})^T, \quad j = 1, 2, \dots, n, \quad (2.44)$$

of the matrices A and A^T corresponding to these eigenvalues form two Markov systems, then some power of the matrix A is TP .

From the relation between TP matrices and oscillatory matrices, we have the following corollary.

Corollary 2.4. [GK02, Corollary, p. 273] *Let $A \in \mathbb{R}^{n,n}$ be TN satisfying the following conditions:*

- 1) *All eigenvalues are positive and distinct, and*
- 2) *the eigenvectors of the matrices A and A^T form two Markov systems.*

Then the matrix A is oscillatory.

By using Theorems 1.2 and 2.46, and applying the methods that are used in the proof of Theorem 2.47 we have the following theorem which characterizes the eigenvalues of a given *t.n.* matrix.

Theorem 2.49. [FD00, Theorem 2.3] *Let $A \in \mathbb{R}^{n,n}$ be *t.n.* Then all the eigenvalues of A are real and distinct. Moreover, exactly one eigenvalue is negative, and this eigenvalue is simple, has the largest modulus and has a positive eigenvector.*

By continuity, it follows from Theorems 2.47 and 2.49 that if $A \in \mathbb{R}^{n,n}$ is *TN* (*t.n.p.*), then all the eigenvalues of A are real and nonnegative (at most one is negative). For more about the spectral structures of *TN* matrices see [Pin10, Chapter 5] and [FJ11, Chapter 5]. The inverse eigenvalue problem of *TP* matrices is solved in [BJ84], and a complete solution for the spectral problem for irreducible *TN* matrices is given in [FGJ00] and [FG05].

3. Totally Nonnegative Cells and the Cauchon Algorithm

In this chapter we present a partition of the set of the totally nonnegative matrices into so-called totally nonnegative cells, i.e., we divide the set of totally nonnegative matrices into classes according to the whether some specified minors vanish. This is done by the so-called Cauchon Algorithm. This algorithm also provides an efficient method for checking total nonnegativity and nonsingular total nonpositivity. The relation between this algorithm and Neville elimination is investigated. Representation of the entries of the resulting matrix that is obtained by the application of the Cauchon Algorithm to nonsingular totally nonnegative and nonsingular totally nonpositive matrices, optimal determinantal criteria for total nonnegativity and nonsingular total nonpositivity, and characterizations of some common subclasses and examples of totally nonnegative matrices by using the Cauchon Algorithm are given.

This chapter consists of eight sections. In Section 3.1, totally nonnegative cells, Cauchon diagrams, and Cauchon matrices are presented. Section 3.2 consists of two subsections in which the Cauchon Algorithm is introduced and its condensed form is derived. In Section 3.3, the connections and relations between totally nonnegative matrices and the matrices obtained by the Cauchon Algorithm are given. In Section 3.4, the relation between Neville elimination and the condensed form of the Cauchon Algorithm is investigated, and a method for writing a given nonsingular totally nonnegative matrix as a product of bidiagonal matrices is proposed. In Section 3.5, the Cauchon Algorithm is applied to totally nonpositive matrices and new characterizations of nonsingular totally nonpositive and totally negative matrices by Cauchon matrices are shown. In Section 3.6, optimal determinantal criteria for testing totally nonnegative and nonsingular totally nonpositive matrices are derived. In Section 3.7, the representations of the entries of a matrix which is obtained by application of the Cauchon Algorithm to nonsingular totally nonnegative and nonsingular totally nonpositive matrices are given; these will serve as the key in the proof of Garloff's Conjecture and analogous results in Sections 4.2 and 4.3. In Section 3.8, the findings that are presented in the previous sections are used to give new characterizations and simple proofs for total nonnegativity of some subclasses and examples of totally nonnegative matrices.

3.1. Totally Nonnegative Cells

In this section we present a partition of the set of totally nonnegative matrices, the Cauchon diagram, and Cauchon matrix that will be used to parametrize the totally nonnegative matrices.

We denote the set of the n -by- m totally nonnegative matrices by $TN_{n,m}$, and any family of minors by \mathcal{F} which is viewed as elements of the form $[\alpha|\beta]$ for some $\alpha \in Q_{k,n}$, $\beta \in Q_{k,m}$, and $k \leq n' := \min\{n, m\}$, i.e.,

$$\mathcal{F} := \{[\alpha|\beta] \mid \alpha \in Q_{k,n}, \beta \in Q_{k,m} \text{ for some } k \leq n'\}.$$

The set $TN_{n,m}$ admits a partition into so-called totally nonnegative cells as follows [GLL11b], [LL14]. The *totally nonnegative cell* (abbreviated *TN cell*) associated with the family of minors \mathcal{F} , denoted by $\mathcal{S}_{\mathcal{F}}$, is defined as

$$\mathcal{S}_{\mathcal{F}} := \{A \in TN_{n,m} \mid \det A[\alpha|\beta] = 0 \text{ if and only if } [\alpha|\beta] \in \mathcal{F}\}. \quad (3.1)$$

The following example shows that not all *TN* cells are nonempty.

Example 3.1. [GLL11b, p.782] *Let the family of minors $\mathcal{F} = \{[2|2]\}$ in $TN_{2,2}$. Then the cell $\mathcal{S}_{\mathcal{F}}$ is empty. To show this, suppose on the contrary that this cell were nonempty. Then there would exist a *TN* matrix A such that*

$$A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}, \quad (3.2)$$

where $0 < a, b, c$, but then we have $0 < \det A = -bc < 0$, which is impossible.

Hence the following definition is meaningful.

Definition 3.1. [GLL11b, Definition 1.1] *A family of minors, \mathcal{F} say, is admissible if the corresponding *TN* cell, $\mathcal{S}_{\mathcal{F}}$, is nonempty.*

The set $TN_{n,m}$ possesses the following partition

$$TN_{n,m} = \cup_{\mathcal{F} \text{ admissible}} \mathcal{S}_{\mathcal{F}}. \quad (3.3)$$

In general, for a given n -by- m matrix there are $\binom{n+m}{n} - 1$ minors and so there are $2^{\binom{n+m}{n} - 1}$ family of minors. Some of these families are admissible and some of them are not. Among these families there are a large number of admissible families. A weak, but easily established, lower bound is that $TN_{n,m}$ has more than n^m nonempty cells, [LL14, p. 373].

Although we have a very large number of *TN* cells and the set of *TP* matrices is dense in the set of *TN* matrices, the set of *TP* matrices equals the *TN* cell that is associated with $\mathcal{F} = \phi$.

The parametrization of nonempty *TN* cells is very important. There is a parametrization of them by using Cauchon diagrams. In fact, there is a one to one correspondence between these diagrams and the *TN* cells as we will see later.

Definition 3.2. [LL14, Definition 2.1] *An n -by- m Cauchon diagram C is an n -by- m grid consisting of $n \cdot m$ squares colored black and white, where each black square has the property that either every square to its left (in the same row) or every square above it (in the same column) is black.*

We denote by $\mathcal{C}_{n,m}$ the set of the n -by- m Cauchon diagrams; when $n = m$ we write \mathcal{C}_n . Following [GLL11b], we identify an n -by- m Cauchon diagram with the set of coordinates of its black squares, i.e., we fix positions in a Cauchon diagram in the following way: For $C \in \mathcal{C}_{n,m}$ and $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, we say that $(i, j) \in C$ if the square in row i and column j is black. Here we use the usual matrix notation for the (i, j) position in a Cauchon diagram, i.e., the square in $(1, 1)$ position of the Cauchon diagram is in its top left corner.

For instance, for the Cauchon diagram C depicted in Figure 3.1, we have $(2, 3) \notin C$, whereas $(3, 2) \in C$.

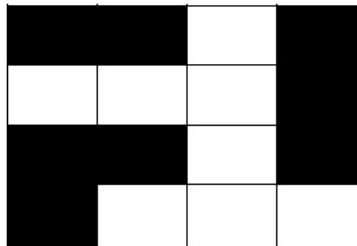


Figure 3.1.: An example of a Cauchon diagram

Definition 3.3. [LL14, Definition 2.2] *Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$ and $C \in \mathcal{C}_{n,m}$. We say that A is a Cauchon matrix associated with the Cauchon diagram C if for all (i, j) , $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, we have $a_{ij} = 0$ if and only if $(i, j) \in C$. If A is a Cauchon matrix associated with an unspecified Cauchon diagram, we just say that A is a Cauchon matrix.*

If A is a Cauchon matrix, then we also say that C is a *Cauchon diagram associated to the Cauchon matrix A* , denoted by C_A , if A is a Cauchon matrix associated with the Cauchon diagram C .

Recall that the zero-nonzero pattern of a given TN matrix has a special form, so one can easily prove that every TN matrix is a Cauchon matrix, see [GLL11b, Lemma B.1], while the converse need not be true, but starting from a given nonnegative Cauchon matrix we can generate a TN matrix as we will see in the next section.

3.2. The Cauchon Algorithm and its Condensed Form

In this section we recall the Cauchon Algorithm which can be used to check a given matrix for total nonnegativity and derive its condensed form by which we reduce the needed work. This algorithm is also used to check the membership of a given totally nonnegative matrix in a specified totally nonnegative cell. The Cauchon Algorithm was originally developed by G. Cauchon [Cau03] while studying quantum matrices. In [GLL11b] and [GLL11a] a

connection has been shown to exist between the cell decomposition of totally nonnegative matrices and the invariant prime spectrum of the algebra of quantum matrices.

3.2.1. Cauchon Algorithm

In this subsection we present the Cauchon Algorithm¹⁾ and its inverse, the Restoration Algorithm. First we need some notations.

Set $E^\circ := \{1, \dots, n\} \times \{1, \dots, m\} \setminus \{(1, 1)\}$, $E := E^\circ \cup \{(n+1, 2)\}$.

Let $(s, t) \in E^\circ$. Then $(s, t)^+ := \min \{(i, j) \in E \mid (s, t) \leq (i, j), (s, t) \neq (i, j)\}$; here the minimum is taken with respect to the lexicographical order.

Algorithm 3.1. [GLL11b], [LL14] (The Cauchon Algorithm) *Let $A \in \mathbb{R}^{n,m}$. As r runs in decreasing order over the set E with respect to the lexicographical order, we define matrices $A^{(r)} = (a_{ij}^{(r)}) \in \mathbb{R}^{n,m}$ as follows:*

1. Set $A^{(n+1,2)} := A$.
2. For $r = (s, t) \in E^\circ$ define the matrix $A^{(r)} = (a_{ij}^{(r)})$ as follows:
 - (a) If $a_{st}^{(r^+)} = 0$, then put $A^{(r)} := A^{(r^+)}$.
 - (b) If $a_{st}^{(r^+)} \neq 0$, then put

$$a_{ij}^{(r)} := \begin{cases} a_{ij}^{(r^+)} - \frac{a_{it}^{(r^+)} a_{sj}^{(r^+)}}{a_{st}^{(r^+)}} & \text{for } i < s \text{ and } j < t, \\ a_{ij}^{(r^+)} & \text{otherwise.} \end{cases} \quad (3.4)$$

3. Set $\tilde{A} := A^{(1,2)}$; \tilde{A} is called the matrix obtained from A (by the Cauchon Algorithm)²⁾.

Example 3.2. *Let*

$$A = \begin{bmatrix} 6 & 3 & 3 & 1 \\ 3 & 2 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \quad (3.5)$$

Then by application of the Cauchon Algorithm to A we obtain

¹⁾ This algorithm is called in [GLL11b] the deleting derivations algorithm (as the inverse of the Restoration Algorithm) and in [LL14] the Cauchon reduction algorithm.

²⁾ Note that $A^{(k,1)} = A^{(k,2)}$, $k = 1, \dots, n-1$, and $A^{(2,2)} = A^{(1,2)}$ so that the algorithm could already be terminated when $A^{(2,2)}$ is computed.

$$\begin{aligned}
A^{(5,2)} = A, \quad A^{(4,4)} &= \begin{bmatrix} 5 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A^{(4,3)} = \begin{bmatrix} 3 & 0 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A^{(4,3)} = A^{(4,2)} = A^{(4,1)}, \\
A^{(3,4)} &= \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A^{(3,3)} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \\
A^{(3,3)} = A^{(3,2)} = A^{(3,1)} = A^{(2,4)} = A^{(2,3)} = A^{(2,2)} = A^{(2,1)} = A^{(1,4)} = A^{(1,3)} = A^{(1,2)} = \tilde{A}.
\end{aligned}$$

The next algorithm represents the inverse algorithm of the Cauchon Algorithm which is introduced in [Lau04].

Algorithm 3.2. [Lau04, Proposition 3.5], [GLL11b] (The Restoration Algorithm) *Let $A \in \mathbb{R}^{n,m}$. As r runs in increasing order over the set E with respect to the lexicographical order, we define matrices $A^{(r)} = (a_{ij}^{(r)}) \in \mathbb{R}^{n,m}$ as follows:*

1. Set $A^{(1,2)} := A$.
2. For $r = (s, t) \in E^\circ$ define the matrix $A^{(r^+)} = (a_{ij}^{(r^+)})$ as follows:
 - (a) If $a_{st}^{(r)} = 0$, then put $A^{(r^+)} := A^{(r)}$.
 - (b) If $a_{st}^{(r)} \neq 0$, then put

$$a_{ij}^{(r^+)} := \begin{cases} a_{ij}^{(r)} + \frac{a_{it}^{(r)} a_{sj}^{(r)}}{a_{st}^{(r)}} & \text{for } i < s \text{ and } j < t, \\ a_{ij}^{(r)} & \text{otherwise.} \end{cases} \quad (3.6)$$

3. Set $\bar{A} := A^{(n+1,2)}$; \bar{A} is called the matrix obtained from A (by the Restoration Algorithm).

The following theorem shows that the application of the Restoration Algorithm to a given nonnegative Cauchon matrix results in a TN matrix.

Theorem 3.1. [GLL11b, Theorem 4.1] *Let $A \in \mathbb{R}^{n,m}$ be a nonnegative Cauchon matrix. Then \bar{A} is TN .*

In the sequel we also employ the following notations that make the presentation more clear, see [GLL11b].

Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$ and $\delta := \det A[\alpha|\beta]$ be a minor of A . If $r = (s, t) \in E$, set

$$\delta^{(r)} := \det A^{(r)}[\alpha|\beta].$$

For $i \in \alpha$, $t \notin \alpha$, and $j \in \beta$, set

$$\begin{aligned}\delta_{\hat{\alpha}_i \hat{\beta}_j}^{(r)} &:= \det A^{(r)}[\alpha_i | \beta_j], \\ \delta_{\hat{\alpha}_i \rightarrow t}^{(r)} &:= \det A^{(r)}[\alpha_i \cup \{t\} | \beta].\end{aligned}$$

We conclude this subsection with two propositions and a lemma, given in [GLL11b], which relate the determinants of some special submatrices of the intermediate matrices during the performance of the Restoration Algorithm (or its inverse, the Cauchon Algorithm). These results will be very useful in the analysis of the intermediate matrices, see Section 3.5.

Proposition 3.1. [GLL11b, Proposition 3.7] *Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$ and $r = (s, t) \in E^\circ$. Assume that $a_{st} \neq 0$. Let $\delta = \det A[\alpha | \beta]$ with $\alpha \in Q_{l,n}$, $\beta \in Q_{l,m}$, and $(\alpha_l, \beta_l) = r$. Then $\delta^{(r^+)} = \delta_{\hat{s}\hat{t}}^{(r)} a_{st}$ holds.*

Proposition 3.2. [GLL11b, Proposition 3.11] *Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$ and $r = (s, t) \in E^\circ$. Let $\delta = \det A[\alpha | \beta]$ with $\alpha \in Q_{l,n}$, $\beta \in Q_{l,m}$, and $(\alpha_l, \beta_l) < r$. If $a_{st} = 0$, or if $\alpha_l = s$, or if $t \in \{\beta_1, \dots, \beta_l\}$, or if $t < \beta_1$, then $\delta^{(r^+)} = \delta^{(r)}$.*

Lemma 3.1. [GLL11b, Lemma B.3] *Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$, and let $r = (s, t) \in E^\circ$. Let $\delta = \det A[\alpha | \beta]$ with $\alpha \in Q_{l,n}$, $\beta \in Q_{l,m}$. Assume that $a_{st} \neq 0$ and that $\alpha_l < s$ while $\beta_h < t < \beta_{h+1}$ for some $h \in \{1, \dots, l\}$ (by convention, $\beta_{l+1} := m + 1$). Then*

$$\delta^{(r^+)} = \delta^{(r)} + \sum_{k=1}^h (-1)^{k+h} \delta_{\beta_k \rightarrow t}^{(r)} a_{s, \beta_k} a_{st}^{-1}. \quad (3.7)$$

3.2.2. Condensed Form of the Cauchon Algorithm

In this subsection we present the condensed form of the Cauchon Algorithm which reduces for a given n -by- n matrix the number of the required arithmetic operations from $O(n^4)$ to $O(n^3)$ (for the exact number of operations, see Section 3.4). We accomplish this by relating the entries of $A^{(k,2)}$ to the entries of $A^{(k+1,2)}$, $k = 2, \dots, n$. The following proposition plays a fundamental role in deriving the condensed form of the Cauchon Algorithm.

Proposition 3.3. [AG14a, Proposition 3.4] *Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$. Then for the entries of the matrices generated by Algorithm 3.1 the following relation holds for $k = n, \dots, 2$:*

$$a_{ij}^{(k,2)} = \begin{cases} \frac{\det A^{(k+1,2)}[i,k|j,u_j]}{a_{ku_j}^{(k+1,2)}} & \text{if } u_j < \infty, \\ a_{ij}^{(k+1,2)} & \text{if } u_j = \infty, \end{cases}$$

where $u_j := \min \{h \in \{j+1, \dots, m\} \mid a_{kh}^{(k+1,2)} \neq 0\}$ (we set $u_j := \infty$ if this set is empty), $j = 1, \dots, m-1$, $i = 1, \dots, k-1$.

Proof. It suffices to prove the relation for $k = n$.

By step 2. of Algorithm 3.1 we have

$$a_{ij}^{(n,2)} = a_{ij}^{(n,j+1)} = \begin{cases} \frac{\det A^{(n,u_j)^+}[i,n|j,u_j]}{a_{n,u_j}} & \text{if } u_j < \infty, \\ a_{ij} & \text{if } u_j = \infty, \end{cases} \quad (3.8)$$

for $i = 1, \dots, n-1$, $j = 1, \dots, m-1$, where u_j is defined as above. By application of Proposition 3.2 (several times if necessary) to $\det A^{(n,u_j)^+}[i,n|j,u_j]$ we obtain

$$\det A^{(n,u_j)^+}[i,n|j,u_j] = \det A[i,n|j,u_j]. \quad (3.9)$$

By substituting (3.9) into (3.8) the result follows. \square

Based on Proposition 3.3 we give the Cauchon Algorithm in its condensed form (note that we use the upper index in a slightly more convenient form).

Algorithm 3.3. [AG14a, Algorithm 3.2](The condensed form of the Cauchon Algorithm)

Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$. Set $A^{(n)} := A$.

For $k = n-1, \dots, 1$ define $A^{(k)} = (a_{ij}^{(k)}) \in \mathbb{R}^{n,m}$ as follows:

For $i = 1, \dots, k$,

for $j = 1, \dots, m-1$

set $u_j := \min \{h \in \{j+1, \dots, m\} \mid a_{k+1,h}^{(k+1)} \neq 0\}$ (we set $u_j := \infty$ if this set is empty)

$$a_{ij}^{(k)} := \begin{cases} a_{ij}^{(k+1)} - \frac{a_{k+1,j}^{(k+1)} a_{iu_j}^{(k+1)}}{a_{k+1,u_j}^{(k+1)}} & \text{if } u_j < \infty, \\ a_{ij}^{(k+1)} & \text{if } u_j = \infty, \end{cases}$$

for $i = k+1, \dots, n$, $j = 1, \dots, m$, and $i = 1, \dots, k$, $j = m$

$$a_{ij}^{(k)} := a_{ij}^{(k+1)}.$$

Put $\hat{A} := A^{(1)}$.

It follows from Proposition 3.3 that $\tilde{A} = \hat{A}$ holds. If A is symmetric then \tilde{A} is symmetric, too, and therefore it is not necessary to consider the entries of \tilde{A} above the main diagonal if $a_{kk}^{(k)} \neq 0$ for $k = n, n-1, \dots, 2$. The Cauchon Algorithm can then be shortened in the way that after the k^{th} step the computations are continued with the submatrix $A^{(k)}[1, \dots, k|1, \dots, k]$, $k = n-1, \dots, 2$, because the k^{th} row of $A^{(k)}$ is identical with the k^{th} column of \hat{A} .

Example 3.3. Let A be given as in Example 3.2. Then by application of the condensed form of the Cauchon Algorithm to A we obtain

$$A^{(4)} = A, \quad A^{(3)} = \begin{bmatrix} 3 & 0 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A^{(2)} = A^{(1)} = \tilde{A}.$$

3.3. Cauchon Algorithm and Totally Nonnegative Matrices

In this section we present the relationships between the totally nonnegative matrices and the Cauchon Algorithm, and a special type of finite sequences which play a fundamental role in reducing the required number of minors that are needed for checking total nonnegativity.

The following theorem describes properties of the matrices that result from the application of the Cauchon Algorithm on a given TN matrix and gives necessary and sufficient conditions for a given matrix to be TN or TP by using the Cauchon Algorithm. In Section 3.5, we will present analogous results in the case of $Ns.t.n.p.$ and $t.n.$ matrices.

Theorem 3.2. [GLL11b, Theorem B4], [LL14, Theorems 2.6 and 2.7] *Let $A \in \mathbb{R}^{n,m}$. Then the following statements hold:*

- (i) *If A is TN and $2 \leq s$, then for all $(s,t) \in E$, $0 \leq A^{(s,t)}$, $A^{(s,t)}$ is a Cauchon matrix, and $A^{(s,t)}[1, \dots, s-1 | 1, \dots, m]$ is TN .*
- (ii) *A is TP (TN) if and only if $0 < \tilde{A}$ ($0 \leq \tilde{A}$ and a Cauchon matrix).*

Before we present a theorem which gives an efficient criterion for partitioning the set of TN matrices into cells we need the following definition.

Let π be the function defined on the set $TN_{n,m}$ into the set of Cauchon diagrams, i.e.,

$$\pi : TN_{n,m} \rightarrow \mathcal{C}_{n,m}, \quad (3.10)$$

such that $\pi(A)$ is the Cauchon diagram associated to the Cauchon matrix \tilde{A} , for any $A \in TN_{n,m}$.

Theorem 3.3. [LL14, Theorem 2.7]

1. *Let $A, B \in TN_{n,m}$. Then A and B belong to the same TN cell if and only if $\pi(A) = \pi(B)$, i.e., \tilde{A} and \tilde{B} are associated to the same Cauchon diagram.*
2. *Nonempty TN cells are parametrized by the Cauchon diagrams, and the nonempty TN cells in $TN_{n,m}$ are precisely the sets*

$$\mathcal{S}_C^0 := \{A \in TN_{n,m} \mid \pi(A) = C\}, \quad (3.11)$$

where C runs over the set $\mathcal{C}_{n,m}$.

3. *Let $A \in TN_{n,m}$. Then $A \in \mathcal{S}_C^0$ if and only if $\tilde{a}_{ij} = 0$ if $(i,j) \in C$ and $\tilde{a}_{ij} > 0$ if $(i,j) \notin C$.*

The number of TN cells in $TN_{n,m}$ is exactly equal to the number of the n -by- m Cauchon diagrams since by Theorem 3.2 (ii) starting from a TN matrix and application of the Cauchon Algorithm yields a nonnegative Cauchon matrix and conversely, by Theorem 3.1 starting from a nonnegative Cauchon matrix and application of the Restoration Algorithm results in a TN matrix.

We define a special type of finite sequences which play a fundamental role in characterizing TN cells.

Definition 3.4. [LL14, Definition 3.1] Let $C \in \mathcal{C}_{n,m}$. We say that a sequence

$$\gamma := ((i_k, j_k), k = 0, 1, \dots, p) \quad (3.12)$$

which is strictly increasing in both arguments is a lacunary sequence with respect to C if the following conditions hold:

1. $(i_k, j_k) \notin C$, $k = 1, \dots, p$;
2. $(i, j) \in C$ for $i_p < i \leq n$ and $j_p < j \leq m$.
3. Let $s \in \{0, \dots, p-1\}$. Then $(i, j) \in C$ if
 - (a) either for all (i, j) , $i_s < i < i_{s+1}$ and $j_s < j$,
or for all (i, j) , $i_s < i < i_{s+1}$ and $j_0 \leq j < j_{s+1}$
 and
 - (b) either for all (i, j) , $i_s < i$ and $j_s < j < j_{s+1}$
or for all (i, j) , $i < i_{s+1}$, and $j_s < j < j_{s+1}$.

Condition 3. of Definition 3.4 is illustrated by Figures 3.2 and 3.3; here the collection of black squares determined by Condition (a) or (b) (displayed in dark gray) is enlarged by taking into account that the underlying diagram is a Cauchon diagram (displayed in light gray). In Figure 3.1, the sequence $((1, 1), (2, 3), (4, 4))$ is a lacunary sequence while the sequence $((1, 1), (2, 2), (4, 4))$ is not.

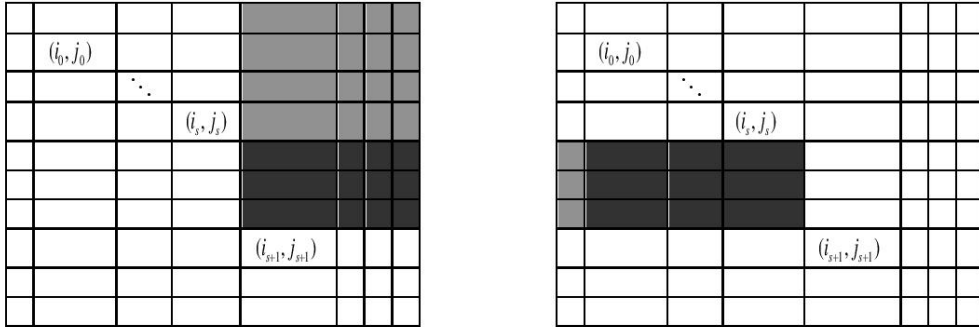


Figure 3.2.: Condition 3.(a) of Definition 3.4

Definition 3.5. For the lacunary sequence γ given by (3.12), p is called its length. If

$$(i_{k+1}, j_{k+1}) = (i_k + 1, j_k + 1), \quad k = 0, 1, \dots, p-1,$$

then we say that the sequence γ is diagonal.

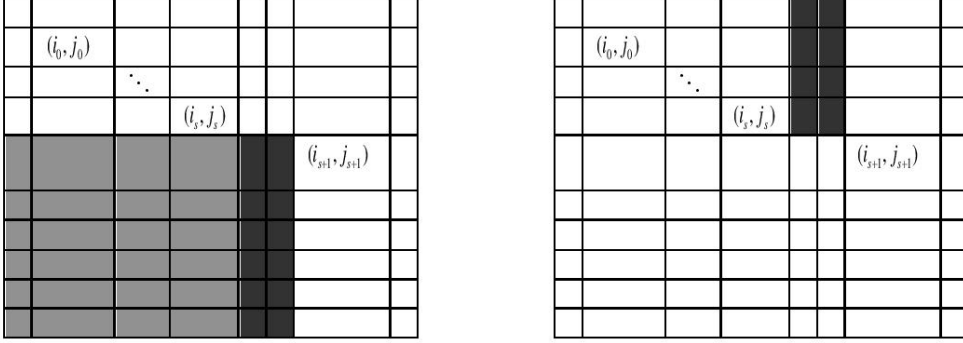


Figure 3.3.: Condition 3.(b) of Definition 3.4

In [LL14] an algorithm for generating lacunary sequences with respect to a given Cauchon diagram is introduced. These sequences play a fundamental role in reducing the number of minors that are needed in order to check whether a given matrix belongs to the cell associated to a given Cauchon diagram to $n \cdot m$ minors. Later we will use these sequences to solve the recognition problem for TN and $Ns.t.n.p.$ matrices of order n by checking only n^2 minors.

Proposition 3.4. [LL14, Proposition 4.1] *Let $A \in \mathbb{R}^{n,m}$ and $C \in \mathcal{C}_{n,m}$. For each $i_0 \in \{1, \dots, n\}$ and $j_0 \in \{1, \dots, m\}$ fix a lacunary sequence γ given by (3.12) with respect to C starting at (i_0, j_0) . Assume that either (a) or (b) below holds.*

(a) *The matrix A is TN and $A \in \mathcal{S}_C^0$.*

(b) *For all (i_0, j_0) , we have*

$$\begin{aligned} 0 &= \det A[i_0, i_1, \dots, i_p | j_0, j_1, \dots, j_p] \quad \text{if } (i_0, j_0) \in C \quad \text{and} \\ 0 &< \det A[i_0, i_1, \dots, i_p | j_0, j_1, \dots, j_p] \quad \text{if } (i_0, j_0) \notin C. \end{aligned}$$

Then

$$\det A[i_0, \dots, i_p | j_0, \dots, j_p] = \tilde{a}_{i_0, j_0} \cdot \tilde{a}_{i_1, j_1} \cdots \tilde{a}_{i_p, j_p} \quad (3.13)$$

for all lacunary sequences γ given by (3.12).

The above proposition plays an important role in the proof of the following theorem which gives an efficient determinantal criterion for testing whether a given matrix is TN and belongs to a given TN cell.

Theorem 3.4. [LL14, Theorem 4.4] *Let $A \in \mathbb{R}^{n,m}$ and $C \in \mathcal{C}_{n,m}$. Then the following two statements are equivalent:*

- (i) *The matrix A is TN and $A \in \mathcal{S}_C^0$.*
- (ii) *For each (i_0, j_0) , $i_0 \in \{1, \dots, n\}$, $j_0 \in \{1, \dots, m\}$, fix a lacunary sequence γ given by (3.12) (with respect to C) starting at (i_0, j_0) . Then*

$$\det A[i_0, i_1, \dots, i_p | j_0, j_1, \dots, j_p] \begin{cases} = 0 & \text{if } (i, j) \in C, \\ > 0 & \text{if } (i, j) \notin C. \end{cases}$$

Remark 3.1. *Let $A \in \mathbb{R}^{n,m}$ be TN and $A \in \mathcal{S}_C^0$. Then by Proposition 3.4 the following relation holds:*

$$\tilde{a}_{i_0, j_0} = \frac{\det A[i_0, i_1, \dots, i_p | j_0, j_1, \dots, j_p]}{\det A[i_1, \dots, i_p | j_1, \dots, j_p]}$$

for any lacunary sequence γ given by (3.12) with respect to C .

We conclude this section with the following result which enables one to decide whether a given square TN matrix A is nonsingular or not.

Proposition 3.5. [AG13, Proposition 2.8] *Let $A \in \mathbb{R}^{n,n}$ be TN. Then A is nonsingular if and only if $\tilde{a}_{ii} > 0$, $i = 1, \dots, n$.*

Proof. Let C be the Cauchon diagram associated with the totally nonnegative cell that contains A . By Lemma 2.6 and the Cauchon Algorithm, $0 < a_{nn} = \tilde{a}_{nn}$. Assume there exists $1 \leq i_0 < n$ such that $\tilde{a}_{i_0, i_0} = 0$ and $\tilde{a}_{ii} > 0$, $i = i_0 + 1, \dots, n$. Consider the lacunary sequence (with respect to C) $((i, i), i = i_0, i_0 + 1, \dots, n)$. Then by Theorem 3.4 it follows by $(i_0, i_0) \in C$ that $\det A[i_0, \dots, n] = 0$, contradicting Lemma 2.6.

Conversely, assume that $\tilde{a}_{ii} > 0$, $i = 1, \dots, n$. Then the sequence $((i, i), i = 1, \dots, n)$ is a lacunary sequence with respect to C and by Theorem 3.4 it follows that $\det A[1, \dots, n] > 0$ since $(1, 1) \notin C$. \square

3.4. Cauchon Algorithm and Neville Elimination

In this section we show that for totally positive (totally nonnegative) matrices the intermediate matrices of the condensed form of the Cauchon Algorithm 3.3 can be represented as matrices generated by the Neville elimination described in Section 2.6. Also we explain how we can find the bidiagonal factorization of a given nonsingular totally nonnegative matrix by using the Cauchon Algorithm.

To investigate the close relationship between both algorithms, we first modify the usual Neville elimination as follows: we do not produce zeros only below the main diagonal but also on it and above it below the first row which remains unchanged. In this way, for

$A \in \mathbb{R}^{n,n}$, we generate a sequence of matrices (here we assume that no exchange of rows is required)

$$A = A^{\dagger(1)} \rightarrow A^{\dagger(2)} \rightarrow \dots \rightarrow A^{\dagger(n)}$$

with

$$A^{\dagger(k)}[2, \dots, n|1, \dots, k-1] = 0, \quad k = 2, \dots, n,$$

$$a_{ij}^{\dagger(k)} := \begin{cases} a_{ij}^{\dagger(k-1)} & \text{for } i = 1, j = 1, \dots, n, \\ a_{ij}^{\dagger(k-1)} - \frac{a_{i,k-1}^{\dagger(k-1)}}{a_{i-1,k-1}^{\dagger(k-1)}} a_{i-1,j}^{\dagger(k-1)} & \text{if } a_{i-1,k-1}^{\dagger(k-1)} \neq 0, \\ a_{ij}^{\dagger(k-1)} & \text{if } a_{i-1,k-1}^{\dagger(k-1)} = 0, \end{cases} \quad \left. \begin{array}{l} \text{for } i = 2, \dots, n, \\ j = k, \dots, n. \end{array} \right\}$$

We call the resulting algorithm *modified Neville elimination*.

Theorem 3.5. [AG14a, Theorem 4.2] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be TP and put $B := (A^\#)^T = (A^T)^\#$. We run Algorithm 3.3 on A and the modified Neville elimination on B . Then we have for $k = 1, \dots, n$*

$$B^{\dagger(k)}[1, \dots, n|k] = T_n(A^{(n-k+1)}[n-k+1|1, \dots, n])^T. \quad (3.14)$$

Proof. The entries of B are given by

$$b_{ij} = a_{n-j+1, n-i+1}, \quad i, j = 1, \dots, n. \quad (3.15)$$

Since $B^{\dagger(1)} = B$ holds, the entries of the first column of $B^{\dagger(1)}$ are identical with the entries of the last row of A in reverse order which are the entries of the right-hand side of (3.14) so that the statement is true for $k = 1$.

To simplify the representation we write $|\alpha|\beta|$ to denote $\det A[\alpha|\beta] / \det A[\alpha_{\hat{\alpha}_1}|\beta_{\hat{\beta}_1}]$.

For general k , the entries of row $n-k$ of $A^{(n-k)}$ are given by, see (3.33),

$$\begin{aligned} & |\delta|1, \dots, k+1|, \dots, |\delta|n-k-1, \dots, n-1|, |\delta|, \dots, \\ & |[n-k, n-k+1, n-k+2|n-2, n-1, n]|, |[n-k, n-k+1|n-1, n]|, a_{n-k, n}, \end{aligned} \quad (3.16)$$

where $\delta := (n-k, \dots, n)$, and similarly for row $n-k-1$ with $\epsilon := (n-k-1, n-k+1, \dots, n)$

$$\begin{aligned} & |[\epsilon|1, \dots, k+1|], \dots, |[\epsilon|n-k-1, \dots, n-1]|, |[\epsilon|n-k, \dots, n]|, \dots, \\ & |[n-k-1, n-k+1, n-k+2|n-2, n-1, n]|, |[n-k-1, n-k+1|n-1, n]|, a_{n-k-1, n}, \end{aligned}$$

which follows by running the steps up to $n-k$ of Algorithm 3.3 with the matrix which is obtained from A by deleting its row $n-k$. We assume that the statement is true for $k+1$. Then the $(k+2)^{\text{th}}$ column of $B^{\dagger(k+1)}$ is given by

$$\begin{aligned} & a_{n-k-1, n}, |[n-k-1, n-k+1|n-1, n]|, |[n-k-1, n-k+1, n-k+2|n-2, n-1, \\ & n]|, \dots, |[\epsilon|n-k, \dots, n]|, \dots, |[\epsilon|n-k-1, \dots, n-1]|, \dots, |[\epsilon|1, \dots, k+1]|, \end{aligned} \quad (3.17)$$

which follows by the induction hypothesis and applying the steps $1, \dots, k$ of the modified Neville elimination on the matrix that is obtained from B by deleting the column $k+1$. We

show that then the statement is also true for $k+2$. The entries of row $n-k-1$ of $A^{(n-k-1)}$ are given by

$$\begin{aligned} & |[\zeta|1, \dots, k+2]|, \dots, |[\zeta|n-k-2, \dots, n-1]|, |[\zeta]|, \dots, \\ & |[n-k-1, n-k, n-k+1|n-2, n-1, n]|, |[n-k-1, n-k|n-1, n]|, a_{n-k-1, n}, \end{aligned} \quad (3.18)$$

where $\zeta := (n-k-1, \dots, n)$. We apply the modified Neville elimination to $B^{\dagger(k+1)}$. Since for the lower triangular part modified Neville elimination is identical with the Neville elimination, we may apply the determinantal representation (2.38) and obtain by (3.15) that the last $n-k-1$ entries in the $(k+2)^{\text{th}}$ column of $B^{\dagger(k+2)}$ are equal to the first $n-k-1$ entries in (3.18). The first entry of this column is $a_{n-k-1, n}$ which is identical with the last entry of (3.18). Coincidence of the second entry in that column and of the last but one of (3.18) can be easily seen from (3.16) and (3.17). Coincidence of the remaining entries above and on the main diagonal in the $(k+2)^{\text{th}}$ column of $B^{\dagger(k+2)}$ with the respective entries of (3.18) can be shown by using Lemma 1.3. This completes the inductive proof. \square

Let A and B be defined as in Theorem 3.5. Then from the relation between the modified Neville elimination and the Neville elimination we have the following identity:

$$\ddot{B}^{(k)}[k, \dots, n|k] = T_{n-k+1}(A^{(n-k+1)}[n-k+1|1, \dots, n-k+1])^T, \quad \text{for } k = 1, \dots, n,$$

where $\ddot{B}^{(k)}$ is defined as in Section 2.6.

The extension of Theorem 3.5 to the case of TN matrices can be accomplished as follows: We proceed as in the TP case but with the following modification. If during the application of the modified Neville elimination a zero row occurs then we leave this row at its place and continue the algorithm by considering this row as deleted. On the other hand, if in the run of the condensed form of the Cauchon Algorithm $A^{(k)}[1, \dots, k|1, \dots, n]$ has a zero column then we may continue the algorithm by deleting this column or leaving it at its position. For the $NsTN$ case we may use the fact that the closure of the set of TP matrices is the set of TN matrices, see Theorem 2.13.

Remark 3.2. [AG14a] *In passing we note that an alternative proof to the existing proofs, see, e.g., [FJ11, p. 62], of the closure property of the TP matrices relies on Algorithm 3.2 (the Restoration Algorithm): Let A be $NsTN$. Then by Theorem 3.2 (ii) and Proposition 3.5 \tilde{A} is a nonnegative Cauchon matrix with positive diagonal entries and therefore all entries in the same row to the left of (in the lower part) or all entries in the same column above (in the upper part) a zero entry in \tilde{A} vanish, too. We replace such zero entries from the right to the left and from the bottom to the top by increasing integral powers of a positive number ϵ . Call the resulting positive matrix \tilde{A}_ϵ . We apply the Restoration Algorithm to \tilde{A}_ϵ and obtain the TP matrix A_ϵ . Since \tilde{A}_ϵ tends to \tilde{A} as ϵ tends to 0, A_ϵ tends to A . So we can approximate the given $NsTN$ matrix A by the TP matrix A_ϵ as closely as desired.*

To extend Theorem 3.5 to the $NsTN$ case, we approximate the given $NsTN$ matrix A by the TP matrix A_ϵ as described, see Remark 3.2. Then we obtain that (after cancellation

of common powers of ϵ) the denominators appearing on both sides of (3.14) do not contain ϵ . Letting ϵ tend to 0 the extension of Theorem 3.5 to the $NsTN$ case follows.

The Cauchon Algorithm provides an efficient test for total nonnegativity and total positivity, cf. Theorem 3.2 (ii). Complete Neville elimination and the condensed form of the Cauchon Algorithm both need the same number of arithmetic operations for a square matrix A of order n^3), viz. $(n+1)(n-1)^2$. Besides the arithmetic operations, the Cauchon Algorithm requires testing whether \tilde{A} is a Cauchon matrix, a test which can be implemented with quadratic complexity. However, this test is very easy in the TP and $NsTN$ cases since in the TP case we have merely to test whether \tilde{A} contains only positive entries and in the $NsTN$ case we have to check whether the diagonal entries of \tilde{A} are positive (due to Proposition 3.5) and in the case of a zero entry all entries to the left of it in the same row or in the same column above it vanish. As for Neville elimination, these tests should already be performed during the running of the algorithm. In the general TN case complete Neville elimination requires in addition that the rows which have been shifted to the bottom are all zero rows, see Theorem 2.43. So the amount of work is comparable for both algorithms. However, in Section 3.6 we will derive determinantal tests for the $NsTN$, TN , and $Ns.t.n.p.$ matrices which are based on the Cauchon Algorithm and require significantly fewer minors to be checked than the known tests.

An important tool for the analysis of a given $NsTN$ matrix A is its bidiagonal factorization, see Section 2.5. In the following we show how one can find this factorization by application of the Cauchon Algorithm.

It is known that the numbers l_i and u_j , given in Theorem 2.38, can be represented as multipliers of Neville elimination (2.37), see [GPn96]. So, by using the relationships between Neville elimination and the condensed form of the Cauchon Algorithm we can obtain these multipliers also by running the condensed form of the Cauchon Algorithm on $G := (A^\#)^T$ and get in this way a bidiagonalization of A . Specifically, we have the following relations:

$$\begin{aligned}
\text{(a)} \quad & l_k = \frac{\tilde{g}_{n1}}{\tilde{g}_{n2}}, l_{k-1} = \frac{\tilde{g}_{n2}}{\tilde{g}_{n3}}, \dots, l_{k-n+2} = \frac{\tilde{g}_{n,n-1}}{\tilde{g}_{nn}}, \\
& l_{k-n+1} = \frac{\tilde{g}_{n-1,1}}{\tilde{g}_{n-1,2}}, l_{k-n} = \frac{\tilde{g}_{n-1,2}}{\tilde{g}_{n-1,3}}, \dots, l_{k-2n+4} = \frac{\tilde{g}_{n-1,n-2}}{\tilde{g}_{n-1,n-1}}, \dots, \\
& l_1 = \frac{\tilde{g}_{21}}{\tilde{g}_{22}}; \\
\text{(b)} \quad & d_{ii} = \tilde{g}_{n-i,n-i}, i = 1, \dots, n; \\
\text{(c)} \quad & u_1 = \frac{\tilde{g}_{12}}{\tilde{g}_{22}}, \\
& u_2 = \frac{\tilde{g}_{13}}{\tilde{g}_{23}}, u_3 = \frac{\tilde{g}_{23}}{\tilde{g}_{33}}, \dots, \\
& u_{k-n+2} = \frac{\tilde{g}_{n-1,n}}{\tilde{g}_{nn}}, \dots, u_{k-1} = \frac{\tilde{g}_{2n}}{\tilde{g}_{3n}}, u_k = \frac{\tilde{g}_{1n}}{\tilde{g}_{2n}}.
\end{aligned}$$

By setting $\frac{0}{0} := 0$, all the above quantities are defined since in the lower (respectively, upper) half of \tilde{G} if one entry vanishes then all of the entries to the left of it (respectively, above it) vanish, too. These relations can be directly verified by using Remark 3.2 and the representations of l_m , u_m , and d_{ii} , $m = 1, \dots, k$, $i = 1, \dots, n$ of a given n -by- n TP matrix

³⁾ If A is symmetric then the condensed form of the Cauchon Algorithm requires $\frac{n(n-1)(4n+1)}{6}$ operations, provided that $a_{kk}^{(k)} \neq 0$ for all $k = 2, \dots, n$.

in terms of its initial minors, see, e.g., [Koe05]. By setting $\epsilon = 0$ in the resulting quantities (after cancellation of common powers of ϵ) the relations (a), (b), and (c) follow.

3.5. Characterization of Nonsingular Totally Nonpositive Matrices through the Cauchon Algorithm

In this section we apply the Cauchon Algorithm to totally nonpositive matrices and present necessary and sufficient conditions for a given matrix to be nonsingular and totally nonpositive or totally negative.

By using Propositions 3.1 and 3.2 we derive a useful representation for the determinant of a nonsingular matrix under certain conditions. The following theorem can be considered as a generalization of Proposition 3.5.

Theorem 3.6. [AG16a, Theorem 4.3] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ and assume that $\tilde{a}_{ii} \neq 0$, $i = 1, \dots, n$. Then the following equality holds*

$$\det A = \tilde{a}_{11} \cdots \tilde{a}_{nn}. \quad (3.19)$$

Proof. Since $a_{nn}^{(n+1,2)} = a_{nn} = \tilde{a}_{nn} \neq 0$ it follows from Proposition 3.1 that

$$\det A = \det A^{(n+1,2)} = \det A^{(n,n)}[1, \dots, n-1] \cdot \tilde{a}_{nn}. \quad (3.20)$$

Furthermore, we have

$$\det A^{(n,n)}[1, \dots, n-1] = \det A^{(n,1)}[1, \dots, n-1] \quad (3.21)$$

because the latter submatrix is obtained from the first one by a sequence of adding a scalar multiple of one column to another column. Now we set $r := (n-1, n)$; then $r^+ = (n, 1)$ and application of Proposition 3.2 to $A^{(n,1)}[1, \dots, n-1|1, \dots, n]$ yields

$$\det A^{(n,1)}[1, \dots, n-1] = \det A^{(n-1,n)}[1, \dots, n-1]. \quad (3.22)$$

By assumption $a_{n-1,n-1}^{(n-1,n)} = \tilde{a}_{n-1,n-1} \neq 0$ holds. Application of Proposition 3.1 to the matrix $A^{(n-1,n)}[1, \dots, n-1|1, \dots, n]$ (as matrix A) with $r := (n-1, n-1)$ results in

$$\det A^{(n-1,n)}[1, \dots, n-1] = \det A^{(n-1,n-1)}[1, \dots, n-2] \cdot \tilde{a}_{n-1,n-1}. \quad (3.23)$$

Plugging (3.23) into (3.22), the resulting identity into (3.21), and finally the obtained identity into (3.20) gives

$$\det A = \det A^{(n-1,n-1)}[1, \dots, n-2] \cdot \tilde{a}_{n-1,n-1} \cdot \tilde{a}_{nn}.$$

Continuing in this way we arrive at (3.19). □

The statement of Theorem 3.6 remains true if $\tilde{a}_{11} = 0$ and $\tilde{a}_{ii} \neq 0$ for $i = 2, \dots, n$ while it fails if we waive the assumption that $\tilde{a}_{ii} \neq 0$, $i = 2, \dots, n$. A counterexample is provided by the following matrix:

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Now we present the changes in the entries and minors of a given *Ns.t.n.p.* matrix with nonzero entry in position (n, n) during the running of the Cauchon Algorithm. By Lemma 2.4 applied to $A^\#$ all the entries of such matrix are negative except possibly the entry in position $(1, 1)$. The following theorem gives the changes for the steps $r = (n, n), \dots, (n, 2)$.

Theorem 3.7. [AG16a, Theorem 4.4] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be Ns.t.n.p. with $a_{nn} < 0$. If we apply the Cauchon Algorithm to A , then we have the following properties:*

- (i) *All entries of $A^{(n,t)}[1, \dots, n-1]$ are nonnegative for all $t = 2, \dots, n$.*
- (ii) *$A^{(n,t)}[1, \dots, n-1 | 1, \dots, t-1]$ is TN for all $t = 2, \dots, n$.*
- (iii) *$A^{(n,t)}[1, \dots, n-1]$ is TN for all $t = 2, \dots, n$.*
- (iv) *$A^{(n,2)}[1, \dots, n-1]$ is NsTN.*
- (v) *$A^{(n,2)}$ is a Cauchon matrix.*
- (vi) *For $t = 2, \dots, n$, $\det A^{(n,t)}[\alpha | \beta] \leq 0$ for all $\alpha \in Q_{l,n-1}, \beta \in Q_{l,n}$ with $\beta_l = n$ and $l = 1, \dots, n-1$.*

Proof. (i) If $t = n$, then let $r = (n, n)$ and by Proposition 3.1 we have $\det A^{(r^+)}[i, n | j, n] = \det A^{(r)}[i | j] \cdot a_{nn}$, hence $\det A[i, n | j, n] = a_{ij}^{(r)} \cdot a_{nn}$. Since A is *t.n.p.* and $a_{nn} < 0$ it follows that $0 \leq a_{ij}^{(r)}$ for all $i, j = 1, \dots, n-1$. This proves the case $t = n$. Proposition 3.2 implies that $\det A^{(r)}[i, n | j, h] = \det A[i, n | j, h] \leq 0$ for all $h \leq n-1$. In the remaining cases we proceed by induction and repeat the above arguments and use the fact that $a_{nj} < 0$ for all $j = 1, \dots, n$.

(ii) We prove this property only for the case $t = n$ since in the other cases we proceed by induction and repeat the arguments.

If $t = n$ then by (i) $A^{(n,n)}[1, \dots, n-1]$ is a nonnegative matrix. It follows from Proposition 3.1 that

$$\begin{aligned} \det A[\alpha_1, \dots, \alpha_k, n | \beta_1, \dots, \beta_k, n] &= \det A^{(n+1,2)}[\alpha_1, \dots, \alpha_k, n | \beta_1, \dots, \beta_k, n] \\ &= \det A^{(n,n)}[\alpha_1, \dots, \alpha_k | \beta_1, \dots, \beta_k] \cdot a_{nn}, \end{aligned}$$

for all $\alpha_k, \beta_k \leq n-1$. Since A is *t.n.p.* and $a_{nn} < 0$, we have

$$0 \leq \det A^{(n,n)}[\alpha_1, \dots, \alpha_k | \beta_1, \dots, \beta_k].$$

Hence $A^{(n,n)}[1, \dots, n-1]$ is TN . This proves the case $t = n$. For the other cases we use the fact that $a_{nj} < 0$ for all $j = 1, \dots, n$ and for $\beta_{k+1} < n$

$$\det A^{(n,n)}[\alpha_1, \dots, \alpha_k, n | \beta_1, \dots, \beta_k, \beta_{k+1}] = \det A[\alpha_1, \dots, \alpha_k, n | \beta_1, \dots, \beta_k, \beta_{k+1}] \leq 0$$

which follows by Proposition 3.2.

- (iii) We proceed by induction on t (primary induction) and l (secondary induction), where l is the order of the minors.

The case $t = n$ is a consequence of (ii).

Suppose that $A^{(n,t+1)}[1, \dots, n-1]$ is TN ; we want to show that $A^{(n,t)}[1, \dots, n-1]$ is TN , i.e., $0 \leq \det A^{(n,t)}[\alpha|\beta]$ for all $\alpha, \beta \in Q_{l,n-1}$.

The case $l = 1$ is a consequence of (i). So, we assume that $2 \leq l$.

If $\beta_l < t$, then the statement follows from (ii).

If $t < \beta_1$ or t is contained in β then by Proposition 3.2, we have

$$\det A^{(n,t+1)}[\alpha|\beta] = \det A^{(n,t)}[\alpha|\beta]$$

which implies by the induction hypothesis on t that $0 \leq \det A^{(n,t)}[\alpha|\beta]$. So, it just remains to consider the case where there exists h , $1 \leq h \leq l-1$, such that $\beta_h < t < \beta_{h+1}$.

In order to prove the statement in this case we simplify the notation and proceed parallel to the proof given in [GLL11b, pp. 822-823]. We set for $\alpha, \beta \in Q_{l,n-1}$

$$[\alpha|\beta] := \det A^{(n,t)}[\alpha|\beta], \quad [\alpha|\beta]^+ := \det A^{(n,t+1)}[\alpha|\beta],$$

and for $k \in \{1, \dots, l\}$, $m \in \{1, \dots, h\}$,

$$\alpha^{(k)} := (\alpha_1, \dots, \hat{\alpha}_k, \dots, \alpha_l), \quad \beta^{(m)} := (\beta_1, \dots, \hat{\beta}_m, \dots, \beta_{l-1}),$$

where the 'hat' over an entry indicates that this entry has to be discarded from the index sequence (note that the sequences $\alpha^{(k)}$ and $\beta^{(m)}$ have different lengths).

By using Lemma 1.7 we have for $k = 1, \dots, l$

$$\begin{aligned} & [\alpha^{(k)} | \beta^{(m)} \cup \{t\}] \cdot [\alpha|\beta] = \\ & [\alpha^{(k)} | \beta^{(m)} \cup \{\beta_l\}] \cdot [\alpha|\beta^{(m)} \cup \{\beta_m, t\}] + [\alpha^{(k)} | \beta^{(m)} \cup \{\beta_m\}] \cdot [\alpha|\beta^{(m)} \cup \{t, \beta_l\}]. \end{aligned} \tag{3.24}$$

It follows from the induction hypothesis on l that the minors $[\alpha^{(k)} | \beta^{(m)} \cup \{t\}]$, $[\alpha^{(k)} | \beta^{(m)} \cup \{\beta_l\}]$, $[\alpha^{(k)} | \beta^{(m)} \cup \{\beta_m\}]$ are nonnegative. Furthermore, it is a consequence of Proposition 3.2 that $[\alpha|\beta^{(m)} \cup \{\beta_m, t\}] = [\alpha|\beta^{(m)} \cup \{\beta_m, t\}]^+$ and $[\alpha|\beta^{(m)} \cup \{t, \beta_l\}] = [\alpha|\beta^{(m)} \cup \{t, \beta_l\}]^+$ and so we deduce by the induction on t that the two minors are nonnegative. Hence all of these inequalities together imply that the left-hand side of (3.24) is nonnegative. If $0 < [\alpha^{(k)} | \beta^{(m)} \cup \{t\}]$ for some k and m , then $0 \leq [\alpha|\beta]$, as desired. If for all k, m , $[\alpha^{(k)} | \beta^{(m)} \cup \{t\}] = 0$ then it follows by Laplace expansion that $[\alpha|\beta^{(m)} \cup \{\beta_l, t\}] = 0$. Then by Lemma 3.1 we have

$$\det A^{(n,t+1)}[\alpha|\beta] = \det A^{(n,t)}[\alpha|\beta].$$

Hence we obtain by induction on t that $0 \leq \det A^{(n,t)}[\alpha|\beta]$, as desired. This completes the induction step for the proof of (iii).

(iv) By (iii) $A^{(n,2)}[1, \dots, n-1]$ is TN . Similarly as in the proof of Theorem 3.6 we obtain

$$\det A = \det A^{(n,2)}[1, \dots, n-1] \cdot a_{nn}.$$

Since A is $Ns.t.n.p.$ and $a_{nn} < 0$ we have that $0 < \det A^{(n,2)}[1, \dots, n-1]$. Hence $A^{(n,2)}[1, \dots, n-1]$ is $NsTN$.

(v) Since the entries in the last row and last column of A are negative (and are not changed when running the Cauchon Algorithm) and since by (iv) $A^{(n,2)}[1, \dots, n-1]$ is $NsTN$, we conclude that $A^{(n,2)}$ is a Cauchon matrix.

(vi) We prove the statement by induction on l and decreasing induction on t .

The case $l = 1$ is a consequence of the negativity of the entries in the last column of $A^{(n,t)}$, $t = 2, \dots, n$.

If $t = n$ then by Proposition 3.2 we have $\det A^{(n,n)}[\alpha|\beta] = \det A[\alpha|\beta]$ since $\beta_l = n$.

Suppose that the statement is true for all minors of order less than l (secondary induction) and for all $t + 1, \dots, n$ (primary induction).

If $t < \beta_1$ or $t = \beta_h$ for some $h = 1, \dots, l$ then by Proposition 3.2 we have $\det A^{(n,t+1)}[\alpha|\beta] = \det A^{(n,t)}[\alpha|\beta]$, and by the induction hypothesis on t we are done.

If $\beta_h < t < \beta_{h+1}$ for some $h = 1, \dots, l-1$ then we consider again (3.24).

The minors $[\alpha^{(k)}|\beta^{(m)} \cup \{t\}]$, $[\alpha|\beta^{(m)} \cup \{\beta_m, t\}]$, $[\alpha^{(k)}|\beta^{(m)} \cup \{\beta_m\}]$ are nonnegative by (iii), $[\alpha^{(k)}|\beta^{(m)} \cup \{\beta_l\}]$ is nonpositive by the induction hypothesis on l , $[\alpha|\beta^{(m)} \cup \{t, \beta_l\}] = [\alpha|\beta^{(m)} \cup \{t, \beta_l\}]^+$ by Proposition 3.2, and by the induction hypothesis on t the latter minor is nonpositive. All of these inequalities yield

$$[\alpha^{(k)}|\beta^{(m)} \cup \{t\}] \cdot [\alpha|\beta] \leq 0.$$

If $0 < [\alpha^{(k)}|\beta^{(m)} \cup \{t\}]$ for some k and m , then we have $[\alpha|\beta] \leq 0$, as desired.

If for all k, m , $[\alpha^{(k)}|\beta^{(m)} \cup \{t\}] = 0$, then proceeding parallel to the last part of (iii) we get

$$\det A^{(n,t+1)}[\alpha|\beta] = \det A^{(n,t)}[\alpha|\beta].$$

Hence by the induction hypothesis on t we obtain $\det A^{(n,t)}[\alpha|\beta] \leq 0$, as desired. \square

By sequentially repeating the steps of the proof of Theorem 3.7 we obtain the following theorem.

Theorem 3.8. [AG16a, Theorem 4.5] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be $Ns.t.n.p.$ with $a_{nn} < 0$. Then the following statements hold:*

(i) $A^{(s,t)}[1, \dots, s-1|1, \dots, t-1]$ is TN for all $s, t = 2, \dots, n$.

(ii) $A^{(s,2)}[1, \dots, s-1]$ is $NsTN$ for all $s = 2, \dots, n$.

(iii) $\tilde{A}[1, \dots, n-1]$ is a nonnegative matrix.

(iv) \tilde{A} is a Cauchon matrix.

Inspection of the proofs of Theorems 3.7 and 3.8 shows that the nonsingularity assumption is only needed for the nonsingularity statements in Theorem 3.7 (iv) and Theorem 3.8 (ii). In the following corollary we present the weakened version of Theorem 3.8. Theorem 3.7 may be weakened accordingly.

Corollary 3.1. [AG16a, Corollary 4.6] *Let $A \in \mathbb{R}^{n,n}$ have all the entries in its bottom row negative and let A be t.n.p. Then the following statements hold*

(i) $A^{(s,t)}[1, \dots, s-1 | 1, \dots, t-1]$ is TN for all $s, t = 2, \dots, n$.

(ii) $\tilde{A}[1, \dots, n-1]$ is a nonnegative matrix.

(iii) \tilde{A} is a Cauchon matrix.

The following proposition provides necessary and sufficient conditions for a given square t.n.p. matrix whose bottom right entry is negative to be nonsingular.

Proposition 3.6. [AG16a, Proposition 4.7] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be t.n.p. with $a_{nn} < 0$. Then A is nonsingular if and only if $0 < \tilde{a}_{ii}$, $i = 1, \dots, n-1$.*

Proof. Let A be Ns.t.n.p. with $a_{nn} < 0$. By Theorem 3.8, $A^{(s,2)}[1, \dots, s-1]$ is NsTN and therefore possesses only positive principal minors, e.g., Lemma 2.6. In particular, $0 < a_{s-1,s-1}^{(s,2)} = \tilde{a}_{s-1,s-1}$, $s = 2, \dots, n$. The converse direction follows from Theorem 3.6. \square

The following theorem provides necessary and sufficient conditions for a given matrix whose entries are all negative except possibly the $(1, 1)$ entry which may be nonpositive to be Ns.t.n.p. by using the Cauchon Algorithm.

Theorem 3.9. [AG16a, Theorem 4.8] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ have all entries negative except possibly $a_{11} \leq 0$. Then the following two properties are equivalent:*

(i) A is a Ns.t.n.p. matrix.

(ii) \tilde{A} is a Cauchon matrix and $\tilde{A}[1, \dots, n-1]$ is a nonnegative matrix with positive diagonal entries.

Proof. The implication (i) \Rightarrow (ii) follows by Theorem 3.8 and Proposition 3.6.

(ii) \Rightarrow (i): By Theorem 3.6, A is nonsingular with $\det A < 0$ since $0 < \tilde{a}_{ii}$, $i = 1, \dots, n-1$, and $a_{nn} < 0$. $\tilde{A}^{(n,n)}$ is the matrix that we obtain after running the Restoration Algorithm applied to \tilde{A} with $r = (n, n-1)$. By the definition of the Restoration Algorithm, the entries of $\tilde{A}^{(n,n)}[1, \dots, n-1]$ are nonnegative⁴. Note that if in step 2(b) in the Restoration

⁴ The negativity of the entries of A and nonpositivity of a_{11} come into play in the last step, i.e., when applying the Restoration Algorithm with $r = (n, n)$.

Algorithm $s = n$ or $t = n$ then the negativity of $\tilde{a}_{nj}^{(r)}$ and $\tilde{a}_{nt}^{(r)}$ and of $\tilde{a}_{in}^{(r)}$ and $\tilde{a}_{sn}^{(r)}$, respectively, results in a nonnegative value of the quotient. In the proof of Theorem 3.1 given in [GLL11b, Theorem 4.1] it is shown that if $N \in \mathbb{R}^{n,n}$ is a nonnegative Cauchon matrix then

$$0 \leq \det N^{(r)}[\alpha|\beta] \quad \text{for all } \alpha, \beta \in Q_{l,n} \text{ with } (\alpha_l, \beta_l) \leq r. \quad (3.25)$$

Again this result carries over to the minors of $\tilde{A}^{(r)}$ such that $\alpha_l, \beta_l < n$, irrespectively of the negativity of the entries in the last column and row of \tilde{A} as long as $r \leq (n, n)$.

Now let $2 \leq l$, $\alpha, \beta \in Q_{l,n}$ with $\alpha_l = n$ and put $t := \beta_l$, $r := (n, t)$. It follows from Proposition 3.1 that for $\delta = \det \tilde{A}[\alpha|\beta]$

$$\det \tilde{A}^{(r^+)}[\alpha|\beta] = \delta^{(r^+)} = \delta_{\hat{n}\hat{t}}^{(r)} \tilde{a}_{nt} = \delta_{\hat{n}\hat{t}}^{(r)} a_{nt}. \quad (3.26)$$

By (3.25) we have $0 \leq \delta_{\hat{n}\hat{t}}^{(r)}$, whence by (3.26) $\det \tilde{A}^{(r^+)}[\alpha|\beta] \leq 0$. By $\tilde{\tilde{A}} = A$ and by repeated application of Proposition 3.2 we obtain

$$\det \tilde{A}^{(r^+)}[\alpha|\beta] = \det A[\alpha|\beta] \leq 0 \quad \text{for all } \alpha, \beta \in Q_{l,n} \text{ with } \alpha_l = n. \quad (3.27)$$

Similarly we can prove that

$$\det A[\alpha|l, \dots, n] \leq 0 \quad \text{for all } \alpha \in Q_{n-l+1,n}, \quad l = 2, \dots, n. \quad (3.28)$$

Finally, since the result of the Cauchon Algorithm applied to $A[l, \dots, n]$ coincides with $\tilde{A}[l, \dots, n]$ Theorem 3.6 implies that

$$\det A[l, \dots, n] = \tilde{a}_{ll} \cdots \tilde{a}_{nn} < 0, \quad l = 1, \dots, n. \quad (3.29)$$

By the condition on the sign of the entries of A and (3.27)-(3.29) it follows from Theorem 2.8 that A is *Ns.t.n.p.* \square

If $A \in \mathbb{R}^{n,n}$ is *Ns.t.n.p.* with $a_{nn} = 0$, replace in Theorem 3.9 A by $B := AG$, $B = (b_{ij})$, where $G = (g_{ij}) \in \mathbb{R}^{n,n}$ is given by

$$g_{ij} = \begin{cases} 1 & \text{if } i = j, \text{ and } i = n - 1 \text{ and } j = n, \\ 0 & \text{otherwise.} \end{cases} \quad (3.30)$$

Then $b_{nn} < 0$ and A is *Ns.t.n.p.* if and only if B is *Ns.t.n.p.*, see, e.g., [Pn03, proof of Theorem 3.1]. Hence A is *Ns.t.n.p.* if and only if \tilde{B} is a Cauchon matrix and $\tilde{B}[1, \dots, n-1]$ is a nonnegative matrix with positive diagonal entries.

If in the proof of Theorem 3.9, $0 < N$ then (3.25) holds with the strict inequality. Combining this with necessary and sufficient conditions for a matrix to be *t.n.* given in Theorem 2.9 we obtain by a similar proof the following corollary.

Corollary 3.2. [AG16a, Corollary 4.9] *Let $A \in \mathbb{R}^{n,n}$ and $A < 0$. Then the following properties are equivalent:*

- (i) A is t.n.,
- (ii) $0 < \tilde{A}[1, \dots, n-1]$.

By proceeding similarly as in the proof of Theorem 3.9 and using [CRU10, Proposition 3.1] instead of Theorem 2.8 we obtain the following corollary.

Corollary 3.3. [AG16a, Corollary 4.10] *Let $A \in \mathbb{R}^{n,m}$ (with $n \leq m$) have all its entries negative except possibly $a_{11} \leq 0$. Then the following properties are equivalent:*

- (i) A is t.n.p. matrix and $A[1, \dots, n | m-n+1, \dots, m]$ is nonsingular.
- (ii) \tilde{A} is a Cauchon matrix, $\tilde{A}[1, \dots, n-1 | 1, \dots, m-1]$ is a nonnegative matrix, and $\tilde{A}[1, \dots, n-1 | m-n+1, \dots, m-1]$ has positive diagonal entries.

3.6. Optimal Determinantal Criteria for Totally Nonnegative and Nonsingular Totally Nonpositive Matrices

In this section we present optimal determinantal criteria for totally nonnegative matrices and nonsingular totally nonpositive matrices.

Our criteria depend on the lacunary sequences. We relate to each entry \tilde{a}_{i_0, j_0} of \tilde{A} a sequence γ given by (3.12). In contrast to Theorem 3.4, we do not start from a fixed Cauchon diagram but sequentially construct the diagram and the lacunary sequences. It is sufficient to describe the construction of the sequence from the starting pair (i_0, j_0) to the next pair (i_1, j_1) with $0 < \tilde{a}_{i_1, j_1}$ (δ_{i_1, j_1} , see below) since for a given matrix A the determinantal test is performed by moving row by row from the bottom to the top row. Once we have found the next index pair (i_1, j_1) we append to (i_0, j_0) the sequence starting at (i_1, j_1) . By construction, the sequence γ is uniquely determined. We start with nonsingular totally nonnegative matrices.

In the sequel let $\delta_{ij} := \det A[i_0, i_1, \dots, i_p | j_0, j_1, \dots, j_p]$ be the minor of A associated to the sequence γ given by (3.12) which starts at position $(i, j) = (i_0, j_0)$ and which is constructed by one of the following procedures.

Procedure 3.1. [AG14a, Procedure 5.1] *Construction of the sequence γ given by (3.12) starting at (i_0, j_0) to the next index pair (i_1, j_1) for the n -by- n $NsTN$ matrix A .*

If $i_0 = n$ or $j_0 = n$ or $\mathcal{U} := \{(i, j) \mid i_0 < i \leq n, j_0 < j \leq n, \text{ and } 0 < \delta_{ij}\}$ is void
then terminate with $p := 0$;

else

put (i_1, j_1) as the minimum of \mathcal{U} with respect to the colexicographic order
and lexicographic order if $j_0 \leq i_0$ and $i_0 < j_0$, respectively;

end if.

Our criteria in this section depend on the fact that for a given n -by- m matrix A , $\tilde{A}[i_0, \dots, n | j_0, \dots, m]$ coincides with the matrix which is obtained by application of the Cauchon Algorithm to $A[i_0, \dots, n | j_0, \dots, m]$. Let $A \in \mathbb{R}^{n,n}$. Then our test starts at the position (n, n) and then continues by moving to position $(1, 1)$ in decreasing order with respect to the lexicographical order. If we proceed from row $i_0 + 1$ to row i_0 we already know the determinantal entries (the numerators of the entries of \tilde{A} in row $i_0 + 1$, see Remark 3.1) which appear in row $i_0 + 1$. Therefore we can easily check when $j_0 < i_0$ whether all $\delta_{i_0+1, j}$, $j = 1, \dots, j_0$, vanish if δ_{i_0+1, j_0+1} vanishes. To check in the case $i_0 < j_0$ whether all determinantal entries in the column $j_0 + 1$, δ_{s, j_0+1} , $s = 1, \dots, i_0$, vanish we have to compute in addition the minors which are associated with the positions $(s, j_0 + 1)$, $s = 1, \dots, i_0$. These minors differ in only one row index from the minor δ_{i_0+1, j_0+1} which is the row $i_0 + 1$ replaced by s , $s = 1, \dots, i_0$. This follows by running the test with the matrix which is obtained from A by deleting its rows $i_0 + 1, \dots, s + 1$. Hence it is easily checked whether the matrix $B := A[i_0 + 1, \dots, n | 1, \dots, n]$ fulfills conditions (i), (ii), and (iii) of Theorem 3.10 below. If one of the conditions is violated for any instance, the test can be terminated since A is not $NsTN$. Otherwise, \tilde{B} is a Cauchon matrix, whence B is TN and we are able to apply Proposition 3.4 and by using Remark 3.1 we obtain the entries of $\tilde{A}[i_0 + 1, \dots, n | 1, \dots, n]$.

Theorem 3.10. [AG14a, Theorem 5.3] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$. Then A is $NsTN$ if and only if for all $i, j = 1, \dots, n$ the quantities δ_{ij} obtained by the sequences that start from positions (i, j) and are constructed by Procedure 3.1 satisfy the following conditions:*

(i) $0 < \delta_{ii}$;

(ii) $0 \leq \delta_{ij}$;

(iii) if $\delta_{qq} = 0$ for some $q, q \in \{1, \dots, n\}$, then $\delta_{q, t_1} = 0$ for all $t_1 < q$ if $g < q$ and $\delta_{t_2, g} = 0$ for all $t_2 < q$ if $q < g$.

Proof. Suppose that A is $NsTN$. For each sequence $((i_0, j_0), (i_1, j_1), \dots, (i_p, j_p))$ that is obtained by Procedure 3.1 set

$$a'_{i_0, j_0} := \begin{cases} a_{i_0, j_0} & \text{if } p = 0, \\ \frac{\delta_{i_0, j_0}}{\delta_{i_1, j_1}} & \text{if } 0 < p. \end{cases} \quad (3.31)$$

By construction a'_{i_0, j_0} is well-defined for each $(i_0, j_0) \in \{1, \dots, n\}^2$. Define $A' := (a'_{i_0, j_0})_{i_0, j_0=1}^n$.

Claim: $A' = \tilde{A}$.

Proof of the claim: We proceed by decreasing induction with respect to the lexicographical order on the pairs (i, j) , $i, j = 1, \dots, n$.

If $i = n$ then by the definition $a'_{nj} = a_{nj} = \tilde{a}_{nj}$ for all $j = 1, \dots, n$. For $j = n$ the claim also holds by the definition. Suppose that we have shown the claim for all pairs (i, j) such that $i = i_0 + 1, \dots, n$, $j = 1, \dots, n$ and $i = i_0$, $j = j_0 + 1, \dots, n$ holds with $j_0 < n$. We want to show the claim for the pair $(i, j) = (i_0, j_0)$. Since A is $NsTN$ then we have by Theorem 3.2 (ii) and Proposition 3.5 that \tilde{A} is a nonnegative Cauchon matrix with positive diagonal entries. By Lemma 2.6, $0 < a_{nn}$ and so $1 \leq p$. Hence by the induction hypothesis we obtain

that the sequence which starts from the position (i_0, j_0) and is constructed by Procedure 3.1 is a lacunary sequence with respect to the Cauchon diagram $C_{\tilde{A}}$. Hence by Proposition 3.4 we have $\det A[i_0, i_1, \dots, i_p | j_0, j_1, \dots, j_p] = \tilde{a}_{i_0, j_0} \cdot \tilde{a}_{i_1, j_1} \cdots \tilde{a}_{i_p, j_p}$. By the induction hypothesis it follows that

$$\begin{aligned} \delta_{i_0, j_0} &= \tilde{a}_{i_0, j_0} \cdot \frac{\delta_{i_1, j_1}}{\delta_{i_2, j_2}} \cdot \frac{\delta_{i_2, j_2}}{\delta_{i_3, j_3}} \cdots \frac{\delta_{i_p, j_p}}{1} \\ &= \tilde{a}_{i_0, j_0} \cdot \delta_{i_1, j_1}. \end{aligned}$$

Hence we obtain $\tilde{a}_{i_0, j_0} = \frac{\delta_{i_0, j_0}}{\delta_{i_1, j_1}}$. Therefore the claim follows and since $A' = \tilde{A}$ is a nonnegative Cauchon matrix with positive diagonal entries (i)-(iii) follow.

Conversely, suppose that (i)-(iii) hold. We want to show that under these conditions $A' = \tilde{A}$. Again we proceed by decreasing induction with respect to the lexicographical order on the pairs (i, j) , $i, j = 1, \dots, n$. For $i = n$ or $j = n$ the claim holds trivially. Suppose that we have shown the claim for all the pairs (i, j) such that $i = i_0 + 1, \dots, n$, $j = 1, \dots, n$ and $i = i_0$, $j = j_0 + 1, \dots, n$ holds with $j_0 < n$. We want to show that the claim holds for the pair $(i, j) = (i_0, j_0)$. By the induction hypothesis and (i)-(iii) $\tilde{A}[i_0 + 1, \dots, n | 1, \dots, n]$ and $\tilde{A}[i_0, \dots, n | j_0 + 1, \dots, n]$ are nonnegative Cauchon matrices and $0 < \tilde{a}_{ii}$ for $i = i_0 + 1, \dots, n$. Hence by the fact that $\tilde{A}[i_0 + 1, \dots, n | 1, \dots, n]$ and $\tilde{A}[i_0, \dots, n | j_0 + 1, \dots, n]$ coincide with the matrices that are obtained by the application of the Cauchon Algorithm to the submatrices $A[i_0 + 1, \dots, n | 1, \dots, n]$ and $A[i_0, \dots, n | j_0 + 1, \dots, n]$, respectively, and by the induction hypothesis we conclude that the latter submatrices are TN . We will also use the fact that $\tilde{A}[i_0, \dots, n | j_0, \dots, n]$ coincides with the matrix that is obtained by the application of the Cauchon Algorithm to $A[i_0, \dots, n | j_0, \dots, n]$. We add a sufficiently large positive number t to the $(1, 1)$ entry of the submatrix $A[i_0, \dots, n | j_0, \dots, n]$ and name the resulting submatrix D_t in order to make \tilde{D}_t a nonnegative matrix. Hence D_t is TN . The sequence that is constructed by Procedure 3.1 is not affected by the addition of t and it is a lacunary sequence with respect to the Cauchon diagram $C_{\tilde{D}_t}$. By application of Proposition 3.4 to D_t and use of the Laplace expansion we obtain

$$\det A[i_0, i_1, \dots, i_p | j_0, j_1, \dots, j_p] + t \det A[i_1, \dots, i_p | j_1, \dots, j_p] = (\tilde{a}_{i_0, j_0} + t) \cdot \tilde{a}_{i_1, j_1} \cdots \tilde{a}_{i_p, j_p}.$$

Hence by using the induction hypothesis we have

$$\begin{aligned} \delta_{i_0, j_0} + t \delta_{i_1, j_1} &= (\tilde{a}_{i_0, j_0} + t) \cdot \frac{\delta_{i_1, j_1}}{\delta_{i_2, j_2}} \cdot \frac{\delta_{i_2, j_2}}{\delta_{i_3, j_3}} \cdots \frac{\delta_{i_p, j_p}}{1} \\ &= (\tilde{a}_{i_0, j_0} + t) \delta_{i_1, j_1}, \end{aligned}$$

from which it follows that $\tilde{a}_{i_0, j_0} = \frac{\delta_{i_0, j_0}}{\delta_{i_1, j_1}}$, whence $A' = \tilde{A}$. Therefore under the conditions (i)-(iii) \tilde{A} is a nonnegative Cauchon matrix with positive diagonal entries. Hence by Theorem 3.2 (ii) and Proposition 3.5 A is $NsTN$. \square

Since a zero column stays a zero column through the performance of the Cauchon Algorithm, the sign of altogether n^2 minors have to be checked (which include also trivial minors

of order 1). The number of minors that are given in Theorem 3.10 is significantly fewer than the number of minors required by the determinantal test which is based on Theorems 2.14 and 2.42 (iii), the number of which is the number of the quasi-initial minors of A minus the number of the leading principal minors (which are twice counted), i.e.,

$$2 \sum_{k=1}^n \binom{n}{k} - n = 2^{n+1} - n - 2.$$

However, the determinantal test given in Theorems 2.14 and 2.42 (iii) is independent of the matrix to be checked in contrast to the test based on Theorem 3.10. If we test a given matrix A for being TP it suffices to check n^2 fixed determinants (independent of A) for positivity. In this case all sequences γ are running diagonally and we obtain just the numerators of the determinantal ratios which are listed in (3.16).

The numerators of the entries in the first column and in the first row of \tilde{A} are the so-called *corner minors*, i.e., minors of the form $\det A[\alpha|\beta]$ in which α consists of the first k and β consists of the last k indices or vice versa, $k = 1, \dots, n$, see (3.16). In the event that it is somehow known that $A \in \mathbb{R}^{n,n}$ is TN and if its corner minors are positive, then by condition (iii) of Theorem 3.10 \tilde{A} does not contain any zero entry and we can conclude that A is TP . This provides a short proof of the fact that positivity of the corner minors of a TN matrix A implies that A is TP , see Theorem 2.20.

Now we extend the results above to the TN case and associate with each entry \tilde{a}_{i_0, j_0} of \tilde{A} a uniquely determined sequence γ given by (3.12). Again we describe only the construction of the sequence from the starting pair (i_0, j_0) to the next pair (i_1, j_1) with $0 < \delta_{i_1, j_1}$.

Procedure 3.2. [AG14a, Procedure 5.2] *Construction of the sequence γ given by (3.12) starting at (i_0, j_0) to the next index pair (i_1, j_1) for the n -by- m TN matrix A .*

If $i_0 = n$ or $j_0 = m$ or $\mathcal{U} := \{(i, j) \mid i_0 < i \leq n, j_0 < j \leq m, \text{ and } 0 < \delta_{ij}\}$ is void **then terminate with** $p := 0$;

else

if $\delta_{i_0, j_0} = 0$ for all $i = i_0 + 1, \dots, n$ **then put** $(i_1, j_1) := \min \mathcal{U}$ with respect to the *colexicographic order*

else

put $i' := \min \{k \mid i_0 < k \leq n \text{ such that } 0 < \delta_{k, j_0}\},$

$J := \{k \mid j_0 < k \leq m \text{ such that } 0 < \delta_{i', k}\};$

if J is not void **then put** $(i_1, j_1) := (i', \min J)$

else put $(i_1, j_1) := \min \mathcal{U}$ with respect to the *lexicographic order*;

end if

end if

end if.

As in the $NsTN$ case, we proceed row by row from the bottom to the top row. After all sequences γ starting in row $i_0 + 1$ are determined it is checked whether the matrix $A[i_0 + 1, \dots, n | 1, \dots, m]$ fulfills conditions (i) and (ii) of Theorem 3.11 below. Its proof is similar to the one of Theorem 3.10 (with the obvious modifications for the rectangular case) and is omitted here with the exception which we show that the sequences which are constructed by Procedure 3.2 are lacunary sequences with respect to related Cauchon diagrams, provided that the conditions (i) and (ii) of Theorem 3.11 are fulfilled.

Theorem 3.11. [AG14a, Theorem 5.4] *Let $A \in \mathbb{R}^{n,m}$. Then A is TN if and only if for all $i = 1, \dots, n$, $j = 1, \dots, m$ the quantities δ_{ij} obtained by the sequences that start from positions (i, j) and are constructed by Procedure 3.2 satisfy the following conditions:*

- (i) $0 \leq \delta_{ij}$;
- (ii) if $\delta_{qg} = 0$ for some $q \in \{1, \dots, n\}$, $g \in \{1, \dots, m\}$, then $\delta_{q,t_1} = 0$ for all $t_1 < g$ or $\delta_{t_2,g} = 0$ for all $t_2 < q$.

Proof. We proceed similarly as in the proof of Theorem 3.10 and only show that each sequence constructed by Procedure 3.2 is lacunary with respect to the Cauchon diagram $C_{\tilde{D}_t}$, where $D := A[i_0, \dots, n | j_0, \dots, m]$ and D_t is defined as in the proof of Theorem 3.10 if we pose in addition conditions (i) and (ii) of Theorem 3.11. As in Procedure 3.2, we consider only the part of the sequence between two adjacent pairs (i_s, j_s) and (i_{s+1}, j_{s+1}) for $s = 0, 1, \dots, p - 1$. We distinguish four cases according to the steps of Procedure 3.2:

If $p = 0$, then $i_0 = n$ or $j_0 = n$ or \mathcal{U} is void and so $((i_0, j_0))$ is a lacunary sequence. If $0 = \delta_{i_s, j_s}$ and hence $0 = \tilde{a}_{i_s, j_s}$ for all $i = i_s + 1, \dots, n$ it follows from the choice of (i_{s+1}, j_{s+1}) that $\delta_{ij} = 0 = \tilde{a}_{ij}$ for all $i_s < i$, $j_s < j < j_{s+1}$, and since $0 < \delta_{i_s, j_s}$ (not necessary for $s = 0$) and hence $0 < \tilde{a}_{i_s, j_s}$ we conclude by (ii) that $\delta_{ij} = 0 = \tilde{a}_{ij}$ for all $i_s < i$, $j < j_s$. If the set J is not void and $j' := \min J$ it follows that $\delta_{i', k} = 0 = \tilde{a}_{i', k}$ for $j_s < k < j'$ and from (ii) by $0 < \delta_{i', j_s}$ and hence $0 < \tilde{a}_{i', j_s}$ we obtain that $\delta_{ij} = 0 = \tilde{a}_{ij}$ for all $i < i'$, $j_s < j < j'$; since $0 < \delta_{i_s, j_s}$ ($0 < \tilde{a}_{i_s, j_s}$) (not necessary for $s = 0$) we conclude from $\delta_{k, j_s} = 0 = \tilde{a}_{k, j_s}$ for $i_s < k < i'$ and (ii) that $\delta_{ij} = 0 = \tilde{a}_{ij}$ for all $i_s < i < i'$, $j < j_s$. Finally, if J is void then $\delta_{i', j} = 0 = \tilde{a}_{i', j}$ for all $j_s < j$ and since $0 < \delta_{i', j_s}$ ($0 < \tilde{a}_{i', j_s}$) it follows by (ii) that $\delta_{ij} = 0 = \tilde{a}_{ij}$ for all $i < i'$, $j_s < j$, and therefore $i' < i_{s+1}$. By choice of (i_{s+1}, j_{s+1}) we have $\delta_{ij} = 0 = \tilde{a}_{ij}$ for all $i_s < i < i_{s+1}$, $j_s < j$ and $i < i_{s+1}$, $j_s < j < j_{s+1}$. Therefore, condition 3. of Definition 3.4 is fulfilled in all four cases. \square

For an n -by- m matrix only $n \cdot m$ minors have to be checked for nonnegativity and satisfaction of condition (ii) of Theorem 3.11. Note that by [FJ11, Example 3.3.1] there cannot be a specified fixed proper subset of all minors which is sufficient for testing a general matrix for being TN .

The sufficient sets of minors presented in this section are only of theoretical value, and they do not lead to efficient methods to check a given matrix for being $NsTN$ or TN , even when some advantage is taken in the overlap of calculation of different minors. More

efficient methods are based on Algorithm 3.3 and Theorems 3.2 (ii), and Theorem 2.42 (i) and (ii). In Section 3.8 we apply our results to special classes of TN matrices.

Example 3.4. Let A be given as in Example 3.2. Then by application of Procedure 3.2 and the arguments below Procedure 3.1 to A we obtain the following sequences which start from the positions (i, j) , $i, j = 1, \dots, 4$:

$i \backslash j$	1	2	3	4
1	$((1, 1), (3, 3), (4, 4))$	$((1, 2), (4, 3))$	$((1, 3), (2, 4))$	$((1, 4))$
2	$((2, 1), (3, 3), (4, 4))$	$((2, 2), (4, 3))$	$((2, 3), (3, 4))$	$((2, 4))$
3	$((3, 1), (4, 2))$	$((3, 2), (4, 3))$	$((3, 3), (4, 4))$	$((3, 4))$
4	$((4, 1))$	$((4, 2))$	$((4, 3))$	$((4, 4))$

By using Theorem 3.11 we conclude that A is TN .

We conclude this section with an efficient determinantal test to check whether a given matrix is $Ns.t.n.p.$ or not.

The conclusion from the hypothesis of Proposition 3.4 (b) depends only on the zero-nonzero values (and not on the positivity) of the involved determinants. Therefore, we obtain the following proposition.

Proposition 3.7. [AG16a, Proposition 4.11] Let $A \in \mathbb{R}^{n,m}$ and $C \in \mathcal{C}_{n,m}$. For each position in C fix a lacunary sequence γ given by (3.12) (with respect to C) starting at this position. Assume that for all (i_0, j_0) , we have $0 = \det A[i_0, i_1, \dots, i_p | j_0, j_1, \dots, j_p]$ if and only if $(i_0, j_0) \in C$. Then

$$\det A[i_0, i_1, \dots, i_p | j_0, j_1, \dots, j_p] = \tilde{a}_{i_0, j_0} \cdot \tilde{a}_{i_1, j_1} \cdots \tilde{a}_{i_p, j_p} \quad (3.32)$$

for all lacunary sequences γ given by (3.12).

Parallel to Procedure 3.1 we have the following procedure in the case of $Ns.t.n.p.$

Procedure 3.3. [AG16a, Procedure 4.12] Construction of the sequence γ given by (3.12) starting at (i_0, j_0) to the next index pair (i_1, j_1) for the n -by- n $Ns.t.n.p.$ matrix A .

If $i_0 = n$ or $j_0 = n$ or $\mathcal{U} := \{(i, j) \mid i_0 < i \leq n, j_0 < j \leq n, \text{ and } \delta_{ij} < 0\}$ is void **then terminate with** $p := 0$;

else

put (i_1, j_1) as the minimum of \mathcal{U} with respect to the colexicographic order and lexicographic order if $j_0 \leq i_0$ and $i_0 < j_0$, respectively;

end if.

As in the *NsTN* case, after all sequences γ starting in row $i_0 + 1$ are determined it is checked whether the matrix $B := A[i_0 + 1, \dots, n | 1, \dots, n]$ fulfills conditions (i), (ii), and (iii) of Theorem 3.12 below. If one of the conditions is violated for any instance, the test can be terminated since A is not *Ns.t.n.p.*

Theorem 3.12. [AG16a, Theorem 4.13] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ with all entries are negative except possibly $a_{11} \leq 0$. Then A is *Ns.t.n.p.* if and only if for all $i, j = 1, \dots, n$ the quantities δ_{ij} obtained by the sequences that start from positions (i, j) and are constructed by Procedure 3.3 satisfy the following conditions:*

(i) $\delta_{ii} < 0$;

(ii) $\delta_{ij} \leq 0$;

(iii) if $\delta_{qg} = 0$ for some $q, g \in \{1, \dots, n\}$, then $\delta_{q,t_1} = 0$ for all $t_1 < g$ if $g < q$ and $\delta_{t_2,g} = 0$ for all $t_2 < q$ if $q < g$.

The proof parallels the proof of Theorem 3.10 with two differences, viz. we use Proposition 3.7 instead of Proposition 3.4 and Theorem 3.9 and Corollary 3.3 instead of Theorem 3.2 (ii) and Proposition 3.5 and is therefore omitted.

3.7. Representation of the Entries of the Matrices Obtained by the Cauchon Algorithm

In this section we show that each entry of the matrix \tilde{A} which is obtained by the application of the Cauchon Algorithm to a given nonsingular totally nonnegative or nonsingular totally nonpositive matrix A can be represented as a ratio of two contiguous minors as in the case of totally positive and totally negative matrices. This representation plays a fundamental role in the proof of the interval property of these classes of matrices. Since the proofs of the representations in the case of nonsingular totally nonnegative matrices and nonsingular totally nonpositive matrices are parallel, we present the proof in details only for the first case and refer to the differences in the latter case.

Let $A \in \mathbb{R}^{n,n}$ be *TP* or *t.n.*, then the entries \tilde{a}_{kj} of the matrix \tilde{A} obtained from A by the Cauchon Algorithm can be represented as $(k, j = 1, \dots, n)$

$$\tilde{a}_{kj} = \frac{\det A[k, \dots, k+w | j, \dots, j+w]}{\det [k+1, \dots, k+w | j+1, \dots, j+w]}, \quad (3.33)$$

where $w := \min \{n - k, n - j\}$, i.e., as ratios of contiguous minors, see [LL14, p. 376] for the *TP* case. If A is *NsTN* or *Ns.t.n.p.*, then some of the minors involved in this representation may be zero. In the following propositions we show that also in these cases the entries of \tilde{A} can be represented as ratios of contiguous minors (with possibly different w).

Proposition 3.8. [AG13, Proposition 2.10] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be NsTN. Then the entries \tilde{a}_{kj} of the matrix \tilde{A} can be represented as $(k, j = 1, \dots, n)$*

$$\tilde{a}_{kj} = \frac{\det A[k, \dots, k+p | j, \dots, j+p]}{\det A[k+1, \dots, k+p | j+1, \dots, j+p]}, \quad (3.34)$$

with a suitable $0 \leq p \leq n-k$, if $j \leq k$ and $0 \leq p \leq n-j$, if $k < j$.

We call p the order of the representation (3.34).

Proof. By Step 2. of Algorithm 3.1 we have

$$\tilde{a}_{nj} = a_{nj}, \quad j = 1, \dots, n, \quad \text{and} \quad \tilde{a}_{in} = a_{in}, \quad i = 1, \dots, n, \quad (3.35)$$

so we can assume that $k, j < n$.

To simplify notation, we will write $[\alpha|\beta]$ to denote $\det A[\alpha|\beta]$.

We will show by decreasing induction on $k = 1, \dots, n-1$ that the following two statements hold for each \tilde{a}_{kj} :

- (i) The entry \tilde{a}_{kj} can be represented in the form (3.34) of order p .
- (ii) The entries $\tilde{a}_{k,j+1}$ for $j < k$ and $\tilde{a}_{k-1,j}$ for $1 < k < j$ can be represented in the form (3.34) of order p , too.

Any deficiency in the order of the representation can only be caused by zero entries in the intermediate matrices, see step 2. of Algorithm 3.1 and Proposition 3.3. Consider $j < k$ and $0 < \tilde{a}_{k+1,j+1} = a_{k+1,j+1}^{(k+2,2)}$. Then we have by Proposition 3.3

$$\tilde{a}_{kj} = a_{kj}^{(k+1,2)} = \frac{\det A^{(k+2,2)}[k, k+1 | j, j+1]}{\tilde{a}_{k+1,j+1}} \quad (3.36)$$

to which Lemma 1.3 can be applied since according to our assumption the representations of $\tilde{a}_{k+1,j}$ and $\tilde{a}_{k+1,j+1}$ have the same order and the representations of the entries $a_{kj}^{(k+2,2)}$ and $a_{k,j+1}^{(k+2,2)}$ coincide with the representations of the entries $\tilde{a}_{k+1,j}$ and $\tilde{a}_{k+1,j+1}$, respectively, with the only exception that the first row index in the numerators, viz. $k+1$, is replaced by k . This follows by running the steps up to $(k+2, 2)$ of Algorithm 3.1 with the matrix which is obtained from A by deleting its row $k+1$. If $0 = \tilde{a}_{k+1,j+1} = a_{k+1,j+1}^{(k+2,2)}$ then by Theorem 3.2 (ii) and Proposition 3.5 $\tilde{a}_{k+1,l} = a_{k+1,l}^{(k+2,2)} = 0$, for $l = 1, \dots, j$. In this case we have $\tilde{a}_{kj} = a_{kj}^{(k+2,2)}$ and so no change in the order occurs. The representation of \tilde{a}_{kj} may also have the same order as the representation of $\tilde{a}_{k+1,j}$ if the latter vanishes and $0 < \tilde{a}_{k+1,j+1}$, see (3.36).

We first consider the case $j < k$ and start with row $n-1$.

By Lemma 2.6 we have $a_{nn} \neq 0$. Set $v := 0$ if $a_{nj} \neq 0$ for all $j = 1, \dots, n-1$; otherwise set

$v := \max \{j \in \{1, \dots, n-1\} \mid a_{nj} = 0\}$.

If $v \geq 1$, then by Lemma 2.8 we have $a_{nl} = 0$, $l = 1, \dots, v$, which yields by Proposition 3.3

$$a_{ij}^{(n,2)} = a_{ij}, \quad i = 1, \dots, n-1, \quad j = 1, \dots, v;$$

in particular, $\tilde{a}_{n-1,j} = a_{n-1,j}$, $j = 1, \dots, v$. For $j = v+1, \dots, n-1$ we have by Proposition 3.3

$$a_{ij}^{(n,2)} = \frac{[i, n \mid j, j+1]}{[n \mid j+1]}.$$

Specification for row $n-1$ of \tilde{A} yields (note footnote 2) and the range of the indices in step 2(b) of Algorithm 3.1)

$$\tilde{a}_{n-1,j} = a_{n-1,j}^{(n,2)} = \frac{[n-1, n \mid j, j+1]}{[n \mid j+1]}, \quad j = v+1, \dots, n-1.$$

Finally, to show (ii), we have by $a_{nv} = 0$

$$\tilde{a}_{n-1,v} = a_{n-1,v} - \frac{a_{n-1,v+1}a_{nv}}{a_{n,v+1}} = \frac{[n-1, n \mid v, v+1]}{[n \mid v+1]}.$$

We assume now that the statements (i) and (ii) are true for all entries that appear in the rows with numbers $n-1, \dots, k+1$. We show (i) and consider first the case $\tilde{a}_{k+1,j+1} \neq 0$. Then by the induction hypothesis there is an integer p such that

$$\begin{aligned} \tilde{a}_{k+1,j} &= \frac{[k+1, \dots, k+p \mid j, \dots, j+p-1]}{[k+2, \dots, k+p \mid j+1, \dots, j+p-1]}, \\ \tilde{a}_{k+1,j+1} &= \frac{[k+1, \dots, k+p \mid j+1, \dots, j+p]}{[k+2, \dots, k+p \mid j+2, \dots, j+p]}. \end{aligned} \quad (3.37)$$

By running the steps up to $(k+2, 2)$ of Algorithm 3.1 with the matrix which is obtained from A by deleting its row $k+1$ we get the representation

$$a_{kj}^{(k+2,2)} = \frac{[k, k+2, \dots, k+p \mid j, \dots, j+p-1]}{[k+2, \dots, k+p \mid j+1, \dots, j+p-1]}, \quad (3.38)$$

and the similar one for $a_{k,j+1}^{(k+2,2)}$. By Proposition 3.3 and application of Lemma 1.3 we obtain

$$\tilde{a}_{kj} = a_{kj}^{(k+2,2)} - \frac{\tilde{a}_{k+1,j} a_{k,j+1}^{(k+2,2)}}{\tilde{a}_{k+1,j+1}} = \frac{[k, \dots, k+p \mid j, \dots, j+p]}{[k+1, \dots, k+p \mid j+1, \dots, j+p]}.$$

We now assume that $\tilde{a}_{k+1,j+1} = 0$. Then it follows by Theorem 3.2 (ii) and Proposition 3.5 that $\tilde{a}_{k+1,j} = 0$. If in (3.37) $p = 1$, then $\tilde{a}_{k+1,j} = a_{k+1,j} = 0$. Hence by Lemma 2.7 we have that $A[k+1, \dots, n \mid 1, \dots, j] = 0$, and so we obtain by the Cauchon Algorithm that $\tilde{a}_{kj} = a_{kj}$. If $1 < p$, then we conclude from (3.37) that

$$[k+1, \dots, k+p \mid j, \dots, j+p-1] = 0 \quad (3.39)$$

and

$$[k+2, \dots, k+p|j+1, \dots, j+p-1] > 0$$

from which it follows by Lemma 2.6 that

$$[k+2, \dots, k+p-1|j+1, \dots, j+p-2] > 0.$$

Application of Lemma 1.3 to (3.39) yields

$$\begin{aligned} & [k+1, \dots, k+p-1|j, \dots, j+p-2][k+2, \dots, k+p|j+1, \dots, j+p-1] \\ &= [k+1, \dots, k+p-1|j+1, \dots, j+p-1][k+2, \dots, k+p|j, \dots, j+p-2], \end{aligned}$$

whence

$$\begin{aligned} & \frac{[k+1, \dots, k+p-1|j, \dots, j+p-2]}{[k+1, \dots, k+p-1|j+1, \dots, j+p-1]} \\ &= \frac{[k+2, \dots, k+p|j, \dots, j+p-2]}{[k+2, \dots, k+p|j+1, \dots, j+p-1]}. \end{aligned} \quad (3.40)$$

Note that $[k+2, \dots, k+p|j+1, \dots, j+p-1] \neq 0$ implies by Lemma 2.7 that $[k+1, \dots, k+p-1|j+1, \dots, j+p-1] \neq 0$.

Now we apply Lemma 1.3 to (3.38), plug (3.40) into the resulting equation, and apply again Lemma 1.3 to obtain a representation of \tilde{a}_{kj} in the form (3.34).

$$\begin{aligned} \tilde{a}_{kj} &= a_{kj}^{(k+2,2)} = \\ & \frac{[k, k+2, \dots, k+p-1|j, \dots, j+p-2]}{[k+2, \dots, k+p-1|j+1, \dots, j+p-2]} \\ &= \frac{[k, k+2, \dots, k+p-1|j+1, \dots, j+p-1]}{[k+2, \dots, k+p-1|j+1, \dots, j+p-2]} \cdot \frac{[k+2, \dots, k+p|j, \dots, j+p-2]}{[k+2, \dots, k+p|j+1, \dots, j+p-1]} \\ &= \frac{[k, \dots, k+p-1|j, \dots, j+p-1]}{[k+1, \dots, k+p-1|j+1, \dots, j+p-1]}. \end{aligned}$$

Now we prove (ii). We assume that the representations of \tilde{a}_{kj} and $\tilde{a}_{k,j+1}$ according to (3.34) are of different orders. The representation of $\tilde{a}_{k,j+1}$ must have the greater order, see the explanation following (ii). Therefore, we have the following representations:

$$\tilde{a}_{kj} = \frac{[k, \dots, k+p|j, \dots, j+p]}{[k+1, \dots, k+p|j+1, \dots, j+p]}, \quad (3.41)$$

$$\tilde{a}_{k,j+1} = \frac{[k, \dots, k+p+t|j+1, \dots, j+p+t+1]}{[k+1, \dots, k+p+t|j+2, \dots, j+p+t+1]},$$

for some $0 < t$. We distinguish the following two cases:

Case (a): $\tilde{a}_{k+1,j+1} = 0$.

By Theorem 3.2 (ii) and Proposition 3.5 we have $\tilde{a}_{k+1,j} = 0$. By Proposition 3.3, $\tilde{a}_{k,j} = a_{k,j}^{(k+2,2)}$ and therefore the order of the representation $\tilde{a}_{k,j}$ equals the order of $\tilde{a}_{k+1,j}$ which can be taken as the common order of $\tilde{a}_{k+1,j}$ and $\tilde{a}_{k+1,j+1}$ by the induction hypothesis. Whence the numerator of $\tilde{a}_{k+1,j+1}$, i.e., $[k+1, \dots, k+p+1|j+1, \dots, j+p+1]$, vanishes. Hence by Lemma 2.6 we obtain that all the leading principal minors of $A[k+1, \dots, k+p+t|j+1, \dots, j+p+t]$ of order greater than p vanish. We apply Lemma 1.3 sequentially to the matrices $A[k, \dots, k+p+t|j+1, \dots, j+p+t+1]$, $A[k, \dots, k+p+t-1|j+1, \dots, j+p+t]$, \dots , $A[k, \dots, k+p+1|j+1, \dots, j+p+2]$ to decrease in each step the order of $\tilde{a}_{k,j+1}$ by one. In this way we obtain

$$\frac{[k, \dots, k+p+t|j+1, \dots, j+p+t+1]}{[k+1, \dots, k+p+t|j+2, \dots, j+p+t+1]} = \frac{[k, \dots, k+p|j+1, \dots, j+p+1]}{[k+1, \dots, k+p|j+2, \dots, j+p+1]},$$

whence the representations of $\tilde{a}_{k,j}$ and $\tilde{a}_{k,j+1}$ are both of order p . By Lemma 2.6 the denominator on the right-hand side is different from zero.

Case (b): $0 < \tilde{a}_{k+1,j+1}$.

The order of the representation of $\tilde{a}_{k,j}$ equals p by assumption which is a result of application of the Cauchon Algorithm to the entries of $A^{(k+2,2)}$ and by using Proposition 3.3 we have (3.36). This minor is calculated by using Lemma 1.3. Hence by the induction hypothesis we may assume that the common order of the representations of $\tilde{a}_{k+1,j}$ and $\tilde{a}_{k+1,j+1}$ is $p-1$ since $0 < \tilde{a}_{k+1,j+1}$. Hence by the induction hypothesis $\tilde{a}_{k+1,j+1}$ has a representation of order $p-1$ and since the order of the representation of $\tilde{a}_{k,j+1}$ equals $p+t$ we may also conclude that the representation of $\tilde{a}_{k+1,j+1}$ has order $p+t-1$, too. Hence by application of Lemma 1.3 to the numerator of the representation of $\tilde{a}_{k+1,j+1}$ of order $p+t-1$ we obtain

$$\begin{aligned} \tilde{a}_{k+1,j+1} &= \frac{[k+1, \dots, k+p|j+1, \dots, j+p]}{[k+2, \dots, k+p|j+2, \dots, j+p]} = \frac{[k+1, \dots, k+p+t|j+1, \dots, j+p+t]}{[k+2, \dots, k+p+t|j+2, \dots, j+p+t]} \\ &= \frac{[k+1, \dots, k+p+t-1|j+1, \dots, j+p+t-1]}{[k+2, \dots, k+p+t-1|j+2, \dots, j+p+t-1]} - \\ &\frac{[k+2, \dots, k+p+t|j+1, \dots, j+p+t-1][k+1, \dots, k+p+t-1|j+2, \dots, j+p+t]}{[k+2, \dots, k+p+t-1|j+2, \dots, j+p+t-1][k+2, \dots, k+p+t|j+2, \dots, j+p+t]}. \end{aligned} \quad (3.42)$$

Application of Lemma 1.3 to the first term of the right-hand side of (3.42) yields

$$\begin{aligned} &\frac{[k+1, \dots, k+p+t-1|j+1, \dots, j+p+t-1]}{[k+2, \dots, k+p+t-1|j+2, \dots, j+p+t-1]} = \\ &\frac{[k+1, \dots, k+p+t-2|j+1, \dots, j+p+t-2]}{[k+2, \dots, k+p+t-2|j+2, \dots, j+p+t-2]} - \\ &\frac{[k+2, \dots, k+p+t-1|j+1, \dots, j+p+t-2]}{[k+2, \dots, k+p+t-2|j+2, \dots, j+p+t-2]} \cdot \\ &\frac{[k+1, \dots, k+p+t-2|j+2, \dots, j+p+t-1]}{[k+2, \dots, k+p+t-1|j+2, \dots, j+p+t-1]}. \end{aligned}$$

Note that since the denominator of the representation of $\tilde{a}_{k+1,j+1}$ of order $p+t-1$ is positive, i.e., $0 < [k+2, \dots, k+p+t|j+2, \dots, j+p+t]$, we have by Lemma 2.6 that all of the minors

that appear in the denominators are positive. Again apply Lemma 1.3 to the first term of the right-hand side of the last equation and continue this procedure till the term

$$\frac{[k+1, \dots, k+p|j+1, \dots, j+p]}{[k+2, \dots, k+p|j+2, \dots, j+p]}$$

appears. Then plug all of these expressions into (3.42) and cancel the latter term from both sides of the resulting equation to obtain that

$$0 = \sum_{h=0}^{t-1} \frac{[k+2, \dots, k+p+h+1|j+1, \dots, j+p+h]}{[k+2, \dots, k+p+h|j+2, \dots, j+p+h]} \cdot \frac{[k+1, \dots, k+p+h|j+2, \dots, j+p+h+1]}{[k+2, \dots, k+p+h+1|j+2, \dots, j+p+h+1]}.$$

Since A is $NsTN$ we conclude that each term of the above sum is zero. In particular, the first summand vanishes which implies by Lemma 2.7

$$[k+2, \dots, k+p+1|j+1, \dots, j+p] = 0.$$

As a consequence of Lemma 2.6 we obtain

$$[k+1, \dots, k+p+1|j, \dots, j+p] = 0. \quad (3.43)$$

We may assume that $0 < [k+1, \dots, k+p+1|j+1, \dots, j+p+1]$ (otherwise we return to Case (a) and reduce the order of $\tilde{a}_{k,j+1}$). Application of Lemma 1.3 yields

$$\begin{aligned} & \frac{[k, \dots, k+p+1|j, \dots, j+p+1]}{[k+1, \dots, k+p+1|j+1, \dots, j+p+1]} = \frac{[k, \dots, k+p|j, \dots, j+p]}{[k+1, \dots, k+p|j+1, \dots, j+p]} \\ & - \frac{[k, \dots, k+p|j+1, \dots, j+p+1][k+1, \dots, k+p+1|j, \dots, j+p]}{[k+1, \dots, k+p|j+1, \dots, j+p][k+1, \dots, k+p+1|j+1, \dots, j+p+1]} \\ & = \frac{[k, \dots, k+p|j, \dots, j+p]}{[k+1, \dots, k+p|j+1, \dots, j+p]} = \tilde{a}_{kj} \end{aligned}$$

by (3.43) and (3.41), whence \tilde{a}_{kj} possesses a representation of order $p+1$, too.

If $t=1$, then we are done. Otherwise we repeat the above arguments and in each step we either decrease the order of the representation of $\tilde{a}_{k,j+1}$ by one (so that we finally arrive at the order of the representation of \tilde{a}_{kj}) or we increase the order of the representation of \tilde{a}_{kj} by one (so that we finally arrive at the order of $\tilde{a}_{k,j+1}$).

In the case $k=j$, it follows by Remark 3.1 since A is $NsTN$ that

$$\tilde{a}_{kk} = \frac{\det A[k, \dots, n]}{\det A[k+1, \dots, n]}.$$

We now consider the case $k < j$. Since the entries \tilde{a}_{kj} with $k < j$ are identical to the entries \tilde{b}_{jk} , where $\tilde{B} = (\tilde{b}_{kj})$ is the matrix obtained from $B := A^T$ by the Cauchon Algorithm, cf. (3.33), we can reduce this case to the case $j < k$, already discussed above. This completes the proof. \square

For the case of *Ns.t.n.p.* matrices the proof parallels the lengthy proof of the above proposition which makes use only of the fact that certain minors are nonzero but not of their common sign. So we may proceed similarly. In the proof of Proposition 3.8 we use Theorem 3.2 (ii), Propositions 3.4 and 3.5, and Lemmata 2.6, 2.7, and 2.8. In the proof of Proposition 3.9 one can use Theorem 3.9, Lemma 2.5, and Propositions 3.7 and 2.3 or 2.4, instead and therefore the proof of Proposition 3.9 is omitted.

Proposition 3.9. [AG16a, Proposition 5.1] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be Ns.t.n.p. with $a_{nn} < 0$. Then the entries \tilde{a}_{kj} of the matrix \tilde{A} can be represented as $(k, j = 1, \dots, n)$*

$$\tilde{a}_{kj} = \frac{\det A[k, \dots, i+p | j, \dots, j+p]}{\det A[k+1, \dots, i+p | j+1, \dots, j+p]}, \quad (3.44)$$

with a suitable $0 \leq p \leq n-k$, if $j \leq k$ and $0 \leq p \leq n-j$, if $k < j$.

3.8. Characterization of Several Subclasses of the Totally Nonnegative Matrices through the Cauchon Algorithm

In this section we present some subclasses and examples of totally nonnegative matrices. We use the Cauchon Algorithm to derive necessary or sufficient conditions for the total nonnegativity of these subclasses and examples.

3.8.1. Oscillatory Matrices

Recall that a given totally nonnegative matrix is oscillatory if and only if it is nonsingular and the entries on its first sub- and superdiagonal are positive, see Theorem 2.28. The following theorem presents a sufficient condition for a given totally nonnegative matrix to be oscillatory.

Theorem 3.13. [AG14a, Theorem 5.5] *Let $A \in \mathbb{R}^{n,n}$ be TN and \tilde{A} be the matrix obtained from A by the Cauchon Algorithm. Then A is oscillatory if $0 < \tilde{a}_{ij}$ whenever $|i-j| \leq 1$.*

Proof. Suppose that A is TN and all entries of \tilde{A} on its main diagonal and on its first sub- and superdiagonal are positive. Then the matrix A is nonsingular by Proposition 3.5. The sequences running along its sub- and superdiagonal are lacunary with respect to the Cauchon diagram $C_{\tilde{A}}$ and by Proposition 3.4 the matrices $A[2, \dots, n | 1, \dots, n-1]$, and $A[1, \dots, n-1 | 2, \dots, n]$ are nonsingular. By Lemma 2.6 the positivity of the entries on their main diagonals follows. \square

The condition in Theorem 3.13 is not necessary as the following example shows.

Example 3.5. [AG14a, Example 5.3] *Choose*

$$A := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 3 & 6 \end{pmatrix}.$$

Then A is oscillatory, however, $\tilde{a}_{21} = 0$.

The following theorem gives necessary and sufficient conditions for a given TN matrix to be oscillatory. Hence by using the results from the relations between bidiagonalization and the Cauchon Algorithm, Theorem 2.39 follows from the next theorem.

Theorem 3.14. [AG14a, Theorem 5.6] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be TN and \tilde{A} be the matrix obtained from A by the Cauchon Algorithm. Then A is oscillatory if and only if $0 < \tilde{a}_{ii}$, $i = 1, \dots, n$, and there is no index k such that $\tilde{a}_{ik} = 0$, $i = k + 1, \dots, n$, or $\tilde{a}_{ki} = 0$, $i = k + 1, \dots, n$.*

Proof. Let A be an oscillatory matrix. Then by Theorem 2.28 and Lemma 2.10 A is irreducible $NsTN$. Hence by Proposition 3.5 $0 < \tilde{a}_{ii}$ for all $i = 1, \dots, n$. Suppose without loss of generality that there exists k_0 such that $\tilde{a}_{i,k_0} = 0$ for all $i = k_0 + 1, \dots, n$. Hence by the Cauchon Algorithm we have that $a_{i,k_0} = 0$ for all $i = k_0 + 1, \dots, n$. Therefore by Lemma 2.7 we have that $A[k_0 + 1, \dots, n | 1, \dots, k_0] = 0$. Hence A is reducible which is a contradiction. Conversely, by Proposition 3.5, A is $NsTN$ since A is TN and $0 < \tilde{a}_{ii}$ for all $i = 1, \dots, n$. Suppose that A is not oscillatory then without loss of generality we conclude by Theorem 2.28 that there exists k_0 such that $a_{k_0+1,k_0} = 0$. Whence by Lemma 2.7 $A[k_0 + 1, \dots, n | 1, \dots, k_0] = 0$. Therefore, by the Cauchon Algorithm we obtain $\tilde{a}_{i,k_0} = 0$ for all $i = k_0 + 1, \dots, n$ which is a contradiction. Hence A is oscillatory. \square

3.8.2. Tridiagonal Totally Nonnegative Matrices

In this subsection we present a short proof of conditions for a nonnegative tridiagonal matrix to be (nonsingular) totally nonnegative. We use these conditions in Chapter 5 to find the largest amount by which the single entries of such a matrix can be perturbed without losing the property of being (nonsingular) totally nonnegative. The following theorem is the content of Propositions 2.10 (b) and 2.11.

Theorem 3.15. [AG14a, Theorem 5.7] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be a nonnegative tridiagonal matrix.*

(a) *Let $2 < n$. Then A is TN if the following conditions hold:*

- (i) $0 \leq \det A$,
- (ii) $0 \leq \det A[1, \dots, n - 1]$,
- (iii) $0 < \det A[1, \dots, i]$, $i = 1, \dots, n - 2$.

(b) *A is $NsTN$ if and only if $0 < \det A[1, \dots, i]$, $i = 1, \dots, n$.*

Proof. Let A be nonnegative tridiagonal and satisfy the conditions (i)-(iii) of (a). Put $B := A^\#$, then the entries of B are given by $b_{ij} = a_{n-i+1, n-j+1}$, $i, j = 1, \dots, n$, and we have $\det B[n - i + 1, \dots, n] = \det A[1, \dots, i]$, $i = 1, \dots, n$. Application of the Cauchon Algorithm to B results in the matrix $\tilde{B} = (\tilde{b}_{ij})$ with

$$\tilde{b}_{ij} = b_{ij} \geq 0 \quad i, j = 1, \dots, n, \quad i \neq j, \quad (3.45)$$

$$\tilde{b}_{ii} = \frac{\det B[i, \dots, n]}{\det B[i+1, \dots, n]} \geq 0, \quad i = 2, \dots, n, \quad (3.46)$$

$$\tilde{b}_{11} = \begin{cases} \frac{\det B}{\det B[2, \dots, n]} \geq 0 & \text{if } \det B[2, \dots, n] > 0, \\ b_{11} \geq 0 & \text{if } \det B[2, \dots, n] = 0. \end{cases} \quad (3.47)$$

Since \tilde{B} is nonnegative, by Theorem 3.2 (ii) it remains to show that \tilde{B} is a Cauchon matrix. The only case we have to consider is the case $0 = \det A[1, \dots, n-1] = \det B[2, \dots, n]$. Then by condition (i) and (3.45)

$$\begin{aligned} 0 \leq \det A &= \det B = b_{11} \det B[2, \dots, n] - b_{12} b_{21} \det B[3, \dots, n] \\ &= -\tilde{b}_{12} \tilde{b}_{21} \det A[1, \dots, n-2], \end{aligned}$$

from which it follows by condition (iii) that $\tilde{b}_{12} = 0$ or $\tilde{b}_{21} = 0$ which completes the proof of (a). If all leading principal minors of A are positive then by (3.46) and (3.47) $0 < \tilde{b}_{ii}$, $i = 1, \dots, n$, so that \tilde{B} is a Cauchon matrix. The necessity follows by Lemma 2.6. \square

3.8.3. Nonsingular Pentadiagonal Totally Nonnegative Matrices

Recall that a matrix $A = (a_{ij}) \in \mathbb{R}^{n,n}$ is pentadiagonal if $a_{ij} = 0$ whenever $|i - j| > 2$. Pentadiagonal totally nonnegative matrices are considered in [Gha06], [Gla04] and banded totally nonnegative matrices in [Met73]. The following theorem presents sufficient conditions for a pentadiagonal matrix with only positive entries inside the band to be nonsingular totally nonnegative.

Theorem 3.16. [AG14a, Theorem 5.8] *Let $2 < n$ and $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be pentadiagonal with $0 < a_{ij}$ if $|i - j| \leq 2$. Then A is NsTN if the following two conditions hold:*

- (i) $0 < \det A[i, \dots, n]$, $i = 1, \dots, n$,
- (ii) $0 < \det A[i, \dots, n | i-1, \dots, n-1]$, $\det A[i-1, \dots, n-1 | i, \dots, n]$, $i = 2, \dots, n$.

Proof. Assume that conditions (i) and (ii) hold. Then A is nonsingular by (i). Application of the Cauchon Algorithm to A results in the matrix \tilde{A} . We show by decreasing induction on the row index i that the entries on the main diagonal and the first subdiagonal of \tilde{A} are positive. By assumption, the entries $\tilde{a}_{n,n-1} = a_{n,n-1}$ and $\tilde{a}_{nn} = a_{nn}$ are positive. Suppose that the assumption holds for the rows with index larger than i . Then the entries in the i^{th} row of \tilde{A} up to position i are as follows

$$0, \dots, a_{i,i-2}, \frac{\det A[i, \dots, n | i-1, \dots, n-1]}{\det A[i+1, \dots, n | i, \dots, n-1]}, \frac{\det A[i, \dots, n]}{\det A[i+1, \dots, n]}$$

and by conditions (i) and (ii) $\tilde{a}_{i,i-1}$ and \tilde{a}_{ii} are positive. Similarly, one shows that the entries on the first superdiagonal of \tilde{A} are positive, too. Therefore, \tilde{A} is a nonnegative Cauchon matrix and we can conclude by Theorem 3.2 (ii) that A is TN. \square

Theorem 3.16 can be easily verified by using Theorem 3.10. Under the hypothesis of this theorem the sequences that will be constructed by using Procedure 3.1 run diagonally and hence by conditions (i) and (ii) of Theorem 3.16 A is $NsTN$.

The matrix A in Example 3.5 shows that condition (ii) is not necessary since $0 = \det A[2, 3|1, 2]$.

The following theorem presents necessary and sufficient conditions for a given pentadiagonal matrix with only positive entries inside the band to be $NsTN$.

Theorem 3.17. [AG14a, Theorem 5.9] *Let $2 < n$ and $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be pentadiagonal with $0 < a_{ij}$ if $|i - j| \leq 2$. Then A is $NsTN$ if and only if the following three sets of conditions hold:*

- (i) $0 < \det A[i, \dots, n], i = 1, \dots, n,$
- (ii) $0 < \det A[i, \dots, n|i - 1, \dots, n - 1], \det A[i - 1, \dots, n - 1|i, \dots, n], i = 3, \dots, n,$
- (iii) $0 \leq \det A[2, \dots, n|1, \dots, n - 1], \det A[1, \dots, n - 1|2, \dots, n].$

Proof. Suppose first that A is $NsTN$ with positive entries on its main diagonal and on its first two sub- and superdiagonals. Then (iii) trivially holds and (i) is satisfied by Lemma 2.6. Let k be the largest index greater than 2 for which $\det A[k, \dots, n|k - 1, \dots, n - 1]$ vanishes. Then by Proposition 3.4 $\tilde{a}_{k,k-1} = 0$ and by Theorem 3.2 (ii) and Proposition 3.5, $0 = \tilde{a}_{k,k-2} = a_{k,k-2}$ which contradicts our assumption. The positivity of $\det A[i - 1, \dots, n - 1|i, \dots, n]$ can be shown analogously. The sufficiency follows as in the proof of Theorem 3.16. \square

Condition (ii) of Theorems 3.16 and 3.17 can somewhat be relaxed for a general pentdiagonal matrix. Suppose that A is a $NsTN$ pentadiagonal matrix with $0 < a_{ij}$ if $|i - j| \leq 1$. Let k be the largest index such that $\det A[k, \dots, n|k - 1, \dots, n - 1] = 0$ and hence $\tilde{a}_{k,k-1} = 0$; then by Theorem 3.2 (ii) and Proposition 3.5 $\tilde{a}_{k,k-2} = a_{k,k-2}$ vanishes, too. Imagine that we want to construct a lacunary sequence starting at position $(i, i - 1)$ and passing through position $(k - 1, k - 2)$. If $\tilde{a}_{k+1,k-1} = a_{k+1,k-1}$ would be zero then according to the zero-nonzero pattern of A we would have

$$0 = \det A[k, \dots, n|k - 1, \dots, n - 1] = a_{k,k-1} \det A[k + 1, \dots, n|k, \dots, n - 1].$$

Which is a contradiction since $0 < a_{ij}$ for $|i - j| \leq 1$ and $0 < \det A[k + 1, \dots, n|k, \dots, n - 1]$. Thus $\tilde{a}_{k+1,k-1} = a_{k+1,k-1} > 0$ and therefore we proceed with $(k + 1, k - 1)$ and continue running diagonally as long as $a_{k+1+l,k-1+l}$ ($1 \leq l$) are nonzero if there exists such an l . If there exists l_0 such that $a_{k+1+l_0,k-1+l_0} = 0$ then we continue with $(k + 1 + l_0, k + l_0)$ and then proceed diagonally with $(k + 1 + l_0 + t, k + l_0 + t)$ for $t = 1, 2, \dots, n - k - 1 - l_0$ since by assumption $0 < \det A[k + 1 + l_0 + t, \dots, n|k + l_0 + t, \dots, n - 1]$ and hence $0 < \tilde{a}_{k+1+l_0+t,k+l_0+t}$. Thus the determinant associated with such a lacunary sequence is equal to the product of the determinant of the submatrix which corresponds to the part of the lacunary sequence above and to the left of the position $(k, k - 1)$ and the determinant that is associated with the

lacunary sequence that starts from the position $(k + 1, k - 1)$.

Hence it suffices to check the sign of the minors $\det A[i, \dots, k - 1 | i - 1, \dots, k - 2]$. A similar procedure applies if there is a second largest index k' with $\det A[k', \dots, k - 1 | k' - 1, \dots, k - 2] = 0$.

By using similar arguments we can generalize the pentadiagonal case to any banded matrix and get results like that given in [GMPn92, Proposition 4.1] and [Pin08, Corollary 4.2, Corollary 4.4].

3.8.4. Almost Totally Positive Matrices

In this subsection we present a necessary and sufficient condition for a given matrix to be almost totally positive by using the Cauchon Algorithm. For details on this subclass of matrices see Subsection 2.3.3.

Theorem 3.18. [AG14a, Theorem 5.10] *Let $A \in \mathbb{R}^{n,n}$ be TN. Then the following two statements are equivalent:*

- (i) *A is NsATP.*
- (ii) *The Cauchon diagrams C_A and $C_{\tilde{A}}$ associated with A and \tilde{A} (obtained from A by the Cauchon Algorithm), respectively, are identical and all squares on the main diagonal of $C_{\tilde{A}}$ are colored white.*

Proof. Let A be NsATP. Then by Proposition 3.5 \tilde{A} has a positive main diagonal, i.e., $C_{\tilde{A}}$ has a white main diagonal. Suppose that C_A and $C_{\tilde{A}}$ differ by the entry in position (i_0, j_0) ; assume first that $0 < a_{i_0, j_0}$ whereas $\tilde{a}_{i_0, j_0} = 0$. Fix a lacunary sequence γ defined by (3.12) with respect to $C_{\tilde{A}}$ starting at the position (i_0, j_0) . According to the definition of a lacunary sequence, the entries \tilde{a}_{i_k, j_k} , $k = 1, \dots, p$, are positive. By Proposition 3.4 it follows that $\det A[i_0, \dots, i_p | j_0, \dots, j_p] = 0$. The entries of A in the respective positions, i.e., a_{i_k, j_k} are positive, too, $k = 1, \dots, p$, since a zero entry stays a zero entry through the performance of the Cauchon Algorithm when it is applied to a NsTN matrix. Since A is ATP it follows that $0 < \det A[i_0, \dots, i_p | j_0, \dots, j_p]$, a contradiction. The case $0 < \tilde{a}_{i_0, j_0}$ and $a_{i_0, j_0} = 0$ is excluded by the above argument of the invariance of zero entries during the performance of the Cauchon Algorithm.

Now suppose that (ii) holds. Then A is nonsingular by Proposition 3.5. Fix a k -by- k submatrix $B := A[\alpha | \beta]$ of A , $1 < k$. Definition 2.3 states that we may restrict the discussion to contiguous α and β . If $0 < \det B$ then by Lemma 2.6 all diagonal entries of B are positive. On the other hand, if all entries of the main diagonal of B are positive then by (ii) the entries of \tilde{A} in the same positions are positive. If $\alpha_k = n$ or $\beta_k = n$ these positions form a lacunary sequence. Otherwise, we fix a lacunary sequence (with respect to $C_{\tilde{A}}$) which starts at position (α_k, β_k) and append it to the sequence $((\alpha_1, \beta_1), \dots, (\alpha_{k-1}, \beta_{k-1}))$. In both cases we obtain a lacunary sequence starting at position (α_1, β_1) . By Proposition 3.4 the associated submatrix D of A has a positive determinant and since B is a principal submatrix of D it follows by Lemma 2.6 that $0 < \det B$ which completes the proof. \square

We conclude this subsection with the following theorem which extends the above theorem to the singular case. The proof uses the same arguments that have been employed in the proof of Theorem 3.18. Note that zero rows and columns as in condition (ii) in the Definition 2.3 of an *ATP* matrix stay zero rows and columns, respectively, during the performance of the Cauchon Algorithm. Also, sequences γ given by (3.12) that are constructed according to Procedure 3.2 coincide with the sequences constructed when zero rows and columns are deleted.

Theorem 3.19. [AGT16, Theorem 2.12] *Let $A \in \mathbb{R}^{n,m}$ be *TN*. Then the following two statements are equivalent:*

- (i) *A is ATP.*
- (ii) *The Cauchon diagrams C_A and $C_{\tilde{A}}$ associated with A and \tilde{A} (obtained from A by the Cauchon Algorithm), respectively, are identical.*

3.8.5. Totally Nonnegative Green's Matrices

In this subsection we consider an example of totally nonnegative matrices which are defined as follows. Let two sequences c_1, \dots, c_n and d_1, \dots, d_n of nonzero real numbers be given. We define the entry a_{ij} of the n -by- n matrix $A = (a_{ij})$ by

$$a_{ij} := c_{\min\{i,j\}} d_{\max\{i,j\}}, \quad i, j = 1, \dots, n. \quad (3.48)$$

The matrix A is called the *Green's matrix* (also referred to as a *single-pair matrix* in [GK02, pp. 78-79]). It is known that the nonsingular Green's matrices are the inverses of symmetric tridiagonal matrices [GK02, p. 82], [Pin10, Section 4.5]. The following theorem presents a necessary and sufficient condition for a given Green's matrix to be *TN*.

Theorem 3.20. [GK02, p. 79], see also [Pin10, Theorem 4.2] *The Green's matrix A defined in (3.48) is *TN* if and only if the c_i and d_j are all of the same strict sign and the inequalities*

$$\frac{c_1}{d_1} \leq \frac{c_2}{d_2} \leq \dots \leq \frac{c_n}{d_n}. \quad (3.49)$$

hold. If l is the number of the strict inequality signs in (3.49) then $\text{rank}(A) = l + 1$.

Proof. Suppose that

$$\frac{c_i}{d_i} = \frac{c_{i+1}}{d_{i+1}}. \quad (3.50)$$

Then row i of A is the $\frac{c_i}{c_{i+1}}$ -multiple of row $i + 1$. We delete in A the i^{th} row and column if (3.50) holds, $i = 1, \dots, n - 1$. The resulting matrix is denoted by B and we have $\text{rank}(B) = \text{rank}(A)$. It is easy to see that A is *TN* if and only if B is *TN*. In the following we assume that the inequalities in (3.49) are strict.

When we compute the matrix $A^{(n-1)}$ from $A^{(n)} = A$ by Algorithm 3.3 all the entries between the main diagonal and the last row of $A^{(n-1)}$ become zero

$$a_{ij}^{(n-1)} = c_j d_i - \frac{c_j d_n c_{j+1} d_i}{c_{j+1} d_n} = 0, \text{ for all } 1 \leq j < i \leq n-1.$$

Furthermore, the entries on the main diagonal are already in their final form, i.e., $a_{ii}^{(n-1)} = \tilde{a}_{ii}$, $i = 1, \dots, n$. Since A is symmetric, \tilde{A} is symmetric, too, whence also the entries between the main diagonal and the last column of \tilde{A} are zero. According to Theorem 3.2 (ii), A is then TN if and only if the diagonal entries of $A^{(n-1)}$ are nonnegative, i.e.,

$$0 < a_{ii}^{(n-1)} = c_i d_i - \frac{c_i d_n c_{i+1} d_{i+1}}{c_{i+1} d_n}, \quad i = 1, \dots, n-1,$$

which is equivalent to (3.49). Hence all diagonal entries of \tilde{A} (obtained from A by the Cauchon Algorithm) are positive and by Proposition 3.5 A is nonsingular which completes the proof. \square

3.8.6. Totally Nonnegative (0,1)-Matrices

In this subsection we present a short proof of a characterization of totally nonnegative (0,1)-matrices, i.e., the matrices whose entries are only 0's and 1's. Totally nonnegative (0,1)-matrices are studied in [BK10] and [FJ11, Section 1.6].

Definition 3.6. [FJ11, Definition 1.6.2] *The matrix $A \in \mathbb{R}^{m,n}$ is said to be in double echelon form if*

(i) *Each row of A has one of the following forms (an asterisk denotes a nonzero entry):*

- (1) $(*, *, \dots, *)$,
- (2) $(*, \dots, *, 0, \dots, 0)$,
- (3) $(0, \dots, 0, *, \dots, *)$, or
- (4) $(0, \dots, 0, *, \dots, *, 0, \dots, 0)$.

(ii) *The first and last nonzero entries in row $i+1$ are not to the left of the first and last nonzero entries in row i , respectively ($i = 1, 2, \dots, n-1$).*

Recall that the zero entries in a given TN matrix are not arbitrary in nature, so it is a necessary condition for a TN matrix to be in double echelon form.

Lemma 3.2. [FJ11, Corollary 1.6.5] *Let $A \in \mathbb{R}^{n,m}$ be a TN matrix with no zero rows or columns. Then A is in double echelon form.*

The following theorem provides necessary and sufficient conditions for a given (0,1)-matrix to be TN based on a single forbidden submatrix and the double echelon form.

Theorem 3.21. [BK10, Theorem 2.2], [FJ11, Theorem 1.6.9] *Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$ be a $(0,1)$ -matrix with no zero rows or columns. Then A is TN if and only if A is in double echelon form and does not contain the submatrix*

$$B := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Proof. Let A be TN . Then by Lemma 3.2, A must be in double echelon form. The necessity is trivial since $\det B = -1$. To prove the sufficiency we run Algorithm 3.3 on A . Then $C := A^{(n-1)}[1, \dots, n-1 | 1, \dots, m]$ cannot contain the entry -1 since A is supposed to be in double echelon form. So the only possible entries are 0's and 1's. The only problematic case is $a_{ij}^{(n-1)} = 0$ resulting from $a_{ij} = a_{nj} = a_{n,u_j} = a_{i,u_j} = 1$, where u_j is defined as in Algorithm 3.3. Then it follows that $a_{ik}^{(n-1)} = 0$, $k = 1, \dots, j-1$, or $a_{kj}^{(n-1)} = 0$, $k = 1, \dots, i-1$, because otherwise C would contain a submatrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. This submatrix would result from a submatrix B in the matrix A which is excluded by our assumption. Moreover, C does not have $D := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ or $E := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as submatrices since otherwise the $(1,1)$ zero entries in D and E imply that $d_{21} = 0$ and $e_{21} = 0$ or $e_{12} = 0$, a contradiction. By the assumption that A has no zero rows and the definition of the Algorithm 3.3 we obtain that C has no zero rows. Since deleting the zero columns of C will not effect in the application of the Algorithm 3.3 on C and hence on $A^{(n-1)}$ (they will remain zero columns among the remaining steps of Algorithm 3.3), we may delete the zero columns of C and therefore the resulting matrix, C' has no zero rows and columns and is in the double echelon form. Moreover, it does not contain the submatrix B because otherwise $b_{13} = 0$ ($b_{31} = 0$) would result from the application of Algorithm 3.3 on A which would imply that $b_{23} = 0$ ($b_{21} = 0$), a contradiction. Now we proceed by induction and obtain that \tilde{A} is a nonnegative Cauchon matrix and by Theorem 3.2 (ii) A is TN . \square

4. Matrix Intervals of Nonsingular Sign Regular Matrices

Matrix intervals of several classes of matrices are studied and investigated by some mathematicians. Matrix intervals of P -matrices, positive definite matrices, inverse positive matrices, M -matrices, inverse M -matrices, and stability of interval matrices are considered in [BG84], [Roh87], [Roh94], [RR96], [JS02], [SH10]. For a survey see [GAT16].

In this chapter we focus on matrix intervals of sign regular matrices. The organization of this chapter is as follows. In the first section, we introduce the required definitions, notations, and known results on matrix intervals of nonsingular sign regular matrices. In Section 4.2, we present the proof of a long standing conjecture, Garloff's Conjecture, on matrix intervals of nonsingular totally nonnegative matrices. In Section 4.3, we give an analogous results for matrix intervals of nonsingular totally nonpositive matrices. Matrix intervals of the nonsingular almost strictly sign regular matrices are considered in Section 4.4. Finally in Section 4.5, we use the results from the previous sections to give some sufficient conditions for a given function matrix to be nonsingular totally nonnegative (nonpositive) or nonsingular almost strictly sign regular.

4.1. Introduction

In this section we present definitions, terminologies, and notations that are used in this chapter. After that known results on the matrix intervals of sign regular matrices are given.

Recall that the set of the real n -by- n matrices can be endowed with the checkerboard partial ordering as follows: for $A, B \in \mathbb{R}^{n,n}$, $A = (a_{ij})$, $B = (b_{ij})$,

$$A \leq^* B \Leftrightarrow (-1)^{i+j} a_{ij} \leq (-1)^{i+j} b_{ij}, \quad i, j = 1, \dots, n. \quad (4.1)$$

We consider *matrix intervals* with respect to this partial ordering, i.e., for $A, B \in \mathbb{R}^{n,n}$ with $A \leq^* B$ let

$$[A, B] := \{Z \in \mathbb{R}^{n,n} \mid A \leq^* Z \leq^* B\}. \quad (4.2)$$

The matrices A and B are called the *corner matrices*. The *vertex matrices* of the matrix interval defined by (4.2) are just the real matrices $Z = (z_{ij})$ with $z_{ij} \in \{a_{ij}, b_{ij}\}$ for $i, j = 1, \dots, n$. By $\mathbb{I}(\mathbb{R}^{n,n})$ we denote the set of all matrix intervals of order n with respect to the checkerboard partial ordering. Equivalently, a matrix interval can be represented as an *interval matrix*, i.e., a matrix with all entries taken from $\mathbb{I}(\mathbb{R})$, the set of the compact and

nonempty real intervals. An element $[a, b] \in \mathbb{I}(\mathbb{R})$ is called *thin* if $a = b$ and is called *thick* if $a < b$.

We extend properties of real matrices to matrix intervals by saying that a *matrix interval has a certain property* if each real matrix contained in that interval possesses this property.

A property which is satisfied by vertices of a given matrix interval and enables us to infer that the matrix interval has this property, too, is called *vertex implication*.

The study of matrix intervals of *SR* matrices with respect to the checkerboard partial ordering was begun in 1982 by J. Garloff. In [Gar82] the following theorem was proved.

Theorem 4.1. [Gar82, Theorem 1] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$. Then the following two statements are equivalent:*

- (i) $[A, B]$ is *SSR*.
- (ii) A and B are *SSR* with the same signature.

The proof of Theorem 4.1 in [Gar82] is based on Theorem 2.1 and Lemma 4.3 below; for a proof in the *TP* case see also [FJ11, pp. 81-82] and [Pin10, pp. 84-85]. We will come back to the proof in Section 4.4.

If in the above theorem the signatures of both corner matrices are $(1, \dots, 1)$, then we have the following special case.

Corollary 4.1. *A matrix interval $[A, B]$ is *TP* if and only if A and B are *TP*.*

In an attempt to extend Theorem 4.1 to matrix intervals of *SR* matrices, the following theorem on the special matrix intervals for which all their thin entries have the same parity was proved in [Gar82].

Theorem 4.2. [Gar82, Theorem 2] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ be such that*

either

$$\forall i, j \in \{1, \dots, n\} \quad a_{ij} = b_{ij} \Rightarrow i + j \text{ is even,}$$

or

$$\forall i, j \in \{1, \dots, n\} \quad a_{ij} = b_{ij} \Rightarrow i + j \text{ is odd.}$$

Then the following two statements are equivalent:

- (i) $[A, B]$ is *SR* (respectively, *NsSR*) and for all $Z, Y \in [A, B]$, Z and Y have the same signature.
- (ii) A and B are *SR* (respectively, *NsSR*) with the same signature.

In the same paper it was conjectured that a matrix interval is *NsTN* if its corner matrices are so.

Conjecture 4.1. (Garloff's Conjecture) [Gar82, Conjecture], see also [Pin10, Section 3.2], [FJ11, Section 3.2] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$. Then the matrix interval $[A, B]$ is $NsTN$ if and only if A and B are $NsTN$.*

Many attempts have been made to prove this conjecture. It was positively settled for some subclasses of $NsTN$ matrices, e.g., for tridiagonal $NsTN$ matrices [Gar82, Theorem 4]. In [Gar96], the conjecture was revisited and proved under the condition that some vertex matrices of the given matrix interval $[A, B]$ are $NsTN$. In this attempt, the set $V([A, B])$ of vertex matrices of the matrix interval $[A, B]$ which is used in Theorem 4.3 below was introduced: An element of $V([A, B])$ has in each row a fixed pattern of the entries of $\{a_{ij}, b_{ij}\}$ which is chosen to be the same pattern as in the first row or its dual. It is easy to see that for $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$, $V([A, B])$ has at most 2^{2n-1} elements. For example, if $[A, B] \in \mathbb{I}(\mathbb{R}^{2,2})$, then

$$V([A, B]) = \left\{ A, B, \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}, \begin{bmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{bmatrix}, \begin{bmatrix} a_{11} & b_{12} \\ b_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & a_{12} \\ a_{21} & b_{22} \end{bmatrix} \right\}.$$

Theorem 4.3. [Gar96, Theorem 4] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$. Then $[A, B]$ is $NsTN$ if and only if all the elements of $V([A, B])$ are $NsTN$.*

In [Gar03], Garloff's conjecture was settled for another subclass of $NsTN$ matrices, namely the $NsATP$ matrices. Therein also the following results were proved which are used in the proof of Theorem 4.4 below.

Lemma 4.1. [Gar03, Lemma 1] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ with A and B are $NsTN$. Then for any $Z \in [A, B]$ we have*

$$0 < \det Z[\alpha] \text{ for all } \alpha \in Q_{k,n} \text{ with } c(\alpha) = 0, \quad k = 1, \dots, n, \quad (4.3)$$

where

$$c(\alpha) := \sum_{i=1}^{k-1} \gamma(\alpha_{i+1} - \alpha_i), \quad \text{where } \gamma(\xi) := \begin{cases} 0 & \text{if } \xi \text{ is odd,} \\ 1 & \text{if } \xi \text{ is even,} \end{cases}$$

with the convention that $c(\alpha) := 0$ for $\alpha \in Q_{1,n}$.

Lemma 4.2. [Gar03, Lemma 2] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ with A and B are $NsTN$. Let $a_{i_0, j_0} = 0$ if $i_0 + j_0$ is even or $b_{i_0, j_0} = 0$ if $i_0 + j_0$ is odd. Then for any $Z \in [A, B]$ it holds that $z_{ij} = 0$*

either for all

$$j_0 \leq j \text{ if } i < i_0 \text{ and } j_0 < j \text{ if } i = i_0, \text{ if } i_0 < j_0,$$

or for all

$$j \leq j_0 \text{ if } i_0 < i \text{ and } j < j_0 \text{ if } i = i_0, \text{ if } j_0 < i_0.$$

Theorem 4.4. [Gar03, Theorem 1] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$. Then $[A, B]$ is $NsATP$ if and only if A and B are $NsATP$.*

In the next section we prove the conjecture completely and therefore we close the book on Garloff's Conjecture. Moreover, we extend the conjecture for some special cases of TN matrices which are not necessarily nonsingular. In Sections 4.3 and 4.4 we give analogous results for some subclasses of the $NsSR$ matrices.

4.2. Matrix Intervals of Nonsingular Totally Nonnegative Matrices

In this section we present the proof of Garloff's Conjecture and extend it to some special cases of totally nonnegative matrices. We first give some auxiliary results which will be used in this section.

Lemma 4.3. [Kut71, Corollary 3.5], [Neu90, Proposition 3.6.6] *Let $A, B, Z \in \mathbb{R}^{n,n}$ and let A and B be nonsingular with $0 \leq A^{-1}, B^{-1}$. If $A \leq Z \leq B$, then Z is nonsingular, and we have $B^{-1} \leq Z^{-1} \leq A^{-1}$.*

The determinantal monotonicity presented in Lemma 4.4 below follows from a similar property given in [Met72, p. 27] for matrices whose leading principal submatrices have nonnegative inverses. We present the proof here since we will refer to it in the proofs of Proposition 4.1 and Corollary 4.2.

Lemma 4.4. [AG13, Lemma 3.2] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$, A be $NsTN$, and B be TN . Then for any $Z \in [A, B]$*

$$\det A \leq \det Z \leq \det B.$$

Proof. We proceed by induction on n . The statement holds trivially for $n = 1$. Assume that the statement is true for fixed n and let $[A, B] \in \mathbb{I}(\mathbb{R}^{n+1,n+1})$, A be $NsTN$, B be TN , and $Z \in [A, B]$. Assume first that B is nonsingular. Then by Lemma 2.6 $A[2, \dots, n+1]$, $B[2, \dots, n+1]$ are $NsTN$ and by the induction hypothesis

$$0 < \det A[2, \dots, n+1] \leq \det Z[2, \dots, n+1] \leq \det B[2, \dots, n+1]. \quad (4.4)$$

Since $0 \leq (A^*)^{-1}$, $(B^*)^{-1}$ and $A^* \leq Z^* \leq B^*$, it follows from Lemma 4.3 that

$$(B^*)^{-1}[1] \leq (Z^*)^{-1}[1] \leq (A^*)^{-1}[1],$$

whence

$$\frac{\det B[2, \dots, n+1]}{\det B} \leq \frac{\det Z[2, \dots, n+1]}{\det Z} \leq \frac{\det A[2, \dots, n+1]}{\det A}. \quad (4.5)$$

From

$$\frac{\det B[2, \dots, n+1]}{\det Z[2, \dots, n+1]} \cdot \det Z \leq \det B$$

and (4.4) we obtain $\det Z \leq \det B$. The remaining inequality follows similarly. If B is singular set $B(\epsilon) := B + \epsilon \mathbf{e}_{n+1} \mathbf{e}_{n+1}^T$ for $0 < \epsilon$, where \mathbf{e}_{n+1} denotes the last unit vector of \mathbb{R}^{n+1} . Then $B(\epsilon)$ is $NsTN$ since by induction

$$0 < \det A[1, \dots, n] \leq \det B[1, \dots, n]$$

and the claim follows now from the case that B is nonsingular and letting ϵ tend to zero. \square

The following proposition provides an interesting property which plays a fundamental role in proving the Garloff's Conjecture. This property shows that the checkerboard partial ordering is preserved among the matrices which result from the application of the Cauchon Algorithm to a given matrix interval whose corner matrices are $NsTN$.

Proposition 4.1. [AG13, Proposition 3.3] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ with $Z \in [A, B]$. If A and B are $NsTN$, then $\tilde{A} \leq^* \tilde{B}$ and $\tilde{Z} \in [\tilde{A}, \tilde{B}]$.*

Proof. Let A and B be $NsTN$. Then by Theorem 3.2 (ii) and Proposition 3.5 \tilde{A} and \tilde{B} are nonnegative Cauchon matrices with positive diagonal entries. We give the proof only for $j \leq k$ since as in the proof of Proposition 3.8 the case $k < j$ can be reduced to the case $j < k$ by replacing A by A^T . We show by decreasing induction on $k = 1, 2, \dots, n$ that the representations (3.34) of \tilde{a}_{kj} , \tilde{z}_{kj} , \tilde{b}_{kj} are of the same order and the following inequalities hold

$$(-1)^{k+j} \tilde{a}_{kj} \leq (-1)^{k+j} \tilde{z}_{kj} \leq (-1)^{k+j} \tilde{b}_{kj}, \quad j = 1, \dots, k.$$

The statement trivially holds for $k = n$ by (3.35).

Assume that the statement is true for fixed $k + 1$, in particular,

$$(-1)^{k+1+j} \tilde{a}_{k+1,j} \leq (-1)^{k+1+j} \tilde{z}_{k+1,j} \leq (-1)^{k+1+j} \tilde{b}_{k+1,j}, \quad j = 1, \dots, k + 1. \quad (4.6)$$

We want to prove the statement for k .

Set $v_A := 0$ if $\tilde{a}_{k+1,j} > 0$, $j = 1, \dots, k$; otherwise set

$$v_A := \max \{j \in \{1, \dots, k\} \mid \tilde{a}_{k+1,j} = 0\}.$$

Define v_B similarly.

Note that by the induction hypothesis and using arguments like that are used in the proof of Theorem 4.6 below we may conclude that $\tilde{Z}[k + 1, \dots, n | 1, \dots, n]$ is a nonnegative Cauchon matrix and so by Theorem 3.2 (ii) $Z[k + 1, \dots, n | 1, \dots, n]$ is TN . Moreover, by using the induction hypothesis, the fact that \tilde{A} and \tilde{B} are nonnegative Cauchon matrices with positive diagonal entries, Lemma 4.1, and following carefully the arguments in the proof of Proposition 3.8 we may assume that (ii) in the proof of the Proposition 3.8 is also fulfilled for the entries of $\tilde{Z}[k + 1, \dots, n | 1, \dots, n]$. If $v_A = v_B = 0$, then by (4.6) $\tilde{z}_{k+1,j} > 0$, $j = 1, \dots, k$. The entries \tilde{a}_{kj} , \tilde{z}_{kj} , \tilde{b}_{kj} can be represented as ratios of contiguous minors for each $j \in \{1, \dots, k\}$. This follows by using the Cauchon Algorithm and Proposition 3.3 since we know at the beginning of the induction step $k + 1 \rightarrow k$ the entries of the matrices $A^{(k+2,2)}$, $Z^{(k+2,2)}$, $B^{(k+2,2)}$. The representations of the entries in row k of the three matrices

coincide with the ones of the entries in their $(k+1)$ -th row with the exception that the first row index in the numerator, viz. $k+1$, is replaced by k . Now the application of (ii) in the proof of Proposition 3.8 and Lemma 1.3 yields that the representations of \tilde{a}_{kj} , \tilde{z}_{kj} , \tilde{b}_{kj} are all of the same order if the pair $(\tilde{a}_{k+1,j}, \tilde{a}_{k+1,j+1})$ has a common order equal to the common order of the pairs $(\tilde{z}_{k+1,j}, \tilde{z}_{k+1,j+1})$ and $(\tilde{b}_{k+1,j}, \tilde{b}_{k+1,j+1})$. Otherwise we proceed as in the proof of (ii) in the proof of Proposition 3.8. Assume without loss of generality that $k+j$ is even.

Case 1: Suppose that the representations of \tilde{a}_{kj} , \tilde{z}_{kj} , and \tilde{b}_{kj} have the same order.

If $\tilde{a}_{kj} > 0$ we get similarly as in (4.5) (taking the reciprocal values) applied to the submatrices corresponding to the numerators of \tilde{a}_{kj} , \tilde{z}_{kj} , and \tilde{b}_{kj}

$$\tilde{a}_{kj} \leq \tilde{z}_{kj} \leq \tilde{b}_{kj}. \quad (4.7)$$

If $\tilde{a}_{kj} = 0$, then by the assumption it has a representation (3.34), where the numerator vanishes and the denominator is positive. We replace a_{kj} by $a_{kj} + \epsilon$, where $0 < \epsilon$, in the submatrix that is associated to the numerator of \tilde{a}_{kj} . Laplace expansion of the determinant of the resulting submatrix along its first row or column shows that its determinant becomes positive. We replace also b_{kj} by $b_{kj} + \epsilon$ and z_{kj} by $z_{kj} + \epsilon$ in the submatrices that are associated to the numerators of \tilde{b}_{kj} and \tilde{z}_{kj} , respectively. By application of Lemma 4.4 ((4.5) taking the reciprocal values) to the resulting submatrices and Laplace expansion along the first row or column of the corresponding submatrices of the numerators in the resulting representations we obtain

$$\tilde{a}_{kj} + \epsilon \leq \tilde{z}_{kj} + \epsilon \leq \tilde{b}_{kj} + \epsilon,$$

which yields (4.7).

Case 2: Suppose that \tilde{a}_{kj} , \tilde{z}_{kj} , and \tilde{b}_{kj} have representations as given in (3.34) but possibly of different orders:

$$\begin{aligned} \tilde{a}_{kj} &= \frac{\det A[k, \dots, k+p_1 | j, \dots, j+p_1]}{\det A[k+1, \dots, k+p_1 | j+1, \dots, j+p_1]}, \\ \tilde{z}_{kj} &= \frac{\det Z[k, \dots, k+p_2 | j, \dots, j+p_2]}{\det Z[k+1, \dots, k+p_2 | j+1, \dots, j+p_2]}, \\ \tilde{b}_{kj} &= \frac{\det B[k, \dots, k+p_3 | j, \dots, j+p_3]}{\det B[k+1, \dots, k+p_3 | j+1, \dots, j+p_3]}, \end{aligned}$$

for some p_1 , p_2 , and p_3 . Then we distinguish the following three subcases:

Case 2.1: $p_1 = \min\{p_1, p_2, p_3\}$.

Since by assumption $0 < \tilde{a}_{k+1,j+1}$ we may conclude by (ii) in the proof of Proposition 3.8 and by Proposition 3.3 that the representation of $\tilde{a}_{k+1,j+1}$ has an order $p_1 - 1$ which is equal to the order of a representation of $\tilde{a}_{k+1,j}$.

In the same way we conclude that $\tilde{z}_{k+1,j+1}$ and $\tilde{b}_{k+1,j+1}$ have representations of orders $p_2 - 1$ and $p_3 - 1$, respectively.

If $\tilde{b}_{k+1,j+1}$ has a representation of order $p_1 - 1$ or less then by proceeding similarly as in Case (b) in the proof of Proposition 3.8 we obtain that all the leading principal minors of

$B[k+1, \dots, k+p_3 | j, \dots, j+p_3-1]$ of order greater than p_1 vanish. Hence by repeated application of Lemma 1.3 to the numerator of the representation of \tilde{b}_{kj} we obtain that \tilde{b}_{kj} has a representation of order p_1 .

If $\tilde{b}_{k+1, j+1}$ has a representation of order greater than p_1-1 then by the induction hypothesis $\tilde{a}_{k+1, j+1}$ has a representation of order greater than p_1-1 . Whence by proceeding as in Case (b) in the proof of Proposition 3.8 we obtain that $\det A[k+1, \dots, k+p_1+1 | j, \dots, j+p_1] = 0$. Let q be the smallest integer such that $\det A[k+1, \dots, k+q+1 | j, \dots, j+q] = 0$. Then it follows that $q \leq p_1$. Hence if $0 < \det B[k+1, \dots, k+q | j, \dots, j+q-1]$, then by application of Lemma 4.4 after adding $0 < \epsilon$ to the bottom right corner entry of each of the following submatrices:

$$B[k+1, \dots, k+q+1 | j, \dots, j+q] \leq^* A[k+1, \dots, k+q+1 | j, \dots, j+q],$$

and letting ϵ tend to zero we arrive at

$$\det B[k+1, \dots, k+q+1 | j, \dots, j+q] = 0. \quad (4.8)$$

Otherwise if $\det B[k+1, \dots, k+q | j, \dots, j+q-1] = 0$, then by Lemma 2.6 we arrive at (4.8). Hence we obtain by Lemma 2.6 that each leading principal minor of $B[k+1, \dots, k+p_3 | j, \dots, j+p_3-1]$ of order greater than q vanishes. Hence by repeated application of Lemma 1.3 to the numerator of the representation of \tilde{b}_{kj} the orders of its representation can be decreased to p_1 . By using verbatim the same arguments we can show that the order of the representation of \tilde{z}_{kj} is p_1 or can be decreased to p_1 .

Case 2.2: $p_3 = \min \{p_1, p_2, p_3\}$.

By Lemma 4.4 we have

$$\begin{aligned} 0 < \det A[k+1, \dots, k+p_1 | j+1, \dots, j+p_1] &\leq \det Z[k+1, \dots, k+p_1 | j+1, \dots, j+p_1] \\ &\leq \det B[k+1, \dots, k+p_1 | j+1, \dots, j+p_1]. \end{aligned}$$

By using Proposition 3.3, $\tilde{a}_{k+1, j+1}$, $\tilde{z}_{k+1, j+1}$, and $\tilde{b}_{k+1, j+1}$ have representations of orders p_1-1 , p_2-1 , and p_3-1 , respectively. If $\tilde{b}_{k+1, j+1}$ has a representation of order different from p_3-1 , then by proceeding as in Case (b) in the proof of Proposition 3.8 we obtain that all the leading principal minors of $B[k+1, \dots, k+p_1 | j, \dots, j+p_1-1]$ of order greater than p_3 vanish and so we can increase the order of \tilde{b}_{kj} to p_1 . If $\tilde{b}_{k+1, j+1}$ has a unique representation of order p_3-1 , then by the induction hypothesis $\tilde{a}_{k+1, j+1}$ has a representation of order p_3-1 . Proceeding as in Case 2.1 we conclude that all the leading principal minors of $B[k+1, \dots, k+p_1 | j, \dots, j+p_1-1]$ of order greater than p_3 vanish. Therefore by using the same arguments we can increase the order of \tilde{b}_{kj} to p_1 . If $p_2 < p_1$, then we repeat the same arguments which we have used in the case of \tilde{b}_{kj} to \tilde{z}_{kj} and increase its order to p_1 . If $p_1 < p_2$, then by Case 2.1 we decrease the order of \tilde{z}_{kj} to p_1 .

Case 2.3: $p_2 = \min \{p_1, p_2, p_3\}$.

We accomplish the case $p_3 < p_1$ as Case 2.2 and increase the orders of the representations to p_1 . Otherwise we proceed as in Case 2.2 to increase the order of the representation of \tilde{z}_{kj} to p_1 and as in Case 2.1 to decrease the order of the representation of \tilde{b}_{kj} to p_1 .

Hence by proceeding as in Case 1 we arrive at (4.7). Now we assume without loss of generality that $v := v_A \geq v_B$ and $v_A \geq 1$. By Theorem 3.2 (ii) and Proposition 3.5 we conclude that $\tilde{a}_{k+1,l} = 0$, $l = 1, \dots, v$, from which it follows by (4.6) that $\tilde{b}_{k+1,l} = \tilde{z}_{k+1,l} = 0$, $l = 1, \dots, v-1$. By the induction hypothesis and using arguments like that have been used in proof of Proposition 3.8, the entries $\tilde{a}_{kl}, \tilde{z}_{kl}, \tilde{b}_{kl}$ can be represented as ratio of contiguous minors of the same order as $\tilde{a}_{k+1,l}, \tilde{z}_{k+1,l}, \tilde{b}_{k+1,l}$ for each $l = 1, 2, \dots, v-1$. Hence by application of Case 1 we obtain (4.7). For $l = v, \dots, k$, application of Proposition 3.3, Lemma 1.3, and the induction hypothesis show that the entries $\tilde{a}_{kl}, \tilde{z}_{kl}, \tilde{b}_{kl}$ can be represented as ratio of contiguous minors of order greater than the corresponding entries $\tilde{a}_{k+1,l}, \tilde{z}_{k+1,l}, \tilde{b}_{k+1,l}$. Now we proceed similarly as in the case $v_A = v_B = 0$ to rewrite the representations of $\tilde{a}_{kl}, \tilde{z}_{kl}, \tilde{b}_{kl}$ as ratios of contiguous minors of the same order as in Case 2. Hence by applying Case 1 we arrive at (4.7). This completes the proof. \square

As a consequence of Proposition 4.1, if the two corners of a given matrix interval are $NsTN$ and belong to the same TN cell then each matrix in this matrix interval belongs to the same TN cell.

Theorem 4.5. [AG13, Theorem 3.4] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ and let A and B be nonsingular and in the same TN cell. If $Z \in [A, B]$, then Z belongs to this cell, too.*

Proof. The statement follows immediately from Proposition 4.1 using Theorem 3.3 part 3. \square

Remark 4.1. *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$, and A and B be $NsTN$ such that $\pi(A) = \pi(B)$. Then $\pi(Z) = \pi(A) = \pi(B)$ for each $Z \in [A, B]$.*

Remark 4.2. [AG13, Remark 3.5] *If A and B are TP , then both matrices are in the cell associated with the empty family of minors which corresponds to the Cauchon diagram with no black squares. It follows from Theorem 4.5 that for each $Z \in [A, B]$, Z is TP . This result is already given in Corollary 4.1.*

Next we settle Conjecture 4.1 which concerns the case that the two corners A and B are not necessarily in the same TN cell.

Theorem 4.6. [AG13, Theorem 3.6] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ with $Z \in [A, B]$. If A and B are $NsTN$, then Z is $NsTN$.*

Proof. Since $0 \leq \tilde{Z}$ by Proposition 4.1 we have to show that \tilde{Z} is a Cauchon matrix with positive diagonal entries, see Theorem 3.2 (ii) and Proposition 3.5. By Proposition 3.5 we have $\tilde{a}_{kk} > 0$, $k = 1, \dots, n$, which implies that $\tilde{z}_{kk} > 0$, $k = 1, \dots, n$. Assume that $\tilde{z}_{kj} = 0$ and $1 < k, j$. Without loss of generality we may assume that $k + j$ is even. Then it follows that $\tilde{a}_{kj} = 0$. Since \tilde{A} is a Cauchon matrix we conclude that all entries of \tilde{A} to the left of the position (k, j) or above it are zero. Without loss of generality we may assume that $\tilde{a}_{ki} = 0$, $i = 1, \dots, j-1$. Since $\tilde{a}_{kk} > 0$, $k = 1, \dots, n$, it follows that $j < k$. By Proposition 4.1 $\tilde{A} \leq^* \tilde{B}$ holds which implies that $\tilde{b}_{k,j-1} = 0$, whence all entries of \tilde{B} to the left of position $(k, j-1)$ or above it must vanish. Again, by Proposition 3.5 we can exclude the latter case. It follows that $\tilde{z}_{ki} = 0$, $i = 1, \dots, j-1$, which concludes the proof. \square

The following examples show that Theorem 4.6 does not hold if at least one of the corner matrices is singular.

Example 4.1. [Gar82, p. 158]

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \leq^* \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \leq^* \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (4.9)$$

Example 4.2. [FJ11, Example 3.2.5]

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \leq^* \begin{bmatrix} 3 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 5 \end{bmatrix} \leq^* \begin{bmatrix} 4 & 0 & 5 \\ 0 & 0 & 0 \\ 5 & 0 & 13 \end{bmatrix}. \quad (4.10)$$

Proceeding similarly as in the proof of the singular case in Lemma 4.4 and using Theorem 5.11 (in the next chapter) we obtain the following corollary as an extension of the nonsingular case.

Corollary 4.2. [AG13, Corollary 3.7] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ with $Z \in [A, B]$. If A and B are TN and $A[2, \dots, n]$ or $A[1, \dots, n-1]$ is nonsingular, then Z is TN .*

We conclude this section with the following result which states that in the tridiagonal case the assumptions can be further weakened.

Corollary 4.3. [AG13, Corollary 3.8] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ with $Z \in [A, B]$. If A and B are tridiagonal TN , then Z is TN .*

Proof. Let $A(\epsilon) := A + \epsilon I$, where $0 < \epsilon$, and define $B(\epsilon), Z(\epsilon)$ analogously. Then by Theorem 5.3 (in the next chapter) $A(\epsilon)$ and $B(\epsilon)$ are TN and by Theorem 5.2

$$0 < \det A + \epsilon^n \leq \det A(\epsilon),$$

whence $A(\epsilon)$ is nonsingular. By Lemma 4.4 $B(\epsilon)$ is also nonsingular and the statement follows now from Theorem 4.6 and letting ϵ tend to zero. \square

4.3. Matrix Intervals of Nonsingular Totally Nonpositive Matrices

In this section we consider matrix intervals of nonsingular totally nonpositive matrices with respect to the checkerboard ordering. We start with the following lemma which is analogous to Lemma 4.4 in the case of totally nonpositive matrices.

Lemma 4.5. [AG16a, Lemma 5.2] *Let $2 \leq n$, $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ with $Z \in [A, B]$ and let A and B be $t.n.p.$ Then the following inequalities hold*

$$\det B \leq \det Z \leq \det A,$$

if

(i) $n = 2$, or

(ii) $2 < n$, A is nonsingular and at least one of the following three conditions is fulfilled:

(a) B is nonsingular,

(b) $b_{11} < 0$,

(c) $b_{nn} < 0$.

Proof. (i) is shown by direct computation.

(ii) We proceed by induction on n . Assume that the statement is true for fixed n and let $A, B, Z \in \mathbb{R}^{n+1, n+1}$, A be *Ns.t.n.p.*, B be *t.n.p.*, and $A \leq^* Z \leq^* B$. Assume first that B is nonsingular. Then by Theorem 2.8 $A[2, \dots, n+1]$, $B[2, \dots, n+1]$ are *Ns.t.n.p.* and by the induction hypothesis

$$\det B[2, \dots, n+1] \leq \det Z[2, \dots, n+1] \leq \det A[2, \dots, n+1] < 0. \quad (4.11)$$

Since $0 \leq (A^*)^{-1}, (B^*)^{-1}$ and $A^* \leq Z^* \leq B^*$, it follows from Lemma 4.3 that

$$(B^*)^{-1}[1] \leq (Z^*)^{-1}[1] \leq (A^*)^{-1}[1],$$

whence

$$\frac{\det B[2, \dots, n+1]}{\det B} \leq \frac{\det Z[2, \dots, n+1]}{\det Z} \leq \frac{\det A[2, \dots, n+1]}{\det A}. \quad (4.12)$$

From

$$\det B \leq \frac{\det B[2, \dots, n+1]}{\det Z[2, \dots, n+1]} \cdot \det Z$$

and (4.11) we obtain $\det B \leq \det Z$. The remaining inequality follows similarly. If B is singular and $b_{11} < 0$ we first show that $b_{22} < 0$. Suppose that $b_{22} = 0$. Then $\det B[2, 3|1, 2] = b_{21} \cdot b_{32} \geq 0$ which implies that $b_{21} = 0$ or $b_{32} = 0$ whence $a_{21} = 0$ or $a_{32} = 0$, a contradiction to Lemma 2.4 (note that $a_{11} < 0$). Therefore (4.11) holds by the induction hypothesis. Set $B(\delta) := B + \delta \mathbf{e}_1 \mathbf{e}_1^T$, $A(\delta) := A + \delta \mathbf{e}_1 \mathbf{e}_1^T$, and $Z(\delta) := Z + \delta \mathbf{e}_1 \mathbf{e}_1^T$ for $0 < \delta < -b_{11}$, where \mathbf{e}_1 denotes the first unit vector of \mathbb{R}^{n+1} . Laplace expansion of $\det B(\delta)$ along its first row or column shows that $B(\delta)$ is *Ns.t.n.p.* and the claim follows now from the case that B is nonsingular and letting δ tend to zero. If B is singular and $b_{nn} < 0$ we proceed similarly. \square

By using Theorem 3.9, Corollary 3.3, Propositions 2.3 or 2.4, 2.5, 3.9, and (4.12) we obtain by an induction proof similarly as in Proposition 4.1 the following result.

Proposition 4.2. [AG16a, Proposition 5.6] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n, n})$ with $Z \in [A, B]$. If A and B are *Ns.t.n.p.* with $b_{nn} < 0$, then $\tilde{A} \leq^* \tilde{B}$ and $\tilde{Z} \in [\tilde{A}, \tilde{B}]$.*

Using Propositions 3.6 and 4.2 and Theorem 3.9 we get by a proof similar to the one given for Theorem 4.6 the following theorem.

Theorem 4.7. [AG16a, Theorem 5.7] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ with $Z \in [A, B]$. If A and B are *Ns.t.n.p.* with $b_{nn} < 0$, then Z is *Ns.t.n.p.**

By passing over to $A^\#$ and vice-versa, Theorem 4.7 remains in force if we replace the condition $b_{nn} < 0$ by $b_{11} < 0$. A similar modification applies to Corollary 4.4 below.

The following example shows that Theorem 4.7 does not hold if both of the corner matrices are singular.

Example 4.3.

$$\begin{bmatrix} -5 & 0 & -4 \\ 0 & 0 & 0 \\ -13 & 0 & -5 \end{bmatrix} \leq^* \begin{bmatrix} -4 & 0 & -3 \\ 0 & 0 & 0 \\ -5 & 0 & -4 \end{bmatrix} \leq^* \begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}. \quad (4.13)$$

Proceeding similarly as in the proof of the singular case in Lemma 4.5 we obtain for a sufficient small positive number δ that $A(\delta) = A + \delta E_{11}$, $Z(\delta) = Z + \delta E_{11}$, and $B(\delta) = B + \delta E_{11}$ in Case (i) or $A(\delta) = A + \delta E_{nn}$, $Z(\delta) = Z + \delta E_{nn}$, and $B(\delta) = B + \delta E_{nn}$ in Case (ii) of the next corollary are nonsingular and by Theorem 4.7 we conclude that $Z(\delta)$ is *Ns.t.n.p.* Hence by letting δ tend to zero and using the fact that the set of *t.n.p.* matrices is closed we obtain the following corollary which provides an extension of the nonsingular case.

Corollary 4.4. [AG16a, Corollary 5.8] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ with $Z \in [A, B]$, and A and B be *t.n.p.* with $b_{nn} < 0$ and*

(i) $A[2, \dots, n]$ *nonsingular* and $b_{11} < 0$,

or

(ii) $A[1, \dots, n-1]$ *nonsingular*.

Then Z is *t.n.p.*

We conclude this section with the following remark which extends Theorem 4.7 to arbitrary intervals of *Ns.t.n.p.* matrices.

Remark 4.3. *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ with $Z \in [A, B]$ and let A and B be *Ns.t.n.p.* with $b_{11} = 0$. If $b_{nn} = 0$, then by Proposition 2.6 there exists a small suitable $0 < \epsilon_0$ such that $B_\epsilon := B - \epsilon E_{nn}$ is *Ns.t.n.p.* for all $0 \leq \epsilon < \epsilon_0$. If $a_{nn} = 0$ then define A_ϵ analogously (with suitable ϵ) otherwise set $A_\epsilon := A$ and by Proposition 2.6 A_ϵ and B_ϵ are *Ns.t.n.p.* matrices. Define the matrices chosen from the matrix interval analogously. Hence we have that for $[A_\epsilon, B_\epsilon]$ Theorem 4.7 holds. By Proposition 2.7 and the definition of *Ns.t.n.p.* matrices $[A, B]$ is *Ns.t.n.p.**

4.4. Matrix Intervals of Nonsingular Almost Strictly Sign Regular Matrices

In this section we consider matrix intervals of nonsingular almost strictly sign regular matrices and nonsingular sign regular matrices with special signatures. We start with the following monotonicity property of the determinant of nonsingular almost strictly sign regular matrices which is analogous to Lemmata 4.4 and 4.5.

Lemma 4.6. [AG16a, Lemma 5.3] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ with $Z \in [A, B]$ and let A and B be $NsASSR$ with the same signature $\epsilon = (\epsilon_1, \dots, \epsilon_n)$. Then $\det Z$ lies between $\det A$ and $\det B$.*

Proof. For $\epsilon_2 = 1$ and $\epsilon_{n-1} \cdot \epsilon_n = 1$ we proceed similarly as in the nonsingular case in the proof of Lemma 4.5. Hereby the nonsingularity of $A[2, \dots, n+1]$ and $B[2, \dots, n+1]$ is assured by Lemma 2.2 (i). The case $\epsilon_{n-1} \cdot \epsilon_n = -1$ can be reduced to the case $\epsilon_{n-1} \cdot \epsilon_n = 1$ by replacing A, Z, B by $-B, -Z, -A$, respectively. The case $\epsilon_2 = -1$ can be reduced to the case $\epsilon_2 = 1$ by multiplication of A, Z, B by T_n , see Lemma 2.2 (iii). \square

The proof of Theorem 4.1 follows easily by using Theorem 2.1 and the above lemma.

Now we are in the position to extend the results of Theorem 4.1 on intervals of SSR matrices and Theorem 4.4 on intervals of $NsATP$ matrices to arbitrary $NsASSR$ matrices.

Theorem 4.8. [AG16a, Theorem 5.5] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ with $Z \in [A, B]$. If A and B are $NsASSR$ with the same signature $\epsilon = (\epsilon_1, \dots, \epsilon_n)$, then Z is $NsASSR$ with the signature ϵ .*

Proof. By Theorem 2.7 it suffices to consider the nontrivial contiguous minors of Z . Let $\det Z[\alpha|\beta]$ be such a minor of order k . We want to show that $0 < \epsilon_k \det Z[\alpha|\beta]$. We proceed by induction on k . The statement trivially holds for $k = 1$. Suppose that the sign condition is true for $k - 1$; we want to show that it is true for k . We have the following two cases:

Case 1: If A and B are both type-I staircase matrices, then obviously Z is also a type-I staircase matrix. Since $Z[\alpha|\beta]$ is contiguous we have

$$A[\alpha|\beta] \leq^* Z[\alpha|\beta] \leq^* B[\alpha|\beta] \quad (4.14)$$

or the reverse inequalities. Without loss of generality suppose that (4.14) holds.

Case 1.1: Suppose that the contiguous minors $\det A[\alpha|\beta]$ and $\det B[\alpha|\beta]$ are both nontrivial and therefore are nonsingular. Hence $A[\alpha|\beta]$ and $B[\alpha|\beta]$ are themselves $NsASSR$ (with common signature $(\epsilon_1, \dots, \epsilon_k)$) and the claim follows by Lemma 4.6.

Case 1.2: Suppose that $\det A[\alpha|\beta]$ or $\det B[\alpha|\beta]$ is trivial. Then Lemma 2.2 (i) implies that $a_{\alpha_1, \beta_1} \cdot a_{\alpha_2, \beta_2} \cdots a_{\alpha_k, \beta_k} = 0$ or $b_{\alpha_1, \beta_1} \cdot b_{\alpha_2, \beta_2} \cdots b_{\alpha_k, \beta_k} = 0$. Let

$$i_0 := \min \{i \in \{1, \dots, k\} \mid a_{\alpha_i, \beta_i} = 0 \text{ or } b_{\alpha_i, \beta_i} = 0\}.$$

Without loss of generality we may assume that $1 < i_0$. By (4.14) we have

$$\det Z[\alpha|\beta] = \det Z[\alpha_1, \dots, \alpha_{i_0-1}|\beta_1, \dots, \beta_{i_0-1}] \cdot \det Z[\alpha_{i_0}, \dots, \alpha_k|\beta_{i_0}, \dots, \beta_k]. \quad (4.15)$$

Since $Z[\alpha|\beta]$ is nontrivial it follows from Lemma 2.2 (i) that $z_{\alpha_1, \beta_1} \cdots z_{\alpha_k, \beta_k} \neq 0$ and $a_{\alpha_1, \beta_1} \cdots a_{\alpha_k, \beta_k} \neq 0$ or $b_{\alpha_1, \beta_1} \cdots b_{\alpha_k, \beta_k} \neq 0$ but not both since $z_{\alpha_1, \beta_1} \cdots z_{\alpha_k, \beta_k} \neq 0$, whence both minors on the right-hand side of (4.15) are nontrivial, too. Lemma 2.3 implies that $\epsilon_j = \epsilon_1^j$, $j = 1, \dots, k$, and we obtain

$$\begin{aligned} \epsilon_k \det Z[\alpha|\beta] &= \epsilon_1^k \det Z[\alpha|\beta] \\ &= \epsilon_1^{i_0-1} \det Z[\alpha_1, \dots, \alpha_{i_0-1}|\beta_1, \dots, \beta_{i_0-1}] \cdot \epsilon_1^{k-i_0+1} \det Z[\alpha_{i_0}, \dots, \alpha_k|\beta_{i_0}, \dots, \beta_k] \\ &= \epsilon_{i_0-1} \det Z[\alpha_1, \dots, \alpha_{i_0-1}|\beta_1, \dots, \beta_{i_0-1}] \cdot \epsilon_{k-i_0+1} \det Z[\alpha_{i_0}, \dots, \alpha_k|\beta_{i_0}, \dots, \beta_k]. \end{aligned}$$

Both signed minors on the right-hand side of the last equation are positive by the induction hypothesis and it follows that $0 < \epsilon_k \det Z[\alpha|\beta]$, as desired. This completes the proof of Case 1.

Case 2: If A and B are type-II staircase matrices, then obviously Z is also a type-II staircase matrix. By Lemma 2.2 (iii) we can reduce Case 2 to Case 1. \square

The following theorem shows that the matrix interval property holds when the corner matrices are tridiagonal and $NsSR$.

Theorem 4.9. [AG16a, Theorem 5.11] *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ with $Z \in [A, B]$ and let A and B be tridiagonal. If A and B are $NsSR$ with the same signature ϵ then Z is $NsSR$ with signature ϵ .*

Proof. Without loss of generality we may assume that $0 \leq A$, otherwise replace A by $-B$ and B by $-A$. Since the statement trivially holds for $n \leq 2$, suppose that $3 \leq n$. It follows from Theorem 2.6 and Corollary 4.3 that $Z[1, \dots, n-1]$ and $Z[2, \dots, n]$ are TN . Since $A_{n-2} := A[1, \dots, n-2]$ and $B_{n-2} := B[1, \dots, n-2]$ are $NsTN$, $0 \leq (A_{n-2}^*)^{-1}, (B_{n-2}^*)^{-1}$, and $A_{n-2}^* \leq Z^*[1, \dots, n-2] \leq B_{n-2}^*$, Lemma 4.3 implies that $Z[1, \dots, n-2]$ is nonsingular, too. Similarly it follows that $Z, Z[2, \dots, n-1]$ are nonsingular. By Theorem 2.6 we obtain that Z is $NsSR$. \square

In Sections 4.2 and 4.3 we investigate the matrix interval property for $NsTN$ matrices and $Ns.t.n.p.$ matrices, respectively. By multiplying each matrix in a given matrix interval from the right or left by the permutation matrix T_n , or by (-1) and using the results on matrix intervals of $NsTN$ and $Ns.t.n.p.$ matrices we obtain the following result.

Theorem 4.10. *Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ with $Z \in [A, B]$ and let A and B be $NsSR$ matrices with the same signature $\epsilon = (\epsilon_1, \dots, \epsilon_n)$. If ϵ is one of the following signatures:*

- (i) $\epsilon_i = (-1)^i$,
- (ii) $\epsilon_i = (-1)^{\frac{i(i-1)}{2}}$,

- (iii) $\epsilon_i = (-1)^{\frac{i(i+1)}{2}}$,
- (iv) $\epsilon_i = (-1)^{i+1}$,
- (v) $\epsilon_i = (-1)^{\frac{i(i-1)}{2}+1}$,
- (vi) $\epsilon_i = (-1)^{\frac{i(i+1)}{2}+1}$,

for $i = 1, \dots, n$, then Z is $NsSR$ with signature ϵ .

Proof. It is easy to show that $D := \text{diag}(-1, -1, \dots, -1)$ and T_n are $NsSR$ matrices with signatures given in (i) and (ii), respectively, and $D^{-1} = D$ and $T_n^{-1} = T_n$. Hence if A and B are $NsSR$ matrices with the same signature which is given in one of (i)-(vi), then by Theorem 2.3 the following hold: If for $i = 1, \dots, n$, $\epsilon_i =$

- (i) $(-1)^i$, then DA and DB are $NsTN$,
- (ii) $(-1)^{\frac{i(i-1)}{2}}$, then T_nA and T_nB are $NsTN$,
- (iii) $(-1)^{\frac{i(i+1)}{2}}$, then DT_nA and DT_nB are $NsTN$,
- (iv) $(-1)^{i+1}$, then DA and DB are $Ns.t.n.p.$,
- (v) $(-1)^{\frac{i(i-1)}{2}+1}$, then T_nA and T_nB are $Ns.t.n.p.$,
- (vi) $(-1)^{\frac{i(i+1)}{2}+1}$, then DT_nA and DT_nB are $Ns.t.n.p.$

Hence by Theorem 4.6 or Theorem 4.7 and Remark 4.3, and Theorem 2.3 Z is a $NsSR$ matrix with the same signature. \square

We have investigated the application of the Cauchon Algorithm to $NsTN$ and $Ns.t.n.p.$ matrices which has lead us to the interval property of these matrices. We also proved that, e.g., the sets of the $NsASSR$ matrices and the tridiagonal $NsSR$ matrices possess this property, too. These results together with the results in the previous sections and in Theorem 4.10 on the interval property of some other classes of $NsSR$ matrices evoke the (open) question whether the interval property holds for general $NsSR$ matrices if the corner matrices are $NsSR$ with the same signature.

4.5. Sign Regular Function Matrices

In this section we consider function matrices and give some sufficient conditions for a function matrix to be totally positive, nonsingular totally nonnegative, totally negative, nonsingular totally nonpositive, strictly sign regular, or nonsingular almost strictly sign regular.

Definition 4.1. [FJ11, Section 10.3] *A function matrix is a matrix function $A(t) = (a_{ij}(t))$ in which $a_{ij}(t)$ is a function in t , for each $i, j = 1, \dots, n$, and each function is real.*

A matrix function may be viewed as a matrix in which each entry is a function of the same collection of variables.

A function matrix $A(t)$ is called *TP* (*TN*, or *Ns.t.n.p.*) if for each t in the domain, $A(t)$ is *TP* (*TN*, or *Ns.t.n.p.*, respectively).

In [FJ11], it was asked under what circumstances is a polynomial matrix *TP*, *TN*, or *NsTN*? Several Toeplitz function matrices with symmetric functions as entries were proved to be *TN* by using a combinatorial approach in [Ska03].

The following theorem presents a characterization of a given *NsTN* polynomial matrix in terms of its bidiagonal factorization.

Theorem 4.11. [FJ11, Theorem 10.3.1] *Let $A(t)$ be a polynomial matrix. Then $A(t)$ is *NsTN* if and only if $A(t)$ has a bidiagonal factorization in which each bidiagonal factor is a polynomial matrix that takes on only nonnegative values and the diagonal factor has polynomial diagonal entries, taking on only positive values. In addition, $A(t)$ is *TP* if and only if $A(t)$ has a bidiagonal factorization in which all factors are polynomial matrices taking on only positive values.*

The following theorem provides a sufficient condition for a given function matrix to be *TP* or *NsTN* in a certain domain.

Theorem 4.12. *Let $A(t) = (a_{ij}(t))$ be an n -by- n function matrix. If $a_{ij}(t)$ are nonnegative continuous functions on $[a, b]$, $(-1)^{i+j}a_{ij}(t)$ are increasing functions for each $i, j = 1, \dots, n$, and $A(a)$ and $A(b)$ are *TP* (*NsTN*), then $A(t)$ is *TP* (*NsTN*).*

Proof. Let $A := A(a)$, $B := A(b)$, and $Z_t := A(t)$ for each $t \in [a, b]$. Then since $(-1)^{i+j}a_{ij}(t)$ are increasing functions for each $i, j = 1, \dots, n$ we have

$$A \leq^* Z_t \leq^* B. \quad (4.16)$$

By Remark 4.2 (Theorem 4.6) Z_t is *TP* (*NsTN*) for each $t \in [a, b]$. Hence $A(t)$ is *TP* (*NsTN*). \square

Theorem 4.5 implies that if both $A(a)$ and $A(b)$ in the above theorem belong to the same *TN* cell, then $A(t)$ belongs to the same *TN* cell that includes $A(a)$ and $A(b)$.

Parallel to Theorem 4.12 and by Theorem 4.7 and Remark 4.3, we have the following result.

Theorem 4.13. *Let $A(t) = (a_{ij}(t))$ be an n -by- n function matrix. If $a_{ij}(t)$ are nonpositive continuous functions on $[a, b]$, $(-1)^{i+j}a_{ij}(t)$ are increasing functions for each $i, j = 1, \dots, n$, and $A(a)$, $A(b)$ are *t.n.* (*Ns.t.n.p.*), then $A(t)$ is *t.n.* (*Ns.t.n.p.*).*

Analogous results hold in the case that $A(a)$ and $A(b)$ in the above theorems are *SSR* or *NsASSR* matrices with the same signature.

5. Perturbation of Totally Nonnegative Matrices

Linear transformations that preserve certain positivity classes of matrices including totally nonnegative and totally positive matrices were studied and investigated in [BHJ85]. In [FFJM14a], [FFJM14b], and [FRS14] a new class of matrices is introduced which is called *totally nonsingular matrices*; these are matrices having all of their minors nonzero, the maximum number of equal entries of a totally nonsingular or totally positive matrix can have is given and the problem of the maximum number of equal minors of totally positive matrices is solved partially. In this chapter we consider the perturbation problem of single entries for totally nonnegative matrices, i.e., we are interested in the largest amount by which the single entries of such a matrix can be varied without losing the property of total nonnegativity.

This chapter consists of five sections. In Section 5.1, the perturbation of single entries of tridiagonal totally nonnegative matrices is considered. In Section 5.2, the perturbation of single entries of totally positive matrices is studied. In Section 5.3, a general description is given on how one should proceed to solve the perturbation problem for totally nonnegative matrices. In Section 5.4, the effects of the perturbation of the entries of an oscillatory matrix on its eigenvalues are presented. Finally in Section 5.5, new results on the Perron complement of totally nonnegative matrices are given.

5.1. Perturbation of Tridiagonal Totally Nonnegative Matrices

The study of tridiagonal matrices is of independent interest since these matrices play an important role in the investigation of small oscillations of various mechanical systems [GK02]. Also, they serve as test cases for hypothesis much along the lines as trees are often test cases in graph theory. In this section we find for each entry of a tridiagonal totally nonnegative matrix the largest amount by which this entry can be perturbed without losing the property of total nonnegativity. Furthermore, we prove some determinantal inequalities and characterizations of irreducible tridiagonal totally nonnegative matrices. This section consists of two subsections. In the first one we study the nonsingular case and in the other the general case.

5.1.1. The Nonsingular Case

In this subsection we consider the variation of single entries of a nonsingular tridiagonal totally nonnegative matrix such that the resulting matrix remains nonsingular totally nonnegative. We may restrict the discussion of the off-diagonal entries to the entries which are lying above the main diagonal since the related statements for the entries below the main

diagonal follow by considering the transpose of the matrix.

Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be tridiagonal $NsTN$. We start with the entries a_{ij} with $j > i + 2$; these zero entries cannot be (strictly) increased because the resulting matrix will not be TN . This can be seen by Lemma 2.8 or by considering the minor

$$\det A[i, i+1 | i+1, j] = -a_{i+1, i+1} a_{i, j}$$

which is zero by $a_{i, j} = 0$. If $a_{i, j} + \tau > 0$ the minor becomes negative because $a_{i+1, i+1} > 0$ by Lemma 2.6.

Hence we treat in the sequel the variation of the entries on the main, the first, and the second diagonals of A , i.e., the entries of the bands (i, i) , $(i, i+1)$, and $(i, i+2)$, respectively. We start with the main diagonal but first we prove the following lemma in order to give the exact bound that allows such variation. This lemma can be derived directly from the Koteljanskii inequality (Theorem 1.4), but we provide here an alternative proof.

Lemma 5.1. [AG14b, Lemma 6] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be tridiagonal $NsTN$. Then the following inequality holds:*

$$\frac{\det A}{\det A(i)} \leq \frac{\det A(n)}{\det A(i, n)}, \quad i = 1, \dots, n-1. \quad (5.1)$$

Proof. By Proposition 1.1 and Lemma 2.6 we have

$$\det A(i) = a \det A[i+1, \dots, n] \quad \text{with} \quad a = \det A[1, \dots, i-1] > 0,$$

and similarly,

$$\det A(i, n) = a \det A[i+1, \dots, n-1], \quad i = 1, \dots, n-2.$$

Again by Proposition 1.1 there exist $b_1, b_2 \geq 0$ with

$$\det A = b_1 \det A[i, \dots, n] - b_2 \det A[i+1, \dots, n]$$

and similarly,

$$\det A(n) = b_1 \det A[i, \dots, n-1] - b_2 \det A[i+1, \dots, n-1].$$

It follows that

$$\begin{aligned} d &:= \det A \det A(i, n) - \det A(n) \det A(i) \\ &= ab_1 (\det A[i, \dots, n] \det A[i+1, \dots, n-1] - \det A[i+1, \dots, n] \det A[i, \dots, n-1]). \end{aligned}$$

Application of (1.10) to $A[i, \dots, n]$ and $A[i, \dots, n-1]$ yields

$$\begin{aligned} d &= ab_1 a_{i, i+1} a_{i+1, i} (\det A[i+1, \dots, n] \det A[i+2, \dots, n-1] \\ &\quad - \det A[i+2, \dots, n] \det A[i+1, \dots, n-1]). \end{aligned}$$

Repeated application of (1.10) results in (where c is a nonnegative constant)

$$\begin{aligned}
d &= c(\det A[n-2, n-1, n] \det A[n-1] - \det A[n-1, n] \det A[n-2, n-1]) \\
&= c[(a_{n-2, n-2}(a_{n-1, n-1}a_{n, n} - a_{n-1, n}a_{n, n-1}) - a_{n-2, n-1}a_{n-1, n-2}a_{n, n})a_{n-1, n-1} \\
&\quad - (a_{n-1, n-1}a_{n, n} - a_{n-1, n}a_{n, n-1})(a_{n-2, n-2}a_{n-1, n-1} - a_{n-2, n-1}a_{n-1, n-2})] \\
&= -ca_{n-1, n}a_{n, n-1}a_{n-2, n-1}a_{n-1, n-2} \leq 0.
\end{aligned}$$

The last inequality and Lemma 2.6 imply inequality (5.1). \square

By using the above lemma we give the exact amount by which the single main diagonal entries can be perturbed without losing the nonsingularity and total nonnegativity property.

Theorem 5.1. [AG14b, Theorem 7] *Let $A = (a_{ij}) \in \mathbb{R}^{n, n}$ be tridiagonal NsTN. Then for any $i \in \{1, \dots, n\}$, the matrix $A + \tau E_{ii}$ is NsTN if and only if*

$$-\frac{\det A}{\det A(i)} < \tau. \quad (5.2)$$

Proof. By Proposition 2.10(b), it suffices to show that condition (5.2) is equivalent to $a_{ii} + \tau \geq 0$ and all leading principal minors of $A_\tau := A + \tau E_{ii}$ are positive. Expansion of $\det A_\tau$ along its i^{th} row (or column) yields

$$\det A_\tau = \det A + \tau \det A(i), \quad (5.3)$$

which is required to be positive for all τ . Therefore condition (5.2) follows from $\det A_\tau > 0$ and Lemma 2.6. Conversely, since $\det A_\tau(n) = \det A(n) + \tau \det A(i, n)$, Lemma 5.1 assures that the leading principal minor of order $n-1$ of A_τ is positive under condition (5.2). By application of Lemma 5.1 to $A(n), A(n-1, n), \dots$ the positivity of the remaining leading principal minors follows. Finally, $a_{ii} + \tau \geq 0$ is guaranteed by Proposition 1.1 and Lemma 2.6 because for $i \geq 2$

$$a_{i, i} + \tau = \frac{\det A_\tau[1, \dots, i] + a_{i-1, i}a_{i, i-1} \det A[1, \dots, i-2]}{\det A[1, \dots, i-1]} \geq 0.$$

\square

Adding any nonnegative quantity to any entry of the main diagonal of a given tridiagonal TN matrix enters positively in each principal minor containing this entry. Hence the following two theorems easily follow by induction and Proposition 2.10.

Theorem 5.2. [And87, Theorem 2.3] *Let $A \in \mathbb{R}^{n, n}$ be a tridiagonal matrix. If A is nonnegative, and all principal minors are nonnegative, then A is TN and for any $0 < \tau_i, i = 1, \dots, n$,*

$$\det A + \prod_{i=1}^n \tau_i \leq \det (A + \text{diag}(\tau_1, \dots, \tau_n)).$$

Theorem 5.3. [And87, Corollary 2.4] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be tridiagonal TN. Then $A + \text{diag}(\tau_1, \dots, \tau_n)$ is TN for any $0 \leq \tau_i, i = 1, \dots, n$.*

Now we move to the entries on the first diagonal. Before we do so we need the following remark and lemma.

Remark 5.1. *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be a tridiagonal matrix. Then by direct computations and Proposition 1.1 we have*

$$\det A(i|i+1) = \begin{cases} a_{21} \det A[3, \dots, n] & \text{if } i = 1, \\ a_{n,n-1} \det A[1, \dots, n-2] & \text{if } i = n-1, \\ a_{i+1,i} \det A[1, \dots, i-1] \det A[i+2, \dots, n] & \text{if } 2 \leq i \leq n-2. \end{cases}$$

Lemma 5.2. [AG14b, Lemma 8] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be tridiagonal NsTN. Then for $i = 1, \dots, n-2$, if $a_{i+1,i} > 0$, the following inequality holds:*

$$\frac{\det A}{\det A(i|i+1)} \leq \frac{\det A(n)}{\det A(i, n|i+1, n)}. \quad (5.4)$$

Proof. First of all by the above remark, $a_{i+1,i} > 0$, and Lemma 2.6 the inequalities (5.4) are defined for all $i = 1, \dots, n-2$. The proof parallels the one of Lemma 5.1. Expansion of $\det A(i|i+1)$ along its i^{th} row yields

$$\det A(i|i+1) = a \det A[i+2, \dots, n], \text{ where } a = a_{i+1,i} \det A[1, \dots, i-1] > 0 \quad (5.5)$$

and similarly,

$$\det A(i, n|i+1, n) = a \det A[i+2, \dots, n-1].$$

As in the proof of Lemma 5.1, we apply Proposition 1.1 (with i replaced by $i+1$) and obtain

$$\begin{aligned} & \det A \det A(i, n|i+1, n) - \det A(n) \det A(i|i+1) \\ &= ab_1 (\det A[i+1, \dots, n] \det A[i+2, \dots, n-1] - \det A[i+2, \dots, n] \det A[i+1, \dots, n-1]). \end{aligned}$$

The claim follows now by proceeding as in the proof of Lemma 5.1. \square

Lemma 5.2 can be proved by using Remark 5.1 and Theorem 1.4.

Theorem 5.4. [AG14b, Theorem 9] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be tridiagonal NsTN. Then for any $i \in \{1, \dots, n-1\}$, if $a_{i+1,i} > 0$, the matrix $A + \tau E_{i,i+1}$ is NsTN if and only if*

$$-a_{i,i+1} \leq \tau < \frac{\det A}{\det A(i|i+1)}.$$

If $a_{i+1,i} = 0$, only the restriction $-a_{i,i+1} \leq \tau$ is required.

Proof. Let $a_{i+1,i} > 0$. Expansion of the determinant of $A_\tau := A + \tau E_{i,i+1}$ along its i^{th} row yields .

$$\det A_\tau = \det A - \tau \det A(i|i+1). \quad (5.6)$$

To show the inequality on the right-hand side, we continue similarly as in the proof of Theorem 5.1 with the application of Lemma 5.2.

If $a_{i+1,i} = 0$, each leading principal minor of A_τ is independent of τ .

For nonpositive τ , $\det A_\tau$ is positive. However, we have to assure that $a_{i,i+1} + \tau$ is nonnegative. \square

We proceed to the entries on the second diagonal. The following theorem gives the bound in this case.

Theorem 5.5. [AG14b, Theorem 10] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be tridiagonal NsTN. For $i = 1, \dots, n-2$, the matrix $A + \tau E_{i,i+2}$ is NsTN if*

$$0 \leq \tau \leq \frac{a_{i,i+1}a_{i+1,i+2}}{a_{i+1,i+1}}. \quad (5.7)$$

Proof. Since $a_{i,i+2} = 0$, τ must be nonnegative if $A_\tau := A + \tau E_{i,i+2}$ is TN. All leading principal minors of order k of A_τ with $k \geq i+2$ are monotonically increasing with respect to τ and the remaining ones are leading principal minors of A . Therefore by Theorem 2.42 (iii), it remains to consider the quasi-initial minors (note that A_τ is no longer tridiagonal if $\tau > 0$).

Let $\alpha = (1, \dots, i+k)$ and $\beta \in Q_{i+k,n}$ be arbitrary. If $k = 0$ it suffices to treat the case $\beta_i = i+2$ since if $\beta_i > i+2$, $A_\tau[\alpha|\beta]$ contains a zero column. If $\beta_{i-1} = i+1$, then $\det A_\tau[\alpha|\beta] = 0$ because all entries of $A[1, \dots, i-1|i+1, i+2]$ are zero. If $\beta_{i-1} \leq i$, then

$$\det A_\tau[\alpha|\beta] = \tau \det A[1, \dots, i-1|\beta_1, \dots, \beta_{i-1}].$$

Therefore, $\det A_\tau[\alpha|\beta] \geq 0$ for all $\tau \geq 0$.

Now let $k > 1$. We can restrict the discussion to $\beta_{i+k} = i+k+1$, because if $\beta_{i+k} > i+k+1$, then $A_\tau[\alpha|\beta]$ contains a zero column and if $\beta_{i+k} = i+k$, then $\det A_\tau[\alpha|\beta]$ is a leading principal minor.

Since in the last column of $A_\tau[\alpha|\beta]$ the only possibly nonzero entry is in the last position it suffices to consider $\alpha = (1, \dots, i+k-1)$. Continuing in this way, we arrive at $\alpha = (1, \dots, i+1)$ and $\beta_{i+1} = i+2$. Therefore it suffices to consider the case $\alpha = (1, \dots, i+1)$.

If $\beta_i = i+1$, we have

$$\begin{aligned} \det A_\tau[\alpha|\beta] &= \det A[1, \dots, i-1|\beta_1, \dots, \beta_{i-1}] \det A_\tau[i, i+1|i+1, i+2] \\ &= \det A[1, \dots, i-1|\beta_1, \dots, \beta_{i-1}](a_{i,i+1}a_{i+1,i+2} - \tau a_{i+1,i+1}). \end{aligned} \quad (5.8)$$

Therefore condition (5.7) guarantees $\det A_\tau[\alpha|\beta] \geq 0$.

If $\beta_i = i$, i.e., $\beta = (1, \dots, i, i+2)$, we have

$$\det A_\tau[\alpha|\beta] = \det A[\alpha|\beta] - \tau \det A[1, \dots, i-1, i+1|1, \dots, i].$$

Then if $\det A[1, \dots, i-1, i+1 | 1, \dots, i] = 0$, then $\det A_\tau[\alpha|\beta] = \det A[\alpha|\beta] \geq 0$.

If

$$\det A[1, \dots, i-1, i+1 | 1, \dots, i] > 0,$$

then $\det A_\tau[\alpha|\beta] \geq 0$ if and only if

$$\tau \leq \frac{\det A[\alpha|\beta]}{\det A[1, \dots, i-1, i+1 | 1, \dots, i]}. \quad (5.9)$$

Therefore to see that (5.7) implies $\det A_\tau[\alpha|\beta] \geq 0$ it remains to show that the right-hand side of inequality (5.9) is not smaller than the right-hand side of (5.7). Since

$$\det A[\alpha|\beta] = a_{i+1, i+2} \det A[1, \dots, i]$$

and

$$\det A[1, \dots, i-1, i+1 | 1, \dots, i] = a_{i+1, i} \det A[1, \dots, i-1],$$

we obtain by using (1.11)

$$\begin{aligned} 0 &\leq a_{i+1, i+2} \det A[1, \dots, i+1] \\ &= a_{i+1, i+2} (a_{i+1, i+1} \det A[1, \dots, i] - a_{i, i+1} a_{i+1, i} \det A[1, \dots, i-1]) \\ &= a_{i+1, i+1} a_{i+1, i+2} \det A[1, \dots, i] - a_{i, i+1} a_{i+1, i+2} a_{i+1, i} \det A[1, \dots, i-1] \\ &= a_{i+1, i+1} \det A[\alpha|\beta] - a_{i, i+1} a_{i+1, i+2} \det A[1, \dots, i-1, i+1 | 1, \dots, i] \end{aligned}$$

from which it follows that

$$\frac{a_{i, i+1} a_{i+1, i+2}}{a_{i+1, i+1}} \leq \frac{\det A[\alpha|\beta]}{\det A[1, \dots, i-1, i+1 | 1, \dots, i]}.$$

Now let $\beta = (1, \dots, i+k)$ and $\alpha \in Q_{i+k, n}$ be arbitrary. If $k = 2$ and $\alpha_{i+2} = i+2$, then $\det A_\tau[\alpha|\beta]$ is a leading principal minor and if $\alpha_{i+2} = i+3$, then

$$\det A_\tau[\alpha|\beta] = a_{i+3, i+2} \det A[\alpha_1, \dots, \alpha_{i+1} | 1, \dots, i+1] \geq 0.$$

If $\alpha_{i+2} > i+3$, then $\det A_\tau[\alpha|\beta] = 0$ because $A_\tau[\alpha|\beta]$ contains a zero row.

If $k > 2$ it suffices to treat only the case $\alpha_{i+k} = i+k+1$ since a submatrix $A_\tau[\alpha|\beta]$ with $\alpha_{i+k} > i+k+1$ contains a zero row and if $\alpha_{i+k} = i+k$ it is a leading principal submatrix. Since for $\alpha_{i+k} = i+k+1$

$$\det A_\tau[\alpha|\beta] = a_{i+k+1, i+k} \det A_\tau[\alpha_1, \dots, \alpha_{i+k-1} | 1, \dots, i+k-1],$$

this case reduces to the case of $\beta = (1, \dots, i+k-1)$. Continuing in this way, we arrive at the case $k = 2$ already treated above. \square

Remark 5.2. [AG14b, Remark 2] *If A is irreducible, i.e., all the entries in its super- and subdiagonal are positive, and the determinant in the second row of (5.8) is positive, so that $\det A_\tau[\alpha|\beta] \geq 0$ implies inequality (5.7). Therefore, condition (5.7) is also necessary.*

The following example provides for each single entry the largest interval for the allowable perturbation.

Example 5.1. [AG14b, Example 1] *We choose A as*

$$A := \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Then A is NsTN ($\det A = 5$). In Table 5.1, we give the largest allowable interval from which τ can be chosen such that the matrix $A(\tau) := A + \tau E_{ij}$ is NsTN, $i, j = 1, 2, 3, 4$. The intervals are given in the (i, j) position ($i \leq j$) of the respective entry. If τ is chosen as the left and right endpoint of the interval for the entries on the diagonal and first diagonal, respectively, the matrix $A(\tau)$ is singular.

$(-\frac{5}{4}, \infty)$	$[-1, \frac{5}{3})$	$[0, \frac{1}{2}]$	$[0, 0]$
	$(-\frac{5}{6}, \infty)$	$[-1, \frac{5}{4})$	$[0, \frac{1}{2}]$
		$(-\frac{5}{6}, \infty)$	$[-1, \frac{5}{3})$
			$(-\frac{5}{4}, \infty)$

Table 5.1.: The largest perturbation intervals in Example 5.1.

5.1.2. The General Case

In this subsection we consider the variation of single entries of a tridiagonal totally nonnegative matrix such that the resulting matrix remains totally nonnegative. By Proposition 2.10 (a), we can restrict the discussion to irreducible tridiagonal totally nonnegative matrices. By considering principal minors of order two, we see that then not only the entries in the super- and subdiagonal are positive, but also the entries on the main diagonal must be positive. The following lemma gives necessary and sufficient conditions for an irreducible tridiagonal nonnegative matrix to be totally nonnegative.

Lemma 5.3. [AG14b, Lemma 11] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be irreducible nonnegative tridiagonal. Then A is TN if and only if*

- (i) $\det A \geq 0$,
- (ii) $\det A[1, \dots, k] > 0$, $k = 1, \dots, n - 1$.

Proof. By Proposition 2.11, it suffices to show that the total nonnegativity of A implies (ii). We proceed by induction on $k = 1, \dots, n - 1$.

For $k = 1$, we have $a_{11} > 0$ (see the discussion above the lemma). Suppose that we have already shown that $\det A[1, \dots, k - 1] > 0$ for some $k < n - 1$. Then we obtain by application of (1.11) to $\det A[1, \dots, k + 1]$

$$\det A[1, \dots, k + 1] = a_{k+1, k+1} \det A[1, \dots, k] - a_{k, k+1} a_{k+1, k} \det A[1, \dots, k - 1].$$

Since $a_{k+1, k+1}, a_{k, k+1}, a_{k+1, k}, \det A[1, \dots, k - 1] > 0$, and $\det A[1, \dots, k + 1] \geq 0$, it follows that $\det A[1, \dots, k] > 0$. \square

The following lemma is very useful in our analysis for the perturbation of single entries of a given n -by- n irreducible tridiagonal TN matrix.

Lemma 5.4. [AG14b, Lemma 12] *Let $A = (a_{ij}) \in \mathbb{R}^{n, n}$ be irreducible tridiagonal TN . Then $\det A(i) > 0, i = 1, \dots, n$.*

Proof. By Lemma 2.6, we have only to consider the case $\det A = 0$. Suppose that $\det A(i) = 0$ for some $i \in \{1, \dots, n\}$. Assume without loss of generality that $i < n$. Then we have

$$0 = \det A(i) = \det A[1, \dots, i - 1] \det A[i + 1, \dots, n].$$

By Lemma 5.3 it follows that $\det A[i + 1, \dots, n] = 0$, whence by (1.12) $\det A[i, \dots, n] = 0$ and by (1.10) applied to the latter determinant we have $\det A[i + 2, \dots, n] = 0$. By repeated application of (1.10) we arrive at $a_{nn} = 0$, a contradiction. \square

By the above lemma and Lemma 2.6 we can easily conclude that for a given n -by- n irreducible tridiagonal TN matrix all principal minors of order less than n are positive.

Now let $A \in \mathbb{R}^{n, n}$ be irreducible tridiagonal TN . From (5.3) and Lemma 5.4 we obtain that $A_\tau := A + \tau E_{ii}, i = 1, \dots, n$, is not TN if $\tau < 0$. On the other hand, A_τ is $NsTN$ for all $\tau > 0$, see Theorem 5.3.

By the proof of Lemma 5.4, we have that $\det A[i, \dots, n] > 0, i = 2, \dots, n$. Therefore, it follows from Remark 5.1 that $\det A(i|i + 1) > 0, i = 1, \dots, n - 1$, and by (5.6) we see that $A_\tau := A + \tau E_{i, i+1}$ is not TN if $\tau > 0$. On the other hand, as in the proof of Theorem 5.4 we obtain that A_τ is $NsTN$ if and only if $-a_{i, i+1} \leq \tau < 0$.

Finally, we extend Theorem 5.5 to the general case. We add $\varepsilon > 0$ to a_{11} . Then the resulting matrix B_ε becomes $NsTN$ and we apply the perturbation result of Theorem 5.5 to B_ε . The bound in (5.7) remains in force when ε tends to 0.

5.2. Perturbation of Totally Positive Matrices

In this section we consider the variation of single entries of a given totally positive matrix. For simplicity we consider here only the square case ($n = m$). As in the previous section

we restrict the discussion of the off-diagonal entries to the entries which are lying above the main diagonal since the related statements for the entries below the main diagonal follow by considering the transpose of the matrix.

Theorem 5.6. [AG16b, Theorem 4.1] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be TP and $0 \leq \tau$. Then for $i \leq j$,*

$$A \pm \tau E_{ij} \text{ is TP if and only if } \tau < \min \mathcal{S}, \quad (5.10)$$

where in each of the following eight cases \mathcal{S} is a set of ratios of minors, where the minor in the denominator is obtained from the minor in the numerator by deleting in A additionally row i and column j . If in an index sequence two indices coincide or i or j appears in one of the index sequences in \mathcal{S} then the respective ratio has to be removed from the listing. In the following cases only the numerator submatrices are listed ¹⁾. The cases $(-)$ and $(+)$ refer to the $--$ and $+-$ sign in (5.10), respectively. In the case that \mathcal{S} is empty put $\min \mathcal{S} := \infty$.

1. $i = 2m, j = 2k$

$$(-) \mathcal{S} : \begin{cases} A, A(n-1, n|1, 2), \dots, A(n-2k+3, \dots, n|1, \dots, 2k-2), \\ A(1, 2|n-1, n), A(1, 2, 3, 4|n-3, n-2, n-1, n), \dots, \\ A(1, \dots, 2m-2|n-2m+3, \dots, n). \end{cases}$$

$$(+) \mathcal{S} : \begin{cases} A(n|1), A(n-2, n-1, n|1, 2, 3), \dots, A(n-2k+2, \dots, n|1, \dots, 2k-1), \\ A(1|n), A(1, 2, 3|n-2, n-1, n), \dots, A(1, \dots, 2m-1|n-2m+2, \dots, n). \end{cases}$$

2. $i = 2m, j = 2k+1$

$$(-) \mathcal{S} : \begin{cases} A(n|1), A(n-2, n-1, n|1, 2, 3), \dots, A(n-2k+2, \dots, n|1, \dots, 2k-1), \\ A(1|n), A(1, 2, 3|n-2, n-1, n), \dots, A(1, \dots, 2m-1|n-2m+2, \dots, n). \end{cases}$$

$$(+) \mathcal{S} : \begin{cases} A, A(n-1, n|1, 2), \dots, A(n-2k+1, \dots, n|1, \dots, 2k), \\ A(1, 2|n-1, n), A(1, 2, 3, 4|n-3, n-2, n-1, n), \dots, \\ A(1, \dots, 2m-2|n-2m+3, \dots, n). \end{cases}$$

3. $i = 2m+1, j = 2k$

$$(-) \mathcal{S} : \begin{cases} A(n|1), A(n-2, n-1, n|1, 2, 3), \dots, A(n-2k+2, \dots, n|1, \dots, 2k-1), \\ A(1|n), A(1, 2, 3|n-2, n-1, n), \dots, A(1, \dots, 2m-1|n-2m+2, \dots, n). \end{cases}$$

¹⁾ E.g., $A(n-1, n|1, 2)$ refers in case 1(-) to the ratio $\frac{\det A(n-1, n|1, 2)}{\det A(2m, n-1, n|1, 2, 2k)}$.

$$(+) \mathcal{S} : \begin{cases} A, A(n-1, n|1, 2), \dots, A(n-2k+3, \dots, n|1, \dots, 2k-2), \\ A(1, 2|n-1, n), A(1, 2, 3, 4|n-3, n-2, n-1, n), \dots, \\ A(1, \dots, 2m|n-2m+1, \dots, n). \end{cases}$$

4. $i = 2m + 1, j = 2k + 1$

$$(-) \mathcal{S} : \begin{cases} A, A(n-1, n|1, 2), \dots, A(n-2k+1, \dots, n|1, \dots, 2k), \\ A(1, 2|n-1, n), A(1, 2, 3, 4|n-3, n-2, n-1, n), \dots, \\ A(1, \dots, 2m|n-2m+1, \dots, n). \end{cases}$$

$$(+) \mathcal{S} : \begin{cases} A(n|1), A(n-2, n-1, n|1, 2, 3), \dots, A(n-2k+2, \dots, n|1, \dots, 2k-1), \\ A(1|n), A(1, 2, 3|n-2, n-1, n), \dots, A(1, \dots, 2m-1|n-2m+2, \dots, n). \end{cases}$$

Proof. The entries in the positions $(1, 1)$ and (n, n) can be increased arbitrarily without losing the property of total positivity because they enter into the top left and bottom right position, respectively, in every submatrix in which they lie. This corresponds to the fact that in the cases 1(+) and 4(+) the set \mathcal{S} is empty for $i = j = 1, n$. In the remaining cases we present the proof here only for $A_\tau = A - \tau E_{2m, 2k}$ (case 1(-)); the proof of the other perturbations is similar. If $2m = n$, then $2k = n$, too. The only initial minor containing $a_{nn} - \tau$ is $\det A_\tau$. By expansion of $\det A_\tau$ along its bottom row we obtain

$$\det A_\tau = \det A - \tau \det A(n)$$

from which the condition

$$0 \leq \tau < \frac{\det A}{\det A(n)}$$

follows. We assume now that $2m, 2k < n$. For $\alpha \in Q_{\kappa, n}$ with $2m, 2k \notin \alpha$ we set

$$\phi(\alpha) := \frac{\det A(\alpha|\alpha)}{\det A(\alpha \cup \{2m\}|\alpha \cup \{2k\})}.$$

We further use the intuitive notation

$$\phi(0) := \frac{\det A}{\det A(2m|2k)}.$$

First we show the inequality

$$\phi(0) \leq \phi(n). \tag{5.11}$$

The inequality follows by Proposition 1.2, setting $\alpha := \alpha' := \{1, 2, \dots, n\}$, $\beta := \beta' := \{1, 2, \dots, n-1\}$, $\gamma := \beta \setminus \{2m\}$, $\gamma' := \beta' \setminus \{2k\}$, $\delta := \alpha \setminus \{2m\}$, $\delta' := \alpha' \setminus \{2k\}$. Then the

assumptions of Proposition 1.2 are fulfilled with $p = p' = n$, $q = q' = n - 2$, and therefore $r = 2$, $\alpha'' = \{1, 2\}$, $\beta'' = \{1, 3\}$. For ω the following four sets can be chosen

$$\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3, 4\}.$$

In all four cases the inequality (1.14) is fulfilled. Applying (5.11) to $A(n), A(n-1, n), \dots, A(2m+1, \dots, n)$, we obtain the chain of inequalities

$$\phi(0) \leq \phi(n) \leq \phi(n-1, n) \leq \dots \leq \phi(2m+1, \dots, n). \quad (5.12)$$

Now we show that all the row-initial minors of A_τ are positive; the proof of the positivity of the column-initial minors is similar. Since by expansion of $\det A_\tau$ along its $2m^{\text{th}}$ row

$$\det A_\tau = \det A - \tau \det A(2m|2k),$$

we obtain the condition $\tau < \phi(0)$. Similarly for $s = 0, 1, \dots, n - 2m - 1$,

$$\det A_\tau(n-s, \dots, n) = \det A(n-s, \dots, n) - \tau \det A(2m, n-s, \dots, n|2k, n-s, \dots, n)$$

is positive if $\tau < \phi(n-s, \dots, n)$. Therefore by (5.12), all leading principal minors of A_τ ($\beta_1 = 1$) are positive if $\tau < \phi(0)$.

Now we consider the row-initial minors $\det A_\tau[\alpha|\beta]$, where $\beta = (\beta_1, \beta_2, \dots, \beta_s)$ with $\beta_1 > 1$. If β_1 is even these minors are constant or strictly increasing with respect to τ so that they remain positive under the perturbation. If β_1 is odd, we apply the proof in the case $\beta_1 = 1$ to the submatrix $A_\tau[1, \dots, n - \beta_1 + 1|\beta_1, \dots, n]$ with $2m \leq n - \beta_1 + 1$ and $\beta_1 \leq 2k$ and obtain the remaining conditions.

For $2k = n$, the required initial minors appear directly and there is no need to compare with other minors as above. By Theorem 2.19 it follows that A_τ is *TP* if τ is taken as the minimum of \mathcal{S} in case 1(-). The necessity follows from the fact that all the initial minors are linear functions in τ and that therefore for $\min \mathcal{S} \leq \tau$ there is an initial minor which is nonpositive. \square

For $i = j = 1, 2$, the bound $\det A / \det A(i)$ on τ of the two cases 1(-) and 4(-) are given in the following two theorems.

Theorem 5.7. [FJS00, Theorem 4.2], [FJ11, Theorem 9.5.4] *Let $A \in \mathbb{R}^{n,n}$ be TP. Then $A - \tau E_{11}$ is a TP_{n-1} matrix, for all $\tau \in \left[0, \frac{\det A}{\det A(1)}\right]$.*

Theorem 5.8. [FJS00, Theorem 4.3], [FJ11, Theorem 9.5.5] *Let $A \in \mathbb{R}^{n,n}$ be TP. Then $A - \tau E_{22}$ is a TP_{n-1} matrix, for all $\tau \in \left[0, \frac{\det A}{\det A(2)}\right]$.*

While for $i = 1$ and $j = 2$, the bound $\det A / \det A(1|2)$ in the case 3(+) is given by the following theorem.

Theorem 5.9. [FJS00, Theorem 4.7], [FJ11, Theorem 9.5.8] *Let $A \in \mathbb{R}^{n,n}$ be TP. Then $A + \tau E_{12}$ is a TP_{n-1} matrix, for all $\tau \in \left[0, \frac{\det A}{\det A(1|2)}\right]$.*

The next theorem shows that the sets \mathcal{S} in Theorem 5.6 are minimal.

Theorem 5.10. [AG16b, Theorem 4.2] *Let $A \in \mathbb{R}^{n,n}$ be TP. Then the set \mathcal{S} of determinantal ratios listed in Theorem 5.6 cannot be reduced in each of the eight cases.*

Proof. We present the proof only for the case 1(-); the proof of the other seven cases is similar.

It suffices to show that the following ratios are not comparable if A runs over the set of the n -by- n TP matrices

$$\begin{aligned} b &:= \frac{\det A}{\det A(2m|2k)}, \\ c_\kappa &:= \frac{\det A(n-2\kappa+3, \dots, n|1, \dots, 2\kappa-2)}{\det A(2m, n-2\kappa+3, \dots, n|1, \dots, 2\kappa-2, 2k)}, \quad \kappa = 2, \dots, k, \\ d_\mu &:= \frac{\det A(1, \dots, 2\mu-2|n-2\mu+3, \dots, n)}{\det A(1, \dots, 2\mu-2, 2m|2k, n-2\mu+3, \dots, n)}, \quad \mu = 2, \dots, m. \end{aligned}$$

We show here only that the ratios c_κ and d_μ are not comparable; the proof of the other cases is similar (and easier). We first prove that the inequality $c_\kappa \leq d_\mu$ does not hold for all n -by- n TP matrices A . To apply Proposition 1.2, we choose

$$\begin{aligned} \alpha &:= \{1, \dots, n-2\kappa+2\}, \alpha' := \{2\kappa-1, \dots, n\}, \\ \delta &:= \{2\mu-1, \dots, n\}, \delta' := \{1, \dots, n-2\mu+2\}, \\ \beta &:= \alpha \setminus \{2m\}, \beta' = \alpha' \setminus \{2k\}, \\ \gamma &:= \delta \setminus \{2m\}, \gamma' = \delta' \setminus \{2k\}. \end{aligned}$$

Then we have

$$\alpha \cup \gamma = \beta \cup \delta \quad \text{and} \quad \alpha' \cup \gamma' = \beta' \cup \delta'$$

by

$$\begin{aligned} 2\mu-1 &< 2m \leq n-2\kappa+2, \\ 2\kappa-1 &< 2k \leq n-2\mu+2, \end{aligned}$$

hence $p = p' = n$;

$$\begin{aligned} \alpha \cap \gamma &= \{2\mu-1, \dots, n-2\kappa+2\} \setminus \{2m\}, \\ \alpha' \cap \gamma' &= \{2\kappa-1, \dots, n-2\mu+2\} \setminus \{2k\}, \end{aligned}$$

hence $q = q' = n-2\kappa-2\mu+3$;

$$\eta : \{1, \dots, 2\mu-2, 2m\} \cup \{n-2\kappa+3, \dots, n\} \rightarrow \{1, \dots, 2\kappa+2\mu-3\},$$

$$\eta' : \{2k, n - 2\mu + 3, \dots, n\} \cup \{1, \dots, 2\kappa - 2\} \rightarrow \{2\kappa + 2\mu - 2, \dots, 4\kappa + 4\mu - 6\},$$

$$\begin{aligned} \alpha'' &= \eta(\{1, \dots, 2\mu - 2, 2m\}) \cup \eta'(\{1, \dots, 2\kappa - 2\}) \\ &= \{1, \dots, 2\mu - 1\} \cup \{2\kappa + 4\mu - 3, \dots, 4\kappa + 4\mu - 6\}, \\ \beta'' &= \eta(\{1, \dots, 2\mu - 2\}) \cup \eta'(\{1, \dots, 2\kappa - 2, 2k\}) \\ &= \{1, \dots, 2\mu - 2\} \cup \{2\kappa + 4\mu - 4, \dots, 4\kappa + 4\mu - 6\}. \end{aligned}$$

Let $w := \{2\kappa + 4\mu - 4, 2\kappa + 4\mu - 3\}$. Then the inequality (1.14) is not fulfilled and by Proposition 1.2 there exists a TN matrix A_1 for which the inequality $c_\kappa > d_\mu$ holds. By interchanging the role of sets $\alpha, \alpha', \gamma, \gamma'$ with the sets $\beta, \beta', \delta, \delta'$, we find by Proposition 1.2 (choosing $w := \{2\mu - 2, 2\mu - 1\}$) a TN matrix A_2 for which the inequality $c_\kappa < d_\mu$ does not hold. So the ratios c_κ and d_μ are not comparable on the set of the TN matrices. By using Proposition 2.13 we find two TP matrices satisfying the respective inequalities which shows that also on the set of TP matrices the ratios c_κ and d_μ are not comparable. \square

We conclude this section by the following example in which we give the largest interval from which τ can be chosen according to Theorem 5.6 such that the matrix A_τ is TP , $i, j = 1, \dots, n, i \leq j$.

Example 5.2. [AG16b, Example 4.4] *We choose A as the Pascal matrix of order 4, i.e.,*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}.$$

Then A is TP , see, e.g., [FJ11, Example 0.1.6]. In Table 5.2, we give the largest allowable interval from which τ can be chosen according to Theorem 5.6 such that the matrix $A_\tau := A + \tau E_{ij}$ is TP , $i, j = 1, \dots, n, i \leq j$. The intervals are given in the (i, j) position of the respective entry. In each case, if τ is chosen as an endpoint of the interval, the respective matrix A_τ contains a vanishing minor.

$$\begin{array}{cccc} (-\frac{1}{4}, \infty) & (-\frac{1}{6}, \frac{1}{6}) & (-\frac{1}{4}, \frac{1}{8}) & (-\frac{1}{3}, \frac{1}{3}) \\ & (-\frac{1}{14}, \frac{1}{4}) & (-\frac{1}{7}, \frac{1}{11}) & (-\frac{1}{3}, \frac{1}{3}) \\ & & (-\frac{1}{10}, \frac{1}{2}) & (-1, \frac{1}{3}) \\ & & & (-1, \infty) \end{array}$$

Table 5.2.: The largest perturbation intervals in Example 5.2.

5.3. Perturbation of Totally Nonnegative Matrices

In this section the perturbation of the single entries of a totally nonnegative matrix is investigated. We will not give explicit bounds as in the previous sections since the situation is different from one totally nonnegative matrix to another. Instead, we describe a general approach on how to find such bounds. The totally nonnegative cells and the determinantal tests that are derived in Section 3.6 play a fundamental role in our analysis.

The entry in position $(1, 1)$ ((n, n)) of a given n -by- n TN matrix is easily handled. It can be arbitrary increased since it enters positively in any minor containing it. The bound for adding a negative value to the $(1, 1)$ entry when the principal minor which is formed by deleting the first row and column is positive is presented in the following lemma.

Lemma 5.5. [FJS00, Lemma 4.1], [FJ11, Lemma 9.5.2] *Let $A \in \mathbb{R}^{n,n}$ be TN with $\det A(1) \neq 0$. Then $A - \tau E_{11}$ is TN for all $\tau \in \left[0, \frac{\det A}{\det A(1)}\right]$.*

The following theorem generalizes the above lemma to the case that the principal minor which is formed by deleting the first row and column of a given TN matrix is nonnegative.

Theorem 5.11. *Let $A \in \mathbb{R}^{n,n}$ be TN . Then $A - \tau E_{11}$ is TN for all $\tau \in [0, \tilde{a}_{11}]$, where $\tilde{A} = (\tilde{a}_{ij})$ is the matrix obtained by the application of the Cauchon Algorithm to A .*

Proof. Let $B := A - \tau E_{11}$. Then the application of the Cauchon Algorithm to B yields the matrix $\tilde{B} = (\tilde{b}_{ij})$ which coincides with \tilde{A} in all of its entries with the exception of $\tilde{b}_{11} = \tilde{a}_{11} - \tau$. Since A is TN we have by Theorem 3.2 (ii) that \tilde{A} is a nonnegative Cauchon matrix, and so \tilde{B} is a Cauchon matrix with nonnegative entries for all $\tau \in [0, \tilde{a}_{11}]$. Hence by Theorem 3.2 (ii) the claim follows. \square

Remark 5.3. *If in the above theorem $0 < \det A(1)$, then by application of Proposition 3.5 to $A(1)$ we obtain that the main diagonal of $\tilde{A}(1)$ is positive. Hence the sequence $((1, 1), (2, 2), \dots, (n, n))$ is a lacunary sequence with respect to the Cauchon diagram $C_{\tilde{A}}$. Therefore by Proposition 3.4 we have*

$$\tilde{a}_{11} = \frac{\det A}{\det A(1)},$$

and so Lemma 5.5 becomes a special case of the above theorem.

Now we turn to the other entries. Let $A \in \mathbb{R}^{n,n}$ be TN . Then by Theorem 3.2 (ii) \tilde{A} is a nonnegative Cauchon matrix. Suppose that we want to perturb the entry in position (i, j) for some $i, j \in \{1, \dots, n\}$ such that the resulting matrix is TN . Let $A_{\tau\pm} := A \pm \tau E_{ij}$. Then we want to find bounds on τ such that $A_{\tau\pm}$ is TN . If we apply Procedure 3.2 to $A_{\tau\pm}$ then the sequences that are starting at the positions (k, l) , $i < k$, $l = 1, \dots, n$, and $k = 1, \dots, n$, $j < l$, coincide with the corresponding lacunary sequences which we obtain when we apply Procedure 3.2 to A and hence they are lacunary. Thus we may concentrate on the sequences that are starting from the positions (k, l) , $k = 1, \dots, i$, $l = 1, \dots, j$.

In order to find the values of τ such that $A_{\tau\pm}$ is *TN*, we distinguish the following two cases:

- (i) A and $A_{\tau\pm}$ belong to the same *TN* cell, or
- (ii) $A_{\tau\pm}$ transits possibly to another *TN* cell.

Case (i): If we want the matrix $A_{\tau\pm}$ to stay in the same *TN* cell as A , i.e., $C_{\tilde{A}} = C_{\tilde{A}_{\tau\pm}}$, then by Theorem 3.4 the lacunary sequences that are starting at the positions (k, l) , $k = 1, \dots, i$, $l = 1, \dots, j$ with respect to $C_{\tilde{A}_{\tau\pm}}$ must coincide with the corresponding lacunary sequences with respect to $C_{\tilde{A}}$. Hence we have to find the values of τ that preserve the positivity or the vanishing of the minors associated with these lacunary sequences. Thus we have at most $i \cdot j$ minors to consider. The following minors can be discarded from further consideration since they do not impose any restriction on the values of τ :

- (1) The minors which do not involve the entry $a_{ij} \pm \tau$, and
- (2) the minors which involve the entry $a_{ij} \pm \tau$ but in such a way that the minors that are obtained from the corresponding submatrices by deleting row i and column j in A are zero.

In Theorem 5.6 we give the minimal conditions that are required for a maximum allowable perturbation of a *TP* matrix. Hence by employing the approach that has been used in the proof of Theorem 5.6 we may discard further superfluous conditions (minors).

Case (ii): If the matrix $A_{\tau\pm}$ is allowed to transit to another *TN* cell, then the analysis is more involved. First we consider the entries in the first column, i.e., $A_{\tau\pm} = A \pm \tau E_{i1}$, $i = 2, \dots, n$. We have to treat the following two cases:

- Suppose that $\tilde{a}_{i1} = 0$. Then we conclude that $\tau = 0$ in $A_{\tau-}$ since the minor that corresponds to the lacunary sequence that starts from the position $(i, 1)$ with respect to $C_{\tilde{A}}$ becomes negative while in $A_{\tau+}$ we have one of the following two cases:
 - $\tilde{a}_{il} = 0$ for all $l = 2, \dots, h$ for some $2 \leq h \leq n$. Then we distinguish the following two cases:
 - * $\tilde{a}_{kl} = 0$ for all $k = 1, \dots, i - 1$ and $l = 2, \dots, h$. Then τ can be calculated by using the minors that correspond to the lacunary sequences which start from the positions $(k, 1)$, $k = 1, \dots, i$, and contain an entry (i, t) for some $t > 1$.
 - * Otherwise there exists $k_0 \in \{1, \dots, i - 1\}$ and $l_0 \in \{2, \dots, h\}$ such that $0 < \tilde{a}_{k_0, l_0}$; then we conclude that $\tau = 0$ which follows from the fact that $\tilde{A}_{\tau+}$ must be a Cauchon matrix.
 - $0 < \tilde{a}_{i2}$. Then τ can be calculated by using the minors that correspond to the lacunary sequences which start from the positions $(k, 1)$, $k = 1, \dots, i$, and contain an entry (i, t) for some $t > 1$.

- $0 < \tilde{a}_{i1}$, then τ in $A_{\tau\pm}$ can be calculated by using the minors that correspond to the lacunary sequences which start from the positions $(k, 1)$, $k = 1, \dots, i$, and contain an entry (i, t) for some $t > 1$.

In the sequel we assume that $1 < i, j < n$ since otherwise we consider A^T for the first row, $A^\#$ for the last column, and $A^{\#T}$ for the last row of A .

Suppose that $\tilde{a}_{ij} = 0$. Then we conclude that $\tau = 0$ in $A_{\tau-}$ while in $A_{\tau+}$ we distinguish the following two cases:

- Suppose that $\tilde{a}_{i,t_1} = 0$ for all $t_1 = j + 1, \dots, h_1$ for some $j + 1 \leq h_1 \leq n$ and there exists $0 < \tilde{a}_{i_0,t_0}$ for some $i_0 \in \{1, \dots, i - 1\}$, $t_0 \in \{j + 1, \dots, h_1\}$, or $\tilde{a}_{t_2,j} = 0$ for all $t_2 = i + 1, \dots, h_2$ for some $i + 1 \leq h_2 \leq n$ and there exists $0 < \tilde{a}_{i',t'}$ for some $i' \in \{i + 1, \dots, h_2\}$, $t' \in \{1, \dots, j - 1\}$. Then $\tau = 0$ which follows from the fact that $\tilde{A}_{\tau+}$ must be a Cauchon matrix.
- Otherwise τ can be calculated by using the minors corresponding to the lacunary sequences that start from the positions (k, l) , $k = 1, \dots, i$, $l = 1, \dots, j$, and contain the entry (i, t) and (h, j) for $t, h \in \{1, \dots, n\}$.

If $0 < \tilde{a}_{ij}$ then we have to consider all the lacunary sequences that start from (k, l) , $k = 1, \dots, i$, $l = 1, \dots, j$, and find the restrictions on the values of τ in $A_{\tau\pm}$ as in the previous section.

5.4. Eigenvalues as Functions of Matrix Elements

In the previous sections we deal with the maximum allowable perturbations of the entries of tridiagonal totally nonnegative, totally positive, and totally nonnegative matrices. In this section we consider eigenvalues of oscillatory matrices as functions of their entries. We start with a general result.

Proposition 5.1. [Pin10, Proposition 5.9] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ with distinct real simple eigenvalues $\lambda_1, \dots, \lambda_n$ and associated eigenvectors $\mathbf{u}^1, \dots, \mathbf{u}^n$. Let $U = (u_{ij}) \in \mathbb{R}^{n,n}$ be the eigenmatrix whose k^{th} column is \mathbf{u}^k . Set $V = (v_{ij})$, where $V = U^{-1}$, and let \mathbf{v}^k denote the k^{th} row of V . Then*

$$\frac{\partial \lambda_k}{\partial a_{ij}} = v_{ki} u_{jk} \quad (5.13)$$

for all $i, j, k \in \{1, \dots, n\}$.

By applying the above theorem to an oscillatory matrix and using Theorem 2.47 we obtain the following theorem.

Theorem 5.12. [Pin10, Theorem 5.10], [GK02, Theorem 16, Theorem 17, pp. 108-109] Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be an oscillatory matrix with eigenvalues $0 < \lambda_n < \dots < \lambda_1$. Then

$$0 < \frac{\partial \lambda_1}{\partial a_{ij}}, \quad 0 < (-1)^{i+j} \frac{\partial \lambda_n}{\partial a_{ij}}, \quad i, j = 1, \dots, n,$$

and

$$0 < \frac{\partial \lambda_k}{\partial a_{11}}, \quad 0 < \frac{\partial \lambda_k}{\partial a_{nn}}, \quad 0 < (-1)^{k+1} \frac{\partial \lambda_k}{\partial a_{1n}}, \quad 0 < (-1)^{k+1} \frac{\partial \lambda_k}{\partial a_{n1}}, \quad k = 1, \dots, n.$$

If entries of the matrix A are subject to perturbations the statements of the above theorem remain only true as long as the perturbed matrix stays oscillatory. Bounds on the allowable perturbation of the single entries can be found by using the results of the previous sections. Hence one can find the maximum and minimum of an eigenvalue of a given oscillatory matrix $A = (a_{ij}) \in \mathbb{R}^{n,n}$ with respect to the perturbation of an entry of the matrix. For example, λ_n is an increasing function in a_{11} and so λ_n tends to its minimum value as a_{11} tends to $a_{11} - \frac{\det A}{\det A[2, \dots, n]}$.

5.5. Extended Perron Complements of Totally Nonnegative Matrices

In this section we investigate the (extended) Perron complement of totally nonnegative matrices by using properties of determinants and determinantal inequalities as well as our results on the perturbation of tridiagonal totally nonnegative matrices. Several interesting properties of the Perron complement of irreducible nonnegative matrices and a method to compute the Perron vector of a given irreducible nonnegative matrix by using Perron complementation and a divide-and-conquer procedure are presented in [Mey89a] and [Mey89b]. Important results on the Perron complement of irreducible totally nonnegative matrices are given in [FN01]. The following theorem is important for the definition of the (extended) Perron complement of an irreducible nonnegative matrix.

Theorem 5.13. [BP94, Corollary (1.5), p. 27] Let $A, B \in \mathbb{R}^{n,n}$ be such that $0 \leq B \leq A$ and $A + B$ is irreducible. Then $\rho(B) < \rho(A)$, where $\rho(\cdot)$ denotes the spectral radius.

Definition 5.1. [FJ11, Section 10.4] Let $A \in \mathbb{R}^{n,n}$ be nonnegative and irreducible. Then the Perron complement of $A[\alpha]$ in A is given by

$$\mathcal{P}(A/A[\alpha]) := A[\beta] + A[\beta|\alpha](\rho(A)I - A[\alpha])^{-1}A[\alpha|\beta], \quad (5.14)$$

where $\alpha \subset \{1, \dots, n\}$ and $\beta := \alpha^c$.

Remark 5.4. The expression on the right-hand side of (5.14) is well defined by Theorem 5.13 since A is irreducible and nonnegative and hence $\rho(A[\alpha]) < \rho(A)$.

The Perron complement is extended in the following way, [FN01, (3)]:
For any $\rho(A) \leq t$, α and β as in Definition 5.1 define

$$\mathcal{P}_t(A/A[\alpha]) := A[\beta] + A[\beta|\alpha](tI - A[\alpha])^{-1}A[\alpha|\beta], \quad (5.15)$$

which is called the *extended Perron complement of $A[\alpha]$ in A at t* . Again $\mathcal{P}_t(A/A[\alpha])$ is well defined, see Remark 5.4.

Remark 5.5. *By setting $t = 0$ in (5.15), we obtain the Schur complement of $A[\alpha]$ in A (1.1), provided that $A[\alpha]$ is nonsingular.*

There is an interesting relationship between Sylvester's Determinantal Identity, see Lemma 1.2, and the Schur complement given by (1.1) for an n -by- n matrix A [And87, p. 175], see also [FJ11, formula (10.4)]. Let $\alpha = \{k, \dots, n\}$. Then

$$\det(A/A[\alpha])[\gamma|\delta] = \frac{\det A[\gamma \cup \alpha|\delta \cup \alpha]}{\det A[\alpha]}, \quad (5.16)$$

where $2 \leq k \leq n$, and $\gamma, \delta \subseteq \alpha^c$ with $|\gamma| = |\delta|$.

From the above equalities we have the following theorem.

Theorem 5.14. [FJ11, Lemma 10.4.1] *Let $A \in \mathbb{R}^{n,n}$ be TN and $\alpha = \{1, \dots, k\}$ or $\alpha = \{k, \dots, n\}$. Then $A/A[\alpha]$ is TN for all $k = 1, \dots, n-1$, provided that $A[\alpha]$ is nonsingular.*

Now we turn to the extended Perron complement. In [FN01], see also [FJ11, Example 10.4.3], an example with $n = 10$, $\alpha = \{7\}$ is given which documents that TN matrices are not closed under arbitrary Perron complementation, even when α is a singleton, except $\alpha = \{1\}$ or $\alpha = \{n\}$. For the following theorem we present a new proof because we will extend this theorem by similar means, see Theorem 5.16. We introduce the following notations which simplify the presentation. For $\gamma = \{\gamma_1, \dots, \gamma_l\} \in \mathcal{Q}_{l, n-1}$ put

$$\gamma + 1 := \{\gamma_1 + 1, \dots, \gamma_l + 1\}, \quad \gamma_{\dot{1}} + 1 := \{\gamma_1, \gamma_2 + 1, \dots, \gamma_l + 1\}.$$

Theorem 5.15. [FN01, Lemma 2.2], [FJ11, Lemma 10.4.2] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be irreducible TN, and $\alpha = \{1\}$ or $\alpha = \{n\}$. Then for any t , $\rho(A) \leq t$, the matrix $\mathcal{P}_t(A/A[\alpha])$ is TN.*

Proof. We give the proof only for the case $\alpha = \{1\}$ since the other case follows by application of the same arguments to $A^\#$. Formula (5.15) specifies for $\alpha = \{1\}$ to

$$\mathcal{P}_t(A/A[1]) = A[2, \dots, n] + \frac{1}{t - a_{11}}A[2, \dots, n|1]A[1|2, \dots, n]. \quad (5.17)$$

By direct computations, it is easy to see that

$$\mathcal{P}_t(A/A[1])[1, \dots, n-1|j] = A[2, \dots, n|j+1] + \frac{a_{1,j+1}}{t - a_{11}}A[2, \dots, n|1], \quad (5.18)$$

for $j = 1, \dots, n - 1$.

For any $\gamma = \{\gamma_1, \dots, \gamma_l\}, \delta = \{\delta_1, \dots, \delta_l\} \in Q_{l, n-1}, l = 1, \dots, n - 1$, it is advantageous to represent $\det \mathcal{P}_t(A/A[1])[\gamma|\delta]$ in the following way

$$\det \mathcal{P}_t(A/A[1])[\gamma|\delta] = \det \begin{bmatrix} 1 & 0 \\ A[\gamma + 1|1] & \mathcal{P}_t(A/A[1])[\gamma|\delta] \end{bmatrix}. \quad (5.19)$$

Then we subtract in the matrix on the right-hand side of (5.19) from the μ^{th} column the first column multiplied by $\frac{a_{1, \delta_{\mu-1}+1}}{t-a_{11}}, \mu = 2, \dots, l + 1$, and extract from the first row the common factor $\frac{1}{t-a_{11}}$ to obtain

$$\begin{aligned} \det \mathcal{P}_t(A/A[1])[\gamma|\delta] &= \frac{1}{t-a_{11}} \det \begin{bmatrix} t-a_{11} & -A[1|\delta+1] \\ A[\gamma+1|1] & A[\gamma+1|\delta+1] \end{bmatrix} \\ &= \frac{1}{t-a_{11}} (t \det A[\gamma+1|\delta+1] - \det A[\{1\} \cup (\gamma+1) | \{1\} \cup (\delta+1)]) \\ &\geq \frac{1}{t-a_{11}} (t \det A[\gamma+1|\delta+1] - a_{11} \det A[\gamma+1|\delta+1]) \\ &= \det A[\gamma+1|\delta+1], \end{aligned} \quad (5.20)$$

$$(5.21)$$

where inequality (5.20) follows by using Theorem 1.4. Hence all minors of $\mathcal{P}_t(A/A[1])$ are nonnegative and so $\mathcal{P}_t(A/A[1])$ is TN . \square

In the next theorem we extend the above statement to two further special singleton sets.

Theorem 5.16. *Let $A = (a_{ij}) \in \mathbb{R}^{n, n}$ be irreducible TN , and $\alpha = \{2\}$ or $\alpha = \{n-1\}$. Then for any $t, \rho(A) \leq t$, the matrix $\mathcal{P}_t(A/A[\alpha])$ is TN .*

Proof. Again we provide the proof only for the case $\alpha = \{2\}$ since the other case follows by application of the same arguments to $A^\#$. Formula (5.15) specifies for $\alpha = \{2\}$ to

$$\mathcal{P}_t(A/A[2]) = A(2) + \frac{1}{t-a_{22}} A[\{2\}^c | 2] A[2 | \{2\}^c]. \quad (5.22)$$

As in the proof of Theorem 5.15, we have

$$\mathcal{P}_t(A/A[2])[1, \dots, n-1|j] = \begin{cases} A[\{2\}^c | 1] + \frac{a_{21}}{t-a_{22}} A[\{2\}^c | 2], & j = 1, \\ A[\{2\}^c | j+1] + \frac{a_{2, j+1}}{t-a_{22}} A[\{2\}^c | 2], & j = 2, \dots, n-1. \end{cases}$$

For any $\gamma = \{\gamma_1, \dots, \gamma_l\}, \delta = \{\delta_1, \dots, \delta_l\} \in Q_{l, n-1}, l = 1, \dots, n - 1$, we distinguish the following four cases. The equalities in the first two cases follow by using properties of determinants as in the proof of Theorem 5.15 and the inequalities follow by using Theorem 1.4.

(i) If $1 \in \gamma \cap \delta$, then

$$\begin{aligned} \det \mathcal{P}_t(A/A[2])[\gamma|\delta] &= \frac{1}{t - a_{22}} (t \det A[\gamma_{\hat{1}} + 1|\delta_{\hat{1}} + 1] \\ &\quad - \det A[\{2\} \cup (\gamma_{\hat{1}} + 1) | \{2\} \cup (\delta_{\hat{1}} + 1)]) \\ &\geq \det A[\gamma_{\hat{1}} + 1|\delta_{\hat{1}} + 1]. \end{aligned} \quad (5.23)$$

(ii) If $1 \notin \gamma \cup \delta$, then

$$\begin{aligned} \det \mathcal{P}_t(A/A[2])[\gamma|\delta] &= \frac{1}{t - a_{22}} (t \det A[\gamma + 1|\delta + 1] - \det A[\{2\} \cup (\gamma + 1) | \{2\} \cup (\delta + 1)]) \\ &\geq \det A[\gamma + 1|\delta + 1]. \end{aligned}$$

(iii) If $1 \in \gamma$ and $1 \notin \delta$, then

$$\begin{aligned} \det \mathcal{P}_t(A/A[2])[\gamma|\delta] &= \frac{1}{t - a_{22}} (t \det A[\gamma_{\hat{1}} + 1|\delta + 1] \\ &\quad + \det A[\{2\} \cup (\gamma_{\hat{1}} + 1) | \{2\} \cup (\delta + 1)]) \\ &\geq 0, \end{aligned}$$

since A is TN .

(iv) If $1 \notin \gamma$ and $1 \in \delta$, then

$$\begin{aligned} \det \mathcal{P}_t(A/A[2])[\gamma|\delta] &= \frac{1}{t - a_{22}} (t \det A[\gamma + 1|\delta_{\hat{1}} + 1] \\ &\quad + \det A[\{2\} \cup (\gamma + 1) | \{2\} \cup (\delta_{\hat{1}} + 1)]) \\ &\geq 0, \end{aligned}$$

since A is TN .

Hence all minors of $\mathcal{P}_t(A/A[2])$ are nonnegative and so $\mathcal{P}_t(A/A[2])$ is TN . \square

Remark 5.6. *By an easy and direct proof one can show that $\mathcal{P}_t(A/A[\alpha])$ in Theorems 5.15 and 5.16 is irreducible. If, in addition, the given matrix in these theorems is nonsingular, then by Lemma 2.6, (5.21), and (5.23) the extended Perron complement is also nonsingular.*

Unfortunately, the above two theorems cannot be extended to any singleton set $\{k\}$, $3 \leq k \leq n - 2$ with $5 \leq n$ as the following example demonstrates.

Example 5.3. *Let*

$$A := \begin{bmatrix} 50 & 25 & 11 & 4 & 1 \\ 35 & 20 & 10 & 4 & 1 \\ 15 & 10 & 6 & 3 & 1 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \quad (5.24)$$

Then A is irreducible and by Theorem 3.11 A is TN ,

$$\mathcal{P}_t(A/A[3]) = \begin{bmatrix} 50 + \frac{165}{t-6} & 25 + \frac{110}{t-6} & 4 + \frac{33}{t-6} & 1 + \frac{11}{t-6} \\ 35 + \frac{150}{t-6} & 20 + \frac{100}{t-6} & 4 + \frac{30}{t-6} & 1 + \frac{10}{t-6} \\ 5 + \frac{45}{t-6} & 4 + \frac{30}{t-6} & 2 + \frac{9}{t-6} & 1 + \frac{3}{t-6} \\ 1 + \frac{15}{t-6} & 1 + \frac{10}{t-6} & 1 + \frac{3}{t-6} & 1 + \frac{1}{t-6} \end{bmatrix}, \quad (5.25)$$

and $\det \mathcal{P}_t(A/A[3])[1, 2|3, 4] = \frac{-1}{t-6} < 0$ for any t , $\rho(A) \approx 72.756 \leq t$.

The following theorem provides a quotient formula for the extended Perron complement.

Theorem 5.17. [FN01, Theorem 2.4], [FJ11, Theorem 10.4.4] *Let $A \in \mathbb{R}^{n,n}$ be an irreducible nonnegative matrix, and fix any nonempty set $\alpha \subset \{1, \dots, n\}$. Then for any nonempty subsets $\alpha_1, \alpha_2 \subset \alpha$ with $\alpha_1 \cup \alpha_2 = \alpha$ and $\alpha_1 \cap \alpha_2 = \emptyset$, we have*

$$\mathcal{P}_t(A/A[\alpha]) = \mathcal{P}_t(\mathcal{P}_t(A/A[\alpha_1])/\mathcal{P}_t(A/A[\alpha_1])[\alpha_2]) \quad (5.26)$$

for any t , $\rho(A) \leq t$.

By using Theorems 5.15, 5.16, and 5.17, we obtain the following theorem.

Theorem 5.18. *Let $A \in \mathbb{R}^{n,n}$ be irreducible TN , and α be any of the following subsets:*

- (i) $\{1, \dots, k\}$ or $\{k, \dots, n\}$;
- (ii) $\{2, \dots, k\}$ or $\{k, \dots, n-1\}$,

for $k = 2, \dots, n-1$. Then for any t , $\rho(A) \leq t$, the matrix $\mathcal{P}_t(A/A[\alpha])$ is TN .

The case (i) in the above theorem is given in [FN01, Theorem 2.5], see also [FJ11, Theorem 10.4.5].

Theorems 5.15 and 5.16 can be extended to any singleton $\{k\}$ for the class of irreducible tridiagonal TN matrices as the following theorem documents. It is a special case of [FN01, Proposition 2.1], [FJ11, Corollary 10.4.6], see Theorem 5.20 below; its proof and the statement on the nonsingularity of $\mathcal{P}_t(A/A[k])$ are new.

Theorem 5.19. *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be irreducible tridiagonal TN . Then for any singleton subset $\alpha = \{k\}$, $k = 1, \dots, n$, the extended Perron complement $\mathcal{P}_t(A/A[k])$ is irreducible tridiagonal $NsTN$ for any t , $\rho(A) \leq t$.*

Proof. For $k = 1, 2, n-1, n$ the total nonnegativity of $\mathcal{P}_t(A/A[k])$ follows by Theorems 5.15 and 5.16, whereas the nonsingularity follows by (5.21), (5.23), and Lemma 5.4. Suppose that $2 < k < n-1$. Then formula (5.15) specifies to

$$\mathcal{P}_t(A/A[k]) = A(k) + \frac{1}{t - a_{kk}} A[\{k\}^c | k] A[k] [\{k\}^c]. \quad (5.27)$$

By direct computations, $\mathcal{P}_t(A/A[k])$ is an irreducible and tridiagonal matrix which coincides with $A(k)$ except in the following four positions:

- (i) $(k-1, k-1), (k, k)$ which become $a_{k-1, k-1} + \frac{1}{t-a_{kk}} a_{k-1, k} \cdot a_{k, k-1}$, $a_{k+1, k+1} + \frac{1}{t-a_{kk}} a_{k+1, k} \cdot a_{k, k+1}$, respectively;
- (ii) $(k, k-1)$ which becomes $\frac{1}{t-a_{kk}} a_{k+1, k} \cdot a_{k, k-1}$;
- (iii) $(k-1, k)$ which becomes $\frac{1}{t-a_{kk}} a_{k-1, k} \cdot a_{k, k+1}$.

We consider $A(k)$ and add first to the diagonal entries the positive quantities that appear in (i). By Theorem 5.3 and Lemma 5.4 the resulting matrix is $NsTN$. Next we add the quantity in (ii) to the position $(k, k-1)$. Since the resulting matrix, called B , has at position $(k-1, k)$ a zero entry it is $NsTN$ by Theorem 5.4. It remains to add the quantity that appears in (iii) to the zero position $(k-1, k)$. By Theorem 5.4 the resulting matrix is $NsTN$ if

$$\frac{1}{t-a_{kk}} a_{k-1, k} \cdot a_{k, k+1} < \frac{\det B}{\det B(k-1|k)}. \quad (5.28)$$

By (1.12) and some simplifications, the right-hand side of (5.28) becomes

$$\frac{(t-a_{kk}) \det A[1, \dots, k-1] + a_{k-1, k} a_{k, k-1} \det A[1, \dots, k-2]}{a_{k, k-1} a_{k+1, k} \det A[1, \dots, k-2] \det A[k+2, \dots, n]} \cdot (\det A[k+1, \dots, n] + \frac{1}{t-a_{kk}} a_{k, k+1} a_{k+1, k} \det A[k+2, \dots, n]),$$

which is a sum consisting of positive terms (by irreducibility of A and Lemma 5.4) and

$$\frac{1}{t-a_{kk}} a_{k-1, k} \cdot a_{k, k+1},$$

whence inequality (5.28) holds. Hence $\mathcal{P}_t(A/A[k])$ is $NsTN$ for all $k = 1, \dots, n$ and $\rho(A) \leq t$. \square

Theorem 5.20. [FN01, Corollary 2.6], [FJ11, Corollary 10.4.6] *Let $A \in \mathbb{R}^{n, n}$ be irreducible tridiagonal TN . Then for any $\alpha \subset \{1, \dots, n\}$, the extended Perron complement is irreducible tridiagonal TN for any t , $\rho(A) \leq t$.*

By using Theorems 5.17, 5.19, and Remark 5.6 we may conclude that under the conditions of Theorem 5.20 the resulting extended Perron complement is, in addition, nonsingular.

6. Totally Nonnegative and Totally Positive Completion Problems

In this chapter we study the completion problem for totally nonnegative and totally positive matrices. A rectangular array is called a *partial matrix* if some of its entries are specified, while the remaining, unspecified, entries are free to be chosen. A *completion* of a partial matrix is a choice of values for the unspecified entries, resulting in a conventional matrix that agrees with the given partial matrix in all of its specified entries. A *matrix completion problem* asks which partial matrices have completions with some desired property. This chapter consists of four sections. In Section 6.1, we present the definition of the totally nonnegative (totally positive) completion problem and some important examples of the totally positive completion problem. In Section 6.2, we consider the totally positive completion problem with one unspecified entry or a complete row or column with unspecified entries. In Section 6.3, we treat partial totally nonnegative and totally positive patterns with some unspecified entries and give some necessary and sufficient conditions for a given partial totally nonnegative or totally positive matrix with a specified pattern to be totally nonnegative or totally positive completable. In Section 6.4, we present some new totally positive completable patterns hereby partially settling two conjectures posed in [JW13].

6.1. Introduction

A partial matrix is called *partial totally nonnegative (totally positive)* if each of its fully specified submatrices is totally nonnegative (totally positive). Total nonnegativity is inherited by submatrices. Therefore, it is a necessary condition that every fully specified submatrix is totally nonnegative. The study of the totally nonnegative completion problem was begun in [JKL98].

Unfortunately, not all partial *TN* and *TP* matrices allow a *TN* and *TP* completion, respectively. The following examples are instructive in this regard. The first example shows that there is a 4-by-4 partial *TP* matrix with the (1, 4) (or (4, 1)) entry unspecified which is not *TP* completable [FJS00, Lemma 2.9]. As usual, the unspecified entries are denoted by ?'s.

Example 6.1. *Let*

$$A := \begin{bmatrix} 100 & 100 & 40 & ? \\ 40 & 100 & 100 & 40 \\ 20 & 80 & 100 & 100 \\ 3 & 20 & 40 & 100 \end{bmatrix}. \quad (6.1)$$

Then A is a partial TP matrix but not TP completable since in order to have $0 < \det A$ one needs to choose the unspecified entry less than $-1144/14$, see [FJ11, p. 196].

The following example shows that there is a 4-by-4 partial TP matrix with the (2, 3) (or (3, 2)) entry unspecified which is not TP completable [FJS00, Lemma 2.10].

Example 6.2. *Let*

$$A := \begin{bmatrix} 1000 & 10 & 10 & 10 \\ 20 & 9 & ? & 10 \\ 2 & 1 & 9 & 10 \\ 1 & 2 & 40 & 1000 \end{bmatrix}. \quad (6.2)$$

Then A is a partial TP matrix but not TP completable since in order to complete A to a TP matrix one needs to choose the unspecified entry by considering the minors $\det A[1, 2|3, 4]$ and $\det A[2, 3, 4|1, 2, 3]$ less than 10 and greater than $199/3$, respectively, which is impossible.

A pattern P of a given partial matrix in which the specified entries have the specific pattern P is called TP (TN) completable if every partial TP (TN) matrix with pattern P has a TP (TN) completion. TP_k completion is defined analogously. In the following sections we present some completable patterns of partial TP and TN matrices.

6.2. Single Entry and Line Insertion Cases of Totally Positive Completion Problems

In this section we consider the totally positive completion problem with only one unspecified entry or a complete row/column with unspecified entries. These particular cases represent the key in solving some totally positive completion problems with more than one unspecified entry.

The following lemma can be easily shown since the (1, 1) entry enters positively in each minor containing it.

Lemma 6.1. [FJS00, Lemma 2.1] *Let A be a partial TP matrix with the only unspecified entry x in position (1, 1). Then, if x is chosen so that all the leading principal minors are positive, then A is TP completable.*

Inserting some rows or columns in a given TP matrix is considered as a TP completion problem. The following lemma states that we can always append rows at the top or bottom and columns to the right or left of a given TP matrix.

Lemma 6.2. [FJS00, Lemma 2.3] *Let $A \in \mathbb{R}^{n,m}$ be TP . Then there exists a positive vector $\mathbf{x} \in \mathbb{R}^n$ such that the augmented matrix $[A|\mathbf{x}]$ is an n -by- $(m+1)$ TP matrix. Similarly, there exist positive vectors $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$ such that $[\mathbf{u}|A]$, $\begin{bmatrix} A \\ \mathbf{v} \end{bmatrix}$, and $\begin{bmatrix} \mathbf{w} \\ A \end{bmatrix}$ are all TP .*

The proof of Lemma 6.2 proceeds by applying Lemma 6.1 sequentially to some special submatrices.

The following lemma plays a fundamental role in solving some problems related to the TP completion problem, in particular, in the proof of the next two theorems.

Lemma 6.3. [JS00, Lemma 2.1], [FJS00, Lemma 1.2] *Let $A = [\mathbf{a}^1 \ \mathbf{a}^2 \ \dots \ \mathbf{a}^n] \in \mathbb{R}^{n-1, n}$ be TP , where \mathbf{a}^i denotes the i^{th} column of A . Then, for $k = 1, \dots, n$,*

$$\mathbf{a}^k = \sum_{i=1, i \neq k}^n y_i \mathbf{a}^i, \quad (6.3)$$

has a unique solution $\mathbf{y} = [y_1 \ y_2 \ \dots \ y_{k-1} \ y_{k+1} \ \dots \ y_n]^T$ for which

$$\text{sgn}(y_i) = \begin{cases} (-1)^i, & \text{if } k \text{ is odd,} \\ (-1)^{i-1}, & \text{if } k \text{ is even.} \end{cases} \quad (6.4)$$

Theorem 6.1. [JS00, Theorem 2.3] *Let A be a TP matrix. Then, a row (column) can be inserted between any pair of adjacent rows (columns) in A so that the resulting matrix is TP .*

Theorem 6.2. [FJS00, Theorem 2.5, Theorem 2.8], [FJ11, Theorem 9.4.4] *Let A be an n -by- m partial TP matrix with only one unspecified entry in the (s, t) position. If $\min\{n, m\} \leq 3$, then A has a TP completion. If $4 \leq \min\{n, m\}$, then any such A has a TP completion if and only if $s + t \leq 4$ or $n + m - 2 \leq s + t$.*

The following theorem is a generalization of Theorem 6.2 to the case that the given matrix is partial TP_k , $4 \leq k$.

Theorem 6.3. [AJN14, Theorem 5.4] *Let A be an n -by- m partial TP_k matrix with $4 \leq \min\{n, m, k\}$ and only one unspecified entry in the (s, t) position. Then any such A has a TP_k completion if and only if $s + t \leq 4$ or $n + m - 2 \leq s + t$.*

The above two theorems play a fundamental role in the proof of our results in Section 6.4. Examples 6.1 and 6.2 can be used to show that the above theorem cannot be extended to any other entry [AJN14, Lemma 5.2].

6.3. Multiple Entries Case of Totally Nonnegative and Totally Positive Completion Problems

In this section we recall some known results concerning the completion problem of totally positive and totally nonnegative matrices. We also present some terminologies from graph theory that help very much to describe in a given matrix the positions the specified entries have.

When a partial TP matrix has more than one unspecified entry then, as in the previous setting, not all patterns have a TP completion, see, e.g., the following example.

Example 6.3. [JKL98, p. 105], [FJ11, Example 9.1.1] *Let*

$$A := \begin{bmatrix} 1 & 1 & 0.4 & ? \\ 0.4 & 1 & 1 & 0.4 \\ 0.2 & 0.8 & 1 & 1 \\ ? & 0.2 & 0.4 & 1 \end{bmatrix}. \quad (6.5)$$

Then A is partial TP but neither TN nor TP completable since for any nonnegative choice of the unspecified entries the determinant of the resulting matrix is negative.

Fortunately, there are some simple patterns which allow a TP completion.

Lemma 6.4. [JK11, Lemma 3.1] *Let*

$$A := \begin{bmatrix} 1 & a_{12} & ? & a_{14} \\ a_{21} & 1 & a_{23} & ? \\ ? & a_{32} & 1 & a_{34} \\ a_{41} & ? & a_{43} & 1 \end{bmatrix} \quad (6.6)$$

be a partial TP matrix such that

$$a_{14} < a_{12}a_{23}a_{34} \quad \text{and} \quad a_{41} < a_{21}a_{32}a_{43}.$$

Then A has a TP completion.

Lemma 6.5. [JK11, Lemma 3.2] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be TP and $a_{11}b_n < b_1a_{1n}$ for some positive numbers b_1, b_n . Then the $(n+1)$ -by- n partial TP matrix*

$$B := \begin{bmatrix} b_1 & ? & \dots & ? & b_n \\ & & & A & \end{bmatrix} \quad (6.7)$$

has a TP completion.

Lemma 6.6. [JK11, Lemma 3.3] *Let $A \in \mathbb{R}^{n,n}$ be TP. Then the $(n+1)$ -by- $(n+1)$ partial TP matrix*

$$\begin{bmatrix} 1 & b_2 & ? & \dots & ? & b_n \\ c_2 & & & & & \\ ? & & & & & \\ \vdots & & & A & & \\ ? & & & & & \\ c_n & & & & & \end{bmatrix} \quad (6.8)$$

has a TP completion.

By using the above lemmata we have the following theorem.

Theorem 6.4. [JK11, Theorem 3.4] *Suppose that*

$$A = \begin{bmatrix} a_{11} & a_{12} & ? & ? & \dots & ? & a_{1n} \\ a_{21} & a_{22} & a_{23} & ? & \dots & ? & ? \\ ? & a_{32} & \ddots & \ddots & \dots & ? & ? \\ ? & ? & \ddots & \ddots & \ddots & ? & ? \\ \vdots & \vdots & & & & \vdots & \vdots \\ ? & ? & \dots & \dots & a_{n-2,n-1} & ? & ? \\ ? & ? & \dots & \dots & a_{n-1,n-1} & a_{n-1,n} & ? \\ a_{n1} & ? & \dots & \dots & a_{n,n-1} & a_{nn} & ? \end{bmatrix} \quad (6.9)$$

is a partial TP matrix. Then A has a TP completion if and only if

$$a_{1n}a_{22}a_{33} \cdots a_{n-1,n-1} < a_{12}a_{23} \cdots a_{n-1,n} \quad (6.10)$$

and

$$a_{n1}a_{22}a_{33} \cdots a_{n-1,n-1} < a_{21}a_{32} \cdots a_{n,n-1}. \quad (6.11)$$

When a partial TN matrix has positive diagonal entries and a pattern like that given in the above theorem, then the necessary and sufficient conditions for this matrix to have a TN completion are the same as in conditions (6.10) and (6.11) but with non-strict inequalities, see [JK11, Theorem 2.1].

Graph theory terminologies help very much in defining and describing the patterns of the specified positions in a given partial matrix. We recall some definitions from graph theory, see e.g., [JKL98], [JT04], [FJ11].

The pattern of the specified entries of an n -by- n partial matrix $A = (a_{ij})$ with specified main diagonal entries may be described by a directed graph $G_A = (V, E)$, where the set of vertices V is $\{1, \dots, n\}$ and E is the set of its edges. We say that there is a directed edge (i, j) , $i \neq j$, from i to j if and only if a_{ij} is specified. In case that the partial matrix is combinatorially symmetric, the graph of the specified entries may be taken to be undirected. A *directed path* is a sequence of edges $((i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k))$ in which all vertices are distinct. If, in addition, $i_k = i_1$, the path is said to be a *directed cycle*. A graph G is said to be *chordal* if there are no minimal cycles of length greater than or equal to 4, where the *length of a cycle* is the number of the edges from which this cycle is composed. A graph is called *complete* if it includes all possible edges between its vertices. A *clique* is an induced subgraph which is complete. A graph G is said to be *block clique* (1-chordal [JT04]) if G is a chordal graph in which every pair of maximal cliques C_i, C_j , $C_i \neq C_j$, intersect in at most one vertex. A *monotonically labeled block clique* (*monotonically labeled 1-chordal*) graph is a labeled block clique (1-chordal) graph in which the maximal cliques are labeled in natural order, that is, for every pair of maximal cliques C_i, C_j in which $i < j$ and $C_i \cap C_j = \{u\}$, the labeling within the two cliques is such that every element of $\{v \mid v \in C_i \setminus \{u\}\}$ is labeled less than u and every element of $\{w \mid w \in C_j \setminus \{u\}\}$ is labeled greater than u . In analogous way, we say that a path, a cycle, is monotonically labeled if it is labeled in natural order.

Definition 6.1. [eGJT08, Definition 1.1] A directed graph $G = (V, E)$ is said to be a double-path if it is made up of two paths, i.e., if $V = \{i_1, i_2, \dots, i_n\}$, and

$$E = \{(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k), (i_1, i_{k+1}), (i_{k+1}, i_{k+2}), \dots, (i_{n-1}, i_n), (i_n, i_k)\},$$

that is, G is made up of two paths C_1 and C_2 from vertex i_1 to i_k .

We say that a double-path $G = (V, E)$ has a *monotonically labeled path* if, without loss of generality, the set of edges is

$$E = \{(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k), (i_1, i_{k+1}), (i_{k+1}, i_{k+2}), \dots, (i_{n-1}, i_n), (i_n, i_k)\}$$

such that $i_j = i_1 + j - 1$ for $j = 2, \dots, k$.

Definition 6.2. [eGJT08, Definition 1.3] Let A be an n -by- n partial TN matrix whose associated graph is a non-monotonically labeled directed cycle:

$$(\{i_1, i_2, \dots, i_n\}, \{(i_1, i_2), (i_2, i_3), \dots, (i_{n-1}, i_n), (i_n, i_1)\}).$$

We say that A satisfies the C -condition if

$$a_{i_1, i_2} a_{i_2, i_3} \cdots a_{i_{n-1}, i_n} a_{i_n, i_1} \leq a_{11} a_{22} \cdots a_{nn}. \quad (6.12)$$

The question: for which graphs G does every partial TN matrix, the graph of whose specified entries is G , have a TN completion? is considered in [JKL98], [JT04], [DJK08], [eGJT08], and [JTeG09].

Since total nonnegativity is not preserved by permutation similarity we must consider *labeled graphs*, i.e., graphs in which the numbering of the vertices is fixed.

The following two partial TN matrices correspond to the same graph but with different labeling [JKL98]

$$A = \begin{bmatrix} 1 & 1 & ? \\ 0 & 1 & 1 \\ ? & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & ? \\ 0 & ? & 1 \end{bmatrix}.$$

The matrix A is TN completable, while B is not TN completable since for any choice of the unspecified entries of B , $\det B[1, 2|2, 3] = -1$. Thus the labeling of the graphs is crucial for the decision whether a given partial TN matrix whose graph is G is TN completable.

A partial TN matrix is called *regular* if every maximal specified principal submatrix is nonsingular and for every pair of maximal specified principal submatrices $A[\alpha]$, $A[\beta]$ such that $\alpha \cap \beta \neq \phi$, then $A[\alpha \cap \beta]$ is also nonsingular. The last condition is satisfied automatically by Lemma 2.6. The following theorem is proved in [JKL98]; the regularity condition is subsequently relaxed in [JT04, Theorem 1.1] and [DJK08, Theorem 1].

Theorem 6.5. [JKL98, Theorem 1] Let G be a connected graph on n vertices. Then every regular partial TN matrix, the labeled graph of whose specified entries is G , has a TN completion if and only if G is a monotonically labeled block clique graph.

The next natural question that should be asked is what other conditions on the specified data are required in order to guarantee a TN or TP completion in the case that the graph is not monotonically labeled block clique. This question is answered in detail for 3-by-3 partial TN matrices and some special graphs in [JT04] and [DJK08], and paths on 4-vertices in [DJK08].

In [JT04] other conditions which are called P -condition (adjacent edge condition [DJK08]), CP -condition, and SS -diagonal condition were presented to guarantee existence of a TN completion for a given partial TN matrix.

Definition 6.3. [JT04, Definition 2.1], [DJK08] *Let G be a path with three vertices that is not monotonically labeled. We say that the partial matrix A , the graph of whose specified entries is G , satisfies the P -condition (adjacent edge condition) if*

$$(a) \ j < i < k \text{ (or } k < i < j), \text{ then } a_{jk}a_{kj} \leq \frac{a_{kk}}{a_{ii}}a_{ij}a_{ji};$$

$$(b) \ i < k < j \text{ (or } j < k < i), \text{ then } a_{ij}a_{ji} \leq \frac{a_{ii}}{a_{kk}}a_{kj}a_{jk}.$$

Definition 6.4. [JT04, Definition 4.1] *Let $A = (a_{ij})$ be an n -by- n partial TN matrix whose associated graph is a monotonically labeled cycle. We say that A satisfies the SS -diagonal condition if*

$$a_{1n} \leq a_{12}a_{23} \cdots a_{n-1,n} \quad \text{and} \quad a_{n1} \leq a_{21}a_{32} \cdots a_{n,n-1}. \quad (6.13)$$

The next lemma presents a necessary and sufficient condition for a given partial TN matrix whose graph contains a clique on the first $n - 1$ vertices and the last vertex is adjacent to one of the first $n - 2$ vertices to have a TN completion.

Lemma 6.7. [JT04, Lemma 2.1], [DJK08, Lemma 1] *Let G be a graph containing a clique on vertices $\{1, 2, \dots, n - 1\}$ and let the vertex n be adjacent to one vertex of $\{1, 2, \dots, n - 2\}$. Every partial TN matrix A , the graph of whose specified entries is G , has a TN completion if and only if A satisfies the P -condition.*

The following theorem is a generalization of the above lemma.

Theorem 6.6. [JT04, Theorem 2.1], [DJK08, Lemma 2] *Let G be a graph containing a clique on the vertices $\{1, 2, \dots, k\}$ and let the vertices $\{k + 1, k + 2, \dots, n\}$ be adjacent to one vertex of $\{1, 2, \dots, k\}$. Every partial TN matrix A , the graph of whose specified entries is G , has a TN completion if and only if A satisfies the P -condition.*

In the case that the graph of the partial TN matrix is a monotonically labeled cycle, the SS -diagonal condition is a necessary and sufficient condition for a TN completion.

Theorem 6.7. [JT04, Theorem 4.1] *Let A be an n -by- n partial TN matrix with $4 \leq n$, whose graph is a monotonically labeled cycle. There exists a TN completion of A if and only if A satisfies the SS -diagonal condition.*

The double-path graphs are considered in [eGJT08]. The authors give the following example which shows that not every partial TN matrix whose associated graph is double-path has a TN completion.

Example 6.4. [eGJT08, Example 1.2] *Let*

$$A := \begin{bmatrix} 1 & 0.5 & ? & 0.5 \\ ? & 1 & 0.5 & ? \\ ? & ? & 1 & ? \\ ? & ? & 1 & 1 \end{bmatrix}. \quad (6.14)$$

Then A is a partial TN matrix whose associated graph is double-path. A has no TN completion since in order to complete A to a TN matrix one needs to choose the unspecified entry in the position $(1, 3)$ by considering the minors $\det A[1, 2|2, 3]$ and $\det A[1, 3|3, 4]$ less than or equal to $1/4$ and greater than or equal to $1/2$, respectively, which is impossible.

The following two theorems give necessary and sufficient conditions for a partial TN matrix whose graph is double-path in the case when at least one of the two paths is non-monotonically labeled to have a TN completion.

Theorem 6.8. [eGJT08, Theorem 2.3] *Let G be a double-path with a monotonically labeled path C_1 and a non-monotonically labeled one C_2 . Then, every partial TN matrix $A = (a_{ij})$, of size $n \times n$, whose associated graph is G , has a TN completion if and only if $P_{C_2} \leq P_{C_1}$, where P_{C_i} is the product of the weights of the edges corresponding to path C_i , $i = 1, 2$, where the weight of an edge (i, j) is a_{ij} .*

Theorem 6.9. [eGJT08, Theorem 2.5] *Let G be a double-path with non-monotonically labeled paths $C_1 = (V_1, E_1)$ and $C_2 = (V_2, E_2)$, whose first and last vertex is not 1. Every partial TN matrix A whose associated graph is G has a TN completion if $P_{C_1} \leq P_{C_2}$ when $1 \in V_1$ or $P_{C_2} \leq P_{C_1}$ when $1 \in V_2$.*

The totally nonpositive completion problem is studied in [ATU06].

6.4. Some Totally Positive Completable Patterns

In this section we consider the totally positive completion problem for some new patterns of the unspecified entries.

We recall from [JW13] the following definitions. We associate with a matrix the directions north, east, south, and west. So the entry in position $(1, 1)$ lies north and west. We call a (possibly rectangular) pattern *jagged* if, whenever a position is unspecified, either all positions north and west of it are unspecified or south and east of it are unspecified, and we call a (possibly rectangular) pattern *echelon* if, whenever a position is unspecified, either all positions north and east of it are unspecified or south and west of it are unspecified; either of these is referred to as *single echelon*, while when both occur, we say *double echelon*. Echelon refers to any of these possibilities. A pattern is called *jagged echelon* if it has both

jagged and echelon patterns. Let A be an n -by- m matrix. We say that A has a P_1 or P'_1 pattern if A has just one unspecified entry, viz. in the $(1, m)$ or $(n, 1)$ position, respectively. The P_2 pattern has just two unspecified entries, viz. in positions $(1, m)$ and $(n, 1)$. Since by Proposition 2.19 a given matrix is TP if and only if all its initial minors are positive, a partial TP matrix with a P_1 pattern has a TP completion if and only if the upper right entry can be chosen so that the contiguous upper right minors are all positive. A similar condition holds for a P'_1 pattern. We introduce two further patterns. We say that A has a P_3 pattern if $3 \leq m$ and the unspecified entries are a_{ij} ,

$$i = 1, \dots, l, j = 3, \dots, k, \text{ and } i = r, \dots, n, j = 1, \dots, m - 2, \\ \text{with } l \in \{1, \dots, n - 1\}, r \in \{l + 1, \dots, n\}.$$

A has a P_4 pattern if $2 \leq n, 4 \leq m$ and the unspecified entries are a_{ij} ,

$$i = 1, \dots, l, j = 1, \dots, k, \text{ and } i = 1, \dots, r, j = k + 3, \dots, m, \text{ and } i = p, \dots, n, \\ j = 1, \dots, h, \text{ and } i = t, \dots, n, j = h + 3, \dots, m, \\ \text{with } 1 \leq r \leq l < t \leq p \leq n, h, k \in \{1, \dots, m - 3\}.$$

Examples 6.1, 6.2, and 6.3 show that the $P_1, P'_1,$ and P_2 patterns are not TP completable if $4 \leq \min\{n, m\}$. This explains why the index ranges will often start at 3 in the sequel.

The following proposition gives a solution for TP completion when a partial TP matrix has a jagged pattern.

Proposition 6.1. [JW13, Theorem 5] *Each jagged pattern is TP completable.*

In [JW13] the following two conjectures are posed.

Conjecture 6.1. [JW13, Conjecture 1] *An echelon pattern is TP completable if and only if it contains no P_1, P'_1 or P_2 as a subpattern.*

Conjecture 6.2. [JW13, Conjecture 2] *A jagged echelon pattern is TP completable if and only if it contains no P_1, P'_1 or P_2 as a subpattern.*

In the sequel we show by using the following series of theorems that any n -by- m partial TP matrix with a $P_3, P_4,$ or related patterns is TP completable.

Theorem 6.10. [AG16b, Theorem 5.1] *Let A be an n -by- m partial TP matrix with the unspecified entries $a_{ij}, i = 1, \dots, l, j = 3, \dots, k,$ where $l \leq n, k \leq m.$ Then A is TP completable.*

Proof. Let $B_{lk} := A[l, \dots, n | 1, 2, k, \dots, m]$. Then by Theorem 6.2, B_{lk} is TP completable. We enter the value for the unspecified entry a_{lk} into the matrix A and call the resulting matrix A_{lk} . If $l > 1$ let $B_{l-1,k} := A_{lk}[l - 1, \dots, n | 1, 2, k, \dots, m]$. Then by Theorem 6.2, $B_{l-1,k}$ is TP completable and similarly as for the entry a_{lk} we obtain the n -by- m partial TP matrix $A_{l-1,k}$ which has one unspecified entry less than A_{lk} . Now we continue in this manner until we find values for all the unspecified entries in column k resulting in the partial TP matrix A_{1k} . If $k > 3$ repeat the above process with the partial TP matrix A_{1k} to find values for the unspecified entries in the columns $k - 1, \dots, 3$. At the end of this process we arrive the matrix A_{13} which is TP . \square

The following corollary easily follows by applying the same arguments as in the proof of the above theorem to $A^\#$.

Corollary 6.1. [AG16b, Corollary 5.2] *Let A be an n -by- m partial TP matrix with the unspecified entries a_{ij} , $i = r, \dots, n$, $j = k, \dots, m - 2$, where $r \leq n$, $k \leq m - 2$. Then A is TP completable.*

By application of arguments that are employed in proof of Theorem 6.10 to A , $A^\#$, or A^T it follows that a partial TP matrix A whose pattern is a single echelon pattern is TP completable if and only if it contains no P_1 or P'_1 as a subpattern. This settles Conjecture 6.1 in a special case.

Theorem 6.11. [AG16b, Theorem 5.3] *Let $3 \leq m$ and A be an n -by- m partial TP matrix with the unspecified entries a_{1j} , $j = 3, \dots, m$, and a_{i1} , $i = l, \dots, n$, with $l \leq 4$. Then A is TP completable.*

Proof. If $l = 1$ or 2 then it is easy to find values for the unspecified entries in the positions $(1, 1)$ and $(2, 1)$, respectively, so let $l \in \{3, 4\}$. Let $B_1 := A[2, \dots, n|1, \dots, m]$. Then by taking the transpose of B_1 and using Theorem 6.10 (with m replaced by n), B_1 is TP completable. We enter the values for the unspecified entries a_{i1} , $i = l, \dots, n$, into the matrix A and call the resulting matrix A_1 . The matrix A_1 is a partial TP matrix with the unspecified entries in the first row. By Theorem 6.10 A_1 is TP completable, and so A is TP completable. \square

Theorem 6.12. [AG16b, Theorem 5.4] *Let $3 \leq m$, $4 \leq n$, and A be an n -by- m partial TP matrix with the unspecified entries a_{ij} , $i = 1, \dots, l$, $j = 3, \dots, m$, and $i = l + 3, \dots, n$, $j = 1, \dots, h$, where $h \leq m$. Then A is TP completable.*

Proof. Let $B_h := [l + 1, \dots, n|h, \dots, m]$. Then B_h is a partial TP matrix and by Theorem 6.10 (taking the transpose), B_h is TP completable. We enter the values for the unspecified entries of B_h into the matrix A and call the resulting matrix A_h . Repeat the last step to find values for the unspecified entries in the lower part of the columns $h - 1, \dots, 2$. At the end of this process we arrive the partial TP matrix A_2 having the unspecified entries in the upper right corner and in the first column below the position $(1, l + 3)$. To find values for the unspecified entries in the first column, we proceed analogously to the proof of Theorem 6.11. Let $C := A_2[l + 1, \dots, n|1, \dots, m]$. Then by Theorem 6.10 C is TP completable. Since only the initial minors of A need to be positive, C can be completed independently of the entries of $A[1, \dots, l|1, 2]$. We enter the values for the unspecified entries of the first column into the matrix A_2 and call the resulting matrix A_1 ; then A_1 is a partial TP matrix with the unspecified entries in the upper right corner. We apply Theorem 6.10 on A_1 and it follows that A_1 is TP completable. Therefore A is TP completable. \square

By combining the patterns that are given in the above theorems we obtain the following theorem which shows that every n -by- m partial TP matrix with a P_3 pattern is TP completable.

Theorem 6.13. [AG16b, Theorem 5.5] *If $A = (a_{ij})$ is an n -by- m partial TP matrix with a P_3 pattern, then A is TP completable.*

Proof. Let A be an n -by- m partial TP matrix with a P_3 pattern. We distinguish two cases. **Case (1):** $k = m$. Let $B_1 := A[l + 1, \dots, n|1, m - 1, m]$. Then by applying Theorem 6.2 successively to $C_\rho := B_1[l + 1, \dots, \rho|1, m - 1, m]$ we find values for the unspecified entries $a_{\rho 1}$, $\rho = r, \dots, n$. Therefore B_1 is TP completable. We enter the values for the unspecified entries of B_1 into the matrix A and call the resulting matrix A_1 . To find values for the unspecified entries in the second column of A_1 in and below position $(r, 2)$ we similarly apply Theorem 6.2 to the submatrix $B_2 := A_1[l + 1, \dots, n|1, 2, m - 1, m]$ if $r > l + 1$. This completion can be accomplished independently of the entries of $A[1, \dots, l|1, 2]$, see the proof of Theorem 6.12. If $r = l + 1$, then we first find a value for the unspecified entry $a_{r,2}$ by considering the partial submatrix $A_1[l, l + 1|1, 2]$ and after that we apply the same arguments as in the case $r > l + 1$. We enter the values for the unspecified entries of B_2 into the matrix A and call the resulting matrix A_2 . Let $B_3 := A_2[l + 1, \dots, n|1, 2, \dots, m]$. Then B_3 is partial TP and by Corollary 6.1 B_3 is TP completable. We enter the values for the unspecified entries of B_3 into the matrix A_2 and call the resulting matrix A_3 . By Theorem 6.10 A_3 is TP completable and hence A is TP completable.

Case (2): $k < m$. We consider first the case $k = m - 1$. If $l + 1 = r$, then we can choose a positive number for the unspecified entry $a_{l,m-1}$ such that the matrix remains partial TP . If $l + 1 < r$, let $B_{l,m-1} := A[l, \dots, r - 1|1, 2, m - 1, m]$ and $B'_{l,m-1} := A[l, \dots, n|m - 1, m]$. Then by Theorem 6.2 both submatrices are TP completable. Moreover, we can choose in both matrices a common value for the unspecified entry $a_{l,m-1}$ because the only nontrivial initial minor of the submatrix $B'_{l,m-1}$ containing this entry, viz. $\det B'_{l,m-1}[l, l + 1|m - 1, m]$, is also an initial minor of the submatrix $B_{l,m-1}$. We enter the value for the unspecified entry $a_{l,m-1}$ into the matrix A and call the resulting matrix $A_{l,m-1}$. Repeating this process we find values for the unspecified entries in column $m - 1$ and finally obtain the partial TP matrix $A_{1,m-1}$. Let $C := A_{1,m-1}[1, \dots, r - 1|1, \dots, m]$. By Theorem 6.10, C is TP completable. We enter the values for the unspecified entries in $A_{1,m-1}$ and call the resulting matrix A' , which is a partial TP matrix. By Corollary 6.1, A' is TP completable and we can conclude that A is also TP completable.

If $k < m - 1$ we follow the proof in the case $k = m - 1$ but we may end the proof already with the definition of the matrix C . \square

By using Proposition 6.1 and Theorem 6.10 the following theorem is proved. It shows that any n -by- m partial TP matrix with a P_4 pattern is TP completable.

Theorem 6.14. [AG16b, Theorem 5.6] *If $A = (a_{ij})$ is an n -by- m partial TP matrix with a P_4 pattern, then A is TP completable.*

Proof. Let A be an n -by- m partial TP matrix with a P_4 pattern.

Without loss of generality we may assume that $l = r$ and $p = t$. Otherwise let $B_0 := A[r + 1, \dots, p - 1|1, \dots, m]$. Then B_0 has a jagged pattern and can be completed by Proposition 6.1 (independently of the entries of $A[1, \dots, r|k + 1, k + 2]$ and $A[p, \dots, n|h + 1, h + 2]$). In

what follows we therefore assume that $l = r$ and $p = t$. Let $B_1 := A[1, \dots, p-1 | k+1, \dots, m]$; then B_1 is TP completable by Theorem 6.10. We enter the values for the unspecified entries of B_1 into the matrix $A[1, \dots, p-1 | 1, \dots, m]$ and call the resulting matrix B_2 . Since B_2 is a partial TP matrix with a jagged pattern it is TP completable by Proposition 6.1. We enter the values for the unspecified entries of B_2 into the matrix A and call the resulting matrix A_1 . Proceeding with $A_1^\#$ we obtain similarly the TP completion of A_1 and in this way of A , too. \square

The following two theorems present related patterns of partial TP matrices which are TP completable.

Theorem 6.15. [AG16b, Theorem 5.7] *If $A = (a_{ij})$ is an n -by- m partial TP matrix with the unspecified entries a_{ij} $i = 1, \dots, l$, $j = 1, \dots, k$, and $i = 1, \dots, r$, $j = k+3, \dots, m$, and $i = t, \dots, n$, $j = 1, \dots, m-2$, with $r \leq l < t$, then A is TP completable.*

Proof. Without loss of generality we may assume that $r = l$, see the proof of Theorem 6.14. Let $C_1 := A[r+1, \dots, n | 1, \dots, k, m-1, m]$. Then C_1 is TP completable by Corollary 6.1. We enter the values for the unspecified entries of C_1 into the matrix A and call the resulting matrix A_1 . Let $C_2 := A_1[r+1, \dots, n | 1, \dots, k+1, m-1, m]$. Then by an argument similar to that used in the proof of Theorem 6.10, by starting from the unspecified entry $a_{t, k+1}$, and proceeding downwards, we can find values for the unspecified entries in C_2 ; so C_2 is TP completable. We enter the values for the unspecified entries of C_2 into the matrix A_1 and call the resulting matrix A_2 . Similarly, we can find values for the unspecified entries in the column $k+2$, then we enter these values into the matrix A_2 and call the resulting matrix A_3 . Let $D := A_3[r+1, \dots, n | 1, \dots, m]$; then D is a partial TP matrix with the same type of pattern as the one treated in Corollary 6.1, thus D is TP completable. We enter the values for the unspecified entries in D into the matrix A_3 and call the resulting matrix A_4 , where A_4 is a partial TP matrix with the same pattern as the one considered in the proof of Theorem 6.14, thus A_4 is TP completable. Hence A is TP completable. \square

Theorem 6.16. [AG16b, Theorem 5.8] *If $A = (a_{ij})$ is an n -by- m partial TP matrix with the unspecified entries a_{ij} $i = 1, \dots, l$, $j = 1, \dots, k$, and $i = 1, \dots, r$, $j = k+3, \dots, m$, and $i = t, \dots, n$, $j = h, \dots, m$, with $r \leq l < t$, and $h < k$, then A is TP completable.*

Proof. We may assume without loss of generality that $r = l$. Otherwise we proceed as follows: Let $B_1 := A[r+1, \dots, t-1 | h, \dots, m]$. Then B_1 is partial TP with a jagged pattern, thus TP completable. If $h > 1$ we have to take into account the entries of $A[t, \dots, n | 1, \dots, h-1]$ when we want to extend the completion to the left. We proceed element-wise by taking successively the entries $a_{l, h-1}$, $a_{l-1, h-1}$, \dots , $a_{r+1, h-1}$, $a_{l, h-2}$, \dots , $a_{r+1, h-2}$, \dots , $a_{r+1, 1}$. For a fixed entry we consider the submatrices which have the chosen entry as the only unspecified entry, viz. in position $(1, 1)$. For each such submatrix we can find a positive number such that the matrix is TP . Then we take the maximum of all these positive numbers (for the chosen entry).

The matrix $C_1 := A[r+1, \dots, n | 1, \dots, k+1]$ is a partial TP matrix with a jagged pattern and by Proposition 6.1 C_1 is TP completable. We enter the values for the unspecified entries

of C_1 into the matrix A and call the resulting matrix A_1 . Let $C_2 := A_1[r+1, \dots, n|1, \dots, k+2]$, and $C_2' := A_1[1, \dots, n|k+1, k+2]$. Then both submatrices are partial TP matrices with jagged patterns, and so by the argument used in the proof of Theorem 6.13, Case (2), common values for the unspecified entries can be found. We enter the values for the unspecified entries of C_2 into the matrix A_1 and call the resulting matrix A_2 . Let $C_3 := A_2[r+1, \dots, n|1, \dots, m]$. Then C_3 is a partial TP matrix with a jagged pattern, thus C_3 is TP completable. We enter the values for the unspecified entries of C_3 into the matrix A_2 and call the resulting matrix A_3 . Since A_3 is a partial TP matrix with the same type of pattern as the one considered in the proof of Theorem 6.14, A_3 is TP completable. Hence A is TP completable. \square

Theorems 6.15 and 6.16 represent some cases of a jagged echelon pattern with no P_1 , P_1' , or P_2 as a subpattern. Thus both theorems and the following remark settle Conjecture 6.2 in some special cases so that this conjecture remains unsettled only in the case when we have a double echelon pattern.

We conclude this section with the following remarks which give more partial TP patterns which are TP completable and extend the results to the TP_k partial matrices case.

Remark 6.1. [AG16b, Remark 5.1] *The following patterns can be proven to be TP completable by using similar methods as in the proofs of the Theorems 6.14, 6.15, and 6.16. The entries a_{ij} are unspecified for*

1. $i = 1, \dots, l, j = 1, \dots, k$, and $i = r, \dots, n, j = 1, \dots, m-2$, with $l < r$;
2. $i = 1, \dots, l, j = 3, \dots, m$, and $i = r, \dots, n, j = k, \dots, m$, with $l < r$;
3. $i = 1, \dots, l, j = 3, \dots, m$, and $i = r, \dots, n, j = 1, \dots, k$, and $i = t, \dots, n, j = k+3, \dots, m$, with $l < t < r$;
4. $i = 1, \dots, l, j = 1, \dots, k$, and $i = r, \dots, n, j = 1, \dots, h$, and $i = t, \dots, n, j = h+3, \dots, m$, with $l < t < r$ and $h < k$.

Remark 6.2. *By using Theorem 6.3 instead of Theorem 6.2 all the results of this section carry over to the case that the given matrix A is partial TP_k , $4 \leq k$.*

7. Hurwitz Polynomials and Hankel Matrices

A real polynomial is called *Hurwitz* or *stable* if all of its roots have negative real parts and is called *quasi-stable* if all of its roots have nonpositive real parts.

In many application areas where stability plays a significant role such as control and systems theory [Bar94], [BCK95], the problem of deciding whether a given polynomial is Hurwitz or not is very important. In [Max68] necessary and sufficient conditions for a given polynomial which has degree at most three to be stable were introduced. In the same paper the question was raised what are the conditions needed for a given polynomial of degree more than three to be stable? Several criteria for stability of a given polynomial have been found, for example the Hermite-Biehler's Theorem [Her56], [Bie79], [Hol03], the Routh's Criterion [Gan59], the Routh-Hurwitz's Criterion [Hur95], the Liénard-Chipart's Criteria [LC14], and the Mihailov's Theorem [Mik38], [AHLVGS13]. A. M. Lyapunov proved in his dissertation a theorem which gives necessary and sufficient conditions for the stability of the characteristic polynomial of a given matrix [Gan59, Chapter XV, Section 5].

Interesting properties of these polynomials were also found, e.g., total nonnegativity of special matrices associated with this kind of polynomials [Asn70], [Kem82], the Hadamard product of Hurwitz polynomials is Hurwitz [GW96], and an interval polynomial is Hurwitz if four specified vertex polynomials are Hurwitz [Kha78], [Bar94], [BCK95] or a certain matrix is almost totally positive [AGT16]. A generalization of Hurwitz polynomials to rational functions is considered in [BT11], [HT12], [AGT16].

The organization of this chapter is as follows. In Section 7.1, the Sturm sequences and some of their properties are recalled. In Section 7.2, the Cauchy index and some of its properties which play an important role in studying polynomials and rational functions are given. In Section 7.3, Routh's Criterion, Routh-Hurwitz's Criterion, and the relationship between them are introduced. Special matrices of Hurwitz type are also given. Finally in Section 7.4, Hankel matrices and their use in the study of polynomials and rational functions are presented.

7.1. Sturm Sequences

In this section we present one of the important tools that are used in the study of properties of a given polynomial, viz. the Sturm sequences.

Let $f(z)$ and $g(z)$ be any two polynomials with the degree of $f(z)$ not less than that of $g(z)$. Then the Sturm Algorithm is defined as follows [Gan59], [Bar71], [Bar08].

Set $f_0(z) := f(z)$, $f_1(z) := g(z)$. Application of the Euclidean Algorithm to $f_0(z)$ and $f_1(z)$ yields the remainder $-f_2(z)$, $-f_3(z)$ is the remainder upon dividing $f_1(z)$ by $f_2(z)$, etc. Then we obtain the following sequence of equations (note that compared to the Euclidean Algorithm the signs of the remainders are reversed):

$$\begin{aligned} f_0(z) &= q_0(z)f_1(z) - f_2(z), \\ f_1(z) &= q_1(z)f_2(z) - f_3(z), \\ &\vdots \\ f_{m-1}(z) &= q_{m-1}(z)f_m(z), \end{aligned} \tag{7.1}$$

where the last remainder $f_m(z)$ that is not identically zero is the greatest common divisor of $f(z)$ and $g(z)$ and also of all the polynomials of the sequence $(f_0, f_1, \dots, f_{m-1})$. The finite sequence (f_0, f_1, \dots, f_m) that is generated in this way is called the *generalized Sturm sequence*.

Definition 7.1. [Gan59, p. 175], [Bar08] *A sequence of polynomials (f_0, f_1, \dots, f_n) is called a Sturm sequence on the interval (a, b) if the following three conditions hold simultaneously:*

- 1) *If $f_0(z_0) = 0$ for some $z_0 \in (a, b)$ then $f_1(z_0) \neq 0$.*
- 2) *If $f_j(z_0) = 0$ for some $z_0 \in (a, b)$ then $f_{j-1}(z_0)f_{j+1}(z_0) < 0$ for $j = 1, 2, \dots, n-1$.*
- 3) *$f_n(z) \neq 0$ for all $z \in (a, b)$.*

A generalized Sturm sequence is easily seen to be a Sturm sequence on any interval where $f_m(z)$ does not vanish. Moreover, if we denote

$$h_j(z) := \frac{f_j(z)}{f_m(z)}, \quad j = 0, 1, \dots, m, \tag{7.2}$$

then $h_m(z) = 1$ and the sequence (h_0, h_1, \dots, h_m) is a Sturm sequence on $(-\infty, \infty)$.

In the following we denote by $V(z)$ the number of variations of signs in the Sturm sequence $(f_0(z), f_1(z), \dots, f_m(z))$ evaluated at a fixed value z , i.e., $V(z) := S^-(f_0(z), f_1(z), \dots, f_m(z))$, and $V(z \pm 0) := V(z \pm \epsilon)$ for some sufficiently small $0 < \epsilon$. We conclude this section with the following lemma which presents a very important property of the Sturm sequences.

Lemma 7.1. [Bar08, Lemma 24] *Let (f_0, f_1, \dots, f_m) be a Sturm sequence on an interval (a, b) . Then if $c \in (a, b)$ is not a zero of odd multiplicity of the initial polynomial $f_0(z)$, then $V(c+0) = V(c-0)$.*

Proof. The proof follows by using properties 2) and 3) of Definition 7.1 and the fact that $c \in (a, b)$ is not a zero of odd multiplicity of $f_0(z)$. \square

7.2. Cauchy Index and its Properties

In this section we present an important quantity in the theory of rational functions which is called Cauchy index and some of its properties.

Definition 7.2. [Gan59, Definition 1, p. 174] *The Cauchy index of a real rational function $R(z)$ between the limits a and b , denoted by $\text{Ind}_a^b R(z)$, where a and b are real numbers or $\pm\infty$, is the difference between the number of jumps of $R(z)$ from $-\infty$ to $+\infty$ and that of jumps from $+\infty$ to $-\infty$ as the argument changes from a to b ¹⁾.*

Remark 7.1. *The jumps of a real rational function from $-\infty$ to $+\infty$ or from $+\infty$ to $-\infty$ occur at its real poles which have odd multiplicities.*

The following definition gives the Cauchy index of a given real rational function at one of its real poles of odd multiplicity.

Definition 7.3. [Bar08, formula (16)] *Let w be a real pole of odd multiplicity of the rational function R . Then the quantity*

$$\text{Ind}_w(R) := \begin{cases} +1, & \text{if } R(w-0) < 0 < R(w+0), \\ -1, & \text{if } R(w+0) < 0 < R(w-0) \end{cases} \quad (7.3)$$

is called the index of the function R at w . If the real pole w has an even multiplicity then we set $\text{Ind}_w(R) := 0$.

Based on Definition 7.3 we can rewrite Definition 7.2 in the following manner.

Definition 7.4. [Bar08], [HT12, formula (2.11)] *Let the real rational function R have m real poles in total:*

$$w_1 < \cdots < w_m. \quad (7.4)$$

Then the following relation holds:

$$\text{Ind}_a^b(R) := \sum_{i: a < w_i < b} \text{Ind}_{w_i}(R). \quad (7.5)$$

If the function R has a pole at the point ∞ , then such a pole is considered as real in [HT12]. In order to introduce the index at ∞ we follow [HT12]. The function R is considered as a map on the projective line $\mathbb{P}\mathbb{R}^1 := \mathbb{R}^1 \cup \{\infty\}$ into itself. Hence if the function R has a pole at ∞ , then by [HT12, formula (2.12)]

$$\text{Ind}_\infty(R) := \begin{cases} +1, & \text{if } R(+\infty) < 0 < R(-\infty), \\ -1, & \text{if } R(-\infty) < 0 < R(+\infty), \\ 0, & \text{if } \text{sign } R(+\infty) = \text{sign } R(-\infty). \end{cases} \quad (7.6)$$

¹⁾ In counting the number of jumps, the extreme values of z , the limits a and b , are excluded.

Thus, the *generalized Cauchy index of the function R on the projective real line* is given by

$$\text{Ind}_{\mathbb{P}\mathbb{R}^1}(R) := \text{Ind}_{-\infty}^{+\infty}(R) + \text{Ind}_{\infty}(R). \quad (7.7)$$

One of the methods of computing the Cauchy index of a given rational function $R(z)$ on an interval (a, b) is based on the Sturm sequence formed from its numerator and denominator.

Lemma 7.2. [Bar08, Lemma 25] *Let (f_0, f_1, \dots, f_m) be a Sturm sequence on a given interval (a, b) . If $c \in (a, b)$ is a zero of odd multiplicity of the initial polynomial f_0 , then*

$$V(c-0) - V(c+0) = \text{Ind}_c \frac{f_1}{f_0}. \quad (7.8)$$

We conclude this section by stating the following theorem which can be obtained by using Lemmata 7.1 and 7.2, and the fact that the full increment of a step function equals to the sum of its increments at the points of discontinuity, see [Bar08].

Theorem 7.1. [Gan59, Theorem 1, (Sturm's Theorem), p. 175], [Bar08, Theorem 26] *Let (f_0, f_1, \dots, f_m) be a Sturm sequence on a given interval (a, b) . Then*

$$\text{Ind}_a^b \frac{f_1}{f_0} = V(a+0) - V(b-0). \quad (7.9)$$

Sturm's Theorem remains valid in the case that the sequence of the polynomials in the theorem is replaced by a generalized Sturm sequence since the multiplication of all the polynomials by one and the same polynomial alters neither the left-hand nor the right-hand side of (7.9) [Gan59, p. 176].

7.3. Routh and Hurwitz Criteria

In this section the Routh and Routh-Hurwitz criteria are presented and the relationship between them is investigated. First of all, we present the Routh scheme which can be derived by using the same approach that has been used in deriving the generalized Sturm sequence. After that we give the Routh's Theorem which can be obtained by using Sturm's Theorem. Routh's Theorem is used to determine the number of roots of a real polynomial that lie in the open right half plane of the complex plane. Next we introduce the Hurwitz matrix and the Routh-Hurwitz's Criterion and its relationship to the Routh's Criterion. We also present the Liénard-Chipart's Criteria that are considered as an improvement of the Routh-Hurwitz's Criterion. Finally, we introduce some matrices which can be regarded as generalizations of the Hurwitz matrix. Some properties of the rational functions associated with this kind of matrices will be investigated in the next chapter.

In many application areas, where stability plays an important role, one is faced by the problem to decide whether a given real polynomial is Hurwitz or not.

It is known, see, e.g., [Gan59, p. 178], that for a given real polynomial

$$f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n, \quad a_0 \neq 0, \quad (7.10)$$

having k and $n - k$ roots with positive and negative real parts, respectively, the following relation holds:

$$\text{Ind}_{-\infty}^{+\infty} \frac{f_1(w)}{f_0(w)} = n - 2k, \quad (7.11)$$

where

$$\begin{aligned} f_0(w) &= a_0 w^n - a_2 w^{n-2} + a_4 w^{n-4} - \dots := r_{0,1} w^n - r_{0,2} w^{n-2} + r_{0,3} w^{n-4} - \dots, \\ f_1(w) &= a_1 w^{n-1} - a_3 w^{n-3} + a_5 w^{n-5} - \dots := r_{1,1} w^{n-1} - r_{1,2} w^{n-3} + r_{1,3} w^{n-5} - \dots, \\ f(iw) &= \begin{cases} (-1)^{\frac{n}{2}} f_0(w) + i(-1)^{\frac{n}{2}-1} f_1(w) & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}} f_1(w) + i(-1)^{\frac{n-1}{2}} f_0(w) & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

In order to determine the Cauchy index on the left-hand side of (7.11), the generalized Sturm sequence formed from $f_0(w)$ and $f_1(w)$,

$$(f_0(w), f_1(w), \dots, f_m(w)), \quad (7.12)$$

is used.

In the case $m = n$, called the *regular case*, the degree of each function in (7.12) is one less than that of the preceding one, and the last function $f_m(w)$ is of degree zero, i.e., $\deg f_j(w) = n - j$, $j = 0, 1, \dots, n$, and $q_i(w) = \frac{r_{i,1}}{r_{i+1,1}} w$ for some constants $r_{i,1}$, $i = 0, 1, \dots, n$. We put the coefficients of each polynomial in the following scheme which is called the *Routh scheme* [Gan59, p. 179]:

$$\begin{array}{ccccccc} f_0(w) : & r_{0,1} & r_{0,2} & r_{0,3} & \dots & & \\ f_1(w) : & r_{1,1} & r_{1,2} & r_{1,3} & \dots & & \\ f_2(w) : & r_{2,1} & r_{2,2} & r_{2,3} & \dots & & \\ f_3(w) : & r_{3,1} & r_{3,2} & r_{3,3} & \dots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array} \quad (7.13)$$

where every row is determined by the two preceding rows according to the following rule: From the entries of the upper row subtract the corresponding entries of the lower row multiplied by the number that makes the first difference zero. Omitting this zero difference, we obtain the required row by

$$r_{i,j} = r_{i-2,j+1} - \frac{r_{i-2,1}}{r_{i-1,1}} r_{i-1,j+1}, \quad i = 2, \dots, n, \quad j = 1, \dots, \left\lceil \frac{n+1-i}{2} \right\rceil. \quad (7.14)$$

In the regular case there is no drop of the degree of the polynomials generated by the above procedure by more than 1, hence repeated application of this procedure never yields a zero in the first column of the Routh scheme, i.e., in the sequence

$$(r_{0,1}, r_{1,1}, r_{2,1}, r_{3,1}, \dots, r_{n,1}). \quad (7.15)$$

The method based on Sturm sequences requires repeated polynomial divisions which are inconvenient for computer implementation. Furthermore, certain non-systematized additional analysis is needed when polynomials which are to be divided are not relatively prime. In contrast, the Routh scheme is executed by a simple recursive table.

By using (7.11), Theorem 7.1, and Lemma 2.11 we have the following important theorem.

Theorem 7.2. [Gan59, Theorem 2 (Routh's Theorem), p. 180] *The number of roots of the real polynomial $f(z)$ given by (7.10) in the open right half of the complex plane is equal to the number of variations of sign changes in the first column of Routh scheme, i.e., $k = S^-(r_{0,1}, r_{1,1}, \dots, r_{n,1})$.*

By using the above theorem we obtain the following criterion which is known as *Routh's Criterion*.

Theorem 7.3. [Gan59, Routh's Criterion, p. 180] *A real polynomial $f(z)$ defined by (7.10) is Hurwitz if and only if in carrying out of the Routh scheme all the elements of the first column of the Routh scheme are different from zero and of like sign.*

In some cases such as $f_0(w)$ and $f_1(w)$ are not relatively prime, the Routh scheme fails due to division by zero. This case is referred to as *singular case*, see, e.g., [Gan59], [BP90], [Mei95].

Remark 7.2. [Gan59, p. 180] *The Routh scheme, Theorem 7.2, and Routh's Criterion are derived under the assumption that $f_0(z)$ and $f_1(z)$ are relatively prime, i.e., $f(z)$ does not have any root on the imaginary axis. Otherwise, formula (7.11) has to be replaced by*

$$\text{Ind}_{-\infty}^{+\infty} \frac{f_1(w)}{f_0(w)} = n - 2k - s, \quad (7.16)$$

where s equals to the number of pure imaginary roots of f .

Remark 7.3. [Gan59, p. 181] *The regular case takes place in the Routh scheme when none of the numbers in the first column of the array vanishes. The singular case arises when one of the following situations occurs:*

- (a) *An entry in the first column becomes zero but not all of the numbers in the corresponding row are zero; or*
- (b) *all the numbers in a row of the array vanish simultaneously.*

In case (a) the given polynomial has some zeros in the right half plane; the Routh scheme is extended in several ways, like the ϵ -method [Gan59], and the Extended Routh's Table [BP90], [Mei95] to cover this case. In case (b) the procedure is modified in the following way. If row $j + 1$ becomes zero, i.e., $f_j(w) = 0$, then $f_{j-1}(w)$ equals to the greatest common divisor of $f_0(w)$ and $f_1(w)$. Hence by (7.16) the s roots of $f(z)$ on the imaginary axis coincide with the real roots of $f_{j-1}(w)$. Thus by (7.5) if these s real roots of $f_{j-1}(w)$ are simple, then we have

$$\text{Ind}_{-\infty}^{+\infty} \frac{f'_{j-1}(w)}{f_{j-1}(w)} = s, \quad (7.17)$$

and so

$$\text{Ind}_{-\infty}^{+\infty} \frac{f_0(w)}{f_1(w)} + \text{Ind}_{-\infty}^{+\infty} \frac{f'_{j-1}(w)}{f_{j-1}(w)} = n - 2k. \quad (7.18)$$

Hence in this case the problem is solved by replacing row $j + 1$ by the coefficients of the derivative of $f_{j-1}(w)$ and then continuing the scheme [Gan59]. If in a following step this situation occurs again the above procedure is applied again.

For a polynomial $f(z)$ given by (7.10), the *Hurwitz matrix* $H(f) = (h_{ij})_{i,j=1}^n$ associated with f is defined by

$$h_{ij} := \begin{cases} a_{2j-i} & \text{for } 0 \leq 2j - i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $H(f)$ is given by

$$H(f) = \begin{bmatrix} a_1 & a_3 & a_5 & a_7 & \dots & 0 & 0 \\ a_0 & a_2 & a_4 & a_6 & \dots & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & \dots & 0 & 0 \\ 0 & a_0 & a_2 & a_4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & a_{n-2} & a_n \end{bmatrix}. \quad (7.19)$$

The *infinite Hurwitz matrix* is defined by $H_\infty(f) = (h_{ij})_{i,j=1}^\infty$, where $h_{ij} = a_{2j-i}$, $i, j = 1, 2, \dots$, with the convention that $a_k := 0$ for $k < 0$ and $n < k$.

It is easy to show that [Gan59, p. 193], [Bar71]

$$r_{11} = \Delta_1, \quad r_{21} = \frac{\Delta_2}{\Delta_1}, \quad r_{31} = \frac{\Delta_3}{\Delta_2}, \quad \dots, \quad r_{n1} = \frac{\Delta_n}{\Delta_{n-1}}, \quad (7.20)$$

where Δ_i is the leading principal minor of order i of the Hurwitz matrix (7.19). These minors are called the *Hurwitz determinants*.

By using Theorem 7.2 and (7.20) we have the following version of Theorem 7.2.

Theorem 7.4. [Gan59, Theorem 4, (Routh-Hurwitz's Theorem), p. 194] *The number of roots, k say, of the real polynomial $f(z)$ given by (7.10) in the open right half plane of the complex plane is determined by the formula*

$$k = S^- \left(a_0, \Delta_1, \frac{\Delta_2}{\Delta_1}, \frac{\Delta_3}{\Delta_2}, \dots, \frac{\Delta_n}{\Delta_{n-1}} \right). \quad (7.21)$$

or equivalently by

$$k = S^- (a_0, \Delta_1, \Delta_3, \dots) + S^- (1, \Delta_2, \Delta_4, \dots). \quad (7.22)$$

Remark 7.4. *In the Routh-Hurwitz's Theorem the regular case is assumed:*

$$\Delta_1 \neq 0, \Delta_2 \neq 0, \dots, \Delta_n \neq 0. \quad (7.23)$$

By using the Routh-Hurwitz's Theorem and considering the case that all of the zeros of the polynomial f given by (7.10) are in the open left half plane we obtain the following criterion.

Theorem 7.5. [Gan59, Criterion of Routh-Hurwitz, p. 194] *All the roots of the real polynomial $f(z)$ defined by (7.10) have negative real parts if and only if the following inequalities hold:*

$$0 < a_0 \Delta_1, 0 < \Delta_2, 0 < a_0 \Delta_3, 0 < \Delta_4, \dots, \begin{array}{ll} 0 < a_0 \Delta_n & \text{for odd } n, \\ 0 < \Delta_n & \text{for even } n. \end{array} \quad (7.24)$$

The following formula relates the Hurwitz determinant Δ_{n-1} of the given polynomial $f(z)$ defined by (7.10) to its leading coefficient and zeros.

Orlando's Formula [Gan59, p. 196]

$$\Delta_{n-1} = (-1)^{\frac{n(n-1)}{2}} a_0^{n-1} \prod_{1 \leq i < k \leq n} (z_i + z_k), \quad (7.25)$$

where $z_i, i = 1, \dots, n$, are the zeros of $f(z)$.

Orlando's formula indicates that $\Delta_{n-1} = 0$ if and only if $f(z)$ has at least two zeros symmetric with respect to the origin, i.e., the given real polynomial has some zeros with at least one of the following properties:

- (i) at least two of its zeros are pure imaginary,
- (ii) at least two of its zeros are real, of equal magnitude, and of opposite sign,
- (iii) at least four of its zeros are complex and located at the vertices of some rectangle whose center is the origin.

Hence the condition $\Delta_{n-1} \neq 0$ is necessary but not sufficient for a given real polynomial to be Hurwitz.

The Routh-Hurwitz's Criterion is valid in the regular case. The following theorem is an extension of this criterion provided that $\Delta_n \neq 0$.

Theorem 7.6. [Gan59, Theorem 5, p. 201] *If some of the Hurwitz determinants are zero, but $\Delta_n \neq 0$, then the number of roots of the real polynomial $f(z)$ given by (7.10) in the open right half plane is determined by the formula*

$$k = V \left(a_0, \Delta_1, \frac{\Delta_2}{\Delta_1}, \frac{\Delta_3}{\Delta_2}, \dots, \frac{\Delta_n}{\Delta_{n-1}} \right) \quad (7.26)$$

in which for the calculation of the value of V for every group of p successive zero determinants

$$\Delta_s \neq 0, \Delta_{s+1} = \dots = \Delta_{s+p} = 0, \Delta_{s+p+1} \neq 0 \quad (7.27)$$

we have to set

$$V \left(\frac{\Delta_s}{\Delta_{s-1}}, \frac{\Delta_{s+1}}{\Delta_s}, \dots, \frac{\Delta_{s+p+2}}{\Delta_{s+p+1}} \right) = h + \frac{1 - (-1)^h \epsilon}{2}, \quad (7.28)$$

where

$$p = 2h - 1, \quad \epsilon = \text{sign} \left(\frac{\Delta_s}{\Delta_{s-1}} \frac{\Delta_{s+p+2}}{\Delta_{s+p+1}} \right), \quad (7.29)$$

$\Delta_0 := a_0$, and $\Delta_{-1} := 1$.

The following theorem provides other criteria for a given real polynomial to be Hurwitz. It shows that about the half of the minors in the Routh-Hurwitz's Criterion are superfluous under the assumption of positivity of certain coefficients of f defined by (7.10).

Theorem 7.7. [LC14], [Gan59, Theorem 11, (Liénard-Chipart Criteria), p. 221] *Let f given by (7.10) be a real polynomial with $0 < a_0$. Then f is stable if and only if one of the following sets of inequalities is fulfilled:*

- 1) $0 < a_n, 0 < a_{n-2}, \dots; 0 < \Delta_1, 0 < \Delta_3, \dots,$
- 2) $0 < a_n, 0 < a_{n-2}, \dots; 0 < \Delta_2, 0 < \Delta_4, \dots,$
- 3) $0 < a_n; 0 < a_{n-1}, 0 < a_{n-3}, \dots; 0 < \Delta_1, 0 < \Delta_3, \dots,$
- 4) $0 < a_n; 0 < a_{n-1}, 0 < a_{n-3}, \dots; 0 < \Delta_2, 0 < \Delta_4, \dots.$

In, e.g., [Gan59], [Bar08], [HT12] some of the above results are extended by replacing $f_0(z)$ and $f_1(z)$ by arbitrary polynomials. We conclude this section by presenting some of

these extensions, see [HT12].

Consider the following rational function

$$R(z) := \frac{q(z)}{p(z)}, \quad (7.30)$$

where $q(z)$ and $p(z)$ are polynomials with real coefficients

$$p(z) := a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n, \quad a_0 \neq 0, \quad (7.31)$$

$$q(z) := b_0 z^n + b_1 z^{n-1} + \dots + b_{n-1} z + b_n. \quad (7.32)$$

Let $g(z)$ be the greatest common divisor of $p(z)$ and $q(z)$. If $\deg g(z) = l$, $0 \leq l \leq n$, then the rational function $R(z)$ has exactly $r := n - l$ poles counting multiplicities.

Define the (in)finite matrices of Hurwitz type as follows depending on the case whether b_0 vanishes or not. We set $a_k := 0$ and $b_k := 0$ for $k > n$.

If $\deg q < \deg p$, that is, if $b_0 = 0$, then

$$H(p, q) := \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \dots \\ 0 & b_1 & b_2 & b_3 & b_4 & b_5 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ 0 & 0 & b_1 & b_2 & b_3 & b_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (7.33)$$

$$H_{2n}(p, q) := H[2, \dots, 2n + 1]; \quad (7.34)$$

if $\deg q = \deg p$, that is, $b_0 \neq 0$, then

$$H(p, q) := \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 & \dots \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (7.35)$$

$$H_{2n}(p, q) := H[2, \dots, 2n + 2]. \quad (7.36)$$

The matrices $H(p, q)$ and $H_{2n}(p, q)$ are called *infinite* and *finite matrices of Hurwitz type*, respectively. We denote by $\nabla_j(p, q)$ the leading principal minor of $H(p, q)$ of order j , $j = 1, 2, \dots$.

Remark 7.5. *It can be easily seen that $H(p, q)(1) = H_\infty(f(z))$, where*

$$f(z) := p(z^2) + zq(z^2).$$

Let p and q be given by (7.31)-(7.32) with $b_0 \neq 0$. Then the following determinant of order $2n$ is called the *resultant of p and q* . The resultant is very useful when we relate the zeros of the both polynomials as shown in the next theorem.

$$R(p, q) := \begin{vmatrix} a_0 & a_1 & \dots & a_{n-1} & a_n & \dots & a_{2n-1} \\ 0 & a_0 & \dots & a_{n-2} & a_{n-1} & \dots & a_{2n-2} \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & a_0 & a_1 & \dots & a_n \\ b_0 & b_1 & \dots & b_{n-1} & b_n & \dots & b_{2n-1} \\ 0 & b_0 & \dots & b_{n-2} & b_{n-1} & \dots & b_{2n-2} \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & b_0 & b_1 & \dots & b_n \end{vmatrix}, \quad (7.37)$$

where $a_i := 0$ and $b_i := 0$ for all $n < i$.

Theorem 7.8. [HT12, Theorem 1.9] *Given two polynomials p and q as in (7.31)-(7.32) with $b_0 \neq 0$, let λ_i and μ_i , $i = 1, \dots, n$, denote the zeros of p and q , respectively. Then*

$$\begin{aligned} R(p, q) &= (-1)^{\frac{n(n-1)}{2}} \frac{1}{b_0} \nabla_{2n+1}(p, q) = a_0^n \prod_{i=1}^n q(\lambda_i) = a_0^n b_0^n \prod_{i,j=1}^n (\lambda_i - \mu_j) \quad (7.38) \\ &= (-1)^n b_0^n \prod_{j=1}^n p(\mu_j). \end{aligned}$$

The following corollary easily follows from Theorem 7.8.

Corollary 7.1. [HT12, Corollary 1.11] *$R(p, q) = 0$ if and only if the polynomials p and q have common roots.*

When the two polynomials given by (7.31)-(7.32) are not relatively prime, the infinite matrix of Hurwitz type $H(p, q)$ can be written as product of two special matrices.

Theorem 7.9. [HT12, Theorem 1.43] *Let $p(z) := \hat{p}(z)g(z)$ and $q(z) := \hat{q}(z)g(z)$ be two polynomials given by (7.31)-(7.32) with the greatest common divisor $g(z) := c_0 z^l + c_1 z^{l-1} + \dots + c_l$. Then*

$$H(p, q) = H(\hat{p}, \hat{q})T(g), \quad (7.39)$$

where $T(g)$ is the infinite upper triangular Toeplitz matrix formed from the coefficients of the polynomial g :

$$T(g) := \begin{bmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & \dots \\ 0 & c_0 & c_1 & c_2 & c_3 & \dots \\ 0 & 0 & c_0 & c_1 & c_2 & \dots \\ 0 & 0 & 0 & c_0 & c_1 & \dots \\ 0 & 0 & 0 & 0 & c_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (7.40)$$

where $c_i := 0$ for all $l < i$.

In the next chapter we will study these types of matrices with respect to the total non-negativity and the location of the zeros of p and q .

Total nonnegativity of $T(p)$ of a given polynomial p and the location of its zeros are related as the following theorem states.

Theorem 7.10. [Pin10, Theorem 4.9] *Let p be given as in (7.31) with $0 < a_0$ and let $T(p)$ be the infinite upper triangular Toeplitz matrix associated to p defined by (7.40). Then the following two statements are equivalent:*

- (i) $T(p)$ is TN.
- (ii) p has n nonpositive zeros.

7.4. Hankel Matrices

In this section another special matrix that is associated with a given rational function is considered, and relationships between the number of its poles and the rank of this matrix and some of its minors are introduced.

Let (s_0, s_1, s_2, \dots) be an infinite sequence. Then the *infinite Hankel* matrix is defined as

$$S := (s_{i+j-2})_{i,j=1}^{\infty} \quad (7.41)$$

The leading principal minors of the matrix S play a fundamental role in the study of its rank and the number and quality of poles of a related rational function as we will see later.

Definition 7.5. [HT12, Definition 1.1] *Let S be an infinite Hankel matrix defined by (7.41). Then The following determinants:*

$$D_j(S) := \det S[1, \dots, j] = \begin{vmatrix} s_0 & s_1 & s_2 & \dots & s_{j-1} \\ s_1 & s_2 & s_3 & \dots & s_j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{j-1} & s_j & s_{j+1} & \dots & s_{2j-2} \end{vmatrix}, \quad j = 1, 2, \dots, \quad (7.42)$$

i.e., the leading principal minors of the infinite Hankel matrix S are called the Hankel minors or Hankel determinants.

We will make also use of the leading principal minors of the matrix obtained from S by deleting its first column:

$$\hat{D}_j(S) := \det S[1, \dots, j|2, \dots, j+1] = \begin{vmatrix} s_1 & s_2 & s_3 & \dots & s_j \\ s_2 & s_3 & s_4 & \dots & s_{j+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_j & s_{j+1} & s_{j+2} & \dots & s_{2j-1} \end{vmatrix}, \quad (7.43)$$

$j = 1, 2, \dots$

The following two theorems relate the rank of the matrix S to its leading principal minors $D_j(S)$ given by (7.42) and to linear combinations of its rows.

Theorem 7.11. [Kro81], [HT12, Theorem 1.2] *An infinite Hankel matrix $S = (s_{i+j-2})_{i,j=1}^{\infty}$ has the finite rank r if and only if*

$$D_r(S) \neq 0, \quad \text{and} \quad D_j(S) = 0, \quad \text{for all } r < j. \quad (7.44)$$

Theorem 7.12. [Gan59, Theorem 7, p. 205] *An infinite Hankel matrix $S = (s_{i+j-2})_{i,j=1}^{\infty}$ has the finite rank r if and only if the row $r + 1$ can be written as a linear combination of the previous r rows and r is the least number having this property.*

By expanding R given by (7.30) into its Laurent series at ∞ , we obtain

$$R(z) = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \dots \quad (7.45)$$

We associate with the sequence of the coefficients of negative powers of z of $R(z)$:

$$(s_0, s_1, s_2, \dots), \quad (7.46)$$

the infinite Hankel matrix $S(R) := S$.

The next theorem provides the relationship between the rank of the Hankel matrix and the number of the poles of $R(z)$ given by (7.30).

Theorem 7.13. [Kro81], [HT12, Theorem 1.3] *An infinite Hankel matrix $S = (s_{i+j-2})_{i,j=1}^{\infty}$ has finite rank if and only if the series*

$$R(z) = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \dots$$

represents a rational function of z . In this case the rank of the matrix S is equal to the number of poles of the function R .

Hence the sequence given by (7.46) which is associated to a given rational function R corresponds to an infinite Hankel matrix of finite rank.

From minors of S one can decide whether the rational function R given by (7.30) has a pole at zero or not as the following corollary states.

Corollary 7.2. [HT12, Corollary 1.4] *A rational function R with exactly r poles represented by the series (7.45) has a pole at the point 0 if and only if*

$$\hat{D}_{r-1}(S(R)) \neq 0 \quad \text{and} \quad \hat{D}_j(S(R)) = 0 \quad \text{for all } j = r, r + 1, \dots \quad (7.47)$$

The signs of the minors given in (7.42)-(7.43) are very important for the study of the location of the zeros and poles of a given rational function. If some of the minors in (7.42) are zero then we assign to them signs according to the following rule.

Remark 7.6. [HT12, Rule 2.4] *If, for some integers i and j ($0 \leq i, 1 \leq j$),*

$$D_i(S(R)) \neq 0, D_{i+1}(S(R)) = D_{i+2}(S(R)) = \dots = D_{i+j}(S(R)) = 0, D_{i+j+1}(S(R)) \neq 0,$$

then the number $V^F(D_0(S(R)), D_1(S(R)), D_2(S(R)), \dots, D_r(S(R)))$ of Frobenius sign changes should be calculated by assigning signs to the vanishing minors as follows:

$$\text{sign } D_{i+\nu}(S(R)) = (-1)^{\frac{\nu(\nu-1)}{2}} \text{sign } D_i(S(R)), \quad \nu = 1, 2, \dots, j, \quad (7.48)$$

where $D_0(S(R)) := 1$.

We conclude this section by considering a special case of $R(z)$. Let $L(z)$ be the *logarithmic derivative* of $p(z)$ given by (7.31), i.e.,

$$L(z) := \frac{d \log(p(z))}{dz} = \frac{p'(z)}{p(z)} = \frac{na_0z^{n-1} + (n-1)a_1z^{n-2} + \dots + a_{n-1}}{a_0z^n + a_1z^{n-1} + \dots + a_n}. \quad (7.49)$$

If

$$p(z) = a_0(z - z_1)^{n_1}(z - z_2)^{n_2} \dots (z - z_t)^{n_t}, \quad n_1 + n_2 + \dots + n_t = n, \quad (7.50)$$

where n_j is the multiplicity of the zero z_j , $j = 1, \dots, t$, then the logarithmic derivative of the polynomial p has the following form:

$$L(z) = \sum_{j=1}^t \frac{n_j}{z - z_j}. \quad (7.51)$$

By expanding the function $L(z)$ into its Laurent series at ∞ , we obtain

$$L(z) = \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \frac{s_3}{z^4} + \dots. \quad (7.52)$$

It is easy to see that, see, e.g., [Gan59, p. 202], the coefficients s_k , $k = 0, 1, \dots$, are the *Newton sums* of the polynomial $p(z)$;

$$s_k = \sum_{j=1}^t n_j z_j^k, \quad k = 0, 1, \dots. \quad (7.53)$$

By (7.50) and (7.51) each zero of $p(z)$ appears as a pole of the function $L(z)$ with multiplicity one. Since each pole is simple and $0 < n_j$, we obtain by (7.5)

$$\text{Ind}_{-\infty}^{+\infty}(L) = \text{number of distinct real zeros of } p(z). \quad (7.54)$$

Moreover by (7.50), (7.51), and Theorem 7.13, the rank of the Hankel matrix $S(L)$ is equal to t . Hence (7.54) can be rewritten in the following form:

$$\text{Ind}_{-\infty}^{+\infty}(L) = t - \text{number of distinct non-real zeros of } p(z). \quad (7.55)$$

Let r , r^+ , and r^- denote the number of distinct real, distinct positive, and distinct negative zeros, respectively, of the polynomial p given by (7.31). A consequence of the connections between the quadratic forms and rank and signature of the Hankel matrix $S(L)$ [Gan59] is the following theorem.

Theorem 7.14. [HT12, Theorem 4.4] *Let the numbers k and l be defined as follows*
 $k = V(1, D_1(S(L)), D_2(S(L)), \dots, D_t(S(L))), \quad l = V(1, \hat{D}_1(S(L)), \hat{D}_2(S(L)), \dots, \hat{D}_t(S(L))).$
Then the number of distinct pairs of non-real zeros of the polynomial p equals k and

$$\begin{aligned} r &= t - 2k; \\ r^- &= l - k; \\ r^+ &= \begin{cases} t - k - l & \text{if } p(0) \neq 0, \\ t - k - l - 1 & \text{if } p(0) = 0. \end{cases} \end{aligned}$$

The next two theorems represent two special extremal cases of Theorem 7.14.

Theorem 7.15. [HT12, Theorem 4.10] *The polynomial p given by (7.31) has exactly t distinct zeros, all of which are real, if and only if*

$$\begin{aligned} 0 < D_j(S(L)), \quad j = 1, 2, \dots, t, \\ D_j(S(L)) = 0, \quad t < j. \end{aligned} \tag{7.56}$$

Theorem 7.16. [HT12, Theorem 4.11] *Let the polynomial p given by (7.31) has exactly t distinct zeros all of which are real. Then the number of its distinct negative zeros equals $V(1, \hat{D}_1(S(L)), \hat{D}_2(S(L)), \dots, \hat{D}_t(S(L)))$. Moreover,*

$$\hat{D}_j(S(L)) = 0, \quad t < j,$$

and

$$\begin{cases} \hat{D}_t(S(L)) \neq 0 & \text{if } p(0) \neq 0, \\ \hat{D}_t(S(L)) = 0, \text{ and } \hat{D}_{t-1}(S(L)) \neq 0 & \text{if } p(0) = 0. \end{cases}$$

All of the above theorems can be restated in terms of the leading principal minors ∇_j of $H(p, q)$ by using the following theorem.

Theorem 7.17. [HT12, Theorem 1.45] *Let $R(z), p(z)$, and $q(z)$ defined by (7.45), (7.31)-(7.32). Then the following relations hold between the the minors $D_j(S(R)), \hat{D}_j(S(R))$, and ∇_j :*

(i) *If $\deg q < \deg p$, then*

$$\begin{aligned} \nabla_{2j} &= a_0^{2j} D_j(S(R)), \quad j = 1, \dots, n, \\ \nabla_{2j+1} &= (-1)^j a_0^{2j+1} \hat{D}_j(S(R)), \quad j = 1, \dots, n. \end{aligned}$$

(ii) *If $\deg q = \deg p$, then*

$$\begin{aligned} \nabla_{2j+1} &= b_0 a_0^{2j} D_j(S(R)), \quad j = 1, \dots, n, \\ \nabla_{2j+2} &= (-1)^j b_0 a_0^{2j+1} \hat{D}_j(S(R)), \quad j = 0, 1, \dots, n, \end{aligned}$$

where $\hat{D}_0(S(R)) := 1$.

8. Total Nonnegativity and Stability

The behavior of polynomials and rational functions can be analyzed by using algebraic constructions involving their coefficients, see, e.g., [Hur95], [Gan59], [Asn70], [Dat76], [Kem82], [Bar08], [BT11], [HT12]. In this chapter we relate the theory of total nonnegativity to the stable polynomials and R -functions of negative type. We derive some properties that the Hurwitz matrices, matrices of Hurwitz type, and Hankel matrices associated with stable polynomials and R -functions of negative type enjoy. Conversely, from the total nonnegativity of these structured matrices several properties of polynomials and rational functions associated with these matrices can be inferred.

The organization of this chapter is as follows. In Section 8.1, the total nonnegativity of infinite Hankel matrices and the almost total positivity of infinite matrices of Hurwitz type as well as of (in)finite Hurwitz matrices are proved under certain conditions. In Section 8.2, R -functions of negative type are considered and related interval problems are investigated. In Section 8.3, the region of stability is introduced and a new and simple proof of the well-known Markov Theorem and a sufficient condition for the stability of an interval polynomial are given.

8.1. Total Nonnegativity of Hankel and Hurwitz Matrices and Stability of Polynomials

In this section we present some characterizations for the total nonnegativity of Hankel matrices and matrices of Hurwitz type, and give, for some known results, uniform and short proofs based on the application of the Cauchon Algorithm, Proposition 3.5, and Theorems 3.2, 3.11, and 3.18.

We start with the infinite Hankel matrix

$$S = (s_{i+j-2})_{i,j=1}^{\infty}, \quad (8.1)$$

where s_i , $i = 0, 1, \dots$, are given real numbers. The following theorem characterizes the total nonnegativity of S (for the equivalence of (i) \Leftrightarrow (ii) see Theorem 4.4 and the references on p. 125 in [Pin10]).

Theorem 8.1. [AGT16, Theorem 3.1] *Let S be a real infinite Hankel matrix of rank n . Furthermore, let $1 < n$ and $0 \leq \det S[1, \dots, n|2, \dots, n+1]$. Put $A := S[1, \dots, n]$ and $B := A[1, \dots, n-1|2, \dots, n]$. Then the following three statements are equivalent:*

- (i) *The matrices A and B are positive definite.*

(ii) The finite Hankel matrix A is TP .

(iii) S is TN .

Proof. (i) \Rightarrow (ii) We first note that any principal minor of A and B is positive since A and B are positive definite matrices. According to Theorem 3.2 (ii) it is sufficient to show that \tilde{A} is positive. The entries in the last row and column of \tilde{A} coincide with the respective entries in the last row and column of A and are positive since they appear on the main diagonal of A or B . Now we turn to the remaining entries of \tilde{A} and assume that there are entries which are not positive. Let (i_0, j_0) be the maximum element of the set $\{1, \dots, n\}^2$ with respect to the lexicographical order such that $\tilde{a}_{i_0, j_0} \leq 0$. We add to \tilde{a}_{i_0, j_0} a sufficiently large positive number t to make $\tilde{a}_{i_0, j_0} + t$ positive and call D the matrix which is obtained from $A[i_0, \dots, n | j_0, \dots, n]$ in this way. Then \tilde{D} is positive and the sequence γ given by (3.12) starting at position (i_0, j_0) which is found by Procedure 3.2 is diagonal. Note that \tilde{d}_{11} is identical with the entry in position $(1, 1)$ of the matrix which is obtained by the application of the Cauchon Algorithm to the matrix which results from $A[i_0, \dots, n | j_0, \dots, n]$ when adding t to a_{i_0, j_0} . Use of Proposition 3.4 and Laplace expansion applied to the minor of D associated to γ by (3.13) yields

$$\det A[i_0, \dots, i_p | j_0, \dots, j_p] + t \det A[i_1, \dots, i_p | j_1, \dots, j_p] = \tilde{a}_{i_0, j_0} \cdot \tilde{a}_{i_1, j_1} \cdots \tilde{a}_{i_p, j_p} + t \tilde{a}_{i_1, j_1} \cdots \tilde{a}_{i_p, j_p}.$$

A further application of Proposition 3.4 gives

$$\det A[i_1, \dots, i_p | j_1, \dots, j_p] = \tilde{a}_{i_1, j_1} \cdots \tilde{a}_{i_p, j_p},$$

whence

$$\det A[i_0, \dots, i_p | j_0, \dots, j_p] = \tilde{a}_{i_0, j_0} \cdot \tilde{a}_{i_1, j_1} \cdots \tilde{a}_{i_p, j_p}. \quad (8.2)$$

By the special pattern of the entries of A the minor on the left-hand side of the equality given in (8.2) is a principal minor of A or B hence it is positive. On the other hand, the product of the right-hand side of the equation (8.2) is nonpositive by our assumption and we have arrived at a contradiction.

(ii) \Rightarrow (iii) Let A be TP and let $A_\mu := S[1, \dots, \mu]$, $\mu = n, n+1, \dots$.

Claim: A_μ is TN and $S[1, \dots, n-1 | \mu-n+2, \dots, \mu]$ is nonsingular for each $\mu = n, n+1, \dots$.

Proof of the claim: The proof proceeds by induction. For $\mu = n$ the claim holds since $A = A_n$ is TP . Suppose the claim holds for μ . We want to show that the claim holds for $\mu + 1$. First of all we prove that $S[1, \dots, n-1 | \mu-n+3, \dots, \mu+1]$ is nonsingular. If it is singular, then let l be the smallest integer less than or equal to $n-1$ such that $S[1, \dots, l | \mu-n+3, \dots, \mu-n+l+2]$ is singular. By the special pattern of S we have

$$S[1, \dots, l | \mu-n+3, \dots, \mu-n+l+2] = S[2, \dots, l+1 | \mu-n+2, \dots, \mu-n+l+1],$$

where the latter submatrix is a submatrix in A_μ . Since by the induction hypothesis A_μ is TN we conclude by Proposition 2.8 that either $S[2, \dots, l+1|1, \dots, l]$ or $S[1, \dots, l|\mu-n+2, \dots, \mu-n+l+1]$ is singular. In either case we have a contradiction since A is TP and by the induction hypothesis $S[1, \dots, n-1|\mu-n+2, \dots, \mu]$ is $NsTN$. If $\det S[1, \dots, n|2, \dots, n+1]$ is positive then by proceeding as above we show that $S[1, \dots, n|\mu-n+2, \dots, \mu+1]$ is nonsingular for each $\mu = n, n+1, \dots$ provided that A_μ is TN . It remains to prove that $A_{\mu+1}$ is TN . We distinguish the following two cases:

Case 1: $\det S[1, \dots, n|2, \dots, n+1]$ is positive.

Let $C \in \mathcal{C}_{\mu+1}$ be defined as $(i, j) \in C$ if and only if $(i, j) \in \{1, \dots, \mu-n+1\}^2$ and let $E := A_{\mu+1}^\#$. We want to show that $E \in \mathcal{S}_C^0$. For each position $(i_0, j_0) \in \{1, \dots, \mu+1\}^2$ fix a lacunary sequence given by (3.12) with respect to C as follows: If $\mu-n+2 \leq i_0$ or $\mu-n+2 \leq j_0$, then $i_{k+1} = i_k + 1$ and $j_{k+1} = j_k + 1$ for $k = 0, 1, \dots, p-1$. Therefore since A is TP and A_μ is TN and hence $S[1, \dots, n|v-n+2, \dots, v+1]$ is $NsTN$ for each $v = n, n+1, \dots, \mu+1$ we obtain by Lemma 2.6 that the minors associated with these lacunary sequences are positive. If $i_0 \leq \mu-n+1$ and $j_0 \leq \mu-n+1$, then set $(i_1, j_1) := (\mu-n+2, \mu-n+2)$ and $i_{k+1} = i_k + 1$, $j_{k+1} = j_k + 1$ for $k = 1, \dots, p-1$. It is easy to see that these sequences are lacunary sequences with respect to C . The length of each such sequence is n and so the minors associated with these sequences are of order $n+1$. Hence the minors associated with such sequences are zero since rank of S is n . Thus by Theorem 3.4 E and hence $A_{\mu+1}$ are TN .

Case 2: $\det S[1, \dots, n|2, \dots, n+1]$ is zero.

We first show that the $(n+1)^{th}$ row (column) can be written as a linear combination of the 2^{nd} , 3^{rd} , \dots , and n^{th} rows (columns). By Theorem 7.12, $(n+1)^{th}$ row can be written as a linear combination of the 1^{st} , 2^{nd} , \dots , and n^{th} rows, i.e., if $\mathbf{R}^1, \mathbf{R}^2, \dots, \mathbf{R}^{n+1}$ represent the first $n+1$ rows of S , then there exists $r_1, r_2, \dots, r_n \in \mathbb{R}$ such that

$$\mathbf{R}^{n+1} = \sum_{i=1}^n r_i \mathbf{R}^i. \quad (8.3)$$

By (8.3) and determinantal properties we obtain

$$\det S[1, \dots, n|2, \dots, n+1] = \det S[2, \dots, n+1|1, \dots, n] = (-1)^{n-1} r_1 \det S[1, \dots, n],$$

by which $r_1 = 0$ since $0 < \det S[1, \dots, n]$ and $\det S[1, \dots, n|2, \dots, n+1] = 0$. Let $C \in \mathcal{C}_{\mu+1}$ be defined as $(i, j) \in C$ if and only if $(i, j) \in \{1, \dots, \mu-n+1\}^2$ or $i = \mu-n+2$, $j = 1, \dots, \mu-n+1$ or $i = 1, \dots, \mu-n+1$, $j = \mu-n+2$ and let $E := A_{\mu+1}^\#$. We want to show that $E \in \mathcal{S}_C^0$. For each position $(i_0, j_0) \in \{1, \dots, \mu+1\}^2$ define a lacunary sequence given by (3.12) as in Case 1. Then it is easy to see that all of these sequences are lacunary with respect to C . The minors that are associated with the lacunary sequences that start from (i_0, j_0) such that $(i_0, j_0) \notin C$ are positive since A is TP and A_μ is TN and hence $S[1, \dots, n-1|v-n+2, \dots, v]$ is $NsTN$ for each $v = n, n+1, \dots, \mu+1$. The minors associated with the lacunary sequences that start from the positions (i_0, j_0) such that $i = \mu-n+2$ and $j = 1, \dots, \mu-n+1$ or $i = 1, \dots, \mu-n+1$ and $j = \mu-n+2$ are

zero since the $(n + 1)^{th}$ row (column) can be written as a linear combination of the 2^{nd} , 3^{rd} , \dots , n^{th} rows (columns) of S . The minors associated with the lacunary sequences that start from the positions (i_0, j_0) such that $(i_0, j_0) \in \{1, \dots, \mu - n + 1\}^2$ are zero for the same reason as in Case 1. Thus by Theorem 3.4 E and hence $A_{\mu+1}$ are TN .

(iii) \Rightarrow (i) Since S is TN and has rank n we may conclude by Theorem 7.11 that A is $NsTN$ and because A is symmetric it is positive definite by Lemma 2.6. The entry $B[1|1] = s_1$ must be positive because otherwise it would follow by $0 < s_2$ that $\det S[1, 2|2, 3] < 0$ contradicting that S is TN . So we may assume that $3 \leq n$. Suppose that B is singular and let r be the smallest integer such that $\det B[1, \dots, r] = 0$. Then it follows that $2 \leq r \leq n - 1$. By application of Proposition 2.8 to $C := S[1, \dots, n + 1]$ at least one of the following holds. Either the rows $1, \dots, r$ or the columns $2, \dots, r + 1$ of C are linearly dependent or one of the matrices $C[1, \dots, n + 1|1, \dots, r]$ and $C[1, \dots, r|2, \dots, n + 1]$ has rank $r - 1$. Since A is nonsingular only the last case could be possible. However, this matrix possesses $C[1, \dots, r|3, \dots, r + 2]$ as a submatrix which is identical with the principal submatrix $A[2, \dots, r + 1]$. By Lemma 2.6 its determinant is positive and so we have arrived at a contradiction. Hence B is nonsingular and because it is furthermore symmetric and TN it is positive definite. \square

For a given infinite Hankel matrix S of rank n , the equivalence of that S is TP_n and $S[1, \dots, n]$ and $S[1, \dots, n|2, \dots, n + 1]$ are positive definite matrices is given in [Gan59, Theorem 19, p. 239]. We give a new proof for this result by using Theorem 8.1 and Proposition 2.8. Under the conditions of Theorem 8.1 and using arguments like in the proof of the following theorem one can show that (i) of Theorem 8.1 is equivalent to that S is TP_{n-1} .

Theorem 8.2. *Let S be a real infinite Hankel matrix of rank n where $1 < n$. Put $A := S[1, \dots, n]$ and $A_1 := S[1, \dots, n|2, \dots, n + 1]$. Then the following two statements are equivalent:*

- (i) *The matrices A and A_1 are positive definite.*
- (ii) *The infinite Hankel matrix S is TP_n .*

Proof. (i) \Rightarrow (ii) Let A and A_1 be positive definite matrices. Then by Theorem 8.1 S is TN . Set $A_k := S[1, \dots, n|k + 1, \dots, k + n]$, $k = 0, 1, \dots$. By the special pattern of S and Theorem 2.21 it suffices to show that A_k is TP for each $k = 0, 1, \dots$. We proceed by increasing induction on k . For $k = 0$, $A_0 = A$ is TP by Theorem 8.1. Suppose we have shown the claim for $k = k_0$. We want to show it for $k = k_0 + 1$. Suppose on the contrary that A_{k_0+1} is not TP . Then by Theorem 2.20 there is a vanishing corner minor and this corner minor lies at the upper part (the part above the main diagonal) of A_{k_0+1} since A_{k_0} is TP . Let l be the smallest integer such that $\det A_{k_0+1}[1, \dots, l|n - l + 1, \dots, n] = 0$. Hence $l \leq n$ and all the leading principal minors of the submatrix that corresponds to the latter minor are positive since A_{k_0} is TP . By the special pattern of S we have that the submatrix $S[2, \dots, l + 1|k_0 + n - l + 1, \dots, k_0 + n] = A_{k_0+1}[1, \dots, l|n - l + 1, \dots, n]$ has rank $l - 1$. Hence by Proposition 2.8 either the rows $2, 3, \dots, (l + 1)$ or the columns $k_0 + n - l + 1,$

$k_0 + n - l + 2, \dots, k_0 + n$ of S are linearly dependent, or the right or left shadow of $S[2, \dots, l+1 | k_0 + n - l + 1, \dots, k_0 + n]$ has rank $l - 1$ which is a contradiction since A_{k_0} is TP . Hence A_{k_0+1} is TP .

(ii) \Rightarrow (i) Let S be TP_n . Then A and A_1 are TP and hence all of their minors are positive. Since A and A_1 are also symmetric they are positive definite. \square

The next theorem presents the relationship between the nonsingularity and total nonnegativity of the matrix $H_{2n}(p, q)$ and the almost total positivity of the matrices $H_{2n}(p, q)$ and $H(p, q)$.

Theorem 8.3. [AGT16, Theorem 3.2] *Let $H(p, q)$ and $H_{2n}(p, q)$ be given as in (7.33)-(7.36). Then the following three statements are equivalent:*

- (i) *The matrix $H_{2n}(p, q)$ is $NsTN$.*
- (ii) *The matrix $H_{2n}(p, q)$ is ATP and $0 < a_i, b_i, i = 1, \dots, n$.*
- (iii) *The matrix $H(p, q)$ is ATP and $0 < a_i, b_i, i = 1, \dots, n$.*

Proof. (i) \Rightarrow (ii) By Theorem 3.2 (ii) and Proposition 3.5, $\tilde{H}_{2n}(p, q)$ is a nonnegative Cauchon matrix with positive diagonal entries. The entries $a_1, \dots, a_n, b_1, \dots, b_n$ appear on the main diagonal of $H_{2n}(p, q)$ and are therefore positive by Lemma 2.6. If $b_0 \neq 0$, then the assumption $H_{2n}(p, q)$ is $NsTN$ shows that b_0 can not be negative.

In order to show that the matrix $H_{2n}(p, q)$ is ATP it is sufficient by Theorem 3.18 to show that the zero-nonzero patterns of $H_{2n}(p, q) = (h_{ij})$ and $\tilde{H}_{2n}(p, q)$ coincide, i.e., by Remark 3.1 it is sufficient to show that the minors associated with lacunary sequences that are constructed by Procedure 3.1 and start at the positions (i, j) such that $h_{ij} = 0$ are zero while those that start at the positions (i, j) such that $0 < h_{ij}$ are positive, $i, j = 1, \dots, 2n$ ($2n + 1$).

By application of Procedure 3.1 we find for each position (i, j) a lacunary sequence with respect to the Cauchon diagram associated to $\tilde{H}_{2n}(p, q)$ starting at this position. Any minor associated with a lacunary sequence that starts from a position (i, j) with $h_{ij} = 0$ vanishes since the corresponding submatrix contains a zero row or column. Any minor associated with a lacunary sequence that starts from a position (i, i) or at a position (i, j) such that $j < i$ and $0 < h_{ij}$ is a principal minor of $H_{2n}(p, q)$ possibly multiplied by a_0 (b_0). While the minors that are associated with the lacunary sequences that start from positions (i, j) such that $i < j$ and $0 < h_{ij}$ are principal minors of $H_{2n}(p, q)$ possibly multiplied by some integer power of a_n . Therefore all minors associated with the lacunary sequences that start from positions (i, j) such that $0 < h_{ij}$ are positive by Lemma 2.6. Hence $H_{2n}(p, q)$ and $\tilde{H}_{2n}(p, q)$ have the same zero-nonzero pattern.

(ii) \Rightarrow (iii) Without loss of generality we consider only the case $b_0 = 0$. Let $A_\nu := H(p, q)[1, \dots, \nu]$, $\nu \geq 2n + 1$. Then A_ν tends to $H(p, q)$ as ν tends to infinity and any submatrix of $H(p, q)$ appears as a submatrix of a suitably chosen A_{ν_0} . We want to show that A_ν is ATP for each $\nu \geq 2n + 1$. The contiguous minors of A_{2n+1} coincide with minors of $H_{2n}(p, q)$ possibly multiplied by $a_0 > 0$ or vanish because the corresponding submatrices contain a zero row or column. Hence A_{2n+1} is ATP since $H_{2n}(p, q)$ is ATP by assumption.

Suppose now that A_ν is *ATP*. Any contiguous minor of $A_{\nu+1}$ appears as a minor of A_ν possibly multiplied by a_n or vanishes since the corresponding submatrix has a zero row or column. Hence $A_{\nu+1}$ is *ATP* and by induction we obtain that $H(p, q)$ is *ATP*.

(iii) \Rightarrow (i) Since $H_{2n}(p, q)$ is a principal submatrix of $H(p, q)$ with positive diagonal entries the claim follows. \square

The following theorem states that the positivity of the leading principal minors of $H_{2n}(p, q)$ is equivalent to the almost total positivity of $H_{2n}(p, q)$ and the positivity of $a_i, b_i, i = 1, \dots, n$.

Theorem 8.4. *Let $a_0 > 0$. Then the matrix $H_{2n}(p, q)$ is *ATP* and $0 < a_i, b_i, i = 1, \dots, n$, if and only if all its leading principal minors are positive.*

Proof. If $H_{2n}(p, q)$ is *ATP* and $0 < a_i, b_i, i = 1, \dots, n$, then all of its principal minors are positive by Theorem 8.3 and Lemma 2.6. The converse is a special case of Theorem 8.5 below. \square

Assume we are given real numbers a_0, a_1, \dots, a_n and $2 \leq M \leq n$. For ease of exposition, set $a_j := 0$ for $j < 0$ and $j > n$. Let us define

$$P = (p_{ij})_{i,j=1}^{\infty}, \quad \text{where } p_{ij} := a_{Mj-i}, \quad i, j = 1, 2, \dots, \quad (8.4)$$

thus

$$P = \begin{bmatrix} a_{M-1} & a_{2M-1} & a_{3M-1} & \dots \\ a_{M-2} & a_{2M-2} & a_{3M-2} & \dots \\ \vdots & \vdots & \vdots & \\ a_0 & a_M & a_{2M} & \dots \\ 0 & a_{M-1} & a_{2M-1} & \dots \\ 0 & a_{M-2} & a_{2M-2} & \dots \\ \vdots & \vdots & \vdots & \\ 0 & a_0 & a_M & \dots \\ 0 & 0 & a_{M-1} & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix}. \quad (8.5)$$

The matrix P is called *generalized Hurwitz matrix* [GS04], [Pin10].

Theorem 8.5. [GS04, Theorem 2.1], [Pin10, Theorem 4.6] *Let P be a generalized Hurwitz matrix with $0 < a_0$. If $0 < \det P[k, \dots, k+r-1|1, \dots, r]$ for $k = 1, \dots, M-1$, and $r = 1, \dots, \lfloor \frac{n+k-1}{M-1} \rfloor$. Then P is *ATP*.*

We note that Theorem 8.5 can be also proven by using the Cauchon Algorithm. However, since our proof is not considerably shorter than the proofs in [GS04] and [Pin10] we will not give it here.

Recall that for a polynomial $f(z)$ given by (7.10) with $0 < a_0$, Routh-Hurwitz's Criterion states that $f(z)$ is stable if and only if all the leading principal minors of the Hurwitz matrix given by (7.19) are positive, i.e.,

$$0 < \det H(f)[1, \dots, k], \quad \text{for all } k = 1, \dots, n. \quad (8.6)$$

In the sequel we shall say that $H(f)$ is *stable* when $f(z)$ is stable.

Asner [Asn70] showed that the Hurwitz matrix $H(f)$ associated with a quasi-stable polynomial given by (7.10) is *TN* and Kemperman [Kem82] proved that the Hurwitz matrix $H(f)$ associated with a stable polynomial (7.10) is *NsATP*. In the following theorems we present these results and give new and simple proofs for them.

Theorem 8.6. [Kem82, Theorem 2] *Let f be given as in (7.10) with $0 < a_0$. If f is stable, then $H(f)$ is *NsATP*.*

Proof. We proceed by induction on n . The statement is obviously true for $n = 2$. Suppose that f given by (7.10) is stable. Consider the polynomial g of degree $n - 1$, defined by

$$g(z) := a_1 z^{n-1} + (a_2 - \frac{a_0}{a_1} a_3) z^{n-2} + a_3 z^{n-3} + (a_4 - \frac{a_0}{a_1} a_5) z^{n-4} + \dots .$$

Then g is the polynomial which corresponds to the second and third rows of the Routh scheme associated to f , see Section 7.3. Since f is stable it follows by the Routh's Criterion, see Theorem 7.3, that g is stable, too, and $0 < a_1$. Hence by the induction hypothesis $H(g)$ is *NsATP*. By Theorem 3.2 (ii) and Proposition 3.5 $\tilde{H}(g)$ is a nonnegative Cauchon matrix with positive diagonal entries and by Theorem 3.18 $H(g)$ and $\tilde{H}(g)$ have the same zero-nonzero pattern. Let $H_1(f)$ be the matrix that is obtained from $H(f)$ by subtracting from each even indexed row the preceding row multiplied by $\frac{a_0}{a_1}$ and storing the resulting row in this even indexed row. It is easy to see that $H(g) = \tilde{H}_1(f)[2, \dots, n]$ and therefore, $H_1(f)[2, \dots, n]$ and $\tilde{H}_1(f)[2, \dots, n]$ have the same zero-nonzero pattern; note that $\tilde{H}_1(f)[2, \dots, n]$ coincides with the matrix which is obtained by application of the Cauchon Algorithm to $H_1(f)[2, \dots, n]$. Since $0 < a_1$ it follows that $H_1(f)$ and hence $H(f)$ are nonsingular.

The minors of $H_1(f)$ that are associated with the sequences which are constructed by Procedure 3.1 and are starting at the positions $(1, l)$, $l = 2, \dots, \lceil \frac{n}{2} \rceil$, are equal to the minors that are associated with the lacunary sequences that are starting from the positions $(3, l+1)$ multiplied by a_n . The minor that is associated with the sequence that starts at the position $(1, 1)$ is equal to $\det H_1(f)$ and hence positive since $H(g)$ is *NsATP* and $0 < a_1$. The minors that are associated with the sequences that are starting from the other positions of the first row and column are zero since the corresponding submatrices have a zero row or column. Hence $\tilde{H}_1(f)$ is a nonnegative Cauchon matrix with positive diagonal entries. Moreover, $H_1(f)$ and $\tilde{H}_1(f)$ have the same zero-nonzero pattern, see Remark 3.1. Thus by Theorem 3.18 $H_1(f)$ is *NsATP*.

We access the entries of $H(f)$ through the entries of $H_1(f)$; by adding to each even indexed row in $H_1(f)$ the preceding row multiplied by $\frac{a_0}{a_1}$. Hence by determinantal properties we

obtain that $H(f)$ is $NsTN$. By application of Procedure 3.1 to $H(f)$ we find for each position (k, l) , $k, l = 1, \dots, n$, a lacunary sequence with respect to the Cauchon diagram associated to $\tilde{H}(f)$ starting from (k, l) . By the special pattern of the entries of $H(f) = (h_{kl})$ the minors associated with the lacunary sequences which start from positions (k, l) such that $h_{kl} \neq 0$ are principal minors in $H(f)$, or principal minors multiplied by some integral power of a_n , or principal minors multiplied by a_0 . Hence by Lemma 2.6 they are positive. The minors that are associated with the lacunary sequences which start from positions (k, l) such that $h_{kl} = 0$ are zero since the corresponding submatrices contain a zero row or column. Hence by Remark 3.1 $H(f)$ and $\tilde{H}(f)$ have the same zero-nonzero pattern. Whence it follows by Theorem 3.18 that $H(f)$ is $NsATP$. \square

The property to be $NsATP$ of $H(f)$ which is associated to a stable polynomial was first shown in [Kem82, Theorem 2]. The converse direction follows by the fact that the leading principal minors of a $NsATP$ matrix are positive and the Routh-Hurwitz's Criterion, see Section 7.3. By Theorem 8.6 the positivity of the coefficients (or the property that all its coefficients have the like sign) of a given polynomial is a necessary condition for its stability, see [Bar08, Theorem 1 (Stodola)]. By using Theorem 8.6 and arguments similar to that have been employed in the proof of the implication (ii) \Rightarrow (iii) of Theorem 8.3 we obtain the following corollary.

Corollary 8.1. *Let f be given as in (7.10) with $0 < a_0$. If f is stable, then the infinite Hurwitz matrix $H_\infty(f)$ is ATP .*

Theorem 8.6 does not hold if the given polynomial is quasi-stable; however, the following weaker result is valid.

Theorem 8.7. [Asn70, Theorem 1], [Kem82, Theorem 1, Remark 2] *Let $H(f)$ be the Hurwitz matrix associated with a quasi-stable polynomial f given by (7.10) with $0 < a_0$. Then $H(f)$ is TN .*

Proof. Let $f(z)$ given by (7.10) be a quasi-stable. If $f(z)$ is stable, then by Theorem 8.6 $H(f)$ is TN . If $f(z) = a_0 z^n$, then it is obvious that $H(f)$ is TN . Suppose now that $f(z)$ has some zeros on the imaginary axis, then $f(z)$ can be written as

$$f(z) = h(z)z^l g(z), \quad \text{for some } l \in \{0, 1, \dots, n-1\}, \quad (8.7)$$

where $h(z)$ is a Hurwitz polynomial and $g(z)$ is a polynomial whose zeros are pure imaginary. By using the fact that $H_\infty(f(z)) = H_\infty(\frac{f(z)}{z^l})$, Remark 7.5, and Theorem 7.9 we obtain

$$H_\infty(f(z)) = H_\infty(h(z)g(z^2)) = H_\infty(h(z))T(g(z^2)). \quad (8.8)$$

$H_\infty(h)$ is TN by Corollary 8.1 since $h(z)$ is stable and $T(g(z^2))$ is TN by Theorem 7.10 since $g(z^2)$ has only real and negative zeros. Hence $H_\infty(f)$ is TN since it can be represented as a product of TN matrices. Therefore $H(f) = H_\infty(f)[1, \dots, n]$ is TN . \square

Remark 8.1. [Asn70] *The converse of Theorem 8.7 is not true. For instance, consider the polynomial*

$$f(z) = z^4 + 198z^2 + (101)^2 \quad (8.9)$$

whose matrix $H(f)$ is TN but f is not quasi-stable. However, Theorem 8.9 below gives under a slightly strengthened hypothesis a converse of Theorem 8.7.

Let $P(\lambda, z)$ be a family of polynomials of fixed degree n whose coefficients are continuous functions of λ on a given interval $I := [a, b]$. In other words, an element of $P(\lambda, z)$ can be written as

$$p(\lambda, z) = a_0(\lambda)z^n + a_1(\lambda)z^{n-1} + \dots + a_n(\lambda), \quad (8.10)$$

where $a_0(\lambda), a_1(\lambda), \dots, a_n(\lambda)$ are continuous functions of λ on I and $a_0(\lambda) \neq 0$ for all $\lambda \in I$. Based on this we recall the following fundamental theorem in stability theory.

Theorem 8.8. [BCK95, Theorem 1.4, (Boundary Crossing Theorem)] *Suppose that $p(a, z)$ has all its zeros in an open set S and $p(b, z)$ has at least one root in the interior of the complement of S . Then there exists at least one real number $\rho \in (a, b)$ such that $p(\rho, z)$ has all its zeros in $S \cup \partial S$ with at least one zero on ∂S , where ∂S denotes the boundary of S .*

Let $H_\sigma(f)$ denotes the Hurwitz matrix of the polynomial $f(z + \sigma)$ for some constant σ . Then, based on Theorem 8.8, we present a proof of the following theorem which was originally due to A. Asner [Asn70, Theorem 2].

Theorem 8.9. *If for some $0 < \epsilon$, $H_\sigma(f)$ is TN for all $\sigma \in [0, \epsilon]$, then $f(z)$ is quasi-stable.*

Proof. If $H(f) = H_0(f)$ is $NsTN$ then $0 < a_0$ and by Lemma 2.6 all the leading principal minors of $H(f)$ are positive. Hence by the Routh-Hurwitz's Criterion (7.24) $f(z)$ is stable. If $H(f)$ is singular and TN , then by the assumption and using Orlando's formula (7.25) it follows that $H_\sigma(f)$ is $NsTN$ for any $\sigma \in (0, \epsilon]$ and so $f(z + \sigma)$ is stable. Suppose on the contrary that $f(z)$ is not quasi-stable. Hence by Theorem 8.8 there exists $0 < \sigma_0 \leq \epsilon$ such that $f(z + \sigma_0)$ is quasi-stable with at least one purely imaginary zero. Hence we have a contradiction to the stability of $f(z + \sigma)$ for any $\sigma \in (0, \epsilon]$. Hence $f(z)$ is quasi-stable. \square

As a related result, we present a test whether a given polynomial has only nonpositive real zero.

Theorem 8.10. [Pin10, Theorem 4.9] (Total nonnegativity criterion for nonpositive zeros) *The polynomial $f(z)$ given in (7.10), $0 < a_0$, has all its zeros nonpositive if and only if the following infinite matrix is TN*

$$D_\infty(f) := \begin{bmatrix} na_0 & (n-1)a_1 & (n-2)a_2 & (n-3)a_3 & (n-4)a_4 & (n-5)a_5 & (n-6)a_6 & \dots \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \dots \\ 0 & na_0 & (n-1)a_1 & (n-2)a_2 & (n-3)a_3 & (n-4)a_4 & (n-5)a_5 & \dots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \dots \\ 0 & 0 & na_0 & (n-1)a_1 & (n-2)a_2 & (n-3)a_3 & (n-4)a_4 & \dots \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We conclude this section with the following two corollaries which give the relationships between total nonnegativity of the Hankel matrices $S(L)$ and $(\text{diag}(1, -1, \dots)S(L)\text{diag}(1, -1, \dots))$ whose entries are given by (7.53) and the zeros of the given polynomial (7.31).

Corollary 8.2. [HT12, Corollary 4.14] *The polynomial p given by (7.31) has only positive zeros and exactly $m \leq n$ of them are distinct if and only if the matrix $S(L)$ is TP_m of rank m .*

Corollary 8.3. [HT12, Corollary 4.16] *The polynomial p given by (7.31) has only negative zeros and exactly $m \leq n$ of them are distinct if and only if the matrix $\text{diag}(1, -1, \dots)S(L)\text{diag}(1, -1, \dots)$ is TP_m of rank m .*

8.2. Total Nonnegativity and R -Functions of Negative Type

In this section we present an important subclass of rational functions which is called R -function of negative type. We investigate some properties of this kind of rational functions and use Theorem 8.11 below and Hankel matrices to study interval problems involving R -functions.

Definition 8.1. *A rational function is called an R -function of negative type (respectively, positive type) if it maps the open upper half-plane of the complex plane into the open lower half-plane (respectively, to itself).*

Concerning R -functions credit is given to Nevanlinna, Herglotz, and Pick. We will present our results only for R -functions of negative type since the corresponding results for R -functions of positive type can be obtained by replacing the function R by $-R$. The following two theorems play an important role in our results.

Theorem 8.11. [HT12, Theorem 3.4] *Let p and q given as in (7.31)-(7.32) be coprime polynomials satisfying*

$$|\deg p - \deg q| \leq 1.$$

Then for the real rational function given by (7.30) with exactly $n = \deg p$ poles, the following statements are equivalent:

- (i) R is an R -function of negative type.
- (ii) The function R can be represented in the form

$$R(z) = -\alpha z + \beta + \sum_{j=1}^n \frac{\gamma_j}{z - w_j}, \quad 0 \leq \alpha, \beta, w_j \in \mathbb{R}, \quad j = 1, 2, \dots, n, \quad (8.11)$$

where

$$\gamma_j = \frac{q(w_j)}{p'(w_j)} > 0, \quad j = 1, 2, \dots, n. \quad (8.12)$$

(iii) The Cauchy index of the function R is maximal:

$$\text{Ind}_{\mathbb{P}_{\mathbb{R}}}(R) = \max(\deg p, \deg q). \quad (8.13)$$

(iv) The polynomials p and q have only real roots and satisfy the inequality

$$p(w)q'(w) - p'(w)q(w) < 0 \quad \text{for all } w \in \mathbb{R}. \quad (8.14)$$

(v) The roots of the polynomials p and q are real, simple, and interlacing, that is, between any two consecutive roots of one of the polynomials there is exactly one root of the other polynomial, and

$$\text{there exists } w \in \mathbb{R} : \quad p(w)q'(w) - p'(w)q(w) < 0. \quad (8.15)$$

(vi) Let the function R be represented by the series

$$R(z) = -\alpha z + \beta + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \dots \quad (8.16)$$

with $0 \leq \alpha$ and $\beta \in \mathbb{R}$. The following inequalities hold:

$$0 < D_j(R), \quad j = 1, 2, \dots, n, \quad (8.17)$$

where the determinants $D_j(R)$ are defined in (7.42).

Theorem 8.12. [HT12, Theorem 3.18] *Let R be expanded as in (7.45). Then the following two statements are equivalent:*

- (i) R is an R -function of negative type and has exactly r poles, all of which are positive.
- (ii) The matrix $S(R)$ is TP_r of rank r .

The above theorem is stated in [HT12] for $s_{-1} = 0$ but it is obviously also true without this assumption. Corollary 8.2 and Theorem 8.12 show that the logarithmic derivative of any polynomial having all its zeros are positive is an R -function of negative type.

The total nonnegativity of the finite and infinite matrices of Hurwitz type $H_{2n}(p, q)$ and $H(p, q)$ plays an important role in characterizing a given rational function as the next two theorems state.

Theorem 8.13. [HT12, Theorem 3.44, (Total nonnegativity of the Hurwitz matrix)] *The following two statements are equivalent:*

- (i) The polynomials p and q defined by (7.31)-(7.32) have only nonpositive zeros and the function $R = q/p$ is either an R -function of negative type or identically zero.
- (ii) The infinite matrix of Hurwitz type $H(p, q)$ defined by (7.33) or (7.35) is TN .

Theorem 8.10 can be shown by using Theorem 8.13. The next theorem can be obtained by using [HT12, Theorem 3.47] and Theorem 8.3.

Theorem 8.14. *Let R , p , and q be given as in (7.30)-(7.32). Then the following two statements are equivalent:*

(i) *The polynomials p and q are coprime and have only negative zeros and the function R is an R -function of negative type.*

(ii) *The matrix $H_{2n}(p, q)$ is ATP with $0 < a_i, b_i, i = 1, \dots, n$.*

Proof. (i) \Rightarrow (ii) By Theorem 8.13 $H_{2n}(p, q)$ is TN since it is a submatrix of $H(p, q)$, and by Theorem 7.17 and Corollary 7.2 $H_{2n}(p, q)$ is nonsingular since R has n negative poles. Hence by Theorem 8.3 the claim follows.

(ii) \Rightarrow (i) By Theorem 8.3 $H(p, q)$ is ATP with $0 < a_i, b_i, i = 1, \dots, n$. Hence by Theorem 8.13 and $a_n, b_n > 0$ the claim follows. \square

The following theorem can be shown directly. However, we give another proof by using Theorems 4.6 and 7.10.

Theorem 8.15. *Let p, q defined by (7.31)-(7.32) and $g(z) = c_0 z^n + c_1 z^{n-1} + \dots + c_n$ be a real polynomial with*

$$(-1)^k a_k \leq (-1)^k c_k \leq (-1)^k b_k, \quad k = 0, 1, \dots, n. \quad (8.18)$$

Then if all the zeros of p and q are real and nonpositive, then all the zeros of g are real and nonpositive, too.

Proof. Let $T(p), T(g)$, and $T(q)$ be the infinite upper triangular Toeplitz matrices associated to p, g , and q , respectively and (8.18) hold. Then as a consequence of (8.18) the inequalities

$$T(p)[1, \dots, \nu] \leq^* T(g)[1, \dots, \nu] \leq^* T(q)[1, \dots, \nu], \quad \text{for all } \nu = 1, 2, \dots \quad (8.19)$$

hold. Since p and q have only real and nonpositive zeros then by Theorem 7.10 and $0 < a_0$, $T(p)[1, \dots, \nu]$ and $T(q)[1, \dots, \nu]$ are $NsTN$ for all $\nu = 1, 2, \dots$. By application of Theorem 4.6 to (8.19) we have that $T(g)[1, \dots, \nu]$ is $NsTN$ for each $\nu = 1, 2, \dots$. Hence again by application of Theorem 7.10, the result follows. \square

Now we turn to interval problems related to R -functions of negative type. The following theorem shows that when the coefficients of the Laurent series of a given rational function lie, in checkerboard fashion, between the coefficients of the Laurent series of two R -functions of negative type and have only positive poles then the given rational function has these properties, too.

Theorem 8.16. [AGT16, Theorem 4.3] *Let R_1, R_2 , and R_3 be rational functions with series (7.45) involving coefficients s_i, t_i , and $d_i, i = -1, 0, 1, \dots$, respectively. Assume that the coefficients satisfy for $i \geq 0$ the following inequalities:*

$$(-1)^i s_i \leq (-1)^i d_i \leq (-1)^i t_i, \quad i = 0, 1, \dots \quad (8.20)$$

If R_1 and R_2 are R -functions of negative type and both functions have exactly r poles all of which are positive, then R_3 is an R -function of negative type and has exactly r poles all of which are positive.

Proof. By Theorem 8.12 $S(R_1)$ and $S(R_2)$ are TP_r of rank r . Let $S(R_3)[\alpha|\beta]$ be any contiguous submatrix of $S(R_3)$ such that $|\alpha| = |\beta| = r$ and (8.20) holds. Then as a consequence of (8.20) the inequalities

$$S(R_1)[\alpha|\beta] \leq^* S(R_3)[\alpha|\beta] \leq^* S(R_2)[\alpha|\beta]$$

or

$$S(R_2)[\alpha|\beta] \leq^* S(R_3)[\alpha|\beta] \leq^* S(R_1)[\alpha|\beta]$$

hold. Hence by Remark 4.2, $S(R_3)[\alpha|\beta]$ is TP , too, and by Theorem 2.21 $S(R_3)$ is TP_r . It remains to show that the rank of $S(R_3)$ equals r . By adding a suitable positive number ϵ to s_{2r}, t_{2r}, d_{2r} we can accomplish that $S(R_1)[1, \dots, r+1]$ and $S(R_2)[1, \dots, r+1]$ become TP . Thus $S(R_3)[1, \dots, r+1]$ is TP . By using Lemma 4.4 and letting ϵ tend to zero we obtain that $\det S(R_3)[1, \dots, r+1] = 0$ since $\det S(R_1)[1, \dots, r+1] = \det S(R_2)[1, \dots, r+1] = 0$. Repeating this process infinitely many times with each $S(R_3)[1, \dots, r+1|\nu, \dots, \nu+r]$, $\nu = 2, 3, \dots$, we arrive at $\det S(R_3)[1, \dots, r+1|\nu, \dots, \nu+r] = 0$ for all $\nu = 1, 2, \dots$. Hence for each $\nu = 1, 2, \dots$ there exist $c'_1, c'_2, \dots, c'_r \in \mathbb{R}$ such that

$$S(R_3)[r+1|\nu, \dots, \nu+r] = [c'_1 \ c'_2 \ \dots \ c'_r] S(R_3)[1, \dots, r|\nu, \dots, \nu+r] \quad (8.21)$$

since $S(R_3)[1, \dots, r+1|\nu, \dots, \nu+r]$ has rank r and $\det S(R_3)[1, \dots, r|\nu, \dots, \nu+r-1]$ is positive. Therefore by (8.21)

$$\begin{aligned} S(R_3)[r+1|2, \dots, r+1] &= [c_1^1 \ c_2^1 \ \dots \ c_r^1] S(R_3)[1, \dots, r|2, \dots, r+1] \\ &= [c_1^2 \ c_2^2 \ \dots \ c_r^2] S(R_3)[1, \dots, r|2, \dots, r+1]. \end{aligned}$$

Whence

$$([c_1^1 \ c_2^1 \ \dots \ c_r^1] - [c_1^2 \ c_2^2 \ \dots \ c_r^2]) S(R_3)[1, \dots, r|2, \dots, r+1] = 0. \quad (8.22)$$

Since $S(R_3)[1, \dots, r|2, \dots, r+1]$ is nonsingular we conclude by (8.22)

$$[c_1^1 \ c_2^1 \ \dots \ c_r^1] = [c_1^2 \ c_2^2 \ \dots \ c_r^2]. \quad (8.23)$$

Repeating the above steps we obtain $c_j^1 = c_j^\nu$ for each $j = 1, \dots, r$, $\nu = 1, 2, \dots$. Hence the row $r+1$ of $S(R_3)$ can be written as a linear combination of the previous rows and $S(R_3)[1, \dots, r]$ is nonsingular. Thus by Theorem 7.12 $S(R_3)$ has rank r . \square

The next theorem shows that if the coefficients of the Laurent series of two R -functions of negative type with only positive poles and are related as in (8.20), then they must have the same number of poles.

Theorem 8.17. [AGT16, Theorem 4.4] *Let R_1 and R_2 be as in Theorem 8.16 with coefficients satisfying (8.20). If R_1 and R_2 are R -functions of negative type and all poles of them are positive, then R_1 and R_2 have the same number of poles.*

Proof. Suppose on the contrary that R_1 has exactly r positive poles and R_2 has exactly k , $k > r$, positive poles. Then by Theorem 8.12, $S(R_1)$ and $S(R_2)$ are TP_r of rank r and TP_k of rank k , respectively. By using Lemma 4.4 we arrive at the contradiction

$$0 < \det S(R_2)[1, \dots, r+1 | 2, \dots, r+2] \leq \det S(R_1)[1, \dots, r+1 | 2, \dots, r+2] = 0.$$

If R_1 has exactly k , $k > r$ positive poles, and R_2 possesses exactly r positive poles, then we arrive with Lemma 4.4 at the contradiction

$$0 < \det S(R_1)[1, \dots, r+1] \leq \det S(R_2)[1, \dots, r+1] = 0.$$

Hence R_1 and R_2 have the same number of positive poles. □

The following two lemmata and theorem provide a further interval property of the rational functions.

Definition 8.2. *We call the rational function $R = q/p$ an R^* -function if R is an R -function of negative type with only negative zeros and poles and q and p are coprime.*

In order to avoid distinguishing the cases that the polynomial degree is even or odd we number in the rest of this section the coefficients of a polynomial in such a way that the coefficient indexed by 0 is the constant term. We affix superscripts to polynomial coefficients for reference to a specific polynomial. Without loss of generality we assume that the leading coefficients of all polynomials appearing in the assumptions of the following statements are positive.

Lemma 8.1. [AGT16, Lemma 4.6] *Let $R_1 = q_1/p$ and $R_2 = q_2/p$ be two R^* -functions, where $\deg q_1 = \deg q_2 = n$ and $\deg p \in \{n-1, n, n+1\}$. Then $R = g_1/p$ is an R^* -function provided that the coefficients of q_1, q_2, g_1 satisfy the following inequalities:*

$$(-1)^k a_k^{q_1} \leq (-1)^k a_k^{g_1} \leq (-1)^k a_k^{q_2}, \quad k = 0, 1, \dots, n. \quad (8.24)$$

Proof. Since R_1 and R_2 are R^* -functions we conclude that all of the coefficients of q_1, q_2 , and p are positive. From the assumption (8.24) it follows that

$$q_1(x) \leq g_1(x) \leq q_2(x) \quad \text{for all } x \leq 0. \quad (8.25)$$

Since q_1 and q_2 have only negative zeros, g_1 and therefore R has only negative zeros, as well. By Theorem 8.11 (i) \Leftrightarrow (v) it is sufficient to show that g_1 and p are coprime and the zeros of g_1 and p are real, simple, and interlacing and there exists a real number v such that

$$p(v)g_1'(v) - p'(v)g_1(v) < 0. \quad (8.26)$$

By Theorem 8.11 the zeros of each pair (q_1, p) and (q_2, p) are real, simple, interlacing, and for each real number w and $i = 1, 2$, we have

$$p(w)q_i'(w) - p'(w)q_i(w) < 0. \quad (8.27)$$

As in the proof of [BCK95, Lemma 5.1], we obtain that the zeros of g_1 and p are real, simple, and interlacing. By setting $w = 0$ in (8.27) and using (8.24), we get

$$a_0^p a_1^{g_1} - a_1^p a_0^{g_1} \leq a_0^p a_1^{q_1} - a_1^p a_0^{q_1} < 0 \quad (8.28)$$

Hence (8.26) is fulfilled. It remains to show that g_1 and p are coprime. Suppose on the contrary that both polynomials have a zero, x_0 say, in common. Then it follows by the interlacing property, (q_1, p) and (q_2, p) are coprime, and (8.25) that $q_1(x_0) < 0 < q_2(x_0)$. We first consider the case that x_0 is greater than the largest zero of q_2 . Let x'_0 be the next (smaller) zero of p . Then $0 < q_1(x'_0) \leq q_2(x'_0)$, a contradiction because q_2 cannot have a simple zero between x'_0 and x_0 . If x_0 is smaller than the largest zero of q_2 we distinguish two cases. If p has a zero x''_0 , $x_0 < x''_0$, then we arrive likewise by $0 < q_1(x''_0) \leq q_2(x''_0)$ at a contradiction. Otherwise we use the fact that $0 < q_2(0)$ to obtain a contradiction. This completes the proof that R is an R^* -function. \square

Using a similar proof technique we obtain the dual of Lemma 8.1.

Lemma 8.2. [AGT16, Lemma 4.7] *Let $R_1 = q/p_1$ and $R_2 = q/p_2$ be two R^* -functions, where $\deg q = n$ and $\deg p_1 = \deg p_2 \in \{n-1, n, n+1\}$. Then $R = q/g_2$ is an R^* -function provided that the coefficients of p_1, p_2, g_2 satisfy the following inequalities*

$$(-1)^k a_k^{p_1} \leq (-1)^k a_k^{g_2} \leq (-1)^k a_k^{p_2}, \quad k = 0, 1, \dots, n. \quad (8.29)$$

By application of the above two lemmata to specific four R^* -functions, we derive the following theorem.

Theorem 8.18. [AGT16, Theorem 4.8] *Let $R_{11} = q_1/p_1$, $R_{12} = q_1/p_2$, $R_{21} = q_2/p_1$, and $R_{22} = q_2/p_2$ be R^* -functions, where $\deg q_1 = \deg q_2 = n$ and $\deg p_1 = \deg p_2 \in \{n-1, n, n+1\}$. Then $R = g_1/g_2$ is an R^* -function provided that the coefficients of q_i, g_1 and p_i, g_2 , $i = 1, 2$, satisfy the inequalities (8.24) and (8.29), respectively.*

Proof. Since R_{11} and R_{21} as well as R_{12} and R_{22} are R^* -functions we conclude from Lemma 8.1 that $R_{11}^g = g_1/p_1$ and $R_{12}^g = g_1/p_2$, respectively, are R^* -functions. By application of Lemma 8.2 to R_{11}^g and R_{12}^g we obtain that R is an R^* -function. \square

The following theorem provides a method to generate R -functions of negative type from four special such functions.

Theorem 8.19. *Let $R_{11} = q_1/p_1$, $R_{12} = q_1/p_2$, $R_{21} = q_2/p_1$, and $R_{22} = q_2/p_2$ be R -functions of negative type. Then*

$$R := (c_1 q_1 + c_2 q_2) / (d_1 p_1 + d_2 p_2)$$

is an R -function of negative type for any $0 < c_1, c_2, d_1, d_2$.

Proof. Let $R_1 := c_1 R_{11} + c_2 R_{21} = (c_1 q_1 + c_2 q_2)/p_1$ and $R_2 := c_1 R_{12} + c_2 R_{22} = (c_1 q_1 + c_2 q_2)/p_2$. Then by the definition of R -functions of negative type we have R_1 and R_2 are R -functions of negative type for any $0 < c_1, c_2$. Again by the definition it is easy to see that $-1/R_1$ and $-1/R_2$ are R -functions of negative type. Let $R_3 := d_1(-1/R_1) + d_2(-1/R_2)$. Then R_3 is an R -function of negative type for any $0 < d_1, d_2$. Hence $R = -1/R_3$ is an R -function of negative type since R_3 is so. \square

We conclude the section by relating the coefficients of the representation of a given rational function R in form of a Stieltjes continued fraction to the entries of the matrix which is obtained by the application of the Cauchon Algorithm to finite section of the infinite Hankel matrix $S(R)$ associated with R .

By [HT12, formulae (1.113)-(1.114), p. 453] the minors $D_i(R)$ and $\hat{D}_i(R)$ given by (7.42)-(7.43) are connected with the coefficients of Stieltjes continued fractions through the relations

$$c_{2j} = \frac{-D_j^2(R)}{\hat{D}_{j-1}(R) \cdot \hat{D}_j(R)}, \quad c_{2j-1} = \frac{\hat{D}_{j-1}^2(R)}{D_{j-1}(R) \cdot D_j(R)}, \quad j = 1, \dots, r, \quad (8.30)$$

where r is the number of poles of the function R .

Under the conditions of Theorem 8.1 we have by Theorem 7.13 that the rational function

$$R(z) = \frac{s_0}{z} + \frac{s_1}{z^2} + \dots$$

has n poles. Let $E := (S[1, \dots, n+1])^\#$. Then by application of the Cauchon Algorithm to E we obtain \tilde{E} which is a nonnegative Cauchon matrix. By Proposition 3.4

$$\begin{aligned} D_i(R) &= \det E[n+2-i, \dots, n+1] = \tilde{e}_{n+2-i, n+2-i} \cdot \tilde{e}_{n+2-i+1, n+2-i+1} \cdots \tilde{e}_{n+1, n+1}, \\ \hat{D}_i(R) &= \det E[n+2-i, \dots, n+1 | n+1-i, \dots, n] \\ &= \tilde{e}_{n+2-i, n+1-i} \cdot \tilde{e}_{n+2-i+1, n+1-i+1} \cdots \tilde{e}_{n+1, n}, \end{aligned}$$

for $i = 1, \dots, n$. Substituting the last expressions into (8.30) we obtain c_i , $i = 1, \dots, 2n$, in terms of the entries of \tilde{E} .

8.3. Region of Stability

In this section we consider the *region of stability*, i.e., a subset of \mathbb{R}^{n+1} whose elements have coordinates which are the coefficients of real stable polynomials of degree n . This region is studied in [Gan59]¹⁾. The study of the region of stability is of great practical interest;

¹⁾ In [Gan59, Chapter XV], Gantmacher called this region a *domain*. We use here 'region' since this set does not form a domain, for instance, this region is not connected because the convex combination of the two stable polynomials given by (8.54) and (8.55) is not stable.

especially in the design of systems in control and systems theory. In this "Coefficient space" all the Hurwitz polynomials of degree n form a certain region which is determined by the Hurwitz inequalities $0 < a_0, 0 < \Delta_1, 0 < \Delta_2, \dots, 0 < \Delta_n$, or by the Liénard-Chipart inequalities $0 < a_n, 0 < a_{n-2}, \dots, 0 < \Delta_1, 0 < \Delta_3, \dots$, or by the Hermite-Biehler Theorem 8.22.

Let the polynomial $f(z)$ defined by (7.10) be written as sum of its even and odd parts, i.e.,

$$f(z) = p(z^2) + zq(z^2). \quad (8.31)$$

Then by expanding the following rational function

$$R(u) = \frac{q(u)}{p(u)}, \quad (8.32)$$

into its Laurent series at ∞ we get

$$R(u) = \frac{q(u)}{p(u)} = s_{-1} + \frac{s_0}{u} - \frac{s_1}{u^2} + \frac{s_2}{u^3} - \frac{s_3}{u^4} + \dots \quad (8.33)$$

The n values $s_0, s_1, \dots, s_{2m-1}$ (for $n = 2m$) or $s_{-1}, s_0, \dots, s_{2m-1}$ (for $n = 2m + 1$) are called the *Markov parameters* of the polynomial $f(z)$. These parameters may be regarded as the coordinates in an n -dimensional space of a point that represents the given polynomial $f(z)$, see [Gan59].

The following theorem gives necessary and sufficient conditions that must be imposed on the Markov parameters in order that the corresponding polynomial is Hurwitz. In this way the region of stability in the space of Markov parameters is determined.

Theorem 8.20. [Gan59, Theorem 17, p. 235, Theorem 19, p. 240] *A real polynomial $f(z) = p(z^2) + zq(z^2)$ of degree $n = 2m$ or $n = 2m + 1$ is a Hurwitz polynomial if and only if*

- (i) *the infinite Hankel matrix $S(R)$ is TP_m of rank m ; and*
- (ii) *(For $n = 2m + 1$)*

$$s_{-1} > 0, \quad (8.34)$$

where $s_0, s_1, \dots, s_{2m-1}$ (for $n = 2m$) or $s_{-1}, s_0, \dots, s_{2m-1}$ (for $n = 2m + 1$) are the Markov parameters of the polynomial $f(z)$.

We follow [Gan59] and say that the point $P = (s_{-1}, s_0, \dots, s_{2m-1})$ precedes the point $R = (t_{-1}, t_0, \dots, t_{2m-1})$, denoted by $P \leq^* R$, if

$$(-1)^i s_i \leq (-1)^i t_i, \quad i = 0, 1, \dots, 2m - 1, \quad (8.35)$$

and (for $n = 2m + 1$)

$$s_{-1} \leq t_{-1} \tag{8.36}$$

and the sign $<$ holds in at least one of these inequalities.

We also say that the point Q *lies between* the points P and R if $P \leq^* Q \leq^* R$.

To every point P there corresponds an infinite Hankel matrix S_P of rank m (the other entries of S_P are defined as linear combinations of $s_0, s_1, \dots, s_{2m-1}$ according to (8.33)). The region of stability in this space will be denoted by \mathcal{G} . The following theorem is known as *Markov's Theorem*. We give a new and simple proof for this theorem.

Theorem 8.21. [Gan59, Theorem 21 (Markov's Theorem), p. 242] *If two points P and R belong to the region of stability \mathcal{G} and P precedes R , then every point Q which lies between P and R also belongs to \mathcal{G} , i.e.,*

$$\text{if } P, R \in \mathcal{G} \text{ and } P \leq^* Q \leq^* R, \text{ it follows that } Q \in \mathcal{G}. \tag{8.37}$$

Proof. The proof follows by using Theorem 8.20 (i) and arguments parallel to that have been used in the proof of Theorem 8.16. \square

The following theorem provides another criterion for stability of a given polynomial by using the behavior of the zeros of two polynomials formed from its even and odd parts.

Theorem 8.22. [Hol03, Theorem 1, (Hermite-Biehler Theorem)] *Let $f(z) = p(z^2) + zq(z^2)$. Then the following two statements are equivalent:*

- (i) *The polynomial f is stable.*
- (ii) *The polynomials $p(u)$ and $q(u)$ have simple, real, negative, and interlacing roots and for some complex number z_0 whose real part is positive, the real part of $\frac{p(z_0^2)}{z_0 q(z_0^2)}$ is positive.*

The next lemma specifies the region of stability with respect to the coefficients of two polynomials sharing the even or odd part. Its proof follows by using Theorem 8.22.

Lemma 8.3. [BCK95, Lemma 5.1, Lemma 5.2] *Let $p(z)$ and $q(z)$ be two stable polynomials defined by (7.31)-(7.32) with $b_0 \neq 0$. Suppose that one of the following two conditions holds:*

(i)

$$a_{2k} = b_{2k} \text{ for all } k = 0, 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor, \text{ and} \tag{8.38}$$

$$(-1)^k a_{2k+1} \leq (-1)^k b_{2k+1} \text{ for all } k = 0, 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor, \text{ or} \tag{8.39}$$

$$(-1)^k a_{2k+1} \geq (-1)^k b_{2k+1} \text{ for all } k = 0, 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor; \tag{8.40}$$

or

(ii)

$$a_{2k+1} = b_{2k+1} \quad \text{for all } k = 0, 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{and} \quad (8.41)$$

$$(-1)^k a_{2k} \leq (-1)^k b_{2k} \quad \text{for all } k = 0, 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor, \quad \text{or} \quad (8.42)$$

$$(-1)^k a_{2k} \geq (-1)^k b_{2k} \quad \text{for all } k = 0, 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor \quad (8.43)$$

and for any polynomial $g(z) = c_0 z^n + c_1 z^{n-1} + \dots + c_{n-1} z + c_n$ the following inequalities hold:

$$\min \{a_i, b_i\} \leq c_i \leq \max \{a_i, b_i\}, \quad \text{for } i = 0, 1, \dots, n.$$

Then g is stable.

An *interval polynomial*, denoted by $\mathcal{I}(z)$, is the set of polynomials whose coefficients vary in closed intervals

$$a_k \in [\underline{a}_k, \bar{a}_k], \quad k = 0, 1, \dots, n. \quad (8.44)$$

These interval coefficients can be also regarded as perturbed parameters. It is, of course, desirable that under perturbations the stability property is maintained and it is advantageous when for the invariance of stability only information on extreme values of the perturbed parameters is needed. Let \mathcal{H}_n denotes the set of Hurwitz polynomials of degree n and let

$$\bar{\mathcal{I}}(z) := \{f(z) \mid a_i = \underline{a}_i, \text{ or } a_i = \bar{a}_i, i = 0, 1, \dots, n\} \quad (8.45)$$

be the set of *vertex polynomials*. Then $\bar{\mathcal{I}}(z)$ has at most 2^{n+1} elements. In [Kha78] it was shown that, see e.g., [MK87]

$$\mathcal{I}(z) \subset \mathcal{H}_n \quad \text{if and only if} \quad \bar{\mathcal{I}}(z) \subset \mathcal{H}_n. \quad (8.46)$$

He proved further the surprising fact that checking the stability of only four members of $\bar{\mathcal{I}}(z)$ is sufficient for $\mathcal{I}(z) \subset \mathcal{H}_n$.

Theorem 8.23. [Kha78], [BCK95, Theorem 5.1, (Kharitonov's Theorem)] *Every polynomial in the family $\mathcal{I}(z)$ is Hurwitz if and only if the following four vertex polynomials are Hurwitz:*

$$K^1(z) = \underline{a}_n + \underline{a}_{n-1}z + \bar{a}_{n-2}z^2 + \bar{a}_{n-3}z^3 + \underline{a}_{n-4}z^4 + \underline{a}_{n-5}z^5 + \bar{a}_{n-6}z^6 + \dots, \quad (8.47)$$

$$K^2(z) = \underline{a}_n + \bar{a}_{n-1}z + \bar{a}_{n-2}z^2 + \underline{a}_{n-3}z^3 + \underline{a}_{n-4}z^4 + \bar{a}_{n-5}z^5 + \bar{a}_{n-6}z^6 + \dots, \quad (8.48)$$

$$K^3(z) = \bar{a}_n + \underline{a}_{n-1}z + \underline{a}_{n-2}z^2 + \bar{a}_{n-3}z^3 + \bar{a}_{n-4}z^4 + \underline{a}_{n-5}z^5 + \underline{a}_{n-6}z^6 + \dots, \quad (8.49)$$

$$K^4(z) = \bar{a}_n + \bar{a}_{n-1}z + \underline{a}_{n-2}z^2 + \underline{a}_{n-3}z^3 + \bar{a}_{n-4}z^4 + \bar{a}_{n-5}z^5 + \underline{a}_{n-6}z^6 + \dots. \quad (8.50)$$

Theorem 8.23 can be shown by using Theorem 8.22 and Lemma 8.3. It holds under the assumption that the degree does not change throughout the interval family. The case of degree dropping is more involved and it is required to test more members of the interval family, see [BCK95, Remark 5.2].

Remark 8.2. *For small n the number of polynomials involved in Theorem 8.23 can be reduced, see, e.g., [Bar94]. In the general case, this number of polynomials cannot be reduced.*

Let $H_{\mathcal{I}}$ be the following $(n+1) \times n$ matrix whose entries are composed from the endpoints of the intervals given by (8.44) ($\underline{a}_k := 0$ and $\bar{a}_k := 0$ for $k > n$)

$$H_{\mathcal{I}} := \begin{bmatrix} \underline{a}_1 & \bar{a}_3 & \underline{a}_5 & \bar{a}_7 & \underline{a}_9 & \bar{a}_{11} & \dots \\ \bar{a}_0 & \underline{a}_2 & \bar{a}_4 & \underline{a}_6 & \bar{a}_8 & \underline{a}_{10} & \dots \\ 0 & \bar{a}_1 & \underline{a}_3 & \bar{a}_5 & \underline{a}_7 & \bar{a}_9 & \dots \\ 0 & \underline{a}_0 & \bar{a}_2 & \underline{a}_4 & \bar{a}_6 & \underline{a}_8 & \dots \\ 0 & 0 & \underline{a}_1 & \bar{a}_3 & \underline{a}_5 & \bar{a}_7 & \dots \\ 0 & 0 & \bar{a}_0 & \underline{a}_2 & \bar{a}_4 & \underline{a}_6 & \dots \\ 0 & 0 & 0 & \bar{a}_1 & \underline{a}_3 & \bar{a}_5 & \dots \\ 0 & 0 & 0 & \underline{a}_0 & \bar{a}_2 & \underline{a}_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \quad (8.51)$$

Let $H_1 := H_{\mathcal{I}}[1, \dots, n-1]$ and $H_2 := H_{\mathcal{I}}[3, \dots, n+1|2, \dots, n]$. Assume that $H_{\mathcal{I}}$ is *ATP* and $\underline{a}_k > 0$, $k = 0, 1, \dots, n$. Then H_1 and H_2 are also *ATP* since they are submatrices of $H_{\mathcal{I}}$. Moreover, the entries on the main diagonal of H_1 and H_2 are positive. Hence H_1 and H_2 are *NsTN*. Furthermore, we see that $H_1 \leq^* H_2$.

Theorem 8.24. [AGT16, Theorem 3.6] *If H_1 and H_2 are NsTN, then all polynomials p given by (7.31) satisfying (8.44) are stable.*

Proof. Let p be any polynomial satisfying (8.44) and $H(p)$ be its Hurwitz matrix. Then it is easy to see that

$$H_1 \leq^* H(p)[1, \dots, n-1] \leq^* H_2. \quad (8.52)$$

By Theorem 4.6 $H(p)[1, \dots, n-1]$ is *NsTN*, hence p is stable by Lemma 2.6 and the Routh-Hurwitz's Criterion. \square

In contrast to Kharitonov's Theorem, Theorem 8.24 also holds if $\underline{a}_0 = 0$ (but $\underline{a}_1 > 0$), i.e., this theorem holds if the interval polynomial contains polynomials of degree $n-1$. However, our condition in Theorem 8.24 is not necessary. A counterexample is provided by the following example.

Example 8.1. [AGT16, Example 3.7] *Let the family of polynomials be given by*

$$\underline{a}_0 = \bar{a}_0 = 1, \underline{a}_1 = 3.5, \underline{a}_2 = \underline{a}_3 = \bar{a}_1 = 6.5, \bar{a}_2 = \bar{a}_3 = 9.5, \underline{a}_4 = 1.5, \bar{a}_4 = 4.5.$$

Then all polynomials satisfying (8.44) are stable, see [Bar84]. However, $\det H_1 < 0$.

The complexity of the stability test based on Theorem 8.24 is $O(n^2)$ for an n^{th} degree polynomial; the test based on checking $H_{\mathcal{I}}$ for being *ATP* requires slightly less operations.

In passing we note that with a matrix of type (8.51) and two submatrices like H_1 and H_2 we easily obtain by using Theorem 8.14 a theorem like Theorem 8.24 on an interval family of *R*-functions given by (7.30) of negative type with exactly n negative poles.

Let p and q be two real Hurwitz polynomials given by (7.31)-(7.32). For $\lambda \in [0, 1]$, define

$$f_\lambda(z) = (1 - \lambda)p(z) + \lambda q(z). \quad (8.53)$$

Then it is not true that f_λ is Hurwitz for all $\lambda \in [0, 1]$. For instance, consider the following two polynomials with $\lambda = 2/3$ which are given in [BG85, Example 1]

$$p(z) = z^3 + z^2 + 2z + 1, \quad (8.54)$$

$$q(z) = z^3 + 0.001z^2 + 0.001z + 10^{-10}. \quad (8.55)$$

The following theorem states that the convex combination of two Hurwitz polynomials of the same degree is Hurwitz whenever they have the same even or odd part.

Theorem 8.25. [BG85, Corollary 1] *Let the real polynomials p and q given by (7.31)-(7.32) be Hurwitz. Then for each $\lambda \in [0, 1]$ f_λ given by (8.53) is Hurwitz if*

$$a_{2k+1} = b_{2k+1}, \quad k = 0, 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor,$$

or

$$a_{2k} = b_{2k}, \quad k = 0, 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor.$$

We conclude this section with the following theorem which considers the convex combination of two Hurwitz polynomials of which the degrees differ by one and the odd indexed coefficients are equal.

Theorem 8.26. [BG85, Theorem II] *Let the real polynomials p, q be given by (7.31)-(7.32), and $b_0 = 0, b_1 \neq 0$ be Hurwitz. Then for each $\lambda \in [0, 1]$ f_λ given by (8.53) is Hurwitz if*

$$a_{2k+1} = b_{2k+1}, \quad k = 0, 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor. \quad (8.56)$$

The condition (8.56) of Theorem 8.26 cannot be replaced by the condition $a_{2k} = b_{2k}$. For instance, consider the polynomial $p(z)$ given by (8.54), $q(z) = z^2 + 2z + 7$, and $\lambda = 1/2$, cf. [BG85].

A. Alternative Proofs

In this appendix we present alternative proofs for Proposition 3.8, Garloff's Conjecture, and analogous results in the case of nonsingular totally nonpositive matrices.

Remark A.1. Let $A \in \mathbb{R}^{n,n}$ be NsTN (Ns.t.n.p.) and let $\alpha = (i, \dots, i+r)$, $\beta = (j, \dots, j+r)$ for some $i, j \in \{1, \dots, n\}$ with $j < i$ and $1 \leq r \leq n-1$. Suppose that $\det A[\alpha|\beta] = 0$ and $\det A[\alpha_i|\beta_j] \neq 0$. Then by Lemma 2.7 (Lemma 2.5) $B := A[i, \dots, n|1, \dots, j+r]$ has rank r . Hence the rows of B are linearly dependent. Since $\det A[\alpha_i|\beta_j] \neq 0$ we have that the 2^{nd} , 3^{rd} , \dots , and $(r+1)^{\text{th}}$ rows of B are linearly independent. As a consequence, the first row of B can be written as a linear combination of the 2^{nd} , 3^{rd} , \dots , and $(r+1)^{\text{th}}$ rows of B , i.e., if $\mathbf{b}^1, \dots, \mathbf{b}^{n-i+1}$ represent the rows of B , then for some $c_2, c_3, \dots, c_{r+1} \in \mathbb{R}$ we have

$$\mathbf{b}^1 = c_2 \mathbf{b}^2 + c_3 \mathbf{b}^3 + \dots + c_{r+1} \mathbf{b}^{r+1}. \quad (\text{A.1})$$

In the representation (A.1) $c_{r+1} \neq 0$ holds. Otherwise row \mathbf{b}^1 depends linearly on the rows $\mathbf{b}^2, \mathbf{b}^3, \dots, \mathbf{b}^r$ and hence $\det A[\alpha_{\widehat{i+r}}|\beta_{\widehat{j}}] = 0$ which implies by Lemma 2.6 and Lemma 2.7 (Proposition 2.3 or 2.4 and Lemma 2.5) that $A[i, \dots, n|1, \dots, j+r]$ has rank $r-1$ if $j+1 < i$ which is a contradiction. If $i = j+1$ then by Lemma 2.6 (Proposition 2.3 or 2.4) we have a contradiction. Hence by using (A.1) and determinantal properties the following equality holds:

$$\det A[\alpha_i|\beta_j] = \frac{(-1)^{r-1}}{c_{r+1}} \det A[\alpha_{\widehat{i+r}}|\beta_j], \quad (\text{A.2})$$

thus $\det A[\alpha_{\widehat{i+r}}|\beta_j] \neq 0$, whence

$$\frac{\det A[\alpha_i|\beta_{\widehat{j+r}}]}{\det A[\alpha_i|\beta_j]} = \frac{\det A[\alpha_{\widehat{i+r}}|\beta_{\widehat{j+r}}]}{\det A[\alpha_{\widehat{i+r}}|\beta_j]}. \quad (\text{A.3})$$

An analogous result holds if $i < j$.

Remark A.2. Let $A \in \mathbb{R}^{n,n}$ be NsTN (Ns.t.n.p. with $a_{nn} < 0$). Then by Procedure 3.1 (Procedure 3.3) we can construct a lacunary sequence $((i_0, j_0), (i_1, j_1), \dots, (i_p, j_p))$ starting from each position (i_0, j_0) , $i_0, j_0 = 1, \dots, n$, with respect to the Cauchon diagram $C_{\bar{A}}$ and by Proposition 3.4 (Proposition 3.7) we have $\det A[i_0, i_1, \dots, i_p|j_0, j_1, \dots, j_p] = \tilde{a}_{i_0, j_0} \cdot \tilde{a}_{i_1, j_1} \cdots \tilde{a}_{i_p, j_p}$. Hence by application of this property to the lacunary sequence $((i_1, j_1), \dots, (i_p, j_p))$ we obtain that

$$\tilde{a}_{i_0, j_0} = \frac{\det A[i_0, i_1, \dots, i_p|j_0, j_1, \dots, j_p]}{\det A[i_1, \dots, i_p|j_1, \dots, j_p]}. \quad (\text{A.4})$$

Therefore each entry of \tilde{A} can be represented as a ratio of two minors. In the following proposition we strengthen this representation in that each entry of \tilde{A} can even be represented as a ratio of two contiguous minors.

We will present the proofs of the following statements only in the case of $NsTN$ matrices since the proofs in the $Ns.t.n.p.$ case are analogous.

Proposition A.1. *Let $A = (a_{kj}) \in \mathbb{R}^{n,n}$ be $NsTN$ ($Ns.t.n.p.$ with $a_{nn} < 0$). Then the entries \tilde{a}_{kj} of the matrix $\tilde{A} = (\tilde{a}_{kj})$ can be represented as*

$$\tilde{a}_{kj} = \frac{\det A[k, k+1, \dots, k+l | j, j+1, \dots, j+l]}{\det A[k+1, \dots, k+l | j+1, \dots, j+l]}, \quad (\text{A.5})$$

where l is called the order of the representation (A.5) and is given by

$$l := \begin{cases} \max \{h \in \{0, 1, \dots, p\} \mid d((j_0, j_1, \dots, j_h)) = 0\} & \text{if } j_0 \leq k_0, \\ \max \{h \in \{0, 1, \dots, p\} \mid d((k_0, k_1, \dots, k_h)) = 0\} & \text{if } k_0 < j_0, \end{cases} \quad (\text{A.6})$$

and $((k_0, j_0), (k_1, j_1), \dots, (k_p, j_p))$ is the lacunary sequence that is constructed by Procedure 3.1 (Procedure 3.3) and which starts from the position $(k_0, j_0) = (k, j)$.

Proof. Let A be $NsTN$. Then by Theorem 3.2 (ii) and Proposition 3.5 \tilde{A} is a nonnegative Cauchon matrix with positive diagonal entries. By Remark A.2, for each position (k, j) , $k, j = 1, \dots, n$, there exists a lacunary sequence $((k_0, j_0), (k_1, j_1), \dots, (k_p, j_p))$ with respect to the Cauchon diagram $C_{\tilde{A}}$ starting from the position $(k, j) = (k_0, j_0)$ such that

$$\tilde{a}_{kj} = \tilde{a}_{k_0, j_0} = \frac{\det A[k_0, k_1, \dots, k_p | j_0, j_1, \dots, j_p]}{\det A[k_1, \dots, k_p | j_1, \dots, j_p]}. \quad (\text{A.7})$$

The proof proceeds by decreasing induction on $k = 1, \dots, n$. For $k = n$ the result holds trivially. Suppose that we have shown the claim for all (k, j) with $k = k_0 + 1, \dots, n$ and $j = 1, \dots, n$, and we want to show it for (k, j) with $k = k_0$ and $j = 1, \dots, n$. We distinguish the following three cases:

Case 1: If $k_0 = j_0$, then the sequence $((k_0, k_0), (k_0 + 1, k_0 + 1), \dots, (n, n))$ is a lacunary sequence and hence by (A.7) we are done.

Case 2: If $j_0 < k_0$, then the following two cases are possible:

Case 2.1: Suppose that $d((j_0, j_1, \dots, j_p)) = 0$. If we also have $d((k_0, k_1, \dots, k_p)) = 0$, then by (A.7) we are done. Suppose now that $d((k_0, k_1, \dots, k_p)) > 0$. Set

$$t := \min \{h \in \{0, 1, \dots, p\} \mid d((k_h, k_{h+1}, \dots, k_p)) = 0\}.$$

Then \tilde{a}_{k_t-1, j_t} and $\tilde{a}_{k_t-1, j_{t-1}}$ vanish and $((k_t - 1, j_{t-1}), (k_t, j_t), \dots, (k_p, j_p))$ is a lacunary sequence. Hence by the induction hypothesis it follows that

$$\tilde{a}_{k_t-1, j_{t-1}} = \frac{\det A[k_t - 1, k_t, \dots, k_p | j_{t-1}, j_t, \dots, j_p]}{\det A[k_t, \dots, k_p | j_t, \dots, j_p]}. \quad (\text{A.8})$$

Note that $k_{t+\kappa} = k_t + \kappa$ and $j_{t+\kappa} = j_t + \kappa$ for $\kappa = 0, 1, \dots, p - t$. Hence by Lemma 2.7, $A[k_t - 1, k_t, \dots, n | 1, 2, \dots, j_p]$ has rank $p - t + 1$. Therefore by using arguments like that have been employed in Remark A.1 it follows that the row $k_t - 1$ of the matrix $B := A[1, \dots, n | 1, \dots, j_p]$ can be written as a linear combination of the rows k_t, k_{t+1}, \dots, k_p of the matrix B with nonzero coefficient c_{k_p} of the row k_p . Whence by substituting the representation of the row k_p in terms of the rows $k_t - 1, k_t, \dots, k_{p-1}$ of the matrix B in (A.7) we obtain

$$\tilde{a}_{k_0, j_0} = \frac{\det A[k_0, \dots, k_{t-1}, k_t - 1, k_t, k_t + 1, \dots, k_t + p - t - 1 | j_0, j_1, \dots, j_p]}{\det A[k_1, \dots, k_{t-1}, k_t - 1, k_t, k_t + 1, \dots, k_t + p - t - 1 | j_1, \dots, j_p]}, \quad (\text{A.9})$$

where

$$\det A[k_1, \dots, k_p | j_1, \dots, j_p] = \frac{(-1)^{p-t}}{c_{k_p}} \cdot \det A[k_1, \dots, k_{t-1}, k_t - 1, k_t, k_t + 1, \dots, k_t + p - t - 1 | j_1, \dots, j_p]. \quad (\text{A.10})$$

Therefore by (A.7) and (A.10) the denominator of (A.9) is nonzero.

By using the induction hypothesis and repeating the above arguments with each gap in $\{k_0, k_1, \dots, k_{t-1}, k_t - 1\}$ we arrive at (A.5), i.e., if $k_{t-1} + 1 < k_t - 1$, then $\tilde{a}_{k_t-2, j_{t-1}} = 0$ and so by Procedure 3.1 $((k_t - 2, j_{t-1}), (k_t, j_t), \dots, (k_p, j_p))$ is a lacunary sequence with respect to $C_{\tilde{A}}$ and by the assumption $d((j_{t-1}, j_t, \dots, j_p)) = 0$. Therefore by the induction hypothesis $\tilde{a}_{k_t-2, j_{t-1}}$ allows the following representation

$$\tilde{a}_{k_t-2, j_{t-1}} = \frac{\det A[k_t - 2, k_t - 1, \dots, k_t + p - t - 1 | j_{t-1}, j_t, \dots, j_p]}{\det A[k_t - 1, \dots, k_t + p - t - 1 | j_t, \dots, j_p]}.$$

Whence repeated application of the arguments that have been used in the case $\tilde{a}_{k_t-1, j_{t-1}} = 0$ yields

$$\tilde{a}_{k_0, j_0} = \frac{\det A[k_0, \dots, k_{t-1}, k_t - 2, k_t - 1, k_t, k_t + 1, \dots, k_t + p - t - 2 | j_0, j_1, \dots, j_p]}{\det A[k_1, \dots, k_{t-1}, k_t - 2, k_t - 1, k_t, k_t + 1, \dots, k_t + p - t - 2 | j_1, \dots, j_p]}.$$

If $k_{t-1} + 1 = k_t - 1$, then set

$$t' := \min \{h \in \{0, 1, \dots, t - 1\} \mid d((k_h, k_{h+1}, \dots, k_{t-1})) = 0\}.$$

Then $\tilde{a}_{k_{t'}-1, j_{t'}}$ and $\tilde{a}_{k_{t'}-1, j_{t'-1}}$ vanish and $((k_{t'} - 1, j_{t'-1}), (k_{t'}, j_{t'}), (k_{t'+1}, j_{t'+1}), \dots, (k_p, j_p))$ is a lacunary sequence. Hence by the induction hypothesis it follows that

$$\tilde{a}_{k_{t'}-1, j_{t'-1}} = \frac{\det A[k_{t'} - 1, k_{t'}, \dots, k_{t'} + p - t' | j_{t'-1}, j_{t'}, \dots, j_p]}{\det A[k_{t'}, \dots, k_{t'} + p - t' | j_{t'}, \dots, j_p]}. \quad (\text{A.11})$$

By employing similar arguments as in the above cases and noting that $k_{t'} + p - t' = k_t + p - t - 1$ we obtain the following representation

$$\tilde{a}_{k_0, j_0} = \frac{\det A[k_0, \dots, k_{t'-1}, k_{t'} - 1, k_{t'}, \dots, k_t - 1, k_t, k_t + 1, \dots, k_t + p - t - 2 | j_0, j_1, \dots, j_p]}{\det A[k_1, \dots, k_{t'-1}, k_{t'} - 1, k_{t'}, \dots, k_t - 1, k_t, k_t + 1, \dots, k_t + p - t - 2 | j_1, \dots, j_p]}.$$

Case 2.2: Suppose that $d((j_0, j_1, \dots, j_p)) > 0$. Then set

$$l := \max \{h \in \{0, 1, \dots, p\} \mid d((j_0, j_1, \dots, j_h)) = 0\}.$$

Since \tilde{A} is a Cauchon matrix with positive diagonal entries we conclude that $\tilde{A}[k_l + 1, k_l + 2, \dots, n \mid 1, 2, \dots, j_l + 1] = 0$ which implies by the Cauchon Algorithm that $A[k_l + 1, k_l + 2, \dots, n \mid 1, 2, \dots, j_l + 1] = 0$. Hence by (A.7) and the zero-nonzero pattern of A we obtain

$$\begin{aligned} \tilde{a}_{k_0, j_0} &= \frac{\det A[k_0, k_1, \dots, k_l \mid j_0, j_1, \dots, j_l] \cdot \det A[k_{l+1}, \dots, k_p \mid j_{l+1}, \dots, j_p]}{\det A[k_1, \dots, k_l \mid j_1, \dots, j_l] \cdot \det A[k_{l+1}, \dots, k_p \mid j_{l+1}, \dots, j_p]} \\ &= \frac{\det A[k_0, k_1, \dots, k_l \mid j_0, j_1, \dots, j_l]}{\det A[k_1, \dots, k_l \mid j_1, \dots, j_l]}, \end{aligned} \quad (\text{A.12})$$

where $d((j_0, j_1, \dots, j_l)) = 0$. The zero-nonzero pattern of A implies that if we have $\tilde{a}_{k_{q+1}, j_{q+1}} = 0$ for some $q < l$ then by the induction hypothesis the representation of $\tilde{a}_{k_{q+1}, j_{q+1}} = 0$ will be a ratio of two contiguous minors in $A[1, \dots, k_l \mid 1, \dots, j_l]$. Now we apply the procedure used in Case 2.1 to the representation (A.12) and arrive finally at (A.5).

Case 3: If $k_0 < j_0$, then we proceed parallel to Case 2 working with the columns. \square

Lemma A.1. *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be $NsTN$ ($Ns.t.n.p.$ with $a_{nn} < 0$) and \tilde{a}_{kj} have a representation (A.5). Then if $k + l, j + l < n$ the following holds*

$$\det A[k, k + 1, \dots, k + l, k + l + 1 \mid j, j + 1, \dots, j + l, j + l + 1] = 0. \quad (\text{A.13})$$

Proof. Let A be $NsTN$. The proof follows by decreasing induction on $k = 1, \dots, n - 1$. If $k = n - 1$ and $\tilde{a}_{n-1, j}$ has a representation (A.5) of order 0 i.e., $\tilde{a}_{n-1, j} = a_{n-1, j}$, then we have that $a_{n, j+1} = 0$. Since A is $NsTN$ then by Lemma 2.7 we have that $a_{ni} = 0$ for all $i = 1, \dots, j + 1$. Whence $\det A[n - 1, n \mid j, j + 1] = 0$. Suppose now that the claim holds for all the entries of \tilde{A} that appear in rows $n - 1, \dots, k + 1$ and we want to show the claim (A.13) for row k . We will present here the proof only for the case $j < k$ since in the case $k < j$ we proceed analogously. If $\tilde{a}_{kj} = 0$ then by (A.5) $\det A[k, k + 1, \dots, k + l \mid j, j + 1, \dots, j + l] = 0$. Hence by Lemma 2.6 the claim follows.

Suppose now that $\tilde{a}_{kj} > 0$. Let $((k, j), (k_1, j_1), \dots, (k_p, j_p))$ be the lacunary sequence with respect to the Cauchon diagram $C_{\tilde{A}}$ that is constructed by Procedure 3.1 and starts from the position $(k, j) = (k_0, j_0)$. Set

$$t := \min \{h \in \{0, 1, \dots, p - 1\} \mid \tilde{a}_{k_{h+1}, j_{h+1}} = 0\}.$$

Such t exists since $k + l, j + l < n$. We distinguish the following two cases:

Case 1: If $j_t + 1 < j_{t+1}$, then $\tilde{A}[k_t + 1, \dots, n \mid 1, \dots, j_t + 1] = 0$ and hence $A[k_t + 1, \dots, n \mid 1, \dots, j_t + 1] = 0$. Thus by (A.6) $t = l$ and by (A.5) $\tilde{a}_{kj} = \frac{\det A[k, k + 1, \dots, k_t \mid j, j + 1, \dots, j_t]}{\det A[k + 1, \dots, k_t \mid j + 1, \dots, j_t]}$ and so in the present case the submatrix $A[k, k + 1, \dots, k_t + 1 \mid j, j + 1, \dots, j_t + 1]$ has a zero row.

Case 2: If $j_{t+1} = j_t + 1$, then if the order of the representation of the entry $\tilde{a}_{k_{t+1}, j_{t+1}}$ is less than or equal to $l - t$, then by Lemma 2.6 we are done. Otherwise we decrease the order of

the representation of $\tilde{a}_{k_{t+1}, j_{t+1}}$ to $l-t$ as follows. The sequence $((k_{t+1}, j_{t+1}), (k_{t+2}, j_{t+2}), \dots, (k_p, j_p))$ is a lacunary sequence and $0 < \tilde{a}_{k_{t+1}, j_{t+1}}$. By Proposition A.1, $\tilde{a}_{k_{t+1}, j_{t+1}}$ can be written as

$$\tilde{a}_{k_{t+1}, j_{t+1}} = \frac{\det A[k_{t+1}, k_{t+1} + 1, \dots, k_{t+1} + l - t - 1 | j_{t+1}, j_{t+1} + 1, \dots, j_{t+1} + l - t - 1]}{\det A[k_{t+1} + 1, \dots, k_{t+1} + l - t - 1 | j_{t+1} + 1, \dots, j_{t+1} + l - t - 1]}.$$

If $k_{t+1} + l - t - 1 < n$, then by the induction hypothesis we conclude that

$$\det A[k_{t+1}, k_{t+1} + 1, \dots, k_{t+1} + l - t | j_{t+1}, j_{t+1} + 1, \dots, j_{t+1} + l - t] = 0. \quad (\text{A.14})$$

If $k_{t+1} + l - t - 1 = n$, then $\tilde{a}_{k_{t+1}-1, j_{t+1}} = 0$ has a representation of order $l-t$. Hence in both cases the order of the representation of $\tilde{a}_{k_{t+1}-1, j_{t+1}} = 0$ can be decreased to $l-t$ if it is greater than $l-t$ by using Lemmata 1.3, 2.6, and (A.14). If $k_{t+1} - 1 = k_t + 1$ then we are done. Otherwise in the same way the order of the representation of $\tilde{a}_{k_{t+1}-2, j_{t+1}} = 0$ can be also decreased to $l-t$ if it is greater than $l-t$ since $\tilde{a}_{k_{t+1}-1, j_{t+1}} = 0$ by the choice of the sequence $((k, j), (k_1, j_1), \dots, (k_p, j_p))$. If $k_{t+1} - 2 = k_t + 1$ then we are done. Otherwise repeat the above arguments till we arrive at the representation of $\tilde{a}_{k_{t+1}, j_{t+1}}$ of order $l-t$. Hence by Lemma 2.6 we are done. \square

Theorem A.1. *Let $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n,n}$ be $NsTN$ (Ns.t.n.p. with $b_{nn} < 0$) and $A \leq^* B$. Then $\tilde{A} = (\tilde{a}_{kj}) \leq^* \tilde{B} = (\tilde{b}_{kj})$ and the entries \tilde{a}_{kj} and \tilde{b}_{kj} can be represented as ratios of contiguous minors of the same order, $k, j = 1, \dots, n$.*

Proof. Let A and B be $NsTN$. Then by Theorem 3.2 (ii) and Proposition 3.5 \tilde{A} and \tilde{B} are nonnegative Cauchon matrices with positive diagonal entries. We proceed by decreasing induction on $k = 1, \dots, n$ and show that for each $k, j = 1, \dots, n$ if \tilde{a}_{kj} and \tilde{b}_{kj} have representations as in (A.5) of order l and l' , respectively, then both of them can be represented as ratios of contiguous minors of order $\min\{l, l'\}$ and $(-1)^{k+j}\tilde{a}_{kj} \leq (-1)^{k+j}\tilde{b}_{kj}$. For $k = n$ the result is trivial and follows by the Cauchon Algorithm and the assumption that $A \leq^* B$. Suppose the claim holds for all the entries that appear in the rows $k+1, \dots, n$. We show that the claim holds for the entries that appear in row k . Let $((k, j), (k_1, j_1), \dots, (k_p, j_p))$ and $((k, j), (k'_1, j'_1), \dots, (k'_{p'}, j'_{p'}))$ be the lacunary sequences that are constructed by Procedure 3.1 and start from the position (k, j) with respect to the Cauchon diagrams $C_{\tilde{A}}$ and $C_{\tilde{B}}$, respectively. Then by Proposition A.1, \tilde{a}_{kj} and \tilde{b}_{kj} allow the following representations

$$\tilde{a}_{kj} = \frac{\det A[k, \dots, k+l | j, \dots, j+l]}{\det A[k+1, \dots, k+l | j+1, \dots, j+l]}, \quad (\text{A.15})$$

$$\tilde{b}_{kj} = \frac{\det B[k, \dots, k+l' | j, \dots, j+l']}{\det B[k+1, \dots, k+l' | j+1, \dots, j+l']}, \quad (\text{A.16})$$

where l and l' are defined by (A.6). We distinguish the following three cases:

Case 1: If $k = j$, then $\tilde{a}_{kk} = \frac{\det A[k, \dots, n]}{\det A[k+1, \dots, n]}$ and $\tilde{b}_{kk} = \frac{\det B[k, \dots, n]}{\det B[k+1, \dots, n]}$. Hence by Lemma 4.4 (formula (4.5) taking the reciprocal values) we conclude that $\tilde{a}_{kk} \leq \tilde{b}_{kk}$ since $A[k, \dots, n] \leq^* B[k, \dots, n]$ and by Lemma 2.6 both of the submatrices are $NsTN$ for each $k = 1, \dots, n-1$.

Case 2: If $j < k$, then we have the following two cases:

Case 2.1: Let $k + j$ be even. Then the following three cases are possible:

Case 2.1.1: Suppose that $l = l'$. Then by (4.5) we conclude that $\tilde{a}_{kj} \leq \tilde{b}_{kj}$ since $k + j$ is even (add $\epsilon > 0$ to the $(1, 1)$ entries of the underlying submatrices if necessary, see Case 1 in the proof of Proposition 4.1).

Case 2.1.2: Suppose that $l < l'$. Then the following two cases are possible:

Case 2.1.2.1: $\tilde{a}_{k+1,j+1} = 0$. If we also have that $\tilde{A}[k+1, \dots, n|j+1] = 0$ then we obtain $l = 0$ and $\tilde{A}[k+1, \dots, n|1, \dots, j+1] = 0$. Therefore $\tilde{a}_{kj} = a_{kj}$ and $A[k+1, \dots, n|1, \dots, j+1] = 0$ since A is $NsTN$. As a consequence of B is $NsTN$, $A \leq^* B$, and $k + j$ is even we conclude that $B[k+1, \dots, n|1, \dots, j] = 0$. Thus by the Cauchon Algorithm we also have that $\tilde{b}_{kj} = b_{kj}$ and the claim follows. Otherwise $\tilde{A}[k+1, \dots, n|j+1] \neq 0$. Hence the sequence $((k+1, j), (k_1, j_1), \dots, (k_p, j_p))$ is lacunary sequence with respect to $C_{\tilde{A}}$ and $\tilde{a}_{k+1,j} = 0$ since $\tilde{a}_{k+1,j+1} = 0$ and $j < k$. By Proposition A.1

$$\tilde{a}_{k+1,j} = \frac{\det A[k+1, \dots, k+l+1|j, j+1, \dots, j+l]}{\det A[k+2, \dots, k+l+1|j+1, \dots, j+l]} = 0.$$

Hence by the induction hypothesis $\tilde{b}_{k+1,j} = 0$ and has a representation of order l or by Proposition A.1 order less than l . Therefore by Lemma 2.6 all the leading principal minors of $B[k+1, \dots, k+l'|j, \dots, j+l'-1]$ of order greater than l vanish. Hence by repeated application of Lemma 1.3 to the numerator of the representation \tilde{b}_{kj} we obtain that $\tilde{b}_{kj} = \frac{\det B[k, k+1, \dots, k+l|j, j+1, \dots, j+l]}{\det B[k+1, \dots, k+l|j+1, \dots, j+l]}$. Hence by proceeding with the representations of \tilde{a}_{kj} and \tilde{b}_{kj} of order l as in Case 2.1.1 we are done.

Case 2.1.2.2: $0 < \tilde{a}_{k+1,j+1}$. Then the representation of $\tilde{a}_{k+1,j}$ given by (A.5) has order t which is less than that of \tilde{a}_{kj} by at least one since the lacunary sequence that is constructed by Procedure 3.1 and starts from the position $(k+1, j)$ has length less than that of the lacunary sequence $((k, j), (k_1, j_1), \dots, (k_p, j_p))$. Suppose that $\tilde{a}_{k+1,j} = \frac{\det A[k+1, \dots, k+t+1|j, j+1, \dots, j+t]}{\det A[k+2, \dots, k+t+1|j+1, \dots, j+t]}$, where $t < l$. If $\tilde{a}_{k+1,j} = 0$, then by the induction hypothesis $\tilde{b}_{k+1,j} = 0$ and by arguments that have been employed in Case 2.1.2.1 all the leading principal minors of $B[k+1, \dots, k+l'|j, \dots, j+l'-1]$ of order greater than t vanish. If $\tilde{a}_{k+1,j} > 0$, then by Lemma A.1 $\det A[k+1, \dots, k+t+2|j, \dots, j+t+1] = 0$. If $\det B[k+1, \dots, k+t+1|j, \dots, j+t] = 0$, then by Lemma 2.6 $\det B[k+1, \dots, k+t+2|j, \dots, j+t+1] = 0$. Otherwise by Lemma 4.4 and adding $\epsilon > 0$ to the bottom right entries of the submatrices $B[k+1, \dots, k+t+2|j, \dots, j+t+1] \leq^* A[k+1, \dots, k+t+2|j, \dots, j+t+1]$ we obtain that $\det B[k+1, \dots, k+t+2|j, \dots, j+t+1] = 0$ by letting ϵ tend to zero. Hence all the leading principal minors of $B[k+1, \dots, k+l'|j, \dots, j+l'-1]$ of order greater than $t+1$ vanish. Since $t < l$ repeated application of Lemma 1.3 to the numerator of the representation of \tilde{b}_{kj} yields the desired result as in the Case 2.1.2.1.

Case 2.1.3: Suppose that $l' < l$. Then the following two cases are possible:

Case 2.1.3.1: $\tilde{b}_{k+1,j+1} = 0$. Then if we also have that $\tilde{B}[k+1, \dots, n|j+1] = 0$ then we conclude like in Case 2.1.2.1 that $\tilde{a}_{kj} = a_{kj}$ and $\tilde{b}_{kj} = b_{kj}$. Hence we are done. Otherwise the sequence $((k+1, j), (k'_1, j'_1), \dots, (k'_{p'}, j'_{p'}))$ is the lacunary sequence that starts from the position $(k+1, j)$ and is constructed by Procedure 3.1 with respect to the Cauchon diagram

$C_{\tilde{B}}$. Therefore $\tilde{b}_{k+1,j} = 0$ allows the following representation:

$$\tilde{b}_{k+1,j} = \frac{\det B[k+1, \dots, k+l'+1 | j, j+1, \dots, j+l']}{\det B[k+2, \dots, k+l'+1 | j+1, \dots, j+l']}.$$

Hence by the induction hypothesis $\tilde{a}_{k+1,j} = 0$ and has a representation of order l' or by Proposition A.1 order less than l' . Hence all leading principal minors of $A[k+1, \dots, k+l | j, \dots, j+l-1]$ of order greater than l' vanish. Whence repeated application of Lemma 1.3 to the numerator of the representation of $\tilde{a}_{k,j}$ will decrease its order to l' . Hence by proceeding with the representations of $\tilde{a}_{k,j}$ and $\tilde{b}_{k,j}$ of order l' as in Case 2.1.1 we are done.

Case 2.1.3.2: $0 < \tilde{b}_{k+1,j+1}$. Then the sequence $((k+1, j+1), (k'_2, j'_2), \dots, (k'_{p'}, j'_{p'}))$ is the lacunary sequence that starts from the position $(k+1, j+1)$ with respect to the Cauchon diagram $C_{\tilde{B}}$ according to Procedure 3.1. Therefore by the induction hypothesis, the lacunary sequence that starts from the position $(k+1, j)$ with respect to the Cauchon diagram $C_{\tilde{A}}$ according to Procedure 3.1 has length at most $p' - 1$. Hence by Lemma A.1 (and Lemma 2.6) $\det A[k+1, \dots, k+l'+1 | j, \dots, j+l'] = 0$. Hence by Lemmata 1.3, 2.6, and proceeding similarly as in Case 2.1.3.1 we decrease the order of $\tilde{a}_{k,j}$ to l' and so by proceeding with the representations of $\tilde{a}_{k,j}$ and $\tilde{b}_{k,j}$ of order l' as in Case 2.1.1 we are done.

Case 2.2: If $k+j$ is odd, then we proceed analogously to Case 2.1.

Case 3: If $k < j$, then we proceed analogously to Case 2. □

Theorem A.2. *Let $A = (a_{ij}), B = (b_{ij}), Z = (z_{ij}) \in \mathbb{R}^{n,n}$ be such that $A \leq^* Z \leq^* B$. If A and B are $NsTN$ ($Ns.t.n.p.$ with $b_{nn} < 0$), then Z is $NsTN$ ($Ns.t.n.p.$ with $z_{nn} < 0$).*

Proof. Let A and B be $NsTN$. Then by Theorem 3.2 (ii) and Proposition 3.5 \tilde{A} and \tilde{B} are nonnegative Cauchon matrices with positive diagonal entries. By Theorem 3.2 (ii) and Proposition 3.5 it suffices to show that \tilde{Z} is a nonnegative Cauchon matrix with positive diagonal entries. By Theorem A.1, $\tilde{a}_{k,j}$ and $\tilde{b}_{k,j}$ can be represented as ratios of contiguous minors of the same order, i.e.,

$$\tilde{a}_{k,j} = \frac{\det A[k, \dots, k+l | j, \dots, j+l]}{\det A[k+1, \dots, k+l | j+1, \dots, j+l]}, \quad (\text{A.17})$$

$$\tilde{b}_{k,j} = \frac{\det B[k, \dots, k+l | j, \dots, j+l]}{\det B[k+1, \dots, k+l | j+1, \dots, j+l]} \quad (\text{A.18})$$

for some l . Define $Z' := (z'_{k,j})$, where $z'_{k,j} = \frac{\det Z[k, \dots, k+l | j, \dots, j+l]}{\det Z[k+1, \dots, k+l | j+1, \dots, j+l]}$. By Lemma 4.4 ((4.5) taking the reciprocal values and add $\epsilon > 0$ to the $(1, 1)$ entries of the underlying submatrices if necessary, see Case 1 in the proof of Proposition 4.1) we obtain that

$$\tilde{A} \leq^* Z' \leq^* \tilde{B}. \quad (\text{A.19})$$

By (A.19) and since \tilde{A} and \tilde{B} are nonnegative Cauchon matrices with positive diagonal entries we have that Z' is a nonnegative Cauchon matrix with positive diagonal entries. If we show that $Z' = \tilde{Z}$ then the result follows.

Claim: $Z' = \tilde{Z}$.

Proof of the claim: Note that by Lemmata 4.4 and 2.6 all contiguous principal minors of Z are positive. We proceed by decreasing induction with respect to the lexicographical order on each pair (k, j) , $k, j = 1, \dots, n$. By definition, $z'_{nj} = z_{nj} = \tilde{z}_{nj}$ for all $j = 1, \dots, n$. Suppose that we have shown the claim holds for each pair (k, j) such that $k = k_0 + 1, \dots, n$, $j = 1, \dots, n$ and $k = k_0$, $j = j_0 + 1, \dots, n$. Hence $Z[k_0 + 1, \dots, n|1, \dots, n]$ and $Z[k_0, \dots, n|j_0 + 1, \dots, n]$ are TN matrices which are consequence of the fact that application of the Cauchon Algorithm to $Z[k_0 + 1, \dots, n|1, \dots, n]$ and $Z[k_0, \dots, n|j_0 + 1, \dots, n]$ results in $\tilde{Z}[k_0 + 1, \dots, n|1, \dots, n]$ and $\tilde{Z}[k_0, \dots, n|j_0 + 1, \dots, n]$, respectively, and by the induction hypothesis the latter matrices are nonnegative Cauchon matrices. Without loss of generality we assume that $j_0 \leq k_0$ and $k_0 + j_0$ is even. If $k_0 = j_0$, then by Case 1 of Proposition A.1 and Case 1 of Theorem A.1 the claim follows. Whence in the following we suppose that $j_0 < k_0$. Let $D := Z[k_0, \dots, n|j_0, \dots, n]$ and $D_x := Z_x[k_0, \dots, n|j_0, \dots, n]$ be the matrix that is obtained from D by adding a sufficiently large positive number x to the $(1, 1)$ entry of D which makes the $(1, 1)$ entry \tilde{d}_{11}^x of \tilde{D}_x positive. Then D_x is a TN matrix and by the Cauchon Algorithm $\tilde{d}_{11}^x = \tilde{d}_{11} + x = \tilde{z}_{k_0, j_0} + x$. Construct a lacunary sequence with respect to the Cauchon diagram $C_{\tilde{D}_x}$ that starts from the position $(1, 1)$ (i.e., (k_0, j_0)) by Procedure 3.1 (note that the choice of this sequence is not affected by the addition of x). Since D_x is a TN matrix we can apply all of the arguments that have been used in the proof of Proposition A.1 and the induction hypothesis in order to write \tilde{d}_{11}^x as a ratio of two contiguous minors; suppose that in D_x a case like Case 2.1 in proof of Proposition A.1 and (A.8) applies. Then (ii) of Proposition 2.8 holds and (i) does not hold since the rows and columns of D_x that are involved in $Z_x[k_t - 1, k_t, \dots, k_p|j_{t-1}, j_t, \dots, j_p]$ are linearly independent which follows from the positivity of the contiguous principal minors of Z and according to the choice of the sequence by Procedure 3.1 there is a white square of the Cauchon diagram $C_{\tilde{D}_x}$ in the column j_{t-1} above the position $(k_t - 1, j_{t-1})$ which has a representation of order greater than $p - t + 1$. Moreover, the left shadow of the submatrix $Z_x[k_t - 1, k_t, \dots, k_p|j_{t-1}, j_t, \dots, j_p]$ has rank $p - t + 1$ since the largest part of this submatrix lies below the main diagonal of Z . Let $((K_0, J_0), (K_1, J_1), \dots, (K_P, J_P))$ be the lacunary sequence with respect to the Cauchon diagram $C_{\tilde{D}_x}$ that is constructed by Procedure 3.1 and starts from the position $(1, 1)$ $((k_0, j_0) = (K_0, J_0))$ and let \tilde{d}_{11}^x have a representation of order L :

$$\tilde{d}_{11}^x = \frac{\det Z_x[k_0, k_0 + 1, \dots, k_0 + L|j_0, j_0 + 1, \dots, j_0 + L]}{\det Z[k_0 + 1, \dots, k_0 + L|j_0 + 1, \dots, j_0 + L]} \quad (\text{A.20})$$

and \tilde{a}_{k_0, j_0} and \tilde{b}_{k_0, j_0} have representations that are given by (A.17) and (A.18) $((k_0, j_0) = (k, j))$, respectively. Then if $l < L$, then $Z[k_0 + 1, \dots, n|1, \dots, n]$ is TN , Lemma 4.4, and the arguments that have been used in Theorem A.1 to decrease the order of the representations (all of these arguments are independent of x) can be applied to decrease the order of \tilde{d}_{11}^x to l . If $L < l$, then define

$$T := \min \{h \in \{0, 1, \dots, P\} \mid \tilde{z}_{K_h+1, J_h+1} = 0\}.$$

Note that such T exists since $L < l$ and $K_T = K_0 + T$ and $J_T = J_0 + T$. Then by the induction hypothesis $\tilde{z}_{K_T+1, J_T} = 0$. If $J_T + 1 < J_{T+1}$, then $\tilde{Z}[K_T + 1, \dots, n|J_T + 1] = 0$. Hence by

the induction hypothesis $\tilde{Z}[K_T + 1, \dots, n | 1, \dots, J_T + 1] = 0$. Therefore by the Cauchon Algorithm $Z[K_T + 1, \dots, n | 1, \dots, J_T + 1] = 0$ and thus by the zero-nonzero pattern of Z and $L < l$ we have $\det Z[k_0 + 1, \dots, k_0 + l | j_0 + 1, \dots, j_0 + l] = 0$. But by (A.17), (A.18), and Lemma 4.4 we obtain

$$0 < \det Z[k_0 + 1, \dots, k_0 + l | j_0 + 1, \dots, j_0 + l] \quad (\text{A.21})$$

which is a contradiction. Therefore $J_T + 1 = J_{T+1}$ and the sequence $((K_T + 1, J_T), (K_{T+1}, J_{T+1}), \dots, (K_P, J_P))$ is lacunary sequence with respect to $C_{\tilde{D}_x}$. Moreover, \tilde{z}_{K_T+1, J_T} can be written as a ratio of two contiguous minors:

$$\tilde{z}_{K_T+1, J_T} = \frac{\det Z[K_T + 1, \dots, K_T + L - T + 1 | J_T, \dots, J_T + L - T]}{\det Z[K_T + 2, \dots, K_T + L - T + 1 | J_T + 1, \dots, J_T + L - T]}.$$

Hence by Lemma 2.6, $\tilde{z}_{K_T+1, J_T} = 0$, $K_0 = k_0$, $J_0 = j_0$, $K_T = K_0 + T$, and $J_T = J_0 + T$ we have

$$\det Z[k_0 + 1, k_0 + 2, \dots, k_0 + L + 1 | j_0, j_0 + 1, \dots, j_0 + L] = 0. \quad (\text{A.22})$$

By (A.20), (A.21), Lemma 2.6, and application of Lemma 1.3 to the numerator of the representation of \tilde{d}_{11}^x we increase its order to l . Hence by Laplace expansion we obtain

$$\frac{\tilde{z}_{k_0, j_0} + x = \det Z[k_0, k_0 + 1, \dots, k_0 + l | j_0, j_0 + 1, \dots, j_0 + l] + x \det Z[k_0 + 1, \dots, k_0 + l | j_0 + 1, \dots, j_0 + l]}{\det Z[k_0 + 1, \dots, k_0 + l | j_0 + 1, \dots, j_0 + l]}.$$

Thus

$$\tilde{z}_{k_0, j_0} = \frac{\det Z[k_0, k_0 + 1, \dots, k_0 + l | j_0, j_0 + 1, \dots, j_0 + l]}{\det Z[k_0 + 1, \dots, k_0 + l | j_0 + 1, \dots, j_0 + l]} = z'_{k_0, j_0}. \quad (\text{A.23})$$

Hence by induction we have that $Z' = \tilde{Z}$. \square

Remark A.3. Suppose that $A = (a_{ij}), B = (b_{ij}), Z = (z_{ij}) \in \mathbb{R}^{n,n}$ be such that $A \leq^* Z \leq^* B$ and A and B are NsTN (Ns.t.n.p. with $b_{nn} < 0$), then by Theorem A.2, Z is NsTN (Ns.t.n.p. with $z_{nn} < 0$) and $\tilde{A} \leq^* \tilde{Z} \leq^* \tilde{B}$.

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List of Symbols

$\mathbb{R}^{n,m}$	the set of n -by- m real matrices
$Q_{\kappa,n}$	the set of all strictly increasing sequences of κ integers chosen from $\{1, \dots, n\}$
$\alpha \cup \beta$	union of the two sequences $\alpha \in Q_{\kappa,n}$ and $\beta \in Q_{\mu,n}$ such that $\alpha \cup \beta \in Q_{\kappa,n}$
$\alpha \setminus \beta$	difference of the two sequences $\alpha \in Q_{\kappa,n}$ and $\beta \in Q_{\mu,n}$ such that $\alpha \setminus \beta \in Q_{\kappa,n}$
$d(\alpha)$	dispersion of α
(a_{ij})	entries of a matrix
$A[\alpha \beta]$	submatrix of A
$A(\alpha \beta)$	complementary submatrix of the submatrix $A[\alpha \beta]$
$A[\alpha]$	principal submatrix of A
$A(\alpha)$	complementary principal submatrix of A
$\det A$	determinant of A
$\det A[\alpha \beta]$	minor of A
$\det A[\alpha]$	principal minor of A
I_n	identity matrix
E_{ij}	standard basis matrix
T_n	the anti-diagonal matrix of order n
A^T	transpose of a matrix A
A^{-1}	inverse of A
$A^\#$	the matrix $T_n A T_n$
N_s	nonsingular
SSR	strictly sign regular matrices
SR	sign regular matrices
$ASSR$	almost strictly sign regular matrices
TP	totally positive matrices
TN	totally nonnegative matrices
ATP	almost totally positive matrices
TP_k	totally positive matrices of order k
TN_k	totally nonnegative matrices of order k
ΔTP	triangular totally positive matrices
P -matrix	a matrix whose all of its principal minors are positive
$t.n.$	totally negative matrices
$t.n.p.$	totally nonpositive matrices
LU	LU factorization
LDU	LDU factorization
$A/A[\alpha]$	Schur complement of $A[\alpha]$ in A
\leq	entry-wise partial ordering or lexicographical ordering
\leq^*	checkerboard partial ordering
\leq_c	colexicographical ordering
$A^{(p)}$	the p -th compound matrix of the matrix A

$S^-(\mathbf{x})$	minimum sign variation of \mathbf{x}
$S^+(\mathbf{x})$	maximum sign variation of \mathbf{x}
EB	set of elementary bidiagonal matrices
GEB	set of generalized elementary bidiagonal matrices
$TN_{n,m}$	the set of n -by- m totally nonnegative matrices
$C_{n,m}$	the set of the n -by- m Cauchon diagrams
C_A	the Cauchon diagram that is associated to the Cauchon matrix A
\tilde{A}	the matrix obtained from A by the Cauchon Algorithm
$\pi(A)$	the Cauchon diagram associated to A
\mathcal{F}	family of minors
$\mathcal{S}_{\mathcal{F}}$	totally nonnegative cell associated with the family of minors \mathcal{F}
\mathcal{S}_C^0	the set of totally nonnegative matrices A such that $\pi(A) = C$
$[A, B]$	matrix interval with respect to checkerboard partial ordering
$\mathbb{I}(\mathbb{R})$	the set of the compact and nonempty real intervals
$\mathbb{I}(\mathbb{R}^{n,n})$	the set of all matrix intervals with respect to the checkerboard partial ordering
$V([A, B])$	a specified subset of the set of vertices of the matrix interval $[A, B]$
$\rho(A)$	spectral radius of A
$\mathcal{P}(A/A[\alpha])$	Perron complement of $A[\alpha]$ in A
$\mathcal{P}_t(A/A[\alpha])$	extended Perron complement of $A[\alpha]$ in A at t
?	unspecified entry
$\text{sgn}(a)$	sign of a
$G = (V, E)$	directed graph
$V(z)$	the number variations of signs in the Sturm sequence $(f_0(z), f_1(z), \dots, f_m(z))$
$R(z)$	real rational function
$\text{Ind}_a^b R$	the Cauchy index of a real rational function R between the limits a and b
$\text{Ind}_w R$	Cauchy index of the function R at its real pole w
$H(f)$	Hurwitz matrix associated with the polynomial f
$H_\infty(f)$	infinite Hurwitz matrix associated with the polynomial f
Δ_i	the leading principal minor of order i of $H(f)$
$H(p, q)$	infinite matrices of Hurwitz type
$H_{2n}(p, q)$	finite matrices of Hurwitz type
$R(p, q)$	resultant of p and q
$T(f)$	infinite upper triangular Toeplitz matrix
$D_j(S)$	the leading principal minors of order j of the infinite Hankel matrix S
$\hat{D}_j(S)$	$\det S[1, \dots, j 2, \dots, j+1]$ of the Hankel matrix S
V^F	Frobenius sign changes of $(D_0(R), D_1(R), D_2(R), \dots, D_r(R))$
$L(z)$	the logarithmic derivative of $p(z)$
∇_j	the leading principal minors of order j of $H(p, q)$
$P(\lambda, z)$	family of polynomials
R -function of negative type	function maps open upper half-plane of the complex plane into the open lower half-plane
R^* -function	R -function of negative type with only negative zeros and the numerator and denominator are coprime

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