Solutions to the Multi-Dimensional Viscous Quantum Hydrodynamic Equations for Semiconductors

Dissertation

zur Erlangung des akademischen Grades
des Doktors der Naturwissenschaften (Dr. rer. nat.)
an der

Universität Konstanz

Mathematisch-Naturwissenschaftliche Sektion
Fachbereich Mathematik und Statistik

vorgelegt von

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Tag der mündlichen Prüfung : 27. Juli 2009

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Preface

This dissertation is designed to investigate some fundamental mathematical questions of "The Viscous Model of Quantum Hydrodynamics for Semiconductors". This project started from the end of 2006 and, is financially supported by the "Juniorprofessorenprogramm des Landes Baden-Württemberg" and Young Scholar Fund at the University of Konstanz.

First of all, I would like to express my thanks wholeheartedly to my supervisor Prof. Dr. Michael Dreher for his providing me the opportunity to achieve this work. I owe him a great deal for his guidance, numerous discussions and suggestions, and also for the careful proof reading of this dissertation. Without his constant support and direction, as well as his wisdom, this work would not be possible

Then I thank Prof. Dr. Robert Denk in all sincerity for his refereeing my PhD thesis, useful suggestions, advice and sharing related references.

Beside, I thank Dr. Li Chen in Tsinghua university and Dr. Peicheng Zhu in Basque Center for Applied Mathematics and Dr. Dewen Xiong in Shanghai Jiaotong university for their useful discussions and encouragement. My thanks also go to those people, from whom valuable comments are supplied in Chinese-German Workshop on Partial Differential Equations and Applications in Geometry and Physics March 2-6 2009.

In addition, I will extend my gratitude to all my friends in Darmstadt, Beijing, Shanghai and Konstanz for their concern, help and support. I will have been keeping each happy moment I spent with them in my memory forever.

Special thanks to my parents for their giving me life and endless love wherever I am. Thanks to my wife for her care, understanding and unconditional support to me. I feel much obliged to them from the bottom of my heart.

Abstract

In this dissertation, we analysis the equations of multi-dimensional viscous model of quantum hydrodynamics (henceforth referred to as QHD) for semiconductors. The whole work will focus on investigations of three analytic results.

First we study the local existence of solutions to the isothermal viscous QHD model. The difficulty consists in dealing with the equations involving third-order derivatives. Here we will overcome this difficulty by using the a-priori estimates of the BVPs of mixed-order. Precisely, we check that the corresponding linear matrix operator is elliptic with parameter in certain closed sector in the complex plane with vertex at the origin. Then the a-priori estimates provide an estimation of the third order derivative. This will lead to a uniform bounds of the approxiamte solutions. Then we study the limits of the approximate solutions from contraction and compactness arguments to get a local solution.

Next, we obtain global existence and asymptotic behavior of solutions to the isothermal viscous QHD on a torus. We consider the situation when the doping profile of background charges is a positive constant and the initial data is close to the steady state. The main steps consists of some proper reformulations, a-priori estimates and application of the usual continuity argument.

Finally, the local existence and uniqueness of solutions to the non-isothermal equations will be studied. It seems that there are hardly any analytic results for this model in multi-dimensions. We consider the situation when the initial data of the current density, the rate of change of the initial data of the particle density and the energy density, the boundary function of the electrostatic potential are small; and the doping profile of background charges is close to the initial data of the particle density. Under these assumptions and with the help of the a-priori estimates from the isothermal equations we then obtain the uniform bounds of approximate solutions. Then we analysis the limit to derive a local solution.

Zusammenfassung

In dieser Arbeit werden lokale Existenz des viskosen Modells der Quantenhydrodynamik aus Halbleitern für allgemeine Randbedingungen und exponentielles Abklingen globaler Lösungen des Modells mit periodischen Randbedingungen untersucht.

Eine der entscheidenden Schwierigkeiten zum mathematischen Analysis besteht in den Termen mit den dritten Ableitungen, weil zum Beispiel Maximumprinzipien oder verwandte Techniken für Gleichungen von dritten Ordnung nicht zur Verfügung stehen.

Zuerst untersucht man die lokale Existenz des isothermischen viskosen systems. Die lokale Existenz des isothermischen viskosen systems basiert auf der A-priori Abschätzungen des Hauptteils als parameter-elliptisches System gemischter Ordnung. Um die Abschätzungen zu bekommen soll insbesondere die Lopatinskii-Shapiro Bedingung überprüft werden. Ausserdem ist das parameter-elliptische System gemischter Ordnung dann lösbar mit der Hilfe Galerkin-Approximation. Mit dem Hauptsatz für parameter-elliptische Randwertprobleme kann man zeigen, dass es eine gleichmässige Beschränkung der dritten Ableitungen existiert. Daraus folgt, dass es eine konvergente Teilfolge der Approximation-Lösungen gibt. Der Grenzwert der Teilfolge im geeigneten Sinne kann dann das isothermische viskose system lösen.

Nächst untersucht man die globale Existenz und exponentielles Abklingen der lokalen Lösungen auf einem Torus. Die hauptliche Schritte zum Beweis bestehen aus Umformulierungen, A-priori Abschätzungen und Verwendung der "Continuity Arguments."

Im letzten Teil der Dissertation analysiert man die lokale Existenz des nicht-isothermischen viskosen systems. Hier benutzt man die Annahme, dass der Anfangswert der Stromdichte, der Gradient des Anfangswerts der Teilchendichte und Energiedichte genügend klein sind. In diesem Fall kann man eine gleimässige Beschränkung der Approximation-Lösungen des nicht-isothermischen viskosen systems bekommen, mit der Hilfe A-priori Abschätzungen der Lösung der isothermischen Gleichungen. Ein weiteres Analysis des Grenzwerts der entsprechenden Approximation-Lösungen bietet eine lokale Lösung. Für nicht-isothermischen Fall stehen kaum analytische Erkenntnisse zur Verfügung. Der Nachweis der lokalen Existenz von Lösungen bei geeigneten Randbedingungen ist dann ein grosse Fortschritt gegenüber der jetzigen Situation.

Contents

Chapter 1

Introduction

1.1 Motivations from Semiconductor

1.1.1 Semiconductor Models

The word ”semiconductor” is used to denote the material that has electrical conductivity between those of a conductor and an insulator. Such materials are so useful because the behavior of a semiconductor can be easily manipulated by the addition of impurities, known as doping.

Semiconductor devices are electronic components made of semiconductor materials, principally silicon, germanium, and gallium arsenide. A main reason why semiconductor technology are so successfully applied in modern consumer electronics, including computers, mobile phones, and digital audio players, is that the device length is much smaller than that of previous electronic devices (like tube transistor).

Nowadays the characteristic lengths of such a device are of nanometers. Thus at such scale a mathematical model describing these physical phenomena should take quantum effects into account. In this research field a main objective is to derive mathematical models which describe electron transport phenomena in a semiconductor device.

Depending upon the device structures there are several different mathematical models describing electronic flow through a semiconductor device. For example, three typical classes of semiconductor models can be distinguished: the drift-diffusion model (DD), the energy-transport model (ET), the hydrodynamic model (HD), where each of them has classcial and quantum mechanical versions according to the sizes of devices under consideration.

The classical DD model is appropriate for semiconductor devices in which the typical length is not much smaller than 10^{-6}m, and the applied voltage is much smaller than 1V. The corresponding mathematical description was formulated by van Roosbroeck in 1950 [79]. The related quantum mechanical analogue of the classical drift-diffusion equations is called the quantum drift-diffusion equations, henceforth it is referred to QDD, which is derived in [21]. The derivation of the quantum energy transport model, denoted by QET, can be found in [21].

From the viewpoint of quantum mechanics, a single electron is considered as a wave described by a complex-valued wave function ψ which is a solution of the Schrödinger equation. Thus the quantum hydrodynamic models (QHD) can be derived from the Schrödinger-Possion system by WKB-ansatz of the wave function. Another practicable way to derive QHD equations is using the Wigner equation combining Fokker-Planck collision operator.

1.1.2 Equations of the Viscous Quantum Hydrodynamics

We consider an ensemble of M Fermions (electrons and holes[1]) in a semiconductor, and wish to derive evolution equations for particle density n, current density J and energy density ne.

The Poisson Equation

The Maxwell's equations

$$\nabla \times E = 0, \qquad \mathrm{div} D = \rho \qquad \text{in } \mathbb{R}^3, \tag{1.1.1}$$

hold in vanishing magnetic fields. Here E denotes the electric field, D is the displacement vector and ρ is the total space charge density. The first equation implies that there exists a potential V such that $E = -\nabla V$. V indicates the electrostatic potential. D is defined by the following relation

$$D = \varepsilon_s E,$$

where ε_s is the semiconductor permittivity.

Let n stand for the electron density, $\mathcal{C}(x)$ denote the so-called doping profile which is a function of the position varible and given by the difference of the number densities of positively charged donor ions and negatively charged acceptor ions. Then the total space charge density ρ is given by

$$\rho = -qn + q\mathcal{C}(x),$$

where q is the elementary charge.

Finally from the second equation of (1.1.1) we infer

$$\varepsilon_s \triangle V = \mathrm{div}(-\varepsilon_s \nabla V) = -\mathrm{div} D = q(n - \mathcal{C}(x)). \tag{1.1.2}$$

This equation is called the *Poisson equation.*

[1] In a bipolar model most of the electrons are valence electrons, i.e., they are responsible for the chemical compound of the semiconductor crystal. When the crytal is electrically neutral, then to each conduction electron there corresponds a "hole" in the valence band. In this case the total particle density is the difference of the electron density and the hole density.

Madelung Equations

From the quantum view a single electron (or hole) is considered as a wave described by a complex-valued wave function φ which is a solution of the Schrödinger equation

$$i\hbar\partial_t\varphi = -\frac{\hbar^2}{2m}\triangle\varphi - qV(x,t)\varphi, \qquad x \in \mathbb{R}^d, \quad t > 0,$$
$$\varphi(x,0) = \varphi_I(x), \qquad x \in \mathbb{R}^d.$$

Here i is the complex unit with $i^2 = -1$, $\hbar = \dfrac{h}{2\pi}$ is the reduced Planck constant (let h denote the Planck constant), m is the electron mass, q is the elementary charge, $V(x)$ is the electrostatic potential.

We introduce some reference values depending only upon the considered device. L is a characteristic length, for instance the device length. The characteristic voltage U is defined by $U := k_B\mathcal{T}_0/q$, where k_B is the Boltzmann constant, $\mathcal{T}_0$ is the lattice temperature. The characteristic time t^* is the time needed by a particle to cross the device. We assume that the thermal energy of a particle is equal to the kinetic energy, then

$$qU = m\left(\frac{L}{t^*}\right)^2.$$

Now we use the scaling

$$t = t^*t_s, \quad x = Lx_s, \quad V = UV_s,$$

then after some calculations it is easy to verify that the scaled wave function

$$\psi(x_s,t_s) := \varphi(x,t) = \varphi(t^*t_s, Lx_s)$$

satisfies the scaled Schrödinger equation for single particle (we omit the index "s")

$$i\epsilon\partial_t\psi = -\frac{\epsilon^2}{2}\triangle\psi - V\psi, \qquad x \in \mathbb{R}^d, \quad t > 0, \tag{1.1.3}$$
$$\psi(x,0) = \psi_I(x), \quad x \in \mathbb{R}^d, \tag{1.1.4}$$

where ϵ is the scaled Planck constant which is given by

$$\epsilon = \frac{\hbar t^*}{mL^2}.$$

Define the measurable quantities $n(x,t)$ and $J(x,t)$ in the following way

$$n(x,t) := |\psi(x,t)|^2, \qquad J(x,t) := -\epsilon\mathrm{Im}(\overline{\psi}\nabla\psi).$$

We make use of the WKB (Wentzel,Kramers,Brillouin) state on initial data:

$$\psi_I = \sqrt{n_I} e^{iS_I/\epsilon},$$

where $n_I(x) \geq 0$, $S_I(x) \in \mathbb{R}$ are some functions. Then the electron density $n(x,t)$ and current density $J(x,t)$ are satisfying (formally) the scaled Madelung equations [60]

$$\begin{aligned} &\partial_t n - \text{div} J = 0, \\ &\partial_t J - \text{div}\left(\frac{J \otimes J}{n}\right) + n\nabla V + \frac{\epsilon^2}{2} n \nabla\left(\frac{\triangle\sqrt{n}}{\sqrt{n}}\right) = 0, \quad x \in \mathbb{R}^d, t > 0 \qquad (1.1.5) \\ &n(x,0) = n_I(x), \qquad J(x,0) = J_I(x) : = -n_I \nabla S_I, \end{aligned}$$

which are formally equivalent to the Schrödinger equation (1.1.3).

The Wigner-Fokker-Planck Equation

We consider a particle ensemble consisting of M electrons, then the motion of the particle ensemble is described by the many-particle Schrödinger equation

$$\begin{aligned} &i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\sum_{j=1}^{M} \triangle_{x_j}\psi - qV(x_1, \cdots, x_M, t)\psi, \quad x_j \in \mathbb{R}^d, t > 0, \\ &\psi(x,0) = \psi_I(x), \end{aligned}$$

where ψ is called the wave function of the particle ensemble, it is a function with variables $(x_1, \cdots, x_M, t)$, $x_j \in \mathbb{R}^d$, $j = 1, \cdots, M$. Corresponding to the wave function ψ we define the density matrix $\varrho := \overline{\psi(r,t)}\psi(s,t)$, $r, s \in \mathbb{R}^{dM}, t > 0$.

Then in order to derive the Wigner equation, we shall use the following assumptions.

- We assume first that the potential V is decomposed into the sum of an external potential and a two-particle interaction potentials:

$$V(x_1, \cdots, x_M, t) = \sum_{l=1}^{M} V_{ext}(x_l, t) + \frac{1}{2}\sum_{l=1}^{M}\sum_{j=1}^{M} V_{int}(x_l, x_j),$$

  and $V_{int}(x_l, x_j) = V_{int}(x_j, x_l)$, i.e., V_{int} is symmetric.

- Next since the particles considered are Fermions, the wave function ψ is antisymmetric, i.e.,

$$\psi(x_1, \cdots, x_M, t) = \text{sgn}(\pi)\psi(x_{\pi(1)}, \cdots, x_{\pi(M)}, t),$$

for any permutation π of $\{1, \cdots, M\}$. This property implies that the density matrix remains invariant under π. Namely, if the ensemble density matrix ϱ is defined by

$$\varrho(r_1, \cdots, r_M, s_1, \cdots, s_M, t) := \overline{\psi(r_1, \cdots, r_M, t)}\psi(s_1, \cdots, s_M, t), \quad r_i, s_i \in \mathbb{R}^d$$

then

$$\varrho(r_1, \cdots, r_M, s_1, \cdots, s_M, t) = \varrho(r_{\pi(1)}, \cdots, r_{\pi(M)}, s_{\pi(1)}, \cdots, s_{\pi(M)}, t).$$

- Finally the density matrices of subensemble consisting of l particles is defined by

$$\begin{aligned}&\rho^l(r_1, \cdots, r_l, s_1, \cdots, s_l, t)\\ :=& \int_{\mathbb{R}^{d(M-l)}} \rho(r_1, \cdots, r_l, u_{l+1}, \cdots, u_M, s_1, \cdots, s_l, u_{l+1}, \cdots, u_M,)du_{l+1} \cdots du_M.\end{aligned}$$

  We assume that the particles in the subensemble move independently from each other. The corresponding mathematical description is the so called Hartree ansatz, i.e., $\rho^l(r_1, \cdots, r_l, s_1, \cdots, s_l, t)$ can be factorized

$$\rho^l(r_1, \cdots, r_l, s_1, \cdots, s_l, t) = \prod_{i=1}^{l} R(r_i, s_i, t).$$

Define $R := \rho^1$, the effective potential $V(x,t)$ reads

$$V(x,t) = V_{ext}(x,t) + \int_{\mathbb{R}^d} MR(y,y,t)V_{int}(x,y)dy.$$

Then after some reformulations and calculations (see [63], Sec. 1.5) it is easy to verify that the function

$$W(x,v,t) := \mathscr{F}^{-1}_{\eta\to v}\left(MR\left(x + \frac{\hbar}{2m}\eta, x - \frac{\hbar}{2m}\eta, t\right)\right)$$

solves the following so-called *Vlasov equation* [16, 59]

$$\partial_t W + v \cdot \nabla_x W + \frac{q}{m}\theta_\hbar[V]W = 0, \quad x, v \in \mathbb{R}^3, \tag{1.1.6}$$

where $\theta_\hbar[V]$ is a pseudo-differential operator defined by

$$\begin{aligned}\theta_\hbar[V]W(x,v,t) := \frac{i}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{m}{\hbar}\left(V\left(x + \frac{\hbar}{2m}\eta, t\right) - V\left(x - \frac{\hbar}{2m}\eta, t\right)\right) \times\\ \times W(x, v', t)e^{i(v-v')\cdot\eta}dv'd\eta,\end{aligned} \tag{1.1.7}$$

$v \in \mathbb{R}^d$ indicates the velocity. Furthermore we refer to [63], pp 62, the quantum electron number density (denoted by $n(x,t)$) can be expressed by

$$n(x,t) = \int_{\mathbb{R}^d} W(x,v,t)dv, \tag{1.1.8}$$

then the macroscopic quantum current density (denoted by $J(x,t)$) is given by

$$J(x,t) = -q \int_{\mathbb{R}^d} vW(x,v,t)dv; \tag{1.1.9}$$

and the energy density (denoted by ne) is infered as

$$ne(x,t) = \frac{m}{2} \int_{\mathbb{R}^d} v^2 W(x,v,t)dv. \tag{1.1.10}$$

The Vlasov equation presented above doesn't involve the impact of the semi-conductor crystal lattice on the motion of the particles and collisions of the charged particles with the background oscillators. In order to take into account these aspacts the energy-band and the collision operator need to be considered.

More precisely, we assume a parabolic energy-band $E(k) = \hbar^2|k|^2/2m$ (k is the wave vector), then it implies

$$v = \frac{1}{\hbar}\nabla_x E(k) = \frac{\hbar k}{m}. \tag{1.1.11}$$

Define the Wigner distribution function $w(x,k,t) := \frac{1}{m}W(x,v,t)$. Recall (1.1.6) $w(x,k,t)$ satisfies the equation

$$\partial_t w + \frac{\hbar k}{m} \cdot \nabla_x w + \frac{q}{m}\theta[V]w = 0, \quad x,k \in \mathbb{R}^3, \tag{1.1.12}$$

where

$$\theta[V]w(x,k,t) = \theta_\hbar[V]W(x,v,t).$$

We use the scaling $\tilde{\eta} = \dfrac{\hbar}{m}\eta$, then substitute $\tilde{\eta}$ into (1.1.7), we infer

$$\theta[V]w(x,k,t) = \frac{i}{(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{m}{\hbar}\left(V\left(x+\frac{\tilde{\eta}}{2},t\right) - V\left(x-\frac{\tilde{\eta}}{2},t\right)\right) \times$$
$$\times\, w(x,k',t)e^{i(k-k')\cdot\tilde{\eta}}dk'd\tilde{\eta}.$$

In order to take into account collisions we introduce the quantum *Fokker-Planck collision operator* [15] which is given by

$$L(w) := \frac{1}{\tau_0}\mathrm{div}_k(kw) + \frac{D_{pp}}{\hbar^2}\triangle_k w + \frac{D_{pq}}{\hbar}\mathrm{div}_x(\nabla_k w) + D_{qq}\triangle_x w, \tag{1.1.13}$$

and models the interaction of the electrons with the phonons of the crystal lattice (oscillators) with constants:

$$D_{pp} = \frac{mk_B\mathcal{T}_0}{\tau_0}, \quad D_{pq} = \frac{\tilde{\Omega}\hbar^2}{6\pi k_B\mathcal{T}_0\tau_0}, \quad D_{qq} = \frac{\hbar^2}{12mk_B\mathcal{T}_0\tau_0}.$$

Here τ_0 is the momentum relaxation time, $\tilde{\Omega}$ is the cut-off frequency of the reservoir oscillators. Combining (1.1.12) and (1.1.13) we obtain the complete *Wigner-Fokker-Planck equation*

$$\partial_t w(x,k,t) + \frac{\hbar k}{m}\cdot\nabla_x w(x,k,t) + \frac{q}{m}\theta[V]w(x,k,t) = L(w(x,k,t)), \quad x,k\in\mathbb{R}^3, t>0. \tag{1.1.14}$$

The Viscous Quantum Hydrodynamic Equations

We are interested in the macroscopic equations with respect to the particle density $n(x,t)$, the current density $J(x,t)$ and the energy density ne, where $n(x,t)$, $J(x,t)$ and $ne(x,t)$ are related to the Wigner function (according to (1.1.8)-(1.1.10) and (1.1.11)) by

$$n(x,t) = \int_{\mathbb{R}^d} w(x,k,t)d(\hbar k), \quad J(x,t) = -\frac{q}{m}\int_{\mathbb{R}^d} w(x,k,t)(\hbar k)d(\hbar k),$$
$$ne(x,t) = \frac{1}{2m}\int_{\mathbb{R}^d} w(x,k,t)|\hbar k|^2 d(\hbar k).$$

In order to derive such equations the moment method as in [40] will be applied. More precisely, the equation (1.1.14) is multiplied by 1, $\hbar k$, and $\frac{1}{2}|\hbar k|^2$ respectively, and then integrated over $\mathbb{R}^d$ with respect to $\hbar k$. The resulting system has to be closed by assuming that the Wigner function w is close to a wave vector displaced equilibrium density which was formulated by Wigner [81]. Next we follow [53] then obtain the following approximate equations of (1.1.14), up to order $O(\epsilon^4)$:

$$\partial_t n - \frac{1}{q}\mathrm{div}J = D_{qq}\triangle n,$$

$$\partial_t J - \frac{1}{q}\mathrm{div}\left(\frac{J\otimes J}{n}\right) - \frac{qk_B}{m}\nabla(nT) + \frac{q^2}{m}n\nabla V$$
$$+\frac{q\hbar^2}{12m^2}\mathrm{div}(n(\nabla\otimes\nabla)\ln n) = -\frac{J}{\tau_0} + \frac{qD_{pq}}{m}\nabla n + D_{qq}\triangle J,$$

$$\partial_t(ne) - \frac{1}{q}\mathrm{div}\left(((ne)E_d + P)\frac{J}{n}\right) + J\cdot\nabla V = -\frac{2}{\tau_0}\left(ne - \frac{d}{2}nk_B\mathcal{T}_0\right) + \frac{2D_{pq}}{q}\mathrm{div}J$$
$$+ D_{qq}\triangle(ne).$$

Here $J \otimes J$ denotes the matrix with component $J_j J_k$, E_d is the $d \times d$ unit matrix, and the stress tensor P and energy density ne are given by

$$P = nk_B T E_d - \frac{\hbar^2}{12m} n (\nabla \otimes \nabla) \ln n,$$

$$ne = \frac{m}{2q^2} \frac{|J|^2}{n} + \frac{d}{2} n k_B T - \frac{\hbar^2}{24m} n \triangle \ln n.$$

Notice that the stress tensor consists of the classical pressure and a quantum "pressure" term. The energy density is the sum of kinetic energy, thermal energy, and quantum energy.

Next we use the following scalings

$$x_s = Lx, \quad t_s = t^* t, \quad \mathcal{C}_s = \sup_{x \in \overline{\Omega}} |\mathcal{C}(x)| \mathcal{C}(x), \quad V_s = \frac{k_B \mathcal{T}_0}{q} V,$$

$$J_s = \frac{q k_B \mathcal{T}_0 (\sup_{x \in \overline{\Omega}} |\mathcal{C}(x)|) t^*}{mL} J, \quad T_s = \mathcal{T}_0 T,$$

then after recalling (1.1.2) we obtain the scaled equations (omitting the index "s")

$$\begin{cases} \partial_t n - \mathrm{div} J = \nu_0 \triangle n, \\ \partial_t J - \mathrm{div}\left(\dfrac{J \otimes J}{n}\right) - \nabla(nT) + n \nabla V \\ \qquad + \dfrac{\epsilon^2}{2} n \nabla \left(\dfrac{\triangle \sqrt{n}}{\sqrt{n}}\right) = \nu_0 \triangle J - \dfrac{J}{\tau} + \mu \nabla n, \\ \partial_t (ne) - \mathrm{div}\left(((ne)E_d + P)\dfrac{J}{n}\right) + J \nabla V = -\dfrac{2}{\tau}(ne) + \dfrac{d}{\tau} n \\ \qquad + \nu_0 \triangle (ne) + \mu \mathrm{div} J, \\ \lambda^2 \triangle V = n - \mathcal{C}(x), \end{cases} \tag{1.1.15}$$

where the scaled parameters are the viscosity constant ν_0, the Planck constant ϵ, the Debye length λ, the relaxation time τ and the interaction constant μ which are given by

$$\nu_0 = \frac{\hbar^2}{12 k_B \mathcal{T}_0 \tau_0 \sqrt{k_B \mathcal{T}_0 m L^2}}, \quad \epsilon^2 = \frac{\hbar^2}{3 k_B \mathcal{T}_0 m L^2}, \quad \lambda^2 = \frac{\varepsilon_s k_B \mathcal{T}_0}{q^2 (\sup_{x \in \overline{\Omega}} |\mathcal{C}(x)|) L^2},$$

$$\tau = \frac{\tau_0}{t^*}, \quad \mu = \frac{\hbar}{6 \pi k_B \mathcal{T}_0 \tau_0};$$

the scaled stress tensor P and energy density have the representations

$$P = nTE_d - \frac{\epsilon^2}{4} n (\nabla \otimes \nabla) \ln n,$$

$$ne = \frac{|J|^2}{2n} + \frac{3}{2} nT - \frac{\epsilon^2}{8} n \triangle \ln n.$$

If the temperature is invariant, i.e., T is a positive constant denoted by T_0, the evolution equation with respect to ne is unnecessary because it is a direct resulting equation from the evolution equations with respect to n, J and the Possion equation. In this case we have *the isothermal viscous QHD model* which is the simplified version of (1.1.15) and reads

$$\begin{cases} \partial_t n - \mathrm{div} J = \nu_0 \triangle n, \\ \partial_t J - \mathrm{div}\left(\dfrac{J \otimes J}{n}\right) - T_0 \nabla n + n \nabla V + \dfrac{\epsilon^2}{2} n \nabla \left(\dfrac{\triangle \sqrt{n}}{\sqrt{n}}\right) = \nu_0 \triangle J - \dfrac{J}{\tau}, \\ \lambda^2 \triangle V = n - \mathcal{C}(x), \end{cases}$$

The complete system (1.1.15) is called *the non-isothermal viscous QHD model*.

1.2 Related Analytic Results

Concerning the QHD model without viscous terms Jüngel and H. Li [49] showed the existence of stationary states for one space dimension. According to this result the exponential decay to a stationary state in one dimensional bounded domains was derived in [50]. H. Li and P.Marcati [62] also investigated the local existence and asymptotic behavior of solutions for several space dimensions.

In case of one dimensional viscous QHD L. Chen and M. Dreher [17] derived the local existence and uniqueness with insulating boundary conditions. A. Jüngel and J.P. [53] proved the existence of steaty states and obtained the corresponding numerical simulations. Additionally the local existence and exponential decay in $\mathbb{R}^1$ was showed in [32].

For the isothermal viscous QHD in multiple dimensions only a few analytic results are available. In [17] the local existence of solutions in the case of higher dimensions under the assumptions of periodic boundary conditions was proved. Later in [32] the local existence of solutions with more general boundary conditons was obtained.

1.3 Scope of the Work

In this dissertation we analysis the equations of viscous QHD model in multiple dimensions. The work is divided into three major parts.

Part I Local Solutions to the Isothermal Viscous QHD Model

Here we investigate the local existence of solutions to

$$\begin{cases} \partial_t n - \mathrm{div} J = \nu_0 \triangle n, \\ \partial_t J - \mathrm{div}\left(\dfrac{J\otimes J}{n}\right) - T_0 \nabla n + n\nabla V + \dfrac{\epsilon^2}{2} n \nabla\left(\dfrac{\triangle\sqrt{n}}{\sqrt{n}}\right) = \nu_0 \triangle J - \dfrac{J}{\tau}, \\ \lambda^2 \triangle V = n - \mathcal{C}(x), \\ (n, J)(0, x) = (n_0, J_0)(x), \end{cases} \tag{1.3.1}$$

subject to the Dirichlet boundary conditions

$$\begin{cases} (n, J, V)(t, x) = (n_\Gamma, J_\Gamma, V_\Gamma) \quad \text{on } \partial\Omega, \\ n_\Gamma \in H^{5/2}(\partial\Omega), \quad J_\Gamma \in H^{3/2}(\partial\Omega), \quad V_\Gamma \in H^{3/2}(\partial\Omega), \end{cases} \tag{1.3.2}$$

where the following regularity and compatibility conditions

$$\begin{cases} n_0 \in H^3(\Omega), \quad J_0 \in H^2(\Omega), \\ (n_0, J_0)\,|_{\partial\Omega} = (n_\Gamma, J_\Gamma), \\ (\nu_0 \triangle n_0 + \mathrm{div} J_0)|_{\partial\Omega} = 0. \end{cases}$$

are also satisfied.

In aspects of mathematics the main difficulties come from the quantum Bohm potential

$$B(n) := \frac{1}{2}\epsilon^2 \frac{\triangle\sqrt{n}}{\sqrt{n}}$$

which brings a third order derivative into the system which otherwise will be considered as a parabolic system coupled to an elliptic equation. Additionally the maximum principle arguments can in general be not applied to third-order equations.

An usual strategy to deal with third-order equations is to introduce a viscous regularization term $\gamma\triangle^2$ $(0 < \gamma < 1)$ [17, 32], where $\gamma\triangle^2$ will vanish as $\gamma \to 0$. However when one attempts to use this method to extend the results of [17] to a system with more general boundary conditions, one is faced with failure because the estimates of the third-order derivative depends always upon γ, and might converges to ∞ as $\gamma \to 0$. Instead we will overcome this difficulty by using the a-priori estimates of the BVPs of mixed-order, which was first defined in [5]. Precisely, we check that the corresponding linear matrix operator of (1.3.1) is elliptic with parameter in certain closed sector in the complex plane with vertex at the origin. Then we use the **Theorem 2.6** in [34] to abtain the a-priori estimates which provide an estimation of the third order derivative. This will lead to a uniform bounds of the approxiamte solutions.

The last step to prove the local existence and uniqueness is to investigate the converges properties of the approximate solutions in certain Sobolev spaces. Namely, the sequence of the approximate solutions in a small time interval should be a Cauchy sequence in certain sense. A further analysis will show that the limit is the local-in-time solution we seek.

Part II Global Existence and Asymptotic Behavior on a Torus

We use the assumption that the doping profile of background charges is a positive constant (denoted by $\mathcal{C}_0$). The long time behavior for one space dimension and the exponential stability in a muli-dimensional box is obtained in [17], where properties of a box was needed for an estimate of the Bohm potential.

Compared to [17] we consider a more general situation. More precisely, according to the result of the local existence on a torus in [17] we will extend the local-in-time solution of (1.3.1) on a torus globally in time, and obtain the exponential decay to the steady state $(\mathcal{C}_0, 0, 0)$ when the initial data is close to $(\mathcal{C}_0, 0, 0)$.

Breafly speaking, we first reformulate the original equations to a fourth-order wave equation. Then we get the a priori estimates. Finally we apply the usual continuity argument to obtain the global existence and large time behavior.

Part III Local Solutions to the Non-isothermal Viscous QHD Model

We investigate the local existence and uniqueness of solutions to the non-isothermal viscous QHD model

$$\begin{cases} \partial_t n - \mathrm{div} J = \nu_0 \triangle n, \\ \partial_t J - \mathrm{div}\left(\dfrac{J \otimes J}{n}\right) - \nabla(nT) + n\nabla V \\ \qquad + \dfrac{\epsilon^2}{2} n \nabla\left(\dfrac{\triangle\sqrt{n}}{\sqrt{n}}\right) = \nu_0 \triangle J - \dfrac{J}{\tau} + \mu \nabla n, \\ \partial_t(ne) - \mathrm{div}\left(((ne)E_d + P)\dfrac{J}{n}\right) + J\nabla V = -\dfrac{2}{\tau}(ne) + \dfrac{3}{\tau} n \\ \qquad + \nu_0 \triangle(ne) + \mu \mathrm{div} J, \\ \lambda^2 \triangle V = n - \mathcal{C}(x), \end{cases} \tag{1.3.3}$$

where μ is the interaction constant; the (scaled) stress tensor P and the temperature T satisfy

$$\begin{cases} P = nTE_d - \dfrac{\epsilon^2}{4} n(\nabla \otimes \nabla)\ln n, \\ ne = \dfrac{|J|^2}{2n} + \dfrac{3}{2} nT - \dfrac{\epsilon^2}{8} n \triangle \ln n, \end{cases} \tag{1.3.4}$$

i.e., the (scaled) stress tensor consists of the classical pressure and a quantum "pressure" term. The energy density is the sum of kinetic energy, thermal energy, and quantum energy.

The initial data and boundary conditions are

$$(n, J, ne)^T(0) = (n_0, J_0, (ne)_0), \tag{1.3.5}$$

$$(n, J, ne, V)^T|_{\partial\Omega} = (n_\Gamma, J_\Gamma, (ne)_\Gamma, V_\Gamma)^T \tag{1.3.6}$$

with

$$\inf_{x\in\Omega} n_0(x) > 0,\ n_0 \in H^3(\Omega),\ J_0 \in H^2(\Omega),\ (ne)_0 \in H^2(\Omega), \tag{1.3.7}$$

$$n_\Gamma \in H^{5/2}(\Omega),\ J_\Gamma \in H^{3/2}(\Omega), (ne)_\Gamma \in H^{3/2}(\Omega),\ , V_\Gamma \in H^{3/2}(\Omega), \tag{1.3.8}$$

$$\left.\begin{aligned} (n_0, J_0, (ne)_0)\,|_{\partial\Omega} &= (n_\Gamma, J_\Gamma, (ne)_\Gamma), \\ (\nu_0 \triangle n_0 + \mathrm{div} J_0)|_{\partial\Omega} &= 0. \end{aligned}\right\} \tag{1.3.9}$$

By far as we know, it seems that there are hardly any analytic results for this model in multi-dimensions. The system (1.3.3) is more complicated than (1.3.1). The main difficulty is that the evolution equation with respect to ne contains a nonlinear term among which a third-order derivative occurs.

In this part, we study the non-isothermal viscous QHD model (1.3.3) under the assumptions that the given data $J_0, \nabla(ne)_0, \nabla n_0, V_\Gamma$ are sufficiently small in

$$(H^2(\Omega))^d \times (L^2(\Omega))^d \times (H^2(\Omega))^d \times H^{3/2}(\Omega))$$

and $\mathcal{C}(x)$ is close to n_0 in $L^2(\Omega)$ norm. Under these assumptions and with the help of the a-priori estimates from **Part I**, we then obtain the uniform bounds of approximate solutions. Finally we analysis the limit to derive a local solution.

Appendices will provide some background material on important calculus, inequalities, functional analysis, Sobolev spaces, etc.

Finally the Bibliography primarily provides a listing of related papers for further informations.

Chapter 2

Preliminaries

In this chapter we first present some notations which will be needed in the remainder of the dissertation. Then some basic materials on elliptic boundary value problems of mixed order will be introduced for convenience. Most results are just recalld without proofs, but the relevant references are given in the end.

2.1 Notations and Some Useful Calculus

The symbol Ω will be reserved for a nonempty open set in n-dimensional real Euclidean space $\mathbb{R}^n$. We stipulate that all function vectors are written in columns, for a scalar-valued function the gradient is a row. Given a vector-valued function f:

$$f : \Omega \to \mathbb{R}^n$$

and matrix-valued function A

$$A : \Omega \to \mathbb{R}^{n\times n}$$

then the following notations are used: $\triangle f = \begin{pmatrix} \triangle f_1 \\ \vdots \\ \triangle f_n \end{pmatrix}$,

$$\nabla f = \begin{pmatrix} \partial_1 f_1 & \dots & \partial_n f_1 \\ \vdots & \ddots & \vdots \\ \partial_1 f_n & \dots & \partial_n f_n \end{pmatrix}, \quad \mathrm{div} A = \begin{pmatrix} \mathrm{div}(A_{11}, \dots, A_{1n}) \\ \vdots \\ \mathrm{div}(A_{n1}, \dots, A_{nn}) \end{pmatrix}.$$

$f \otimes f$ is a $n \times n$ matrix-valued function with entry $f_i f_j$ at position (i, j).

Let f, g be vector-valued functions, ϕ, ψ scalar-valued functions, A a matrix-valued function. Then we have the following calculus facts.

$$
\begin{aligned}
(\mathrm{grad} f)_{ij} &= \partial_j f_i, \\
\mathrm{div}(\phi f) &= \sum_j \partial_j(\phi f_j) = \sum_j (\partial_j \phi) f_j + \sum_j \phi \partial_j f = \langle \mathrm{grad}\phi, f\rangle + \phi \mathrm{div} f, \\
(\mathrm{grad}(\phi f))_{ij} &= \partial_j(\phi f_i) = (\partial_j \phi) f_i + \phi \partial_j f_i = (f \otimes \mathrm{grad}\phi)_{ij} + \phi(\mathrm{grad} f)_{ij}, \\
(\mathrm{div} A)_j &= \sum_i \partial_i A_{ji}, \\
(\mathrm{div}(f \otimes g))_j &= \sum_i \partial_i (f \otimes g)_{ji} = \sum_i \partial_i (f_j g_i) = f_j \sum_i \partial_i g_i + \sum_i (\partial_i f_j) g_i \\
&= f_j(\mathrm{div} g) + \langle \mathrm{grad} f_j, g\rangle, \\
\mathrm{div}(f \otimes g) &= (\mathrm{div} g) f + (\mathrm{grad} f) g, \\
(\mathrm{div}(\phi f \otimes g))_j &= \phi f_j (\mathrm{div} g) + \phi \langle \mathrm{grad} f_j, g\rangle + f_j \langle \mathrm{grad}\phi, g\rangle, \\
\mathrm{div}(\phi f \otimes g) &= \phi(\mathrm{div} g) f + ((\mathrm{grad}\phi) g) f + \phi(\mathrm{grad} f) g \\
(\mathrm{grad}\langle f, g\rangle)_j &= \partial_j \langle f, g\rangle = \langle \partial_j f, g\rangle + \langle f, \partial_j g\rangle, \\
\mathrm{grad}(\phi(\psi)) &= \psi \nabla \phi + \phi \nabla \psi, \\
\triangle(\phi(\psi)) &= \psi \triangle \phi + \phi \triangle \psi + 2\langle \nabla \phi, \nabla \psi\rangle.
\end{aligned}
$$

Additionally let $(\cdot, \cdots, \cdot)^T$ denote the transpose, $H_0^k(\Omega)$ the Sobolev space of functions with square integrable weak derivatives of order k, whose trace on the boundary up to the order $k-1$ are 0, i.e., $H_0^k(\Omega)$ consists of all functions from $\{f \in L^2(\Omega) \mid D^\alpha f \in L^2(\Omega) \text{ for } 0 \le |\alpha| \le k\}$ with the following property

$$
\frac{\partial^j f}{\partial \nu^j}(x) = 0, \quad x \in \partial\Omega, j = 0, \cdots, k-1,
$$

where $D^\alpha f$ is the weak (or distributional) partial derivative and let $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_d)$ denote the unit outward normal vector field on $\partial\Omega$, $\dfrac{\partial^j f}{\partial \nu^j}$ the j-th (outward) normal derivative of f.

For two $n \times n$ matrix-valued functions $A = (a_{ij}(x))$ and $B = (b_{ij}(x))$ whose entries $a_{ij}(x)$ and $b_{ij}(x)$ are square integrable funcions on Ω ($a_{ij}(x), b_{ij}(x) \in L^2(\Omega)$, $i, j = 1, \cdots, n$) let $(\cdot, \cdot)$ denote the function:

$$
\begin{aligned}
& L(\mathbb{C}^n) \times L(\mathbb{C}^n) \to \mathbb{C}, \\
& (A, B) \mapsto \sum_{i=1}^n \sum_{j=1}^n (a_{ij}, b_{ij})_{L^2(\Omega)}
\end{aligned}
\tag{2.1.1}
$$

where $(\cdot, \cdot)_{L^2(\Omega)}$ is the inner product in $L^2(\Omega)$.

The norm of $L^2(\Omega)$ is denoted by $\|\cdot\|$. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be an n-tuple of nonnegative integers α_i, we call α a *multiindex* and denote by x^α the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n} (x \in \mathbb{R}^n)$ with degree $|\alpha| = \sum_{j=1}^n \alpha_j$. Similary, if $D_j = -i\dfrac{\partial}{\partial_{x_j}}$, then

$$D := D_1 \cdots D_n,$$

and

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n},$$

denotes a differential operator of order $|\alpha|$. For two multi-indices α and β, we say that $\beta \leq \alpha$ provided $\beta_j \leq \alpha_j$ for $1 \leq j \leq n$. In this case $\alpha - \beta$ is also a multi-index, and $|\alpha - \beta| + |\beta| = |\alpha|$. We also denote

$$\alpha! = \alpha_1! \cdots \alpha_n!$$

and if $\beta \leq \alpha$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

We have also the Leibniz formula

$$D^\alpha(uv)(x) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u(x) D^{\alpha-\beta} v(x)$$

valid for functions u and v that are $|\alpha|$ times (weakly) differentiable near x.

For $m \in \mathbb{N}_0 \cup \{\infty\}$ let

$$C^m(\Omega) = \{f : \Omega \to \mathbb{R} \mid D^\alpha f \text{ exists and is continuous for all } \alpha \in \mathbb{N}_0^n \text{ such that } |\alpha| \leq m\}$$

be the space of all m-times continuously differentiable functions. Let $C^m(\overline{\Omega})$ denote the topological vector space which consists of all those functions $\phi \in C^m(\Omega)$ for which $D^\alpha \phi$ is bounded and uniformly continuous on Ω for all $0 \leq |\alpha| \leq m$. Notice that $D^\alpha \phi$ possesses a unique, bounded, continuous extension to the closure $\overline{\Omega}$ of Ω.

$$C_0^\infty(\Omega) = \{\varphi \in C^\infty(\mathbb{R}^n) \mid \text{supp}\varphi \text{ is a compact subset of } \Omega\}.$$

A sequence $\{f_j\}$ of functions in $C_0^\infty(\Omega)$ is said to converge to the function $f \in C_0^\infty(\Omega)$ provided

- there exists $K \Subset \Omega$ ($\overline{K} \subset \Omega$ and $\overline{K}$ is compact) such that the supports of all f_j and f lie in K,
- $\lim_{j\to\infty} D^\alpha f_j(x) = D^\alpha f(x)$ uniformly for each multi-index α.

Let X be a Banach space, I an interval in $\mathbb{R}$. Define $C(I;X)$ to be the bounded continuous functions of the form

$$u : I \to X,$$
$$t \mapsto u(x) \in X$$

which is equipped with the norm $\|u\|_{C(I;X)} = \sup_{t\in I} \|u(t)\|_X$. The space $C^n(I;X)$ contains functions whose classical derivatives up to order n are in $C(I;X)$ where the classical derivative is defined as a limit of difference quotients.

2.2 Elliptic BVPs of Mixed Order

2.2.1 Systems of Agmon-Douglis-Nirenberg Type

First consider a polynomial matrix $A(\xi) = (A_{ij}(\xi))_{i,j=1,\cdots,n}$ with $\text{ord}A_{ij} \leq r$. Let $A^0_{ij}(\xi)$ denote the homogeneous part of order r (this part is identically zero if the order is less than r), $A^0(\xi) := A^0_{ij}(\xi)_{i,j=1,\cdots,n}$ the principal part. The matrix $A(\xi)$ is called ellipic if

$$\text{det}A^0(\xi) \neq 0, \quad |\xi| \neq 0.$$

In the case of matrices elliptic in the sense of Douglis-Nirenberg (mixed order systems) we consider systems of Agmon-Douglis-Nirenberg type (henceforth referred to as ADN-systems) which will be represented (here we use the summation convention) as

$$\sum_{j=1}^{n} l_{ij}(x,D)u_j(x) = F_i(x) \tag{2.2.1}$$

which is a system of n equations for an equal number of dependent variables. Here $l_{ij}(x,D)$ are linear differential operators which are polynomials in D with coefficients depending on x over the domain Ω. The orders of these operators will be assumed to depend on two systems of integers $\{s_j\}_{j=1}^n$ and $\{m_j\}_{j=1}^n$. More precisely, the dependence is expressed by the inequality

$$\text{deg}l_{ij}(x,D) \leq s_i + m_j, \quad i,j = 1,\cdots,n,$$

and it is to be understood that $l_{ij} = 0$ if $s_i + m_j < 0$. By a normalization we can select the integers in such a way that

$$s_i \leq 0, \quad m_j \geq 0.$$

According to [5] we have the definition of ellipticity for ADN-systems

Definition 2.2.1. *The system* (2.2.1) *is called elliptic if*

$$det\left(l'_{ij}(x,\xi)\right) \neq 0 \qquad \left(x \in \overline{\Omega}, \xi \in \mathbb{R}^n, \xi \neq 0\right)$$

where $l'_{ij}(x,\xi)$ *consists of the terms in* $l_{ij}(x,\xi)$ *(the symbol of* $l_{ij}(x,D)$*) which are just of the order* $s_i + m_j$*.*

Such equations with constant coefficients in a half-space have been studied in [5] where explicit solution of boundary problems for homogeneous equations and a representation formula in the inhomogeneous boundary problem have been derived. Furthermore the schauder estimates and L^p-estimates for systems of equations with constant coefficients and with special variable coefficients were also investigated. More details can be found in [5].

2.2.2 Ellipticity with Parameter

The solvability and a priori estimate of solutions to scalar, generally nonselfadjoint elliptic boundary value problems under limited smoothness assumptions and under an ellipticity with parameter condition was derived in [7]. The same method was used in [22] and it extended the results of [7] for a scalar case to that for a system involing a discontinuous weight function where the system is elliptic in usual sense(i.e. the highest order of each entry of the matrix operator is a given even integer).

It is pointed out in [22] that the definition of ellipticity with parameter was different from that in [7] since the prolem mentioned in [22] involved a weight function. In this case one allowed the weight function to have discontinuities at certain hypersurfaces lying in Ω, and therefore further transmission boundary conditions had to be supplemented. Very general results for more general elliptic systems of Agmon-Douglis-Nirenberg type were derived in [34] by making minor modifications of the arguments of [22].

We shall use norms depending upon a parameter $\eta \in \mathbb{C}/\{0\}$. Precisely let s, m be integers satisfying $1 \leq s \leq m$, $v \in W_p^s(\Omega)$, assume that the boundary $\partial\Omega$ of Ω is of class $C^{m-1,1}$. Then we define

$$|||v|||_{s,p,\Omega} := \|v\|_{W_p^s(\Omega)} + |\eta|^{\frac{s}{m}} \|v\|_{L^p(\Omega)}. \tag{2.2.2}$$

The vectors $v \in W_p^s(\Omega)$ have boundary values $g = v|_{\partial\Omega}$ and we denote the space of these boundary values by $W_p^{s-\frac{1}{p}}(\partial\Omega)$ and the infimum is taken over those $v \in W_p^s(\Omega)$ for which $v|_{\partial\Omega} = g$. Additionally we use norms depending upon a parameter $\eta \in \mathbb{C}/\{0\}$:

$$|||g|||_{s-\frac{1}{p},p,\partial\Omega} := \|g\|_{W_p^{s-\frac{1}{p}}(\partial\Omega)} + |\eta|^{\frac{s-\frac{1}{p}}{m}} \|g\|_{L^p(\partial\Omega)}.$$

Now let $\mathcal{L}$ be a closed sector in the complex plane with vertex at the origin, let $m, N \in \mathbb{N}, N > 1$, $\{s_j\}_1^N$, $\{m_j\}_1^N$, $\{r_j\}_1^{\frac{mN}{2}}$ denote sequences of integers such that $s_j + m_j = m$ for $j = 1, \cdots, N$ and we suppose mN is even. Furthermore we assume that $0 = m_1 \leq m_2 \leq \cdots \leq m_N$ and $s_1 \geq s_2 \geq \cdots \geq s_N$. Then we consider the following boundary value problem of ADN type with constant coefficients.

$$\begin{cases} (A(D) - \eta)u(x) = f(x) \quad \text{in } \Omega \subset \mathbb{R}^n, \\ \\ \qquad B_j(D) = g_j(x) \quad \text{on } \partial\Omega \quad \text{for } j = 1, \cdots, \dfrac{mN}{2}, \end{cases} \tag{2.2.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$. $u(x) = (u_1(x), \cdots, u_N(x))^T$ and $f(x) = (f_1(x), \cdots, f_N(x))^T$ are $N \times 1$ matrix functions defined in Ω, $g_j(x)$ are scalar functions dedined on $\partial\Omega$. The entries $A_{jk}(D)$ of the $N \times N$ matrix operator $A(D)$ are linear differential operators of order not exceeding $s_j + m_k$ and defined to be 0 if $s_j + m_k < 0$. $B_j(D)$, $1 \leq j \leq \dfrac{mN}{2}$, is a $1 \times N$ matrix operator whose entries are linear differential operators defined on $\partial\Omega$ of order not exceeding $r_j + m_k$ with $r_j < m$ and defined to be 0 if $r_j + m_k < 0$. We suppose further that $A(D)$ has constant coefficients. Then according to [7] and [34] we obtain the following definition and theorem.

Definition 2.2.2. *Let $\mathcal{L}$ be a closed sector in the complex plane with vertex at the origin, $\partial\Omega \in C^{2,1}$. The boundary problem* (2.2.3) *will be called elliptic with parameter in $\mathcal{L}$ if the following conditions are satisfied.*

1. *det$(\mathring{A}(\xi) - \eta I) \neq 0$ for $\xi \in \mathbb{R}^n$ and $\eta \in \mathcal{L}$ if $|\xi| + |\eta| \neq 0$.*

2. *Lopatinskii-Shapiro condition holds. More precisely, fix $x_0 \in \partial\Omega$, assume that the boundary problem* (2.2.3) *is rewritten in a local coordinate system associated with x_0; i.e., we establish a new coordinate system first by rotation after which the positive x_n-axis has the direction of the interior normal to $\partial\Omega$ at x_0 and then translation such that $x_0 \mapsto 0$. Let Φ_{x_0} denote the coordinate transformation with $\Phi_{x_0}(0) = x_0$, $\xi' \in \mathbb{R}^{n-1}$ and $\eta \in \mathcal{L}$, then the boundary problem on the half-line*

$$\begin{cases} \mathring{A}(\xi', D_n))v(t) - \eta v(t) = 0 \quad \textit{for} \quad t = x_n > 0, \\ \qquad\qquad v(t) = 0 \quad \textit{at} \quad t = 0, \\ \qquad\quad |v(t)| \longrightarrow 0 \quad \textit{as} \quad t \longrightarrow \infty \end{cases} \tag{2.2.4}$$

has only the trivial solution for $\xi' \in \mathbb{R}^{n-1}$ and $\eta \in \mathcal{L}$ if $|\xi'| + |\eta| \neq 0$.

Remark 2.2.1. *From Proposition 2.2 in [8] it follows that when Condition 1 of Definition 2.2.2 is satisfied, then mN is even. In [34] the matrix valued case with a diagonal discontinuous weight matrix was studied. For this the given region is subdivided into subregions on which the weights are continuous and* *the transmission conditions at the boundaries of the subregions are needed. It is redundant in our case since there is no weight function in* (2.2.3)*, i.e. the weight functions* $w_j(x) \equiv 1$ *for* $j = 1, \cdots, N$*.*

In (2.2.4) *just the pullback of the original operator with respect to the new coordinate system (after rotation and translation) needs to be considered. Thus according to theorem 2.6 in [34] the definition above gives just the sufficient conditions of the following result.*

Theorem 2.2.1. *[34] Assume the boundary problem* (2.2.3) *is elliptic with parameter in the sector* $\mathcal{L}$*. Then there exists a* $\eta_0 = \eta_0(p) > 0$ *such that for* $\eta \in \mathcal{L}$ *with* $|\eta| \geq \eta_0$*, the boundary problem* (2.2.3) *admits a unique solution* $u = (u_1, \cdots, u_N) \in \prod_{j=1}^N W_p^{m_j+m}(\Omega)$ *for any* $f = (f_1, \cdots, f_N) \in \prod_{j=1}^N W_p^{m_j}(\Omega)$ *and* $g = (g_1, \cdots, g_{\frac{mN}{2}}) \in \prod_{j=1}^{\frac{mN}{2}} W_p^{m-r_j-\frac{1}{p}}(\partial\Omega)$*, and the a priori estimate*

$$\sum_{j=1}^{N} |||u_j|||_{m_j+m,p,\Omega} \leq c \left(\sum_{j=1}^{N} |||f_j|||_{m_j,p,\Omega} + \sum_{j=1}^{\frac{mN}{2}} |||g_j|||_{m-r_j-\frac{1}{p},p,\partial\Omega} \right) \tag{2.2.5}$$

holds, where the constant c *does not depend upon* f, g, η *and we use the norms depending on a parameter* $\eta \in \mathbb{C}/\{0\}$*, namely let* $v \in W_p^s(\Omega), g \in W_p^{s-\frac{1}{p}}(\partial\Omega)$ *where* s *is an integer satisfying* $1 \leq s \leq m$*,*

$$|||v|||_{s,p,\Omega} := \|v\|_{W_p^s(\Omega)} + |\eta|^{\frac{s}{m}} \|v\|_{L^p(\Omega)}, \tag{2.2.6}$$

$$|||g|||_{s-\frac{1}{p},p,\partial\Omega} := \|g\|_{W_p^{s-\frac{1}{p}}(\partial\Omega)} + |\eta|^{\frac{s-\frac{1}{p}}{m}} \|g\|_{L^p(\partial\Omega)}. \tag{2.2.7}$$

Chapter 3

Local Solutions to the Isothermal Viscous QHD

Concerning the *inviscid* model (the viscosity constant is zero) of quantum hydrodynamics several results are derived. For instance in [62] the asymptotic behavior for a constant profile of background charges in a torus[1] is derived. For viscous model only a few results are available. The local existence and uniqueness of solutions for one-dimensional interval with insulating boundary conditions and for higher dimensions on a torus were derived in [17]. Afterwards M. Dreher proved the local existence and uniqueness of solutions of viscous model with a general boundary conditions in [32] where a method of introducing a viscous regularization term was used. In this chapter we study the local existence of solutions to viscous quantum hydrodynamics

$$\begin{cases} \partial_t n - \mathrm{div} J = \nu_0 \triangle n, \\ \partial_t J - \mathrm{div}\left(\frac{J\otimes J}{n}\right) - T_0 \nabla n + n\nabla V + \frac{\epsilon^2}{2} n \nabla\left(\frac{\triangle\sqrt{n}}{\sqrt{n}}\right) = \nu_0 \triangle J - \frac{J}{\tau}, \\ \lambda^2 \triangle V = n - \mathcal{C}(x), \\ (n,J)(0,x) = (n_0, J_0)(x), \end{cases} \tag{3.0.1}$$

where the initial data (n_0, J_0) satisfies

$$n_0 \in H^3(\Omega), \quad J_0 \in (H^2(\Omega))^d, \tag{3.0.2}$$

and the boundary conditions read

$$\begin{cases} (n,J,V)(t,x) = (n_\Gamma, J_\Gamma, V_\Gamma) \quad \text{on } \partial\Omega, \\ n_\Gamma \in H^{5/2}(\partial\Omega), \quad J_\Gamma \in H^{3/2}(\partial\Omega), \quad V_\Gamma \in H^{3/2}(\partial\Omega), \end{cases} \tag{3.0.3}$$

[1]In geometry, a torus (pl. tori) is a surface of revolution generated by revolving a circle in three dimensional space about an axis coplanar with the circle, which does not touch the circle. PDEs on a torus satisfy the periodic boundary conditions.

i.e. the Dirichlet boundary conditions (3.0.3) are given. Furthermore the following compatibility conditions are also required:

$$\begin{cases} (n_0, J_0)\,|_{\partial\Omega} = (n_\Gamma, J_\Gamma), \\ (\nu_0 \triangle n_0 + \mathrm{div} J_0)|_{\partial\Omega} = 0. \end{cases} \tag{3.0.4}$$

Here $(t,x) \in (0,\infty) \times \Omega$ denotes the temporal and spatial variables, $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$ with boundary $\partial\Omega$. The unknown functions are n, J, V which model the particle density, the momentum and the electrostatic potential respectively. The given function $\mathcal{C}(x)$ models the given profile of background charges. The scaled constants are the temperature T_0, the Planck constant ϵ, the Debye length λ, and a viscosity constant ν_0, the momentum relaxation time τ. $J \otimes J$ denotes the tensor product with components $J_i J_k$ and the i-th component of the convective term $\mathrm{div}\left(\frac{J \otimes J}{n}\right)$ equals

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k} \left(\frac{J_i J_k}{n} \right).$$

The idea to derive the local existence result is first to linearize the system (3.0.1) and then to study the corresponding linear system. Next construct approximate solutions (n_k, J_k, V_k) from a fixed-point procedure, which are expected to converge to a solution of the original problem (n, J, V). Namely we shall derive a contraction mapping and then use Banach fixed point theorem. For this a uniform estimate of the approximate solutions (n_k, J_k, V_k) which is the key task is indispensable.

3.1 Linearization of the Original Problem

A reformulation of the Bohm potential term is derived in [32] as follows

$$\frac{\epsilon^2}{2} n \nabla \left(\frac{\triangle\sqrt{n}}{\sqrt{n}} \right) = \frac{\epsilon^2}{4} \nabla \triangle n - \epsilon^2 \mathrm{div} \left((\nabla\sqrt{n}) \otimes (\nabla\sqrt{n}) \right). \tag{3.1.1}$$

Putting $U := (n, J)^T$, $J := (J_1, \dots, J_d)^T$ we can reformulate the equations for n and J from (3.0.1) as

$$\partial_t U + A(\partial_x) U + \begin{pmatrix} 0 \\ G \end{pmatrix} = 0, \tag{3.1.2}$$

where

$$A(\partial_x) := \begin{pmatrix} -\nu_0 \triangle & -\mathrm{div} \\ -T_0 \nabla + \frac{\epsilon^2}{4} \nabla \triangle & -\nu_0 \triangle + \tau^{-1} \end{pmatrix}, \tag{3.1.3}$$

$$G := -\mathrm{div} \left(\frac{J \otimes J}{n} \right) + n \nabla V - \epsilon^2 \mathrm{div} \left((\nabla\sqrt{n}) \otimes (\nabla\sqrt{n}) \right). \tag{3.1.4}$$

By this reformulation we have successfully linearized (3.0.1), i.e., separated the linear term $A(\partial_x)$ and the nonlinear term G where G has lower order than $A(\partial_x)$.

3.2 Solutions to the Linear System

We study the following IBVP

$$\begin{cases} \partial_t \mathbf{u}(t,x) + A(\partial_x)\mathbf{u}(t,x) = F(t,x), \quad (t,x) \in [0,T) \times \Omega \\ \mathbf{u}(0,x) = \mathbf{g}(x), \\ \mathbf{u}|_{\partial\Omega} = 0 \quad \text{for a.e. } t \in [0,T]. \end{cases} \tag{3.2.1}$$

Here

$$A(\partial_x) := \begin{pmatrix} -\nu_0 \triangle & -\text{div} \\ -T_0 \nabla + \dfrac{\epsilon^2}{4} \nabla \triangle & -\nu_0 \triangle + \tau^{-1} \end{pmatrix},$$

defined as (3.1.3). $[0,T)$ is an arbitrary time interval, Ω is a bounded domain with boundary $\partial\Omega$ of regularity C^4. Given $F(t,x) = (F^0(t,x), F^{\mathbf{d}}(t,x))$ for $F^{\mathbf{d}}(t,x) := (F^1(t,x), \cdots, F^d(t,x))$ and $\mathbf{g}(x) := (g^0(x), g^{\mathbf{d}}(x))$ with $g^{\mathbf{d}}(x) := (g^1(t,x), \cdots, g^d(t,x))$ the unknown function $\mathbf{u} = (u^0, u^{\mathbf{d}})$ for $u^{\mathbf{d}} := (u^1, \cdots, u^d)$ is a vector-valued function with $1+d$ components.

In order to solve (3.2.1) we study first the BVP

$$A(\partial_x)\mathbf{u} = \mathbf{f}, \tag{3.2.2}$$

subject to a boundary condition

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_\Gamma. \tag{3.2.3}$$

3.2.1 Existence of Weak Solutions to (3.2.2)-(3.2.3)

Theorem 3.2.1. *The BVP*

$$\begin{cases} -\dfrac{\epsilon^2}{4\nu_0} \nabla \mathit{div} \mathbf{u} - \nu_0 \triangle \mathbf{u} + \dfrac{1}{\tau}\mathbf{u} = \mathbf{f}, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{u}_\Gamma \end{cases} \tag{3.2.4}$$

has a unique solution $\mathbf{u} \in (H^1(\Omega))^d$ *for any* $\mathbf{f} \in (H^{-1}(\Omega))^d$ *and* $\mathbf{u}_\Gamma \in H^{1/2}(\Omega)$.

Proof. Select a function $\mathbf{v} \in (H^1(\Omega))^d$ with $\mathbf{v}|_{\partial\Omega} = \mathbf{u}_\Gamma$. Then it is equivalent to solve the BVP with respect to $\mathbf{s} := \mathbf{u} - \mathbf{v}$:

$$\begin{cases} -\dfrac{\epsilon^2}{4\nu_0} \nabla \text{div} \mathbf{s} - \nu_0 \triangle \mathbf{s} + \dfrac{1}{\tau}\mathbf{s} = \mathbf{f} + \dfrac{\epsilon^2}{4\nu_0} \nabla \text{div} \mathbf{v} + \nu_0 \triangle \mathbf{v} - \dfrac{1}{\tau}\mathbf{v}, \\ \mathbf{s}|_{\partial\Omega} = 0. \end{cases} \tag{3.2.5}$$

For this we first define a bilinear mapping B which reads

$$B : (H_0^1(\Omega))^d \times (H_0^1(\Omega))^d \longrightarrow \mathbb{R},$$
$$(x, y) \longmapsto \frac{\epsilon^2}{4\nu_0}(\mathrm{div}x, \mathrm{div}y) + \nu_0(\nabla x, \nabla y) + \frac{1}{\tau}(x, y).$$

Then from Hölder's inequality it is easy to verify that there exists $c_1 > 0$ such that

$$|B(x, y)| \leq c_1 \|x\|_{(H_0^1(\Omega))^d} \|y\|_{(H_0^1(\Omega))^d}.$$

Furthermore there exists $c_2 > 0$ such that

$$B(x, x) \geq c_2 \|x\|^2_{(H_0^1(\Omega))^d}.$$

Notice that c_1, c_2 are independent of $x, y \in (H_0^1(\Omega))^d$. Since

$$\left(\mathbf{f} + \frac{\epsilon^2}{4\nu_0}\nabla \mathrm{div}\mathbf{v} + \nu_0 \triangle \mathbf{v} - \frac{1}{\tau}\mathbf{v}\right) \in (H^{-1}(\Omega))^d,$$

from Lax-Milgram lemma we conclude that there exists a unique $\mathbf{s} \in (H_0^1(\Omega))^d$ such that

$$B(s, x) = \left(\mathbf{f} + \frac{\epsilon^2}{4\nu_0}\nabla \mathrm{div}\mathbf{v} + \nu_0 \triangle \mathbf{v} - \frac{1}{\tau}\mathbf{v}\right)(x)$$

for all $x \in (H_0^1(\Omega))^d$. Then it follows that (3.2.5) has a unique solution $\mathbf{s} \in (H_0^1(\Omega))^d$. □

Consequently we obtain further

Theorem 3.2.2. *The BVP*

$$-\nu_0 \triangle u_1 - div\mathbf{u}_2 = f, \tag{3.2.6}$$

$$-\nu_0 \triangle \mathbf{u}_2 + \frac{1}{\tau}\mathbf{u}_2 + \frac{\epsilon^2}{4}\nabla \triangle u_1 = \mathbf{f}, \tag{3.2.7}$$

$$(u_1, \mathbf{u}_2)|_{\partial\Omega} = (u_b, \mathbf{u}_b) \tag{3.2.8}$$

has a unique solution $(u_1, \mathbf{u}_2) \in H^2(\Omega) \times (H^1(\Omega))^d$ *for any* $f \in L^2(\Omega), \mathbf{f} \in (H^{-1}(\Omega))$, $u_b \in H^{3/2}(\Omega)$, $\mathbf{u}_b \in H^{1/2}(\Omega)$.

Proof. First from (3.2.6) we obtain

$$-\nu_0 \nabla \triangle u_1 - \nabla \mathrm{div}\mathbf{u}_2 = \nabla f. \tag{3.2.9}$$

Substitute (3.2.9) into (3.2.7) we find

$$-\frac{\epsilon^2}{4\nu_0}\nabla \mathrm{div}\mathbf{u}_2 - \nu_0\triangle\mathbf{u}_2 + \frac{1}{\tau}\mathbf{u}_2 = \mathbf{f} + \frac{\epsilon^2}{4\nu_0}\nabla f, \qquad (3.2.10)$$
$$\mathbf{u}_2|_{\partial\Omega} = \mathbf{u}_b.$$

From **Theorem** 3.2.1 the BVP (3.2.10) admits a unique solution in $(H_0^1(\Omega))^d$. Then going back to (3.2.6) and (3.2.8), u_1 can be solved in $H^2(\Omega)$. □

Theorem 3.2.3. *The BVP*

$$-\nu_0\triangle u_1 - \mathit{div}\mathbf{u}_2 = f, \qquad (3.2.11)$$

$$-\nu_0\triangle\mathbf{u}_2 + \frac{1}{\tau}\mathbf{u}_2 + \frac{\epsilon^2}{4}\nabla\triangle u_1 - T_0\nabla u_1 = \mathbf{f}, \qquad (3.2.12)$$

$$(u_1, \mathbf{u}_2)|_{\partial\Omega} = (u_b, \mathbf{u}_b) \qquad (3.2.13)$$

has a unique solution $(u_1, \mathbf{u}_2) \in H^2(\Omega) \times (H^1(\Omega))^d$ *for any* $f \in L^2(\Omega), \mathbf{f} \in (H^{-1}(\Omega))^d$, $u_b \in H^{3/2}(\Omega)$, $\mathbf{u}_b \in (H^{1/2}(\Omega))^d$.

Proof. It sufficies to prove that

$$-\nu_0\triangle u_1 - \mathrm{div}\mathbf{u}_2 = f, \qquad (3.2.14)$$

$$-\nu_0\triangle\mathbf{u}_2 + \frac{1}{\tau}\mathbf{u}_2 + \frac{\epsilon^2}{4}\nabla\triangle u_1 - T_0\nabla u_1 = \mathbf{f}, \qquad (3.2.15)$$

has a unique solution $(u_1, \mathbf{u}_2) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H_0^1(\Omega))^d$ for any $f \in L^2(\Omega), \mathbf{f} \in (H^{-1}(\Omega))^d$.

Given $v_1 \in L^2(\Omega)$. Owing to the assumptions we see that

$$\mathbf{f} + T_0\nabla v_1 \in (H^{-1}(\Omega))^d.$$

By **Theorem** 3.2.2 we obtain

$$(u_1, \mathbf{u}_2) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H_0^1(\Omega))^d$$

where $(u_1, \mathbf{u}_2)$ are derived from v_1 via

$$-\nu_0\triangle u_1 - \mathrm{div}\mathbf{u}_2 = f, \qquad (3.2.16)$$

$$-\nu_0\triangle\mathbf{u}_2 + \frac{1}{\tau}\mathbf{u}_2 + \frac{\epsilon^2}{4}\nabla\triangle u_1 = \mathbf{f} + T_0\nabla v_1. \qquad (3.2.17)$$

Let us henceforth write $A[v_1] = u_1$. We now assert that $A : L^2(\Omega) \to L^2(\Omega)$ is continuous and compact. Given further $\tilde{v}_1 \in L^2(\Omega)$, then we obtain

$$(\tilde{u}_1, \tilde{\mathbf{u}}_2) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H_0^1(\Omega))^d$$

which solves

$$-\nu_0\triangle\tilde{u}_1 - \mathrm{div}\tilde{\mathbf{u}}_2 = f, \tag{3.2.18}$$

$$-\nu_0\triangle\tilde{\mathbf{u}}_2 + \frac{1}{\tau}\tilde{\mathbf{u}}_2 + \frac{\epsilon^2}{4}\nabla\triangle\tilde{u}_1 = \mathbf{f} + T_0\nabla\tilde{v}_1, \tag{3.2.19}$$

by **Theorem** 3.2.2. Then we obtain

$$-\nu_0\triangle(u_1 - \tilde{u}_1) - \mathrm{div}(\mathbf{u}_2 - \tilde{\mathbf{u}}_2) = 0, \tag{3.2.20}$$

$$-\nu_0\triangle(\mathbf{u}_2 - \tilde{\mathbf{u}}_2) + \frac{1}{\tau}(\mathbf{u}_2 - \tilde{\mathbf{u}}_2) + \frac{\epsilon^2}{4}\nabla\triangle(u_1 - \tilde{u}_1) = T_0(\nabla v_1 - \nabla\tilde{v}_1). \tag{3.2.21}$$

Taking the $L^2(\Omega)$ inner product between $-\frac{\epsilon^2}{4}\triangle(u_1 - \tilde{u}_1)$ and (3.2.20) yields

$$\frac{\epsilon^2}{4}\nu_0\|\triangle(u_1 - \tilde{u}_1)\|^2 - \frac{\epsilon^2}{4}(\nabla\triangle(u_1 - \tilde{u}_1), \mathbf{u}_2 - \tilde{\mathbf{u}}_2) = 0. \tag{3.2.22}$$

Then take the $L^2(\Omega)$ inner product of the second equation of (3.2.21) with $\mathbf{u}_2 - \tilde{\mathbf{u}}_2$, we obtain

$$\begin{aligned} &\frac{\epsilon^2}{4}(\nabla\triangle(u_1 - \tilde{u}_1), \mathbf{u}_2 - \tilde{\mathbf{u}}_2) + \nu_0\|\nabla(\mathbf{u}_2 - \tilde{\mathbf{u}}_2)\|^2 + \frac{1}{\tau}\|\mathbf{u}_2 - \tilde{\mathbf{u}}_2\|^2 \\ &= T_0(\nabla v_1 - \nabla\tilde{v}_1, \mathbf{u}_2 - \tilde{\mathbf{u}}_2). \end{aligned} \tag{3.2.23}$$

From (3.2.22) and (3.2.23) we deduce

$$\begin{aligned} &\frac{\epsilon^2}{4}\nu_0\|\triangle(u_1 - \tilde{u}_1)\|^2 + \nu_0\|\nabla(\mathbf{u}_2 - \tilde{\mathbf{u}}_2)\|^2 + \frac{1}{\tau}\|\mathbf{u}_2 - \tilde{\mathbf{u}}_2\|^2 \\ &= T_0(\nabla v_1 - \nabla\tilde{v}_1, \mathbf{u}_2 - \tilde{\mathbf{u}}_2) \le T_0\|v_1 - \tilde{v}_1\|_{L^2(\Omega)}\|\nabla(\mathbf{u}_2 - \tilde{\mathbf{u}}_2)\|_{L^2(\Omega)}, \end{aligned}$$

from which we conclude

$$\|u_1 - \tilde{u}_1\|_{H^2(\Omega)} \le C\|v_1 - \tilde{v}_1\|_{L^2(\Omega)}.$$

Hence A is continuous. Furthermore by a similar reasoning we find

$$\begin{aligned} &\frac{\epsilon^2}{4}\nu_0\|\triangle u_1\|^2 + \nu_0\|\nabla\mathbf{u}_2\|^2 + \frac{1}{\tau}\|\mathbf{u}_2\|^2 = -\frac{\epsilon^2}{4}(\triangle u_1, f) + T_0(\nabla v_1, \mathbf{u}_2) + (\mathbf{f}, \mathbf{u}_2) \\ &\le C\left(\|\triangle u_1\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)} + \|v_1\|_{L^2(\Omega)}\|\mathrm{div}\mathbf{u}_2\|_{L^2(\Omega)} + \|\mathbf{f}\|_{(H^{-1}(\Omega))^d}\|\mathbf{u}_2\|_{H_0^1(\Omega)}\right). \end{aligned}$$

Then by Cauchy's inequality we see

$$\|u_1\|_{H^2(\Omega)}^2 \le C\left(\|f\|_{L^2(\Omega)}^2 + \|\mathbf{f}\|_{(H^{-1}(\Omega))^d}^2 + \|v_1\|_{L^2(\Omega)}^2\right),$$

which implies the compactness of A from Sobolev compact embedding theorem.

Finally we must show that the set

$$\{u_1 \in L^2(\Omega) \mid u_1 = \lambda A(u_1),\ 0 \le \lambda \le 1\}$$

is bounded in $L^2(\Omega)$. In other words, we shall obtain the a priori estimates of u_1 which satisfies

$$-\nu_0 \triangle u_1 - \mathrm{div}\mathbf{u}_2 = \lambda f, \tag{3.2.24}$$

$$-\nu_0 \triangle \mathbf{u}_2 + \frac{1}{\tau}\mathbf{u}_2 + \frac{\epsilon^2}{4}\nabla \triangle u_1 = \lambda \mathbf{f} + \lambda T_0 \nabla u_1. \tag{3.2.25}$$

Taking the $L^2(\Omega)$ scalar product of (3.2.24) with $-\frac{\epsilon^2}{4}\triangle u_1$ yields

$$\frac{\epsilon^2}{4}\nu_0\|\triangle u_1\|^2 - \frac{\epsilon^2}{4}(\nabla\triangle u_1, \mathbf{u}_2) = -\frac{\epsilon^2}{4}(\lambda f, \triangle u_1). \tag{3.2.26}$$

Take the $L^2(\Omega)$ scalar product of (3.2.24), (3.2.25) with $\lambda T_0 u_1$, $\mathbf{u}_2$ respectively we infer

$$\lambda\nu_0 T_0\|\nabla u_1\|^2 + \lambda T_0(\nabla u_1, \mathbf{u}_2) = \lambda T_0(\lambda f, u_1), \tag{3.2.27}$$

$$\nu_0\|\nabla \mathbf{u}_2\|^2 + \frac{1}{\tau}\|\mathbf{u}_2\|^2 + \frac{\epsilon^2}{4}(\nabla\triangle u_1, \mathbf{u}_2) - \lambda T_0(\nabla u_1, \mathbf{u}_2) = (\lambda \mathbf{f}, \mathbf{u}_2). \tag{3.2.28}$$

(3.2.26)+(3.2.27)+(3.2.28) provides

$$\begin{aligned}&\frac{\epsilon^2}{4}\nu_0\|\triangle u_1\|^2 + \lambda\nu_0 T_0\|\nabla u_1\|^2 + \nu_0\|\nabla\mathbf{u}_2\|^2 + \frac{1}{\tau}\|\mathbf{u}_2\|^2 \\ &= \lambda T_0(\lambda f, u_1) + (\lambda \mathbf{f}, \mathbf{u}_2) - \frac{\epsilon^2}{4}(\lambda f, \triangle u_1).\end{aligned}$$

Then

$$\begin{aligned}&\frac{\epsilon^2}{4}\nu_0\|\triangle u_1\|^2 + \nu_0\|\nabla\mathbf{u}_2\|^2 + \frac{1}{\tau}\|\mathbf{u}_2\|^2 \le \lambda^2 T_0(f, u_1) + \lambda(\mathbf{f}, \mathbf{u}_2) - \lambda\frac{\epsilon^2}{4}(f, \triangle u_1) \\ &\le T_0|(f, u_1)| + |(\mathbf{f}, \mathbf{u}_2)| + \frac{\epsilon^2}{4}|(f, \triangle u_1)|.\end{aligned}$$

Using Hölder's and Cauchy's inequalities the boundedness of u_1 follows immediately. By **Schaefer's Fixed Point Theorem** A has a fixed point in $L^2(\Omega)$. Furthermore from **Theorem** 3.2.2

$$u_1 \in H^2(\Omega), \quad \mathbf{u}_2 \in (H_0^1(\Omega))^d.$$

□

In order to derive a strong solution of (3.2.11)-(3.2.13), i.e., a solution in $H^3(\Omega) \times (H^2(\Omega))^d$, we use the property of parameter-ellipticity of mixed order [5, 34]. The next subsectin will focus on this topic.

3.2.2 Analysis of a Mixed-Order BVP

We consider the spatial operator $A(\partial_x)$. In order to match the assumptions in [34] which are necessary later we rewrite $A(\partial_x)$ to an equivalent operator $A(D)$ with respect to $(u^d, \cdots, u^1, u^0)$ in the following way

$$A(D) := \begin{pmatrix} \tau^{-1} - \nu_0 \triangle & 0 & \cdots & 0 & A_d \\ 0 & \tau^{-1} - \nu_0 \triangle & \cdots & \vdots & A_{d-1} \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & \tau^{-1} - \nu_0 \triangle & A_1 \\ -\partial_d & -\partial_{d-1} & \cdots & -\partial_1 & -\nu_0 \triangle \end{pmatrix} \tag{3.2.29}$$

where

$$A_j := \frac{\epsilon^2}{4} \partial_j \triangle - T_0 \partial_j, \quad j = 1, \cdots, d.$$

Select sequences of integers $\{s_j\}_1^{d+1}$, $\{m_j\}_1^{d+1}$ such that

$$\begin{aligned} s_1 = s_2 = \cdots = s_d = 2, s_{d+1} = 1, \\ m_1 = m_2 = \cdots = m_d = 0, m_{d+1} = 1, \end{aligned} \tag{3.2.30}$$

then it is easy to verify that $A(D)$ is an $(d+1) \times (d+1)$ matrix operator with constant coefficients and the entries $A_{jk}(D)$ are linear differential operators defined on Ω of order not exceeding $s_j + m_k$. For $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d$ let $A(\xi)$ denote the symbol, $\mathring{A}(\xi)$ the principal symbol of $A(D)$ respectively($\mathring{A}_{jk}(\xi)$ consists of the terms in $A(\xi)$ which are of the order $s_j + m_k$), then

$$\mathring{A}(\xi) = \begin{pmatrix} \nu_0 |\xi|^2 & 0 & \cdots & 0 & -i \frac{\epsilon^2}{4} \xi_d |\xi|^2 \\ 0 & \nu_0 |\xi|^2 & \cdots & \vdots & -i \frac{\epsilon^2}{4} \xi_{d-1} |\xi|^2 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & \nu_0 |\xi|^2 & -i \frac{\epsilon^2}{4} \xi_1 |\xi|^2 \\ -i\xi_d & -i\xi_{d-1} & \cdots & -i\xi_1 & \nu_0 |\xi|^2 \end{pmatrix}. \tag{3.2.31}$$

We compute $\det \mathring{A}(\xi)$ and infer

$$\det \mathring{A}(\xi) = 0 \Longleftrightarrow \xi = 0.$$

According to [5] $A(D)$ is an elliptic differential matrix operator of mixed-order(a general system of Agmon-Douglis-Nirenberg type).

Next we shall be concerned with the following boundary value problem with a parameter η in some closed sector in the complex plane with vertex at the origin with respect to the unknown functions

$$\mathbf{u}(x) := (u^d, u^{d-1}, \cdots, u^1, u^0)$$

$$\begin{cases} A(D)\mathbf{u}(x) - \eta\mathbf{u}(x) = f(x) & \text{in } \Omega, \\ B(D)\mathbf{u}(x) = g(x) & \text{on } \partial\Omega, \end{cases} \tag{3.2.32}$$

where $f(x) = (f_1(x), \cdots, f_{d+1}(x))^T$ are $(d+1) \times 1$ matrix functions defined in Ω, $g(x) = (g_1(x), \cdots, g_{d+1}(x))^T$ defined on $\partial\Omega$. The $(d+1) \times (d+1)$ matrix operator $B(D)$ describes the Dirichlet boundary conditions and is defined as follows

$$B(D) := I_{d+1},$$

here let I_{d+1} denote the $(d+1) \times (d+1)$ identity matrix. As for $B(D)$ we define another sequence of integers $\{r_j\}_1^{d+1}$ such that

$$r_1 = r_2 = \cdots = r_d = 0, r_{d+1} = -1, \tag{3.2.33}$$

then it is easy to see that the entries of $B(D)$ are linear differential operators defined on $\partial\Omega$ of order not exceeding $r_j + m_k$ with $r_j < m := s_j + m_j = 2$ and defined to be zero if $r_j + m_k < 0$.

Before starting our main result we study the *pullback* of a linear differential operator. We first consider the linear differential operator whose domain of definition and range are scalar-valued functions.

Definition 3.2.1. *For two nonempty bounded open sets $\Omega, G \subset \mathbb{R}^n$ let $\Phi : \Omega \to G$ be a bijection. Given a function $u : G \to \mathbb{C}$ then the function $\Phi^* u := u \circ \Phi : \Omega \to \mathbb{C}$ is called the pullback of u. Assume $\Phi : \Omega \to G$ is a C^m-diffeomorphism $1 < p < \infty$ and let $A(x, D)$ be a linear differential operator on G of order m with domain $W_p^m(G)$, then the operator $B(x, D) := \Phi^* A := A(\cdot \circ \Phi^{-1}) \circ \Phi$ defined on Ω with domain $W_p^m(\Omega)$ is called the pullback of $A(x, D)$.*

Theorem 3.2.4. *In definition 3.2.1 assume the Jacobimatrix Φ' of Φ is constant in $x \in \Omega$, i.e., $\partial_j \Phi' \equiv 0 (j = 1, \cdots, n)$. Then*

$$B(x, D) = A(\Phi(x), [\Phi'(x)]^{-T} D); \tag{3.2.34}$$

$$\mathring{B}(x, D) = \mathring{A}(\Phi(x), [\Phi'(x)]^{-T} D), \tag{3.2.35}$$

where $[\cdot]^{-T} := ([\cdot]^{-1})^T$, $(\cdot)^T$ denotes the transpose, $\mathring{B}(x, D)$, $\mathring{A}(x, D)$ denote the principal parts of $B(x, D)$ and $A(x, D)$ respectively.

Proof. Without loss of generality assume $(A(y, D)u)(y) = a(y)(D_{i_1} \cdots D_{i_s} u)(y)$ with a function $a : G \to \mathbb{C}$ and $i_1, \cdots, i_s \in \{1, \cdots, n\}$. Let $v \in W_p^m(\Omega)$, then

$v \circ \Phi^{-1} \in W_p^m(G)$(it follows from theorem 3.14 in [1]) and

$$\partial_{i_1}(v \circ \Phi^{-1})(y) = \sum_{j_1=1}^{n} (\partial_{j_1} v)(\Phi^{-1}(y)) S_{j_1 i_1}, \tag{3.2.36}$$

where $(S_{jk})_{j,k=1,\cdots,n} := (\Phi^{-1})'$. By product rule:

$$\partial_{i_2}\partial_{i_1}(v \circ \Phi^{-1})(y) = \sum_{j_2=1}^{n} \sum_{j_1=1}^{n} (\partial_{j_2}\partial_{j_1} v)(\Phi^{-1}(y)) S_{j_2 i_2} S_{j_1 i_1}. \tag{3.2.37}$$

Then it follows from iteration:

$$\partial_{i_1} \cdots \partial_{i_m}(v \circ \Phi^{-1})(y) = \sum_{j_m=1}^{n} \cdots \sum_{j_1=1}^{n} (\partial_{j_1} \cdots \partial_{j_m} v)(\Phi^{-1}(y)) \prod_{l=1}^{m} S_{j_l i_l}, \tag{3.2.38}$$

which implies

$$\begin{aligned}
&(A(y,D)(v \circ \Phi^{-1}))(y) \\
&= a(y) \sum_{j_m=1}^{n} \cdots \sum_{j_1=1}^{n} (D_{j_1} \cdots D_{j_m} v)(\Phi^{-1}(y)) \prod_{l=1}^{m} S_{j_l i_l} \\
&= a(\Phi(x)) \left(\left(\prod_{l=1}^{m} \left(((S_{jk})_{j,k=1,\cdots,n})^T (D_1 \quad D_2 \cdots D_n)^T \right)_{i_l} \right) v \right)(x).
\end{aligned}$$

Thus we obtain (3.2.34), and (3.2.35) follows as a direct consequence. □

Compared to the scalar case the *pullback* of a vector field is defined by a different way. We consider only the special case that the transform $\Phi : \Omega \to G$ is a rotation and therefore Φ is a rotation matrix. Let $u : G \to \mathbb{C}^n$ be a vector field, then the *pullback* of u is $\Phi^* u := \Phi^{-1} \circ u \circ \Phi : \Omega \to \mathbb{C}^n$. Now we give two important examples of the *pullback* of relevant operators on a vector field (or the range is a vector field).

Example 3.2.1. *Let $\Phi : \Omega \to G$ be a rotation, $u : G \to \mathbb{C}$ a scarlar field, then we obtain the pullback of u $u \circ \Phi : \Omega \to \mathbb{C}$ from definition 3.2.1. Let A be the gradient of u, then the corresponding differential operator on $u \circ \Phi$ is also gradient, i.e.,*

$$\nabla(u \circ \Phi)(x) = \Phi^{-1} \nabla u(y)$$

with $y = \Phi(x)$.

Example 3.2.2. *Suppose $u : G \to \mathbb{C}^n$ is a vecor field, A is the divergence of u, then the divergence of $\Phi^{-1} \circ u \circ \Phi : \Omega \to \mathbb{C}^n$ equals to $A(u)$, i.e.,*

$$\mathtt{div}(\Phi^{-1} \circ u \circ \Phi)(x) = \mathtt{div} u(y)$$

with $y = \Phi(x)$.

Remark 3.2.1. *From the aspects of theory of fields the gradient of a scalar field and the divergence of a vector field describe the feature of the scalar field and the vector field respectively and they are independent of the choice of the coordinate system (via rotation).*

After these preparations mentioned above we have the following

Theorem 3.2.5. *The boundary problem* (3.2.32) *is elliptic with parameter in any sector $\mathcal{L}$ in the complex plane with vertex at the origin and*

$$\mathcal{L} \subset S_1 \cup S_2$$

where

$$S_1 := \left\{ \eta \in \mathbb{C}, \quad -\pi + arctg\frac{\epsilon}{2\nu_0} < arg\eta < -arctg\frac{\epsilon}{2\nu_0} \right\},$$

$$S_2 := \left\{ \eta \in \mathbb{C}, \quad arctg\frac{\epsilon}{2\nu_0} < arg\eta < \pi - arctg\frac{\epsilon}{2\nu_0} \right\}.$$

Assume

$$\frac{\epsilon}{2\nu_0} \leq \sqrt{3},$$

then the boundary problem (3.2.32) *is elliptic with parameter in any sector $\mathcal{L}$ where*

$$\mathcal{L} \subset \left\{ z \in \mathbb{C}, \quad argz \neq 0, \pm arctg\frac{\epsilon}{2\nu_0} \right\}$$

in the sense of Definition 2.2.2.

Proof. For $d = 1$,

$$\det(\mathring{A}(\xi) - \eta I) = 0 \Longrightarrow \eta = \nu_0|\xi|^2 \pm i\frac{\epsilon}{2}|\xi|^2;$$

for $d > 1$,

$$\det(\mathring{A}(\xi) - \eta I) = 0 \Longrightarrow \eta = \nu_0|\xi|^2 \quad \text{or} \quad \eta = \nu_0|\xi|^2 \pm i\frac{\epsilon}{2}|\xi|^2.$$

Then the first condition of defintion 2.2.2 follows immediately.

Fix now $x_0 \in \partial\Omega$, let Φ_{x_0} denote the coordinate transformation from the original coordinate system into a local coordinate system at x_0 (here $\Phi_{x_0}(0) = x_0$ and $\nu \mapsto e_n$, where ν is the interior normal to $\partial\Omega$ at x_0 and $(e_1, \cdots, e_n)$ denotes the standard basis in $\mathbb{R}^n$). We achieve this coordinate transformation via translation and rotation. Notice that the considered operator reads

$$\begin{pmatrix} -\nu_0\triangle & -\text{div} \\ -T_0\nabla + \frac{\epsilon^2}{4}\nabla\triangle & -\nu_0\triangle + \tau^{-1} \end{pmatrix}, \tag{3.2.39}$$

then from the examples 3.2.1, 3.2.2 and $\triangle = \nabla \cdot \nabla$ we can rewrite this operator in the local coordinate system associated with x_0 in the same form as (3.2.39). Thus the principal parts of the *pullback* of $A(D)$ reads the same as $A(D)$.

Checking Lopatinskii-Shapiro condition (the second condition of Definition 2.2.2)will be carried out with the help of [5](45-51) since we consider the following problem

$$\begin{cases} \mathring{A}(\xi', D_d)v(t) - \eta v(t) = 0 \quad \text{for} \quad t = x_d > 0, \\ v(t) = 0 \quad \text{at} \quad t = 0, \\ |v(t)| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty \end{cases} \tag{3.2.40}$$

which is a system of linear ordinary differential equations with constant (complex) coefficients and

$$\mathring{A}(\xi', D_d) := \begin{pmatrix} \nu_0|\xi'|^2 + \nu_0 D_d^2 & 0 & \cdots & 0 & \mathring{A}_d \\ 0 & \nu_0|\xi'|^2 + \nu_0 D_d^2 & \cdots & \vdots & \mathring{A}_{d-1} \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & \nu_0|\xi'|^2 + \nu_0 D_d^2 & \mathring{A}_1 \\ -iD_d & -i\xi_{d-1} & \cdots & -i\xi_1 & \nu_0|\xi'|^2 + \nu_0 D_d^2 \end{pmatrix}$$

with

$$\mathring{A}_d : = -i\frac{\epsilon^2}{4}|\xi'|^2 D_d - i\frac{\epsilon^2}{4}D_d^3,$$
$$\mathring{A}_j : = -i\frac{\epsilon^2}{4}\xi_j|\xi'|^2 - i\frac{\epsilon^2}{4}\xi_j D_d^2, \quad j = 1, \cdots, d-1.$$

We apply the results of S. Agmon, A. Douglis, and L. Nirenberg in [5] pp. 45-51 and record the corresponding theorem here for easy reference.

Theorem(**see** [5]) *Let $L(D_d)$, $L^{jk}(D_d)$ denote* $\det\Big(\mathring{A}(\xi', D_d) - \eta I_{d+1}\Big)$, *and the adjoint to the matrix $\mathring{A}(\xi', D_d) - \eta I_{d+1}$ respectively. Assume the polynomial $L(r)$ has exactly p^+ complex roots with* $\mathrm{Im} r > 0$ *and $L(r) = L^+(r)L^-(r)$ where the roots of $L^+(r)$* are *those of $L(r)$ for which* $\mathrm{Im} r > 0$. Let $\xi' \in \mathbb{R}^{d-1}$, $\eta \in \mathcal{L}$ *and* $|\xi'| + |\eta| \neq 0$. *Then the following statements are all equivalent to one another*:

1. *Complementing Condition 2 in* [5](48) *holds*, *i.e.*, *The lines of the $p^+ \times (d+1)$ matrix*

$$B_{\sigma j}(r)L^{jk}(r)$$

are linear independent modulo the polynomial $L^+(r)$; *i.e., the equations*

$$\sum_\sigma C_\sigma B_{\sigma j}(r)L^{jk}(r) \equiv 0 \quad (\bmod L^+(r))$$

imply the constants C_σ *are all zero, where* $B_{\sigma j}(r)$ *is a submatrix of* $B(r)$ *with* $\sigma = 1, \cdots, p^+$, $j = 1, \cdots, d+1$.

2. *The problem*

$$\begin{cases} \mathring{A}(\xi', D_d)v(t) - \eta v(t) = 0 & \text{for} \quad t = x_d > 0, \\ B_{\sigma j}(D)v_j(t) = a_\sigma & \text{at} \quad t = 0(\sigma = 1, \cdots, p^+), \\ |v(t)| \longrightarrow 0 & \text{as} \quad t \longrightarrow \infty(\text{exponentially}) \end{cases} \tag{3.2.41}$$

has a solution for arbitrary $a_\sigma \in \mathbb{C}$.

3. *The problem*

$$\begin{cases} \mathring{A}(\xi', D_d)v(t) - \eta v(t) = 0 & \text{for} \quad t = x_d > 0, \\ B_{\sigma j}(D)v_j(t) = 0 & \text{at} \quad t = 0(\sigma = 1, \cdots, p^+), \\ |v(t)| \longrightarrow 0 & \text{as} \quad t \longrightarrow \infty(\text{exponentially}) \end{cases} \tag{3.2.42}$$

has a unique solution: $v_j = 0$ *is the only exponentially decaying solution.*

We will apply this theorem to complete the proof.

After some calculations we obtain

$$L(r) = (\nu_0(|\xi'|^2 + r^2) - \eta)^{d-1}\left((\nu_0(|\xi'|^2 + r^2) - \eta)^2 + \frac{\epsilon^2}{4}(|\xi'|^2 + r^2)^2\right)$$

and

$$L(r) = 0 \Longleftrightarrow r^2 = \frac{\eta}{\nu_0} - |\xi'|^2 \text{ or } r^2 = \frac{4}{4\nu_0^2 + \epsilon^2}\left(\nu_0 - i\frac{\epsilon}{2}\right)\eta - |\xi'|^2$$
$$\text{or } r^2 = \frac{4}{4\nu_0^2 + \epsilon^2}\left(\nu_0 + i\frac{\epsilon}{2}\right)\eta - |\xi'|^2.$$

Since according to the assumptions the arguments of η don't equal to zero and $\pm\text{arctg}\dfrac{\epsilon}{2\nu_0}$ $L(r)$ has exactly $d+1$ complex roots with positive imaginary part. Define $y := r^2 + |\xi'|^2$, let q_1, q_2, q_3 denote the roots of $L(r)$ with positive imaginary part where $q_1^2 = \dfrac{\eta}{\nu_0} - |\xi'|^2$, $q_2^2 = \dfrac{4}{4\nu_0^2 + \epsilon^2}\left(\nu_0 - i\dfrac{\epsilon}{2}\right)\eta - |\xi'|^2$, $q_3^2 = \dfrac{4}{4\nu_0^2 + \epsilon^2}\left(\nu_0 + i\dfrac{\epsilon}{2}\right)\eta - |\xi'|^2$, then

$$L^+(r) = (r - q_1)^{d-1}(r - q_2)(r - q_3).$$

Now we compute the $(d+1)\times(d+1)$ matrix $B_{\sigma j}(r)L^{jk}(r)$ for $d=2$ and $d=3$ separately.

$d=2$:

$$B_{\sigma j}(r)L^{jk}(r) = \begin{pmatrix} (\nu_0 y-\eta)^2+\frac{\epsilon^2}{4}|\xi'|^2 y & -\frac{\epsilon^2}{4}r\xi_1 y & i(\nu_0 y-\eta)\frac{\epsilon^2}{4}ry \\ -\frac{\epsilon^2}{4}\xi_1 ry & (\nu_0 y-\eta)^2+\frac{\epsilon^2}{4}r^2 y & i(\nu_0 y-\eta)\frac{\epsilon^2}{4}\xi_1 y \\ ir(\nu_0 y-\eta) & i\xi_1(\nu_0 y-\eta) & (\nu_0 y-\eta)^2 \end{pmatrix}$$

$d=3$:

$$B_{\sigma j}(r)L^{jk}(r) = (B_{\sigma j}(r)L^{jk}(r))_{ij}(i,j=1,\cdots,4)$$

here the entries $(B_{\sigma j}(r)L^{jk}(r))_{ij}$ read as follows:

$$(B_{\sigma j}(r)L^{jk}(r))_{11} = (\nu_0 y-\eta)\left((\nu_0 y-\eta)^2+\frac{\epsilon^2}{4}|\xi'|^2 y\right),$$

$$(B_{\sigma j}(r)L^{jk}(r))_{12} = -(\nu_0 y-\eta)\frac{\epsilon^2}{4}\xi_2 ry, \quad (B_{\sigma j}(r)L^{jk}(r))_{13} = -(\nu_0 y-\eta)\frac{\epsilon^2}{4}\xi_1 ry,$$

$$(B_{\sigma j}(r)L^{jk}(r))_{14} = i(\nu_0 y-\eta)^2\frac{\epsilon^2}{4}ry, \quad (B_{\sigma j}(r)L^{jk}(r))_{21} = -(\nu_0 y-\eta)\frac{\epsilon^2}{4}\xi_2 ry,$$

$$(B_{\sigma j}(r)L^{jk}(r))_{22} = (\nu_0 y-\eta)\left((\nu_0 y-\eta)^2+\frac{\epsilon^2}{4}y(r^2+\xi_1^2)\right),$$

$$(B_{\sigma j}(r)L^{jk}(r))_{23} = -(\nu_0 y-\eta)\frac{\epsilon^2}{4}\xi_1\xi_2 y, \quad (B_{\sigma j}(r)L^{jk}(r))_{24} = i(\nu_0 y-\eta)^2\frac{\epsilon^2}{4}\xi_2 y,$$

$$(B_{\sigma j}(r)L^{jk}(r))_{31} = -(\nu_0 y-\eta)\frac{\epsilon^2}{4}\xi_1 ry, \quad (B_{\sigma j}(r)L^{jk}(r))_{32} = -(\nu_0 y-\eta)\frac{\epsilon^2}{4}\xi_1\xi_2 y,$$

$$(B_{\sigma j}(r)L^{jk}(r))_{33} = (\nu_0 y-\eta)\left((\nu_0 y-\eta)^2+\frac{\epsilon^2}{4}(r^2+\xi_2^2)y\right),$$

$$(B_{\sigma j}(r)L^{jk}(r))_{34} = i(\nu_0 y-\eta)^2\frac{\epsilon^2}{4}\xi_1 y, \quad (B_{\sigma j}(r)L^{jk}(r))_{41} = ir(\nu_0 y-\eta)^2,$$

$$(B_{\sigma j}(r)L^{jk}(r))_{42} = i\xi_2(\nu_0 y-\eta)^2, \quad (B_{\sigma j}(r)L^{jk}(r))_{43} = i\xi_1(\nu_0 y-\eta)^2,$$

$$(B_{\sigma j}(r)L^{jk}(r))_{44} = (\nu_0 y-\eta)^3.$$

The following cases need to be considered.

1. **Case 1:** $\eta=0$.

   According to the assumption $|\xi'|+|\eta|\neq 0$ we have $|\xi'|\neq 0$. From the calculations above we obtain $q_1=q_2=q_3=i|\xi'|$ and

   $$L^+(r) = (r-i|\xi'|)^{d+1}.$$

We consider two cases:

(a) $d = 2$:
The equations

$$\sum_{\sigma} a_\sigma B_{\sigma j}(r) L^{jk}(r) \equiv 0 \bmod L^+(r) \quad (a_\sigma \in \mathbb{C}, \sigma = 1,2,3)$$

imply that $i|\xi'|$ is a root of multiplicity 3 of the linear combination of the first column of $B_{\sigma j}(r)L^{jk}(r)$:

$$\begin{aligned} l_1(r) :&= a_1\nu_0^2 y^2 + a_1\frac{\epsilon^2}{4}|\xi'|^2 y - a_2\frac{\epsilon^2}{4}\xi_1 r y + i a_3 r \nu_0 y, \\ &= y\left(a_1\nu_0^2 y + a_1\frac{\epsilon^2}{4}|\xi'|^2 - a_2\frac{\epsilon^2}{4}\xi_1 r + i a_3 r\nu_0\right), \\ &= (r - i|\xi'|)(r + i|\xi'|)\times \\ &\quad \times \left(a_1\nu_0^2 y + a_1\frac{\epsilon^2}{4}|\xi'|^2 - a_2\frac{\epsilon^2}{4}\xi_1 r + i a_3 r\nu_0\right). \end{aligned}$$

Since $a_1\nu_0^2 y + a_1\frac{\epsilon^2}{4}|\xi'|^2 - a_2\frac{\epsilon^2}{4}\xi_1 r + i a_3 r\nu_0$ is a polynomial of order 2 with respect to r and $a_1\nu_0^2 y + a_1\frac{\epsilon^2}{4}|\xi'|^2 - a_2\frac{\epsilon^2}{4}\xi_1 r + i a_3 r\nu_0$ has a factor $(r - i|\xi'|)^2$ it follows

$$a_1\nu_0^2 y + a_1\frac{\epsilon^2}{4}|\xi'|^2 - a_2\frac{\epsilon^2}{4}\xi_1 r + i a_3 r\nu_0 = a_1\nu_0^2 (r - i|\xi'|)^2,$$

which is equivalent to

$$\begin{aligned} &a_1\nu_0^2 r^2 - i2a_1\nu_0^2|\xi'|r - a_1\nu_0^2|\xi'|^2 \\ &= a_1\nu_0^2 r^2 + \left(i a_3\nu_0 - a_2\frac{\epsilon^2}{4}\xi_1\right) r + a_1|\xi'|^2\left(\nu_0^2 + \frac{\epsilon^2}{4}\right). \end{aligned}$$

By coefficient comparison we obtain

$$-a_1\nu_0^2|\xi'|^2 = a_1|\xi'|^2\left(\nu_0^2 + \frac{\epsilon^2}{4}\right), \tag{3.2.43}$$

which implies

$$a_1 = 0. \tag{3.2.44}$$

Then the linear combination of the second column of $B_{\sigma j}(r)L^{jk}(r)$ is reduced as:

$$\begin{aligned} l_2(r) :&= a_2\nu_0^2 y^2 + a_2\frac{\epsilon^2}{4} r^2 y + i a_3\xi_1\nu_0 y, \\ &= (r - i|\xi'|)(r + i|\xi'|)\left(\left(a_2\nu_0^2 + a_2\frac{\epsilon^2}{4}\right) r^2 + a_2\nu_0^2|\xi'|^2 + i a_3\xi_1\nu_0\right). \end{aligned}$$

By a same reasoning

$$\left(a_2\nu_0^2 + a_2\frac{\epsilon^2}{4}\right) r^2 + a_2\nu_0^2|\xi'|^2 + ia_3\xi_1\nu_0 = \left(a_2\nu_0^2 + a_2\frac{\epsilon^2}{4}\right)(r - i|\xi'|)^2,$$

from which it follows

$$a_2 = 0. \tag{3.2.45}$$

Then $a_3 = 0$ follows immediately. Combining (3.2.44) and (3.2.45) the lines of $B_{\sigma j}(r)L^{jk}(r)$ are linear independent modulo the polynomial $L^+(r)$.

(b) $d = 3$:

On the basis of the same procedure as case of $d = 2$ the equations

$$\sum_\sigma a_\sigma B_{\sigma j}(r)L^{jk}(r) \equiv 0 \bmod L^+(r) \quad (a_\sigma \in \mathbb{C}, \sigma = 1, 2, 3, 4)$$

imply that $i|\xi'|$ is a root of multiplicity 4 of the linear combination of the first column of $B_{\sigma j}(r)L^{jk}(r)$:

$$\begin{aligned} L_1(r) :&= a_1\nu_0 y\left(\nu_0^2 y^2 + \frac{\epsilon^2}{4}|\xi'|^2 y\right) - a_2\nu_0\frac{\epsilon^2}{4}\xi_2 r y^2 - a_3\nu_0\frac{\epsilon^2}{4}\xi_1 r y^2 + \\ &\quad + ia_4\nu_0^2 r y^2 \\ &= \left(a_1\nu_0^3 r^2 + \left(ia_4\nu_0^2 - a_2\nu_0\frac{\epsilon^2}{4}\xi_2 - a_3\nu_0\frac{\epsilon^2}{4}\xi_1\right) r + a_1\nu_0^3|\xi'|^2 + \right. \\ &\quad \left. + a_1\nu_0\frac{\epsilon^2}{4}|\xi'|^2\right)(r - i|\xi'|)^2(r + i|\xi'|)^2. \end{aligned} \tag{3.2.46}$$

The first factor of $L_1(r)$ is a polynomial of order 2 and has a factor $(r - i|\xi'|)^2$ thus

$$\begin{aligned} &a_1\nu_0^3 r^2 + \left(ia_4\nu_0^2 - a_2\nu_0\frac{\epsilon^2}{4}\xi_2 - a_3\nu_0\frac{\epsilon^2}{4}\xi_1\right) r + a_1\nu_0^3|\xi'|^2 + a_1\nu_0\frac{\epsilon^2}{4}|\xi'|^2 \\ &= a_1\nu_0^3(r - i|\xi'|)^2 \\ &= a_1\nu_0^3 r^2 - i2a_1\nu_0^3|\xi'|r - a_1\nu_0^3|\xi'|^2. \end{aligned}$$

Coefficient comparison yields

$$a_1 = 0. \tag{3.2.47}$$

Then the linear combination of the second column of $B_{\sigma j}(r)L^{jk}(r)$ is reduced as:

$$\begin{aligned} L_2(r) :&= a_2\nu_0 y\left(\nu_0^2 y^2 + \frac{\epsilon^2}{4}y(r^2+\xi_1^2)\right) - a_3\nu_0\frac{\epsilon^2}{4}\xi_1\xi_2 y^2 + ia_4\xi_2\nu_0^2 y^2 \\ &= (r-i|\xi'|)^2(r+i|\xi'|)^2\times \\ &\quad\times\left(a_2\nu_0^3 y + a_2\nu_0\frac{\epsilon^2}{4}(r^2+\xi_1^2) - a_3\nu_0\frac{\epsilon^2}{4}\xi_1\xi_2 + ia_4\xi_2\nu_0^2\right) \\ &= (r-i|\xi'|)^2(r+i|\xi'|)^2\mathcal{R}, \end{aligned}$$

where

$$\mathcal{R} := \left(a_2\nu_0^3 + a_2\nu_0\frac{\epsilon^2}{4}\right)r^2 + a_2\nu_0^3|\xi'|^2 + a_2\nu_0\frac{\epsilon^2}{4}\xi_1^2 - a_3\nu_0\frac{\epsilon^2}{4}\xi_1\xi_2 + ia_4\xi_2\nu_0^2.$$

$\mathcal{R}$ is a polynomial in r of order 2 and it has a factor $(r-i|\xi'|)^2$ thus

$$\mathcal{R} = \left(a_2\nu_0^3 + a_2\nu_0\frac{\epsilon^2}{4}\right)(r-i|\xi'|)^2.$$

Coefficient comparison yields

$$a_2 = 0. \tag{3.2.48}$$

Then the linear combination of the third column of $B_{\sigma j}(r)L^{jk}(r)$ is reduced as:

$$\begin{aligned} L_3(r) :&= a_3\nu_0 y\left(\nu_0^2 y^2 + \frac{\epsilon^2}{4}(r^2+\xi_2^2)y\right) + ia_4\xi_1\nu_0^2 y^2 \\ &= (r-i|\xi'|)^2(r+i|\xi'|)^2\times \\ &\quad\times\left(\left(a_3\nu_0^3 + a_3\nu_0\frac{\epsilon^2}{4}\right)r^2 + a_3\nu_0^3|\xi'|^2 + a_3\nu_0\frac{\epsilon^2}{4}\xi_2^2 + ia_4\xi_1\nu_0^2\right). \end{aligned}$$

A same reasoning provides

$$\begin{aligned} &\left(a_3\nu_0^3 + a_3\nu_0\frac{\epsilon^2}{4}\right)r^2 + a_3\nu_0^3|\xi'|^2 + a_3\nu_0\frac{\epsilon^2}{4}\xi_2^2 + ia_4\xi_1\nu_0^2 \\ &= \left(a_3\nu_0^3 + a_3\nu_0\frac{\epsilon^2}{4}\right)(r-i|\xi'|)^2. \end{aligned}$$

Coefficient comparison yields

$$a_3 = 0. \tag{3.2.49}$$

Then the linear combination of the fourth column of $B_{\sigma j}(r)L^{jk}(r)$ is

$$L_4(r) := a_4\nu_0^3(r - i|\xi'|)^3(r + i|\xi'|)^3.$$

That $L_4(r)$ has a factor $(r - i|\xi'|)^4$ implies

$$a_4 = 0. \tag{3.2.50}$$

Thus (3.2.47)-(3.2.50) yield the lines of $B_{\sigma j}(r)L^{jk}(r)$ are linear independent modulo the polynomial $L^+(r)$.

2. **Case 2:** $\eta \neq 0$.

   In this case q_1, q_2, q_3 are pairwise unequal and we still consider two cases.

   (a) $d = 2$:
   The equations

$$\sum_\sigma a_\sigma B_{\sigma j}(r)L^{jk}(r) \equiv 0 \text{ mod } L^+(r) \quad (a_\sigma \in \mathbb{C}, \sigma = 1, 2, 3)$$

are equivalent to that $l_i(q_j) = 0 (i, j = 1, 2, 3)$ where let $l_i(r)$ denote the corresponding linear combination of the ith column. Since q_1 is a root of the linear combination of the second column of $B_{\sigma j}(r)L^{jk}(r)$

$$\begin{aligned} l_2(r) := -a_1\frac{\epsilon^2}{4}r\xi_1 y + a_2\left((\nu_0 y - \eta)^2 + \frac{\epsilon^2}{4}r^2 y\right) \\ + a_3 i\xi_1(\nu_0 y - \eta). \end{aligned} \tag{3.2.51}$$

Then $l_2(q_1) = 0$ is equivalent to

$$a_1\xi_1 - a_2 q_1 = 0. \tag{3.2.52}$$

By the same reasoning q_2 q_3 are roots of the linear combination of the third column of $B_{\sigma j}(r)L^{jk}(r)$

$$l_3(r) := a_1 i(\nu_0 y - \eta)\frac{\epsilon^2}{4}ry + a_2 i(\nu_0 y - \eta)\frac{\epsilon^2}{4}\xi_1 y + a_3(\nu_0 y - \eta)^2.$$

$l_3(q_2) = 0$ is equivalent to

$$a_1\frac{\epsilon}{2}q_2 + a_2\frac{\epsilon}{2}\xi_1 - a_3 = 0; \tag{3.2.53}$$

$l_3(q_3) = 0$ is equivalent to

$$a_1\frac{\epsilon}{2}q_3 + a_2\frac{\epsilon}{2}\xi_1 + a_3 = 0. \tag{3.2.54}$$

Now we compute $l_1(q_1)=0, l_1(q_2)=0, l_1(q_3)=0, l_2(q_2)=0, l_2(q_3)=0, l_3(q_1)=0$ then we obtain

$$\begin{aligned}
l_1(q_1)=0 &\Longleftrightarrow \xi_1(a_1\xi_1-a_2q_1)=0;\\
l_1(q_2)=0 &\Longleftrightarrow a_1\frac{\epsilon}{2}q_2+a_2\frac{\epsilon}{2}\xi_1-a_3=0;\\
l_1(q_3)=0 &\Longleftrightarrow a_1\frac{\epsilon}{2}q_3+a_2\frac{\epsilon}{2}\xi_1+a_3=0;\\
l_2(q_2)=0 &\Longleftrightarrow \left(a_1\frac{\epsilon}{2}q_2+a_2\frac{\epsilon}{2}\xi_1-a_3\right)\xi_1=0;\\
l_2(q_3)=0 &\Longleftrightarrow \left(a_1\frac{\epsilon}{2}q_3+a_2\frac{\epsilon}{2}\xi_1+a_3\right)\xi_1=0;\\
l_3(q_1)=0 &\Longleftrightarrow (a_1,a_2,a_3)^T\in\mathbb{R}^3 \text{ are arbitrary.}
\end{aligned}$$

Finally it is easy to verify that $l_i(q_j)=0$ $(i,j=1,2,3)$ are equivalent to (3.2.52), (3.2.53) and (3.2.54).

The system of equations (3.2.52), (3.2.53) and (3.2.54) with respect to $(a_1,a_2,a_3)^T$ has only the trivial solution iff.

$$\det\begin{pmatrix} \xi_1 & -q_1 & 0\\ \frac{\epsilon}{2}q_2 & \frac{\epsilon}{2}\xi_1 & -1\\ \frac{\epsilon}{2}q_3 & \frac{\epsilon}{2}\xi_1 & 1\end{pmatrix}\neq 0,$$

which is equivalent to

$$\xi_1^2+\frac{1}{2}q_1(q_2+q_3)\neq 0.$$

Thus we obtain that the following two statements are equivalent

i. $\sum_\sigma a_\sigma B_{\sigma j}(r)L^{jk}(r)\equiv 0 \bmod L^+(r)\quad (a_\sigma\in\mathbb{C},\sigma=1,2,3)$ imply that the constants a_σ are all zero,

ii. $\xi_1^2+\frac{1}{2}q_1(q_2+q_3)\neq 0.$

(b) $d=3$:

Since each entry of $B_{\sigma j}(r)L^{jk}(r)$ has a factor $\nu_0 y-\eta=\nu_0(r+q_1)(r-q_1)$ the equations $\sum_\sigma a_\sigma B_{\sigma j}(r)L^{jk}(r)\equiv 0 \bmod L^+(r)\quad (a_\sigma\in\mathbb{C},\sigma=1,2,3,4)$ are equivalent to that $L_i(q_j)=0(i=1,2,3,4\quad j=1,2,3)$ where $L_i(r):=\dfrac{l_i(r)}{\nu_0 y-\eta}$ and let $l_i(r)$ denote the corresponding linear combination of the ith column of $B_{\sigma j}(r)L^{jk}(r)$. Some calculations yield

$$
\begin{aligned}
L_1(q_1) = 0 &\Longleftrightarrow a_1|\xi'|^2 - a_2\xi_2 q_1 - a_3\xi_1 q_1 = 0,\\
L_1(q_2) = 0 &\Longleftrightarrow a_1\frac{\epsilon}{2}q_2 + a_2\frac{\epsilon}{2}\xi_2 + a_3\frac{\epsilon}{2}\xi_1 - a_4 = 0,\\
L_1(q_3) = 0 &\Longleftrightarrow a_1\frac{\epsilon}{2}q_3 + a_2\frac{\epsilon}{2}\xi_2 + a_3\frac{\epsilon}{2}\xi_1 + a_4 = 0,\\
L_2(q_1) = 0 &\Longleftrightarrow a_1\xi_2 q_1 - a_2(q_1^2 + \xi_1^2) + a_3\xi_1\xi_2 = 0,\\
L_2(q_2) = 0 &\Longleftrightarrow \xi_2\left(a_1\frac{\epsilon}{2}q_2 + a_2\frac{\epsilon}{2}\xi_2 + a_3\frac{\epsilon}{2}\xi_1 - a_4\right) = 0,\\
L_2(q_3) = 0 &\Longleftrightarrow \xi_2\left(a_1\frac{\epsilon}{2}q_3 + a_2\frac{\epsilon}{2}\xi_2 + a_3\frac{\epsilon}{2}\xi_1 + a_4\right) = 0,\\
L_3(q_1) = 0 &\Longleftrightarrow a_1\xi_1 q_1 + a_2\xi_1\xi_2 - a_3(q_1^2 + \xi_2^2) = 0,\\
L_3(q_2) = 0 &\Longleftrightarrow \xi_1\left(a_1\frac{\epsilon}{2}q_2 + a_2\frac{\epsilon}{2}\xi_2 + a_3\frac{\epsilon}{2}\xi_1 - a_4\right) = 0,\\
L_3(q_3) = 0 &\Longleftrightarrow \xi_1\left(a_1\frac{\epsilon}{2}q_3 + a_2\frac{\epsilon}{2}\xi_2 + a_3\frac{\epsilon}{2}\xi_1 + a_4\right) = 0,\\
L_4(q_1) = 0 &\Longleftrightarrow (a_1, \cdots, a_4)^T \in \mathbb{C}^4 \text{ are arbitraty},\\
L_4(q_2) = 0 &\Longleftrightarrow a_1\frac{\epsilon}{2}q_2 + a_2\frac{\epsilon}{2}\xi_2 + a_3\frac{\epsilon}{2}\xi_1 - a_4 = 0,\\
L_4(q_3) = 0 &\Longleftrightarrow a_1\frac{\epsilon}{2}q_3 + a_2\frac{\epsilon}{2}\xi_2 + a_3\frac{\epsilon}{2}\xi_1 + a_4 = 0.
\end{aligned}
$$

Thus we obtain that

$$L_i(q_j) = 0(i = 1,2,3,4 \quad j = 1,2,3)$$

are equivalent to the equations

$$
\begin{aligned}
a_1|\xi'|^2 - a_2\xi_2 q_1 - a_3\xi_1 q_1 &= 0,\\
a_1\xi_2 q_1 - a_2(q_1^2 + \xi_1^2) + a_3\xi_1\xi_2 &= 0,\\
a_1\xi_1 q_1 + a_2\xi_1\xi_2 - a_3(q_1^2 + \xi_2^2) &= 0,\\
a_1\frac{\epsilon}{2}q_2 + a_2\frac{\epsilon}{2}\xi_2 + a_3\frac{\epsilon}{2}\xi_1 - a_4 &= 0,\\
a_1\frac{\epsilon}{2}q_3 + a_2\frac{\epsilon}{2}\xi_2 + a_3\frac{\epsilon}{2}\xi_1 + a_4 &= 0.
\end{aligned}
$$

We consider the following linear equations:

$$B(a_1, a_2, a_3, a_4)^T = \mathbf{0},$$

here

$$B := \begin{pmatrix} |\xi'|^2 & -\xi_2 q_1 & -\xi_1 q_1 & 0 \\ \xi_2 q_1 & -q_1^2 - \xi_1^2 & \xi_1\xi_2 & 0 \\ \xi_1 q_1 & \xi_1\xi_2 & -q_1^2 - \xi_2^2 & 0 \\ \frac{\epsilon}{2} q_2 & \frac{\epsilon}{2}\xi_2 & \frac{\epsilon}{2}\xi_1 & -1 \\ \frac{\epsilon}{2} q_3 & \frac{\epsilon}{2}\xi_2 & \frac{\epsilon}{2}\xi_1 & 1 \end{pmatrix}.$$

Let $w_j (j = 1, \cdots, 4)$ denote the jth column of B, then it is easy to check that w_2, w_3, w_4 are linear independent. Assume w_1 can be written as a linear combination of w_2, w_3, w_4, namely

$$w_1 = \sum_{j=2}^{4} \alpha_j w_j$$

for $\alpha_j \in \mathbb{C}$, after some calculations we find

$$\alpha_2 = -\frac{\xi_2}{q_1}, \quad \alpha_3 = -\frac{\xi_1}{q_1}, \quad \alpha_4 = -\frac{\epsilon}{2q_1}|\xi'|^2 - \frac{\epsilon}{2} q_2 = \frac{\epsilon}{2q_1}|\xi'|^2 + \frac{\epsilon}{2} q_3,$$

which yields

$$|\xi'|^2 + \frac{1}{2} q_1(q_2 + q_3) = 0. \tag{3.2.55}$$

Then $Bx = 0$ has only the trivial solution iff. $|\xi'|^2 + \frac{1}{2} q_1(q_2 + q_3) \neq 0$.

Recall the case of $d = 2$ we obtain the conclusion that for both cases($d = 2, 3$) and $\eta \neq 0$ the lines of the $p^+ \times (d + 1)$ matrix

$$B_{\sigma j}(r) L^{jk}(r)$$

are linear independent modulo the polynomial $L^+(r)$; i.e., the equations

$$\sum_\sigma C_\sigma B_{\sigma j}(r) L^{jk}(r) \equiv 0 \quad (\bmod L^+(r))$$

imply the constants C_σ are all zero iff.

$$|\xi'|^2 + \frac{1}{2} q_1(q_2 + q_3) \neq 0. \tag{3.2.56}$$

In case of $\xi' = \mathbf{0}$ (3.2.56) holds for all $\eta \in \mathcal{L}$ since $q_1 \neq 0, q_2 + q_3 \neq 0$ thus we consider only the case of $\xi' \neq \mathbf{0}$. We have the reasoning

$$\begin{aligned} & |\xi'|^2 + \frac{1}{2} q_1(q_2 + q_3) = 0 \\ \Longrightarrow\ & (2|\xi'|^2 + q_1(q_2 + q_3))(2|\xi'|^2 - q_1(q_2 + q_3)) \times \\ & \times (2|\xi'|^2 + q_1(q_2 - q_3))(2|\xi'|^2 - q_1(q_2 - q_3)) = 0, \end{aligned} \tag{3.2.57}$$

and then expand the right equation of (3.2.57) with respect to η to find

$$(2|\xi'|^2 + q_1(q_2 + q_3))(2|\xi'|^2 - q_1(q_2 + q_3))(2|\xi'|^2 + q_1(q_2 - q_3))\times$$
$$\times(2|\xi'|^2 - q_1(q_2 - q_3)) = 0,$$

which implies

$$\frac{\epsilon^2 c^2}{\nu_0^2}\eta^3 - \frac{2\epsilon^2 c^2|\xi'|^2}{\nu_0}\eta^2 + |\xi'|^4(\epsilon^2 c^2 + 16c)\eta - 16|\xi'|^6\left(\nu_0 c + \frac{1}{\nu_0}\right) = 0,$$

which is equivalent to

$$\begin{aligned} P(\eta) := \eta^3 - 2\nu_0|\xi'|^2\eta^2 + \nu_0^2|\xi'|^4\left(5 + \frac{16\nu_0^2}{\epsilon^2}\right)\eta \\ -\frac{16|\xi'|^6\nu_0^2}{c\epsilon^2}\left(2\nu_0 + \frac{\epsilon^2}{4\nu_0}\right) = 0 \end{aligned} \tag{3.2.58}$$

where $c := \dfrac{4}{4\nu_0^2 + \epsilon^2}$.

$P(\eta)$ is a polynomial in η of order 3 and we check that

$$P(0) = -\frac{16|\xi'|^6\nu_0^2}{c\epsilon^2}\left(2\nu_0 + \frac{\epsilon^2}{4\nu_0}\right) < 0$$

then $P(\eta)$ has a positive real root. The discriminant Δ of $P(\eta)$ reads

$$\Delta = \frac{p^3}{27} + \frac{q^2}{4},$$

with

$$p := \nu_0^2|\xi'|^4\left(\frac{11}{3} + \frac{16\nu_0^2}{\epsilon^2}\right)$$
$$q := \nu_0^2|\xi'|^6\left(-\frac{16}{27}\nu_0 + \frac{2\nu_0}{3}\left(5 + \frac{16\nu_0^2}{\epsilon^2}\right) - \frac{16}{c\epsilon^2}\left(2\nu_0 + \frac{\epsilon^2}{4\nu_0}\right)\right).$$

Then $\Delta > 0$ and from algebraic laws $P(\eta) = 0$ has one real and a pair of complex conjugate roots. Factor $P(\eta)$ as

$$P(\eta) = (\eta - \alpha)(\eta^2 + \beta\eta + \gamma)$$

where let α denote the positive real root of $P(\eta) = 0$ and β, γ denote the well-founded coefficients. Coefficient comparison yields

$$\beta = \alpha - 2\nu_0|\xi'|^2.$$

Substitute $2\nu_0|\xi'|^2$ into $P(\eta)$ we obtain $P(2\nu_0|\xi'|^2) < 0$; since $\alpha > 0$ is the only real root of $P(\eta) = 0$ we conclude $2\nu_0|\xi'|^2 < \alpha$ thus $\beta > 0$. Then the two complex conjugate roots of $P(\eta) = 0$ are situated in the second and the third quadrants in the complex plane respectively.

Notice that (3.2.56) holds for any

$$\eta \in S_1 \cup S_2$$

where

$$S_1 := \left\{ x \in \mathbb{C} | -\pi + \mathrm{arctg}\frac{\epsilon}{2\nu_0} < \arg x < -\mathrm{arctg}\frac{\epsilon}{2\nu_0} \right\},$$

$$S_2 := \left\{ x \in \mathbb{C} | \mathrm{arctg}\frac{\epsilon}{2\nu_0} < \arg x < \pi - \mathrm{arctg}\frac{\epsilon}{2\nu_0} \right\}$$

since $\frac{1}{2}q_1(q_2 + q_3)$ always has a negative imaginary part for

$$\eta \in \left\{ x \in \mathbb{C} | -\pi + \mathrm{arctg}\frac{\epsilon}{2\nu_0} < \arg x < -\mathrm{arctg}\frac{\epsilon}{2\nu_0} \right\}$$

and a positive imaginary part for

$$\eta \in \left\{ x \in \mathbb{C} | \mathrm{arctg}\frac{\epsilon}{2\nu_0} < \arg x < \pi - \mathrm{arctg}\frac{\epsilon}{2\nu_0} \right\}.$$

Furthermore from $P(\eta) = 0$ we see

$$\eta \left(\frac{\epsilon^2 c^2}{\nu_0^2}\eta^2 + |\xi'|^4(\epsilon^2 c^2 + 16c) \right) = 2|\xi'|^2 \left(\frac{\epsilon^2 c^2}{\nu_0^2}\eta^2 + 8|\xi'|^4 \left(\nu_0 c + \frac{1}{\nu_0} \right) \right).$$

A necessary condition of $P(\eta) = 0$ is the argument of the complex root of $P(\eta) = 0$ with positive imaginary part is greater equal than 0 and less than $\frac{2}{3}\pi$ otherwise the lie of

$$\eta \left(\frac{\epsilon^2 c^2}{\nu_0^2}\eta^2 + |\xi'|^4(\epsilon^2 c^2 + 16c) \right)$$

is in the first or the second quadrant, and

$$2|\xi'|^2 \left(\frac{\epsilon^2 c^2}{\nu_0^2}\eta^2 + 8|\xi'|^4 \left(\nu_0 c + \frac{1}{\nu_0} \right) \right)$$

lies in the third or the fourth quadrant, i.e. will be situated in different quadrants.

Assume $\frac{\epsilon}{2\nu_0} \leq \sqrt{3}$ then the complex conjugate roots of $P(\eta) = 0$ are in

$$S_1 \cup S_2,$$

consequently (3.2.56) holds for all $\eta \in \mathbb{C}$ exclusive the nonnegative real axis.

□

Theorem 3.2.6. *Let $1 < p < \infty$, there exists a $\eta_0 > 0$ such that for all $\eta \in \mathcal{L}$ with $|\eta| \geq \eta_0$ where $\mathcal{L}$ is defined in theorem 3.2.5 the boundary problem* (3.2.32) *has a unique solution $\mathbf{u}(x) = (u^d, u^{d-1}, \cdots, u^1, u^0) \in \prod_{j=1}^{d+1} W_p^{m_j+m}(\Omega)$ for any $f = (f_1, \cdots, f_{d+1}) \in \prod_{j=1}^{d+1} W_p^{m_j}(\Omega)$ and $g = (g_1, \cdots, g_{d+1}) \prod_{j=1}^{d+1} W_p^{3/2-r_j}(\Omega)$ where the sequences of integers $\{m_j\}_1^{d+1}$ and $\{r_j\}_1^{d+1}$ are defined in* (3.2.30) *and* (3.2.33) *respectively. Further the a priori estimate*

$$\begin{aligned}
&\sum_{j=0}^{d} |||u_j|||_{m_{d+1-j}+2,p,\Omega} \\
&\leq c \left(\sum_{j=1}^{d+1} |||f_{d+2-j}|||_{m_{d+2-j},p,\Omega} + \sum_{j=1}^{d+1} |||g_{d+2-j}|||_{2-r_{d+2-j}-\frac{1}{p},p,\partial\Omega} \right)
\end{aligned} \tag{3.2.59}$$

holds where the constant c does not depend upon f, g, η and we use the norms depending on a parameter $\eta \in \mathbb{C}/\{0\}$, namely let $v \in W_p^s(\Omega), g \in W_p^{s-\frac{1}{p}}(\partial\Omega)$ where s is an integer satisfying $1 \leq s \leq m$,

$$|||v|||_{s,p,\Omega} := \|v\|_{W_p^s(\Omega)} + |\eta|^{\frac{s}{m}} \|v\|_{L^p(\Omega)}, \tag{3.2.60}$$

$$|||g|||_{s-\frac{1}{p},p,\partial\Omega} := \|g\|_{W_p^{s-\frac{1}{p}}(\partial\Omega)} + |\eta|^{\frac{s-\frac{1}{p}}{m}} \|g\|_{L^p(\partial\Omega)}. \tag{3.2.61}$$

Proof. Since the boundary problem (3.2.32) is elliptic with parameter in $\mathcal{L}$ from **Theorem** 3.2.5, the existence and the a priori estimate follow directly from **Theorem** 2.2.1. □

3.2.3 Approach by Galerkin-Approximation

We use Galerkin's method to solve the linear system (3.2.1). The idea of this method is that one first constructs a family of approximate problems whose solutions satisfy certain a priori estimates. This yields a sequence of approximate solutions that have uniform bounds. Uniform bounds imply the existence of a weakly convergent subsequence. One then shows that the weak limit is the solution we seek.

Preliminaries

We introduce some basic materials on functional analysis and Sobolev spaces which will be needed in the later procedure.

Theorem 3.2.7. *(Rellich selection theorem)*

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open subset. Every bounded sequence in $H_0^1(\Omega)$ has a subsequence, which converges in the norm of $L^2(\Omega)$.

Theorem 3.2.8. *(Eigenvectors of a compact, symmetric operator)*

Let H be a separable Hilbert space, and suppose $S : H \to H$ is a compact and symmetric operator. Then there exists a countable orthonormal basis of H which consists of eigenvectors of S.

We define a bilinear form in $H_0^1(\Omega) \cap H^2(\Omega)$

$$B : \big(H_0^1(\Omega) \cap H^2(\Omega)\big) \times \big(H_0^1(\Omega) \cap H^2(\Omega)\big) \to \mathbb{R}$$

with

$$B(u,v) = (u,v)_{L^2} + (\nabla u, \nabla v)_{L^2} + (\triangle u, \triangle v)_{L^2}. \tag{3.2.62}$$

It is easy to check that $B(u,v)$ is an inner product in $H_0^1(\Omega) \cap H^2(\Omega)$. The norm defined by this inner product in $H_0^1(\Omega) \cap H^2(\Omega)$ is equivalent to the norm defined by

$$(u,v)_{H^2(\Omega)} = \sum_{|\alpha| \le 2} (D^\alpha u, D^\alpha v)$$

from the following fact:

Theorem 3.2.9. *Suppose that $f \in H^{-1}(\Omega)$, Ω is C^1, $u \in H_0^1(\Omega)$ is a weak solution of the elliptic boundary-value problem*

$$\begin{cases} \triangle u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

Assume finally $f \in H^m(\Omega)$, $\partial\Omega$ is C^{m+2} $(m = 0, 1, 2, \cdots)$. Then

$$u \in H^{m+2}(\Omega)$$

and we have the estimate

$$\|u\|_{H^{m+2}(\Omega)} \le C_{\triangle,m,\Omega} \|f\|_{H^m(\Omega)},$$

the constant $C_{\triangle,m,\Omega}$ depends only on m, Ω.

Proof. The proof can be found in [33] pp. 323. □

Additionally we record the following embedding from [33] pp. 288

Theorem 3.2.10. *Let Ω be open, bounded in $\mathbb{R}^d$, $T > 0$, m be a nonnegative interger. Assume the boundary $\partial\Omega$ is smooth.*

Suppose $u \in L^2(0,T; H^{m+2}(\Omega))$, with $\partial_t u \in L^2(0,T; H^m(\Omega))$, then

$$u \in C([0,T]; H^{m+1}(\Omega)),$$

and the estimates

$$\|u\|_{C([0,T];H^{m+1}(\Omega))} \leq C\left(\|u\|_{L^2(0,T;H^{m+2}(\Omega))} + \|\partial_t u\|_{L^2(0,T;H^m(\Omega))}\right)$$

holds, where $C > 0$ *depends only on* T, Ω, *and* m.

Then we obtain the following corollary:

Theorem 3.2.11. *Let* $\mathbf{u} \in L^2(0,T;H_0^1(\Omega) \cap H^2(\Omega))$, $\partial_t \mathbf{u} \in L^2(0,T;L^2(\Omega))$, *then the function*

$$t \mapsto \|\nabla \mathbf{u}\|^2$$

is absolutely continuous and

$$\frac{d}{dt}\|\nabla \mathbf{u}\|^2 = -2(\partial_t \mathbf{u}, \triangle \mathbf{u})_{L^2(\Omega)} \tag{3.2.63}$$

for a.e $0 \leq t \leq T$.

Proof. From **Theorem** 3.2.10 it follows direct that $\mathbf{u} \in C([0,T];H^1(\Omega))$.

Fix any point t in $(0,T)$ there exists $\sigma > 0$ such that $t \in (\varepsilon, T-\varepsilon)$ for all $\varepsilon \leq \sigma$. Set $\mathbf{u}^\varepsilon = \eta_\varepsilon * \mathbf{u}$, η_ε denoting the usual mollifier on $\mathbb{R}^1$, i.e., $\eta_\varepsilon = \varepsilon^{-1}\eta(t/\varepsilon)$ where

$$\eta(t) = \begin{cases} ke^{-1/(1-|t|^2)} & \text{if } |t| < 1, \\ 0 & \text{if } |t| \geq 1, \end{cases}$$

$k > 0$ is selected so that $\int_{\mathbb{R}} \eta(t)dt = 1$. We find first that

$$\eta_\varepsilon(t) = 0 \text{ if } |t| \geq \varepsilon, \qquad \int_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(t)dt = 1.$$

Furthermore

$$\mathbf{u}^\varepsilon(t) = \int_0^T \eta_\varepsilon(t-y)\mathbf{u}(y)dy = \int_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(y)\mathbf{u}(t-y)dy, \quad t \in (\varepsilon, T-\varepsilon)$$

in the sense of Bochner integral with respect to $H_0^1(\Omega) \cap H^2(\Omega)$. It is easy to calculate that

$$\frac{d^n \mathbf{u}^\varepsilon(t)}{dt^n} = \int_0^T \frac{d^n \eta_\varepsilon}{dt^n}(t-y)\mathbf{u}(y)dy \qquad \text{for any } n \in \mathbb{N},$$

which implies

$$\mathbf{u}^\varepsilon \in C^\infty((\varepsilon, T-\varepsilon); H_0^1(\Omega) \cap H^2(\Omega))).$$

Choose a $s \in (\varepsilon, T-\varepsilon)$, without loss of generality, let $s < t$. Then $\mathbf{u}^\varepsilon \in C^\infty([s,t]; H^1(\Omega))$. Now for $x \in [s,t]$

$$\begin{aligned}
\|\mathbf{u}^\varepsilon(x) - \mathbf{u}(x)\|_{H^1(\Omega)} &= \left\| \int_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(y)\mathbf{u}(x-y)dy - \int_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(y)\mathbf{u}(x)dy \right\|_{H^1(\Omega)} \\
&= \left\| \int_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(y)(\mathbf{u}(x-y) - \mathbf{u}(x))dy \right\|_{H^1(\Omega)} \\
&\leq \sup_{y \in (-\varepsilon,\varepsilon)} \|\mathbf{u}(x-y) - \mathbf{u}(x)\|_{H^1(\Omega)}.
\end{aligned}$$

Since $\mathbf{u}$ is uniformly continuous on $[s,t]$ in $H^1(\Omega)$, i.e., $u \in C([s,t]; H^1(\Omega))$, then for any $\delta > 0$ there exists $\theta > 0$ independent of $x \in [s,t]$ such that for all $x_1, x_2 \in [s,t]$ with $|x_1 - x_2| < \theta$ $\|\mathbf{u}(x_1) - \mathbf{u}(x_2)\|_{H^1(\Omega)} < \delta$. Thus if $\varepsilon < \theta$, we obtain

$$\sup_{y \in (-\varepsilon,\varepsilon)} \|\mathbf{u}(x-y) - \mathbf{u}(x)\|_{H^1(\Omega)} < \delta,$$

which implies

$$\|\mathbf{u}^\varepsilon(x) - \mathbf{u}(x)\|_{H^1(\Omega)} < \delta$$

for all $x \in [s,t]$. Then we conclude that $\mathbf{u}^\varepsilon$ converges to $\mathbf{u}$ in $C([s,t]; H^1(\Omega))$ as $\varepsilon \to 0$. Furthermore from properties of mollifiers we obtain also

$$\begin{cases} \mathbf{u}^\varepsilon \longrightarrow \mathbf{u} & \text{in} \quad L^2_{loc}(0,T; H^1_0(\Omega) \cap H^2(\Omega))), \\ (\mathbf{u}^\varepsilon)' \longrightarrow \mathbf{u}' & \text{in} \quad L^2_{loc}(0,T; L^2(\Omega)) \end{cases}$$

as $\varepsilon \longrightarrow 0$. Then

$$\frac{d}{dt}\|\nabla \mathbf{u}^\varepsilon(t)\|^2 = 2(\nabla \mathbf{u}^\varepsilon(t), \nabla(\mathbf{u}^\varepsilon(t))')_{L^2(\Omega)} = -2((\mathbf{u}^\varepsilon(t))', \triangle \mathbf{u}^\varepsilon(t))_{L^2(\Omega)},$$

consequently for any $s \in (\varepsilon, T-\varepsilon)$

$$\|\nabla \mathbf{u}^\varepsilon(s)\|^2 = \|\nabla \mathbf{u}^\varepsilon(t)\|^2 + \int_t^s \left(-2((\mathbf{u}^\varepsilon(\tau))', \triangle \mathbf{u}^\varepsilon(\tau))_{L^2(\Omega)}\right) d\tau. \tag{3.2.64}$$

Next we find

$$\begin{aligned}
\|\nabla \mathbf{u}^\varepsilon(s)\|^2 - \|\nabla \mathbf{u}(s)\|^2 &= \int_\Omega (\nabla \mathbf{u}^\varepsilon(s) - \nabla \mathbf{u}(s))(\nabla \mathbf{u}^\varepsilon(s) + \nabla \mathbf{u}(s))dx \\
&\leq \|\nabla \mathbf{u}^\varepsilon(s) - \nabla \mathbf{u}(s)\|_{L^2(\Omega)} \|\nabla \mathbf{u}^\varepsilon(s) + \nabla \mathbf{u}(s)\|_{L^2(\Omega)} \\
&\leq \|\nabla \mathbf{u}^\varepsilon(s) - \nabla \mathbf{u}(s)\|^2_{L^2(\Omega)} \\
&\quad + 2\|\nabla \mathbf{u}(s)\|_{L^2(\Omega)} \|\nabla \mathbf{u}^\varepsilon(s) - \nabla \mathbf{u}(s)\|_{L^2(\Omega)}.
\end{aligned}$$

Since $\mathbf{u}^\varepsilon(s)$ converges to $\mathbf{u}(s)$ in $H^1(\Omega)$ as $\varepsilon \to 0$, then $\|\nabla \mathbf{u}^\varepsilon(s)\|^2 \to \|\nabla \mathbf{u}(s)\|^2$ as $\varepsilon \to 0$. Recalling (3.2.64), it follows

$$\|\nabla \mathbf{u}(s)\|^2 = \|\nabla \mathbf{u}(t)\|^2 + \int_t^s \left(-2((\mathbf{u}(\tau))', \triangle \mathbf{u}(\tau))_{L^2(\Omega)}\right) d\tau.$$

□

Definition of Weak Solutions

Assume that $\mathbf{u}$ is a classical solution to (3.2.1), then multiply (3.2.1) by

$$(v^0, v^1, \cdots, v^d) =: (v^0, v^{\mathbf{d}}) =: v \in (C_0^\infty(\Omega_T))^{1+d}$$

and integrate by parts, then we obtain the following equations

$$\begin{cases} \displaystyle\int_0^T B_1(\mathbf{u}, v) dt = \int_0^T (F^0, v^0) dt, \\ \displaystyle\int_0^T B_2(\mathbf{u}, v) dt = \int_0^T (F^{\mathbf{d}}, v^{\mathbf{d}}) dt, \end{cases} \tag{3.2.65}$$

with

$$B_1(\mathbf{u}, v) := (\partial_t u^0, v^0) - \nu_0(v^0, \triangle u^0) - (v^0, \mathrm{div} u^{\mathbf{d}}),$$

$$B_2(\mathbf{u}, v) := (\partial_t u^{\mathbf{d}}, v^{\mathbf{d}}) + \nu_0(\nabla v^{\mathbf{d}}, \nabla u^{\mathbf{d}}) + \frac{1}{\tau}(u^{\mathbf{d}}, v^{\mathbf{d}}) - T_0(\nabla u^0, v^{\mathbf{d}}) - \frac{\epsilon^2}{4}(\triangle u^0, \mathrm{div} v^{\mathbf{d}}),$$

which give us the motivation for definition of weak solutions to (3.2.1).

Definition 3.2.2. *Assume that*

$$F^0 \in L^2(0,T; L^2(\Omega)), \qquad F^{\mathbf{d}} \in L^2(0,T; (H^{-1}(\Omega))^d),$$
$$g^0 \in H_0^1(\Omega), \qquad g^{\mathbf{d}} \in L^2(\Omega)$$

We say a function

$$\mathbf{u} \in L^2(0,T; (H_0^1(\Omega) \cap H^2(\Omega)) \times (H_0^1(\Omega))^d) \cap H^1(0,T; L^2(\Omega) \times (H^{-1}(\Omega))^d)$$

is a weak solution of the system (3.2.1) *provided* (3.2.65) *holds for each* $v \in L^2(0,T; (H_0^1(\Omega))^{1+d})$*, and* $\mathbf{u}(0) = (g^0, g^{\mathbf{d}})^T$*.*

Existence and Uniqueness

Theorem 3.2.12. *Assume that the given functions*

$$F^0 \in L^2(0,T; L^2(\Omega)), \qquad F^{\mathbf{d}} \in L^2(0,T; (H^{-1}(\Omega))^d),$$
$$g^0 \in H_0^1(\Omega), \qquad g^{\mathbf{d}} \in L^2(\Omega)$$

then (3.2.1) *has a unique weak solution which satisfies the following a priori estimates*

$$\begin{aligned}
&\|u^0\|^2_{L^2(0,T;H^1_0(\Omega)\cap H^2(\Omega))} + \|u^{\mathbf{d}}\|^2_{L^2(0,T;H^1_0(\Omega))} + \|\partial_t \mathbf{u}\|^2_{L^2(0,T;L^2(\Omega)\times(H^{-1}(\Omega))^d)} \\
&\leq C\left(\|F^0\|^2_{L^2(0,T;L^2(\Omega))} + \|F^{\mathbf{d}}\|^2_{L^2(0,T;(H^{-1}(\Omega))^d)} + \|g^0\|^2_{H^1_0(\Omega))} + \|g^{\mathbf{d}}\|^2_{L^2(\Omega)}\right),
\end{aligned}$$

where the constant C does not depend upon F.

Remark 3.2.2. *From **Definition** (3.2.2) and **Theorem** 3.2.10 it follows $\mathbf{u} \in C([0,T], H^1(\Omega)\times(L^2(\Omega))^d)$, which allows us to interpret the initial condition.*

Proof. Our idea is the Faedo-Galerkin method. More precisely, we expect a set of functions $\phi_j = \phi_j(x)$ $(j = 1, 2, ...)$ in $H^1_0(\Omega)\cap H^2(\Omega)$ such that $\{\phi_j\}_{j=1}^\infty$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis in $H^1_0(\Omega)$ and $H^1_0(\Omega)\cap H^2(\Omega)$ respectively. We take $\{\phi_j\}_{j=1}^\infty$ to be the complete set of normalized eigenfunctions for the operator $-\triangle$ in $H^1_0(\Omega)$. Set S the solution operator

$$\begin{aligned}
S : L^2(\Omega) &\to L^2(\Omega) \\
f &\mapsto u = Sf,
\end{aligned}$$

u is the weak solution of

$$\begin{aligned}
-\triangle u &= f, \\
u|_{\partial\Omega} &= 0.
\end{aligned}$$

It is obvious that S is symmetric. Furthermore from Rellich selection theorem we check that S is compact, which implies all the eigenvalues of S are real, nonzero and there are corresponding eigenfunctions suitably normalized, which make up an orthonormal basis in $L^2(\Omega)$. But observe as well that for $\eta \neq 0$, we have $Sf = \eta f$ if and only if $-\triangle f = \lambda f$ for $\lambda = \dfrac{1}{\eta}$. Let $\{\eta_j\}_{j=1}^\infty$ comprise the sequence of distinct eigenvalues of S, H_j the eigenspace with respect to η_j, we obtain thereby an orthonormal basis $\phi_j = \phi_j(x)$ $(j = 1, \cdots)$ of normalized eigenvectors in H_j of $L^2(\Omega)$. Actually $\phi_j \in H^1_0(\Omega)\cap C^\infty(\Omega)$, $\triangle\phi_j \in H^1_0(\Omega)$ and we have the following calculations

$$\begin{aligned}
&(\nabla\phi_i, \nabla\phi_j)_{L^2} = \eta_i(\phi_i, \phi_j)_{L^2} = 0, \\
&-\triangle\phi_i = \eta_i\phi_i \Longrightarrow \triangle^2\phi_i = \eta_i(-\triangle\phi_i) = \eta_i^2\phi_i, \\
&\qquad\qquad\Longrightarrow (\triangle\phi_i, \triangle\phi_j)_{L^2} = \eta_i^2(\phi_i, \phi_j)_{L^2} = 0.
\end{aligned}$$

Therefore $\{\phi_j\}_{j=1}^\infty$ is orthogonal in $H^1_0(\Omega)\cap H^2(\Omega)$ with respect to (3.2.62). We claim further that $\{\phi_j\}_{j=1}^\infty$ is complete in $H^1_0(\Omega)\cap H^2(\Omega)$. To see this, it suffices to verify that for $u \in H^1_0(\Omega)\cap H^2(\Omega)$, $B(\phi_j, u) = 0$ implies $u \equiv 0$. From the properties of ϕ_j, $B(\phi_j, u) = (1 + \eta_j + \eta_j^2)(\phi_j, u)_{L^2}$. The completeness follows immediately since $\{\phi_j\}_{j=1}^{\infty}$ is complete in $L^2(\Omega)$. That $\{\phi_j\}_{j=1}^{\infty}$ is complete in $H_0^1(\Omega)$ follows from the same reason. Via normalization of $\{\phi_j\}_{j=1}^{\infty}$ in $H_0^1(\Omega)$ $\left(\omega_j := \dfrac{\phi_j}{\|\phi_j\|_{H_0^1(\Omega)}}\right)$ we obtain an orthonormal basis $\{\omega_j\}_{j=1}^{\infty}$ in $H_0^1(\Omega)$.

Fix now a positive integer m. We will look for $1 + d$ functions u_m^0, $u_m^{\mathbf{d}} := (u_m^1, \cdots, u_m^d)^T$ with

$$u_m^0 : [0, T] \to H_0^1(\Omega) \cap H^2(\Omega), \tag{3.2.66}$$

$$u_m^{\mathbf{d}} : [0, T] \to (H_0^1(\Omega))^d, \tag{3.2.67}$$

of the form

$$u_m^0(t) := \sum_{j=1}^{m} d_{u_m^0}^j(t)\omega_j, \tag{3.2.68}$$

$$u_m^{\mathbf{d}}(t) := \left(\sum_{j=1}^{m} d_{u_m^1}^j(t)\phi_j, \sum_{j=1}^{m} d_{u_m^2}^j(t)\phi_j, \cdots, \sum_{j=1}^{m} d_{u_m^d}^j(t)\phi_j\right)^T, \tag{3.2.69}$$

where we hope to select the coefficients $d_{u_m^0}^j(t)$, $d_{u_m^l}^j(t)$ $(0 \le t \le T,\ l = 1, ..., d,\ j = 1, ..., m)$ so that

$$d_{u_m^0}^j(0) = (g^0, \omega_j)_{H_0^1(\Omega)}, \quad d_{u_m^l}^j(0) = (g^l, \phi_j) \tag{3.2.70}$$

and

$$\begin{cases} (\partial_t u_m^0, \omega_j) + \nu_0(\nabla\omega_j, \nabla u_m^0) + (\nabla\omega_j, u_m^{\mathbf{d}}) = (F^0, \omega_j), \\ (\partial_t u_m^l, \phi_j) + \nu_0(\nabla\phi_j, \nabla u_m^l) + \dfrac{1}{\tau}(u_m^l, \phi_j) \\ \qquad -T_0(\partial_{x_l} u_m^0, \phi_j) + \dfrac{\epsilon^2}{4}(\partial_{x_l} \triangle u_m^0, \phi_j) = (F^l, \phi_j). \end{cases} \tag{3.2.71}$$

Thus we seek functions u_m^0, $u_m^{\mathbf{d}}$ of the form (3.2.68) and (3.2.69) respectively, that satisfy the "projection" (3.2.71) of problem (3.2.1) onto the finite dimensional subspaces spanned by $\{\phi_j\}_{j=1}^{m}$. Assuming u_m^0, $u_m^{\mathbf{d}}$ have the structures (3.2.68), (3.2.69) respectively, we first note that

$$(\partial_t u_m^0(t), \omega_j) = \frac{1}{\|\phi_j\|_{H_0^1(\Omega)}^2}(d_{u_m^0}^j(t))', \tag{3.2.72}$$

$$(\partial_t u_m^l(t), \phi_j) = (d_{u_m^l}^j(t))', \tag{3.2.73}$$

since $\{\phi_j\}_{j=1}^{\infty}$ forms an orthonormal basis in $L^2(\Omega)$. Then (3.2.71) becomes a linear system of ODE subject to the initial conditions (3.2.70), which may be written in the form

$$\mathbf{d}'(t) = A\mathbf{d}(t) + b(t),$$
$$\mathbf{d}(0) = \mathbf{a},$$

where $\mathbf{a}$ is via (3.2.70) solvable. For $b(t) \in L^2(0,T)$ this equation has a unique solution in $\{v \in H^1(0,T) | v(0) = 0\}$, more precisely, $\mathbf{d}(t)$ has the form

$$\mathbf{d}(t) = \int_0^t e^{A(t-s)} b(s) ds + e^{At}\mathbf{a}.$$

Multiplying (3.2.71) by $d^j_{u^0_m}(t)$, $d^j_{u^l_m}(t)$ respectively, sum for $j = 1, \cdots, m$, $l = 1, \cdots, d$, and then recall (3.2.68) and (3.2.69), we obtain

$$\begin{cases} (\partial_t u^0_m, u^0_m) + \nu_0(\nabla u^0_m, \nabla u^0_m) + (\nabla u^0_m, u^{\mathbf{d}}_m) = (F^0, u^0_m), \\ (\partial_t u^{\mathbf{d}}_m, u^{\mathbf{d}}_m) + \nu_0(\nabla u^{\mathbf{d}}_m, \nabla u^{\mathbf{d}}_m) + \frac{1}{\tau}(u^{\mathbf{d}}_m, u^{\mathbf{d}}_m) \\ \qquad -T_0(\nabla u^0_m, u^{\mathbf{d}}_m) + \frac{\epsilon^2}{4}(\nabla \triangle u^0_m, u^{\mathbf{d}}_m) = (F^{\mathbf{d}}, u^{\mathbf{d}}_m). \end{cases} \tag{3.2.74}$$

Next we will get the energy estimates of $(u^0_m, u^{\mathbf{d}}_m)^T$. Since $-\triangle\omega_j = \eta_j\omega_j$, then

$$-\triangle u^0_m = \sum_{j=1}^m d^j_{u^0_m}(-\triangle\omega_j) = \sum_{j=1}^m d^j_{u^0_m}\eta_j\omega_j.$$

Multiplying the first equation of (3.2.71) by $d^j_{u^0_m}\eta_j$, summing up for $j = 1, \cdots, m$ yields

$$(\partial_t \nabla u^0_m, \nabla u^0_m) + \nu_0(\triangle u^0_m, \triangle u^0_m) - (\nabla\triangle u^0_m, u^{\mathbf{d}}_m) = -(F^0, \triangle u^0_m). \tag{3.2.75}$$

Combining (3.2.74) and (3.2.75) we obtain the following equation

$$\begin{aligned} &\frac{1}{2}\partial_t\big(T_0\|u^0_m\|^2 + \frac{\epsilon^2}{4}\|\nabla u^0_m\|^2 + \|u^{\mathbf{d}}_m\|^2\big) + \nu_0 T_0\|\nabla u^0_m\|^2 \\ &\qquad + \frac{\epsilon^2}{4}\nu_0\|\triangle u^0_m\|^2 + \nu_0\|\nabla u^{\mathbf{d}}_m\|^2 + \frac{1}{\tau}\|u^{\mathbf{d}}_m\|^2 \\ &= T_0(F^0, u^0_m) + (F^{\mathbf{d}}, u^{\mathbf{d}}_m) - \frac{\epsilon^2}{4}(F^0, \triangle u^0_m). \end{aligned} \tag{3.2.76}$$

Integrating (3.2.76) from 0 to T yields

$$\begin{aligned}
&\frac{T_0}{2}\|u_m^0(T)\|^2 + \frac{1}{2}\|u_m^{\mathbf{d}}(T)\|^2 + \frac{\epsilon^2}{8}\|\nabla u_m^0(T)\|^2 \\
&\quad + \nu_0 T_0\|\nabla u_m^0\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\epsilon^2}{4}\nu_0\|\triangle u_m^0\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + \nu_0\|\nabla u_m^{\mathbf{d}}\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{\tau}\|u_m^{\mathbf{d}}\|^2_{L^2(0,T;L^2(\Omega))} \\
&= \int_0^T \left(T_0(F^0, u_m^0) + (F^{\mathbf{d}}, u_m^{\mathbf{d}}) - \frac{\epsilon^2}{4}(F^0, \triangle u_m^0)\right) dt \\
&\quad + \frac{T_0}{2}\|u_m^0(0)\|^2 + \frac{1}{2}\|u_m^{\mathbf{d}}(0)\|^2 + \frac{\epsilon^2}{8}\|\nabla u_m^0(0)\|^2,
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{T_0}{2}\|u_m^0(T)\|^2 + \frac{1}{2}\|u_m^{\mathbf{d}}(T)\|^2 + \frac{\epsilon^2}{8}\|\nabla u_m^0(T)\|^2 \\
&\quad + \nu_0 T_0\|\nabla u_m^0\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\epsilon^2}{4}\nu_0\|\triangle u_m^0\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + \nu_0\|\nabla u_m^{\mathbf{d}}\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{\tau}\|u_m^{\mathbf{d}}\|^2_{L^2(0,T;L^2(\Omega))} \\
&\le c_1\int_0^T \|F^0\|_{H^{-1}(\Omega)}\|u_m^0\|_{H^1(\Omega)} + c_2\int_0^T \|F^{\mathbf{d}}\|_{H^{-1}(\Omega)}\|u_m^{\mathbf{d}}\|_{H^1(\Omega)} \\
&\quad + c_3\int_0^T \|F^0\|_{L^2(\Omega)}\|\triangle u_m^0\|_{L^2(\Omega)} \\
&\quad + \max\left(\frac{T_0}{2}, \frac{1}{2}, \frac{\epsilon^2}{8}\right)(\|g^0\|^2_{H_0^1(\Omega)} + \|g^{\mathbf{d}}\|^2_{L^2(\Omega)})
\end{aligned} \tag{3.2.77}$$

since $\|u_m^0(0)\|^2_{H_0^1(\Omega)} \le \|g^0\|^2_{H_0^1(\Omega)}$, $\|u_m^{\mathbf{d}}(0)\|^2_{L^2(\Omega)} \le \|g^{\mathbf{d}}\|^2_{L^2(\Omega)}$ by (3.2.70). Applying Poincare's inequality

$$\|u_m^0\|^2_{L^2(\Omega)} \le c_4\|\nabla u_m^0\|^2_{L^2(\Omega)}, \quad (c_4 \text{ is a constant independent of } m)$$

and Cauchy's inequality we obtain

$$\begin{aligned}
&\|u_m^0\|^2_{L^2(0,T;H^2(\Omega))} + \|u_m^{\mathbf{d}}\|^2_{L^2(0,T;H^1(\Omega))} \\
&\le c_5\Big(\|F^0\|^2_{L^2(0,T;L^2(\Omega))} + \|F^{\mathbf{d}}\|^2_{L^2(0,T;(H^{-1}(\Omega))^d)} + \|g^0\|^2_{H_0^1(\Omega)} + \|g^{\mathbf{d}}\|^2_{L^2(\Omega)}\Big).
\end{aligned} \tag{3.2.78}$$

Fix any $h \in L^2(\Omega)$, with $\|h\|_{L^2(\Omega)} \le 1$, and write $h = h^1 + h^2$, where $h^1 \in \text{span}\{\omega_j\}_{j=1}^m$ and $(h^2, \omega_j) = 0$ $(j = 1, \cdots, m)$. Since the functions $\{\omega_j\}_{j=1}^\infty$ are orthogonal in $L^2(\Omega)$, $\|h^1\|_{L^2(\Omega)} \le \|h\|_{L^2(\Omega)} \le 1$. Utilizing (3.2.71), we deduce for a.e. $0 \le t \le T$ that

$$(\partial_t u_m^0, h^1) - \nu_0(h^1, \triangle u_m^0) - (h^1, \mathrm{div} u_m^{\mathbf{d}}) = (F^0, h^1) \tag{3.2.79}$$

then we obtain

$$\begin{aligned}|(\partial_t u_m^0, h)| = |(\partial_t u_m^0, h^1)| = |(F^0, h^1) + \nu_0(h^1, \triangle u_m^0) + (h^1, \mathrm{div} u_m^{\mathbf{d}})|\\ \le c_7(\|F^0\|_{L^2(\Omega)} + \|\triangle u_m^0\|_{L^2(\Omega)} + \|\mathrm{div} u_m^{\mathbf{d}}\|_{L^2(\Omega)}),\end{aligned}$$

which yields

$$\|\partial_t u_m^0\|_{L^2(\Omega)} \le c_7(\|F^0\|_{L^2(\Omega)} + \|u_m^0\|_{H^2(\Omega)} + \|u_m^{\mathbf{d}}\|_{H^1(\Omega)}),$$

and

$$\int_0^T \|\partial_t u_m^0\|^2_{L^2(\Omega)} dt \le c_8 \int_0^T (\|F^0\|^2_{L^2(\Omega)} + \|u_m^0\|^2_{H^2(\Omega)} + \|u_m^{\mathbf{d}}\|^2_{H^1(\Omega)}) dt,$$

since (3.2.78) holds, we obtain the uniform bounds of $\partial_t u_m^0$ in $L^2\left(0,T;L^2(\Omega)\right)$

$$\begin{aligned}&\|\partial_t u_m^0\|^2_{L^2(0,T;L^2(\Omega))}\\ &\le c_9\big(\|F^0\|^2_{L^2(0,T;L^2(\Omega))} + \|F^{\mathbf{d}}\|^2_{L^2(0,T;(H^{-1}(\Omega))^d)} + \|g^0\|^2_{H_0^1(\Omega)} + \|g^{\mathbf{d}}\|^2_{L^2(\Omega)}\big).\end{aligned} \tag{3.2.80}$$

Next fix any $h_* \in H_0^1(\Omega)$, with $\|h_*\|_{H_0^1(\Omega)} \le 1$, and write $h = h_*^1 + h_*^2$, where $h_*^1 \in \mathrm{span}\{\omega_j\}_{j=1}^m$ and $(h_*^2, \omega_j) = 0$ $(j = 1, \cdots, m)$. Since the functions $\{\omega_j\}_{j=1}^\infty$ are orthogonal in $H_0^1(\Omega)$, $\|h_*^1\|_{H_0^1(\Omega)} \le \|h_*\|_{H_0^1(\Omega)} \le 1$. By an analogous reason we find

$$\begin{aligned}&\|\partial_t u_m^{\mathbf{d}}\|^2_{L^2(0,T;(H^{-1}(\Omega))^d)}\\ &\le c_{10}\big(\|F^0\|^2_{L^2(0,T;L^2(\Omega))} + \|F^{\mathbf{d}}\|^2_{L^2(0,T;(H^{-1}(\Omega))^d)} + \|g^0\|^2_{H_0^1(\Omega)} + \|g^{\mathbf{d}}\|^2_{L^2(\Omega)}\big).\end{aligned} \tag{3.2.81}$$

Then (3.2.78), (3.2.80), and (3.2.81) are sufficient to ensure the existence of a subsequence $\{u_{m_s}^0\}_{s=1}^\infty \subset \{u_m^0\}_{m=1}^\infty$ and a subsequence $\{u_{m_s}^{\mathbf{d}}\}_{s=1}^\infty \subset \{u_m^{\mathbf{d}}\}_{m=1}^\infty$ and functions

$$u^0 \in L^2(0,T; H_0^1(\Omega) \cap H^2(\Omega)), \tag{3.2.82}$$

$$\mathbf{v}^0 \in L^2\left(0,T;L^2(\Omega)\right), \tag{3.2.83}$$

$$u^{\mathbf{d}} \in L^2(0,T;(H_0^1(\Omega))^d), \tag{3.2.84}$$

$$\mathbf{v}^{\mathbf{d}} \in L^2(0,T;(H^{-1}(\Omega))^d), \tag{3.2.85}$$

such that

$$\begin{cases} u^0_{m_s} \rightharpoonup u^0 & \text{weakly in} \quad L^2(0,T;H^1_0(\Omega)\cap H^2(\Omega)), \\ \partial_t u^0_{m_s} \rightharpoonup \mathbf{v}^0 & \text{weakly in} \quad L^2\left(0,T;L^2(\Omega)\right), \\ u^{\mathbf{d}}_{m_s} \rightharpoonup u^{\mathbf{d}} & \text{weakly in} \quad L^2(0,T;(H^1_0(\Omega))^d), \\ \partial_t u^{\mathbf{d}}_{m_s} \rightharpoonup^* \mathbf{v}^{\mathbf{d}} & \text{weakly in} \quad L^2(0,T;(H^{-1}(\Omega))^d). \end{cases} \tag{3.2.86}$$

Let $\phi \in C_0^\infty((0,T))$, $\omega \in L^2(\Omega)$, then

$$\int_0^T (\partial_t u^0_{m_s}, \phi\omega)dt = -\int_0^T (u^0_{m_s}, (\partial_t\phi)\omega)dt.$$

By (3.2.86) we obtain

$$\int_0^T (\mathbf{v}^0, \phi\omega)dt = -\int_0^T (u^0, \partial_t\phi\omega)dt, \qquad \text{as } s \to \infty.$$

Since $\omega \in L^2(\Omega) = (L^2(\Omega))^*$ from the properties of the Bochner integral (see [82] Section 5)

$$\omega\left(\int_0^T (\mathbf{v}^0, \phi)dt\right) = \int_0^T (\mathbf{v}^0, \phi\omega)dt,$$

$$\omega\left(-\int_0^T (u^0, \partial_t\phi)dt\right) = -\int_0^T (u^0, \partial_t\phi\omega)dt,$$

which implies that

$$\partial_t u^0 = \mathbf{v}^0.$$

Similarly, by choosing $\omega \in (H^1_0(\Omega))^d$ we obtain

$$\partial_t u^{\mathbf{d}} = \mathbf{v}^{\mathbf{d}}.$$

Fix an integer N and choose functions

$$f \in C^1([0,T];H^1_0(\Omega))$$

and

$$(h^1, h^2, \cdots, h^d)^T =: h^{\mathbf{d}} \in C^1([0,T];(H^1_0(\Omega))^d)$$

having the form

$$f(t) = \sum_{j=1}^N d^j_f(t)\omega_j \tag{3.2.87}$$

$$h^{\mathbf{d}}(t) = \left(\sum_{j=1}^N d^j_{h^1}(t)\phi_j, \sum_{j=1}^N d^j_{h^2}(t)\phi_j, \cdots, \sum_{j=1}^N d^j_{h^d}(t)\phi_j\right)^T \tag{3.2.88}$$

where $\{d_f^j(t)\}_{j=1}^N$, $\{d_{h^s}^j(t)\}_{j=1}^N$ $(s = 1, 2, \cdots, d)$ are given smooth functions. We choose $m \geq N$, multiply (3.2.71) by $d_f^j(t)$, $d_{h^s}^j(t)$ respectively, sum up for $j = 1, \cdots, N$, $s = 1, \cdots, d$, and then integrate with respect to $0 \leq t \leq T$ to find

$$\begin{cases} \int_0^T \Big((\partial_t u_m^0, f) + \nu_0(\nabla f, \nabla u_m^0) + (\nabla f, u_m^{\mathbf{d}})\Big)dt = \int_0^T (F^0, f)dt, \\ \int_0^T \Big((\partial_t u_m^{\mathbf{d}}, h^{\mathbf{d}}) + \nu_0(\nabla h^{\mathbf{d}}, \nabla u_m^{\mathbf{d}}) + \frac{1}{\tau}(u_m^{\mathbf{d}}, h^{\mathbf{d}}) \\ \qquad -T_0(\nabla u_m^0, h^{\mathbf{d}}) - \frac{\epsilon^2}{4}(\triangle u_m^0, \mathrm{div} h^{\mathbf{d}})\Big)dt = \int_0^T (F^{\mathbf{d}}, h^{\mathbf{d}})dt. \end{cases}$$

We set $u_m^0 = u_{m_s}^0$, $u_m^{\mathbf{d}} = u_{m_s}^{\mathbf{d}}$, and recall (3.2.86), to find upon passing to weak limits that

$$\begin{cases} \int_0^T \Big((\partial_t u^0, f) + \nu_0(\nabla f, \nabla u^0) + (\nabla f, u^{\mathbf{d}})\Big)dt = \int_0^T (F^0, f)dt, \\ \int_0^T \Big((\partial_t u^{\mathbf{d}}, h^{\mathbf{d}}) + \nu_0(\nabla h^{\mathbf{d}}, \nabla u^{\mathbf{d}}) + \frac{1}{\tau}(u^{\mathbf{d}}, h^{\mathbf{d}}) \\ \qquad -T_0(\nabla u^0, h^{\mathbf{d}}) - \frac{\epsilon^2}{4}(\triangle u^0, \mathrm{div} h^{\mathbf{d}})\Big)dt = \int_0^T (F^{\mathbf{d}}, h^{\mathbf{d}})dt. \end{cases} \tag{3.2.89}$$

We claim that functions of the form (3.2.87), (3.2.88) are dense in

$$L^2\left(0, T; H_0^1(\Omega)\right) \quad \text{and} \quad L^2(0, T; (H_0^1(\Omega))^d)$$

respectively. To prove this let $g \in L^2([0, T], H_0^1(\Omega))$, $\vartheta > 0$. The assertion follows if we can show that there is a function p of the form (3.2.87) with

$$\|p - g\|_{L^2\left(0,T;H_0^1(\Omega)\right)} < \vartheta.$$

Since $\{\omega_j\}_{j=1}^\infty$ is an orthonormal basis in $H_0^1(\Omega)$, then

$$t \mapsto a_j(t) = (g(t), \omega_j)_{H_0^1(\Omega)} : \quad [0, T] \to \mathbb{R}$$

belongs to $L^2((0, T))$ because

$$\int_0^T a_j(t)^2 dt = \int_0^T (g(t), \omega_j)_{H_0^1(\Omega)}^2 dt \leq \int_0^T \|g(t)\|_{H_0^1(\Omega)}^2 dt < \infty.$$

Let $g_n := \sum_{j=1}^{n} a_j\omega_j$, then

$$\int_0^T \left\| \sum_{j=l}^{m} a_j(t) w_j \right\|_{H_0^1(\Omega)}^2 dt$$

$$= \int_0^T \left(\left(\sum_{j=l}^{m} a_j\omega_j, \sum_{j=l}^{m} a_j\omega_j \right)_{L^2(\Omega)} + \left(\sum_{j=l}^{m} a_j\nabla\omega_j, \sum_{j=l}^{m} a_j\nabla\omega_j \right)_{L^2(\Omega)} \right) dt$$

$$= \int_0^T \left(\sum_{j=l}^{m} a_j^2 \|w_j\|_{L^2(\Omega)}^2 + \sum_{j=l}^{m} a_j^2 \|\nabla w_j\|_{L^2(\Omega)}^2 \right) dt = \int_0^T \sum_{j=l}^{m} a_j^2 dt$$

$$= \sum_{j=l}^{m} \|a_j\|_{L^2((0,T))}^2.$$

The sequence of functions $\left(\sum_{j=1}^{n} a_j^2(t) \right)_n$ converges pointwise to $\|g(t)\|_{H_0^1(\Omega)}^2$ with

$$\int_0^T \|g(t)\|_{H_0^1(\Omega)}^2 dt < \infty, \qquad \sum_{j=1}^{n} a_j^2(t) \le \|g(t)\|_{H_0^1(\Omega)}^2$$

for a.e. $t \in [0,T]$ and all $n \in \mathbb{N}$, then Lebesgue's dominated convergence theorem provides

$$\int_0^T \|g(t)\|_{H_0^1(\Omega)}^2 dt = \lim_{n\to\infty} \int_0^T \sum_{j=1}^{n} a_j^2(t) dt = \lim_{n\to\infty} \sum_{j=1}^{n} \|a_j\|_{L^2((0,T))}^2, \tag{3.2.90}$$

from which we conclude g_n is a Cauchy sequence in $L^2(0,T;H_0^1(\Omega))$. Assume g_n converges to v in $L^2(0,T;H_0^1(\Omega))$, since g_n converges to g pointwise for a.e. $t \in [0,T]$ it follows that $v = g$.

Then there exists $N \in \mathbb{N}$ such that for all $n \ge N$

$$g_n \in \{f \in L^2(0,T;H_0^1(\Omega)) | \|f - g\|_{L^2(0,T;H_0^1(\Omega))} < \vartheta\}.$$

Fix $m > N$ then $\exists r > 0$ such that

$$H_1 \subset H_2 \tag{3.2.91}$$

where

$$H_1 := \{f \in L^2(0,T;H_0^1(\Omega)) | \|f - g_m\|_{L^2(0,T;H_0^1(\Omega))} < r\},$$

$$H_2 := \{f \in L^2(0,T;H_0^1(\Omega)) | \|f - g\|_{L^2(0,T;H_0^1(\Omega))} < \vartheta\},$$

since g_m is an interior point of $\{f \in L^2(0,T;H_0^1(\Omega)) | \|f-g\|_{L^2(0,T;H_0^1(\Omega))} < \vartheta\}$. Using the dense embedding $C_0^\infty((0,T)) \hookrightarrow L^2((0,T))$ then for each a_j ($j = 1,\cdots,m$) we can select a sequence $(a_{j,n})_n \subset C_0^\infty((0,T))$ such that $a_{j,n} \to a_j$ in $L^2((0,T))$ for $n \to \infty$.

$$f_n := \sum_{j=1}^{m} a_{j,n}(t) w_j \longrightarrow g_m$$

in $L^2(0,T;H_0^1(\Omega))$ since

$$\|f_n - g_m\|_{L^2(0,T;H_0^1(\Omega))} = \left\| \sum_{j=1}^{m} (a_{j,n} - a_j) w_j \right\|_{L^2(0,T;H_0^1(\Omega))}$$

$$\leq \sum_{j=1}^{m} \|(a_{j,n} - a_j) w_j\|_{L^2(0,T;H_0^1(\Omega))} = \sum_{j=1}^{m} \|a_{j,n} - a_j\|_{L^2((0,T))}.$$

Thus there exists $M \in \mathbb{N}$ such that for all $n \geq M$ f_n belongs to

$$\{f \in L^2(0,T;H_0^1(\Omega)) | \|f - g_m\|_{L^2(0,T;H_0^1(\Omega))} < r\}.$$

Consequently from (3.2.91) $\|f_n - g\|_{L^2(0,T;H_0^1(\Omega))} < \vartheta$.

The system (3.2.89) holds for all

$$f \in L^2\left(0,T;H_0^1(\Omega)\right) \quad \text{and} \quad h^{\mathbf{d}} \in L^2(0,T;(H_0^1(\Omega))^d),$$

since functions of the form (3.2.87), (3.2.88) are dense in

$$L^2\left(0,T;H_0^1(\Omega)\right) \quad \text{and} \quad L^2(0,T;(H_0^1(\Omega))^d)$$

respectively.

In order to prove $u^0(0) = u^{\mathbf{d}}(0) = \mathbf{g}$, we first note from (3.2.86) that

$$\begin{cases} u_{m_s}^0 \rightharpoonup^* u^0 & \text{weakly star in} \quad L^2(0,T;H^{-1}(\Omega)), \\ \partial_t u_{m_s}^0 \rightharpoonup^* \partial_t u^0 & \text{weakly star in} \quad L^2\left(0,T;H^{-1}(\Omega)\right), \\ u_{m_s}^{\mathbf{d}} \rightharpoonup^* u^{\mathbf{d}} & \text{weakly star in} \quad L^2(0,T;(H^{-1}(\Omega))^d), \\ \partial_t u_{m_s}^{\mathbf{d}} \rightharpoonup^* \partial_t u^{\mathbf{d}} & \text{weakly star in} \quad L^2(0,T;(H^{-1}(\Omega))^d), \end{cases} \tag{3.2.92}$$

then from **Theorem** B.0.7 we obtain

$$(u_{m_s}^0, u_{m_s}^{\mathbf{d}})(0) \rightharpoonup^* (u^0, u^{\mathbf{d}})(0) \quad \text{weakly star in} \quad H^{-1}(\Omega) \times (H^{-1}(\Omega))^d.$$

By the uniqueness of the limit, we get

$$\mathbf{u}(0) = \mathbf{g}.$$

Let $(u_1^0, u_1^{\mathbf{d}})$ and $(u_2^0, u_2^{\mathbf{d}})$ be two weak solutions of (3.2.1), define

$$u_\triangle^0 = u_1^0 - u_2^0, \qquad u_\triangle^{\mathbf{d}} = u_1^{\mathbf{d}} - u_2^{\mathbf{d}}$$

then we get the system

$$\partial_t u_\triangle^0 - \nu_0 \triangle u_\triangle^0 - \mathrm{div} u_\triangle^{\mathbf{d}} = 0, \tag{3.2.93}$$

$$\partial_t u_\triangle^{\mathbf{d}} - \nu_0 \triangle u_\triangle^{\mathbf{d}} + \frac{\epsilon^2}{4} \nabla \triangle u_\triangle^0 - T_0 \nabla u_\triangle^0 + \frac{1}{\tau} u_\triangle^{\mathbf{d}} = 0 \tag{3.2.94}$$

with

$$(u_\triangle^0, u_\triangle^{\mathbf{d}})(0) = \mathbf{0}, \quad (u_\triangle^0, u_\triangle^{\mathbf{d}})|_{\partial\Omega} = \mathbf{0}.$$

To get the uniqueness we first note $-\int_\Omega \partial_t u_\triangle^0 \triangle u_\triangle^0 dx = \frac{1}{2}\partial_t \|\nabla u_\triangle^0\|^2$ by theorem 3.2.11.

Taking the $L^2(\Omega)$ scalar product of (3.2.93) with $-\frac{\epsilon^2}{4}\triangle u_\triangle^0$ then for a.e. $0 \leq t \leq T$

$$\frac{1}{2}\frac{\epsilon^2}{4}\partial_t \|\nabla u_\triangle^0\|^2 + \frac{\epsilon^2}{4}\nu_0 \|\triangle u_\triangle^0\|^2 - \frac{\epsilon^2}{4}(\nabla \triangle u_\triangle^0, u_\triangle^{\mathbf{d}}) = 0. \tag{3.2.95}$$

Taking the $L^2(\Omega)$ scalar product of (3.2.93) with $T_0 u_\triangle^0$; (3.2.94) with $u_\triangle^{\mathbf{d}}$ we obtain for a.e. $0 \leq t \leq T$

$$\frac{1}{2} T_0 \partial_t \|u_\triangle^0\|^2 + \nu_0 T_0 \|\nabla u_\triangle^0\|^2 - T_0(\mathrm{div} u_\triangle^{\mathbf{d}}, u_\triangle^0) = 0, \tag{3.2.96}$$

and

$$\begin{aligned} \frac{1}{2}\partial_t \|u_\triangle^{\mathbf{d}}\|^2 + \frac{1}{\tau}\|u_\triangle^{\mathbf{d}}\| + \nu_0 \|\nabla u_\triangle^{\mathbf{d}}\|^2 + T_0(\mathrm{div} u_\triangle^{\mathbf{d}}, u_\triangle^0) \\ + \frac{\epsilon^2}{4}(\nabla \triangle u_\triangle^0, u_\triangle^{\mathbf{d}}) = 0. \end{aligned} \tag{3.2.97}$$

Summing (3.2.95)-(3.2.97) yields

$$\begin{aligned} \frac{1}{2}\partial_t \left(T_0 \|u_\triangle^0\|^2 + \frac{\epsilon^2}{4} \|\nabla u_\triangle^0\|^2 + \|u_\triangle^{\mathbf{d}}\|^2 \right) + \nu_0 T_0 \|\nabla u_\triangle^0\|^2 + \frac{\epsilon^2}{4}\nu_0 \|\triangle u_\triangle^0\|^2 \\ + \nu_0 \|\nabla u_\triangle^{\mathbf{d}}\|^2 + \frac{1}{\tau}\|u_\triangle^{\mathbf{d}}\|^2 = 0, \end{aligned} \tag{3.2.98}$$

then $u_\triangle^0 = u_\triangle^{\mathbf{d}} = 0$. □

Regularity of the Weak Solution

Theorem 3.2.13. *Concerning the linear problem*

$$\begin{cases} \partial_t \mathbf{u}(t,x) + A(\partial_x)\mathbf{u}(t,x) = F(t,x), \quad (t,x) \in [0,T) \times \Omega \\ \mathbf{u}(0,x) = 0, \\ \mathbf{u}|_{\partial\Omega} = 0 \quad \text{for a.e. } t \in [0,T]. \end{cases} \tag{3.2.99}$$

Let $T > 0$ and

$$\begin{cases} F^0 \in L^\infty(0,T;H^1(\Omega)), F^{\mathbf{d}} \in L^\infty(0,T;(L^2(\Omega))^d); \\ \dot{F}^0 \in L^\infty(0,T;L^2(\Omega)), \dot{F}^{\mathbf{d}} \in L^\infty(0,T;(H^{-1}(\Omega))^d); \\ F^0(0,x) \in H_0^1(\Omega), F^{\mathbf{d}}(0,x) \in L^2(\Omega), \end{cases} \tag{3.2.100}$$

then the weak solution $\mathbf{u} = (u^0, u^{\mathbf{d}})$ of (3.2.99) has the following global regularity

$$\begin{cases} u^0 \in L^\infty(0,T;H^3(\Omega)) \cap C([0,T],C^1(\overline{\Omega})) \cap C([0,T],H^2(\Omega)), \\ u^{\mathbf{d}} \in L^\infty(0,T;H^2(\Omega)) \cap C([0,T],C(\overline{\Omega})) \cap C([0,T],H_0^1(\Omega)), \\ \partial_t u^0 \in L^\infty(0,T;H_0^1(\Omega)) \cap C([0,T],L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)), \\ \partial_t u^{\mathbf{d}} \in L^\infty(0,T;L^2(\Omega)) \cap C([0,T],L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)), \end{cases} \tag{3.2.101}$$

and the a priori estimates

$$\begin{aligned} &\|u^0\|^2_{L^\infty(0,T;H^3(\Omega))} + \|u^{\mathbf{d}}\|^2_{L^\infty(0,T;H^2(\Omega))} \\ &\le C\Bigg(\|F^0\|^2_{L^\infty(0,T;H^1(\Omega))} + \|F^{\mathbf{d}}\|^2_{L^\infty(0,T;L^2(\Omega))} \\ &\qquad\qquad + \|F^0(0,\cdot)\|^2_{H^1(\Omega)} + \|F^{\mathbf{d}}(0,\cdot)\|^2_{L^2(\Omega)} \Bigg) \\ &\quad + C\left(\|\dot{F}^0\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\dot{F}^{\mathbf{d}}\|^2_{L^\infty(0,T;H^{-1}(\Omega))} \right) \int_0^T e^{Cs} ds, \end{aligned} \tag{3.2.102}$$

and

$$\begin{aligned} &\|\dot{u}^0\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\dot{u}^{\mathbf{d}}\|^2_{L^\infty(0,T;L^2(\Omega))} \\ &\le C\left(\|F^0(0,\cdot)\|^2_{H^1(\Omega)} + \|F^{\mathbf{d}}(0,\cdot)\|^2_{L^2(\Omega)} \right) \\ &\quad + C\left(\|\dot{F}^0\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\dot{F}^{\mathbf{d}}\|^2_{L^\infty(0,T;H^{-1}(\Omega))} \right) \int_0^T e^{Cs} ds \end{aligned} \tag{3.2.103}$$

hold, where the constant C depends only on Ω and the coefficients of $A(\partial_x)$, but doesn't depend upon F.

Proof. The usual way of proving regularity for solutions of problem (3.2.99) is to first establish temporal regularity and then use elliptic estimates for the operator $A(\partial_x)$ to show spatial regularity. More precisely, we can formally differentiate equation (3.2.99) with respect to time. This yields

$$\begin{cases} \partial_t \dot{\mathbf{u}} + A(\partial_x)\dot{\mathbf{u}} = \dot{F}, \\ \dot{\mathbf{u}}(0,x) = F(0,x), \\ \dot{\mathbf{u}}|_{\partial\Omega} = 0 \quad \text{for a.e. } t \in [0,T]. \end{cases} \tag{3.2.104}$$

We now consider $\dot{\mathbf{u}}$ as a new variable $Q = (Q^0, Q^{\mathbf{d}})$, then from theorem 3.2.12 $Q \in L^2(0,T;V) \cap H^1(0,T;(H^{-1}(\Omega))^{1+d}) \cap C([0,T], L^2(\Omega))$ solves

$$\begin{cases} \partial_t Q + A(\partial_x)Q = \dot{F}, \\ Q(0,x) = F(0,x), \\ Q|_{\partial\Omega} = 0 \quad \text{for a.e. } t \in [0,T]. \end{cases} \tag{3.2.105}$$

Below we shall prove that actually $Q = \dot{\mathbf{u}}$. For this set

$$Z(t) := \int_0^t Q d\tau \tag{3.2.106}$$

in the sense of the Bochner integral. Then we have

$$\int_0^T \|Z(t)\|_V^2 dt = \int_0^T \left\| \int_0^t Q d\tau \right\|_V^2 dt \le \int_0^T \int_0^T \|Q\|_V^2 d\tau dt < \infty$$
$$\Longrightarrow Z(t) \in L^2(0,T;V).$$

We conclude that $\dot{Z} = Q$ in the sense of distribution and

$$\begin{aligned} A(\partial_x)Z(t) &= \int_0^t A(\partial_x)Q d\tau \\ &= F(t) - Q(t). \end{aligned}$$

With W denoting $Z - \mathbf{u}$, then $W \in L^2(0,T;V)$. We have $\dot{Z}(t) = \dot{W}(t) + \dot{\mathbf{u}}(t)$, combining $\dot{Z}(t) = Q(t)$ we thus obtain

$$\begin{cases} \dot{W} + A(\partial_x)W = 0, \\ W(0) = 0. \end{cases} \tag{3.2.107}$$

Since (3.2.107) has only one solution and 0 is a solution, we conclude $W = 0$, which implies that the times derivatives of $\mathbf{u}$ satisfies the same boundary conditions as $\mathbf{u}$, namely $\dot{\mathbf{u}}|_{\partial\Omega} = \mathbf{u}|_{\partial\Omega} = 0$ for a.e. $0 \le t \le T$. Using a same calculation as (3.2.76) on (3.2.105):

$$\begin{aligned}\frac{1}{2}\partial_t\big(T_0\|Q^0\|^2 + \frac{\epsilon^2}{4}\|\nabla Q^0\|^2 + \|Q^{\mathbf{d}}\|^2\big) + \nu_0 T_0\|\nabla Q^0\|^2 \\ + \frac{\epsilon^2}{4}\nu_0\|\triangle Q^0\|^2 + \nu_0\|\nabla Q^{\mathbf{d}}\|^2 + \frac{1}{\tau}\|Q^{\mathbf{d}}\|^2 \\ = T_0(\dot{F}^0, Q^0) + (\dot{F}^{\mathbf{d}}, Q^{\mathbf{d}}) - \frac{\epsilon^2}{4}(\dot{F}^0, \triangle Q^0).\end{aligned} \tag{3.2.108}$$

Define the energy

$$E(t) := \frac{1}{2}\left(T_0\|Q^0\|^2 + \frac{\epsilon^2}{4}\|\nabla Q^0\|^2 + \|Q^{\mathbf{d}}\|^2\right),$$

then by Hölder's inequality and Cauchy's inequality and the assumption that

$$\dot{F}^0 \in L^\infty(0,T;L^2(\Omega)), \quad \dot{F}^{\mathbf{d}} \in L^\infty(0,T;(H^{-1}(\Omega))^d)$$

we obtain the following type of differential inequality:

$$\frac{d}{dt}E(t) + C_1E(t) \le C_2, \quad \forall t \ge 0,$$

where $C_2 > 0$ depends only on

$$\|\dot{F}^0\|_{L^\infty(0,T;L^2(\Omega))} + \|\dot{F}^{\mathbf{d}}\|_{L^\infty(0,T;(H^{-1}(\Omega))^d)}.$$

Solving this differential inequality leads to

$$E(t) \le E(0)e^{-C_1t} + \frac{C_2}{C_1}, \quad \forall t \ge 0.$$

Since

$$\begin{aligned}E(0) &= \frac{1}{2}\left(T_0\|Q^0(0)\|^2 + \frac{\epsilon^2}{4}\|\nabla Q^0(0)\|^2 + \|Q^{\mathbf{d}}(0)\|^2\right) \\ &= \frac{1}{2}\left(T_0\|F^0(0,\cdot)\|^2 + \frac{\epsilon^2}{4}\|\nabla F^0(0,\cdot)\|^2 + \|F^{\mathbf{d}}(0,\cdot)\|^2\right),\end{aligned}$$

we conclude

$$\|\partial_t u^0\|_{L^\infty(0,T;H_0^1(\Omega))} + \|\partial_t u^{\mathbf{d}}\|_{L^\infty(0,T;L^2(\Omega))} \le C,$$

$C > 0$ depends only on $F(0,x)$ and $\dot{F}$.

From theorem 3.2.5 and 2.2.1 we have the estimate

$$\begin{aligned}
&\|u^0\|_{L^\infty(0,T;H^3(\Omega))} + |\eta|^{3/2}\|u^0\|_{L^\infty(0,T;L^2(\Omega))} + \|u^{\mathbf{d}}\|_{L^\infty(0,T;H^2(\Omega))} \\
&\quad + |\eta|\|u^{\mathbf{d}}\|_{L^\infty(0,T;L^2(\Omega))} \\
\le& c\big(\|F^0\|_{L^\infty(0,T;H^1(\Omega))} + |\eta|^{1/2}\|F^0\|_{L^\infty(0,T;L^2(\Omega))} + \|F^{\mathbf{d}}\|_{L^\infty(0,T;L^2(\Omega))} \\
&\quad + \|\partial_t u^0\|_{L^\infty(0,T;H_0^1(\Omega))} + |\eta|^{1/2}\|\partial_t u^0\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t u^{\mathbf{d}}\|_{L^\infty(0,T;L^2(\Omega))}\big) \\
\le& C,
\end{aligned}$$

$C > 0$ depends only on F, $F(0,x)$ and $\dot{F}$.

For $0 \le t' \le t'' \le T$, the Hölder estimates

$$\begin{aligned}
\|u^0(t',\cdot) - u^0(t'',\cdot)\|_{H^2(\Omega)} &\le \int_{t'}^{t''} \|\partial_t u^0\|_{H^2(\Omega)} dt \\
&\le |t' - t''|^{1/2}\|\partial_t u^0\|_{L^2(0,T;H^2(\Omega))}
\end{aligned}$$

holds. Fix a number β with $0 < \beta < \frac{1}{2}$ then by Sobolev's embedding theorem

$$\|u^{\mathbf{d}}(t') - u^{\mathbf{d}}(t'')\|_{C(\overline{\Omega})} \le C\|u^{\mathbf{d}}(t') - u^{\mathbf{d}}(t'')\|_{H^{2-\beta}(\Omega)}, \tag{3.2.109}$$

here let C denote the embedding constant. Interpolation yields

$$\begin{aligned}
&\|u^{\mathbf{d}}(t') - u^{\mathbf{d}}(t'')\|_{H^{2-\beta}(\Omega)} \\
&\le C\|u^{\mathbf{d}}(t') - u^{\mathbf{d}}(t'')\|_{L^2(\Omega)}^{\beta/2}\|u^{\mathbf{d}}(t') - u^{\mathbf{d}}(t'')\|_{H^2(\Omega)}^{(2-\beta)/2}.
\end{aligned} \tag{3.2.110}$$

Finally

$$\begin{aligned}
\|u^{\mathbf{d}}(t') - u^{\mathbf{d}}(t'')\|_{L^2(\Omega)} &\le \int_{t'}^{t''} \|\partial_t u^{\mathbf{d}}\|_{L^2(\Omega)} dt \\
&\le |t' - t''|\|\partial_t u^{\mathbf{d}}\|_{L^\infty(0,T;L^2(\Omega))}
\end{aligned} \tag{3.2.111}$$

together with (3.2.109) and (3.2.110) yields

$$\begin{aligned}
&\|u^{\mathbf{d}}(t') - u^{\mathbf{d}}(t'')\|_{C(\overline{\Omega})} \\
&\le C|t' - t''|^{\beta/2}\|\partial_t u^{\mathbf{d}}\|_{L^\infty(0,T;L^2(\Omega))}^{\beta/2}\|u^{\mathbf{d}}\|_{L^\infty(0,T;H^2(\Omega))}^{(2-\beta)/2}.
\end{aligned} \tag{3.2.112}$$

In like manner

$$\begin{aligned}
&\|\nabla u^0(t') - \nabla u^0(t'')\|_{C(\overline{\Omega})} \\
&\le C|t' - t''|^{\beta/2}\|\partial_t u^0\|_{L^\infty(0,T;H_0^1(\Omega))}^{\beta/2}\|u^0\|_{L^\infty(0,T;H^3(\Omega))}^{(2-\beta)/2}.
\end{aligned} \tag{3.2.113}$$

Then

$$\begin{cases} u^0 \in C([0,T], C^1(\overline{\Omega})), \\ u^{\mathbf{d}} \in C([0,T], C(\overline{\Omega})). \end{cases}$$

Thus we have completed the proof of the theorem. □

3.3 Local Existence and Uniqueness

3.3.1 The Main Result

Our conclusion is the following local existence result for the IBVP (3.0.1)-(3.0.3):

Theorem 3.3.1. *There is a number $t_* > 0$ such that the model of viscous quantum hydrodynamics, i.e. the IBVP* (3.0.1)-(3.0.3) *with the compatibility conditions* (3.0.4) *has a unique local solution on $[0, t_*)$ with*

$$
\begin{aligned}
n &\in L^\infty(0, t_*; H^3(\Omega)), & J &\in L^\infty(0, t_*; H^2(\Omega)),\\
\partial_t n &\in L^2(0, t_*; H^2(\Omega)), & \partial_t J &\in L^2(0, t_*; H^1(\Omega)),\\
(n, \nabla n, J) &\in C([0, t_*) \times \bar{\Omega}), & &\\
V &\in C(0, t_*; H^2(\Omega)), & &\\
\partial_t V &\in L^2(0, t_*; H^1(\Omega)). & &
\end{aligned}
$$

This solution persists as long as n stays positive and $(n, \nabla n, J)$ are bounded in $L^\infty(\Omega)$.

3.3.2 Proof of Theorem 3.3.1

Local Existence

To prove local existence in **Theorem 3.3.1** we will use the iteration method and compactness arguments. The main task is to construct a sequence of approximate solutions which is uniformly bounded in a certain Sobolev space in a fixed time interval. Compactness arguments then imply that there exists a limit which proves to be a local-in-time solution of (3.0.1). The first step is to linearize the system (3.0.1) around its initial state (n_0, J_0, V_0), where V_0 solves the Dirichlet problem

$$
\begin{cases}
\lambda^2 \triangle V_0 = n_0(x) - \mathcal{C}(x),\\
V_0(x)|_{\partial\Omega} = V_\Gamma.
\end{cases}
\tag{3.3.1}
$$

We study the equations for the perturbation $P := (P^0, P^{\mathbf{d}}) := (n - n_0, J - J_0)$, and first construct approximate solutions $P_k := (P_k^0, P_k^{\mathbf{d}})$ for $P_k^0 := n_k - n_0$, $P_k^{\mathbf{d}} := J_k - J_0$ $(k \geq 1)$ from a fixed-point procedure, which are expected to converge to a solution P of the perturbed problem as $k \to \infty$. For this, we shall derive uniform bounds in certain Sobolev spaces on a uniform time interval and apply standard compactness arguments. A further analysis will show that $(n, J) = (P^0 + n_0, P^{\mathbf{d}} + J_0)$ with $n > 0$ is the expected local (in time) solution of the original problem (3.0.1)(V is via (3.3.1) uniquely determinate). For given (P_{k-1}, V_{k-1})[2] we obtain the following linearized problems for P_k $(k \geq 1)$

$$\begin{cases} \partial_t P_k + A(\partial_x) P_k = F_{k-1}, \\ P_k(0,x) = 0, \\ P_k(t,x) = 0, \quad \text{on } \partial\Omega \text{ for a.e. } 0 \leq t \leq T, \end{cases} \tag{3.3.2}$$

where

$$F_{k-1} := \begin{pmatrix} 0 \\ S(P_{k-1}) \end{pmatrix} - A(\partial_x) \begin{pmatrix} n_0 \\ J_0 \end{pmatrix} \ (k \geq 2),$$

$$S(P_{k-1}) := \operatorname{div}\left(\frac{(P^{\mathbf{d}}_{k-1} + J_0) \otimes (P^{\mathbf{d}}_{k-1} + J_0)}{P^0_{k-1} + n_0} \right) - (P^0_{k-1} + n_0)\nabla V_{k-1}$$
$$+ \epsilon^2 \operatorname{div}\left(\left(\nabla\sqrt{P^0_{k-1} + n_0}\right) \otimes \left(\nabla\sqrt{P^0_{k-1} + n_0}\right)\right) \ (k \geq 2),$$

$$\lambda^2 \triangle V_{k-1} = P^0_{k-1} + n_0 - \mathcal{C}(x), \quad V_{k-1}(t,x)|_{\partial\Omega} = V_\Gamma(x).$$

In the next lemma we will show that for $k \geq 2$ and any time interval $[0,T]$, if P_{k-1} satisfies (3.2.101) then F_{k-1} satisfies (3.2.100).

Lemma 3.3.1. *Let $T > 0, k \geq 1$, if $P_{k-1} = (P^0_{k-1}, P^{\mathbf{d}}_{k-1})$ satisfies*

$$\begin{cases} P^0_{k-1} \in L^\infty(0,T; H^3(\Omega)) \cap C([0,T], C^1(\overline{\Omega})) \cap C([0,T], H^2(\Omega)), \\ P^{\mathbf{d}}_{k-1} \in L^\infty(0,T; H^2(\Omega)) \cap C([0,T], C(\overline{\Omega})) \cap C([0,T], H^1_0(\Omega)), \\ \partial_t P^0_{k-1} \in L^\infty(0,T; H^1_0(\Omega)) \cap C([0,T], L^2(\Omega)) \cap L^2(0,T; H^2(\Omega)), \\ \partial_t P^{\mathbf{d}}_{k-1} \in L^\infty(0,T; L^2(\Omega)) \cap C([0,T], L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega)), \end{cases} \tag{3.3.3}$$

and

$$P_{k-1}(0,x) = (P^0_{k-1}, P^{\mathbf{d}}_{k-1})(0,x) = 0, \tag{3.3.4}$$

then F_{k-1} satisfies

$$\begin{cases} F^0_{k-1} \in L^\infty(0,T; H^1(\Omega)), & F^{\mathbf{d}}_{k-1} \in L^\infty(0,T; (L^2(\Omega))^d); \\ \dot{F}^0_{k-1} \in L^\infty(0,T; L^2(\Omega)), & \dot{F}^{\mathbf{d}}_{k-1} \in L^\infty(0,T; (H^{-1}(\Omega))^d); \\ F^0_{k-1}(0,x) \in H^1_0(\Omega), & F^{\mathbf{d}}_{k-1}(0,x) \in L^2(\Omega). \end{cases} \tag{3.3.5}$$

[2]Actually it is sufficient to give P_{k-1} as the known function since (3.3.1) holds, where V_{k-1} can be considered as the unique solution of (3.3.1) for a given P^0_{k-1}.

Proof. Since $F^0_{k-1} = \nu_0 \triangle n_0 + \mathrm{div} J_0$ and thereby $\dot{F}^0_{k-1} = 0$, then from (3.0.4)

$$\begin{cases} F^0_{k-1} \in L^\infty(0,T;H^1(\Omega)), \\ \dot{F}^0_{k-1} \in L^\infty(0,T;L^2(\Omega)), \\ F^0_{k-1}(0,x) \in H^1_0(\Omega), \end{cases}$$

follows immediately. Next

$$F^{\mathbf{d}}_{k-1}(0,x) = S(P_{k-1})(0) - \frac{\epsilon^2}{4}\nabla\triangle n_0 + T_0 \nabla n_0 + \nu_0 \triangle J_0 - \frac{1}{\tau} J_0 \tag{3.3.6}$$

and

$$S(P_{k-1})(0) = \mathrm{div}\left(\frac{J_0 \otimes J_0}{n_0}\right) - n_0 \nabla V_0 + \epsilon^2 \mathrm{div}\left((\nabla\sqrt{n_0}) \otimes (\nabla\sqrt{n_0})\right). \tag{3.3.7}$$

Let $g \in (L^2(\Omega))^d$ and

$$\sup_{j\in\{1,\cdots,d\}} \|g_j\|_{L^2(\Omega)} = 1,$$

where g_j is the j-th component of g. Then for a.e. $0 \le t \le T$

$$\begin{aligned}
&(S(P_{k-1})(0), g) \\
&= \left(\mathrm{div}\left(\frac{J_0 \otimes J_0}{n_0}\right), g\right) - (n_0 \nabla V_0, g) + \epsilon^2 \left(\mathrm{div}\left((\nabla\sqrt{n_0}) \otimes (\nabla\sqrt{n_0})\right), g\right) \\
&= \sum_{l=1}^{d}\sum_{k=1}^{d}\Bigg(\left(\frac{(\partial_{x_k}(J_0)_l)(J_0)_k + (J_0)_l(\partial_{x_k}(J_0)_k)}{n_0}, g_l\right) \\
&\qquad - \left(\frac{(\partial_{x_k} n_0)(J_0)_l(J_0)_k}{n_0^2}, g_l\right)\Bigg) - \sum_{l=1}^{d}(n_0 \partial_{x_l} V_0, g_l) \\
&\quad + \frac{\epsilon^2}{4}\sum_{l=1}^{d}\sum_{k=1}^{d}\Bigg(\left(\frac{(\partial_{x_k}\partial_{x_l} n_0)(\partial_{x_k} n_0) + (\partial_{x_l} n_0)\left(\partial^2_{x_k} n_0\right)}{n_0}, g_l\right) \\
&\qquad - \left(\frac{(\partial_{x_k} n_0)(\partial_{x_l} n_0)(\partial_{x_k} n_0)}{n_0^2}, g_l\right)\Bigg),
\end{aligned}$$

thus

$$|(S(P_{k-1})(0), g)| \le K_1 + K_2 + K_3, \tag{3.3.8}$$

where

$$K_1 := \sum_{l=1}^{d}\sum_{k=1}^{d}\Bigg(\left\|n_0^{-1}\right\|_{L^\infty(\Omega)} \left\|\partial_{x_k}(J_0)_l\right\|_{L^4(\Omega)} \left\|(J_0)_k\right\|_{L^4(\Omega)} + \left\|n_0^{-1}\right\|_{L^\infty(\Omega)} \left\|\partial_{x_k}(J_0)_k\right\|_{L^4(\Omega)} \left\|(J_0)_l\right\|_{L^4(\Omega)} + \left\|n_0^{-2}\right\|_{L^\infty(\Omega)} \left\|\partial_{x_k} n_0\right\|_{L^6(\Omega)} \left\|(J_0)_l\right\|_{L^6(\Omega)} \left\|(J_0)_k\right\|_{L^6(\Omega)} \Bigg),$$

$$K_2 := \sum_{l=1}^{d} \left\|n_0\right\|_{L^4(\Omega)} \left\|\partial_{x_l} V_0\right\|_{L^4(\Omega)},$$

$$K_3 := \frac{\epsilon^2}{4}\sum_{l=1}^{d}\sum_{k=1}^{d}\Bigg(\left\|n_0^{-1}\right\|_{L^\infty(\Omega)} \left\|\partial_{x_k}\partial_{x_l} n_0\right\|_{L^4(\Omega)} \left\|\partial_{x_k} n_0\right\|_{L^4(\Omega)} + \left\|n_0^{-1}\right\|_{L^\infty(\Omega)} \left\|\partial_{x_k}^2 n_0\right\|_{L^4(\Omega)} \left\|\partial_{x_l} n_0\right\|_{L^4(\Omega)} + \left\|n_0^{-2}\right\|_{L^\infty(\Omega)} \left\|\partial_{x_k} n_0\right\|_{L^6(\Omega)} \left\|\partial_{x_l} n_0\right\|_{L^6(\Omega)} \left\|\partial_{x_k} n_0\right\|_{L^6(\Omega)} \Bigg).$$

Since Ω satisfies the cone condition under our assumptions then from the Sobolev embedding theorem the following embedding

$$H^1(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^4(\Omega) \tag{3.3.9}$$

holds. From (3.3.9) we estimate K_1 and K_3 as follows.

$$K_1 \le C\sum_{l=1}^{d}\sum_{k=1}^{d}\Bigg(\left\|n_0^{-1}\right\|_{L^\infty(\Omega)} \left\|\partial_{x_k}(J_0)_l\right\|_{H^1(\Omega)} \left\|(J_0)_k\right\|_{H^1(\Omega)} + \left\|n_0^{-1}\right\|_{L^\infty(\Omega)} \left\|\partial_{x_k}(J_0)_k\right\|_{H^1(\Omega)} \left\|(J_0)_l\right\|_{H^1(\Omega)} + \left\|n_0^{-2}\right\|_{L^\infty(\Omega)} \left\|\partial_{x_k} n_0\right\|_{H^1(\Omega)} \left\|(J_0)_l\right\|_{H^1(\Omega)} \left\|(J_0)_k\right\|_{H^1(\Omega)} \Bigg),$$

which yields

$$K_1 \le C\Bigg(\left\|n_0^{-1}\right\|_{L^\infty(\Omega)} \left\|J_0\right\|_{H^2(\Omega)} \left\|J_0\right\|_{H^1(\Omega)} + \left\|n_0^{-1}\right\|_{L^\infty(\Omega)} \left\|J_0\right\|_{H^2(\Omega)} \left\|J_0\right\|_{H^1(\Omega)} + \left\|n_0^{-2}\right\|_{L^\infty(\Omega)} \left\|n_0\right\|_{H^2(\Omega)} \left\|J_0\right\|_{H^1(\Omega)}^2 \Bigg) < \infty.$$

$$K_3 \le C \sum_{l=1}^{d} \sum_{k=1}^{d} \Bigg(\left\| n_0^{-1} \right\|_{L^\infty(\Omega)} \left\| \partial_{x_k} \partial_{x_l} n_0 \right\|_{H^1(\Omega)} \left\| \partial_{x_k} n_0 \right\|_{H^1(\Omega)} + \left\| n_0^{-1} \right\|_{L^\infty(\Omega)} \left\| \partial_{x_k}^2 n_0 \right\|_{H^1(\Omega)} \left\| \partial_{x_l} n_0 \right\|_{H^1(\Omega)} + \left\| n_0^{-2} \right\|_{L^\infty(\Omega)} \left\| \partial_{x_k} n_0 \right\|_{H^1(\Omega)} \left\| \partial_{x_l} n_0 \right\|_{H^1(\Omega)} \left\| \partial_{x_k} n_0 \right\|_{H^1(\Omega)} \Bigg),$$

from which we obtain

$$K_3 \le C \Bigg(\left\| n_0^{-1} \right\|_{L^\infty(\Omega)} \left\| n_0 \right\|_{H^3(\Omega)} \left\| n_0 \right\|_{H^2(\Omega)} + \left\| n_0^{-1} \right\|_{L^\infty(\Omega)} \left\| n_0 \right\|_{H^3(\Omega)} \left\| n_0 \right\|_{H^2(\Omega)} + \left\| n_0^{-2} \right\|_{L^\infty(\Omega)} \left\| n_0 \right\|_{H^1(\Omega)}^3 \Bigg) < \infty.$$

To estimate K_2 we go back to (3.3.1) then with (3.3.9) to find

$$\begin{aligned} K_2 &\le \sum_{l=1}^{d} \left\| n_0 \right\|_{H^1(\Omega)} \left\| \partial_{x_l} V_0 \right\|_{H^1(\Omega)} \\ &\le C \left\| n_0 \right\|_{H^1(\Omega)} \left\| V_0 \right\|_{H^2(\Omega)} \\ &\le C \left\| n_0 \right\|_{H^1(\Omega)} \Big(\left\| n_0 \right\|_{L^2(\Omega)} + \left\| \mathcal{C}(x) \right\|_{L^2(\Omega)} + \left\| V_\Gamma \right\|_{H^{3/2}(\Omega)} \Big) < \infty. \end{aligned} \tag{3.3.10}$$

(3.3.6), (3.3.7) and (3.3.8) together with the estimations of K_1, K_2, K_3 and our assumptions that $n_0 \in H^3(\Omega)$, $J_0 \in (H^2(\Omega))^d$ yield that

$$F_{k-1}^{\mathbf{d}}(0, x) \in L^2(\Omega).$$

In order to prove $F_{k-1}^{\mathbf{d}} \in L^\infty(0, T; (L^2(\Omega))^d)$, it is sufficient to show that

$$S(P_{k-1}) \in L^\infty(0, T; (L^2(\Omega))^d).$$

To see this we use our assumptions on P_{k-1} then find for a.e. $t \in [0, T]$

$$\begin{aligned} &\int_\Omega \left(\operatorname{div} \left(\frac{(P_{k-1}^{\mathbf{d}} + J_0) \otimes (P_{k-1}^{\mathbf{d}} + J_0)}{P_{k-1}^0 + n_0} \right) \right)^2 dx \\ &\le C \frac{\sup_{[0,T]\times\overline{\Omega}} (P_{k-1}^{\mathbf{d}} + J_0)^4}{\inf_{[0,T]\times\overline{\Omega}} (P_{k-1}^0 + n_0)^4} \| P_{k-1}^0 + n_0 \|_{H^1(\Omega)}^2 \\ &\quad + C \frac{\sup_{[0,T]\times\overline{\Omega}} (P_{k-1}^{\mathbf{d}} + J_0)^2}{\inf_{[0,T]\times\overline{\Omega}} (P_{k-1}^0 + n_0)^2} \| P_{k-1}^{\mathbf{d}} + J_0 \|_{H^1(\Omega)}^2 < \infty, \end{aligned}$$

from which it follows

$$\operatorname{div}\left(\frac{(P_{k-1}^{\mathbf{d}}+J_0)\otimes(P_{k-1}^{\mathbf{d}}+J_0)}{P_{k-1}^0+n_0}\right)\in L^\infty(0,T;(L^2(\Omega))^d). \tag{3.3.11}$$

Concerning the second term of $S(P_{k-1})$ we have

$$\begin{aligned}
&\int_\Omega \left((P_{k-1}^0+n_0)\nabla V_{k-1}\right)^2 dx = \int_\Omega (P_{k-1}^0+n_0)^2(\nabla V_{k-1})^2 dx\\
&\leq \sup_{[0,T]\times\overline{\Omega}} (P_{k-1}^0+n_0)^2\|V_{k-1}\|_{H^1(\Omega)}^2\\
&\leq C \sup_{[0,T]\times\overline{\Omega}} (P_{k-1}^0+n_0)^2\big(\|P_{k-1}^0\|_{L^2(\Omega)}+\|n_0\|_{L^2(\Omega)}\|\mathcal{C}(x)\|_{L^2(\Omega)}+\|V_\Gamma\|_{H^{3/2}(\Omega)}\big),
\end{aligned}$$

which yields

$$(P_{k-1}^0+n_0)\nabla V_{k-1}\in L^\infty(0,T;(L^2(\Omega))^d). \tag{3.3.12}$$

It remains to give a estimation of the third term of $S(P_{k-1})$. For this we find

$$\begin{aligned}
&\int_\Omega \left(\epsilon^2\operatorname{div}\left(\left(\nabla\sqrt{P_{k-1}^0+n_0}\right)\otimes\left(\nabla\sqrt{P_{k-1}^0+n_0}\right)\right)\right)^2 dx\\
&\leq C\frac{\sup\limits_{[0,T]\times\overline{\Omega}}(\nabla(P_{k-1}^0+n_0))^6}{\inf\limits_{[0,T]\times\overline{\Omega}}(P_{k-1}^0+n_0)^4}+C\frac{\sup\limits_{[0,T]\times\overline{\Omega}}(\nabla(P_{k-1}^0+n_0))^2}{\inf\limits_{[0,T]\times\overline{\Omega}}(P_{k-1}^0+n_0)^2}\|P_{k-1}^0+n_0\|_{H^2(\Omega)}^2,
\end{aligned}$$

which infers

$$\epsilon^2\operatorname{div}\left(\left(\nabla\sqrt{P_{k-1}^0+n_0}\right)\otimes\left(\nabla\sqrt{P_{k-1}^0+n_0}\right)\right)\in L^\infty(0,T;(L^2(\Omega))^d). \tag{3.3.13}$$

Then we obtain $S(P_{k-1})\in L^\infty(0,T;(L^2(\Omega))^d)$, therefore $F_{k-1}^{\mathbf{d}}\in L^\infty(0,T;(L^2(\Omega))^d)$.

Since $\dot F_{k-1}^{\mathbf{d}}=\dot S(P_{k-1})$, it is sufficient to show

$$\dot S(P_{k-1})\in L^\infty(0,T;(H^{-1}(\Omega))^d).$$

Let $g\in(H_0^1(\Omega))^d$ and $\sup_{j\in\{1,\cdots,d\}}\|g_j\|_{H_0^1(\Omega)}=1$, where g_j is the j-th component of g. Then for a.e. $0\leq t\leq T$

$$\begin{aligned}
(-\dot S(P_{k-1}),g)=&\left(\left(\frac{(P_{k-1}^{\mathbf{d}}+J_0)\otimes(P_{k-1}^{\mathbf{d}}+J_0)}{P_{k-1}^0+n_0}\right)',\nabla g\right)+\left(\left((P_{k-1}^0+n_0)\nabla V_{k-1}\right)',g\right)\\
&+\epsilon^2\left(\left(\left(\nabla\sqrt{P_{k-1}^0+n_0}\right)\otimes\left(\nabla\sqrt{P_{k-1}^0+n_0}\right)\right)',\nabla g\right).
\end{aligned}$$

Notice that in this equation we use the notation of (2.1.1). More precisely,

$$\begin{aligned}
&\left(\left(\frac{(P^{\mathbf{d}}_{k-1}+J_0)\otimes(P^{\mathbf{d}}_{k-1}+J_0)}{P^0_{k-1}+n_0}\right)', \nabla g\right)\\
&=\sum_{i,j=1,\cdots,d}\left(\left(\frac{(P^{\mathbf{d}}_{k-1}+J_0)_i(P^{\mathbf{d}}_{k-1}+J_0)_j}{P^0_{k-1}+n_0}\right)', \partial_{x_j} g_i\right)\\
&=\sum_{i,j=1,\cdots,d}\left(\left(\frac{(P^{\mathbf{d}}_{k-1})'_i(P^{\mathbf{d}}_{k-1}+J_0)_j}{P^0_{k-1}+n_0}, \partial_{x_j} g_i\right)+\left(\frac{(P^{\mathbf{d}}_{k-1}+J_0)_i(P^{\mathbf{d}}_{k-1})'_j}{P^0_{k-1}+n_0}, \partial_{x_j} g_i\right)\right.\\
&\qquad\left.-\left(\frac{(P^0_{k-1})'(P^{\mathbf{d}}_{k-1}+J_0)_i(P^{\mathbf{d}}_{k-1}+J_0)_j}{(P^0_{k-1}+n_0)^2}, \partial_{x_j} g_i\right)\right),
\end{aligned}$$

from which we infer

$$\begin{aligned}
&\left|\left(\left(\frac{(P^{\mathbf{d}}_{k-1}+J_0)\otimes(P^{\mathbf{d}}_{k-1}+J_0)}{P^0_{k-1}+n_0}\right)', \nabla g\right)\right|\\
&\le C\sum_{i,j=1,\cdots,d}\left(\left\|\frac{1}{P^0_{k-1}+n_0}\right\|_{L^\infty(\Omega)}\left\|(P^{\mathbf{d}}_{k-1})'_i\right\|_{L^2(\Omega)}\left\|(P^{\mathbf{d}}_{k-1}+J_0)_j\right\|_{L^\infty(\Omega)}\right.\\
&\qquad+\left\|\frac{1}{P^0_{k-1}+n_0}\right\|_{L^\infty(\Omega)}\left\|(P^{\mathbf{d}}_{k-1})'_j\right\|_{L^2(\Omega)}\left\|(P^{\mathbf{d}}_{k-1}+J_0)_i\right\|_{L^\infty(\Omega)}\\
&\qquad\left.+\left\|\frac{(P^{\mathbf{d}}_{k-1}+J_0)_i(P^{\mathbf{d}}_{k-1}+J_0)_j}{(P^0_{k-1}+n_0)^2}\right\|_{L^\infty(\Omega)}\left\|(P^0_{k-1})'\right\|_{L^2(\Omega)}\right)<\infty.
\end{aligned}\tag{3.3.14}$$

Next we deduce

$$\begin{aligned}
&\left(\left((P^0_{k-1}+n_0)\nabla V_{k-1}\right)', g\right)\\
&=\left(\left((P^0_{k-1}+n_0)'\nabla V_{k-1}\right), g\right)+\left(\left((P^0_{k-1}+n_0)\nabla\dot V_{k-1}\right), g\right)\\
&\le\|\dot P^0_{k-1}\|_{L^4(\Omega)}\|\nabla V_{k-1}\|_{L^4(\Omega)}+\|P^0_{k-1}+n_0\|_{L^\infty(\Omega)}\|\nabla\dot V_{k-1}\|_{L^2(\Omega)}\\
&\le C\|\dot P^0_{k-1}\|_{H^1(\Omega)}\|V_{k-1}\|_{H^2(\Omega)}\\
&\quad+C\sup_{[0,T]\times\overline\Omega}|P^0_{k-1}+n_0|\left(\sup_{[0,T]}\|\dot P^0_{k-1}\|_{L^2(\Omega)}\right)<\infty.
\end{aligned}\tag{3.3.15}$$

By a similar calculation,

$$\begin{aligned}
&\epsilon^2 \left(\left(\left(\nabla \sqrt{P^0_{k-1} + n_0} \right) \otimes \left(\nabla \sqrt{P^0_{k-1} + n_0} \right) \right)', \nabla g \right) \\
&= \frac{\epsilon^2}{4} \left(\left(\frac{\nabla (P^0_{k-1} + n_0) \otimes \nabla (P^0_{k-1} + n_0)}{P^0_{k-1} + n_0} \right)', \nabla g \right) \\
&= \frac{\epsilon^2}{4} \sum_{i,j=1,\cdots,d} \left(\left(\frac{(\nabla (P^0_{k-1} + n_0))_i (\nabla (P^0_{k-1} + n_0))_j}{P^0_{k-1} + n_0} \right)', \partial_j g_i \right),
\end{aligned}$$

which is equal to

$$\begin{aligned}
\frac{\epsilon^2}{4} \sum_{i,j=1,\cdots,d} \Bigg(&\left(\frac{(\nabla P^0_{k-1})'_i (\nabla (P^0_{k-1} + n_0))_j}{P^0_{k-1} + n_0}, \partial_j g_i \right) \\
&+ \left(\frac{(\nabla (P^0_{k-1} + n_0))_i (\nabla P^0_{k-1})'_j}{P^0_{k-1} + n_0}, \partial_j g_i \right) \\
&- \left(\frac{(P^0_{k-1})' (\nabla (P^0_{k-1} + n_0))_i (\nabla (P^0_{k-1} + n_0))_j}{(P^0_{k-1} + n_0)^2}, \partial_j g_i \right) \Bigg).
\end{aligned}$$

This yields

$$\begin{aligned}
&\left| \left(\left(\nabla \sqrt{P^0_{k-1} + n_0} \right) \otimes \left(\nabla \sqrt{P^0_{k-1} + n_0} \right) \right)', \nabla g \right| \\
&= \frac{1}{4} \left| \left(\left(\frac{(\nabla (P^0_{k-1} + n_0)) \otimes (\nabla (P^0_{k-1} + n_0))}{P^0_{k-1} + n_0} \right)', \nabla g \right) \right| \\
&\le C \sum_{i,j=1,\cdots,d} \Bigg(\left\| \frac{1}{P^0_{k-1} + n_0} \right\|_{L^\infty(\Omega)} \left\| (\nabla P^0_{k-1})'_i \right\|_{L^2(\Omega)} \left\| (\nabla (P^0_{k-1} + n_0))_j \right\|_{L^\infty(\Omega)} \\
&\quad + \left\| \frac{1}{P^0_{k-1} + n_0} \right\|_{L^\infty(\Omega)} \left\| (\nabla P^0_{k-1})'_j \right\|_{L^2(\Omega)} \left\| (\nabla (P^0_{k-1} + n_0))_i \right\|_{L^\infty(\Omega)} \\
&\quad + \left\| \frac{(\nabla (P^0_{k-1} + n_0))_i (\nabla (P^0_{k-1} + n_0))_j}{(P^0_{k-1} + n_0)^2} \right\|_{L^\infty(\Omega)} \left\| (P^0_{k-1})' \right\|_{L^2(\Omega)} \Bigg) \\
&< \infty.
\end{aligned} \tag{3.3.16}$$

Combining (3.3.14), (3.3.15) and (3.3.16) we conclude

$$\dot{F}^{\mathbf{d}}_{k-1} \in L^\infty(0, T; (H^{-1}(\Omega))^d).$$

□

Now select two positive numbers $\delta_0, M > 0$ such that

$$\delta_0 < \inf_{x\in\Omega} n_0 \tag{3.3.17}$$

$$\max\left(\|\nabla n_0\|_{L^\infty(\Omega)}, \|n_0\|_{L^\infty(\Omega)}, \|J_0\|_{L^\infty(\Omega)}\right) < \delta_0^{-1}, \tag{3.3.18}$$

and

$$\|n_0\|_{H^3(\Omega)} < M, \quad \|J_0\|_{H^2(\Omega)} < M. \tag{3.3.19}$$

Next we shall use iteration method to obtain a sequence of approximate solutions.

Lemma 3.3.2. *Let $k \geq 0$ and $P_0 = 0$, there exist $t' > 0$, $C_0 > 0$, $C^* > 0$ and $C' > 0$ independent of k such that in the interval $[0, t']$ P_k satisfies*

$$\begin{cases} P_k^0 \in L^\infty(0, t'; H^3(\Omega)) \cap C([0, t'], C^1(\overline{\Omega})) \cap C([0, t'], H^2(\Omega)), \\ P_k^{\mathbf{d}} \in L^\infty(0, t'; H^2(\Omega)) \cap C([0, t'], C(\overline{\Omega})) \cap C([0, t'], H_0^1(\Omega)), \\ \partial_t P_k^0 \in L^\infty(0, t'; H_0^1(\Omega)) \cap C([0, t'], L^2(\Omega)) \cap L^2(0, t'; H^2(\Omega)), \\ \partial_t P_k^{\mathbf{d}} \in L^\infty(0, t'; L^2(\Omega)) \cap C([0, t'], L^2(\Omega)) \cap L^2(0, t'; H_0^1(\Omega)), \end{cases} \tag{3.3.20}$$

and the following uniform bounds

$$\begin{cases} \|P_k^0\|_{L^\infty(0,t';H^3(\Omega))} \leq C', & \|P_k^{\mathbf{d}}\|_{L^\infty(0,t';H^2(\Omega))} \leq C', \\ \|P_k^0\|_{L^\infty(0,t';H^2(\Omega))} \leq C^*, & \|P_k^{\mathbf{d}}\|_{L^\infty(0,t';H_0^1(\Omega))} \leq C^*, \\ \|P_k^0\|_{L^\infty(0,t';H_0^1(\Omega))} \leq C_0, & \|P_k^{\mathbf{d}}\|_{L^\infty(0,t';L^2(\Omega))} \leq C_0, \\ \|\dot{P}_k^0\|_{L^\infty(0,t';H_0^1(\Omega))} \leq C_0, & \|\dot{P}_k^{\mathbf{d}}\|_{L^\infty(0,t';L^2(\Omega))} \leq C_0, \\ \|\dot{P}_k^0\|_{L^2(0,t';H^2(\Omega))} \leq C_0, & \|\dot{P}_k^{\mathbf{d}}\|_{L^2(0,t';H_0^1(\Omega))} \leq C_0, \end{cases} \tag{3.3.21}$$

and

$$\begin{cases} \inf_{[0,t']} \inf_{x\in\overline{\Omega}} (P_k^0 + n_0) > \delta_0, \\ \sup_{[0,t']} \max\Big(\|\nabla(P_k^0 + n_0)\|_{L^\infty(\Omega)}, \|P_k^0 + n_0\|_{L^\infty(\Omega)}, \\ \qquad \|P_k^{\mathbf{d}} + J_0\|_{L^\infty(\Omega)}\Big) < \delta_0^{-1}, \end{cases} \tag{3.3.22}$$

hold.

Remark 3.3.1. *Lemma 3.3.2 interprets a uniform bounds of P_k in given reflexive spaces, that guarantees the existence of a convergent subsequence of P_k in certain weak senses.*

Proof. Throughout the proof we will use C as a generic constant which may change from line to line and depend on δ_0, Ω and all physical constants, but is independent of k.

Obviously, for $P_0 = 0$, F_0 satisfies (3.2.100). Starting with $P_0 = 0$, by theorem 3.2.13 we obtain a solution P_1 of (3.3.2) on any time interval $[0, T]$, $T > 0$, i.e., P_1 solves

$$\begin{cases} \partial_t P_1 + A(\partial_x) P_1 = -\begin{pmatrix} 0 \\ S(P_0) \end{pmatrix} - A(\partial_x) \begin{pmatrix} n_0 \\ J_0 \end{pmatrix}, \\ \qquad P_1(0, x) = 0, \\ \qquad P_1(t, x) = 0, \quad \text{on } \partial\Omega \text{ for a.e. } 0 \le t \le T, \end{cases}$$

with

$$\begin{cases} S(P_0) = \operatorname{div}\left(\dfrac{J_0 \otimes J_0}{n_0}\right) - n_0 \nabla V_0 + \epsilon^2 \operatorname{div}\left((\nabla\sqrt{n_0}) \otimes (\nabla\sqrt{n_0})\right), \\ \lambda^2 \triangle V_0 = n_0(x) - \mathcal{C}(x), \\ V_0(x)|_{\partial\Omega} = V_\Gamma. \end{cases}$$

Since obviously $P_0 = 0$ satisfies (3.3.3) and (3.3.4) then F_0 satisfies(3.3.5). By Theorem 3.2.13 P_1 satisfies (3.2.101).Thus we can shrink the interval $[0, T]$ to $[0, t_1]$ such that on this interval

$$\delta_0 < \inf_{[0,t_1]} \inf_{x\in\Omega} n_1$$

and

$$\sup_{[0,t_1]} \max\left(\|\nabla n_1\|_{L^\infty(\Omega)}, \|n_1\|_{L^\infty(\Omega)}, \|J_1\|_{L^\infty(\Omega)}\right) < \delta_0^{-1}.$$

Then we use mathematical induction to complete the proof. Let $c^* > 0$, $T > 0$ assume $P_{k-1} (k \ge 1)$ satisfies

$$\begin{cases} P^0_{k-1} \in L^\infty(0, T; H^3(\Omega)) \cap C([0, T], C^1(\overline{\Omega})) \cap C([0, T], H^2(\Omega)), \\ P^{\mathbf{d}}_{k-1} \in L^\infty(0, T; H^2(\Omega)) \cap C([0, T], C(\overline{\Omega})) \cap C([0, T], H^1_0(\Omega)), \\ \dot{P}^0_{k-1} \in L^\infty(0, T; H^1_0(\Omega)) \cap C([0, T], L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)), \\ \dot{P}^{\mathbf{d}}_{k-1} \in L^\infty(0, T; L^2(\Omega)) \cap C([0, T], L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)), \end{cases} \tag{3.3.23}$$

$$P_{k-1}(0, x) = 0, \tag{3.3.24}$$

and

$$\begin{cases} \|P^0_{k-1}\|_{L^\infty(0,T;H^1_0(\Omega))} \le c^*, \qquad \|P^{\mathbf{d}}_{k-1}\|_{L^\infty(0,T;L^2(\Omega))} \le c^*, \\ \|\dot{P}^0_{k-1}\|_{L^\infty(0,T;H^1_0(\Omega))} \le c^*, \qquad \|\dot{P}^{\mathbf{d}}_{k-1}\|_{L^\infty(0,T;L^2(\Omega))} \le c^*, \end{cases} \tag{3.3.25}$$

$$\begin{cases} \inf_{[0,T]} \inf_{x\in\overline{\Omega}} (P^0_{k-1} + n_0) > \delta_0, \\ \sup_{[0,T]} \max \Big(\|\nabla(P^0_{k-1} + n_0)\|_{L^\infty(\Omega)}, \|P^0_{k-1} + n_0\|_{L^\infty(\Omega)}, \\ \qquad \|P^{\mathbf{d}}_{k-1} + J_0\|_{L^\infty(\Omega)} \Big) < \delta_0^{-1}. \end{cases} \tag{3.3.26}$$

From lemma 3.3.1 F_{k-1} satisfies (3.3.5). Then by theorem 3.2.13 we obtain P_k in $[0,T)$ as the solution of (3.3.2) which satisfies the regularity mentioned in (3.2.101) with $\mathbf{u}$ replaced by P_k. We expand (3.3.2) then obtain two equations for P^0_k and $P^{\mathbf{d}}_k$ for a.e. $t \in [0,T]$

$$\begin{cases} \partial_t P^0_k - \nu_0 \triangle P^0_k - \mathrm{div} P^{\mathbf{d}}_k = \nu_0 \triangle n_0 + \mathrm{div} J_0, \\ \partial_t P^{\mathbf{d}}_k - \nu_0 \triangle P^{\mathbf{d}}_k + \frac{1}{\tau} P^{\mathbf{d}}_k \\ \quad -T_0 \nabla P^0_k + \frac{\epsilon^2}{4} \nabla \triangle P^0_k = S(P_{k-1}) + \nu_0 \triangle J_0 - \frac{1}{\tau} J_0 + T_0 \nabla n_0 \\ \qquad - \frac{\epsilon^2}{4} \nabla \triangle n_0. \end{cases} \tag{3.3.27}$$

Setting $Q_k := \nu_0 \triangle n_0 + \mathrm{div} J_0$, $R_k := S(P_{k-1}) + \nu_0 \triangle J_0 - \frac{1}{\tau} J_0 + T_0 \nabla n_0 - \frac{\epsilon^2}{4} \nabla \triangle n_0$. Take the inner product between (3.3.27) and $(T_0 P^0_k, P^{\mathbf{d}}_k)$, then integrate by parts:

$$\frac{1}{2} \partial_t T_0 \|P^0_k\|^2 + \nu_0 T_0 \|\nabla P^0_k\|^2 - T_0 (\mathrm{div} P^{\mathbf{d}}_k, P^0_k) = T_0 (Q_k, P^0_k), \tag{3.3.28}$$

$$\begin{aligned} &\frac{1}{2} \partial_t \|P^{\mathbf{d}}_k\|^2 + \frac{1}{\tau} \|P^{\mathbf{d}}_k\|^2 + \nu_0 \|\nabla P^{\mathbf{d}}_k\|^2 \\ &\qquad + T_0 (P^0_k, \mathrm{div} P^{\mathbf{d}}_k) + \frac{\epsilon^2}{4} (\nabla \triangle P^0_k, P^{\mathbf{d}}_k) = (R_k, P^{\mathbf{d}}_k), \end{aligned} \tag{3.3.29}$$

where the following calculations

$$(Q_k, P^0_k) = -\nu_0 (\nabla n_0, \nabla P^0_k) + (\mathrm{div} J_0, P^0_k), \tag{3.3.30}$$

$$\begin{aligned} (R_k, P^{\mathbf{d}}_k) = &-\nu_0 (\nabla J_0, \nabla P^{\mathbf{d}}_k) - \frac{1}{\tau} (J_0, P^{\mathbf{d}}_k) - T_0 (n_0, \mathrm{div} P^{\mathbf{d}}_k) \\ &(S(P_{k-1}), P^{\mathbf{d}}_k) - \frac{\epsilon^2}{4} (\nabla \triangle n_0, P^{\mathbf{d}}_k). \end{aligned} \tag{3.3.31}$$

hold.

Taking the $L^2(\Omega)$ scalar product of the first equation in (3.3.27) with $-\frac{\epsilon^2}{4} \triangle P^0_k$, then for a.e. $t \in [0,T]$ we obtain

$$\frac{1}{2} \partial_t \frac{\epsilon^2}{4} \|\nabla P^0_k\|^2 + \frac{\epsilon^2}{4} \nu_0 \|\triangle P^0_k\|^2 - \frac{\epsilon^2}{4} (\nabla \triangle P^0_k, P^{\mathbf{d}}_k) = -\frac{\epsilon^2}{4} (Q_k, \triangle P^0_k) \tag{3.3.32}$$

since $\partial_t P_k^0 \in L^\infty(0,T;H_0^1(\Omega))$ from theorem 3.2.13.

Summing up (3.3.28), (3.3.29) and (3.3.32) yields

$$\begin{aligned}\frac{1}{2}\partial_t\Big(T_0\|P_k^0\|^2+\frac{\epsilon^2}{4}\|\nabla P_k^0\|^2+\|P_k^{\mathbf{d}}\|^2\Big)+\nu_0T_0\|\nabla P_k^0\|^2&\\ +\frac{\epsilon^2}{4}\nu_0\|\triangle P_k^0\|^2+\nu_0\|\nabla P_k^{\mathbf{d}}\|^2+\frac{1}{\tau}\|P_k^{\mathbf{d}}\|^2&\\ =T_0(Q_k,P_k^0)+(R_k,P_k^{\mathbf{d}})-\frac{\epsilon^2}{4}(Q_k,\triangle P_k^0),&\end{aligned}\tag{3.3.33}$$

for a.e. $t\in[0,T]$, here

$$\begin{aligned}&(R_k,P_k^{\mathbf{d}})\\ =&\ I_1+I_2+I_3+\left(\nu_0\triangle J_0-\frac{1}{\tau}J_0+T_0\nabla n_0-\frac{\epsilon^2}{4}\nabla\triangle n_0,P_k^{\mathbf{d}}\right),\end{aligned}\tag{3.3.34}$$

and we define

$$I_1:=-\left(\frac{(P_{k-1}^{\mathbf{d}}+J_0)\otimes(P_{k-1}^{\mathbf{d}}+J_0)}{P_{k-1}^0+n_0},\nabla P_k^{\mathbf{d}}\right),\tag{3.3.35}$$

$$I_2:=-\int_\Omega(P_{k-1}^0+n_0)\nabla V_{k-1}P_k^{\mathbf{d}}dx,\tag{3.3.36}$$

$$I_3:=-\epsilon^2\left(\left(\nabla\sqrt{P_{k-1}^0+n_0}\right)\otimes\left(\nabla\sqrt{P_{k-1}^0+n_0}\right),\nabla P_k^{\mathbf{d}}\right),\tag{3.3.37}$$

where I_1, I_3 are defined as (2.1.1). The items I_k can be estimated as follows.

First by Hölder's inequality

$$\begin{aligned}|I_1|=&\left|\sum_{l=1}^d\int_\Omega(P_{k-1}^0+n_0)^{-1}(P_{k-1}^{\mathbf{d}}+J_0)_l(P_{k-1}^{\mathbf{d}}+J_0)\nabla(P_k^{\mathbf{d}})_ldx\right|\\ =&\left|\sum_{l=1}^d\sum_{s=1}^d\int_\Omega(P_{k-1}^0+n_0)^{-1}(P_{k-1}^{\mathbf{d}}+J_0)_l(P_{k-1}^{\mathbf{d}}+J_0)_s\partial_{x_s}(P_k^{\mathbf{d}})_ldx\right|\\ \le&\sum_{l=1}^d\sum_{s=1}^d\left\|\frac{(P_{k-1}^{\mathbf{d}}+J_0)_l}{(P_{k-1}^0+n_0)}\right\|_{L^\infty(\Omega)}\left\|(P_{k-1}^{\mathbf{d}}+J_0)_s\right\|_{L^2(\Omega)}\left\|\partial_{x_s}(P_k^{\mathbf{d}})_l\right\|_{L^2(\Omega)}.\end{aligned}$$

We use the assumptions (3.3.25) and (3.3.26) and Cauchy's inequality then infer for a.e. $t\in[0,T]$

$$|I_1|\le d^2\delta_0^{-2}(c^*+M)\|\nabla P_k^{\mathbf{d}}\|_{L^2(\Omega)}\le\epsilon_1\|\nabla P_k^{\mathbf{d}}\|_{L^2(\Omega)}^2+\frac{d^4\delta_0^{-4}(c^*+M)^2}{4\epsilon_1},\tag{3.3.38}$$

here $\left\|\nabla P_k^{\mathbf{d}}\right\|_{L^2(\Omega)}$ is the norm:

$$\left\|\nabla P_k^{\mathbf{d}}\right\|_{L^2(\Omega)} = \sup_{s,l=1,\cdots,d} \left\|\partial_{x_s}(P_k^{\mathbf{d}})_l\right\|_{L^2(\Omega)}.$$

To estimate I_2, we have

$$|I_2| \le \left\|P_{k-1}^0 + n_0\right\|_{L^4(\Omega)} \|\nabla V_{k-1}\|_{L^4(\Omega)} \|P_k^{\mathbf{d}}\|_{L^2(\Omega)}. \tag{3.3.39}$$

Since $V_{k-1} \in H^2(\Omega)$ solves

$$\begin{cases} \lambda^2 \triangle V_{k-1} = P_{k-1}^0 + n_0 - \mathcal{C}(x), \\ \quad V_{k-1}\big| = V_\Gamma \in H^{3/2}(\partial\Omega), \end{cases} \tag{3.3.40}$$

we deduce for $t \in [0, T]$

$$\begin{aligned} &\|V_{k-1}(t,\cdot)\|_{H^2(\Omega)} \\ &\le C_{\lambda,\Omega}(\|P_{k-1}^0 + n_0 - \mathcal{C}(x)\|_{L^2(\Omega)} + \|V_\Gamma\|_{H^{3/2}(\partial\Omega)}), \\ &\le C_{\lambda,\Omega,c^*,M,\mathcal{C}(x),V_\Gamma}, \end{aligned} \tag{3.3.41}$$

where $C_{\lambda,\Omega}$ and $C_{\lambda,\Omega,c^*,M,\mathcal{C}(x),V_\Gamma}$ depend only on what are mentioned in the subscript respectively.

Then from $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and the assumption (3.3.25) we infer for a.e $t \in [0, T]$

$$\begin{aligned} |I_2| &\le C_{14}^2 \left\|P_{k-1}^0 + n_0\right\|_{H^1(\Omega)} \|V_{k-1}\|_{H^2(\Omega)} \|P_k^{\mathbf{d}}\|_{L^2(\Omega)} \\ &\le C_{14}^2 (c^* + M) C_{\lambda,\Omega,c^*,M,\mathcal{C}(x),V_\Gamma} \|P_k^{\mathbf{d}}\|_{L^2(\Omega)} \\ &\le \epsilon_2 \|P_k^{\mathbf{d}}\|_{L^2(\Omega)}^2 + \frac{C_{14}^4 (c^* + M)^2 C_{\lambda,\Omega,c^*,M,\mathcal{C}(x),V_\Gamma}^2}{4\epsilon_2}, \end{aligned} \tag{3.3.42}$$

here let C_{14} denote the imbedding constant of $H^1(\Omega) \hookrightarrow L^4(\Omega)$.

As for I_3 we first find for a.e. $t \in [0, T]$ that

$$\begin{aligned} |I_3| \le\ & \epsilon^2 \sum_{l=1}^d \sum_{s=1}^d \int_\Omega \left| \left(\partial_{x_l} \sqrt{P_{k-1}^0 + n_0}\right) \left(\partial_{x_s} \sqrt{P_{k-1}^0 + n_0}\right) \partial_{x_s} \left(P_k^{\mathbf{d}}\right)_l \right| dx \\ \le\ & \epsilon^2 \sum_{l=1}^d \sum_{s=1}^d \left\| \left(\partial_{x_l} \sqrt{P_{k-1}^0 + n_0}\right) \right\|_{L^\infty(\Omega)} \left\| \left(\partial_{x_s} \sqrt{P_{k-1}^0 + n_0}\right) \right\|_{L^2(\Omega)} \times \\ & \times \left\| \partial_{x_s} \left(P_k^{\mathbf{d}}\right)_l \right\|_{L^2(\Omega)} \\ \le\ & \frac{\epsilon^2}{4} \sum_{l=1}^d \sum_{s=1}^d \left\| \frac{\partial_{x_l}(P_{k-1}^0 + n_0)}{\sqrt{P_{k-1}^0 + n_0}} \right\|_{L^\infty(\Omega)} \left\| \frac{\partial_{x_s}(P_{k-1}^0 + n_0)}{\sqrt{P_{k-1}^0 + n_0}} \right\|_{L^2(\Omega)} \left\| \partial_{x_s} \left(P_k^{\mathbf{d}}\right)_l \right\|_{L^2(\Omega)}. \end{aligned}$$

Using the assumptions 3.3.25 and (3.3.26) again we obtain

$$\begin{aligned} |I_3| &\leq \frac{\epsilon^2}{4} d^2 \delta_0^{-2} (c^* + M) \|\nabla P_k^{\mathbf{d}}\|_{L^2(\Omega)} \\ &\leq \epsilon_3 \|\nabla P_k^{\mathbf{d}}\|_{L^2(\Omega)}^2 + \frac{\frac{\epsilon^4}{16} d^4 \delta_0^{-4} (c^* + M)^2}{4\epsilon_3}. \end{aligned} \tag{3.3.43}$$

In the following, we will prove uniform in k estimates of P_k, which will imply that the life span can not tend to zero for k going to infinity. Then we will have a uniform existence interval as well as uniform estimates and can prove the convergence of a subsequence P_{k_i} by compactness argument.

Recall (3.3.30), we have

$$\begin{aligned} & T_0 |(Q_k, P_k^0)| \\ \leq\; & T_0 \nu_0 \|\nabla n_0\|_{L^2(\Omega)} \|\nabla P_k^0\|_{L^2(\Omega)} + T_0 \|J_0\|_{L^2(\Omega)} \|\nabla P_k^0\|_{L^2(\Omega)} \\ \leq\; & T_0 M (\nu_0 + 1) \|\nabla P_k^0\|_{L^2(\Omega)} \\ \leq\; & \epsilon_4 \|\nabla P_k^0\|_{L^2(\Omega)}^2 + \frac{T_0^2 M^2 (\nu_0 + 1)^2}{4\epsilon_4}. \end{aligned} \tag{3.3.44}$$

Recall (3.3.34):

$$\begin{aligned} |(R_k, P_k^{\mathbf{d}})| \leq \nu_0 |(\triangle J_0, P_k^{\mathbf{d}})| + \frac{1}{\tau} |(J_0, P_k^{\mathbf{d}})| + T_0 |(\nabla n_0, P_k^{\mathbf{d}})| \\ + \frac{\epsilon^2}{4} |(\nabla \triangle n_0, P_k^{\mathbf{d}})| + |I_1| + |I_2| + |I_3|. \end{aligned}$$

Using Hölder's inequality and (3.3.19)

$$\begin{aligned} |(R_k, P_k^{\mathbf{d}})| \leq & \nu_0 \|\triangle J_0\|_{L^2(\Omega)} \|P_k^{\mathbf{d}}\|_{L^2(\Omega)} + \frac{1}{\tau} \|J_0\|_{L^2(\Omega)} \|P_k^{\mathbf{d}}\|_{L^2(\Omega)} \\ & + T_0 \|\nabla n_0\|_{L^2(\Omega)} \|P_k^{\mathbf{d}}\|_{L^2(\Omega)} + \frac{\epsilon^2}{4} \|\nabla \triangle n_0\|_{L^2(\Omega)} \|P_k^{\mathbf{d}}\|_{L^2(\Omega)} \\ & + |I_1| + |I_2| + |I_3| \\ \leq & \left(\nu_0 + \frac{1}{\tau} + T_0 + \frac{\epsilon^2}{4} \right) M \|P_k^{\mathbf{d}}\|_{L^2(\Omega)} + |I_1| + |I_2| + |I_3|. \end{aligned}$$

Then from (3.3.38), (3.3.42), (3.3.43) and Cauchy's inequality we conclude

$$\begin{aligned}|(R_k,P_k^{\mathbf{d}})|\le&\epsilon_5\|P_k^{\mathbf{d}}\|^2_{L^2(\Omega)}+\frac{\left(\nu_0+\frac{1}{\tau}+T_0+\frac{\epsilon^2}{4}\right)^2M^2}{4\epsilon_5}\\&+\epsilon_1\|\nabla P_k^{\mathbf{d}}\|^2_{L^2(\Omega)}+\frac{d^4\delta_0^{-4}(c^*+M)^2}{4\epsilon_1}\\&+\epsilon_2\|P_k^{\mathbf{d}}\|^2_{L^2(\Omega)}+\frac{(c^*+M)^2C^2_{\lambda,\Omega,c^*,M,\mathcal{C}(x),V_\Gamma}}{4\epsilon_2}\\&+\epsilon_3\|\nabla P_k^{\mathbf{d}}\|^2_{L^2(\Omega)}+\frac{\frac{\epsilon^4}{16}d^4\delta_0^{-4}(c^*+M)^2}{4\epsilon_3}.\end{aligned}\tag{3.3.45}$$

It is easy to verify

$$\begin{aligned}&\frac{\epsilon^2}{4}|(Q_k,\triangle P_k^0)|\le\frac{\epsilon^2}{4}\nu_0|(\triangle n_0,\triangle P_k^0)|+\frac{\epsilon^2}{4}|(\mathrm{div}J_0,\triangle P_k^0)|\\&\le\frac{\epsilon^2}{4}\nu_0\|\triangle n_0\|_{L^2(\Omega)}\|\triangle P_k^0\|_{L^2(\Omega)}+\frac{\epsilon^2}{4}\|\mathrm{div}J_0\|_{L^2(\Omega)}\|\triangle P_k^0\|_{L^2(\Omega)}.\end{aligned}\tag{3.3.46}$$

By (3.3.19):

$$\begin{aligned}\frac{\epsilon^2}{4}|(Q_k,\triangle P_k^0)|\le&\frac{\epsilon^2}{4}M(\nu_0+1)\|\triangle P_k^0\|_{L^2(\Omega)}\\&\le\epsilon_6\|\triangle P_k^0\|^2_{L^2(\Omega)}+\frac{\frac{\epsilon^4}{16}M^2(\nu_0+1)^2}{4\epsilon_6}.\end{aligned}\tag{3.3.47}$$

Combining (3.3.33), (3.3.44), (3.3.45) and (3.3.47) we obtain the following inequality

$$\begin{aligned}&\frac{1}{2}\partial_t\big(T_0\|P_k^0\|^2+\frac{\epsilon^2}{4}\|\nabla P_k^0\|^2+\|P_k^{\mathbf{d}}\|^2\big)+\nu_0T_0\|\nabla P_k^0\|^2\\&\qquad+\frac{\epsilon^2}{4}\nu_0\|\triangle P_k^0\|^2+\nu_0\|\nabla P_k^{\mathbf{d}}\|^2+\frac{1}{\tau}\|P_k^{\mathbf{d}}\|^2\\&\quad\le\epsilon_4\|\nabla P_k^0\|^2+(\epsilon_2+\epsilon_5)\|P_k^{\mathbf{d}}\|^2+(\epsilon_1+\epsilon_3)\|\nabla P_k^{\mathbf{d}}\|^2\\&\qquad+\epsilon_6\|\triangle P_k^0\|^2+C_{\epsilon_1,\cdots,\epsilon_6,\nu_0,T_0,\tau,\epsilon,\lambda,V_\Gamma,\mathcal{C}(x),\Omega,\delta_0,M,c^*,C_{14}},\end{aligned}\tag{3.3.48}$$

$C_{\epsilon_1,\cdots,\epsilon_6,\nu_0,T_0,\tau,\epsilon,\lambda,V_\Gamma,\mathcal{C}(x),\Omega,\delta_0,M,c^*}>0$ depends only upon the constants that occur in the subscript. Notice that $C_{\epsilon_1,\cdots,\epsilon_6,\nu_0,T_0,\tau,\epsilon,\lambda,V_\Gamma,\mathcal{C}(x),\Omega,\delta_0,M,c^*}>0$ doesn't depend on k.

Select ϵ_i, $i=1,\cdots,6$ sufficiently small such that

$$\epsilon_4<\nu_0T_0,\quad(\epsilon_2+\epsilon_5)<\frac{1}{\tau},\quad(\epsilon_1+\epsilon_3)<\nu_0,\quad\epsilon_6<\frac{\epsilon^2}{4}\nu_0.$$

Then we obtain

$$\begin{aligned}&\partial_t\big(T_0\|P_k^0\|^2+\frac{\epsilon^2}{4}\|\nabla P_k^0\|^2+\|P_k^{\mathbf{d}}\|^2\big)\\&+\mathcal{C}_1\left(T_0\|P_k^0\|^2+\frac{\epsilon^2}{4}\|\nabla P_k^0\|^2+\|\triangle P_k^0\|^2+\|\nabla P_k^{\mathbf{d}}\|^2+\|P_k^{\mathbf{d}}\|^2\right)\le\mathcal{C}_2,\end{aligned}\tag{3.3.49}$$

where $\mathcal{C}_1>0$ is independent of c^*, $\mathcal{C}_2$ depends on all physical known quantities, the imbedding constants and c^*, δ_0 and M. Via solving the differential inequality (3.3.49) we infer the following estimation.

$$\begin{aligned}&T_0\|P_k^0\|^2_{L^\infty(0,T;L^2(\Omega))}+\frac{\epsilon^2}{4}\|\nabla P_k^0\|^2_{L^\infty(0,T;L^2(\Omega))}+\|P_k^{\mathbf{d}}\|^2_{L^\infty(0,T;L^2(\Omega))}\\&\le\mathcal{C}_2\int_0^T e^{\mathcal{C}_1 s}ds.\end{aligned}\tag{3.3.50}$$

Integrating (3.3.49) from 0 to T yields

$$\|P_k^0\|^2_{L^2(0,T;H^2(\Omega))}+\|P_k^{\mathbf{d}}\|^2_{L^2(0,T;H^1(\Omega))}\le\mathcal{C}_3T,\tag{3.3.51}$$

$\mathcal{C}_3>0$ depends on all physical known quantities and c^*, δ_0 and M.

Next we shall estimate $\dot{P}_k^0$ and $\dot{P}_k^{\mathbf{d}}$. Since P_{k-1} satisfies the assumptions (3.3.23) and (3.3.24), then from **Lemma** 3.3.1 F_{k-1} satisfies (3.3.5), thus the time derivatives of $(P_k^0,P_k^{\mathbf{d}})$ satisfy the same boundary conditions as $(P_k^0,P_k^{\mathbf{d}})$ from **Theorem** 3.2.13. Differentiating formally (3.3.27) with respect to $0\le t\le T$, then we obtain for a.e. $0\le t\le T$

$$\begin{cases}\partial_t\dot{P}_k^0-\nu_0\triangle\dot{P}_k^0-\mathrm{div}\dot{P}_k^{\mathbf{d}}=0,\\ \partial_t\dot{P}_k^{\mathbf{d}}-\nu_0\triangle\dot{P}_k^{\mathbf{d}}+\frac{1}{\tau}\dot{P}_k^{\mathbf{d}}-T_0\nabla\dot{P}_k^0+\frac{\epsilon^2}{4}\nabla\triangle\dot{P}_k^0=-\dot{S}(P_{k-1}).\end{cases}\tag{3.3.52}$$

Notice that $-\int_\Omega\partial_t\dot{P}_k^0\triangle\dot{P}_k^0dx=\frac{1}{2}\partial_t\|\nabla\dot{P}_k^0\|^2$ by **Theorem** 3.2.11. Then we deduce, by a similar computation as before(see (3.3.33)), that

$$\begin{aligned}&\frac{1}{2}\partial_t\big(T_0\|\dot{P}_k^0\|^2+\frac{\epsilon^2}{4}\|\nabla\dot{P}_k^0\|^2+\|\dot{P}_k^{\mathbf{d}}\|^2\big)+\nu_0T_0\|\nabla\dot{P}_k^0\|^2\\&\qquad+\frac{\epsilon^2}{4}\nu_0\|\triangle\dot{P}_k^0\|^2+\nu_0\|\nabla\dot{P}_k^{\mathbf{d}}\|^2+\frac{1}{\tau}\|\dot{P}_k^{\mathbf{d}}\|^2\\&=(\dot{S}(P_{k-1}),\dot{P}_k^{\mathbf{d}}).\end{aligned}\tag{3.3.53}$$

The scalar product $-(\dot{S}(P_{k-1}),\dot{P}_k^{\mathbf{d}})$ has the representation

$$-(\dot{S}(P_{k-1}),\dot{P}_k^{\mathbf{d}})=I_1'+I_2'+I_3',\tag{3.3.54}$$

where

$$I_1' := \left(\left(\frac{(P_{k-1}^{\mathbf{d}} + J_0) \otimes (P_{k-1}^{\mathbf{d}} + J_0)}{P_{k-1}^0 + n_0} \right)', \nabla \dot{P}_k^{\mathbf{d}} \right), \tag{3.3.55}$$

$$I_2' := \left(\left((P_{k-1}^0 + n_0) \nabla V_{k-1} \right)', \dot{P}_k^{\mathbf{d}} \right), \tag{3.3.56}$$

$$I_3' := \epsilon^2 \left(\left((\nabla \sqrt{P_{k-1}^0 + n_0}) \otimes (\nabla \sqrt{P_{k-1}^0 + n_0}) \right)', \nabla \dot{P}_k^{\mathbf{d}} \right). \tag{3.3.57}$$

Concerning I_k', $k = 1, 3$ we use the same notation as I_k, $k = 1, 3$ which are defined in (2.1.1).

First:

$$\begin{aligned} I_1' &= \sum_{i,j=1,\cdots,d} \left(\left(\frac{(P_{k-1}^{\mathbf{d}} + J_0)_i (P_{k-1}^{\mathbf{d}} + J_0)_j}{P_{k-1}^0 + n_0} \right)', \partial_{x_j} \left(\dot{P}_k^{\mathbf{d}} \right)_i \right) \\ &= Z_1 + Z_2 + Z_3, \end{aligned}$$

where we define

$$Z_1 := \sum_{i,j=1,\cdots,d} \left(\frac{(P_{k-1}^{\mathbf{d}})_i' (P_{k-1}^{\mathbf{d}} + J_0)_j}{P_{k-1}^0 + n_0}, \partial_{x_j} \left(\dot{P}_k^{\mathbf{d}} \right)_i \right), \tag{3.3.58}$$

$$Z_2 := \sum_{i,j=1,\cdots,d} \left(\frac{(P_{k-1}^{\mathbf{d}} + J_0)_i (P_{k-1}^{\mathbf{d}})_j'}{P_{k-1}^0 + n_0}, \partial_{x_j} \left(\dot{P}_k^{\mathbf{d}} \right)_i \right), \tag{3.3.59}$$

$$Z_3 := \sum_{i,j=1,\cdots,d} - \left(\frac{(P_{k-1}^0)' (P_{k-1}^{\mathbf{d}} + J_0)_i (P_{k-1}^{\mathbf{d}} + J_0)_j}{(P_{k-1}^0 + n_0)^2}, \partial_{x_j} \left(\dot{P}_k^{\mathbf{d}} \right)_i \right). \tag{3.3.60}$$

Under the assumptions (3.3.25) and (3.3.26) the items $Z_l, l = 1, 2, 3$ can be estimated as follows.

$$\begin{aligned} |Z_1| &\le \sum_{i,j=1,\cdots,d} \left\| \frac{(P_{k-1}^{\mathbf{d}} + J_0)_j}{P_{k-1}^0 + n_0} \right\|_\infty \left\| (P_{k-1}^{\mathbf{d}})_i' \right\|_{L^2(\Omega)} \left\| \partial_{x_j} \left(\dot{P}_k^{\mathbf{d}} \right)_i \right\|_{L^2(\Omega)} \\ &\le \sum_{i,j=1,\cdots,d} \delta_0^{-2} c^* \left\| \nabla \dot{P}_k^{\mathbf{d}} \right\|_{L^2(\Omega)} \le d^2 \delta_0^{-2} c^* \left\| \nabla \dot{P}_k^{\mathbf{d}} \right\|_{L^2(\Omega)}. \end{aligned}$$

Then by Cauchy's inequality

$$|Z_1| \le \epsilon_1' \left\| \nabla \dot{P}_k^{\mathbf{d}} \right\|_{L^2(\Omega)}^2 + \frac{d^4 \delta_0^{-4} (c^*)^2}{4 \epsilon_1'}. \tag{3.3.61}$$

Furthermore

$$|Z_2| \le \sum_{i,j=1,\cdots,d} \left\| \frac{(P^{\mathbf{d}}_{k-1}+J_0)_i}{P^0_{k-1}+n_0} \right\|_\infty \left\| (P^{\mathbf{d}}_{k-1})'_j \right\|_{L^2(\Omega)} \left\| \partial_{x_j} \left(\dot{P}^{\mathbf{d}}_k \right)_i \right\|_{L^2(\Omega)}$$
$$\le \sum_{i,j=1,\cdots,d} \delta_0^{-2} c^* \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|_{L^2(\Omega)} \le d^2 \delta_0^{-2} c^* \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|_{L^2(\Omega)}.$$

Cauchy's inequality yields

$$|Z_2| \le \epsilon'_2 \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|^2_{L^2(\Omega)} + \frac{d^4 \delta_0^{-4} (c^*)^2}{4\epsilon'_2}. \tag{3.3.62}$$

Finally

$$|Z_3| \le \sum_{i,j=1,\cdots,d} \left\| \frac{(P^{\mathbf{d}}_{k-1}+J_0)_i (P^{\mathbf{d}}_{k-1}+J_0)_j}{(P^0_{k-1}+n_0)^2} \right\|_{L^\infty(\Omega)} \left\| (P^0_{k-1})' \right\|_{L^2(\Omega)} \times$$
$$\times \left\| \partial_{x_j} \left(\dot{P}^{\mathbf{d}}_k \right)_i \right\|_{L^2(\Omega)}$$
$$\le \sum_{i,j=1,\cdots,d} \delta_0^{-4} c^* \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|_{L^2(\Omega)} \le d^2 \delta_0^{-4} c^* \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|_{L^2(\Omega)}.$$

Using Cauchy's inequality

$$|Z_3| \le \epsilon'_3 \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|^2_{L^2(\Omega)} + \frac{d^4 \delta_0^{-8} (c^*)^2}{4\epsilon'_3}. \tag{3.3.63}$$

Combining (3.3.61), (3.3.62) and (3.3.63) we obtain

$$|I'_1| \le (\epsilon'_1 + \epsilon'_2 + \epsilon'_3) \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|^2_{L^2(\Omega)} + C_{\epsilon'_1,\epsilon'_2,\epsilon'_3,\Omega,\delta_0,c^*}, \tag{3.3.64}$$

where the constant $C_{\epsilon'_1,\epsilon'_2,\epsilon'_3,\Omega,\delta_0,c^*} > 0$ depends only on $\epsilon'_1, \epsilon'_2, \epsilon'_3, \Omega, \delta_0, c^*$.

In order to estimate I'_2, we find for a.e. $t \in [0,T]$

$$|I'_2| \le |(\dot{P}^0_{k-1} \nabla V_{k-1}, \dot{P}^{\mathbf{d}}_k)| + |((P^0_{k-1}+n_0) \nabla \dot{V}_{k-1}, \dot{P}^{\mathbf{d}}_k)|$$
$$\le \|\dot{P}^{\mathbf{d}}_k\|_{L^2(\Omega)} \|\nabla V_{k-1}\|_{L^4(\Omega)} \|\dot{P}^0_{k-1}\|_{L^4(\Omega)}$$
$$+ \|\dot{P}^{\mathbf{d}}_k\|_{L^2(\Omega)} \|P^0_{k-1}+n_0\|_{L^4(\Omega)} \|\nabla \dot{V}_{k-1}\|_{L^4(\Omega)}.$$

Notice that $\dot{V}_{k-1} \in H^2(\Omega) \cap H^1_0(\Omega)$ solves

$$\lambda^2 \triangle \dot{V}_{k-1} = \dot{P}^0_{k-1},$$

then

$$\|\dot{V}_{k-1}\|_{H^2(\Omega)} \le C_{\Omega,\lambda} \|\dot{P}^0_{k-1}\|_{L^2(\Omega)}.$$

Recall (3.3.40), (3.3.41) and use the imbedding

$$H^1(\Omega) \hookrightarrow L^4(\Omega) \tag{3.3.65}$$

where let C_{14} denote the imbedding constant of (3.3.65), then under the assumptions (3.3.25) we find

$$\begin{aligned}
|I_2'| \leq & C_{14}^2 \|\dot{P}_k^{\mathbf{d}}\|_{L^2(\Omega)} \|V_{k-1}\|_{H^2(\Omega)} \|\dot{P}_{k-1}^0\|_{H^1(\Omega)} \\
& + C_{14}^2 \|\dot{P}_k^{\mathbf{d}}\|_{L^2(\Omega)} \|P_{k-1}^0 + n_0\|_{H^1(\Omega)} \|\dot{V}_{k-1}\|_{H^2(\Omega)} \\
\leq & c^* C_{14}^2 C_{\lambda,\Omega,c^*,M,\mathcal{C}(x),V_\Gamma} \|\dot{P}_k^{\mathbf{d}}\|_{L^2(\Omega)} \\
& + c^*(c^* + M) C_{14}^2 C_{\Omega,\lambda} \|\dot{P}_k^{\mathbf{d}}\|_{L^2(\Omega)},
\end{aligned}$$

which implies

$$\begin{aligned}
|I_2'| \leq & \epsilon_4' \|\dot{P}_k^{\mathbf{d}}\|_{L^2(\Omega)}^2 + \frac{(c^*)^2 C_{14}^4 C_{\lambda,\Omega,c^*,M,\mathcal{C}(x),V_\Gamma}^2}{4\epsilon_4'} \\
& + \epsilon_5' \|\dot{P}_k^{\mathbf{d}}\|_{L^2(\Omega)}^2 + \frac{(c^*(c^* + M))^2 C_{14}^4 C_{\Omega,\lambda}^2}{4\epsilon_5'} \\
\leq & (\epsilon_4' + \epsilon_5') \|\dot{P}_k^{\mathbf{d}}\|_{L^2(\Omega)}^2 + C_{\epsilon_4',\epsilon_5',\lambda,\Omega,c^*,M,\mathcal{C}(x),V_\Gamma,C_{14}}.
\end{aligned} \tag{3.3.66}$$

Now we consider I_3'. By a similar computation as I_1' we obtain

$$\begin{aligned}
I_3' = & \frac{\epsilon^2}{4} \sum_{i,j=1,\cdots,d} \left(\left(\frac{\partial_i (P_{k-1}^0 + n_0) \partial_j (P_{k-1}^0 + n_0)}{P_{k-1}^0 + n_0} \right)', \partial_{x_j} \left(\dot{P}_k^{\mathbf{d}} \right)_i \right) \\
= & \frac{\epsilon^2}{4} (\mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Z}_3),
\end{aligned}$$

where we define

$$\mathcal{Z}_1 := \sum_{i,j=1,\cdots,d} \left(\frac{(\partial_{x_i} P_{k-1}^0)' \partial_{x_j} (P_{k-1}^0 + n_0)}{P_{k-1}^0 + n_0}, \partial_{x_j} \left(\dot{P}_k^{\mathbf{d}} \right)_i \right), \tag{3.3.67}$$

$$\mathcal{Z}_2 := \sum_{i,j=1,\cdots,d} \left(\frac{\partial_{x_i} (P_{k-1}^0 + n_0) (\partial_{x_j} P_{k-1}^0)'}{P_{k-1}^0 + n_0}, \partial_{x_j} \left(\dot{P}_k^{\mathbf{d}} \right)_i \right), \tag{3.3.68}$$

$$\mathcal{Z}_3 := - \sum_{i,j=1,\cdots,d} \left(\frac{(P_{k-1}^0)' \partial_{x_i} (P_{k-1}^0 + n_0) \partial_{x_j} (P_{k-1}^0 + n_0)}{(P_{k-1}^0 + n_0)^2}, \partial_{x_j} \left(\dot{P}_k^{\mathbf{d}} \right)_i \right). \tag{3.3.69}$$

Under the assumptions (3.3.25) and (3.3.26) the items $\mathcal{Z}_l, l = 1, 2, 3$ can be estimated as follows.

$$
\begin{aligned}
|\mathcal{Z}_1| &\le \sum_{i,j=1,\cdots,d} \left\| \frac{\partial_{x_j}(P^0_{k-1}+n_0)}{P^0_{k-1}+n_0} \right\|_\infty \left\| \partial_{x_i} \dot{P}^0_{k-1} \right\|_{L^2(\Omega)} \left\| \partial_{x_j} \left(\dot{P}^{\mathbf{d}}_k \right)_i \right\|_{L^2(\Omega)} \\
&\le \sum_{i,j=1,\cdots,d} \delta_0^{-2} c^* \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|_{L^2(\Omega)} \le d^2 \delta_0^{-2} c^* \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|_{L^2(\Omega)}.
\end{aligned}
$$

Then by Cauchy's inequality

$$
|\mathcal{Z}_1| \le \epsilon_6' \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|^2_{L^2(\Omega)} + \frac{d^4 \delta_0^{-4} (c^*)^2}{4\epsilon_6'}. \tag{3.3.70}
$$

Next we see

$$
\begin{aligned}
|\mathcal{Z}_2| &\le \sum_{i,j=1,\cdots,d} \left\| \frac{\partial_{x_i}(P^0_{k-1}+n_0)}{P^0_{k-1}+n_0} \right\|_\infty \left\| \partial_{x_j} \dot{P}^0_{k-1} \right\|_{L^2(\Omega)} \left\| \partial_{x_j} \left(\dot{P}^{\mathbf{d}}_k \right)_i \right\|_{L^2(\Omega)} \\
&\le \sum_{i,j=1,\cdots,d} \delta_0^{-2} c^* \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|_{L^2(\Omega)} \le d^2 \delta_0^{-2} c^* \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|_{L^2(\Omega)}.
\end{aligned}
$$

Cauchy's inequality yields

$$
|\mathcal{Z}_2| \le \epsilon_7' \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|^2_{L^2(\Omega)} + \frac{d^4 \delta_0^{-4} (c^*)^2}{4\epsilon_7'}. \tag{3.3.71}
$$

Finally we obtain

$$
\begin{aligned}
|\mathcal{Z}_3| \le & \sum_{i,j=1,\cdots,d} \left\| \frac{\partial_{x_i}(P^0_{k-1}+n_0)\partial_{x_j}(P^0_{k-1}+n_0)}{(P^0_{k-1}+n_0)^2} \right\|_{L^\infty(\Omega)} \left\| (P^0_{k-1})' \right\|_{L^2(\Omega)} \times \\
& \qquad \times \left\| \partial_{x_j} \left(\dot{P}^{\mathbf{d}}_k \right)_i \right\|_{L^2(\Omega)} \\
\le & \sum_{i,j=1,\cdots,d} \delta_0^{-4} c^* \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|_{L^2(\Omega)} \le d^2 \delta_0^{-4} c^* \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|_{L^2(\Omega)}.
\end{aligned}
$$

Using Cauchy's inequality yields

$$
|\mathcal{Z}_3| \le \epsilon_8' \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|^2_{L^2(\Omega)} + \frac{d^4 \delta_0^{-8} (c^*)^2}{4\epsilon_8'}. \tag{3.3.72}
$$

Combining (3.3.70), (3.3.71) and (3.3.72) we obtain

$$
|I_3'| \le \frac{\epsilon^2}{4} (\epsilon_6' + \epsilon_7' + \epsilon_8') \left\| \nabla \dot{P}^{\mathbf{d}}_k \right\|^2_{L^2(\Omega)} + C_{\epsilon_6', \epsilon_7', \epsilon_8', \epsilon, \Omega, \delta_0, c^*}, \tag{3.3.73}
$$

where the constant $C_{\epsilon_6',\epsilon_7',\epsilon_8',\epsilon,\Omega,\delta_0,c^*} > 0$ depends only on $\epsilon_6',\epsilon_7',\epsilon_8',\epsilon,\Omega,\delta_0,c^*$. Recall (3.3.53), (3.3.54), (3.3.64), (3.3.66) and (3.3.73) then

$$\begin{aligned}
&\frac{1}{2}\partial_t\big(T_0\|\dot P_k^0\|^2+\frac{\epsilon^2}{4}\|\nabla\dot P_k^0\|^2+\|\dot P_k^{\mathbf d}\|^2\big)+\nu_0T_0\|\nabla\dot P_k^0\|^2\\
&\qquad+\frac{\epsilon^2}{4}\nu_0\|\triangle\dot P_k^0\|^2+\nu_0\|\nabla\dot P_k^{\mathbf d}\|^2+\frac{1}{\tau}\|\dot P_k^{\mathbf d}\|^2\\
&\qquad\le\left(\epsilon_1'+\epsilon_2'+\epsilon_3'+\frac{\epsilon^2}{4}(\epsilon_6'+\epsilon_7'+\epsilon_8')\right)\left\|\nabla\dot P_k^{\mathbf d}\right\|^2_{L^2(\Omega)}\\
&\qquad+(\epsilon_4'+\epsilon_5')\left\|\dot P_k^{\mathbf d}\right\|^2_{L^2(\Omega)}+\mathcal C',
\end{aligned}\tag{3.3.74}$$

where $\mathcal C'>0$ is a constant depending upon $\epsilon_l',(l=1,\cdots,8)$, Ω, δ_0, M, c^*, the imbedding constants and all physical known quantities.

Select $\epsilon_l',(l=1,\cdots,8)$ sufficiently small such that

$$\epsilon_1'+\epsilon_2'+\epsilon_3'+\frac{\epsilon^2}{4}(\epsilon_6'+\epsilon_7'+\epsilon_8')<\nu_0,\quad \epsilon_4'+\epsilon_5'<\frac{1}{\tau},$$

then we obtain

$$\begin{aligned}
&\partial_t\big(T_0\|\dot P_k^0\|^2+\frac{\epsilon^2}{4}\|\nabla\dot P_k^0\|^2+\|\dot P_k^{\mathbf d}\|^2\big)\\
&+\mathcal C_1'\left(T_0\|\dot P_k^0\|^2+\frac{\epsilon^2}{4}\|\nabla\dot P_k^0\|^2+\|\triangle\dot P_k^0\|^2+\|\nabla\dot P_k^{\mathbf d}\|^2+\|\dot P_k^{\mathbf d}\|^2\right)\le\mathcal C_2'.
\end{aligned}\tag{3.3.75}$$

where $\mathcal C_1'>0$ is independent of c^*, $\mathcal C_2'$ depends on all physical known quantities, the imbedding constants and c^*, δ_0 and M. Via solving the differential inequality (3.3.75) we infer the following estimation.

$$\begin{aligned}
&T_0\|\dot P_k^0\|^2_{L^\infty(0,T;L^2(\Omega))}+\frac{\epsilon^2}{4}\|\nabla\dot P_k^0\|^2_{L^\infty(0,T;L^2(\Omega))}+\|\dot P_k^{\mathbf d}\|^2_{L^\infty(0,T;L^2(\Omega))}\\
&\le T_0\|\dot P_k^0(0)\|^2+\frac{\epsilon^2}{4}\|\nabla\dot P_k^0(0)\|^2+\|\dot P_k^{\mathbf d}(0)\|^2+\mathcal C_2'\int_0^T e^{\mathcal C_1's}ds.
\end{aligned}\tag{3.3.76}$$

From (3.3.27)

$$\dot P_k^0(0,\cdot)=Q_k,\quad \dot P_k^{\mathbf d}(0,\cdot)=R_k(0,\cdot),$$

then the estimations

$$\begin{aligned}
&T_0\|\dot P_k^0(0)\|^2+\frac{\epsilon^2}{4}\|\nabla\dot P_k^0(0)\|^2\le\max\left(T_0,\frac{\epsilon^2}{4}\right)\|\dot P_k^0(0)\|^2_{H_0^1(\Omega)}\\
&\le\max\left(T_0,\frac{\epsilon^2}{4}\right)(\nu_0\|n_0\|_{H^3(\Omega)}+\|J_0\|_{H^2(\Omega)})^2\\
&\le\max\left(T_0,\frac{\epsilon^2}{4}\right)M^2(\nu_0+1)^2,
\end{aligned}\tag{3.3.77}$$

and

$$\|\dot{P}_k^{\mathbf{d}}(0,\cdot)\|^2 \le \|R_k(0,\cdot)\|^2$$
$$\le \left(\|S(P_{k-1})(0)\| + \nu_0\|\triangle J_0\| + \frac{1}{\tau}\|J_0\| + T_0\|\nabla n_0\| + \frac{\epsilon^2}{4}\|\nabla\triangle n_0\|\right)^2$$
$$\le \left(\|S(P_{k-1})(0)\| + \left(\nu_0 + \frac{1}{\tau} + T_0 + \frac{\epsilon^2}{4}\right) M\right)^2,$$

hold. $\|S(P_{k-1})(0)\|$ can be estimated via a same procedure as (3.3.7)-(3.3.10) from which we obtain that $\|S(P_{k-1})(0)\|$ is bounded by a constant depending only upon $\delta_0, M, C_{14}, C_{16}$ where $C_{1s}, s = 4, 6$ denotes the imbedding constant of $H^1(\Omega) \hookrightarrow L^s(\Omega)$ respectively. Thus

$$\|\dot{P}_k^{\mathbf{d}}(0,\cdot)\|^2 \le C_{\nu_0,T_0,\tau,\epsilon,\delta_0,M,C_{14},C_{16}}. \tag{3.3.78}$$

Together with (3.3.76) and (3.3.77) we obtain

$$\begin{aligned} &T_0\|\dot{P}_k^0\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\epsilon^2}{4}\|\nabla\dot{P}_k^0\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\dot{P}_k^{\mathbf{d}}\|^2_{L^\infty(0,T;L^2(\Omega))} \\ &\le \mathcal{C}_{\nu_0,T_0,\tau,\epsilon,\delta_0,M,C_{14},C_{16}} + \mathcal{C}_2'\int_0^T e^{\mathcal{C}_1' s} ds, \end{aligned} \tag{3.3.79}$$

which implies

$$\|\dot{P}_k^0\|_{L^\infty(0,T;H_0^1(\Omega))} \le \sqrt{\frac{\mathcal{C}_{\nu_0,T_0,\tau,\epsilon,\delta_0,M,C_{14},C_{16}} + \mathcal{C}_2'\int_0^T e^{\mathcal{C}_1' s} ds}{\min\left(T_0, \frac{\epsilon^2}{4}, 1\right)}}, \tag{3.3.80}$$

$$\|\dot{P}_k^{\mathbf{d}}\|_{L^\infty(0,T;L^2(\Omega))} \le \sqrt{\frac{\mathcal{C}_{\nu_0,T_0,\tau,\epsilon,\delta_0,M,C_{14},C_{16}} + \mathcal{C}_2'\int_0^T e^{\mathcal{C}_1' s} ds}{\min\left(T_0, \frac{\epsilon^2}{4}, 1\right)}}. \tag{3.3.81}$$

Integrate (3.3.75) from 0 to T:

$$\begin{aligned} &\mathcal{C}_1'\Bigg(T_0\|\dot{P}_k^0\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\epsilon^2}{4}\|\nabla\dot{P}_k^0\|^2_{L^2(0,T;L^2(\Omega))} \\ &\qquad + \|\triangle\dot{P}_k^0\|^2_{L^2(0,T;L^2(\Omega))} + \|\dot{P}_k^{\mathbf{d}}\|^2_{L^2(0,T;H_0^1(\Omega))}\Bigg) \\ &\le T_0\|\dot{P}_k^0(0)\|^2 + \frac{\epsilon^2}{4}\|\nabla\dot{P}_k^0(0)\|^2 + \|\dot{P}_k^{\mathbf{d}}(0)\|^2 + \mathcal{C}_2'T \\ &\le C_{\nu_0,T_0,\tau,\epsilon,\delta_0,M,C_{14},C_{16}} + \mathcal{C}_2'T. \end{aligned} \tag{3.3.82}$$

Recall **Theorem** 3.2.9

$$\|\dot{P}_k^0\|^2_{L^2(0,T;H^2(\Omega))} \leq C^2_{\triangle,0,\Omega}\|\triangle\dot{P}_k^0\|^2_{L^2(0,T;L^2(\Omega))},$$

from which

$$\|\dot{P}_k^0\|^2_{L^2(0,T;H^2(\Omega))} \leq \frac{C^2_{\triangle,0,\Omega}}{\mathcal{C}_1'}C_{\nu_0,T_0,\tau,\epsilon,\delta_0,M,C_{14},C_{16}} + \frac{C^2_{\triangle,0,\Omega}}{\mathcal{C}_1'}\mathcal{C}_2'T =: \mathcal{M}_1 + \frac{C^2_{\triangle,0,\Omega}}{\mathcal{C}_1'}\mathcal{C}_2'T,$$

$$\|\dot{P}_k^{\mathbf{d}}\|^2_{L^2(0,T;H_0^1(\Omega))} \leq \frac{1}{\mathcal{C}_1'}C_{\nu_0,T_0,\tau,\epsilon,\delta_0,M,C_{14},C_{16}} + \frac{1}{\mathcal{C}_1'}\mathcal{C}_2'T =: \mathcal{M}_2 + \frac{1}{\mathcal{C}_1'}\mathcal{C}_2'T.$$

Fix now a positive number $\gamma > 0$ and define

$$C_0 := \max\left(\sqrt{\frac{\mathcal{C}_{\nu_0,T_0,\tau,\epsilon,\delta_0,M,C_{14},C_{16}} + \gamma}{\min\left(T_0, \frac{\epsilon^2}{4}, 1\right)}}, \sqrt{\mathcal{M}_1 + \gamma}, \sqrt{\mathcal{M}_2 + \gamma}\right). \tag{3.3.83}$$

Recall (3.3.50) then select $c^* = C_0$, and a $t^* > 0$ such that

$$\begin{cases} \max\left(\mathcal{C}_2'\int_0^{t^*} e^{\mathcal{C}_1's}ds, \frac{C^2_{\triangle,0,\Omega}}{\mathcal{C}_1'}\mathcal{C}_2't^*, \frac{1}{\mathcal{C}_1'}\mathcal{C}_2't^*\right) < \gamma, \\ \max\left(\sqrt{\frac{\mathcal{C}_2\int_0^{t^*} e^{\mathcal{C}_1s}ds}{\min\left(T_0, \frac{\epsilon^2}{4}\right)}}, \sqrt{\mathcal{C}_2\int_0^{t^*} e^{\mathcal{C}_1s}ds}\right) < C_0. \end{cases} \tag{3.3.84}$$

We recall (3.3.23)-(3.3.26) and conclude that if P_{k-1} defined in $[0,t^*)$ satisfies

$$\begin{cases} P_{k-1}^0 \in L^\infty(0,t^*;H^3(\Omega)) \cap C([0,t^*],C^1(\overline{\Omega})) \cap C([0,t^*],H^2(\Omega)), \\ P_{k-1}^{\mathbf{d}} \in L^\infty(0,t^*;H^2(\Omega)) \cap C([0,t^*],C(\overline{\Omega})) \cap C([0,t^*],H_0^1(\Omega)), \\ \dot{P}_{k-1}^0 \in L^\infty(0,t^*;H_0^1(\Omega)) \cap C([0,t^*],L^2(\Omega)) \cap L^2(0,t^*;H^2(\Omega)), \\ \dot{P}_{k-1}^{\mathbf{d}} \in L^\infty(0,t^*;L^2(\Omega)) \cap C([0,t^*],L^2(\Omega)) \cap L^2(0,t^*;H_0^1(\Omega)), \end{cases} \tag{3.3.85}$$

$$P_{k-1}(0,x) = 0, \tag{3.3.86}$$

and

$$\begin{cases} \|P_{k-1}^0\|_{L^\infty(0,t^*;H_0^1(\Omega))} \leq C_0, \qquad \|P_{k-1}^{\mathbf{d}}\|_{L^\infty(0,t^*;L^2(\Omega))} \leq C_0, \\ \|\dot{P}_{k-1}^0\|_{L^\infty(0,t^*;H_0^1(\Omega))} \leq C_0, \qquad \|\dot{P}_{k-1}^{\mathbf{d}}\|_{L^\infty(0,t^*;L^2(\Omega))} \leq C_0, \end{cases} \tag{3.3.87}$$

$$\begin{cases} \inf_{[0,t^*]} \inf_{x\in\overline{\Omega}} (P_{k-1}^0 + n_0) > \delta_0, \\ \sup_{[0,t^*]} \max \Big(\|\nabla(P_{k-1}^0 + n_0)\|_{L^\infty(\Omega)}, \|P_k^0 + n_0\|_{L^\infty(\Omega)}, \\ \qquad \|P_{k-1}^{\mathbf{d}} + J_0\|_{L^\infty(\Omega)} \Big) < \delta_0^{-1}, \end{cases} \tag{3.3.88}$$

then first from (3.3.85) and (3.3.86) together with **Lemma** 3.3.1 and **Theorem** 3.2.13 the solution P_k of (3.3.2) in $[0, t^*)$ satisfies

$$\begin{cases} P_k^0 \in L^\infty(0,t^*; H^3(\Omega)) \cap C([0,t^*], C^1(\overline{\Omega})) \cap C([0,t^*], H^2(\Omega)), \\ P_k^{\mathbf{d}} \in L^\infty(0,t^*; H^2(\Omega)) \cap C([0,t^*], C(\overline{\Omega})) \cap C([0,t^*], H_0^1(\Omega)), \\ \dot{P}_k^0 \in L^\infty(0,t^*; H_0^1(\Omega)) \cap C([0,t^*], L^2(\Omega)) \cap L^2(0,t^*; H^2(\Omega)), \\ \dot{P}_k^{\mathbf{d}} \in L^\infty(0,t^*; L^2(\Omega)) \cap C([0,t^*], L^2(\Omega)) \cap L^2(0,t^*; H_0^1(\Omega)), \end{cases} \tag{3.3.89}$$

$$P_k(0,x) = 0; \tag{3.3.90}$$

second from (3.3.50), (3.3.80), (3.3.81), (3.3.83) and (3.3.84) P_k satisfies

$$\begin{cases} \|P_k^0\|_{L^\infty(0,t^*;H_0^1(\Omega))} \le C_0, \qquad \|P_k^{\mathbf{d}}\|_{L^\infty(0,t^*;L^2(\Omega))} \le C_0, \\ \|\dot{P}_k^0\|_{L^\infty(0,t^*;H_0^1(\Omega))} \le C_0, \qquad \|\dot{P}_k^{\mathbf{d}}\|_{L^\infty(0,t^*;L^2(\Omega))} \le C_0, \\ \|\dot{P}_k^0\|_{L^2(0,t^*;H^2(\Omega))} \le C_0, \qquad \|\dot{P}_k^{\mathbf{d}}\|_{L^2(0,t^*;H_0^1(\Omega))} \le C_0. \end{cases} \tag{3.3.91}$$

Recall (3.3.49) there exist $\mathcal{C}_1^* > 0$ independent of C_0; $\mathcal{C}_2^* > 0$ dependent of C_0 such that

$$\begin{aligned} &\partial_t \big(T_0 \|P_k^0\|^2 + \frac{\epsilon^2}{4} \|\nabla P_k^0\|^2 + \|P_k^{\mathbf{d}}\|^2 \big) \\ &\qquad + \mathcal{C}_1^* \Big(\|P_k^0\|_{H^2(\Omega)}^2 + \|P_k^{\mathbf{d}}\|_{H_0^1(\Omega)}^2 \Big) \le \mathcal{C}_2^* \end{aligned} \tag{3.3.92}$$

for a.e. $t \in [0,t^*]$. Obviously that $\mathcal{C}_1^*, \mathcal{C}_2^*$ don't depend upon t^*. Then for a.e. $t \in [0,t^*]$ (3.3.92) and (3.3.91) yield

$$\|P_k^0\|_{H^2(\Omega)}^2 + \|P_k^{\mathbf{d}}\|_{H_0^1(\Omega)}^2 \tag{3.3.93}$$

$$\le \frac{1}{\mathcal{C}_1^*} \Bigg(2T_0 \|P_k^0\|_{L^2(\Omega)} \|\dot{P}_k^0\|_{L^2(\Omega)} + \frac{\epsilon^2}{2} \|\nabla P_k^0\|_{L^2(\Omega)} \|\nabla \dot{P}_k^0\|_{L^2(\Omega)} \tag{3.3.94}$$

$$+ 2\|P_k^{\mathbf{d}}\|_{L^2(\Omega)} \|\dot{P}_k^{\mathbf{d}}\|_{L^2(\Omega)} \Bigg) + \frac{\mathcal{C}_2^*}{\mathcal{C}_1^*} \tag{3.3.95}$$

$$\le \frac{C_0^2}{\mathcal{C}_1^*} \left(2T_0 + \frac{\epsilon^2}{2} + 2 \right) + \frac{\mathcal{C}_2^*}{\mathcal{C}_1^*} =: (C^*)^2, \tag{3.3.96}$$

which implies

$$\|P_k^0\|_{L^\infty(0,t^*;H^2(\Omega))} \leq C^*, \quad \|P_k^{\mathbf{d}}\|_{L^\infty(0,t^*;H_0^1(\Omega))} \leq C^*. \tag{3.3.97}$$

To get more estimates first from (3.3.2) we conclude that for a.e. $t \in [0, t^*]$ P_k solves

$$\begin{cases} A(\partial_x)P_k = F_{k-1} - \dot{P}_k, \\ \quad P_k(x) = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{3.3.98}$$

where $F_{k-1} - \dot{P}_k \in H_0^1(\Omega) \times (L^2(\Omega))^d$ because of the assumptions (3.3.85), (3.3.86) together with **Lemma** 3.3.1 and (3.3.89). Using **Theorem** 3.2.6 we have the following a priori estimate for P_k:

$$\begin{aligned} &\|P_k^0\|_{L^\infty(0,t^*;H^3(\Omega))} + \|P_k^{\mathbf{d}}\|_{L^\infty(0,t^*;(H^2(\Omega))^d)} \\ &\leq C_{A(\partial_x)}\big(\|F_{k-1}^0\|_{L^\infty(0,t^*;H_0^1(\Omega))} + \|F_{k-1}^{\mathbf{d}}\|_{L^\infty(0,t^*;L^2(\Omega))} \\ &\qquad + \|\dot{P}_k^0\|_{L^\infty(0,t^*;H_0^1(\Omega))} + \|\dot{P}_k^{\mathbf{d}}\|_{L^\infty(0,t^*;(L^2(\Omega))^d)}\big), \end{aligned} \tag{3.3.99}$$

the constant $C_{A(\partial_x)} > 0$ depends only on the coefficients of operator $A(\partial_x)$.

Since $F_{k-1}^0 = \nu_0 \triangle n_0 + \mathrm{div} J_0$ then

$$\|F_{k-1}^0\|_{L^\infty(0,t^*;H_0^1(\Omega))} \leq \nu_0 \|n_0\|_{H^3(\Omega)} + \|J_0\|_{H^2(\Omega)} \leq M(\nu_0 + 1). \tag{3.3.100}$$

Moreover

$$F_{k-1}^{\mathbf{d}} = S(P_{k-1}) + \nu_0 \triangle J_0 - \frac{1}{\tau} J_0 + T_0 \nabla n_0 - \frac{\epsilon^2}{4} \nabla \triangle n_0,$$

which can be estimated as

$$\begin{aligned} &\|F_{k-1}^{\mathbf{d}}\|_{L^\infty(0,t^*;L^2(\Omega))} \\ &\leq \|S(P_{k-1})\|_{L^\infty(0,t^*;L^2(\Omega))} + \nu_0 \|J_0\|_{H^2(\Omega)} + \frac{1}{\tau}\|J_0\|_{L^2(\Omega)} \\ &\quad + T_0 \|n_0\|_{H^1(\Omega)} + \frac{\epsilon^2}{4}\|n_0\|_{H^3(\Omega)} \\ &\leq M\left(\nu_0 + \frac{1}{\tau} + T_0 + \frac{\epsilon^2}{4}\right) + \|S(P_{k-1})\|_{L^\infty(0,t^*;L^2(\Omega))}. \end{aligned} \tag{3.3.101}$$

For a.e. $0 \leq t \leq t^*$,

$$\begin{aligned} \|S(P_{k-1})\|_{L^2(\Omega)} \leq &\left\| \mathrm{div}\left(\frac{(P_{k-1}^{\mathbf{d}} + J_0) \otimes (P_{k-1}^{\mathbf{d}} + J_0)}{P_{k-1}^0 + n_0} \right) \right\|_{L^2(\Omega)} \\ &+ \left\| (P_{k-1}^0 + n_0) \nabla V_{k-1} \right\|_{L^2(\Omega)} \\ &+ \left\| \epsilon^2 \mathrm{div}\left(\left(\nabla \sqrt{P_{k-1}^0 + n_0}\right) \otimes \left(\nabla \sqrt{P_{k-1}^0 + n_0}\right) \right) \right\|_{L^2(\Omega)} \\ =: &\ \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3. \end{aligned}$$

- **Estimates of $\mathcal{M}_1$.**

$$\begin{aligned}\mathcal{M}_1 &= \sup_{l\in\{1,\cdots,d\}} \left\| \operatorname{div}\left(\frac{(P^{\mathbf{d}}_{k-1}+J_0)_l (P^{\mathbf{d}}_{k-1}+J_0)}{P^0_{k-1}+n_0} \right) \right\|_{L^2(\Omega)} \\ &\leq \sup_{l\in\{1,\cdots,d\}} \left\| \sum_{k=1}^{d} \frac{(P^{\mathbf{d}}_{k-1}+J_0)_k}{P^0_{k-1}+n_0} \partial_{x_k} (P^{\mathbf{d}}_{k-1}+J_0)_l \right\|_{L^2(\Omega)} \\ &\quad + \sup_{l\in\{1,\cdots,d\}} \left\| \sum_{k=1}^{d} \frac{(P^{\mathbf{d}}_{k-1}+J_0)_l}{P^0_{k-1}+n_0} \partial_{x_k} (P^{\mathbf{d}}_{k-1}+J_0)_k \right\|_{L^2(\Omega)} \\ &\quad + \sup_{l\in\{1,\cdots,d\}} \left\| \sum_{k=1}^{d} \frac{\partial_{x_k}(P^0_{k-1}+n_0)(P^{\mathbf{d}}_{k-1}+J_0)_l (P^{\mathbf{d}}_{k-1}+J_0)_k}{\left(P^0_{k-1}+n_0\right)^2} \right\|_{L^2(\Omega)},\end{aligned}$$

then we obtain

$$\begin{aligned}\mathcal{M}_1 &\leq \sup_{l\in\{1,\cdots,d\}} \sum_{k=1}^{d} \left\| \frac{(P^{\mathbf{d}}_{k-1}+J_0)_k}{P^0_{k-1}+n_0} \right\|_{L^\infty(\Omega)} \left\| \partial_{x_k} (P^{\mathbf{d}}_{k-1}+J_0)_l \right\|_{L^2(\Omega)} \\ &\quad + \sup_{l\in\{1,\cdots,d\}} \sum_{k=1}^{d} \left\| \frac{(P^{\mathbf{d}}_{k-1}+J_0)_l}{P^0_{k-1}+n_0} \right\|_{L^\infty(\Omega)} \left\| \partial_{x_k} (P^{\mathbf{d}}_{k-1}+J_0)_k \right\|_{L^2(\Omega)} \\ &\quad + \sup_{l\in\{1,\cdots,d\}} \sum_{k=1}^{d} \left\| \frac{(P^{\mathbf{d}}_{k-1}+J_0)_l (P^{\mathbf{d}}_{k-1}+J_0)_k}{(P^0_{k-1}+n_0)^2} \right\|_{L^\infty(\Omega)} \times \\ &\qquad\qquad \times \left\| \partial_{x_k} (P^0_{k-1}+n_0) \right\|_{L^2(\Omega)}.\end{aligned}$$

Assume

$$\|P^0_{k-1}\|_{L^\infty(0,t^*;H^2(\Omega))} \leq C^*, \quad \|P^{\mathbf{d}}_{k-1}\|_{L^\infty(0,t^*;H^1_0(\Omega))} \leq C^*. \tag{3.3.102}$$

Combining (3.3.19), (3.3.87), (3.3.88) and (3.3.102) we decuce

$$\mathcal{M}_1 \leq 2d\delta_0^{-2}(C^*+M) + d\delta_0^{-4}(C_0+M). \tag{3.3.103}$$

- **Estimates of $\mathcal{M}_2$.**

$$\begin{aligned}\mathcal{M}_2 &\leq \left\| P^0_{k-1}+n_0 \right\|_{L^4(\Omega)} \|\nabla V_{k-1}\|_{L^4(\Omega)} \\ &\leq C_{14}^2 \left\| P^0_{k-1}+n_0 \right\|_{H^1(\Omega)} \|V_{k-1}\|_{H^2(\Omega)}.\end{aligned}$$

From (3.3.87) and (3.3.41) we infer

$$\mathcal{M}_2 \leq C_{14}^2 (C_0+M) C_{\lambda,\Omega,C_0,M,\mathcal{C}(x),V_\Gamma}. \tag{3.3.104}$$

- **Estimates of $\mathcal{M}_3$.**

  Since

$$\mathcal{M}_3 = \frac{\epsilon^2}{4} \left\| \mathrm{div} \left(\frac{\nabla(P^0_{k-1} + n_0) \otimes \nabla(P^0_{k-1} + n_0)}{P^0_{k-1} + n_0} \right) \right\|_{L^2(\Omega)},$$

  we use a similar reasoning as $\mathcal{M}_1$ to find

$$\begin{aligned} \frac{4}{\epsilon^2}\mathcal{M}_3 \leq & \sup_{l\in\{1,\cdots,d\}} \sum_{k=1}^{d} \left\| \frac{\partial_{x_k}(P^0_{k-1} + n_0)}{P^0_{k-1} + n_0} \right\|_{L^\infty(\Omega)} \left\| \partial_{x_k}\partial_{x_l}(P^0_{k-1} + n_0) \right\|_{L^2(\Omega)} \\ & + \sup_{l\in\{1,\cdots,d\}} \sum_{k=1}^{d} \left\| \frac{\partial_{x_l}(P^0_{k-1} + n_0)}{P^0_{k-1} + n_0} \right\|_{L^\infty(\Omega)} \left\| \partial^2_{x_k}(P^0_{k-1} + n_0) \right\|_{L^2(\Omega)} \\ & + \sup_{l\in\{1,\cdots,d\}} \sum_{k=1}^{d} \left\| \frac{\partial_{x_l}(P^0_{k-1} + n_0)\partial_{x_k}(P^0_{k-1} + n_0)}{(P^0_{k-1} + n_0)^2} \right\|_{L^\infty(\Omega)} \times \\ & \times \left\| \partial_{x_k}(P^0_{k-1} + n_0) \right\|_{L^2(\Omega)}, \end{aligned}$$

  which implies

$$\mathcal{M}_3 \leq \frac{\epsilon^2}{4} \left(2d\delta_0^{-2}(C^* + M) + d\delta_0^{-4}(C_0 + M) \right). \tag{3.3.105}$$

Thus combining (3.3.103)-(3.3.105) we conclude that for a.e. $t \in [0, t^*]$

$$\|S(P_{k-1})\|_{L^2(\Omega)} \leq C_{\lambda,\epsilon,\Omega,C_0,C^*,\delta_0,M,\mathcal{C}(x),V_\Gamma,C_{14}}. \tag{3.3.106}$$

Together with (3.3.91), (3.3.99), (3.3.100) and (3.3.101) we deduce the following estimation.

$$\begin{aligned} & \|P^0_k\|_{L^\infty(0,t^*;H^3(\Omega))} + \|P^{\mathbf{d}}_k\|_{L^\infty(0,t^*;(H^2(\Omega))^d)} \\ & \leq C_{\nu_0,\lambda,\epsilon,\tau,\Omega,C_0,C^*,\delta_0,M,\mathcal{C}(x),V_\Gamma,C_{14}} =: C', \end{aligned} \tag{3.3.107}$$

$C' > 0$ depends upon all known physical quantities, the imbedding constant of $H^1(\Omega) \hookrightarrow L^4(\Omega)$, δ_0, M and C_0, C^*, but is independent of k and t^*.

Now we make a summary. Given a function P_{k-1}; three positive constants C_0, C^*, C' which are defined in (3.3.83), (3.3.96), (3.3.107) respectively and depend only upon all physical constants, the corresponding imbedding constants C_{14}, C_{16}, the initial functions n_0, J_0 and the boundary conditions $(n_\Gamma, J_\Gamma, V_\Gamma)$; a time interval $[0, t^*)$ which satisfies (3.3.84). If P_{k-1} satisfies (3.3.85), (3.3.86), (3.3.87), (3.3.88), (3.3.102) and

$$\begin{cases} \|P^0_{k-1}\|_{L^\infty(0,t^*;H^3(\Omega))} + \|P^{\mathbf{d}}_{k-1}\|_{L^\infty(0,t^*;(H^2(\Omega))^d)} \leq C', \\ \|\dot{P}^0_k\|_{L^2(0,t^*;H^2(\Omega))} \leq C_0, \quad \|\dot{P}^{\mathbf{d}}_k\|_{L^2(0,t^*;H^1_0(\Omega))} \leq C_0, \end{cases} \tag{3.3.108}$$

then we obtain P_k via solving (3.3.2), and P_k satisfies

$$\begin{cases} P_k^0 \in L^\infty(0,t^*;H^3(\Omega)) \cap C([0,t^*],C^1(\overline{\Omega})) \cap C([0,t^*],H^2(\Omega)), \\ P_k^{\mathbf{d}} \in L^\infty(0,t^*;H^2(\Omega)) \cap C([0,t^*],C(\overline{\Omega})) \cap C([0,t^*],H_0^1(\Omega)), \\ \partial_t P_k^0 \in L^\infty(0,t^*;H_0^1(\Omega)) \cap C([0,t^*],L^2(\Omega)) \cap L^2(0,t^*;H^2(\Omega)), \\ \partial_t P_k^{\mathbf{d}} \in L^\infty(0,t^*;L^2(\Omega)) \cap C([0,t^*],L^2(\Omega)) \cap L^2(0,t^*;H_0^1(\Omega)), \end{cases} \tag{3.3.109}$$

and

$$\begin{cases} \|P_k^0\|_{L^\infty(0,t^*;H^3(\Omega))} \le C', \qquad \|P_k^{\mathbf{d}}\|_{L^\infty(0,t^*;H^2(\Omega))} \le C', \\ \|P_k^0\|_{L^\infty(0,t^*;H^2(\Omega))} \le C^*, \qquad \|P_k^{\mathbf{d}}\|_{L^\infty(0,t^*;H_0^1(\Omega))} \le C^*, \\ \|P_k^0\|_{L^\infty(0,t^*;H_0^1(\Omega))} \le C_0, \qquad \|P_k^{\mathbf{d}}\|_{L^\infty(0,t^*;L^2(\Omega))} \le C_0, \\ \|\dot{P}_k^0\|_{L^\infty(0,t^*;H_0^1(\Omega))} \le C_0, \qquad \|\dot{P}_k^{\mathbf{d}}\|_{L^\infty(0,t^*;L^2(\Omega))} \le C_0, \\ \|\dot{P}_k^0\|_{L^2(0,t^*;H^2(\Omega))} \le C_0, \qquad \|\dot{P}_k^{\mathbf{d}}\|_{L^2(0,t^*;H_0^1(\Omega))} \le C_0. \end{cases} \tag{3.3.110}$$

It is pointed out that up to now we have not derived (3.3.22) yet. In order to let P_k satisfy (3.3.22) we can shrink the time interval $[0,t^*)$ into $[0,t_k)$ such that

$$\begin{cases} \inf_{[0,t_k]} \inf_{x\in\overline{\Omega}} (P_k^0 + n_0) > \delta_0, \\ \sup_{[0,t_k]} \max \Big(\|\nabla(P_k^0+n_0)\|_{L^\infty(\Omega)}, \|P_k^0+n_0\|_{L^\infty(\Omega)}, \\ \qquad \|P_k^{\mathbf{d}} + J_0\|_{L^\infty(\Omega)} \Big) < \delta_0^{-1}. \end{cases} \tag{3.3.111}$$

This process can be realized because of (3.3.17), (3.3.18) and (3.3.109). Under the conditions (3.3.109), (3.3.110), (3.3.111) and via shrinking the time interval into $[0,t_{k+1})$ the solution P_{k+1} satisfies the same estimates as P_k but in $[0,t_{k+1})$. More precisely, we have derived a sequence $\{P_k\}_{k=1}^\infty$ in $[0,t_k)$ which satisfies the uniform bounds (3.3.110) and (3.3.111).

Next we are able to show that to guarantee (3.3.111) t_k will not converge to zero for $k\to\infty$. For $0\le t_1\le t_2\le t_k$, recall $n_k = P_k^0 + n_0$, $J_k = P_k^{\mathbf{d}} + J_0$, we deduce the Hölder estimates

$$\begin{aligned} \|n_k(t_1,\cdot) - n_k(t_2,\cdot)\|_{H^2(\Omega)} &\le \int_{t_1}^{t_2} \|n_k'(s,\cdot)\|_{H^2(\Omega)} ds \\ &\le |t_1-t_2|^{1/2} \|\dot{P}_k^0\|_{L^2(0,t_k;H^2(\Omega))}, \end{aligned}$$

which allows us to estimate P_k in $C^{1/2}([0,t_k],C(\overline{\Omega}))$. Fix a number β with $0<\beta<\frac{1}{2}$ then by Sobolev's embedding theorem

$$\|P_k(t_1) - P_k(t_2)\|_{C(\overline{\Omega})} \le C_\beta \|P_k(t_1) - P_k(t_2)\|_{H^{2-\beta}(\Omega)}, \tag{3.3.112}$$

here let C_β denote the imbedding constant depending only on Ω. Interpolation yields

$$\begin{aligned}&\|P_k(t_1)-P_k(t_2)\|_{H^{2-\beta}(\Omega)}\\ \le\ & C_{inter}\|P_k(t_1)-P_k(t_2)\|_{L^2(\Omega)}^{\beta/2}\|P_k(t_1)-P_k(t_2)\|_{H^2(\Omega)}^{(2-\beta)/2}.\end{aligned} \tag{3.3.113}$$

Finally

$$\begin{aligned}\|P_k(t_1)-P_k(t_2)\|_{L^2(\Omega)} &\le \int_{t_1}^{t_2}\|\partial_t P_k\|_{L^2(\Omega)}dt\\ &\le |t_1-t_2|\|\partial_t P_k\|_{L^\infty(0,t_k;L^2(\Omega))}\end{aligned} \tag{3.3.114}$$

together with (3.3.112) and (3.3.113) yields

$$\begin{aligned}&\|P_k(t_1)-P_k(t_2)\|_{C(\overline{\Omega})}\\ \le\ & 2^{(2-\beta)/2}C_\beta C_{inter}\|\partial_t P_k\|_{L^\infty(0,t_k;L^2(\Omega))}^{\beta/2}\|P_k\|_{L^\infty(0,t_k;H^2(\Omega))}^{(2-\beta)/2}|t_1-t_2|^{\beta/2}.\end{aligned} \tag{3.3.115}$$

Thus under (3.3.110)

$$\begin{aligned}&\|n_k(t_1,\cdot)-n_k(t_2,\cdot)\|_{C(\overline{\Omega})}\\ \le\ & 2^{(2-\beta)/2}C_\beta C_{inter}\|\dot{P}_k^0\|_{L^\infty(0,t_k;L^2(\Omega))}^{\beta/2}\|n_k\|_{L^\infty(0,t_k;H^2(\Omega))}^{(2-\beta)/2}|t_1-t_2|^{\beta/2}\\ \le\ & 2^{(2-\beta)/2}C_\beta C_{inter}C_0^{\beta/2}(C^*+M)^{(2-\beta)/2}|t_1-t_2|^{\beta/2}.\end{aligned} \tag{3.3.116}$$

$$\begin{aligned}&\|J_k(t_1,\cdot)-J_k(t_2,\cdot)\|_{C(\overline{\Omega})}\\ \le\ & 2^{(2-\beta)/2}C_\beta C_{inter}\|\dot{P}_k^{\mathbf{d}}\|_{L^\infty(0,t_k;L^2(\Omega))}^{\beta/2}\|J_k\|_{L^\infty(0,t_k;H^2(\Omega))}^{(2-\beta)/2}|t_1-t_2|^{\beta/2}\\ \le\ & 2^{(2-\beta)/2}C_\beta C_{inter}C_0^{\beta/2}(C'+M)^{(2-\beta)/2}|t_1-t_2|^{\beta/2}.\end{aligned} \tag{3.3.117}$$

By a similar reasoning

$$\begin{aligned}&\|\nabla n_k(t_1,\cdot)-\nabla n_k(t_2,\cdot)\|_{C(\overline{\Omega})}\\ \le\ & 2^{(2-\beta)/2}C_\beta C_{inter}\|\nabla\dot{P}_k^0\|_{L^\infty(0,t_k;L^2(\Omega))}^{\beta/2}\|n_k\|_{L^\infty(0,t_k;H^3(\Omega))}^{(2-\beta)/2}|t_1-t_2|^{\beta/2}\\ \le\ & 2^{(2-\beta)/2}C_\beta C_{inter}C_0^{\beta/2}(C'+M)^{(2-\beta)/2}|t_1-t_2|^{\beta/2}.\end{aligned} \tag{3.3.118}$$

Since the right-hand sides of (3.3.116), (3.3.117) and (3.3.118) are uniformly bounded with respect to k, there is a time interval $[0,t']$ with $0<t'\le t^*$ such that for all $k=1,2,...$, (3.3.20), (3.3.21) and (3.3.22) hold. We have finished the proof of **Lemma** 3.3.2. □

To obtain a local-in-time solution the convergence property of the total sequence $\{P_k\}_{k=0}^{\infty}$ needs to be considered. For this we have

Lemma 3.3.3. *There exits $t_* \in (0, t')$ such that $\{P_k\}_{k=0}^{\infty}$ is a Cauchy sequence in $L^\infty(0, t_*; H^1(\Omega) \times (L^2(\Omega))^d)$. Moreover there exists a $P^* \in L^\infty(0, t_*; H^1(\Omega) \times (L^2(\Omega))^d)$ such that P_k converges to P^* in $L^\infty(0, t_*; H^1(\Omega) \times (L^2(\Omega))^d)$.*

Proof. Recall (3.3.2) the following system

$$\begin{cases} \partial_t(P_{k+1} - P_k) + A(\partial_x)(P_{k+1} - P_k) = F_k - F_{k-1}, \\ \qquad (P_{k+1} - P_k)(0, x) = 0, \\ \qquad (P_{k+1} - P_k)(t, x) = 0, \quad \text{on } \partial\Omega \text{ for a.e. } 0 \le t \le t', \end{cases}$$

holds in the time interval $[0, t']$. By a similar calculations as (3.3.28)-(3.3.33):

$$\begin{aligned} &\frac{1}{2}\partial_t\big(T_0\|P^0_{k+1} - P^0_k\|^2 + \frac{\epsilon^2}{4}\|\nabla(P^0_{k+1} - P^0_k)\|^2 + \|P^{\mathbf{d}}_{k+1} - P^{\mathbf{d}}_k\|^2\big) \\ &\quad + \nu_0 T_0\|\nabla(P^0_{k+1} - P^0_k)\|^2 + \frac{\epsilon^2}{4}\nu_0\|\triangle(P^0_{k+1} - P^0_k)\|^2 \\ &\quad + \nu_0\|\nabla(P^{\mathbf{d}}_{k+1} - P^{\mathbf{d}}_k)\|^2 + \frac{1}{\tau}\|P^{\mathbf{d}}_{k+1} - P^{\mathbf{d}}_k\|^2 \\ =&T_0(F^0_k - F^0_{k-1}, P^0_{k+1} - P^0_k) + (F^{\mathbf{d}}_k - F^{\mathbf{d}}_{k-1}, P^{\mathbf{d}}_{k+1} - P^{\mathbf{d}}_k) \\ &\quad - \frac{\epsilon^2}{4}(F^0_k - F^0_{k-1}, \triangle(P^0_{k+1} - P^0_k)). \end{aligned}$$

By Hölder's inequality

$$\begin{aligned} &\frac{1}{2}\partial_t\big(T_0\|P^0_{k+1} - P^0_k\|^2 + \frac{\epsilon^2}{4}\|\nabla(P^0_{k+1} - P^0_k)\|^2 + \|P^{\mathbf{d}}_{k+1} - P^{\mathbf{d}}_k\|^2\big) \\ &\quad + \nu_0 T_0\|\nabla(P^0_{k+1} - P^0_k)\|^2 + \frac{\epsilon^2}{4}\nu_0\|\triangle(P^0_{k+1} - P^0_k)\|^2 \\ &\quad + \nu_0\|\nabla(P^{\mathbf{d}}_{k+1} - P^{\mathbf{d}}_k)\|^2 + \frac{1}{\tau}\|P^{\mathbf{d}}_{k+1} - P^{\mathbf{d}}_k\|^2 \\ \le&T_0\|F^0_k - F^0_{k-1}\|_{L^2(\Omega)}\|P^0_{k+1} - P^0_k\|_{L^2(\Omega)} + \|F^{\mathbf{d}}_k - F^{\mathbf{d}}_{k-1}\|_{H^{-1}(\Omega)} \times \\ &\times \|P^{\mathbf{d}}_{k+1} - P^{\mathbf{d}}_k\|_{H^1_0(\Omega)} + \frac{\epsilon^2}{4}\|F^0_k - F^0_{k-1}\|_{L^2(\Omega)}\|\triangle(P^0_{k+1} - P^0_k)\|_{L^2(\Omega)}, \end{aligned}$$

then by Cauchy's inequality

$$\begin{aligned} &\partial_t\big(T_0\|P^0_{k+1} - P^0_k\|^2 + \frac{\epsilon^2}{4}\|\nabla(P^0_{k+1} - P^0_k)\|^2 + \|P^{\mathbf{d}}_{k+1} - P^{\mathbf{d}}_k\|^2\big) \\ \le& C\Big(\|F^0_k - F^0_{k-1}\|^2_{L^2(\Omega)} + \|F^{\mathbf{d}}_k - F^{\mathbf{d}}_{k-1}\|^2_{H^{-1}(\Omega)}\Big) \end{aligned}$$

for some $C > 0$ depending only on the coefficients of operator $A(\partial_x)$. Then we infer for any $t \in (0, t')$

$$\begin{aligned}
&\|(P_{k+1}^0 - P_k^0)(t,\cdot)\|^2 + \|\nabla(P_{k+1}^0 - P_k^0)(t,\cdot)\|^2 + \|(P_{k+1}^{\mathbf{d}} - P_k^{\mathbf{d}})(t,\cdot)\|^2 \\
\le & C \int_0^t \Big(\|(F_k^0 - F_{k-1}^0)(s,\cdot)\|_{L^2(\Omega)}^2 + \|(F_k^{\mathbf{d}} - F_{k-1}^{\mathbf{d}})(s,\cdot)\|_{H^{-1}(\Omega)}^2 \Big) ds \\
\le & Ct' \Big(\|F_k^0 - F_{k-1}^0\|_{L^\infty(0,t';L^2(\Omega))}^2 + \|F_k^{\mathbf{d}} - F_{k-1}^{\mathbf{d}}\|_{L^\infty(0,t';H^{-1}(\Omega))}^2 \Big),
\end{aligned}$$

which implies

$$\begin{aligned}
&\|P_{k+1}^0 - P_k^0\|_{L^\infty(0,t';H^1(\Omega))}^2 + \|P_{k+1}^{\mathbf{d}} - P_k^{\mathbf{d}}\|_{L^\infty(0,t';(L^2(\Omega))^d)}^2 \\
\le & Ct' \Big(\|F_k^0 - F_{k-1}^0\|_{L^\infty(0,t';L^2(\Omega))}^2 + \|F_k^{\mathbf{d}} - F_{k-1}^{\mathbf{d}}\|_{L^\infty(0,t';H^{-1}(\Omega))}^2 \Big).
\end{aligned} \tag{3.3.119}$$

Here

$$F_k^0 - F_{k-1}^0 = \nu_0 \triangle(n_0 - n_0) + \operatorname{div}(J_0 - J_0) = 0 \tag{3.3.120}$$

$F_k^{\mathbf{d}} - F_{k-1}^{\mathbf{d}}$ has the representation

$$F_k^{\mathbf{d}} - F_{k-1}^{\mathbf{d}} = S(P_k) - S(P_{k-1})),$$

which implies

$$\|F_k^{\mathbf{d}} - F_{k-1}^{\mathbf{d}}\|_{L^\infty(0,t';H^{-1}(\Omega))}^2 = \|S(P_k) - S(P_{k-1})\|_{L^\infty(0,t' H^{-1}(\Omega))}^2. \tag{3.3.121}$$

Using the notations $n_k = P_k^0 + n_0$, $J_k = P_k^{\mathbf{d}} + J_0$ gives

$$\begin{aligned}
&\|S(P_k) - S(P_{k-1})\|_{H^{-1}(\Omega)} \\
= & \Bigg\| \operatorname{div}\left(\frac{J_k \otimes J_k}{n_k} - \frac{J_{k-1} \otimes J_{k-1}}{n_{k-1}} \right) - (n_k \nabla V_k - n_{k-1} \nabla V_{k-1}) \\
& + \frac{\epsilon^2}{4} \operatorname{div}\left(\frac{\nabla n_k \otimes \nabla n_k}{n_k} - \frac{\nabla n_{k-1} \otimes \nabla n_{k-1}}{n_{k-1}} \right) \Bigg\|_{H^{-1}(\Omega)} \\
\le & \left\| \operatorname{div}\left(\frac{J_k \otimes J_k}{n_k} - \frac{J_{k-1} \otimes J_{k-1}}{n_{k-1}} \right) \right\|_{H^{-1}(\Omega)} + \|n_k \nabla V_k - n_{k-1} \nabla V_{k-1}\|_{H^{-1}(\Omega)} \\
& + \frac{\epsilon^2}{4} \left\| \operatorname{div}\left(\frac{\nabla n_k \otimes \nabla n_k}{n_k} - \frac{\nabla n_{k-1} \otimes \nabla n_{k-1}}{n_{k-1}} \right) \right\|_{H^{-1}(\Omega)} \\
=: & L_1 + L_2 + \frac{\epsilon^2}{4} L_3.
\end{aligned}$$

To estimate L_1 we select $g \in (H_0^1(\Omega))^d$ with $\|g\|_{(H_0^1(\Omega))^d} \leq 1$, then

$$\begin{aligned}
&\left|\left(\operatorname{div}\left(\frac{J_k \otimes J_k}{n_k} - \frac{J_{k-1} \otimes J_{k-1}}{n_{k-1}}\right), g\right)\right| \\
&= \left|\left(\left(\frac{J_k \otimes J_k}{n_k} - \frac{J_{k-1} \otimes J_{k-1}}{n_{k-1}}\right), \nabla g\right)\right| \\
&\leq \left\|\frac{J_k \otimes J_k}{n_k} - \frac{J_{k-1} \otimes J_{k-1}}{n_{k-1}}\right\|_{L^2(\Omega)} \\
&= \sup_{l,s=1,\cdots,d} \left\|\frac{(J_k)_l (J_k)_s}{n_k} - \frac{(J_{k-1})_l (J_{k-1})_s}{n_{k-1}}\right\|_{L^2(\Omega)}.
\end{aligned} \tag{3.3.122}$$

Set $\triangle_1 := J_k - J_{k-1}$, $\triangle_2 := n_k - n_{k-1}$ then

$$\begin{aligned}
\frac{(J_k)_l (J_k)_s}{n_k} - \frac{(J_{k-1})_l (J_{k-1})_s}{n_{k-1}} &= \frac{(J_k)_l (J_k)_s n_{k-1} - (J_{k-1})_l (J_{k-1})_s n_k}{n_k n_{k-1}} \\
&=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}_1 &= \frac{(J_k)_l (J_k)_s n_{k-1} - (J_{k-1})_l (J_k)_s n_{k-1}}{n_k n_{k-1}} = \frac{(J_k)_s n_{k-1}}{n_k n_{k-1}}((J_k)_l - (J_{k-1})_l), \\
\mathcal{I}_2 &= \frac{(J_{k-1})_l (J_k)_s n_{k-1} - (J_{k-1})_l (J_{k-1})_s n_{k-1}}{n_k n_{k-1}} = \frac{(J_{k-1})_l n_{k-1}}{n_k n_{k-1}}((J_k)_s - (J_{k-1})_s), \\
\mathcal{I}_3 &= \frac{(J_{k-1})_l (J_{k-1})_s n_{k-1} - (J_{k-1})_l (J_{k-1})_s n_k}{n_k n_{k-1}} = \frac{(J_{k-1})_l (J_{k-1})_s}{n_k n_{k-1}}(n_{k-1} - n_k).
\end{aligned}$$

Since P_k in $[0, t')$ satisfies (3.3.21) and (3.3.22) from **Lemma** 3.3.2

$$\begin{aligned}
\|\mathcal{I}_1\|_{L^2(\Omega)} &\leq \left\|\frac{(J_k)_s n_{k-1}}{n_k n_{k-1}}\right\|_{L^\infty(\Omega)} \|\triangle_1\|_{L^2(\Omega)} \leq \delta_0^{-4} \|\triangle_1\|_{L^2(\Omega)}, \\
\|\mathcal{I}_2\|_{L^2(\Omega)} &\leq \left\|\frac{(J_{k-1})_l n_{k-1}}{n_k n_{k-1}}\right\|_{L^\infty(\Omega)} \|\triangle_1\|_{L^2(\Omega)} \leq \delta_0^{-4} \|\triangle_1\|_{L^2(\Omega)}, \\
\|\mathcal{I}_3\|_{L^2(\Omega)} &\leq \left\|\frac{(J_{k-1})_l (J_{k-1})_s}{n_k n_{k-1}}\right\|_{L^\infty(\Omega)} \|\triangle_2\|_{L^2(\Omega)} \leq \delta_0^{-4} \|\triangle_2\|_{L^2(\Omega)}.
\end{aligned}$$

Thus

$$\begin{aligned}
L_1 &\leq 2\delta_0^{-4}\Big(\|P_k^{\mathbf{d}} - P_{k-1}^{\mathbf{d}}\|_{L^2(\Omega)} + \|P_k^0 - P_{k-1}^0\|_{L^2(\Omega)}\Big) \\
&\leq 2\delta_0^{-4}\Big(\|P_k^{\mathbf{d}} - P_{k-1}^{\mathbf{d}}\|_{L^2(\Omega)} + \|P_k^0 - P_{k-1}^0\|_{H^1(\Omega)}\Big)
\end{aligned} \tag{3.3.123}$$

Now consider L_2. Set $\triangle_3 := V_k - V_{k-1}$, then

$$\begin{aligned} n_k \nabla V_k - n_{k-1} \nabla V_{k-1} &= n_k \nabla V_k - n_k \nabla V_{k-1} + n_k \nabla V_{k-1} - n_{k-1} \nabla V_{k-1} \\ &= n_k \nabla \triangle_3 + \nabla V_{k-1} \triangle_2, \end{aligned}$$

which yields

$$\begin{aligned} L_2 &\leq \|n_k \nabla \triangle_3\|_{L^2(\Omega)} + \|\nabla V_{k-1} \triangle_2\|_{L^2(\Omega)} \\ &\leq \|n_k\|_{L^\infty(\Omega)} \|\triangle_3\|_{H^1(\Omega)} + \|\nabla V_{k-1}\|_{L^4(\Omega)} \|\triangle_2\|_{L^4(\Omega)} \\ &\leq \delta_0^{-1} \|\triangle_3\|_{H^1(\Omega)} + C_{14}^2 \|V_{k-1}\|_{H^2(\Omega)} \|\triangle_2\|_{H^1(\Omega)} . \end{aligned}$$

Since $\triangle_3$ solves

$$\begin{cases} \lambda^2 \triangle \triangle_3 = \triangle_2, \\ \triangle_3|_{\partial\Omega} = 0 \end{cases}$$

recall (3.3.41) we obtain

$$L_2 \leq C \|\triangle_2\|_{H^1(\Omega)} , \tag{3.3.124}$$

$C > 0$ is independent of k, t', but depends upon C_0. The representation and estimates of L_3 follows from a similar process as L_1. Precisely, let $\partial_{x_l} n_k$ ($\partial_{x_s} n_k$,$\partial_{x_l} n_{k-1}$,$\partial_{x_s} n_{k-1}$) replace $(J_k)_l$ ($(J_k)_s$,$(J_{k-1})_l$,$(J_{k-1})_s$) in (3.3.122) then from (3.3.123)

$$\begin{aligned} L_3 &\leq 2\delta_0^{-4} \Big(\|\nabla P_k^0 - \nabla P_{k-1}^0\|_{L^2(\Omega)} + \|P_k^0 - P_{k-1}^0\|_{L^2(\Omega)} \Big) \\ &\leq 4\delta_0^{-4} \|P_k^0 - P_{k-1}^0\|_{H^1(\Omega)}. \end{aligned} \tag{3.3.125}$$

Combining (3.3.123), (3.3.124) and (3.3.125) we conclude

$$\begin{aligned} &\|S(P_k) - S(P_{k-1})\|^2_{L^\infty(0,t';H^{-1}(\Omega))} \\ \leq\, & C \left(\|\triangle_1\|^2_{L^2(\Omega)} + \|\triangle_2\|^2_{H^1(\Omega)} \right) \\ \leq\, & C \Big(\|P_k^0 - P_{k-1}^0\|^2_{L^\infty(0,t';H^1(\Omega))} + \|P_k^{\mathbf{d}} - P_{k-1}^{\mathbf{d}}\|^2_{L^\infty(0,t';L^2(\Omega))} \Big), \end{aligned} \tag{3.3.126}$$

where $C > 0$ is independent of k and t'.

Recall (3.3.119), (3.3.120) and (3.3.121):

$$\begin{aligned} &\|P_{k+1}^0 - P_k^0\|^2_{L^\infty(0,t';H^1(\Omega))} + \|P_{k+1}^{\mathbf{d}} - P_k^{\mathbf{d}}\|^2_{L^\infty(0,t';L^2(\Omega))} \\ \leq\, & Ct' \Big(\|P_k^0 - P_{k-1}^0\|^2_{L^\infty(0,t';H^1(\Omega))} + \|P_k^{\mathbf{d}} - P_{k-1}^{\mathbf{d}}\|^2_{L^\infty(0,t';L^2(\Omega))} \Big). \end{aligned}$$

Define a Banach space $\mathcal{B} := L^\infty(0,t';H^1(\Omega)) \times L^\infty(0,t';(L^2(\Omega))^d)$ with norm

$$\|(x_1,x_2)\|_{\mathcal{B}} = \sqrt{\|x_1\|^2_{L^\infty(0,t';H^1(\Omega))} + \|x_2\|^2_{L^\infty(0,t';(L^2(\Omega))^d}},$$

then

$$\|P_{k+1} - P_k\|_{\mathcal{B}} \leq C(t')^{1/2} \|P_k - P_{k-1}\|_{\mathcal{B}}.$$

We shrink $[0, t']$ into $[0, t_*]$ such that $C(t_*)^{1/2} < 1$ since C is independent of the time interval. Consequently let $k > l$,

$$\|P_k - P_l\|_{\mathcal{B}} \leq \|P_1 - P_0\|_{\mathcal{B}} \sum_{j=l}^{k-1} (C(t_*)^{1/2})^j.$$

Hence $\{P_k = (P_k^0, P_k^{\mathbf{d}})\}_{k=0}^{\infty}$ is a Cauchy sequence in $\mathcal{B}$, and therefore there exists a point $P^* = ((P^*)^0, (P^*)^{\mathbf{d}}) \in L^\infty(0, t_*; H^1(\Omega)) \times L^\infty(0, t_*; (L^2(\Omega))^d)$ with $P_k \to P^*$ in $L^\infty(0, t_*; H^1(\Omega)) \times L^\infty(0, t_*; (L^2(\Omega))^d)$. □

Recall the uniform bounds of $\{P_k^0\}_{k=0}^{\infty}$ it follows

Lemma 3.3.4. *The limit P^* in **Lemma** 3.3.3 satisfies*

$$P^* \in A_1^* \cap A_2^* \cap A_3^*$$

with

$$\partial_t P^* \in L^2(0, t_*; H^2(\Omega)) \times L^2(0, t_*; (H^1(\Omega))^d)$$

where

$$\begin{aligned} A_1^* &:= C([0, t_*]; H^2(\Omega)) \times C([0, t_*]; (H^1(\Omega))^d), \\ A_2^* &:= L^\infty(0, t_*; H^3(\Omega)) \times L^\infty(0, t_*; (H^2(\Omega))^d), \\ A_3^* &:= C([0, t_*]; C^1(\overline{\Omega})) \times C([0, t_*]; (C(\overline{\Omega}))^d). \end{aligned}$$

Proof. Since the embeddings $H^3(\Omega) \hookrightarrow H^2(\Omega), H^2(\Omega) \hookrightarrow H^1(\Omega)$ are compact, $(P_k^0, P_k^{\mathbf{d}})$ $(k = 0, 1, \cdots)$ are bounded in $L^\infty(0, t_*; H^3(\Omega) \times (H^2(\Omega))^d)$ and $(\dot{P}_k^0, \dot{P}_k^{\mathbf{d}})$ $(k = 0, 1, \cdots)$ are bounded in $L^2(0, t_*; H^1(\Omega) \times (L^2(\Omega))^d)$ from **Lemma** 3.3.2, then Aubin's Lemma (Corollary 4 in [72]) yields $\{P_k^0\}_{k=0}^{\infty}$ and $\{P_k^{\mathbf{d}}\}_{k=0}^{\infty}$ are relatively compact in $C([0, t_*],\ H^2(\Omega))$ and $C([0, t_*], H^1(\Omega))$ respectively, i.e., there are a subsequence

$$\{(P_{k_s}^0, P_{k_s}^{\mathbf{d}})\}_{s=0}^{\infty} \subseteq \{(P_k^0, P_k^{\mathbf{d}})\}_{k=0}^{\infty}$$

and a function

$$P_S = (P_S^0, P_S^{\mathbf{d}}) \in C([0, t_*]; H^2(\Omega)) \times C([0, t_*]; (H^1(\Omega))^d) \tag{3.3.127}$$

such that

$$(P_{k_s}^0, P_{k_s}^{\mathbf{d}}) \to (P_S^0, P_S^{\mathbf{d}}) \text{ in } C([0, t_*]; H^2(\Omega)) \times C([0, t_*]; (H^1(\Omega))^d). \tag{3.3.128}$$

Furthermore from the uniform bounds of P_k (see **Lemma** 3.3.2) there exists a subsequence $\{(P^0_{k_{s_m}}, P^{\mathbf{d}}_{k_{s_m}})\}_{m=0}^{\infty}$ of $\{(P^0_{k_s}, P^{\mathbf{d}}_{k_s})\}_{s=0}^{\infty}$ and functions

$$P^0_M \in L^\infty(0,t_*;H^3(\Omega)), \qquad P^{\mathbf{d}}_M \in L^\infty(0,t_*;(H^2(\Omega))^d),$$
$$v^0_M \in L^2(0,t_*;H^2(\Omega)), \qquad v^{\mathbf{d}}_M \in L^2(0,t_*;(H^1(\Omega))^d)$$

such that

$$\begin{cases} \dot{P}^0_{k_{s_m}} \rightharpoonup v^0_M & \text{in} \quad L^2(0,t_*;H^2(\Omega)), \\ \dot{P}^{\mathbf{d}}_{k_{s_m}} \rightharpoonup v^{\mathbf{d}}_M & \text{in} \quad L^2(0,t_*;(H^1(\Omega))^d), \\ P^0_{k_{s_m}} \rightharpoonup^* P^0_M & \text{in} \quad L^\infty(0,t_*;H^3(\Omega)), \\ P^{\mathbf{d}}_{k_{s_m}} \rightharpoonup^* P^{\mathbf{d}}_M & \text{in} \quad L^\infty(0,t_*;(H^2(\Omega))^d) \end{cases} \tag{3.3.129}$$

as $m \to \infty$.

We claim that $v^0_M = \dot{P}^0_M$, $v^{\mathbf{d}}_M = \dot{P}^{\mathbf{d}}_M$. In order to show this assertion let us first select two distributions $\phi \in C_0^\infty(0,t_*)$, $\omega \in (H^2(\Omega))^*$. Then

$$\int_0^{t_*} \omega\left(\phi \dot{P}^0_{k_{s_m}}\right) dt = -\int_0^{t_*} \omega\left(\dot{\phi} P^0_{k_{s_m}}\right) dt. \tag{3.3.130}$$

As $m \to \infty$ we obtain

$$\int_0^{t_*} \omega\left(\phi v^0_M\right) dt = -\int_0^{t_*} \omega\left(\dot{\phi} P^0_M\right) dt.$$

Since $\phi v^0_M \in L^2(0,t_*;H^2(\Omega))$ is Bochner integrable on $(0,t_*)$, there exits a sequence of simple functions $\{f_j\}$ such that

$$\lim_{j\to\infty} \int_0^{t_*} \|f_j(t) - \phi v^0_M(t)\|_{H^2(\Omega)} dt = 0, \tag{3.3.131}$$

and

$$\int_0^{t_*} \phi v^0_M(t) dt = \lim_{j\to\infty} \int_0^{t_*} f_j(t) dt. \tag{3.3.132}$$

From $\omega \in (H^2(\Omega))^*$ we infer

$$\omega\left(\int_0^{t_*} \phi v^0_M(t) dt\right) = \lim_{j\to\infty} \omega\left(\int_0^{t_*} f_j(t) dt\right). \tag{3.3.133}$$

Assume the simple function $f_j(t) : (0,t_*) \to H^2(\Omega)$ is defined by

$$f_j(t) = \sum_{i=1}^{k} \mathcal{X}_{A_i^j} x_i^j,$$

where $\{A_1^j, \cdots, A_k^j\}$ is a finite collection of mutually disjoint subsets of $(0, t_*)$ each have finite $\mu-$measure, and $\{x_1^j, \cdots, x_k^j\}$ is a corresponding set of elements of $H^2(\Omega)$; $\mathcal{X}_{A_i^j} : (0, t_*) \to \{0, 1\}$ is a function with $\mathcal{X}_{A_i^j}(x) = 1$ if $x \in A_i^j$, $\mathcal{X}_{A_i^j}(x) = 0$ if $x \in (0, t_*)/A_i^j$. Then the corresponding integration is defined by

$$\int_0^{t_*} f_j(t)dt = \sum_{i=1}^k \mu(A_i^j)x_i^j.$$

Therefore

$$\omega\left(\int_0^{t_*} f_j(t)dt\right) = \int_0^{t_*} \omega(f_j(t))dt. \tag{3.3.134}$$

Recall (3.3.131) it follows

$$\lim_{j\to\infty} \int_0^{t_*} \left|\omega(f_j(t)) - \omega(\phi v_M^0(t))\right| dt = 0, \tag{3.3.135}$$

which implies

$$\int_0^{t_*} \omega(\phi v_M^0(t))dt = \lim_{j\to\infty} \int_0^{t_*} \omega(f_j(t))dt. \tag{3.3.136}$$

Combining (3.3.133) and (3.3.134) we obtain

$$\omega\left(\int_0^{t_*} \phi v_M^0 dt\right) = \int_0^{t_*} \omega\left(\phi v_M^0\right) dt. \tag{3.3.137}$$

From a same reasoning

$$\omega\left(\int_0^{t_*} \dot{\phi} P_M^0 dt\right) = \int_0^{t_*} \omega\left(\dot{\phi} P_M^0\right) dt.$$

From (3.3.130) then it follows

$$\omega\left(\int_0^{t_*} \phi v_M^0 dt\right) = -\omega\left(\int_0^{t_*} \dot{\phi} P_M^0 dt\right), \tag{3.3.138}$$

i.e., (3.3.138) holds for all $\omega \in (H^2(\Omega))^*$, then the Bochner integral $\int_0^{t_*} \phi v_M^0 dt$ equals to $-\int_0^{t_*} \dot{\phi} P_M^0 dt$, which implies that $v_M^0 = \dot{P}_M^0$. It follows $v_M^{\mathbf{d}} = \dot{P}_M^{\mathbf{d}}$ now from a similar reasoning.

As derived above

$$P_{k_{s_m}}^0 \longrightarrow P_S^0, \text{ in } C([0, t_*]; H^2(\Omega)) \text{ as } m \to \infty,$$

$$P_{k_{s_m}}^{\mathbf{d}} \longrightarrow P_S^{\mathbf{d}}, \text{ in } C([0, t_*]; (H^1(\Omega))^d) \text{ as } m \to \infty,$$

we infer

$$P^0_{k_{s_m}} \rightharpoonup^* P^0_S, \text{ in } L^\infty(0,t_*;H^2(\Omega)) \text{ as } m\to\infty,$$

$$P^{\mathbf{d}}_{k_{s_m}} \rightharpoonup^* P^{\mathbf{d}}_S, \text{ in } L^\infty(0,t_*;(H^1(\Omega))^d) \text{ as } m\to\infty.$$

Since $(H^1(\Omega))^* \subset (H^2(\Omega))^* \subset (H^3(\Omega))^*$, then from (3.3.129)

$$P^0_{k_{s_m}} \rightharpoonup^* P^0_M, \text{ in } L^\infty(0,t_*;H^2(\Omega)) \text{ as } m\to\infty,$$

$$P^{\mathbf{d}}_{k_{s_m}} \rightharpoonup^* P^{\mathbf{d}}_M, \text{ in } L^\infty(0,t_*;(H^1(\Omega))^d) \text{ as } m\to\infty.$$

This implies

$$P^0_S = P^0_M, \quad P^{\mathbf{d}}_S = P^{\mathbf{d}}_M. \tag{3.3.139}$$

Fix $0<\gamma<\dfrac{1}{2}$ and any $t\in(0,t_*)$ we deduce from the interpolation inequality (C.0.1) that

$$\|P^0_{k_{s_m}} - P^0_M\|_{H^{3-\gamma}(\Omega)} \le C_{\text{in}}\|P^0_{k_{s_m}} - P^0_M\|^{(3-\gamma)/3}_{H^3(\Omega)}\|P^0_{k_{s_m}} - P^0_M\|^{\gamma/3}_{L^2(\Omega)},$$

then by Lemma 3.3.2

$$\|P^0_{k_{s_m}} - P^0_M\|_{H^{3-\gamma}(\Omega)} \le C_{\text{in}}2^{(3-\gamma)/3}C'^{(3-\gamma)/3}\|P^0_{k_{s_m}} - P^0_M\|^{\gamma/3}_{L^2(\Omega)}. \tag{3.3.140}$$

Similarly

$$\|P^{\mathbf{d}}_{k_{s_m}} - P^{\mathbf{d}}_M\|_{H^{2-\gamma}(\Omega)} \le C_{\text{in}}2^{(2-\gamma)/2}C'^{(2-\gamma)/2}\|P^{\mathbf{d}}_{k_{s_m}} - P^{\mathbf{d}}_M\|^{\gamma/2}_{L^2(\Omega)}. \tag{3.3.141}$$

From (3.3.140), (3.3.141) and Sobolev embedding theorem we infer

$$(P^0_{k_{s_m}}, \nabla P^0_{k_{s_m}}, P^{\mathbf{d}}_{k_{s_m}}) \to (P^0_M, \nabla P^0_M, P^{\mathbf{d}}_M) \quad \text{in} \quad C([0,t_*]\times\overline{\Omega}). \tag{3.3.142}$$

From **Lemma** 3.3.3 the subsequence $\{P_{k_{s_m}}\}_{m=1}^\infty$ converges to P^* in

$$L^\infty(0,t_*;H^1(\Omega))\times L^\infty(0,t_*;(L^2(\Omega))^d).$$

This implies

$$P_{k_{s_m}} \rightharpoonup^* P^* \text{ in } L^\infty(0,t_*;H^1(\Omega))\times L^\infty(0,t_*;(L^2(\Omega))^d).$$

From (3.3.129) and the embeddings $L^2(\Omega)\subset(H^2(\Omega))^*$, $(H^1(\Omega))^*\subset(H^3(\Omega))^*$ we conclude

$$P_{k_{s_m}} \rightharpoonup^* (P^0_M, P^{\mathbf{d}}_M) \text{ in } L^\infty(0,t_*;H^1(\Omega))\times L^\infty(0,t_*;(L^2(\Omega))^d).$$

By the uniqueness of the limit

$$P^* = (P^0_M, P^{\mathbf{d}}_M).$$

Furthermore using (3.3.127), (3.3.139), (3.3.129) and (3.3.142) we complete the proof. □

Remark 3.3.2. *This lemma ensures that the limit function* $(n, J) := P^* + (n_0, J_0)$ *satisfies the initial conditions* (3.0.1) *and the boundary conditions from* (3.0.3).

It is also obvious that the sequence $\{V_k\}_{m=0}^{\infty}$ *converges to a limit* V *in the space* $C([0, t_*]; H^2(\Omega))$ *which solves the Possion equation* $\lambda^2 \triangle V = n - \mathcal{C}(x)$ *subject to the boundary codition* $V|_{\partial\Omega} = V_\Gamma$.

Let $P_0 = 0$, consider

$$\begin{cases} \partial_t P_k + A(\partial_x) P_k = F_{k-1}, \\ \qquad P_k(0, x) = 0, \\ \qquad P_k(t, x) = 0, \quad \text{on } \partial\Omega \text{ for a.e. } 0 \le t \le t_*. \end{cases} \tag{3.3.143}$$

We choose a test function $\varphi \in C_0^\infty(Q_*)$ where $Q_* := [0, t_*] \times \overline{\Omega}$, then it follows

$$\iint_{Q_*} (-\varphi_t P_k^0 - \nu_0 (\triangle\varphi) P_k^0 + (\nabla\varphi) P_k^0) dxdt = \iint_{Q_*} \varphi\, (\nu_0 \triangle n_0 + \mathrm{div} J_0)) dxdt,$$

and

$$\begin{aligned} &\iint_{Q_*} \Big(-\varphi_t P_k^{\mathbf{d}} - \nu_0 (\triangle\varphi) P_k^{\mathbf{d}} + \tau^{-1} \varphi P_k^{\mathbf{d}} - \frac{\epsilon^2}{4} (\nabla\triangle\varphi) P_k^0 + T_0 \varphi P_k^0 \Big) dxdt \\ = &\iint_{Q_*} -\varphi S(P_{k-1}) dxdt + \iint_{Q_*} \varphi \big(\nu_0 \triangle J_0 - \tau^{-1} J_0 + T_0 \nabla n_0 - \frac{\epsilon^2}{4} \nabla \triangle n_0 \big) dxdt, \end{aligned}$$

where

$$\begin{aligned} &\iint_{Q_*} -\varphi S(P_{k-1}) dxdt \\ = &\iint_{Q_*} \left(\frac{\left(P_{k-1}^{\mathbf{d}} + J_0\right) \otimes \left(P_{k-1}^{\mathbf{d}} + J_0\right)}{P_{k-1}^0 + n_0} \right) \nabla\varphi dxdt + \iint_{Q_*} V_{k-1} \nabla \left(P_{k-1}^0 + n_0\right) \varphi dxdt \\ &+ \iint_{Q_*} V_{k-1} \left(P_{k-1}^0 + n_0\right) \nabla\varphi dxdt \\ &+ \frac{\epsilon^2}{4} \iint_{Q_*} \left(\frac{\nabla \left(P_{k-1}^0 + n_0\right) \otimes \nabla \left(P_{k-1}^0 + n_0\right)}{P_{k-1}^0 + n_0} \right) \nabla\varphi dxdt \\ =: &\iint_{Q_*} \mathcal{X}_1(P_{k-1}) \nabla\varphi dxdt + \mathcal{X}_2(P_{k-1}) + \mathcal{X}_3(P_{k-1}) + \frac{\epsilon^2}{4} \iint_{Q_*} \mathcal{X}_4(P_{k-1}) \nabla\varphi dxdt. \end{aligned}$$

Notice that $\mathcal{X}_1(P_{k-1})$ and $\mathcal{X}_4(P_{k-1})$ are $d \times d$ matrices, and let $(\mathcal{X}_1(P_{k-1}))_{ls}$ $(l, s = 1, \cdots, d)$, $(\mathcal{X}_4(P_{k-1}))_{ls}$ denote the entry in position (l, s) respectively.

Then we find

$$\begin{aligned}
&\left|\iint_{Q_*}\mathcal{X}_1(P_{k-1})\nabla\varphi dxdt-\iint_{Q_*}\mathcal{X}_1(P^*)\nabla\varphi dxdt\right|\\
=&\left|\iint_{Q_*}\left(\mathcal{X}_1(P_{k-1})-\mathcal{X}_1(P^*)\right)\nabla\varphi dxdt\right|\\
\leq&\sup_{l\in\{1,\cdots,d\}}\left|\sum_{s=1}^{d}\iint_{Q_*}\left(\left((\mathcal{X}_1(P_{k-1}))_{ls}-(\mathcal{X}_1(P^*))_{ls}\right)\partial_{x_s}\varphi\right)dxdt\right|.
\end{aligned}\tag{3.3.144}$$

We compute $(\mathcal{X}_1(P_{k-1}))_{ls}-(\mathcal{X}_1(P^*))_{ls}$:

$$(\mathcal{X}_1(P_{k-1}))_{ls}-(\mathcal{X}_1(P^*))_{ls}=\mathcal{G}_1+\mathcal{G}_2+\mathcal{G}_3$$

where

$$\begin{aligned}
\mathcal{G}_1&:=\frac{(P_{k-1}^{\mathbf{d}}+J_0)_s((P^*)^0+n_0)}{(P_{k-1}^0+n_0)((P^*)^0+n_0)}\left(P_{k-1}^{\mathbf{d}}-(P^*)^{\mathbf{d}}\right)_l,\\
\mathcal{G}_2&:=\frac{((P^*)^{\mathbf{d}}+J_0)_l((P^*)^0+n_0)}{(P_{k-1}^0+n_0)((P^*)^0+n_0)}\left(P_{k-1}^{\mathbf{d}}-(P^*)^{\mathbf{d}}\right)_s,\\
\mathcal{G}_3&:=\frac{((P^*)^{\mathbf{d}}+J_0)_l((P^*)^{\mathbf{d}}+J_0)_s}{(P_{k-1}^0+n_0)((P^*)^0+n_0)}\left((P^*)^0-P_{k-1}^0\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\|\mathcal{G}_1\|_{L^2(\Omega\times(0,t_*))}&\leq\delta_0^{-4}\|P_{k-1}^{\mathbf{d}}-(P^*)^{\mathbf{d}}\|_{L^2(\Omega\times(0,t_*))},\\
\|\mathcal{G}_2\|_{L^2(\Omega\times(0,t_*))}&\leq\delta_0^{-4}\|P_{k-1}^{\mathbf{d}}-(P^*)^{\mathbf{d}}\|_{L^2(\Omega\times(0,t_*))},\\
\|\mathcal{G}_2\|_{L^2(\Omega\times(0,t_*))}&\leq\delta_0^{-4}\|(P^*)^0-P_{k-1}^0\|_{L^2(\Omega\times(0,t_*))}.
\end{aligned}$$

Then we deduce

$$\begin{aligned}
&\|(\mathcal{X}_1(P_{k-1}))_{ls}-(\mathcal{X}_1(P^*))_{ls}\|_{L^2(\Omega\times(0,t_*))}\\
\leq&\,\delta_0^{-4}\left(2\|P_{k-1}^{\mathbf{d}}-(P^*)^{\mathbf{d}}\|_{L^2(\Omega\times(0,t_*))}+\|(P^*)^0-P_{k-1}^0\|_{L^2(\Omega\times(0,t_*))}\right).
\end{aligned}\tag{3.3.145}$$

Going back to (3.3.144) we find

$$\begin{aligned}
&\left|\iint_{Q_*}\mathcal{X}_1(P_{k-1})\nabla\varphi dxdt-\iint_{Q_*}\mathcal{X}_1(P^*)\nabla\varphi dxdt\right|\\
\leq&\,C(\varphi)\sup_{l\in\{1,\cdots,d\}}\sum_{s=1}^{d}\|(\mathcal{X}_1(P_{k-1}))_{ls}-(\mathcal{X}_1(P^*))_{ls}\|_{L^2(\Omega\times(0,t_*))},
\end{aligned}\tag{3.3.146}$$

$C(\varphi) > 0$ is a constant depending upon φ. Substitute (3.3.145) into (3.3.146), it follows

$$\left| \iint_{Q_*} \mathcal{X}_1(P_{k-1}) \nabla \varphi dxdt - \iint_{Q_*} \mathcal{X}_1(P^*) \nabla \varphi dxdt \right|$$
$$\leq d\delta_0^{-4} C(\varphi) \left(2\|P_{k-1}^{\mathbf{d}} - (P^*)^{\mathbf{d}}\|_{L^2(\Omega\times(0,t_*))} + \|(P^*)^0 - P_{k-1}^0\|_{L^2(\Omega\times(0,t_*))} \right).$$

Using **Lemma** 3.3.3 we conclude

$$\left| \iint_{Q_*} \mathcal{X}_1(P_{k-1}) \nabla \varphi dxdt - \iint_{Q_*} \mathcal{X}_1(P^*) \nabla \varphi dxdt \right| \longrightarrow 0, \text{ as } k \to \infty.$$

By a similar calculation it is easy to verify that

$$\frac{\epsilon^2}{4} \iint_{Q_*} \mathcal{X}_4(P_{k-1}) \nabla \varphi dxdt \longrightarrow \frac{\epsilon^2}{4} \iint_{Q_*} \mathcal{X}_4(P^*) \nabla \varphi dxdt, \text{ as } k \to \infty.$$

Finally from **Lemma** 3.3.3 it is easy to infer that $\mathcal{X}_2(P_{k-1})$, $\mathcal{X}_3(P_{k-1})$ converges to $\mathcal{X}_2(P^*)$ and $\mathcal{X}_3(P^*)$ respectively as k tends to infinity.

Now it follows

$$\iint_{Q_*} -\varphi S(P_{k-1}) dxdt \longrightarrow \iint_{Q_*} -\varphi S(P^*) dxdt.$$

Thus we conclude that P^* is the solution we seek.

Uniqueness

Let (n^1, J^1, V^1) and (n^2, J^2, V^2) be two solutions of (3.0.1)-(3.0.4). Put

$$n_\triangle = n^1 - n^2, \qquad J_\triangle = J^1 - J^2, \qquad V_\triangle = V^1 - V^2,$$

then we obtain the system

$$\begin{cases} \partial_t n_\triangle - \nu_0 \triangle n_\triangle = \operatorname{div} J_\triangle, \\ \partial_t J_\triangle - \nu_0 \triangle J_\triangle + \frac{1}{\tau} J_\triangle - T_0 \nabla n_\triangle + \frac{\epsilon^2}{4} \nabla \triangle n_\triangle = F_1 - F_2, \\ \lambda^2 \triangle V_\triangle = n_\triangle, \\ (n_\triangle, J_\triangle, V_\triangle)(t,x) = 0 \quad \text{on } [0, t_*) \times \partial\Omega, \\ (n_\triangle, J_\triangle)(0,x) = 0, \end{cases}$$

where

$$F_1 := -\operatorname{div}\left(\frac{J^1 \otimes J^1}{n^1} \right) + n^1 \nabla V^1 - \epsilon^2 \operatorname{div}\left((\nabla\sqrt{n^1}) \otimes (\nabla\sqrt{n^1}) \right),$$

$$F_2 := -\operatorname{div}\left(\frac{J^2 \otimes J^2}{n^2} \right) + n^2 \nabla V^2 - \epsilon^2 \operatorname{div}\left((\nabla\sqrt{n^2}) \otimes (\nabla\sqrt{n^2}) \right).$$

Similarly as (3.3.28)-(3.3.33) we obtain

$$\begin{aligned}&\frac{1}{2}\partial_t\Big(T_0\|n_\triangle\|^2+\frac{\epsilon^2}{4}\|\nabla n_\triangle\|^2+\|J_\triangle\|^2\Big)+\nu_0T_0\|\nabla n_\triangle\|^2\\&\qquad+\frac{\epsilon^2}{4}\nu_0\|\triangle n_\triangle\|^2+\nu_0\|\nabla J_\triangle\|^2+\frac{1}{\tau}\|J_\triangle\|^2\\&=(F_1-F_2,J_\triangle)\le\|F_1-F_2\|_{H^{-1}(\Omega)}\|J_\triangle\|_{H^1_0(\Omega)}.\end{aligned}\tag{3.3.147}$$

We go back to the proof of **Lemma** 3.3.3. By a similar calculations as (3.3.121)-(3.3.126) we deduce

$$\|F_1-F_2\|^2_{H^{-1}(\Omega)}\le C(\|n_\triangle\|^2_{H^1(\Omega)}+\|J_\triangle\|^2_{L^2(\Omega)}).\tag{3.3.148}$$

(3.3.147) and (3.3.148) together with Cauchy's inequality yield

$$\begin{aligned}&\frac{1}{2}\partial_t\left(T_0\|n_\triangle\|^2+\frac{\epsilon^2}{4}\|\nabla n_\triangle\|^2+\|J_\triangle\|^2\right)\\&\le C(\|n_\triangle\|^2_{H^1(\Omega)}+\|J_\triangle\|^2_{L^2(\Omega)}).\end{aligned}\tag{3.3.149}$$

Applying Gronwall's lemma on (3.3.149) then

$$n_\triangle\equiv J_\triangle\equiv 0.$$

Chapter 4

Global Existence and Exponential Decay on a Torus

We assume that the doping profile of background charges is constant, denoted by $\mathcal{C}_0$. Then it is easy to verify that $(\mathcal{C}_0, 0, 0)$ is a steady state of (3.0.1) on a torus $\mathbb{T}^d$ or on a bounded domain Ω under insulating boundary conditions. In order to extend the local solution of (3.0.1)-(3.0.3) globally in time we need to establish uniform bounds. Here we only consider the situation when the initial data is close to the steady state $(\mathcal{C}_0, 0, 0)$.

4.1 Reformulation

We consider the viscous QHD model (3.0.1) on a d-dimensional torus $\mathbb{T}^d$. The local existence of solutions to the viscous model of quantum hydrodynamics on a torus is proved in [17]. We will study the situation when the initial data are assumed in a small neighborhood of the steady state $(\mathcal{C}_0, 0, 0)$. The idea of extending the local classical solution globally in time is first to establish uniform estimates, and then to use the usual continuity argument.

Suppose

$$\inf_{x\in\Omega} n_0(x) > 0, \tag{4.1.1}$$

$$\int_\Omega (n_0 - \mathcal{C}_0)dx = 0. \tag{4.1.2}$$

The second condition (4.1.2) is necessary, otherwise the Poisson equation for V would not be solvable from Green's formulas. Note that (4.1.1) and (4.1.2) imply $\mathcal{C}_0 > 0$.

In the following, we use the abbreviation

$$u = n - \mathcal{C}_0, \quad u_0 = n_0 - \mathcal{C}_0. \tag{4.1.3}$$

Now we reformulate the original problem (3.0.1) into an equivalent one with respect to the classical solution (u, J, V) which describes the perturbation to the steady state $(\mathcal{C}_0, 0, 0)$.

$$u_t - \nu_0 \triangle u = \mathrm{div} J, \tag{4.1.4}$$

$$J_t - \nu_0 \triangle J + \frac{1}{\tau} J - T_0 \nabla u + (u + \mathcal{C}_0)\nabla V + \frac{\epsilon^2}{4} \nabla \triangle u = G, \tag{4.1.5}$$

$$\lambda^2 \triangle V = u, \tag{4.1.6}$$

$$(u, J)(0, x) = (u_0, J_0), \tag{4.1.7}$$

here

$$\begin{aligned} G := &\mathrm{div}\left(\frac{J \otimes J}{u + \mathcal{C}_0}\right) + \epsilon^2 \mathrm{div}\left(\left(\nabla \sqrt{u + \mathcal{C}_0}\right) \otimes \left(\nabla \sqrt{u + \mathcal{C}_0}\right)\right) \\ = &\mathrm{div}\left(\frac{J \otimes J}{u + \mathcal{C}_0}\right) + \frac{\epsilon^2}{4} \mathrm{div}\left(\frac{\nabla u \otimes \nabla u}{u + \mathcal{C}_0}\right) \end{aligned}$$

Furthermore we deduce

$$(4.1.4) \Longrightarrow -\nu_0 \triangle u_t + \nu_0^2 \triangle^2 u = -\nu_0 \mathrm{div} \triangle J, \tag{4.1.8}$$

$$(4.1.4) \Longrightarrow \frac{1}{\tau} - \frac{1}{\tau} \nu_0 \triangle u = \frac{1}{\tau} \mathrm{div} J. \tag{4.1.9}$$

Differentiate (4.1.4) with respect to t, take divergence of (4.1.5), add the resultants, then substitute (4.1.8) (4.1.9) into it, we obtain a nonlinear fourth-order wave equation for u

$$\begin{aligned} u_{tt} + \frac{1}{\tau} u_t - 2\nu_0 \triangle u_t + &\frac{\mathcal{C}_0}{\lambda^2} u - (T_0 + \frac{\nu_0}{\tau}) \triangle u \\ &+ \left(\frac{\epsilon^2}{4} + \nu_0^2\right) \triangle^2 u = \mathrm{div} G. \end{aligned} \tag{4.1.10}$$

Introduce the perturbations of the steady-state $(\mathcal{C}_0, 0, 0)$

$$u = n - \mathcal{C}_0, \quad J = J - 0, \quad V = V - 0.$$

Then, using (4.1.4)-(4.1.6) and (4.1.10), the evolution equations for (u, J, V) read as follows

$$\begin{cases} u_{tt} + \frac{1}{\tau} u_t - 2\nu_0 \triangle u_t + \frac{\mathcal{C}_0}{\lambda^2} u \\ \quad -(T_0 + \frac{\nu_0}{\tau}) \triangle u + \left(\frac{\epsilon^2}{4} + \nu_0^2\right) \triangle^2 u = \mathrm{div} G, \\ \qquad J_t - \nu_0 \triangle J + \frac{1}{\tau} J = \mathcal{H}, \\ \qquad \lambda^2 \triangle V = u, \end{cases} \tag{4.1.11}$$

where

$$\mathcal{H} := G + T_0\nabla u - (u+\mathcal{C}_0)\nabla V - \frac{\epsilon^2}{4}\nabla\triangle u.$$

The initial data are given:

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad J(0,x) = J_0(x) \tag{4.1.12}$$

with

$$u_1(x) := \mathrm{div}J_0 + \nu_0\triangle n_0.$$

4.2 The A-priori Estimates

In view of the uniform a priori estimates of (u, J, V), we are able to extend the local classical solution globally in time and prove its exponential convergence to the stationary solution $(\mathcal{C}_0, 0, 0)$.

Assume that for any given $T > 0$, there is a classical solution (u, J, V) of the IVP (4.1.11)-(4.1.12) satisfying the regularity conditions

$$(u, J, V) \in S_u \times S_J \times S_V, \tag{4.2.1}$$

where

$$S_u := H^1(0,T;H^5(\mathbb{T}^d))\bigcap L^2(0,T;,H^6(\mathbb{T}^d)),$$

$$S_J := C([0,T],H^4(\mathbb{T}^d))\bigcap L^2(0,T;,H^5(\mathbb{T}^d)),$$

$$S_V := C([0,T],H^4(\mathbb{T}^d)).$$

Sobolev embedding theorem provides

$$H^1(0,T;H^5(\mathbb{T}^d)) \hookrightarrow C([0,T],H^5(\mathbb{T}^d)).$$

Assume that

$$\delta_T := \max_{0\le t\le T}\left(\|u\|_{H^5(\mathbb{T}^d)} + \|J\|_{H^4(\mathbb{T}^d)}\right). \tag{4.2.2}$$

By the Sobolev embedding theorem

$$H^5(\mathbb{T}^d)\times H^4(\mathbb{T}^d)\times H^4(\mathbb{T}^d) \longrightarrow C^3(\overline{\mathbb{T}^d})\times C^2(\overline{\mathbb{T}^d})\times C^2(\overline{\mathbb{T}^d}),$$

it is easy to verify that if δ_T is sufficiently small, there are constants ϕ_-, ϕ_+ such that

$$0 < \phi_- \le u + \mathcal{C}_0 \le \phi_+.$$

In the following we assume that δ_T is sufficiently small such that the above estimate holds. And we will use C as a generic constant which may change from line to line and is not allowed to depend on time $t \in [0, T]$.

Lemma 4.2.1. *Given the multi-index* α *satisfies* $0 \le \alpha \le 4$*, then the following inequality holds*

$$\frac{d}{dt}\|J\|_{|\alpha|}^2 + C\|J\|_{|\alpha|+1}^2 \le C\|(u, u_t, \nabla u, \triangle u)\|_{|\alpha|}^2. \tag{4.2.3}$$

Proof. Setting $\hat{u} := D^\alpha u$, $\hat{J} := D^\alpha J$, one obtains the following equation

$$\hat{J}_t - \nu_0 \triangle \hat{J} + \frac{1}{\tau}\hat{J} = D^\alpha G + T\nabla\hat{u} - D^\alpha\left((u+\mathcal{C}_0)\nabla V\right) - \frac{\epsilon^2}{4}\nabla\triangle\hat{u}. \tag{4.2.4}$$

Take the inner product between (4.2.4) and $\hat{J}$, and integrate by parts over $\mathbb{T}^d$:

$$\frac{1}{2}\frac{d}{dt}\|\hat{J}\|^2 + \frac{1}{\tau}\|\hat{J}\|^2 + \nu_0\|\nabla\hat{J}\|^2 \tag{4.2.5}$$

$$= T(\nabla\hat{u}, \hat{J}) + \frac{\epsilon^2}{4}(\triangle\hat{u}, \mathrm{div}\hat{J}) + L_1 + L_2 + L_3, \tag{4.2.6}$$

here define

$$L_1 : = -\sum_{k,l}\int_{\mathbb{T}^d}(\partial_k\hat{J}_l)D^\alpha\left(\frac{J_l J_k}{u+\mathcal{C}_0}\right)dx,$$

$$L_2 : = -\frac{\epsilon^2}{4}\sum_{k,l}\int_{\mathbb{T}^d}(\partial_k\hat{J}_l)D^\alpha\left(\frac{(\partial_k u)(\partial_l u)}{u+\mathcal{C}_0}\right)dx,$$

$$L_3 : = -\int_{\mathbb{T}^d}\hat{J}(D^\alpha((u+\mathcal{C}_0)\nabla V))dx.$$

Next we estimate the integrals L_1, L_2, L_3. The constants in the following computations may change from one line to another, and can depend on the order of differentiation $|\alpha|$, the space dimension d and δ_T, but are independent of the time T. Recall the embedding $H^{|\alpha|}(\mathbb{T}^d) \subset L^\infty(\mathbb{T}^d)$, and We will make free use of the following Moser-type calculus inequalities [55, 61, 76]

$$\|D^\alpha(fg)\| \le C(\|f\|_{L^\infty}\|D^\alpha g\| + \|D^\alpha f\|\|g\|_{L^\infty}),$$

$$\|D^\alpha(fg) - fD^\alpha g\| \le C(\|g\|_{L^\infty}\|D^\alpha f\| + \|D^{\alpha-1}g\|\|f\|_{L^\infty}).$$

Then we conclude that

$$|L_1| \le \sum_{k,l}\|\partial_k\hat{J}_l\|_{L^2}\|(u+\mathcal{C}_0)^{-1}J_l J_k\|_{H^{|\alpha|}}$$

$$\le C\|\nabla\hat{J}\|_{L^2}\sum_{k,l}\left(\|(u+\mathcal{C}_0)^{-1}J_l\|_{L^\infty}\|J_k\|_{H^{|\alpha|}} + \|(u+\mathcal{C}_0)^{-1}J_l\|_{H^{|\alpha|}}\|J_k\|_{L^\infty}\right).$$

Since the following estimates hold

$$
\begin{aligned}
\|(u+\mathcal{C}_0)^{-1}J_l\|_{L^\infty} &\le C\phi_-\delta_T, \\
\|(u+\mathcal{C}_0)^{-1}J_l\|^2_{H^{|\alpha|}} &= \sum_{\beta\le\alpha}\left\|\sum_{\gamma\le\beta}\frac{\gamma!}{\beta!(\beta-\gamma)!}D^\gamma\left(\frac{1}{u+\mathcal{C}_0}\right)D^{\beta-\gamma}J_l\right\|^2_{L^2} \\
&\le C\sum_{\beta\le\alpha}\sum_{\gamma\le\beta}\left\|D^\gamma\left(\frac{1}{u+\mathcal{C}_0}\right)D^{\beta-\gamma}J_l\right\|^2_{L^2} \\
&\le C\delta_T^2\,\|J\|^2_{H^{|\alpha|+1}}\,,
\end{aligned}
$$

we infer

$$
|L_1| \le C\sigma_1\|\nabla J\|^2_{H^{|\alpha|}} + \frac{C(\delta_T^2+\delta_T^4)}{\sigma_1}\|J\|^2_{H^{|\alpha|+1}}. \tag{4.2.7}
$$

Concerning the second integral, we have

$$
\begin{aligned}
|L_2| &\le C\sum_{k,l}\|\partial_k\hat{J}_l\|_{L^2}\left\|(u+\mathcal{C}_0)^{-1}(\partial_l u)(\partial_k u)\right\|_{H^{|\alpha|}} \\
&\le C\|\nabla\hat{J}\|_{L^2}\sum_{k,l}\left(\|(u+\mathcal{C}_0)^{-1}\partial_l u\|_{L^\infty}\|\partial_k u\|_{H^{|\alpha|}} + \|(u+\mathcal{C}_0)^{-1}\partial_l u\|_{H^{|\alpha|}}\|\partial_k u\|_{L^\infty}\right).
\end{aligned}
$$

Similarly to the computations of L_1, we deduce

$$
|L_2| \le C\sigma_2\|\nabla J\|^2_{H^{|\alpha|}} + \frac{C(\delta_T^2+\delta_T^4)}{\sigma_2}\|\nabla u\|^2_{H^{|\alpha|}}. \tag{4.2.8}
$$

Concerning the third integral, we have

$$
\begin{aligned}
|L_3| &\le \|\hat{J}\|_{L^2}\|(u+\mathcal{C}_0)\nabla V\|_{H^{|\alpha|}} \\
&\le C\|\hat{J}\|_{L^2}\left(\|u+\mathcal{C}_0\|_{L^\infty}\|\nabla V\|_{H^{|\alpha|}} + \|u+\mathcal{C}_0\|_{H^{|\alpha|}}\|\nabla V\|_{L^\infty}\right) \\
&\le C\|\hat{J}\|_{L^2}\left(\|V\|_{H^{|\alpha|+1}} + \|u+\mathcal{C}_0\|_{H^{|\alpha|}}\|V\|_{H^{|\alpha|+1}}\right) \\
&\le \frac{C}{\lambda^2}\|\hat{J}\|_{L^2}\left(\|u\|_{H^{|\alpha|-1}} + \|u+\mathcal{C}_0\|_{H^{|\alpha|}}\|u\|_{H^{|\alpha|-1}}\right) \\
&\le C\sigma_3\|\hat{J}\|^2_{L^2} + \frac{C}{\sigma_3}\|u\|^2_{H^{|\alpha|}}.
\end{aligned}
$$

Therefore we obtain

$$
|L_3| \le C\sigma_3\|J\|^2_{H^{|\alpha|}} + \frac{C}{\sigma_3}\|u\|^2_{H^{|\alpha|}}. \tag{4.2.9}
$$

The items σ_i are arbitrarily selected positive numbers.

Recall (4.2.5), substitute (4.2.7), (4.2.8) and (4.2.9) into (4.2.5), then sum up for all derivatives whose order are less than $|\alpha|$, we obtain (4.2.3), that completes the proof of lemma 4.2. □

Next we estimate $\|u\|_{H^{|\alpha|+2}}$ in terms of $\|J\|_{H^{|\alpha|}}$

Lemma 4.2.2. *We set*

$$Y(x,t) := \left(\left\| \sqrt{\frac{\tau}{2}} u_t + \sqrt{\frac{1}{2\tau}} u \right\|_{|\alpha|}^2 + \frac{(\tau+1)\mathcal{C}_0}{2\lambda^2} \|u\|_{|\alpha|}^2 \right.$$
$$\left. + \frac{(\tau+1)(4\nu_0^2+\epsilon^2)}{8} \|\triangle u\|_{|\alpha|}^2 + \frac{1}{2}\|u_t\|_{|\alpha|}^2 + c' \|\nabla u\|_{|\alpha|}^2 \right)$$

here

$$c' := \frac{(\tau+1)(T+\frac{\nu_0}{\tau}) + 2\nu_0}{2}.$$

Then

$$\frac{d}{dt}Y + CY \le C\delta_T \|J\|_{H^{|\alpha|+1}}^2. \tag{4.2.10}$$

Proof. From (4.1.10) $\hat{u}$ satisfies the equation

$$\begin{aligned} &\hat{u}_{tt} + \frac{1}{\tau}\hat{u}_t - 2\nu_0 \triangle \hat{u}_t + \frac{\mathcal{C}_0}{\lambda^2}\hat{u} - (T+\frac{\nu_0}{\tau})\triangle\hat{u} + \left(\frac{\epsilon^2}{4}+\nu_0^2\right)\triangle^2\hat{u} \\ &= \mathrm{div} D^\alpha G. \end{aligned} \tag{4.2.11}$$

Taking the inner product between (4.2.11) and $\hat{u} + (\tau+1)\hat{u}_t$, then integrating the resulting equation by parts over $\mathbb{T}^d$ yields

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{T}^d} \left(\left(\sqrt{\frac{\tau}{2}}\hat{u}_t + \sqrt{\frac{1}{2\tau}}\hat{u} \right)^2 + \frac{(\tau+1)\mathcal{C}_0}{2\lambda^2}\hat{u}^2 \right. \\ &\qquad \left. + \frac{(\tau+1)(4\nu_0^2+\epsilon^2)}{8}|\triangle\hat{u}|^2 + \frac{1}{2}\hat{u}_t^2 + c'|\nabla\hat{u}|^2 \right) dx \\ &\qquad + \left(\frac{1}{4}\epsilon^2+\nu_0^2\right)\|\triangle\hat{u}\|^2 + \frac{\mathcal{C}_0}{\lambda^2}\|\hat{u}\|^2 + \frac{1}{\tau}\|\hat{u}_t\|^2 \\ &\qquad + (T+\frac{\nu_0}{\tau})\|\nabla\hat{u}\|^2 + 2\nu_0(\tau+1)\|\nabla\hat{u}_t\|^2 \\ &= -(D^\alpha G, \nabla(\hat{u}+(\tau+1)\hat{u}_t)) \\ &\le C\left(\frac{1}{\sigma_*}\|D^\alpha G\|^2 + \sigma_*(\|\nabla\hat{u}\|^2 + \|\nabla\hat{u}_t\|^2) \right). \end{aligned}$$

Concerning $\|D^\alpha G\|^2$, we have

$$\begin{aligned} \|D^\alpha G\|^2 &\le C\left(\left\| \mathrm{div} D^\alpha \left(\frac{J\otimes J}{u+\mathcal{C}_0} \right) \right\|^2 + \left\| \mathrm{div} D^\alpha \left(\frac{(\nabla u)\otimes(\nabla u)}{u+\mathcal{C}_0} \right) \right\|^2 \right) \\ &\le C(\delta_T^2+\delta_T^4)\left(\|J\|_{H^{|\alpha|+1}}^2 + \|(\nabla u, \triangle u)\|_{H^{|\alpha|}}^2 \right), \end{aligned}$$

then we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^d}\Bigg(\left(\sqrt{\frac{\tau}{2}}\hat{u}_t+\sqrt{\frac{1}{2\tau}}\hat{u}\right)^2+\frac{(\tau+1)\mathcal{C}_0}{2\lambda^2}\hat{u}^2\\
&\qquad+\frac{(\tau+1)(4\nu_0^2+\epsilon^2)}{8}|\triangle\hat{u}|^2+\frac{1}{2}\hat{u}_t^2+c'|\nabla\hat{u}|^2\Bigg)dx\\
&\qquad\quad+\left(\frac{1}{4}\epsilon^2+\nu_0^2\right)\|\triangle\hat{u}\|^2+\frac{\mathcal{C}_0}{\lambda^2}\|\hat{u}\|^2+\frac{1}{\tau}\|\hat{u}_t\|^2\\
&\qquad\quad+(T+\frac{\nu_0}{\tau})\|\nabla\hat{u}\|^2+2\nu_0(\tau+1)\|\nabla\hat{u}_t\|^2\\
&\quad\le\frac{C}{\sigma_*}(\delta_T^2+\delta_T^4)\left(\|J\|^2_{H^{|\alpha|+1}}+\|(\nabla u+\triangle u)\|^2_{H^{|\alpha|}}\right)+C\sigma_*(\|\nabla\hat{u}\|^2+\|\nabla\hat{u}_t\|^2)
\end{aligned}
$$

for $\sigma_*>0$. Sum up for all orders less equal to $|\alpha|$

$$
\begin{aligned}
&\frac{d}{dt}Y+\left(\frac{1}{4}\epsilon^2+\nu_0^2\right)\|\triangle u\|^2_{|\alpha|}+\frac{\mathcal{C}_0}{\lambda^2}\|u\|^2_{|\alpha|}+\frac{1}{\tau}\|u_t\|^2_{|\alpha|}\\
&\qquad+(T+\frac{\nu_0}{\tau})\|\nabla u\|^2_{|\alpha|}+2\nu_0(\tau+1)\|\nabla u_t\|^2_{|\alpha|}\\
&\quad\le\frac{C}{\sigma_*}(\delta_T^2+\delta_T^4)\left(\|J\|^2_{|\alpha|+1}+\|(\nabla u+\triangle u)\|^2_{|\alpha|}\right)+C\sigma_*(\|\nabla u\|^2_{|\alpha|}+\|\nabla u_t\|^2_{|\alpha|}).
\end{aligned}
$$

Note that there exists a constant κ, such that

$$
\begin{aligned}
&\left\|\sqrt{\frac{\tau}{2}}u_t+\sqrt{\frac{1}{2\tau}}u\right\|^2_{|\alpha|}+\frac{(\tau+1)\mathcal{C}_0}{2\lambda^2}\|u\|^2_{|\alpha|}+\frac{(\tau+1)(4\nu_0^2+\epsilon^2)}{8}\|\triangle u\|^2_{|\alpha|}\\
&\qquad+\frac{1}{2}\|u_t\|^2_{|\alpha|}+c'\|\nabla u\|^2_{|\alpha|}\\
&\quad\le\kappa\|(u,u_t,\nabla u,\triangle u)\|^2_{|\alpha|}.
\end{aligned}
$$

Select δ_T, σ_* sufficiently small such that

$$
\begin{aligned}
&\min\Bigg(T+\frac{\nu_0}{\tau}-C\sigma_*-\frac{C}{\sigma_*}(\delta_T^2+\delta_T^4),\\
&\qquad\frac{\epsilon^2}{4}+\nu_0^2-\frac{C}{\sigma_*}(\delta_T^2+\delta_T^4),2\nu_0(\tau+1)-C\sigma_*\Bigg)>0,\quad\delta_T\le1,
\end{aligned}
$$

then there is a positive constant β_1, such that

$$
\begin{aligned}
&\frac{d}{dt}Y+\beta_1\Bigg(\left(\frac{1}{4}\epsilon^2+\nu_0^2\right)\|\triangle u\|^2_{|\alpha|}+\frac{\mathcal{C}_0}{\lambda^2}\|u\|^2_{|\alpha|}+\frac{1}{\tau}\|u_t\|^2_{|\alpha|}\\
&\qquad+(T+\frac{\nu_0}{\tau})\|\nabla u\|^2_{|\alpha|}+2\nu_0(\tau+1)\|\nabla u_t\|^2_{|\alpha|}\Bigg)\\
&\quad\le C\delta_T\|J\|^2_{|\alpha|+1}.
\end{aligned}
\tag{4.2.12}
$$

which implies (4.2.10). □

4.3 Global Existence and Exponential Decay

Now we are in a position to show the following

Theorem 4.3.1. *Assume that* (4.1.1) *and* (4.1.2) *hold, Let* $(\mathcal{C}_0, 0, 0)$ *be a stationary solution of the initial-value problem* (3.0.1) *on a torus* $\mathbb{T}^d$. *Assum that the initial datum* $(u_0, J_0) \in H^6(\mathbb{T}^d) \times H^5(\mathbb{T}^d)$, *then there is a number* $m_1 > 0$ *such that if*

$$\|u_0\|^2_{H^6(\mathbb{T}^d)} + \|J_0\|^2_{H^5(\mathbb{T}^d)} \leq m_1,$$

the (classical) solution (u, J, V) *of* (4.1.11)-(4.1.12) *exists globally in time and satisfies*

$$\|u\|^2_{H^5(\mathbb{T}^d)} + \|J\|^2_{H^4(\mathbb{T}^d)} + \|V\|^2_{H^4(\mathbb{T}^d)} \leq C\left(\|u_0\|^2_{H^6(\mathbb{T}^d)} + \|J_0\|^2_{H^5(\mathbb{T}^d)}\right)e^{-c_0 t}$$

for all $t \geq 0$. *Here,* $C > 0$ *and* $c_0 > 0$ *are constants independent of time* t.

Proof. lemma 4.2 yields

$$C\delta_T\|J\|^2_{|\alpha|+1} \leq C\delta_T\|(u, u_t, \nabla u, \triangle u)\|^2_{|\alpha|} - C\delta_T\frac{d}{dt}\|J\|^2_{|\alpha|}. \tag{4.3.1}$$

Substituting (4.3.1) into (4.2.10) yields

$$\frac{d}{dt}(Y + C\delta_T\|J\|^2_{|\alpha|}) + CY \leq C\delta_T\|(u, u_t, \nabla u, \triangle u)\|^2_{|\alpha|}.$$

Choose δ_T sufficiently small such that

$$\frac{d}{dt}(Y + C\delta_T\|J\|^2_{|\alpha|}) + CY \leq 0, \tag{4.3.2}$$

then add (4.3.1) and (4.3.2), we obtain

$$\frac{d}{dt}(Y + C\delta_T\|J\|^2_{|\alpha|}) + CY + C\delta_T\|J\|^2_{|\alpha|+1} \leq C\delta_T\|(u, u_t, \nabla u, \triangle u)\|^2_{|\alpha|},$$

$$\Longrightarrow \frac{d}{dt}(Y + C\delta_T\|J\|^2_{|\alpha|}) + CY + C\delta_T\|J\|^2_{|\alpha|} \leq C\delta_T\|(u, u_t, \nabla u, \triangle u)\|^2_{|\alpha|}.$$

Choosing δ_T sufficiently small, then there is a constant C such that

$$\frac{d}{dt}(Y + C\delta_T\|J\|^2_{|\alpha|}) + CY + C\delta_T\|J\|^2_{|\alpha|} \leq 0. \tag{4.3.3}$$

Via solving the differential inequality (3.2.74) we deduce

$$Y + C\delta_T\|J\|^2_{|\alpha|} \leq (Y_0 + C\delta_T\|J_0\|^2_{|\alpha|})e^{-\beta_2 t}, \tag{4.3.4}$$

here $Y_0 := Y(0,x)$, the initial value of Y. Recall (4.1.12) we infer

$$\|u_t(0,\cdot)\|_{|\alpha|}^2 \le C(\|J_0\|_{|\alpha|+1}^2 + \|u_0\|_{|\alpha|+2}^2). \tag{4.3.5}$$

Hence (4.3.4) and (4.3.5) yield

$$\|u\|_{|\alpha|+2}^2 + \|J\|_{|\alpha|}^2 \le C_{\delta_T}(\|J_0\|_{|\alpha|+1}^2 + \|u_0\|_{|\alpha|+2}^2)e^{-\beta_2 t}, \tag{4.3.6}$$

here C_{δ_T} depends upon δ_T. Thus

$$\|n - \mathcal{C}_0\|_5^2 + \|J\|_4^2 + \|V\|_4^2 \le C(\|J_0\|_5^2 + \|u_0\|_6^2)e^{-c_0 t}, \tag{4.3.7}$$

the constants C and c_0 are independent of t. Integrate (4.2.12) from 0 to T we deduce

$$\begin{aligned} &\|u\|_{L^2(0,T;H^6(\mathbb{T}^d))}^2 + \|u\|_{H^1(0,T;H^5(\mathbb{T}^d))}^2 \\ &\le C\delta_T\|J\|_{L^2(0,T;H^5(\mathbb{T}^d))}^2 + CY(0) \end{aligned} \tag{4.3.8}$$

Integrate (4.2.3) from 0 to T we deduce

$$\begin{aligned} \|J\|_{L^2(0,T;H^5(\mathbb{T}^d))}^2 \le &C\left(\|u\|_{L^2(0,T;H^6(\mathbb{T}^d))}^2 + \|u\|_{H^1(0,T;H^5(\mathbb{T}^d))}^2\right) \\ &+ \|J_0\|_{H^4(\mathbb{T}^d)}^2. \end{aligned} \tag{4.3.9}$$

From (4.3.8) and (4.3.9) we obtain

$$\begin{aligned} &\|u\|_{L^2(0,T;H^6(\mathbb{T}^d))}^2 + \|u\|_{H^1(0,T;H^5(\mathbb{T}^d))}^2 \\ &\le C\delta_T\left(\|u\|_{L^2(0,T;H^6(\mathbb{T}^d))}^2 + \|u\|_{H^1(0,T;H^5(\mathbb{T}^d))}^2\right) + C\delta_T\|J_0\|_{H^4(\mathbb{T}^d)}^2 + CY(0). \end{aligned}$$

For sufficiently small δ_T

$$\|u\|_{L^2(0,T;H^6(\mathbb{T}^d))}^2 + \|u\|_{H^1(0,T;H^5(\mathbb{T}^d))}^2 \le C\delta_T\|J_0\|_{H^4(\mathbb{T}^d)}^2 + CY(0). \tag{4.3.10}$$

Go back to (4.3.9) we obtain

$$\|J\|_{L^2(0,T;H^5(\mathbb{T}^d))}^2 \le C\delta_T\|J_0\|_{H^4(\mathbb{T}^d)}^2 + CY(0) + \|J_0\|_{H^4(\mathbb{T}^d)}^2. \tag{4.3.11}$$

By Theorem 2.4 in [17] there exists a local-in-time solution (n, J, V) of the IVP (3.0.1). Assume the initial data (n_0, J_0, V_0) is sufficiently close to $(\mathcal{C}_0, 0, 0)$ then there exists a time interval $[0, T^*]$ such that (3.0.1) admits a local solution (n, J, V) in $[0, T^*]$ and δ_{T^*} is so small that $(n - \mathcal{C}_0, J, V)$ in $[0, T^*]$ satisfies (4.3.7), (4.3.10) and (4.3.11) where T, δ_T is replaced by T^*, δ_{T^*} respectively.

Further choosing the initial data $\|J_0\|_5^2 + \|u_0\|_6^2$ so small that

$$C(\|J_0\|_5^2 + \|u_0\|_6^2) < \delta_{T^*}.$$

By Sobolev embedding theorem and (4.3.7) we conclude first $n > 0$ in $[0, T^*] \times \overline{\mathbb{T}^d}$; and second by the usual continuity argument (n, J, V) exists globally in time and satisfies (4.3.7). □

Chapter 5

Local Solutions to the Non-Isothermal Viscous QHD

5.1 Introduction and the Main Result

Compared to the situation of constant temperature the non-isothermal system is on equations of n, J, V, ne which stands for particle density, the current density, the electrostatic potential and energy density respectively.

In this chapter we investigate the local existence of solutions to the following non-isothermal viscous model of QHD

$$\begin{cases} \partial_t n - \mathrm{div} J = \nu_0 \triangle n, \\ \partial_t J - \mathrm{div}\left(\dfrac{J \otimes J}{n}\right) - \nabla(nT) + n\nabla V \\ \qquad + \dfrac{\epsilon^2}{2} n \nabla\left(\dfrac{\triangle\sqrt{n}}{\sqrt{n}}\right) = \nu_0 \triangle J - \dfrac{J}{\tau} + \mu \nabla n, \\ \partial_t(ne) - \mathrm{div}\left(((ne)E_d + P)\dfrac{J}{n}\right) + J\nabla V = -\dfrac{2}{\tau}(ne) + \dfrac{3}{\tau} n \\ \qquad + \nu_0 \triangle(ne) + \mu \mathrm{div} J, \\ \lambda^2 \triangle V = n - \mathcal{C}(x), \end{cases} \tag{5.1.1}$$

where μ is the interaction constant; the (scaled) stress tensor P and the temperature T satisfy

$$\begin{cases} P = nTE_d - \dfrac{\epsilon^2}{4} n(\nabla \otimes \nabla) \ln n, \\ ne = \dfrac{|J|^2}{2n} + \dfrac{3}{2} nT - \dfrac{\epsilon^2}{8} n \triangle \ln n. \end{cases} \tag{5.1.2}$$

The (scaled) stress tensor consists of the classical pressure and a quantum "pressure" term. The energy density is the sum of kinetic energy, thermal energy, and quantum energy.

The initial data and boundary conditions are

$$(n, J, ne)^T(0) = (n_0, J_0, (ne)_0), \tag{5.1.3}$$

$$(n, J, ne, V)^T|_{\partial\Omega} = (n_\Gamma, J_\Gamma, (ne)_\Gamma, V_\Gamma)^T \tag{5.1.4}$$

with

$$\inf_{x\in\Omega} n_0(x) > 0,\ n_0 \in H^3(\Omega),\ J_0 \in H^2(\Omega),\ (ne)_0 \in H^2(\Omega), \tag{5.1.5}$$

$$n_\Gamma \in H^{5/2}(\Omega),\ J_\Gamma \in H^{3/2}(\Omega), (ne)_\Gamma \in H^{3/2}(\Omega),\ , V_\Gamma \in H^{3/2}(\Omega). \tag{5.1.6}$$

We also require the following compatibility conditions.

$$\begin{cases} (n_0, J_0, (ne)_0)\,|_{\partial\Omega} = (n_\Gamma, J_\Gamma, (ne)_\Gamma), \\ (\nu_0 \triangle n_0 + \mathrm{div} J_0)|_{\partial\Omega} = 0. \end{cases} \tag{5.1.7}$$

Furthermore the given data $J_0, \nabla(ne)_0, \nabla n_0, V_\Gamma$ are supposed to be sufficiently small in

$$(H^2(\Omega))^d \times (L^2(\Omega))^d \times (H^2(\Omega))^d \times H^{3/2}(\Omega))$$

and $\mathcal{C}(x)$ is close to n_0 in $L^2(\Omega)$ norm, which means physically that the initial data of the current density, the spatial change of the initial data of the particle density and the energy density, the boundary function of the electrostatic potential are small; and the doping profile of background charges is close to the initial data of the particle density. Under these assumptions we conclude

Theorem 5.1.1. *There exist* $\mathcal{T}_* > 0$, $\mathcal{C}^* > 0$ *such that the system* (5.1.1) *with the initial data* (5.1.3) *and the boundary conditions* (5.1.4) *under the assumptions* (5.1.5)*,* (5.1.6)*,* (5.1.7) *and*

$$\begin{aligned} &\|\nabla n_0\|_{H^2(\Omega)} + \|J_0\|_{H^2(\Omega)} \\ &\quad + \|\nabla(ne)_0\|_{L^2(\Omega)} + \|n_0 - \mathcal{C}(x)\|_{L^2(\Omega)} + \|V_\Gamma\|_{H^{3/2}(\Omega)} \le \mathcal{C}^*, \end{aligned} \tag{5.1.8}$$

has a unique local-in-time solution (n, J, ne, V) *in* $[0, \mathcal{T}_*)$ *with*

$$\begin{aligned} n &\in L^\infty(0, \mathcal{T}_*; H^3(\Omega)), & J &\in L^\infty(0, \mathcal{T}_*; H^2(\Omega)), \\ \partial_t n &\in L^2(0, \mathcal{T}_*; H^2(\Omega)), & \partial_t J &\in L^2(0, \mathcal{T}_*; H^1(\Omega)), \\ (n, \nabla n, J) &\in C([0, \mathcal{T}_*) \times \bar{\Omega}), & ne &\in L^\infty(0, \mathcal{T}_*; H^2(\Omega)), \\ V &\in C(0, \mathcal{T}_*; H^2(\Omega)), & \partial_t(ne) &\in L^2(0, \mathcal{T}_*; H^1(\Omega)), \\ \partial_t V &\in L^2(0, \mathcal{T}_*; H^1(\Omega)). \end{aligned}$$

5.2 Proof of Theorem 5.1.1

We also study the equations for the perturbation $(n-n_0, J-J_0, (ne)-(ne)_0)$. The main steps to derive a local solution to (5.1.1) are first to construct approximate solutions, second to derive a uniform time interval and their uniform bounds, finally to analysis the limit of the approximate solutions. For uniform bounds we refer to the a-priori estimates of solutions to the isothermal equations in previous chapters.

5.2.1 Reformulation

First the calculus

$$n(\nabla\otimes\nabla)\ln n = (\nabla\otimes\nabla)n - \frac{\nabla n\otimes\nabla n}{n},$$

$$n\triangle\ln n = \triangle n - \frac{|\nabla n|^2}{n},$$

hold. From (5.1.2), we have the represention of the (scaled) stress tensor P and nT:

$$\begin{aligned} nT &= \frac{2}{3}(ne) - \frac{|J|^2}{3n} + \frac{\epsilon^2}{12}\triangle n - \frac{\epsilon^2}{12}\frac{|\nabla n|^2}{n},\\ P &= \left(\frac{2}{3}(ne) - \frac{|J|^2}{3n} + \frac{\epsilon^2}{12}\triangle n - \frac{\epsilon^2}{12}\frac{|\nabla n|^2}{n}\right)E_d - \frac{\epsilon^2}{4}(\nabla\otimes\nabla)n + \frac{\epsilon^2}{4}\frac{\nabla n\otimes\nabla n}{n}\\ &=: P(n-n_0, J-J_0, ne-(ne)_0). \end{aligned}$$

After an equivalent reformulation of (5.1.1) we obtain the following system.

$$\begin{cases} \partial_t n - \nu_0\triangle n - \mathrm{div}J = 0,\\ \partial_t J - \nu_0\triangle J + \dfrac{J}{\tau} + \dfrac{\epsilon^2}{6}\nabla\triangle n - \mu\nabla n = \mathcal{F}_{\mathbf{d}}(n,J,ne),\\ \partial_t(ne) - \nu_0\triangle(ne) + \dfrac{2}{\tau}(ne) = \mathcal{F}(n,J,ne),\\ \lambda^2\triangle V = n - \mathcal{C}(x), \end{cases} \tag{5.2.1}$$

where

$$\begin{aligned} \mathcal{F}_{\mathbf{d}}(n,J,ne) &:= \frac{2}{3}\nabla(ne) + \mathrm{div}\left(\frac{J\otimes J}{n}\right) - n\nabla V + \epsilon^2\mathrm{div}\left((\nabla\sqrt{n})\otimes(\nabla\sqrt{n})\right)\\ &\quad - \frac{1}{3}\nabla\frac{|J|^2}{n} - \frac{\epsilon^2}{12}\nabla\frac{|\nabla n|^2}{n},\\ \mathcal{F}(n,J,ne) &:= \mathrm{div}\left(\frac{J}{n}(ne + P(n-n_0, J-J_0, ne-(ne)_0))\right) - J\nabla V + \frac{3}{\tau}n + \mu\mathrm{div}J. \end{aligned}$$

Set $\mathcal{P} := (\mathcal{P}^n, \mathcal{P}^J, \mathcal{P}^{ne}) := (n - n_0, J - J_0, ne - (ne)_0)$, then from (5.2.1) we obtain the following IBVP with respect to $\mathcal{P}$.

$$\begin{cases} \partial_t \mathcal{P}^n - \nu_0 \triangle \mathcal{P}^n - \mathrm{div} \mathcal{P}^J = \mathfrak{F}_0, \\ \partial_t \mathcal{P}^J - \nu_0 \triangle \mathcal{P}^J + \dfrac{\mathcal{P}^J}{\tau} + \dfrac{\epsilon^2}{6} \nabla \triangle \mathcal{P}^n - \mu \nabla \mathcal{P}^n = \mathfrak{F}_{\mathbf{d}}(\mathcal{P}^n, \mathcal{P}^J, \mathcal{P}^{ne}), \\ \partial_t \mathcal{P}^{ne} - \nu_0 \triangle \mathcal{P}^{ne} + \dfrac{2}{\tau} \mathcal{P}^{ne} = \mathfrak{F}(\mathcal{P}^n, \mathcal{P}^J, \mathcal{P}^{ne}), \\ \lambda^2 \triangle V = \mathcal{P}^n + n_0 - \mathcal{C}(x), \\ \mathcal{P}(0, \cdot) = 0, \ \mathcal{P}|_{\partial\Omega} = 0, \ V|_{\partial\Omega} = V_\Gamma, \end{cases} \tag{5.2.2}$$

where

$$\begin{aligned} \mathfrak{F}_0 &:= \nu_0 \triangle n_0 + \mathrm{div} J_0, \\ \mathfrak{F}_{\mathbf{d}}(\mathcal{P}^n, \mathcal{P}^J, \mathcal{P}^{ne}) &:= \frac{2}{3} \nabla \mathcal{P}^{ne} + \frac{2}{3} \nabla (ne)_0 + \mathrm{div} \left(\frac{(\mathcal{P}^J + J_0) \otimes (\mathcal{P}^J + J_0)}{\mathcal{P}^n + n_0} \right) \\ &\quad - (\mathcal{P}^n + n_0) \nabla V + \epsilon^2 \mathrm{div} \left((\nabla \sqrt{\mathcal{P}^n + n_0}) \otimes (\nabla \sqrt{\mathcal{P}^n + n_0}) \right) \\ &\quad - \frac{1}{3} \nabla \frac{|\mathcal{P}^J + J_0|^2}{\mathcal{P}^n + n_0} - \frac{\epsilon^2}{12} \nabla \frac{|\nabla (\mathcal{P}^n + n_0)|^2}{\mathcal{P}^n + n_0} \\ &\quad + \nu_0 \triangle J_0 - \frac{1}{\tau} J_0 + \mu \nabla n_0 - \frac{\epsilon^2}{6} \nabla \triangle n_0, \\ \mathfrak{F}(\mathcal{P}^n, \mathcal{P}^J, \mathcal{P}^{ne}) &:= \mathrm{div} \left(\frac{\mathcal{P}^J + J_0}{\mathcal{P}^n + n_0} (\mathcal{P}^{ne} + (ne)_0 + P(\mathcal{P}^n, \mathcal{P}^J, \mathcal{P}^{ne})) \right) \\ &\quad + \nu_0 \triangle (ne)_0 - \frac{2}{\tau} (ne)_0 + \frac{3}{\tau} (\mathcal{P}^n + n_0) + \mu \mathrm{div} (\mathcal{P}^J + J_0) \\ &\quad - (\mathcal{P}^J + J_0) \nabla V. \end{aligned}$$

5.2.2 Construction of Approximate Solutions and their Uniform Bounds

Fix $\varphi_1, \varphi_2 > 0$ define

$$c^* := \sqrt{\|\mathfrak{F}(0,0,0)\|^2 + \varphi_1 + \varphi_2}. \tag{5.2.3}$$

Lemma 5.2.1. *Let $T > 0$, $q^{ne} \in L^\infty(0,T; H_0^1(\Omega))$, $\partial_t q^{ne} \in L^\infty(0,T; L^2(\Omega))$, with*

$$\|q^{ne}\|_{L^\infty(0,T;H_0^1(\Omega))} \le c^*, \quad \|\partial_t q^{ne}\|_{L^\infty(0,T;L^2(\Omega))} \le c^*, \tag{5.2.4}$$

then for any $\gamma > 0$ *there exist* $\mathcal{T} \in (0, T]$, $C_{\mu,\nu_0,\epsilon,\tau} > 0$ *depending only on* $\mu, \nu_0, \epsilon, \tau$; $C^*, C' > 0$ *depending only upon* $C_0 := \max(\mathcal{A}_1, \mathcal{A}_2)$ *with*

$$\mathcal{A}_1 := \sqrt{\frac{\left(\mu\|\mathfrak{F}_0\|^2 + \frac{\epsilon^2}{6}\|\nabla\mathfrak{F}_0\|^2 + \|\mathfrak{F}_{\boldsymbol{d}}(0,0,0)\|^2\right) + \gamma}{\min\left(\mu, \frac{\epsilon^2}{6}, 1\right)}},$$

$$\mathcal{A}_2 := \sqrt{\frac{\left(\mu\|\mathfrak{F}_0\|^2 + \frac{\epsilon^2}{6}\|\nabla\mathfrak{F}_0\|^2 + \|\mathfrak{F}_{\boldsymbol{d}}(0,0,0)\|^2\right) + \gamma}{C_{\mu,\nu_0,\epsilon,\tau}}}$$

c^*, *the initial data and all physical constants such that the IBVP*

$$\begin{cases} \partial_t q^n - \nu_0 \triangle q^n - div q^J = \nu_0 \triangle n_0 + div J_0, \\ \partial_t q^J - \nu_0 \triangle q^J + \frac{q^J}{\tau} + \frac{\epsilon^2}{6}\nabla\triangle q^n - \mu\nabla q^n = \mathfrak{F}_{\boldsymbol{d}}(q^n, q^J, q^{ne}), \\ \lambda^2 \triangle q^v = q^n + n_0 - \mathcal{C}(x), \\ (q^n, q^J)(0,\cdot) = 0, \quad (q^n, q^J)|_{\partial\Omega} = 0, \quad q^v|_{\partial\Omega} = V_\Gamma, \end{cases} \tag{5.2.5}$$

has a unique local solution $(q^n, q^J, q^v)^T$ *in* $[0, \mathcal{T})$ *with*

$$\begin{aligned} q^n &\in L^\infty(0,\mathcal{T}; H^3(\Omega)), & q^J &\in L^\infty(0,\mathcal{T}; H^2(\Omega)), \\ \partial_t q^n &\in L^2(0,\mathcal{T}; H^2(\Omega)), & \partial_t q^J &\in L^2(0,\mathcal{T}; H^1(\Omega)), \\ (q^n, \nabla q^n, q^J) &\in C([0,\mathcal{T}) \times \bar{\Omega}), \\ q^v &\in C(0,\mathcal{T}; H^2(\Omega)), \\ \partial_t q^v &\in L^2(0,\mathcal{T}; H^1(\Omega)), \end{aligned}$$

and the following estimates

$$\begin{cases} \|q^n\|_{L^\infty(0,\mathcal{T};H^3(\Omega))} \le C', & \|q^J\|_{L^\infty(0,\mathcal{T};H^2(\Omega))} \le C', \\ \|q^n\|_{L^\infty(0,\mathcal{T};H^2(\Omega))} \le C^*, & \|q^J\|_{L^\infty(0,\mathcal{T};H_0^1(\Omega))} \le C^*, \\ \|q^n\|_{L^\infty(0,\mathcal{T};H_0^1(\Omega))} \le C_0, & \|q^J\|_{L^\infty(0,\mathcal{T};L^2(\Omega))} \le C_0, \\ \|\dot{q}^n\|_{L^\infty(0,\mathcal{T};H_0^1(\Omega))} \le C_0, & \|\dot{q}^J\|_{L^\infty(0,\mathcal{T};L^2(\Omega))} \le C_0, \\ \|\dot{q}^n\|_{L^2(0,\mathcal{T};H^2(\Omega))} \le C_0, & \|\dot{q}^J\|_{L^2(0,\mathcal{T};H_0^1(\Omega))} \le C_0, \end{cases} \tag{5.2.6}$$

and

$$\begin{cases} \inf_{[0,\mathcal{T}]} \inf_{x\in\overline{\Omega}} (q^n + n_0) \geq \delta_0, \\ \sup_{[0,\mathcal{T}]} \max \Big(\|\nabla(q^n + n_0)\|_{L^\infty(\Omega)}, \|q^n + n_0\|_{L^\infty(\Omega)}, \\ \qquad \|q^J + J_0\|_{L^\infty(\Omega)} \Big) \leq \delta_0^{-1}, \end{cases} \tag{5.2.7}$$

hold, where δ_0 *is defined in* (3.3.17).

Proof. This is a direct consequence by making minor modifications of the proof of **Theorem** 3.3.1. □

Additionally we have the following conclusion which is useful for the remainder of the proof.

Lemma 5.2.2. *Let* $T > 0$, f_1, f_2, f_3 *be functions satisfying*

$$\begin{aligned} f_1 &\in L^\infty(0,T;H^3(\Omega)) \cap C([0,t'];C^1(\overline{\Omega})) \cap C([0,T];H^2(\Omega)), \\ f_2 &\in L^\infty(0,T;H^2(\Omega)) \cap C([0,T];C(\overline{\Omega})) \cap C([0,T];H_0^1(\Omega)), \\ \partial_t f_1 &\in L^\infty(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)), \\ \partial_t f_2 &\in L^\infty(0,T;L^2(\Omega)) \cap C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)), \\ f_3 &\in L^\infty(0,T;H^1(\Omega)), \quad \partial_t f_3 \in L^\infty(0,T;L^2(\Omega)), \end{aligned}$$

and

$$\inf_{[0,T]} \inf_{x\in\overline{\Omega}} (f_1 + n_0) > 0, \quad (f_1, f_2, f_3)(0) = 0.$$

Then

$$\begin{cases} \mathfrak{F}(f_1, f_2, f_3) \in L^\infty(0,T,L^2(\Omega)), \\ \partial_t \mathfrak{F}(f_1, f_2, f_3) \in L^2(0,T,H^{-1}(\Omega)), \\ \mathfrak{F}(f_1, f_2, f_3)(0,\cdot) \in L^2(\Omega). \end{cases}$$

Proof. At first

$$\mathfrak{F}(f_1, f_2, f_3) := \mathcal{Y}_1 + \mathcal{Y}_2 + \mathcal{Y}_3 + \mathcal{Y}_4,$$

where

$$\mathcal{Y}_1 := \operatorname{div}\left(((f_3 + (ne)_0)E_d) \frac{f_2 + J_0}{f_1 + n_0} \right), \qquad \mathcal{Y}_2 := \operatorname{div}\left(P(f_1, f_2, f_3) \frac{f_2 + J_0}{f_1 + n_0} \right)$$

$$\mathcal{Y}_3 := -(f_2 + J_0)\nabla g,$$

$$\mathcal{Y}_4 := \nu_0 \triangle (ne)_0 - 2\tau^{-1}(ne)_0 + 3\tau^{-1}(f_1 + n_0) + \mu \operatorname{div}(f_2 + J_0),$$

$$\lambda^2 \triangle g = f_1 + n_0 - \mathcal{C}(x), \quad g|_{\partial\Omega} = V_\Gamma.$$

From our assumptions it is easy to verify that $\mathcal{Y}_1, \mathcal{Y}_4 \in L^\infty(0,T,L^2(\Omega))$ and $\partial_t \mathcal{Y}_1, \partial_t \mathcal{Y}_4 \in L^2(0,T,H^{-1}(\Omega))$.

The item $P(f_1,f_2,f_3)$ has the representation

$$\begin{aligned} P(f_1,f_2,f_3) = \Big(\frac{2}{3}(f_3+(ne)_0) - \frac{|f_2+J_0|^2}{3(f_1+n_0)} + \frac{\epsilon^2}{12}\triangle(f_1+n_0) \\ - \frac{\epsilon^2}{12}\frac{|\nabla(f_1+n_0)|^2}{f_1+n_0}\Big)E_d - \frac{\epsilon^2}{4}(\nabla\otimes\nabla)(f_1+n_0) \\ + \frac{\epsilon^2}{4}\frac{\nabla(f_1+n_0)\otimes\nabla(f_1+n_0)}{f_1+n_0}, \end{aligned}$$

from which we obtain $\|P(f_1,f_2,f_3)\|_{L^\infty(0,T;H^1(\Omega))} < \infty$, which implies $\mathcal{Y}_2 \in L^\infty(0,T,L^2(\Omega))$. Furthermore it is easy to see $\|\partial_t P(f_1,f_2,f_3)\|_{L^2(0,T;L^2(\Omega))} < \infty$ according to our assumptions. Then $\|\partial_t \mathcal{Y}_2\|_{L^2(0,T;H^{-1}(\Omega))} < \infty$.

Now it remains to estimate $\mathcal{Y}_3$.

$$\begin{aligned} &\|\mathcal{Y}_3\|_{L^\infty(0,T;L^2(\Omega))} \le C\|f_2+J_0\|_{L^\infty(0,T;H^1(\Omega))}\|g\|_{L^\infty(0,T;H^2(\Omega))} \\ &\le C\|f_2+J_0\|_{L^\infty(0,T;H^1(\Omega))}\left(\|f_1+n_0-\mathcal{C}(x)\|_{L^\infty(0,T;L^2(\Omega))} + \|V_\Gamma\|_{H^{3/2}(\Omega)}\right), \end{aligned}$$

then $\mathcal{Y}_3 \in L^\infty(0,T;L^2(\Omega))$. Next for a.e. $0 \le t \le T$

$$\begin{aligned} \|\partial_t\mathcal{Y}_3\|_{L^2(\Omega)} &\le C\|\partial_t f_2\|_{H^1(\Omega)}\|g\|_{H^2(\Omega)} + C\|f_2+J_0\|_{H^1(\Omega)}\|\partial_t g\|_{H^2(\Omega)} \\ &\le C\left(\|f_1+n_0-\mathcal{C}(x)\|_{L^\infty(0,T;L^2(\Omega))} + \|V_\Gamma\|_{H^{3/2}(\Omega)}\right)\|\partial_t f_2\|_{H^1(\Omega)} \\ &\quad + C\|\partial_t f_1\|_{L^\infty(0,T;L^2(\Omega))}\|f_2+J_0\|_{L^\infty(0,T;H^1(\Omega))}. \end{aligned}$$

Obviously, $\partial_t\mathcal{Y}_3 \in L^2(0,T;H^{-1}(\Omega))$. □

Let $T>0$, $q_0^{ne}=0$. Then it is trivial that

$$\|q_0^{ne}\|_{L^\infty(0,T;H_0^1(\Omega))} \le c^*, \quad \|\partial_t q_0^{ne}\|_{L^\infty(0,T;L^2(\Omega))} \le c^*.$$

By **Lemma** 5.2.1 there exists a time interval $[0,t_0) \subseteq [0,T)$ such that the system

$$\begin{cases} \partial_t q_0^n - \nu_0\triangle q_0^n - \mathrm{div} q_0^J = \nu_0\triangle n_0 + \mathrm{div}J_0, \\ \partial_t q_0^J - \nu_0\triangle q_0^J + \dfrac{q_0^J}{\tau} + \dfrac{\epsilon^2}{6}\nabla\triangle q_0^n - \mu\nabla q_0^n = \mathfrak{F}_{\mathbf{d}}(q_0^n, q_0^J, q_0^{ne}), \\ \lambda^2\triangle q_0^v = q_0^n + n_0 - \mathcal{C}(x), \\ (q_0^n, q_0^J)(0,\cdot) = 0, \quad (q_0^n,q_0^J)|_{\partial\Omega} = 0, \quad q_0^v|_{\partial\Omega} = V_\Gamma, \end{cases} \tag{5.2.8}$$

admits a unique local-in-time solution (q_0^n, q_0^J, q_0^v) with

$$
\begin{aligned}
q_0^n &\in L^\infty(0,t_0;H^3(\Omega)), & q_0^J &\in L^\infty(0,t_0;H^2(\Omega)),\\
\partial_t q_0^n &\in L^2(0,t_0;H^2(\Omega)), & \partial_t q_0^J &\in L^2(0,t_0;H^1(\Omega)),\\
(q_0^n, \nabla q_0^n, q_0^J) &\in C([0,t_0)\times\bar\Omega),\\
q_0^v &\in C(0,t_0;H^2(\Omega)),\\
\partial_t q_0^v &\in L^2(0,t_0;H^1(\Omega)),
\end{aligned}
$$

$$
\begin{cases}
\|q_0^n\|_{L^\infty(0,t_0;H^3(\Omega))} \le C', & \|q_0^J\|_{L^\infty(0,t_0;H^2(\Omega))} \le C',\\
\|q_0^n\|_{L^\infty(0,t_0;H^2(\Omega))} \le C^*, & \|q_0^J\|_{L^\infty(0,t_0;H_0^1(\Omega))} \le C^*,\\
\|q_0^n\|_{L^\infty(0,t_0;H_0^1(\Omega))} \le C_0, & \|q_0^J\|_{L^\infty(0,t_0;L^2(\Omega))} \le C_0,\\
\|\dot q_0^n\|_{L^\infty(0,t_0;H_0^1(\Omega))} \le C_0, & \|\dot q_0^J\|_{L^\infty(0,t_0;L^2(\Omega))} \le C_0,\\
\|\dot q_0^n\|_{L^2(0,t_0;H^2(\Omega))} \le C_0, & \|\dot q_0^J\|_{L^2(0,t_0;H_0^1(\Omega))} \le C_0,
\end{cases}
\tag{5.2.9}
$$

and

$$
\begin{cases}
\inf_{[0,t_0]} \inf_{x\in\bar\Omega} (q_0^n + n_0) \ge \delta_0,\\
\sup_{[0,t_0]} \max\Big(\|\nabla(q_0^n+n_0)\|_{L^\infty(\Omega)}, \|q_0^n+n_0\|_{L^\infty(\Omega)},\\
\qquad \|q_0^J + J_0\|_{L^\infty(\Omega)}\Big) \le \delta_0^{-1}.
\end{cases}
\tag{5.2.10}
$$

Now we consider the third equation of (5.2.2). More precisely, we shall study the problem

$$
\begin{cases}
\partial_t q_1^{ne} - \nu_0 \triangle q_1^{ne} + \dfrac{2}{\tau} q_1^{ne} = \mathfrak{F}(q_0^n, q_0^J, q_0^{ne}),\\
q_1^{ne}(0,\cdot) = 0, \quad q_1^{ne}|_{\partial\Omega} = 0.
\end{cases}
\tag{5.2.11}
$$

This BVP admits a unique solution q_1^{ne} with $q_1^{ne} \in L^\infty(0,t_0,H_0^1(\Omega))$, $\partial_t q_1^{ne} \in L^\infty(0,t_0,L^2(\Omega)) \cap L^2(0,t_0,H_0^1(\Omega))$ provided

$$
\begin{cases}
\mathfrak{F}(q_0^n, q_0^J, q_0^{ne}) \in L^\infty(0,t_0,L^2(\Omega)),\\
\partial_t \mathfrak{F}(q_0^n, q_0^J, q_0^{ne}) \in L^2(0,t_0,H^{-1}(\Omega)),\\
\mathfrak{F}(q_0^n, q_0^J, q_0^{ne})(0,\cdot) \in L^2(\Omega).
\end{cases}
\tag{5.2.12}
$$

Check that since (q_0^n, q_0^J) satisfies (5.2.9) and it is obviously true that $\mathfrak{F}(q_0^n, q_0^J, q_0^{ne})$ satisfies (5.2.12) from **Lemma** 5.2.12.

Let $k \ge 1$, $t_{k-1} > 0$. Assume

$$
q_k^{ne} \in L^\infty(0,t_{k-1};H_0^1(\Omega)), \quad \partial_t q_k^{ne} \in L^\infty(0,t_{k-1};L^2(\Omega))
$$

with

$$\|q_k^{ne}\|_{L^\infty(0,t_{k-1};H_0^1(\Omega))} \le c^*, \quad \|\partial_t q_k^{ne}\|_{L^\infty(0,t_{k-1};L^2(\Omega))} \le c^*, \; q_k^{ne}(0) = 0. \tag{5.2.13}$$

Then by **Lemma** 5.2.1 there exists a time interval $[0, t_k) \subseteq [0, t_{k-1})$ such that the system

$$\begin{cases} \partial_t q_k^n - \nu_0 \triangle q_k^n - \mathrm{div} q_k^J = \nu_0 \triangle n_0 + \mathrm{div} J_0, \\ \partial_t q_k^J - \nu_0 \triangle q_k^J + \dfrac{q_k^J}{\tau} + \dfrac{\epsilon^2}{6} \nabla \triangle q_k^n - \mu \nabla q_k^n = \mathfrak{F}_{\mathbf{d}}(q_k^n, q_k^J, q_k^{ne}), \\ \lambda^2 \triangle q_k^v = q_k^n + n_0 - \mathcal{C}(x), \\ (q_k^n, q_k^J)(0, \cdot) = 0, \quad (q_k^n, q_k^J)|_{\partial\Omega} = 0, q_k^v|_{\partial\Omega} = V_\Gamma, \end{cases} \tag{5.2.14}$$

admits a unique local-in-time solution (q_k^n, q_k^J, q_k^v) with

$$\begin{aligned} q_k^n &\in L^\infty(0, t_k; H^3(\Omega)), \quad q_k^J \in L^\infty(0, t_k; H^2(\Omega)), \\ \partial_t q_k^n &\in L^2(0, t_k; H^2(\Omega)), \quad \partial_t q_k^J \in L^2(0, t_k; H^1(\Omega)), \\ (q_k^n, \nabla q_k^n, q_k^J) &\in C([0, t_k) \times \bar{\Omega}), \\ q_k^v &\in C(0, t_k; H^2(\Omega)), \\ \partial_t q_k^v &\in L^2(0, t_k; H^1(\Omega)), \end{aligned} \tag{5.2.15}$$

$$\begin{cases} \|q_k^n\|_{L^\infty(0,t_k;H^3(\Omega))} \le C', \quad \|q_k^J\|_{L^\infty(0,t_k;H^2(\Omega))} \le C', \\ \|q_k^n\|_{L^\infty(0,t_k;H^2(\Omega))} \le C^*, \quad \|q_k^J\|_{L^\infty(0,t_k;H_0^1(\Omega))} \le C^*, \\ \|q_k^n\|_{L^\infty(0,t_k;H_0^1(\Omega))} \le C_0, \quad \|q_k^J\|_{L^\infty(0,t_k;L^2(\Omega))} \le C_0, \\ \|\dot{q}_k^n\|_{L^\infty(0,t_k;H_0^1(\Omega))} \le C_0, \quad \|\dot{q}_k^J\|_{L^\infty(0,t_k;L^2(\Omega))} \le C_0, \\ \|\dot{q}_k^n\|_{L^2(0,t_k;H^2(\Omega))} \le C_0, \quad \|\dot{q}_k^J\|_{L^2(0,t_k;H_0^1(\Omega))} \le C_0, \end{cases} \tag{5.2.16}$$

and

$$\begin{cases} \inf\limits_{[0,t_k]} \inf\limits_{x \in \overline{\Omega}} (q_k^n + n_0) \ge \delta_0, \\ \sup\limits_{[0,t_k]} \max \Big(\|\nabla(q_k^n + n_0)\|_{L^\infty(\Omega)}, \|q_k^n + n_0\|_{L^\infty(\Omega)}, \\ \qquad \|q_k^J + J_0\|_{L^\infty(\Omega)} \Big) \le \delta_0^{-1}, \end{cases} \tag{5.2.17}$$

Next consider the problem

$$\begin{cases} \partial_t q_{k+1}^{ne} - \nu_0 \triangle q_{k+1}^{ne} + \dfrac{2}{\tau} q_{k+1}^{ne} = \mathfrak{F}(q_k^n, q_k^J, q_k^{ne}), \\ q_{k+1}^{ne}(0, \cdot) = 0, q_{k+1}^{ne}|_{\partial\Omega} = 0. \end{cases} \tag{5.2.18}$$

This BVP admits a unique solution q_{k+1}^{ne} with $q_{k+1}^{ne} \in L^\infty(0,t_k,H_0^1(\Omega))$, $\partial_t q_{k+1}^{ne} \in L^\infty(0,t_k,L^2(\Omega)) \cap L^2(0,t_k,H_0^1(\Omega))$ provided

$$\begin{cases} \mathfrak{F}(q_k^n, q_k^J, q_k^{ne}) \in L^\infty(0,t_k,L^2(\Omega)), \\ \partial_t \mathfrak{F}(q_k^n, q_k^J, q_k^{ne}) \in L^2(0,t_k,H^{-1}(\Omega)), \\ \mathfrak{F}(q_k^n, q_k^J, q_k^{ne})(0,\cdot) \in L^2(\Omega). \end{cases} \tag{5.2.19}$$

It is easy to verify that (5.2.19) is satisfied from **Lemma** 5.2.2.

Using (5.2.11), we take the inner product with $-\triangle q_{k+1}^{ne}$ for a.e. $t \in [0,t_k]$. This yields

$$\frac{1}{2}\partial_t \|\nabla q_{k+1}^{ne}\|^2 + \nu_0 \|\triangle q_{k+1}^{ne}\|^2 + \frac{2}{\tau}\|\nabla q_{k+1}^{ne}\|^2 = -\left(\mathfrak{F}(q_k^n, q_k^J, q_k^{ne}), \triangle q_{k+1}^{ne}\right),$$

by which we use Hölder's inequality and Cauchy's inequality then obtain

$$\begin{aligned} &\frac{1}{2}\partial_t \|\nabla q_{k+1}^{ne}\|^2 + \nu_0 \|\triangle q_{k+1}^{ne}\|^2 + \frac{2}{\tau}\|\nabla q_{k+1}^{ne}\|^2 \\ \le& \left\|\mathfrak{F}(q_k^n, q_k^J, q_k^{ne})\right\|_{L^2(\Omega)} \left\|\triangle q_{k+1}^{ne}\right\|_{L^2(\Omega)} \\ \le& \sigma_1 \|\triangle q_{k+1}^{ne}\|^2 + \frac{1}{4\sigma_1}\left\|\mathfrak{F}(q_k^n, q_k^J, q_k^{ne})\right\|^2. \end{aligned} \tag{5.2.20}$$

Using (5.2.13), (5.2.9) and (5.2.10) $\left\|\mathfrak{F}(q_k^n, q_k^J, q_k^{ne})\right\|_{L^2(\Omega)}$ can be estimated as follows.

$$\begin{aligned} \left\|\mathfrak{F}(q_k^n, q_k^J, q_k^{ne})\right\| \le& \left\|\operatorname{div}\left(\frac{q_k^J + J_0}{q_k^n + n_0}(q_k^{ne} + (ne)_0 + P(q_k^n, q_k^J, q_k^{ne}))\right)\right\| \\ &+ \left\|\nu_0 \triangle (ne)_0 - \frac{2}{\tau}(ne)_0 + \frac{3}{\tau}(q_k^n + n_0) + \mu \operatorname{div}(q_k^J + J_0)\right\| \\ &+ \left\|(q_k^J + J_0)\nabla q_k^v\right\| \le C_{c^*,C_0}. \end{aligned}$$

Let $\sigma_1 < \nu_0$ we obtain

$$\begin{aligned} &\frac{1}{2}\partial_t \|\nabla q_{k+1}^{ne}\|^2 + (\nu_0 - \sigma_1)\|\triangle q_{k+1}^{ne}\|^2 + \frac{2}{\tau}\|\nabla q_{k+1}^{ne}\|^2 \\ \le& \left\|\mathfrak{F}(q_k^n, q_k^J, q_k^{ne})\right\|_{L^2(\Omega)} \left\|\triangle q_{k+1}^{ne}\right\|_{L^2(\Omega)} \le \frac{C_{c^*,C_0}}{4\sigma_1}, \end{aligned} \tag{5.2.21}$$

which implies

$$\|q_{k+1}^{ne}\|_{L^\infty(0,t_k;H_0^1(\Omega))} \le \sqrt{C_2 \int_0^{t_k} e^{C_1 s} ds}. \tag{5.2.22}$$

C_1 doesn't depend upon c^*, but C_2 is allowed to be depend on c^*. In the sequal we will use the same notations, i.e., let $C_i, i = 1, 2$ denote the generic constants which may change from line to line.

Differentiate formally (5.2.11) with respect to $0 \le t \le t_k$, then take the $L^2(\Omega)$ scalar product of the resultant with $\dot q^{ne}_{k+1}$ we obtain for a.e. $t \in [0, t_k]$

$$\begin{aligned}&\frac{1}{2}\partial_t\|\dot q^{ne}_{k+1}\|^2 + \nu_0\|\nabla \dot q^{ne}_{k+1}\|^2 + \frac{2}{\tau}\|\dot q^{ne}_{k+1}\|^2\\ &= \left(\dot{\mathfrak{F}}(q^n_k, q^J_k, q^{ne}_k), \dot q^{ne}_{k+1}\right) \le \left\|\dot{\mathfrak{F}}(q^n_k, q^J_k, q^{ne}_k)\right\|_{H^{-1}(\Omega)} \left\|\dot q^{ne}_{k+1}\right\|_{H^1_0(\Omega)},\end{aligned} \tag{5.2.23}$$

among which

$$\begin{aligned}\left\|\dot{\mathfrak{F}}(q^n_k, q^J_k, q^{ne}_k)\right\|_{H^{-1}(\Omega)} \le &\left\|\left(\frac{q^J_k + J_0}{q^n_k + n_0}(q^{ne}_k + (ne)_0 + P(q^n_k, q^J_k, q^{ne}_k))\right)'\right\|_{L^2(\Omega)}\\ &+ \frac{3}{\tau}\|\dot q^n_k\|_{L^2(\Omega)} + \mu\|\dot q^J_k\|_{L^2(\Omega)} + C^2_{14}\|\dot q^J_k\|_{L^2(\Omega)}\|q^v_k\|_{H^2(\Omega)}\\ &+ C^2_{14}\|q^J_k + J_0\|_{L^2(\Omega)}\|\dot q^v_k\|_{H^2(\Omega)}.\end{aligned}$$

Obviously,

$$\begin{aligned}&\left(\frac{q^J_k + J_0}{q^n_k + n_0}(q^{ne}_k + (ne)_0 + P(q^n_k, q^J_k, q^{ne}_k))\right)'\\ &= \left(\frac{q^J_k + J_0}{q^n_k + n_0}\right)'(q^{ne}_k + (ne)_0) + \left(\frac{q^J_k + J_0}{q^n_k + n_0}\right)' P(q^n_k, q^J_k, q^{ne}_k)\\ &\quad + \frac{q^J_k + J_0}{q^n_k + n_0}\dot q^{ne}_k + \frac{q^J_k + J_0}{q^n_k + n_0}\dot P(q^n_k, q^J_k, q^{ne}_k) =: \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4.\end{aligned}$$

- **Estimates of $\mathcal{S}_1$.**

$$\begin{aligned}\|\mathcal{S}_1\|_{L^2(\Omega)} &\le \left\|\frac{\dot q^J_k}{q^n_k + n_0}(q^{ne}_k + (ne)_0)\right\| + \left\|\frac{\dot q^n_k(q^J_k + J_0)}{(q^n_k + n_0)^2}(q^{ne}_k + (ne)_0)\right\|\\ &\le \|q^{ne}_k + (ne)_0\|_{L^4(\Omega)}\left(\delta_0^{-1}\|\dot q^J_k\|_{L^4(\Omega)} + \delta_0^{-3}\|\dot q^n_k\|_{L^4(\Omega)}\right)\\ &\le C_2\|\dot q^J_k\|_{H^1(\Omega)} + C_2.\end{aligned} \tag{5.2.24}$$

- **Estimates of $\mathcal{S}_2$.**

$$\begin{aligned}\|\mathcal{S}_2\|_{L^2(\Omega)} &\le \left\|\frac{\dot q^J_k}{q^n_k + n_0}P(q^n_k, q^J_k, q^{ne}_k)\right\| + \left\|\frac{\dot q^n_k(q^J_k + J_0)}{(q^n_k + n_0)^2}P(q^n_k, q^J_k, q^{ne}_k)\right\|\\ &\le \|P(q^n_k, q^J_k, q^{ne}_k)\|_{L^4(\Omega)}\left(\delta_0^{-1}\|\dot q^J_k\|_{L^4(\Omega)} + \delta_0^{-3}\|\dot q^n_k\|_{L^4(\Omega)}\right).\end{aligned}$$

By Sobolev embedding theorem $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and under (5.2.13), (5.2.9) and (5.2.10) we estimate

$$\|P(q^n_k, q^J_k, q^{ne}_k)\|_{L^4(\Omega)} \le C_2.$$

Thus

$$\|\mathcal{S}_2\|_{L^2(\Omega)} \le C_2\|\dot{q}_k^J\|_{H^1(\Omega)} + C_2. \tag{5.2.25}$$

- **Estimates of $\mathcal{S}_3$.**

It is easy to verify that

$$\|\mathcal{S}_3\|_{L^2(\Omega)} \le C_2. \tag{5.2.26}$$

- **Estimates of $\mathcal{S}_4$.**

$$\|\mathcal{S}_4\|_{L^2(\Omega)} \le \delta_0^{-2}\left\|\dot{P}(q_k^n, q_k^J, q_k^{ne})\right\|_{L^2(\Omega)}.$$

By (5.2.9) and (5.2.10) we obtain

$$\|\mathcal{S}_4\|_{L^2(\Omega)} \le C_2\|\dot{q}_k^n\|_{H^2(\Omega)} + C_2. \tag{5.2.27}$$

Combining (5.2.23)-(5.2.27) we deduce

$$\begin{aligned}&\frac{1}{2}\partial_t\|\dot{q}_{k+1}^{ne}\|^2 + \nu_0\|\nabla\dot{q}_{k+1}^{ne}\|^2 + \frac{2}{\tau}\|\dot{q}_{k+1}^{ne}\|^2\\ &\le C_2 + C_2\|\dot{q}_k^n\|^2_{H^2(\Omega)} + C_2\|\dot{q}_k^J\|^2_{H^1(\Omega)},\end{aligned} \tag{5.2.28}$$

from which via integration from 0 to $t \in (0, t_k]$ we conclude from (5.2.9)

$$\begin{aligned}&\|\dot{q}_{k+1}^{ne}\|_{L^\infty(0,t_k;L^2(\Omega))}\\ &\le \sqrt{\|\dot{q}_{k+1}^{ne}(0)\|^2 + C_2 t_k + C_2\|\dot{q}_k^n\|^2_{L^2(0,t_k;H^2(\Omega))} + C_2\|\dot{q}_k^J\|^2_{L^2(0,t_k;H^1(\Omega))}}\\ &\le \sqrt{\|\dot{q}_{k+1}^{ne}(0)\|^2 + C_2 t_k + C_2 C_0^2}.\end{aligned} \tag{5.2.29}$$

It is obvious that

$$\dot{q}_{k+1}^{ne}(0) = \mathfrak{F}(0,0,0),$$

then (5.2.29) yields that there exit $\mathcal{C}, t_{\mathcal{C}} > 0$ independent of k such that if $\inf_{x\in\Omega} n_0 > \delta_0$, $t_k \le t_{\mathcal{C}}$ and

$$\begin{aligned}&\|\nabla n_0\|_{H^2(\Omega)} + \|J_0\|_{H^2(\Omega)}\\ &\quad + \|\nabla(ne)_0\|_{L^2(\Omega)} + \gamma + \|n_0 - \mathcal{C}(x)\|_{L^2(\Omega)} + \|V_\Gamma\|_{H^{3/2}(\Omega)} \le \mathcal{C},\end{aligned} \tag{5.2.30}$$

then after recalling (5.2.13)

$$\begin{aligned}&\|\dot{q}_{k+1}^{ne}\|_{L^\infty(0,t_k;L^2(\Omega))} \le \sqrt{\|\mathfrak{F}(0,0,0)\|^2 + \varphi_1 + \varphi_2} = c^*,\\ &\|q_{k+1}^{ne}\|_{L^\infty(0,t_k;H_0^1(\Omega))} \le c^*.\end{aligned} \tag{5.2.31}$$

Recursively using Lemma 5.2.1 once more we obtain $[0, t_{k+1})$ which guarantees that γ satisfying (5.2.30) and $t_{k+1} \le t_{\mathcal{C}}$, and functions (q^n_{k+1}, q^J_{k+1}) in $[0, t_{k+1})$ satisfying

$$
\begin{aligned}
q^n_{k+1} &\in L^\infty(0, t_{k+1}; H^3(\Omega)), & q^J_{k+1} &\in L^\infty(0, t_{k+1}; H^2(\Omega)),\\
\partial_t q^n_{k+1} &\in L^2(0, t_{k+1}; H^2(\Omega)), & \partial_t q^J_{k+1} &\in L^2(0, t_{k+1}; H^1(\Omega)),\\
(q^n_{k+1}, \nabla q^n_{k+1}, q^J_{k+1}) &\in C([0, t_{k+1}) \times \bar{\Omega}), & &\\
q^v_{k+1} &\in C(0, t_{k+1}; H^2(\Omega)), & &\\
\partial_t q^v_{k+1} &\in L^2(0, t_{k+1}; H^1(\Omega)), & &
\end{aligned}
$$

$$
\begin{cases}
\|q^n_{k+1}\|_{L^\infty(0,t_{k+1};H^3(\Omega))} \le C', & \|q^J_{k+1}\|_{L^\infty(0,t_{k+1};H^2(\Omega))} \le C',\\
\|q^n_{k+1}\|_{L^\infty(0,t_{k+1};H^2(\Omega))} \le C^*, & \|q^J_{k+1}\|_{L^\infty(0,t_{k+1};H^1_0(\Omega))} \le C^*,\\
\|q^n_{k+1}\|_{L^\infty(0,t_{k+1};H^1_0(\Omega))} \le C_0, & \|q^J_{k+1}\|_{L^\infty(0,t_{k+1};L^2(\Omega))} \le C_0,\\
\|\dot{q}^n_{k+1}\|_{L^\infty(0,t_{k+1};H^1_0(\Omega))} \le C_0, & \|\dot{q}^J_{k+1}\|_{L^\infty(0,t_{k+1};L^2(\Omega))} \le C_0,\\
\|\dot{q}^n_{k+1}\|_{L^2(0,t_{k+1};H^2(\Omega))} \le C_0, & \|\dot{q}^J_{k+1}\|_{L^2(0,t_{k+1};H^1_0(\Omega))} \le C_0,
\end{cases}
\tag{5.2.32}
$$

and

$$
\begin{cases}
\inf_{[0,t_{k+1}]} \inf_{x\in\overline{\Omega}} (q^n_{k+1} + n_0) \ge \delta_0,\\
\sup_{[0,t_{k+1}]} \max \Big(\|\nabla(q^n_{k+1} + n_0)\|_{L^\infty(\Omega)}, \|q^n_{k+1} + n_0\|_{L^\infty(\Omega)},\\
\qquad \|q^J_{k+1} + J_0\|_{L^\infty(\Omega)} \Big) \le \delta_0^{-1}.
\end{cases}
\tag{5.2.33}
$$

In the next lemma we will show that t_k tends not to zero as $k \to \infty$.

Lemma 5.2.3. *There exits an in k uniform time interval $[0, t_u)$ with $t_u \le t_{\mathcal{C}}$ such that for γ satisfying (5.2.30) and all $k = 0, 1, \cdots$*

$$
\begin{aligned}
q^n_k &\in L^\infty(0, t_u; H^3(\Omega)), & q^J_k &\in L^\infty(0, t_u; H^2(\Omega)),\\
\partial_t q^n_k &\in L^2(0, t_u; H^2(\Omega)), & \partial_t q^J_k &\in L^2(0, t_u; H^1(\Omega)),\\
(q^n_k, \nabla q^n_k, q^J_k) &\in C([0, t_u) \times \bar{\Omega}), & q^{ne}_k &\in L^\infty(0, t_u; H^1_0(\Omega)),\\
q^v_k &\in C(0, t_u; H^2(\Omega)), & \partial_t q^{ne}_k &\in L^\infty(0, t_u; L^2(\Omega)),\\
\partial_t q^v_k &\in L^2(0, t_u; H^1(\Omega)), & &
\end{aligned}
$$

$$\begin{cases} \|q_k^n\|_{L^\infty(0,t_u;H^3(\Omega))} \le C', & \|q_k^J\|_{L^\infty(0,t_u;H^2(\Omega))} \le C', \\ \|q_k^n\|_{L^\infty(0,t_u;H^2(\Omega))} \le C^*, & \|q_k^J\|_{L^\infty(0,t_u;H_0^1(\Omega))} \le C^*, \\ \|q_k^n\|_{L^\infty(0,t_u;H_0^1(\Omega))} \le C_0, & \|q_k^J\|_{L^\infty(0,t_u;L^2(\Omega))} \le C_0, \\ \|\dot{q}_k^n\|_{L^\infty(0,t_u;H_0^1(\Omega))} \le C_0, & \|\dot{q}_k^J\|_{L^\infty(0,t_u;L^2(\Omega))} \le C_0, \\ \|\dot{q}_k^n\|_{L^2(0,t_u;H^2(\Omega))} \le C_0, & \|\dot{q}_k^J\|_{L^2(0,t_u;H_0^1(\Omega))} \le C_0, \\ \|q_k^{ne}\|_{L^\infty(0,t_u;H_0^1(\Omega))} \le c^*, & \|\dot{q}_k^{ne}\|_{L^\infty(0,t_u;L^2(\Omega))} \le c^*, \end{cases} \tag{5.2.34}$$

and

$$\begin{cases} \inf\limits_{[0,t_u]} \inf\limits_{x\in\bar{\Omega}} (q_k^n + n_0) \ge \delta_0, \\ \sup\limits_{[0,t_u]} \max \Big(\|\nabla(q_k^n + n_0)\|_{L^\infty(\Omega)}, \|q_k^n + n_0\|_{L^\infty(\Omega)}, \\ \qquad \|q_k^J + J_0\|_{L^\infty(\Omega)} \Big) \le \delta_0^{-1}, \end{cases} \tag{5.2.35}$$

hold.

Proof. Using reduction to absurdity, assume there exists no such time interval, then for any sufficiently small $[0, t_s)$ with $t_s \le t_{\mathcal{C}}$ we can find an earliest $k \in \mathbb{N}$ such that though $q_k^{ne} \in L^\infty(0, t_s; H_0^1(\Omega))$ and $\partial_t q_k^{ne} \in L^\infty(0, t_s; L^2(\Omega))$ with

$$\|q_k^{ne}\|_{L^\infty(0,t_s;H_0^1(\Omega))} \le c^*, \quad \|\partial_t q_k^{ne}\|_{L^\infty(0,t_s;L^2(\Omega))} \le c^*,$$

the system (5.2.14) has no slutions in $[0, t_s)$ satisfying

$$\begin{aligned} q_k^n &\in L^\infty(0, t_s; H^3(\Omega)), & q_k^J &\in L^\infty(0, t_s; H^2(\Omega)), \\ \partial_t q_k^n &\in L^2(0, t_s; H^2(\Omega)), & \partial_t q_k^J &\in L^2(0, t_s; H^1(\Omega)), \\ (q_k^n, \nabla q_k^n, q_k^J) &\in C([0, t_s) \times \bar{\Omega}), \\ q_k^v &\in C(0, t_s; H^2(\Omega)), \\ \partial_t q_k^v &\in L^2(0, t_s; H^1(\Omega)), \end{aligned}$$

$$\begin{cases} \|q_k^n\|_{L^\infty(0,t_s;H^3(\Omega))} \le C', & \|q_k^J\|_{L^\infty(0,t_s;H^2(\Omega))} \le C', \\ \|q_k^n\|_{L^\infty(0,t_s;H^2(\Omega))} \le C^*, & \|q_k^J\|_{L^\infty(0,t_s;H_0^1(\Omega))} \le C^*, \\ \|q_k^n\|_{L^\infty(0,t_s;H_0^1(\Omega))} \le C_0, & \|q_k^J\|_{L^\infty(0,t_s;L^2(\Omega))} \le C_0, \\ \|\dot{q}_k^n\|_{L^\infty(0,t_s;H_0^1(\Omega))} \le C_0, & \|\dot{q}_k^J\|_{L^\infty(0,t_s;L^2(\Omega))} \le C_0, \\ \|\dot{q}_k^n\|_{L^2(0,t_s;H^2(\Omega))} \le C_0, & \|\dot{q}_k^J\|_{L^2(0,t_s;H_0^1(\Omega))} \le C_0, \end{cases} \tag{5.2.36}$$

and

$$\begin{cases} \inf\limits_{[0,t_s]} \inf\limits_{x\in\overline{\Omega}} (q_k^n + n_0) \geq \delta_0, \\ \sup\limits_{[0,t_s]} \max \Big(\|\nabla(q_k^n + n_0)\|_{L^\infty(\Omega)}, \|q_k^n + n_0\|_{L^\infty(\Omega)}, \\ \qquad \|q_k^J + J_0\|_{L^\infty(\Omega)} \Big) \leq \delta_0^{-1}. \end{cases} \tag{5.2.37}$$

Now we go back to the proof of **Theorem** 3.3.1. In order to solve (5.2.14) with respect to (q_k^n, q_k^J, q_k^v) we first construct approximate solutions $\{(q_{k,l}^n, q_{k,l}^J, q_{k,l}^v)\}_{l=1}^\infty$. From Lemma 3.3.2 and 3.3.3 we claim there exists $\mathcal{C}_c > 0$ independent of k such that if $t_s \leq \mathcal{C}_c$ (in the sequal we will always use this assumption of t_s), then that $(q_{k,l}^n, q_{k,l}^J, q_{k,l}^v)$ satisfies (5.2.36) and

$$\begin{cases} \inf\limits_{[0,t_s]} \inf\limits_{x\in\overline{\Omega}} (q_{k,l}^n + n_0) > \delta_0, \\ \sup\limits_{[0,t_s]} \max \Big(\|\nabla(q_{k,l}^n + n_0)\|_{L^\infty(\Omega)}, \|q_{k,l}^n + n_0\|_{L^\infty(\Omega)}, \\ \qquad \|q_{k,l}^J + J_0\|_{L^\infty(\Omega)} \Big) < \delta_0^{-1} \end{cases} \tag{5.2.38}$$

in $[0, t_s)$ for all $l = 1, 2, \cdots$ yields $\{(q_{k,l}^n, q_{k,l}^J, q_{k,l}^v)\}_{l=1}^\infty$ is a Cauchy sequence in

$$L^\infty(0, t_s; H_0^1(\Omega)) \times L^\infty(0, t_s; (L^2(\Omega))^d) \times L^\infty(0, t_s; (H^2(\Omega))^d).$$

From this the local-in-time existence of solutions to (5.2.14) in $[0, t_s)$ can be obtained where the solution satisfies (5.2.36) and (5.2.37).

Recall our premise that the system (5.2.14) has no slutions satisfying (5.2.36) and (5.2.37) in $[0, t_s)$, there must be an earliest l such that $(q_{k,l}^n, q_{k,l}^J, q_{k,l}^v)$ violate (5.2.36) or (5.2.38). From the proof of **Theorem** 3.3.1 there exists $t_\gamma > 0$ independent of k such that if $t_s \leq t_\gamma$ (in the sequal we will always use this assumption of t_s) and $(q_{k,l-1}^n, q_{k,l-1}^J)$ satisfies (5.2.36) and

$$\begin{cases} \inf\limits_{[0,t_s]} \inf\limits_{x\in\overline{\Omega}} (q_{k,l-1}^n + n_0) > \delta_0, \\ \sup\limits_{[0,t_s]} \max \Big(\|\nabla(q_{k,l-1}^n + n_0)\|_{L^\infty(\Omega)}, \|q_{k,l-1}^n + n_0\|_{L^\infty(\Omega)}, \\ \qquad \|q_{k,l-1}^J + J_0\|_{L^\infty(\Omega)} \Big) < \delta_0^{-1}, \end{cases} \tag{5.2.39}$$

then $(q_{k,l}^n, q_{k,l}^J)$ satisfies (5.2.36). This implies $(q_{k,l}^n, q_{k,l}^J)$ violate (5.2.38). Finally we obtain the interpolation inequalities, i.e., for $t_1, t_2 \in [0, t_s)$,

$$\begin{aligned} &\|q_{k,l}^n(t_1, \cdot) - q_{k,l}^n(t_2, \cdot)\|_{C(\overline{\Omega})} \\ &\leq 2^{(2-\beta)/2} C_\beta C_{inter} \|\partial_t q_{k,l}^n\|_{L^\infty(0,t_k;L^2(\Omega))}^{\beta/2} \|q_{k,l}^n\|_{L^\infty(0,t_k;H^2(\Omega))}^{(2-\beta)/2} |t_1 - t_2|^{\beta/2}, \end{aligned}$$

$$
\begin{aligned}
&\|q^J_{k,l}(t_1,\cdot)-q^J_{k,l}(t_2,\cdot)\|_{C(\overline{\Omega})}\\
\le\ & 2^{(2-\beta)/2}C_\beta C_{inter}\|\partial_t q^J_{k,l}\|^{\beta/2}_{L^\infty(0,t_k;L^2(\Omega))}\|q^J_{k,l}\|^{(2-\beta)/2}_{L^\infty(0,t_k;H^2(\Omega))}|t_1-t_2|^{\beta/2}\\
\le\ & 2^{(2-\beta)/2}C_\beta C_{inter}C_0^{\beta/2}(C'+M)^{(2-\beta)/2}|t_1-t_2|^{\beta/2}.
\end{aligned}
$$

By a similar reasoning

$$
\begin{aligned}
&\|\nabla q^n_{k,l}(t_1,\cdot)-\nabla q^n_{k,l}(t_2,\cdot)\|_{C(\overline{\Omega})}\\
\le\ & 2^{(2-\beta)/2}C_\beta C_{inter}\|\nabla\partial_t q^n_{k,l}\|^{\beta/2}_{L^\infty(0,t_k;L^2(\Omega))}\|q^n_{k,l}\|^{(2-\beta)/2}_{L^\infty(0,t_k;H^3(\Omega))}|t_1-t_2|^{\beta/2}\\
\le\ & 2^{(2-\beta)/2}C_\beta C_{inter}C_0^{\beta/2}(C'+M)^{(2-\beta)/2}|t_1-t_2|^{\beta/2}.
\end{aligned}
$$

Since t_s is sufficiently small then from the interpolations mentioned above, that $(q^n_{k,l}, q^J_{k,l})$ violate (5.2.38) produces a contradiction. □

Remark 5.2.1. *Recall* (5.2.18),

$$\|q^{ne}_{k+1}\|_{L^\infty(0,t_u;H^2(\Omega))}\le C\left(\|\partial_t q^{ne}_{k+1}\|_{L^\infty(0,t_u;L^2(\Omega))}+\|\mathfrak{F}(q^n_k,q^J_k,q^{ne}_k)\|_{L^\infty(0,t_u;L^2(\Omega))}\right).$$

Integrate (5.2.28) *from* 0 *to* t_u *then recall* (5.2.29) *we infer*

$$
\begin{aligned}
&\|\partial_t q^{ne}_{k+1}\|_{L^2(0,t_u;H^1(\Omega))}\\
\le\ & \sqrt{\|\dot q^{ne}_{k+1}(0)\|^2+C_2t_k+C_2\|\dot q^n_k\|^2_{L^2(0,t_k;H^2(\Omega))}+C_2\|\dot q^J_k\|^2_{L^2(0,t_k;H^1(\Omega))}}\\
\le\ & \sqrt{\|\dot q^{ne}_{k+1}(0)\|^2+C_2t_k+C_2C_0^2}.
\end{aligned}
$$

From (5.2.18), (5.2.23), (5.2.29), (5.2.34) *and* (5.2.35) *we conclude that there exists* $\tilde{\mathcal{C}}$ *independent of* k *such that for all* $k=0,1,\cdots$

$$\|q^{ne}_k\|_{L^\infty(0,t_u;H^2(\Omega))}+\|\partial_t q^{ne}_k\|_{L^2(0,t_u;H^1(\Omega))}\le\tilde{\mathcal{C}}. \tag{5.2.40}$$

5.2.3 Analysis of the Limit of the Approximate Solutions

After having obtained the uniform bounds we deduce the following

Lemma 5.2.4. *There exits* $\mathcal{C}_*>0$ *independent of* k *such that if* $t_u\le\mathcal{C}_*$, $\{q^{ne}_k\}_{k=0}^\infty$ *is a Cauchy sequence in* $L^\infty(0,t_u;L^2(\Omega)$. *Namely there exists a* $q^{ne}_*\in L^\infty(0,t_u;L^2(\Omega)$ *such that* q^{ne}_k *converges to* q^{ne}_* *in* $L^\infty(0,t_u;L^2(\Omega)$.

Proof. Since (q^n_k,q^J_k) solves

$$
\begin{cases}
\partial_t q^n_k-\nu_0\triangle q^n_k-\mathrm{div}q^J_k=\nu_0\triangle n_0+\mathrm{div}J_0,\\
\partial_t q^J_k-\nu_0\triangle q^J_k+\dfrac{q^J_k}{\tau}+\dfrac{\epsilon^2}{6}\nabla\triangle q^n_k-\mu\nabla q^n_k=\mathfrak{F}_{\mathbf{d}}(q^n_k,q^J_k,q^{ne}_k),\\
\lambda^2\triangle q^v_k=q^n_k+n_0-\mathcal{C}(x),\\
(q^n_k,q^J_k)(0,\cdot)=0,(q^n_k,q^J_k)|_{\partial\Omega}=0,q^v_k|_{\partial\Omega}=V_\Gamma,
\end{cases}
\tag{5.2.41}
$$

and (q^n_{k-1}, q^J_{k-1}) solves

$$\begin{cases} \partial_t q^n_{k-1} - \nu_0 \triangle q^n_{k-1} - \mathrm{div} q^J_{k-1} = \nu_0 \triangle n_0 + \mathrm{div} J_0, \\ \partial_t q^J_{k-1} - \nu_0 \triangle q^J_{k-1} + \dfrac{q^J_{k-1}}{\tau} + \dfrac{\epsilon^2}{6}\nabla \triangle q^n_{k-1} - \mu \nabla q^n_{k-1} = \mathfrak{F}_{\mathbf{d}}(q^n_{k-1}, q^J_{k-1}, q^{ne}_{k-1}), \\ \lambda^2 \triangle q^v_{k-1} = q^n_{k-1} + n_0 - \mathcal{C}(x), \\ (q^n_{k-1}, q^J_{k-1})(0,\cdot) = 0, (q^n_{k-1}, q^J_{k-1})|_{\partial\Omega} = 0, q^v_{k-1}|_{\partial\Omega} = V_\Gamma. \end{cases} \tag{5.2.42}$$

(5.2.41)-(5.2.42) yields

$$\begin{cases} \partial_t (\mathbf{q}_k - \mathbf{q}_{k-1})^T + \mathcal{A}(\partial_x)(\mathbf{q}_k - \mathbf{q}_{k-1})^T = \left(0, \mathfrak{F}_{\mathbf{d}}(\mathbf{q}_k, q^{ne}_k) - \mathfrak{F}_{\mathbf{d}}(\mathbf{q}_{k-1}, q^{ne}_{k-1})\right)^T, \\ \lambda^2 \triangle (q^v_k - q^v_{k-1}) = q^n_k - q^n_{k-1}, \\ (\mathbf{q}_k - \mathbf{q}_{k-1})^T(0,\cdot) = 0, \\ (\mathbf{q}_k - \mathbf{q}_{k-1}, q^v_k - q^v_{k-1})^T|_{\partial\Omega} = 0, \end{cases} \tag{5.2.43}$$

where $\mathbf{q}_i := (q^n_i, q^J_i)^T, i = k, k-1,$

$$\mathcal{A}(\partial_x) := \begin{pmatrix} -\nu_0 \triangle & -\mathrm{div} \\ -\mu\nabla + \dfrac{\epsilon^2}{6}\nabla\triangle & -\nu_0\triangle + \tau^{-1} \end{pmatrix}.$$

By a similar calculations as (3.3.28)-(3.3.33) we find

$$\begin{aligned} &\frac{1}{2}\partial_t \Big(\mu \|q^n_k - q^n_{k-1}\|^2 + \frac{\epsilon^2}{6}\|\nabla(q^n_k - q^n_{k-1})\|^2 + \|q^J_k - q^J_{k-1}\|^2 \Big) \\ &\quad + \nu_0 \mu \|\nabla(q^n_k - q^n_{k-1})\|^2 + \frac{\epsilon^2}{6}\nu_0 \|\triangle(q^n_k - q^n_{k-1})\|^2 \\ &\quad + \nu_0 \|\nabla(q^J_k - q^J_{k-1})\|^2 + \frac{1}{\tau}\|q^J_k - q^J_{k-1}\|^2 \\ =& (\mathfrak{F}_{\mathbf{d}}(\mathbf{q}_k, q^{ne}_k) - \mathfrak{F}_{\mathbf{d}}(\mathbf{q}_{k-1}, q^{ne}_{k-1}), q^J_k - q^J_{k-1}). \end{aligned}$$

By Hölder's inequality and Cauchy's inequlity we obtain

$$\begin{aligned} &\partial_t \Big(\mu \|q^n_k - q^n_{k-1}\|^2 + \frac{\epsilon^2}{6}\|\nabla(q^n_k - q^n_{k-1})\|^2 + \|q^J_k - q^J_{k-1}\|^2 \Big) \\ &\quad + C\big(\|\nabla(q^n_k - q^n_{k-1})\|^2 + \|\triangle(q^n_k - q^n_{k-1})\|^2 \\ &\qquad + \|\nabla(q^J_k - q^J_{k-1})\|^2 + \|q^J_k - q^J_{k-1}\|^2 \big) \\ \le& C\|\mathfrak{F}_{\mathbf{d}}(\mathbf{q}_k, q^{ne}_k) - \mathfrak{F}_{\mathbf{d}}(\mathbf{q}_{k-1}, q^{ne}_{k-1})\|^2_{H^{-1}(\Omega)}. \end{aligned} \tag{5.2.44}$$

Using (5.2.9), (5.2.10) we calculate similarly as (3.3.121)-(3.3.126) to obtain for a.e. $t \in [0, t')$

$$\begin{aligned}&\|\mathfrak{F}_{\mathbf{d}}(\mathbf{q}_k, q_k^{ne}) - \mathfrak{F}_{\mathbf{d}}(\mathbf{q}_{k-1}, q_{k-1}^{ne})\|_{H^{-1}(\Omega)}^2 \\ &\leq C \left(\|q_k^n - q_{k-1}^n\|_{H^1(\Omega)}^2 + \|q_k^J - q_{k-1}^J\|_{L^2(\Omega)}^2 + \|q_k^{ne} - q_{k-1}^{ne}\|_{L^2(\Omega)}^2 \right).\end{aligned} \tag{5.2.45}$$

Then we have

$$\begin{aligned}&\|q_k^n - q_{k-1}^n\|_{L^\infty(0,t_u;H_0^1(\Omega))}^2 + \|q_k^J - q_{k-1}^J\|_{L^\infty(0,t_u;L^2(\Omega))}^2 \\ &\leq Ct_u \Big(\|q_k^n - q_{k-1}^n\|_{L^\infty(0,t_u;H_0^1(\Omega))}^2 + \|q_k^J - q_{k-1}^J\|_{L^\infty(0,t_u;L^2(\Omega))}^2 \\ &\qquad + \|q_k^{ne} - q_{k-1}^{ne}\|_{L^\infty(0,t_u;L^2(\Omega))}^2 \Big).\end{aligned}$$

Assume $[0, t_u)$ is sufficiently small such that $Ct_u < 1$, then

$$\begin{aligned}&\|q_k^n - q_{k-1}^n\|_{L^\infty(0,tv;H_0^1(\Omega))}^2 + \|q_k^J - q_{k-1}^J\|_{L^\infty(0,t_u;L^2(\Omega))}^2 \\ &\leq \frac{Ct_u}{1 - Ct_u} \|q_k^{ne} - q_{k-1}^{ne}\|_{L^\infty(0,t_u;L^2(\Omega))}^2.\end{aligned} \tag{5.2.46}$$

Integrate (5.2.44) from 0 to t_u we obtain

$$\begin{aligned}&\|q_k^n - q_{k-1}^n\|_{L^2(0,t_u;H^2(\Omega))}^2 + \|q_k^J - q_{k-1}^J\|_{L^2(0,t_u;H^1(\Omega))}^2 \\ &\leq Ct_u \Big(\|q_k^n - q_{k-1}^n\|_{L^\infty(0,t_u;H_0^1(\Omega))}^2 + \|q_k^J - q_{k-1}^J\|_{L^\infty(0,t_u;L^2(\Omega))}^2 \\ &\qquad + \|q_k^{ne} - q_{k-1}^{ne}\|_{L^\infty(0,t_u;L^2(\Omega))}^2 \Big).\end{aligned}$$

Together with (5.2.46) we conclude

$$\begin{aligned}&\|q_k^n - q_{k-1}^n\|_{L^2(0,t_u;H^2(\Omega))}^2 + \|q_k^J - q_{k-1}^J\|_{L^2(0,t_u;H^1(\Omega))}^2 \\ &\leq \frac{Ct_u}{1 - Ct_u} \|q_k^{ne} - q_{k-1}^{ne}\|_{L^\infty(0,t_u;L^2(\Omega))}^2.\end{aligned} \tag{5.2.47}$$

From (5.2.11) we find

$$\begin{cases} \quad \partial_t (q_{k+1}^{ne} - q_k^{ne}) - \nu_0 \triangle (q_{k+1}^{ne} - q_k^{ne}) + \dfrac{2}{\tau} (q_{k+1}^{ne} - q_k^{ne}) \\ = \mathfrak{F}(q_k^n, q_k^J, q_k^{ne}) - \mathfrak{F}(q_{k-1}^n, q_{k-1}^J, q_{k-1}^{ne}), \\ (q_{k+1}^{ne} - q_k^{ne})(0, \cdot) = 0, (q_{k+1}^{ne} - q_k^{ne})|_{\partial\Omega} = 0. \end{cases}$$

Taking the inner product with $q_{k+1}^{ne} - q_k^{ne}$ for a.e. $t \in [0, t_u)$ yields

$$\begin{aligned}&\partial_t \frac{1}{2} \|q_{k+1}^{ne} - q_k^{ne}\|^2 + \nu_0 \|\nabla (q_{k+1}^{ne} - q_k^{ne})\|^2 + \frac{2}{\tau} \|q_{k+1}^{ne} - q_k^{ne}\|^2 \\ &\leq \|\mathfrak{F}(q_k^n, q_k^J, q_k^{ne}) - \mathfrak{F}(q_{k-1}^n, q_{k-1}^J, q_{k-1}^{ne})\|_{H^{-1}(\Omega)} \|q_{k+1}^{ne} - q_k^{ne}\|_{H_0^1(\Omega)}.\end{aligned}$$

Cauchy's inequality provides

$$\begin{aligned}&\partial_t\|q_{k+1}^{ne}-q_k^{ne}\|^2+C\|q_{k+1}^{ne}-q_k^{ne}\|_{H_0^1(\Omega)}^2\\ \le\ & C\|\mathfrak{F}(q_k^n,q_k^J,q_k^{ne})-\mathfrak{F}(q_{k-1}^n,q_{k-1}^J,q_{k-1}^{ne})\|_{H^{-1}(\Omega)}^2,\end{aligned}\tag{5.2.48}$$

among which

$$\begin{aligned}&\|\mathfrak{F}(q_k^n,q_k^J,q_k^{ne})-\mathfrak{F}(q_{k-1}^n,q_{k-1}^J,q_{k-1}^{ne})\|_{H^{-1}(\Omega)}\\ \le\ & \left\|\frac{J_k((ne)_k+P_k)}{n_k}-\frac{J_{k-1}((ne)_{k-1}+P_{k-1})}{n_{k-1}}\right\|_{L^2(\Omega)}\\ &+\left\|J_k\nabla q_k^v-J_{k-1}\nabla q_{k-1}^v\right\|_{L^2(\Omega)}\\ &+\frac{3}{\tau}\left\|n_k-n_{k-1}\right\|_{L^2(\Omega)}+\mu\left\|J_k-J_{k-1}\right\|_{L^2(\Omega)},\end{aligned}\tag{5.2.49}$$

where

$$n_i:=q_i^n+n_0,\quad J_i:=q_i^J+J_0,\quad (ne)_i:=q_i^{ne}+(ne)_0,(i=k,k-1),$$

$$P_i:=\left(\frac{2}{3}(ne)_i-\frac{|J_i|^2}{3n_i}+\frac{\epsilon^2}{12}n_i\triangle\ln n_i\right)E_d-\frac{\epsilon^2}{4}n_i(\nabla\otimes\nabla)\ln n_i,(i=k,k-1).$$

Setting $\triangle_n^k:=n_k-n_{k-1}$, $\triangle_J^k:=J_k-J_{k-1}$, $\triangle_{ne}^k:=(ne)_k-(ne)_{k-1}$, $\triangle_V^k:=q_k^v-q_{k-1}^v$, we first have

$$\begin{aligned}P_k-P_{k-1}=&\Bigg(\frac{2}{3}\triangle_{ne}^k+\frac{\epsilon^2}{12}\triangle\triangle_n^k-\frac{\epsilon^2}{12}\left(\frac{|\nabla n_k|^2}{n_k}-\frac{|\nabla n_{k-1}|^2}{n_{k-1}}\right)\\ &-\frac{1}{3}\left(\frac{|J_k|^2}{n_k}-\frac{|J_{k-1}|^2}{n_{k-1}}\right)\Bigg)E_d\\ &+\frac{\epsilon^2}{4}\left(\frac{\nabla n_k\otimes\nabla n_k}{n_k}-\frac{\nabla n_{k-1}\otimes\nabla n_{k-1}}{n_{k-1}}\right)-\frac{\epsilon^2}{4}(\nabla\otimes\nabla)\triangle_n^k.\end{aligned}$$

By a similar reasoning as (3.3.122)-(3.3.123):

$$\left\|\frac{\nabla n_k\otimes\nabla n_k}{n_k}-\frac{\nabla n_{k-1}\otimes\nabla n_{k-1}}{n_{k-1}}\right\|_{L^2(\Omega)}\le C\|\triangle_n^k\|_{H^1(\Omega)},$$

$$\left\|\frac{|\nabla n_k|^2}{n_k}-\frac{|\nabla n_{k-1}|^2}{n_{k-1}}\right\|_{L^2(\Omega)}\le C\|\triangle_n^k\|_{H^1(\Omega)},$$

$$\left\|\frac{|J_k|^2}{n_k}-\frac{|J_{k-1}|^2}{n_{k-1}}\right\|_{L^2(\Omega)}\le C(\|\triangle_n^k\|_{H^1(\Omega)}+\|\triangle_J^k\|_{L^2(\Omega)}).$$

Thus we conclude

$$\|P_k-P_{k-1}\|_{L^2(\Omega)}\le C\left(\left\|\triangle_n^k\right\|_{H^2(\Omega)}+\left\|\triangle_{ne}^k\right\|_{L^2(\Omega)}+\left\|\triangle_J^k\right\|_{L^2(\Omega)}\right).\tag{5.2.50}$$

Further,

$$\begin{aligned}&\frac{J_k((ne)_k+P_k)}{n_k}-\frac{J_{k-1}((ne)_{k-1}+P_{k-1})}{n_{k-1}}\\=&\left(\frac{J_k(ne)_k}{n_k}-\frac{J_{k-1}(ne)_{k-1}}{n_{k-1}}\right)+\left(\frac{J_kP_k}{n_k}-\frac{J_{k-1}P_{k-1}}{n_{k-1}}\right)=:\mathcal{K}_1+\mathcal{K}_2.\end{aligned}$$

Here

$$\begin{aligned}\mathcal{K}_1&=\frac{J_k}{n_k}\triangle_{ne}^k+\frac{(ne)_{k-1}}{n_k}\triangle_J^k-\frac{J_{k-1}(ne)_{k-1}}{n_kn_{k-1}}\triangle_n^k,\\ \mathcal{K}_2&=\frac{J_k}{n_k}(P_k-P_{k-1})+\frac{P_{k-1}}{n_k}\triangle_J^k-\frac{J_{k-1}P_{k-1}}{n_kn_{k-1}}\triangle_n^k,\end{aligned}$$

from which we infer

$$\begin{aligned}\|\mathcal{K}_1\|_{L^2(\Omega)}\le&\left\|J_kn_k^{-1}\right\|_{L^\infty(\Omega)}\|\triangle_{ne}^k\|_{L^2(\Omega)}+\left\|n_k^{-1}\right\|_{L^\infty(\Omega)}\|(ne)_{k-1}\|_{L^4(\Omega)}\|\triangle_J^k\|_{L^4(\Omega)}\\&+\left\|J_{k-1}(n_kn_{k-1})^{-1}\right\|_{L^\infty(\Omega)}\|(ne)_{k-1}\|_{L^4(\Omega)}\|\triangle_n^k\|_{L^4(\Omega)}\\ \le&C\left(\|\triangle_{ne}^k\|_{L^2(\Omega)}+\|\triangle_J^k\|_{H^1(\Omega)}+\|\triangle_n^k\|_{H^1(\Omega)}\right).\end{aligned}$$

In like manner

$$\begin{aligned}\|\mathcal{K}_2\|_{L^2(\Omega)}\le&\left\|J_kn_k^{-1}\right\|_{L^\infty(\Omega)}\|P_k-P_{k-1}\|_{L^2(\Omega)}+\left\|n_k^{-1}\right\|_{L^\infty(\Omega)}\|P_{k-1}\|_{L^4(\Omega)}\|\triangle_J^k\|_{L^4(\Omega)}\\&+\left\|J_{k-1}(n_kn_{k-1})^{-1}\right\|_{L^\infty(\Omega)}\|P_{k-1}\|_{L^4(\Omega)}\|\triangle_n^k\|_{L^4(\Omega)}\\ \le&C\left(\|P_k-P_{k-1}\|_{L^2(\Omega)}+\|\triangle_J^k\|_{H^1(\Omega)}+\|\triangle_n^k\|_{H^1(\Omega)}\right).\end{aligned}$$

Recall (5.2.50) we obtain

$$\|\mathcal{K}_2\|_{L^2(\Omega)}\le C\left(\|\triangle_{ne}^k\|_{L^2(\Omega)}+\|\triangle_J^k\|_{H^1(\Omega)}+\|\triangle_n^k\|_{H^2(\Omega)}\right).$$

Thus

$$\begin{aligned}&\left\|\frac{J_k((ne)_k+P_k)}{n_k}-\frac{J_{k-1}((ne)_{k-1}+P_{k-1})}{n_{k-1}}\right\|_{L^2(\Omega)}\\ \le&\ C\left(\|\triangle_{ne}^k\|_{L^2(\Omega)}+\|\triangle_J^k\|_{H^1(\Omega)}+\|\triangle_n^k\|_{H^2(\Omega)}\right).\end{aligned}\tag{5.2.51}$$

It remains to estimate $\left\|J_k\nabla q_k^v-J_{k-1}\nabla q_{k-1}^v\right\|_{L^2(\Omega)}$. We first have the reformulation:

$$J_k\nabla q_k^v-J_{k-1}\nabla q_{k-1}^v=J_k\nabla\triangle_V^k+\nabla q_{k-1}^v\triangle_J^k.$$

Since $\triangle_V^k$ is the unique solution of

$$\begin{cases} \lambda^2 \triangle \triangle_V^k = \triangle_n^k, \\ \triangle_V^k|_{\partial\Omega} = 0, \end{cases} \tag{5.2.52}$$

then $\|\triangle_V^k\|_{H^2(\Omega)} \leq C\|\triangle_n^k\|_{L^2(\Omega)}$. Thus

$$\left\|J_k \nabla q_k^v - J_{k-1}\nabla q_{k-1}^v\right\|_{L^2(\Omega)} \leq C\left(\|\triangle_J^k\|_{H^1(\Omega)} + \|\triangle_n^k\|_{L^2(\Omega)}\right). \tag{5.2.53}$$

Recalling (5.2.48), combining (5.2.49), (5.2.51) and (5.2.53) we obtain

$$\begin{aligned} &\partial_t \|q_{k+1}^{ne} - q_k^{ne}\|^2 + C\|q_{k+1}^{ne} - q_k^{ne}\|^2_{H_0^1(\Omega)} \\ &\leq C\left(\|\triangle_{ne}^k\|^2_{L^2(\Omega)} + \|\triangle_J^k\|^2_{H^1(\Omega)} + \|\triangle_n^k\|^2_{H^2(\Omega)}\right). \end{aligned} \tag{5.2.54}$$

Integrating (5.2.54) from 0 to any $t \in [0, t_u)$ yields

$$\begin{aligned} &\|(q_{k+1}^{ne} - q_k^{ne})(t,\cdot)\|^2 \\ &\leq Ct_u\|\triangle_{ne}^k\|^2_{L^\infty(0,t_u;L^2(\Omega))} + C\left(\|\triangle_J^k\|^2_{L^2(0,t_u;H^1(\Omega))} + \|\triangle_n^k\|^2_{L^2(0,t_u;H^2(\Omega))}\right). \end{aligned}$$

From (5.2.47) we decuce

$$\|q_{k+1}^{ne} - q_k^{ne}\|^2_{L^\infty(0,t_u;L^2(\Omega))} \leq \left(Ct_u + \frac{Ct_u}{1-Ct_u}\right)\|\triangle_{ne}^k\|^2_{L^\infty(0,t_u;L^2(\Omega))},$$

which implies $\{q_k^{ne}\}_{k=0}^\infty$ is a Cauchy sequence in $L^\infty(0,t_u;L^2(\Omega))$ assuming

$$\left(Ct_u + \frac{Ct_u}{1-Ct_u}\right) < 1,$$

thus there exists $q_*^{ne} \in L^\infty(0,t_u;L^2(\Omega))$ such that

$$q_k^{ne} \longrightarrow q_*^{ne} \text{ in } L^\infty(0,t_u;L^2(\Omega)).$$

□

Now we are in a position to study the convergence behavior of the sequence $\{(q_k^n, q_k^J, q_k^v, q_k^{ne})\}_{k=0}^\infty$. Select $0 < \mathcal{T}_* < \min(t_u, \mathcal{C}_*)$. Since the embeddings $H^3(\Omega) \hookrightarrow H^2(\Omega)$, $H^2(\Omega) \hookrightarrow H^1(\Omega)$ are compact, $\{(q_k^n, q_k^J, q_k^{ne})\}_{k=0}^\infty$ are bounded in $L^\infty(0,\mathcal{T}_*;H^3(\Omega)\times(H^2(\Omega))^{d+1})$ and $\{(\partial_t q_k^n, \partial_t q_k^J, \partial_t q_k^{ne})\}_{k=0}^\infty$ are bounded in $L^2(0,\mathcal{T}_*;H^1(\Omega)\times(L^2(\Omega))^{d+1})$ from **Lemma** 5.2.3 and **Remark** 5.2.1, then Aubin's Lemma (Corollary 4 in [72]) yields $\{q_k^n\}_{k=0}^\infty$, $\{q_k^J\}_{k=0}^\infty$ and $\{q_k^{ne}\}_{k=0}^\infty$ are relatively compact in $C([0,\mathcal{T}_*],H^2(\Omega))$, $C([0,\mathcal{T}_*];(H^1(\Omega))^d)$ and $C([0,\mathcal{T}_*];H^1(\Omega))$ respectively, i.e., there is a subsequence of (q_k^n, q_k^J, q_k^{ne}) and functions $(\mathcal{P}^n, \mathcal{P}^J, V, \mathcal{P}^{ne})$ which satisfy (maybe after extracting a subsequence)

$$(q_k^n, q_k^J) \to (\mathcal{P}^n, \mathcal{P}^J) \text{ in } C([0,\mathcal{T}_*];H^2(\Omega)\times(H^1(\Omega))^d), \tag{5.2.55}$$

$$(q_k^v, q_k^{ne}) \to (V, \mathcal{P}^{ne}) \text{ in } C([0,\mathcal{T}_*]; H^2(\Omega)) \times C([0,\mathcal{T}_*]; H^1(\Omega)). \tag{5.2.56}$$

Furthermore we also obtain

$$\begin{cases} \partial_t q_k^n \rightharpoonup \partial_t \mathcal{P}^n & \text{in} \quad L^2(0,\mathcal{T}_*; H^2(\Omega)), \\ \partial_t q_k^J \rightharpoonup \partial_t \mathcal{P}^J & \text{in} \quad L^2(0,\mathcal{T}_*; H^1(\Omega)), \\ \partial_t q_k^{ne} \rightharpoonup \partial_t \mathcal{P}^{ne} & \text{in} \quad L^2(0,\mathcal{T}_*; H^1(\Omega)), \\ q_k^n \rightharpoonup^* \mathcal{P}^n & \text{in} \quad L^\infty(0,\mathcal{T}_*; H^3(\Omega)), \\ q_k^J \rightharpoonup^* \mathcal{P}^J & \text{in} \quad L^\infty(0,\mathcal{T}_*; H^2(\Omega)), \\ q_k^{ne} \rightharpoonup^* \mathcal{P}^{ne} & \text{in} \quad L^\infty(0,\mathcal{T}_*; H^2(\Omega)). \end{cases} \tag{5.2.57}$$

Fix now $0 < \gamma < \dfrac{1}{2}$ and any $t \in (0, \mathcal{T}_*)$ we deduce the interpolation (after extracting a subsequence)

$$\|q_k^n - \mathcal{P}^n\|_{H^{3-\gamma}(\Omega)} \le C_{\text{in}} \|q_k^n - \mathcal{P}^n\|_{H^3(\Omega)}^{(3-\gamma)/3} \|q_k^n - \mathcal{P}^n\|_{L^2(\Omega)}^{\gamma/3},$$

then by **Lemma** 5.2.3

$$\|q_k^n - \mathcal{P}^n\|_{H^{3-\gamma}(\Omega)} \le C_{\text{in}} 2^{(3-\gamma)/3} C'^{(3-\gamma)/3} \|q_k^n - \mathcal{P}^n\|_{L^2(\Omega)}^{\gamma/3}. \tag{5.2.58}$$

Similarly

$$\|q_k^J - \mathcal{P}^J\|_{H^{2-\gamma}(\Omega)} \le C_{\text{in}} 2^{(2-\gamma)/2} C'^{(2-\gamma)/2} \|q_k^J - \mathcal{P}^J\|_{L^2(\Omega)}^{\gamma/2}, \tag{5.2.59}$$

$$\|q_k^{ne} - \mathcal{P}^{ne}\|_{H^{2-\gamma}(\Omega)} \le C_{\text{in}} 2^{(2-\gamma)/2} \tilde{C}^{(2-\gamma)/2} \|q_k^{ne} - \mathcal{P}^{ne}\|_{L^2(\Omega)}^{\gamma/2}. \tag{5.2.60}$$

By Sobolev embedding theorem we deduce

$$(q_k^n, \nabla q_k^n, q_k^J, q_k^{ne}) \to (\mathcal{P}^n, \nabla \mathcal{P}^n, \mathcal{P}^J, \mathcal{P}^{ne}) \quad \text{in} \quad C([0,\mathcal{T}_*] \times \overline{\Omega}). \tag{5.2.61}$$

Notice that the functions $(\mathcal{P}^n, \mathcal{P}^J, V, \mathcal{P}^{ne})$ depend upon the selection of the subsequence.

Then we can find a subsequence

$$\{(q_{k_j}^n, q_{k_j}^J, q_{k_j}^v, q_{k_j}^{ne})\}_{j=0}^\infty \subseteq \{(q_k^n, q_k^J, q_k^v, q_k^{ne})\}_{k=0}^\infty$$

such that there are functions $(\mathcal{P}_a^n, \mathcal{P}_a^J, V_a, \mathcal{P}_a^{ne})$ and $\mathcal{P}_b^{ne}$ with $(q_{k_j}^n, q_{k_j}^J, q_{k_j}^v, q_{k_j}^{ne})$ converges to $(\mathcal{P}_a^n, \mathcal{P}_a^J, V_a, \mathcal{P}_a^{ne})$ in the sense of (5.2.55)-(5.2.61) as $j \to \infty$; $q_{k_j+1}^{ne}$ converges to $\mathcal{P}_b^{ne}$ in the sense of (5.2.55) and (5.2.61) as $j \to \infty$. Then from **Lemma** 5.2.4 $\mathcal{P}_a^{ne}$ coincides with $\mathcal{P}_b^{ne}$.

We consider the system

$$
\begin{cases}
\partial_t q^n_{k_j} - \nu_0 \triangle q^n_{k_j} - \operatorname{div} q^J_{k_j} = \nu_0 \triangle n_0 + \operatorname{div} J_0, \\
\partial_t q^J_{k_j} - \nu_0 \triangle q^J_{k_j} + \dfrac{q^J_{k_j}}{\tau} + \dfrac{\epsilon^2}{6} \nabla \triangle q^n_{k_j} - \mu \nabla q^n_{k_j} = \mathfrak{F}_{\mathbf{d}}(q^n_{k_j}, q^J_{k_j}, q^{ne}_{k_j}), \\
\lambda^2 \triangle q^v_{k_j} = q^n_{k_j} + n_0 - \mathcal{C}(x), \\
\partial_t q^{ne}_{k_j+1} - \nu_0 \triangle q^{ne}_{k_j+1} + \dfrac{2}{\tau} q^{ne}_{k_j+1} = \mathfrak{F}(q^n_{k_j}, q^J_{k_j}, q^{ne}_{k_j}), \\
q^{ne}_{k_j+1}(0,\cdot) = 0, \quad q^{ne}_{k_j+1}|_{\partial\Omega} = 0, \\
(q^n_{k_j}, q^J_{k_j}, q^{ne}_{k_j+1})(0,\cdot) = 0, \quad (q^n_{k_j}, q^J_{k_j}, q^{ne}_{k_j+1})|_{\partial\Omega} = 0, \quad q^v_{k_j}|_{\partial\Omega} = V_\Gamma,
\end{cases}
$$

We choose a test function $\varphi \in C_0^\infty(Q_*)$ where $Q_* := [0, t_*] \times \overline{\Omega}$, then it follows

$$
\iint_{Q_*} (-\varphi_t q^n_{k_j} - \nu_0 (\triangle \varphi) q^n_{k_j} + (\nabla \varphi) q^J_{k_j}) dxdt = \iint_{Q_*} \varphi\, (\nu_0 \triangle n_0 + \operatorname{div} J_0)) dxdt,
$$

$$
\iint_{Q_*} \left(-\varphi_t q^J_{k_j} - \nu_0 (\triangle \varphi) q^J_{k_j} + \tau^{-1} \varphi q^J_{k_j} - \frac{\epsilon^2}{6} (\nabla \triangle \varphi) q^n_{k_j} + \mu \varphi q^n_{k_j} \right) dxdt
$$
$$
= \iint_{Q_*} \varphi \mathfrak{F}_{\mathbf{d}}(q^n_{k_j}, q^J_{k_j}, q^{ne}_{k_j}) dxdt,
$$

$$
\iint_{Q_*} \lambda^2 \triangle \varphi q^v_{k_j} dxdt = \iint_{Q_*} \varphi (q^n_{k_j} + n_0 - \mathcal{C}(x)) dxdt,
$$

$$
\iint_{Q_*} \left(-\varphi_t q^{ne}_{k_j+1} - \nu_0 (\triangle \varphi) q^{ne}_{k_j+1} + 2\tau^{-1} \varphi q^{ne}_{k_j+1} \right) dxdt = \iint_{Q_*} \varphi \mathfrak{F}(q^n_{k_j}, q^J_{k_j}, q^{ne}_{k_j}) dxdt.
$$

Here

$$
\iint_{Q_*} \varphi \mathfrak{F}_{\mathbf{d}}(q^n_{k_j}, q^J_{k_j}, q^{ne}_{k_j}) dxdt
$$
$$
= \iint_{Q_*} \varphi \left(\frac{2}{3} \nabla (ne)_0 + \nu_0 \triangle J_0 - \tau^{-1} J_0 + \mu \nabla n_0 - \frac{\epsilon^2}{6} \nabla \triangle n_0 \right) dxdt
$$
$$
- \iint_{Q_*} \nabla \varphi \left(\frac{2}{3} q^{ne}_{k_j} + \left(\frac{(q^J_{k_j} + J_0) \otimes (q^J_{k_j} + J_0)}{q^n_{k_j} + n_0} \right) - \frac{1}{3} \frac{|q^J_{k_j} + J_0|^2}{q^n_{k_j} + n_0} \right) dxdt
$$
$$
- \iint_{Q_*} \nabla \varphi \left(\frac{\epsilon^2}{4} \left(\frac{\nabla \left(q^n_{k_j} + n_0 \right) \otimes \nabla \left(q^n_{k_j} + n_0 \right)}{q^n_{k_j} + n_0} \right) - \frac{\epsilon^2}{12} \frac{|\nabla (q^n_{k_j} + n_0)|^2}{q^n_{k_j} + n_0} \right) dxdt
$$
$$
+ \iint_{Q_*} q^v_{k_j} \nabla \left(q^n_{k_j} + n_0 \right) \varphi dxdt + \iint_{Q_*} q^v_{k_j} \left(q^n_{k_j} + n_0 \right) \nabla \varphi dxdt.
$$

According to (5.2.61)-(5.2.57) it is easy to verify that

$$\iint_{Q_*} \varphi \mathfrak{F}_{\mathbf{d}}(q^n_{k_j}, q^J_{k_j}, q^{ne}_{k_j}) dxdt \longrightarrow \iint_{Q_*} \varphi \mathfrak{F}_{\mathbf{d}}(\mathcal{P}^n_a, \mathcal{P}^J_a, \mathcal{P}^{ne}_a) dxdt, \quad \text{as} \quad j \to \infty.$$

Furthermore we have the calculus

$$\begin{aligned}
&\iint_{Q_*} \varphi \mathfrak{F}(q^n_{k_j}, q^J_{k_j}, q^{ne}_{k_j}) dxdt \\
&= \iint_{Q_*} \varphi \Big(\nu_0 \triangle (ne)_0 - 2\tau^{-1}(ne)_0 + 3\tau^{-1}(q^n_{k_j} + n_0) + \mu \operatorname{div}(q^J_{k_j} + J_0) \Big) dxdt \\
&+ \iint_{Q_*} \nabla \varphi q^v_{k_j} (q^J_{k_j} + J_0) dxdt + \iint_{Q_*} \varphi q^v_{k_j} \operatorname{div}(q^J_{k_j} + J_0) dxdt \\
&- \iint_{Q_*} \nabla \varphi \left(\frac{q^J_{k_j} + J_0}{q^n_{k_j} + n_0} \right) (q^{ne}_{k_j} + (ne)_0) dxdt \\
&- \iint_{Q_*} \nabla \varphi \left(P(q^n_{k_j}, q^J_{k_j}, q^{ne}_{k_j}) \left(\frac{q^J_{k_j} + J_0}{q^n_{k_j} + n_0} \right) \right) dxdt.
\end{aligned}$$

Using (5.2.61)-(5.2.57) we obtain the convergence

$$\iint_{Q_*} \nabla \varphi \left(\frac{q^J_{k_j} + J_0}{q^n_{k_j} + n_0} \right) (q^{ne}_{k_j} + (ne)_0) dxdt \to \iint_{Q_*} \nabla \varphi \left(\frac{\mathcal{P}^J_a + J_0}{\mathcal{P}^n_a + n_0} \right) (\mathcal{P}^{ne}_a + (ne)_0) dxdt,$$

$$\iint_{Q_*} \nabla \varphi \left(\frac{q^J_{k_j} + J_0}{q^n_{k_j} + n_0} \right) \triangle \left(q^n_{k_j} + n_0 \right) dxdt \to \iint_{Q_*} \nabla \varphi \left(\frac{\mathcal{P}^J_a + J_0}{\mathcal{P}^n_a + n_0} \right) \triangle \left(\mathcal{P}^n_a + n_0 \right) dxdt,$$

$$\iint_{Q_*} \nabla \varphi \left(\frac{q^J_{k_j} + J_0}{q^n_{k_j} + n_0} \right) \frac{|q^J_{k_j} + J_0|^2}{q^n_{k_j} + n_0} dxdt \to \iint_{Q_*} \nabla \varphi \left(\frac{\mathcal{P}^J_a + J_0}{\mathcal{P}^n_a + n_0} \right) \frac{|\mathcal{P}^J_a + J_0|^2}{\mathcal{P}^n_a + n_0} dxdt,$$

$$\iint_{Q_*} \nabla \varphi \left(\frac{q^J_{k_j} + J_0}{q^n_{k_j} + n_0} \right) \frac{|\nabla (q^n_{k_j} + n_0)|^2}{q^n_{k_j} + n_0} dxdt \to \iint_{Q_*} \nabla \varphi \left(\frac{\mathcal{P}^J_a + J_0}{\mathcal{P}^n_a + n_0} \right) \frac{|\nabla (\mathcal{P}^n_a + n_0)|^2}{\mathcal{P}^n_a + n_0} dxdt,$$

$$\iint_{Q_*} \nabla \varphi \Big((\nabla \otimes \nabla) \left(q^n_{k_j} + n_0 \right) \Big) \left(\frac{q^J_{k_j} + J_0}{q^n_{k_j} + n_0} \right) dxdt$$

converges to

$$\iint_{Q_*} \nabla \varphi \left((\nabla \otimes \nabla) \left(\mathcal{P}^n_a + n_0 \right) \right) \left(\frac{\mathcal{P}^J_a + J_0}{\mathcal{P}^n_a + n_0} \right) dxdt, \quad \text{as } j \to \infty,$$

and

$$\iint_{Q_*} \nabla \varphi \left(\frac{\nabla (q^n_{k_j} + n_0) \otimes \nabla (q^n_{k_j} + n_0)}{q^n_{k_j} + n_0} \right) \left(\frac{q^J_{k_j} + J_0}{q^n_{k_j} + n_0} \right) dxdt$$

converges to

$$\iint_{Q_*} \nabla\varphi \left(\frac{\nabla(\mathcal{P}_a^n + n_0) \otimes \nabla(\mathcal{P}_a^n + n_0)}{\mathcal{P}_a^n + n_0} \right) \left(\frac{\mathcal{P}_a^J + J_0}{\mathcal{P}_a^J + n_0} \right) dxdt, \quad \text{as } j \to \infty,$$

which implies

$$\iint_{Q_*} \varphi \mathfrak{F}(q_{k_j}^n, q_{k_j}^J, q_{k_j}^{ne}) dxdt \longrightarrow \iint_{Q_*} \varphi \mathfrak{F}(\mathcal{P}_a^n, \mathcal{P}_a^J, \mathcal{P}_a^{ne}) dxdt, \quad \text{as} \quad j \to \infty.$$

Thus we conclude $(\mathcal{P}_a^n, \mathcal{P}_a^J, V_a, \mathcal{P}_a^{ne})$ is the solution we seek.

5.2.4 Uniqueness

Let $(n^1, J^1, V^1, (ne)^1)$ and $(n^2, J^2, V^2, (ne)^2)$ be two solutions of (5.1.1)-(5.1.4). Put

$$n_\triangle = n^1 - n^2, \quad J_\triangle = J^1 - J^2, \quad V_\triangle = V^1 - V^2, \quad (ne)_\triangle = (ne)^1 - (ne)^2,$$

then we obtain the system

$$\begin{cases} \partial_t n_\triangle - \nu_0 \triangle n_\triangle = \operatorname{div} J_\triangle, \\ \partial_t J_\triangle - \nu_0 \triangle J_\triangle + \dfrac{1}{\tau} J_\triangle - \mu \nabla n_\triangle + \dfrac{\epsilon^2}{6} \nabla \triangle n_\triangle = \triangle_{\mathcal{F}_\mathbf{d}}, \\ \partial_t (ne)_\triangle - \nu_0 \triangle (ne)_\triangle + \dfrac{2}{\tau} (ne)_\triangle = \triangle_{\mathcal{F}}, \\ \lambda^2 \triangle V_\triangle = n_\triangle, \\ (n_\triangle, J_\triangle, V_\triangle, (ne)_\triangle)(t,x) = 0 \quad \text{on } [0, \mathcal{T}_*) \times \partial\Omega, \\ (n_\triangle, J_\triangle, (ne)_\triangle)(0,x) = 0, \end{cases}$$

where

$$\begin{aligned} \triangle_{\mathcal{F}_\mathbf{d}} &:= \mathcal{F}_\mathbf{d}(n^1, J^1, (ne)^1) - \mathcal{F}_\mathbf{d}(n^2, J^2, (ne)^2), \\ \triangle_{\mathcal{F}} &:= \mathcal{F}(n^1, J^1, (ne)^1) - \mathcal{F}(n^2, J^2, (ne)^2). \end{aligned}$$

Similarly as (5.2.44) we obtain for a.e. $t \in [0, \mathcal{T}_*]$

$$\begin{aligned} \frac{1}{2} \partial_t \Big(\mu \|n_\triangle\|^2 + \frac{\epsilon^2}{6} \|\nabla n_\triangle\|^2 + \|J_\triangle\|^2 \Big) + \nu_0 T_0 \|\nabla n_\triangle\|^2 \\ + \frac{\epsilon^2}{6} \nu_0 \|\triangle n_\triangle\|^2 + \nu_0 \|\nabla J_\triangle\|^2 + \frac{1}{\tau} \|J_\triangle\|^2 \\ = (\triangle_{\mathcal{F}_\mathbf{d}}, J_\triangle) \le \|\triangle_{\mathcal{F}_\mathbf{d}}\|_{H^{-1}(\Omega)} \|J_\triangle\|_{H_0^1(\Omega)} . \end{aligned}$$

Using Young's inequality

$$\begin{aligned}\frac{1}{2}\partial_t\Big(\mu\|n_\triangle\|^2+\frac{\epsilon^2}{6}\|\nabla n_\triangle\|^2+\|J_\triangle\|^2\Big)+C\Big(&\nu_0T_0\|\nabla n_\triangle\|^2\\ &+\frac{\epsilon^2}{6}\nu_0\|\triangle n_\triangle\|^2+\nu_0\|\nabla J_\triangle\|^2+\frac{1}{\tau}\|J_\triangle\|^2\Big)\\ \leq & C\,\|\triangle_{\mathcal{F}_\mathbf{d}}\|^2_{L^\infty(0,\mathcal{T}_*;H^{-1}(\Omega))}\,,\end{aligned}\tag{5.2.62}$$

which implies

$$\|n_\triangle\|^2_{L^\infty(0,\mathcal{T}_*;H^1_0(\Omega))}+\|J_\triangle\|^2_{L^\infty(0,\mathcal{T}_*;L^2(\Omega))}\leq C\mathcal{T}_*\|\triangle_{\mathcal{F}_\mathbf{d}}\|^2_{L^\infty(0,\mathcal{T}_*;H^{-1}(\Omega))}.\tag{5.2.63}$$

We go back to the proof of **Lemma** 5.2.4. According to (5.2.45) we deduce for a.e. $t\in[0,\mathcal{T}_*]$

$$\|\triangle_{\mathcal{F}_\mathbf{d}}\|^2_{H^{-1}(\Omega)}\leq C\left(\|n_\triangle\|^2_{H^1(\Omega)}+\|J_\triangle\|^2_{L^2(\Omega)}+\|(ne)_\triangle\|^2_{L^2(\Omega)}\right).\tag{5.2.64}$$

Integrate (5.2.62) from 0 to $t\in[0,\mathcal{T}_*)$ then we infer

$$\begin{aligned}&\|n_\triangle\|^2_{L^\infty(0,\mathcal{T}_*;H^1_0(\Omega))}+\|J_\triangle\|^2_{L^\infty(0,\mathcal{T}_*;L^2(\Omega))}\\ &\quad\leq\frac{C\mathcal{T}_*}{1-C\mathcal{T}_*}\|(ne)_\triangle\|^2_{L^\infty(0,\mathcal{T}_*;L^2(\Omega))},\end{aligned}\tag{5.2.65}$$

which implies

$$\begin{aligned}&\|n_\triangle\|^2_{L^2(0,\mathcal{T}_*;H^2(\Omega))}+\|J_\triangle\|^2_{L^2(0,\mathcal{T}_*;H^1(\Omega))}\\ &\quad\leq\frac{C\mathcal{T}_*}{1-C\mathcal{T}_*}\|(ne)_\triangle\|^2_{L^\infty(0,\mathcal{T}_*;L^2(\Omega))}.\end{aligned}\tag{5.2.66}$$

Next we have

$$\begin{aligned}&\partial_t\frac{1}{2}\|(ne)_\triangle\|^2+\nu_0\|\nabla((ne)_\triangle)\|^2+\frac{2}{\tau}\|(ne)_\triangle\|^2\\ \leq&\ \|\triangle_{\mathcal{F}}\|_{H^{-1}(\Omega)}\|(ne)_\triangle\|_{H^1_0(\Omega)}.\end{aligned}\tag{5.2.67}$$

By a similar reasoning as (5.2.49)-(5.2.53) we obtain for a.e. $t\in[0,\mathcal{T}_*)$

$$\|\triangle_{\mathcal{F}}\|^2_{H^{-1}(\Omega)}\leq C\left(\|n_\triangle\|^2_{H^2(\Omega)}+\|J_\triangle\|^2_{H^1(\Omega)}+\|(ne)_\triangle\|^2_{L^2(\Omega)}\right).\tag{5.2.68}$$

Integrate (5.2.67) from 0 to any $t\in[0,\mathcal{T}_*)$ we see

$$\begin{aligned}&\|(ne)_\triangle\|^2_{L^\infty(0,\mathcal{T}_*;L^2(\Omega))}\\ \leq&C\mathcal{T}_*\|(ne)_\triangle\|^2_{L^\infty(0,\mathcal{T}_*;L^2(\Omega))}\\ &+C\left(\|n_\triangle\|^2_{L^2(0,\mathcal{T}_*;H^2(\Omega))}+\|J_\triangle\|^2_{L^2(0,\mathcal{T}_*;H^1(\Omega))}\right).\end{aligned}\tag{5.2.69}$$

Finally let us substitute (5.2.66) into (5.2.69), then we conclude

$$\|(ne)_{\triangle}\|^2_{L^\infty(0,\mathcal{T}_*;L^2(\Omega))} \le \left(C\mathcal{T}_* + \frac{C\mathcal{T}_*}{1 - C\mathcal{T}_*} \right) \|(ne)_{\triangle}\|^2_{L^\infty(0,\mathcal{T}_*;L^2(\Omega))}.$$

Since $\mathcal{T}_* \le t_u$,

$$C\mathcal{T}_* + \frac{C\mathcal{T}_*}{1 - C\mathcal{T}_*} < 1,$$

from which it follows $(ne)_{\triangle} \equiv 0$. Furthermore from (5.2.66) we have

$$n_{\triangle} \equiv J_{\triangle} \equiv 0.$$

Thus we have completed the proof of the uniqueness of local solutions.

Appendix A

Calculus Facts

A.1 Gauss-Green Theorem

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and $\partial\Omega$ is C^1.

- **Gauss-Green Theorem**

  Suppose $u \in H^1(\Omega)$. Then

$$\int_\Omega u_{x_i} dx = \int_{\partial\Omega} u\nu_i dS \quad (i = 1, \cdots, n).$$

- **Integration-by-parts formula**

  Let $u, v \in H^1(\Omega)$. Then

$$\int_\Omega u_{x_i} v dx = -\int_\Omega u v_{x_i} dx + \int_{\partial\Omega} uv\nu_i dS \quad (i = 1, \cdots, n).$$

- **Green's formulas** Let $u, v \in H^2(\Omega)$. Then

  1. $\displaystyle\int_\Omega \triangle u dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS,$
  2. $\displaystyle\int_\Omega \nabla u \cdot \nabla v dx = -\int_\Omega u \triangle v dx + \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} dS,$
  3. $\displaystyle\int_\Omega (u\triangle v - v\triangle u) dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS.$

A.2 Convolution

We introduce tools which provide smooth approximations to given functions.

Let $\Omega \subset \mathbb{R}^n$ be open, write $\Omega_\varepsilon := \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$.

Definition A.2.1. *Define* $\eta \in C^{\infty}(\mathbb{R}^n)$ *by*

$$\eta(x) := \begin{cases} C\,exp\left(\dfrac{1}{|x|^2-1}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

the constant $C > 0$ *is selected such that* $\int_{\mathbb{R}^n} \eta dx = 1$. *For each* $\varepsilon > 0$, *set*

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right).$$

We call η *the standard mollifier. Obviously, the functions* $\eta_\varepsilon(x)$ *are in* $C^{\infty}(\mathbb{R}^n)$ *and satisfy*

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x)dx = 1, \quad supp\eta_\varepsilon \subset B(0,\varepsilon),$$

where $B(0,\varepsilon)$ *is a closed ball with center* 0, *radius* ε.

Definition A.2.2. *If* $f : \Omega \to \mathbb{R}$ *is locally integrable, we define its mollification*

$$f^\varepsilon := \eta_\varepsilon * f \quad \text{in } \Omega_\varepsilon,$$

where

$$\eta_\varepsilon * f = \int_\Omega \eta_\varepsilon(x-y)f(y)dy = \int_{B(0,\varepsilon)} \eta_\varepsilon(y)f(x-y)dy \qquad \text{for } x \in \Omega_\varepsilon.$$

Theorem A.2.1. *(Properties of mollifiers)*

1. $f^\varepsilon \in C^{\infty}(\Omega_\varepsilon)$.
2. $f^\varepsilon \to f$ *a.e. as* $\varepsilon \to 0$.
3. *If* $f \in C(\Omega)$, *then* $f^\varepsilon \to f$ *uniformly on compact subsets of* Ω.
4. *If* $1 \leq p < \infty$ *and* $f \in L^p_{loc}(\Omega)$, *then* $f^\varepsilon \to f$ *in* $L^p_{loc}(\Omega)$.
5. *If* $f \in C(\overline{\Omega})$, *then* $f^\varepsilon \to f$ *uniformly on* Ω.

Appendix B

Convergence and Compactness

Theorem B.0.2. *Any bounded set in a reflexive Banach space is weakly compact, i.e., any sequence in a bounded set has a weakly converging subsequence.*

Theorem B.0.3. *Any bounded set in $L^p(\Omega)$ with $1 < p \leq \infty$ is weakly star compact.*

Theorem B.0.4. *(J.P. Aubin [12])*

Let $T > 0$, X, B, Y be Banach spaces with continuous embeddings $X \subset B \subset Y$, the first embedding being compact. Then we have

1. *Let $1 \leq p \leq \infty$, F be a bounded set of $L^p(0,T;X)$ and be precompact in $L^p(0,T;Y)$, then F is precompact in $L^p(0,T;B)$.*

2. *Let $1 \leq p < \infty$, F be a bounded set of $L^p(0,T;X)$, $\partial_t F := \{\partial_t f \mid f \in F\}$ be bounded in $L^1(0,T;Y)$. Then F is precompact in $L^p(0,T;B)$.*

3. *Let F be bounded of $L^\infty(0,T;X)$, $\partial_t F$ be bounded in $L^r(0,T;Y)$ where $r > 1$. Then F is precompact in $C([0,T];B)$.*

Theorem B.0.5. *(almost everywhere Point-wise Convergence)*

1. *Suppose that Ω is a bounded or unbounded domain in $\mathbb{R}^n$, $u_k(x), u(x)$ are real functions in $L^p(\Omega), (1 \leq p \leq \infty)$ such that u_k converges to u in $L^p(\Omega)$. Then if $1 \leq p < \infty$, u_k has a subsequence a.e. converging to u; if $p = \infty$, then u_k itself converges to u a.e. in Ω.*

2. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$, $\{u_k(x)\}_{k=1}^{\infty}$ is a bounded sequence in $L^p(\Omega), (1 \leq p < \infty)$ such that u_k converges to a function u a.e. in Ω. Then u also belongs to $L^p(\Omega)$, and u_k converges weakly to u in $L^p(\Omega)$. If $p = \infty$, the conclusion becomes that u_k converges to u weakly star in $L^\infty(\Omega)$.*

Definition B.0.3. *A Banach space B is called uniformly conves if for any $\phi, \varphi \in B, \varepsilon > 0$ such that $\|\phi\| = \|\varphi\| = 1$, $\|\phi - \varphi\| \geq \varepsilon$, then there exists a constant $\delta > 0$ depending only upon ε and $\|\phi + \varphi\| \leq 2(1-\delta) < 2$.*

Remark B.0.1. *A Hilbert space is uniformly convex. A uniformly convex Banach space is reflexive. If $1 < p < \infty$, then $L^p(\Omega)$ is uniformly convex. $L^p(\Omega)$ is reflexive if and only if $1 < p < \infty$. Thus in $L^p(\Omega)$*

$$\textit{Reflexivity} \Leftrightarrow \textit{Uniform Convexity} \Leftrightarrow 1 < p < \infty.$$

Theorem B.0.6. *Let B be a uniformly convex Banach space. Suppose that $u_k, u \in B$, $\|u_k\| \to \|u\|$, and u_k converges weakly to u. Then u_k converges strongly to u, i.e., $\|u_k - u\| \to 0$ as $k \to \infty$.*

Theorem B.0.7. *Suppose $1 < p \leq \infty$, B is a Banach space, B^* is the dual space of B, and*

$$\begin{cases} u_k \rightharpoonup^* u & \textit{in } L^p(0,T;B^*), \\ u_k' \rightharpoonup^* u' & \textit{in } L^p(0,T;B^*), \end{cases}$$

then $u_k(0) \rightharpoonup^* u(0)$ in B^*.

Furthermore suppose $1 < p < \infty$, B^* is a reflexive Banach space. Then the weakly star convergence above is equivalent to the weak convergence.

Appendix C

Inequalities

- **Young's inequalities.** Let a,b and ε be positive numbers and $p, q \geq 1$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then

$$ab \leq \frac{\varepsilon^p a^p}{p} + \frac{b^q}{q\varepsilon^q}.$$

- **Hölder's inequality.** Let $1 \leq p_1, \cdots, p_m \leq \infty$, with $\sum_{j=1}^{m} \dfrac{1}{p_j} = 1$, and assume $u_k \in L^{p_k}(\Omega)$ for $k = 1, \cdots, m$. Then

$$\int_\Omega |u_1 \cdots u_m| dx \leq \prod_{k=1}^{m} \|u_i\|_{L^{p_k}(\Omega)}.$$

- **Gronwall's inequality**

  1. Suppose that a, b are nonnegative constants $T > 0$, and $u(t)$ is a nonnegative integrable function. Suppose the inequality

  $$u(t) \leq a + b \int_0^t u(s) ds$$

  holds for $t \in [0, T]$. Then for $0 \leq t \leq T$,

  $$u(t) \leq a e^{bt}.$$

  2. Let $y(\cdot)$ be a nonnegative, absolutely continuous function on $[0, T]$, which satisfies for a.e. $t \in [0, T]$ the differential inequality

  $$y'(t) \leq \phi(t) y(t) + \psi(t),$$

  where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0, T]$. Then

  $$y(t) \leq e^{\int_0^t \phi(s) ds} \left(y(0) + \int_0^t \psi(s) ds \right).$$

- **Interpolation inequalities.**

1. Assume $1 \le s \le r \le t \le \infty$ and $\dfrac{1}{r} = \dfrac{\gamma}{s} + \dfrac{1-\gamma}{t}$ for $\gamma \in [0,1]$. Suppose also $u \in L^s(\Omega) \cap L^t(\Omega)$, then $u \in L^r(\Omega)$, and

$$\|u\|_{L^r(\Omega)} \le \|u\|_{L^s(\Omega)}^{\gamma} \|u\|_{L^t(\Omega)}^{1-\gamma}.$$

2. Suppose $\Omega \subset \mathbb{R}^n$ satisfies the cone condition and $0 \le j \le m, 1 \le p < \infty$, $u \in W_p^m(\Omega)$. Then there exists $K > 0$ depending only on n, m, p, Ω such that

$$\|u\|_{W_p^j(\Omega)} \le K \|u\|_{W_p^m(\Omega)}^{j/m} \|u\|_{L^p(\Omega)}^{(m-j)/m}. \tag{C.0.1}$$

3. Suppose $\Omega \subset \mathbb{R}^n$ satisfies the cone condition and $m \ge 0, 1 \le p < \infty$, $u \in W_p^m(\Omega)$. If $mp > n$, let $q \in [p, \infty]$; if $mp = n$, let $q \in [p, \infty)$; if $mp < n$, let $q \in [p, \frac{np}{n-mp}]$. Then there exists $K > 0$ depending only on n, m, p, q, Ω such that

$$\|u\|_{L^q(\Omega)} \le K \|u\|_{W_p^m(\Omega)}^{\theta} \|u\|_{L^p(\Omega)}^{1-\theta}$$

with $\theta = \frac{n}{mp} - \frac{n}{mq}$.

4. Suppose $\Omega \subset \mathbb{R}^n$ satisfies the cone condition, $p > 1, mp > n, u \in W_p^m(\Omega)$. Suppose either $1 \le q \le p$ or $q > p$ and $mp - p < n$. Then there exists $K > 0$ depending only on n, m, p, q, Ω such that

$$\|u\|_{L^\infty(\Omega)} \le K \|u\|_{W_p^m(\Omega)}^{\theta} \|u\|_{L^q(\Omega)}^{1-\theta}.$$

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