Asymptotic stability of homogeneous states in the relativistic dynamics of viscous, heat-conductive fluids

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Abstract

This paper shows global-in-time existence and asymptotic decay of small solutions to the Navier-Stokes-Fourier equations for a class of viscous, heat-conductive fluids. As this second-order system is symmetric-hyperbolic, existence and uniqueness on a short time interval follow from work of Hughes, Kato, and Marsden. Here it is proven that solutions which are close to a homogeneous reference state can be extended globally and decay to the reference state. The proof combines decay results for the linearization with refined Kawashima-type estimates of the nonlinear terms.

1. Introduction

In relativistic fluid dynamics, stresses in perfect fluids are described by the inviscid energy-momentum-tensor ("relativistic inviscid Cauchy Tensor")

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + pg^{\alpha\beta}, \tag{1.1}$$

where ρ and p are the internal energy and the pressure of the fluid and u^α is its 4-velocity. In this paper we will exclusively consider causal barotropic fluids, a class defined by the property that there exists a one-to-one relation between ρ and p,

$$p = \hat{p}(\rho), \tag{1.2}$$

with a smooth function $\hat{p} : (0, \infty) \to (0, \infty)$ that satisfies $0 < \hat{p}' < 1$. One way to describe the dynamics of dissipative barotropic fluids is via a system

$$\frac{\partial}{\partial x^\beta} \left(T^{\alpha\beta} + \Delta T^{\alpha\beta} \right) = 0, \ \ \alpha = 0, 1, 2, 3 \tag{1.3}$$

of partial differential equations - the conservation laws of energy and momentum -, in which the "dissipation tensor" $\Delta T^{\alpha\beta}$ is linear in the gradients of the state variables determined by coefficients η, ζ of viscosity and χ of heat conduction. Freistühler and Temple have recently proposed a particular new way of choosing $\Delta T^{\alpha\beta}$ such that basic requirements, notably of causuality, are met; see [3] for this and also for a discussion of the interesting history of the causuality problem. According to [3], $\Delta T^{\alpha\beta}$ is given as

$$-\Delta T^{\alpha\beta} = B^{\alpha\beta\gamma\delta}(\psi) \frac{\partial \psi_\gamma}{\partial x^\delta},$$

where ψ denotes the so-called Godunov variables

$$\psi_\gamma = \frac{u_\gamma}{f}$$

with f the Lichnerowicz index of the fluid. The key property of Godunov variables is that in these, the first-order term of a system of conservation laws, here

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta},$$

becomes *symmetric* hyperbolic [4].[1] Now, the requirement that also

$$-\frac{\partial}{\partial x^\beta} \left(\Delta T^{\alpha\beta} \right)$$

should be symmetric hyberbolic when written in the same variables determines a set of coefficient fields $B^{\alpha\beta\gamma\delta}(\psi)$ which make (1.3) an element of a class of systems that was introduced by Hughes, Kato and Marsden and shown to be well-posed in Sobolev spaces [5]. As shown in [3], the requirements of equivariance (isotropicity) and other physical necessities indeed make $B^{\alpha\beta\gamma\delta}(\psi)$ determined by the coefficients η, ζ, χ.

The purpose of this paper is to provide a global-in-time solution theory of these relativistic Navier-Stokes-Fourier equations (1.3). To this end, we analyze first the linearization of (1.3) at some homogeneous reference

[1]See [2] for details and the history of the use of such variables in relativistic fluid dynamics

state and then the nonlinear problem as a pertubation of the linear one, both with techniques that were developed, or are simililar to techniques developed by Kawashima and co-authors notably in [6], [1].

To have a clear setting, we carry out the whole argument under the additional assumption that the fluid is indeed thermobarotropic, which means that that not only its pressure but also its temperature is a function of its internal energy,

$$\theta = \hat{\theta}(\rho). \tag{1.4}$$

In this case, the Lichnerowicz index is identical with the temperature,

$$f = \theta. \tag{1.5}$$

and actual heat conduction can be an integrated part of a four-field theory, see [2]. An important physical example of this is given by the case of the pure radiation fluid [7], whose internal energy as function of particle number, density and specific entropy is given by

$$\rho(n, s) = kn^{\frac{4}{3}} s^{\frac{4}{3}}.$$

2. Preliminaries and Notation

As stated in the introduction, the goal of this paper is to prove existence and asymptotic decay of global-in-time solutions of (1.3) near homogeneous reference states. First, writting (1.3) in Godunov variables gives

$$-B^{\alpha\beta\gamma\delta}(\psi)\frac{\partial\psi_\gamma}{\partial x^\beta \partial^\delta_x} + \frac{\partial}{\partial x^\beta} T^{\alpha\beta}(\psi) - \frac{\partial}{\partial x^\beta}\left(B^{\alpha\beta\gamma\delta}(\psi)\right)\frac{\partial\psi_\gamma}{\partial x^\delta} = 0, \qquad \alpha = 0, 1, 2, 3. \tag{2.1}$$

In our case of a thermobarotropic fluid the "dissipation tensor" and the inviscid energy-momentum-tensor are given by [2]

$$\begin{aligned} B^{\alpha\beta\gamma\delta} = \chi\theta^2 u^\alpha u^\gamma g^{\beta\delta} - \sigma\theta u^\beta u^\delta \Pi^{\alpha\gamma} + \tilde{\zeta}\theta\Pi^{\alpha\beta}\Pi^{\gamma\delta} \\ + \eta\theta(\Pi^{\alpha\gamma}\Pi^{\beta\delta} + \Pi^{\alpha\delta}\Pi^{\beta\gamma} - \frac{2}{3}\Pi^{\alpha\beta}\Pi^{\gamma\delta}) \\ + \sigma\theta(u^\alpha u^\beta \eta^{\gamma\delta} - u^\alpha u^\delta \eta^{\beta\gamma}) + \chi\theta^2(u^\beta u^\gamma \eta^{\alpha\delta} - u^\gamma u^\delta \eta^{\alpha\beta}), \end{aligned}$$

[2] We use the Minkowski metric $g^{\alpha\beta}$ with signature $-+++$ and the standard projection $\Pi^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$

with $\sigma = (\frac{4}{3}\eta + \zeta)/(1 - c_s^2) - c_s^2\chi\theta$, $\tilde{\zeta} = \zeta + c_s^2\sigma - c_s^2(1 - c_s^2)\chi\theta$, where $c_s^2 = p'(\rho) \in (0, 1)$ is the speed of sound (cf. [3]), and

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta} = sn\theta^2 \left[u^\alpha g^{\beta\gamma} + u^\beta g^{\alpha\gamma} + u^\gamma g^{\alpha\beta} + (3 + c_s^{-2}) u^\alpha u^\beta u^\gamma \right] \frac{\partial \psi_\gamma}{\partial x^\beta},$$

with particle number n and specific entropy s. It was shown in [3] that (2.1) is symmetric hyperbolic in the sense of Hughes-Kato-Marsden [5]. Thus, when replacing

$$B^{\alpha\beta\gamma\delta}(\psi) \frac{\partial \psi_\gamma}{\partial x^\beta \partial_x^\delta}$$

by

$$\tilde{B}^{\alpha\beta\gamma\delta}(\psi) \frac{\partial \psi_\gamma}{\partial x^\beta \partial_x^\delta} = \frac{1}{2} \left(B^{\alpha\beta\gamma\delta}(\psi) + B^{\alpha\delta\gamma\beta}(\psi) \right) \frac{\partial \psi_\gamma}{\partial x^\beta \partial_x^\delta},$$

we can write (2.1) as

$$A(\psi)\psi_{tt} - \sum_{i,j=1}^{3} B_{ij}(\psi)\psi_{x_i x_j} + \sum_{j=1}^{3} D_j(\psi)\psi_{t x_j} + f(\psi, \psi_t, \partial_x \psi) = 0, \quad (2.2)$$

where

$$A = (-\tilde{B}^{\alpha 0 \gamma 0})_{0 \le \alpha, \gamma \le 3}, \quad B_{ij} = (-\tilde{B}^{\alpha i \gamma j})_{0 \le \alpha, \gamma \le 3},$$
$$D_j = (-\tilde{B}^{\alpha 0 \gamma j})_{0 \le \alpha, \gamma \le 3},$$

are symmetric 4×4 matrices, $A(\psi)$ is positive definite, $\sum_{i,j=1}^{3} \xi_i B_{ij}(\psi) \xi_j$ is positive definite for arbitrary $\xi \in \mathbb{R}^3 \setminus \{0\}$, and

$$f^\alpha = \frac{\partial}{\partial x^\beta} T^{\alpha\beta}(\psi) - \frac{\partial}{\partial x^\beta} \left(B^{\alpha\beta\gamma\delta}(\psi) \right) \frac{\partial \psi_\gamma}{\partial x^\delta} = 0, \ \alpha = 0, 1, 2, 3.$$

Throughout the paper we will consider the Cauchy problem associated with (2.2):

$$A\psi_{tt} - \sum_{i,j=1}^{3} B_{ij}\psi_{x_i x_j} + \sum_{j=1}^{3} D_j \psi_{t x_j} + f = 0 \text{ on } (0, T] \times \mathbb{R}^3, \quad (2.3)$$

$$\psi(0) = {}^0\psi \text{ on } \mathbb{R}^3, \quad (2.4)$$

$$\psi_t(0) = {}^1\psi \text{ on } \mathbb{R}^3, \quad (2.5)$$

The main result is the following:

2.1 Theorem. *Let $s \geq 4$, $\theta > 0$ and $\bar{\psi} = (\theta^{-1}, 0, 0, 0,)^t$. Then there exist $\delta_0 > 0, C_0 = C_0(\delta) > 0$ such that for all initial data $({}^0\psi, {}^1\psi_1) \in (H^{s+1} \times H^s) \cap (L^1 \times L^1)$ satisfying $\|({}^0\psi - \bar{\psi}, {}^1\psi)\|^2_{s+1,s,1} < \delta_0$ there exists a unique solution ψ of the Cauchy problem* (2.3)-(2.5) *such that*

$$\psi - \bar{\psi} \in \bigcap_{j=1}^{s} C^j \left([0, \infty), H^{s+1-j} \right).$$

ψ satisfies the decay estimates

$$\|(\psi(t) - \bar{\psi}, \psi_t(t)\|^2_{s+1,s} + \int_0^t \|(\psi(\tau), \psi_t(\tau))\|^2_{s+1,s} d\tau \leq C_0 \|({}^0\psi - \bar{\psi}, {}^1\psi)\|^2_{s+1,s}, \tag{2.6}$$

$$\|(\psi(t) - \bar{\psi}, \psi_t(t))\|_{s,s-1} \leq C_0 (1+t)^{-\frac{3}{4}} \|({}^0\psi - \bar{\psi}, {}^1\psi)\|_{s,s-1,1} \tag{2.7}$$

for all $t \in [0, \infty)$.

In the last part of this section we introduce some notation. For $p \in [1, \infty]$ and some $m \in \mathbb{N}$ just write L^p for $L^p(\mathbb{R}^3, \mathbb{R}^m)$. For $s \in \mathbb{N}_0$ we denote by H^s the L^2-Sobolev-space of order s, namely

$$H^s := \{u \in L^2 : \forall\ \alpha \in \mathbb{N}_0^n\ (|\alpha| \leq s) : \|\partial_x^\alpha u\|_{L^2} < \infty\}$$

with norm

$$\|u\|_s = \left(\sum_{0 \leq |\alpha| \leq s} \|\partial_x^\alpha u\|_{L^2} \right)^{\frac{1}{2}}.$$

We just write $\|u\|$ instead of $\|u\|_0$. For $s, k \in \mathbb{N}_0$ and $U = (u_1, u_2) \in H^s \times H^k$ set

$$\|U\|_{s,k} = \left(\|u_1\|_s^2 + \|u_2\|_k^2 \right)^{\frac{1}{2}}$$

and for $U \in (H^s \times H^k) \cap (L^p)^2$ set

$$\|U\|_{s,k,p} = \|U\|_{s,k} + \|U\|_{(L^p)^2}.$$

For $u \in H^s$ and integers $0 \leq k \leq s$, ∂_x^k shall denote the vector in $\mathbb{R}^N$, $N = m\#\{\alpha \in \mathbb{N}_0^n : |\alpha| = k\}$, whose entries are the partial derivatives of u of order k.

For $u \in H^s$, $v \in H^{l-1}$ $(0 \leq l \leq s)$ and $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq s$ set

$$[\partial_x^\alpha, u]v = \partial_x^\alpha(uv) - u\partial_x^\alpha v.$$

For $\delta > 0$ let ϕ_δ denote the Friedrichs mollifier and set

$$[\phi_\delta *, u]v = \phi_\delta * (uv) - u(\phi_\delta * v).$$

3. Decay Estimates for the Linearized System

In this section we study the linearization of (2.2) in the fluids restframe $u^\alpha = (1,0,0,0)^t$ and for a constant temperature $\theta > 0$. The resulting equations read

$$A^{(1)}\psi_{tt} - \sum_{i,j=1}^{3} B_{ij}^{(1)}\psi_{x_i x_j} + a^{(1)}\psi_t + \sum_{j=1}^{3} b_j^{(1)}\psi_{x_j} = 0, \tag{3.1}$$

where

$$A^{(1)} = \begin{pmatrix} \chi\theta^2 & 0 \\ 0 & \sigma\theta I_3 \end{pmatrix},$$

$$B_{ij}^{(1)} = \begin{pmatrix} \chi\theta^2 & 0 \\ 0 & \theta\eta I_3\delta_{ij} + \theta(\tilde{\zeta} + \frac{1}{3}\eta)(e_i \otimes e_j + e_j \otimes e_i) \end{pmatrix},$$

$$a^{(1)} = ns\theta^2 \begin{pmatrix} c_s^{-2} & 0 \\ 0 & I_3 \end{pmatrix}, \quad b_j^{(1)} = ns\theta^2(e_j \otimes e_0 + e_0 \otimes e_j),$$

First, multiply (3.1) by $(ns)^{-1}\theta^{-2}$ and set $\bar{\chi} = \chi(ns)^{-1}$, $\bar{\eta} = \eta(ns\theta)^{-1}$, $\bar{\zeta} = \tilde{\zeta}(ns\theta)^{-1}$, $\bar{\sigma} = \bar{\zeta} + \frac{1}{3}\bar{\eta}$. We arrive at the equivalent system

$$A^{(2)}\psi_{tt} - \sum_{i,j=1}^{3} B_{ij}^{(2)}\psi_{x_i x_j} + a^{(2)}\psi_t + \sum_{j=1}^{3} b_j^{(2)}\psi_{x_j} = 0, \tag{3.2}$$

Ã¼ where

$$A^{(2)} = \begin{pmatrix} \bar{\chi} & 0 \\ 0 & \bar{\sigma} I_3 \end{pmatrix}, \quad B_{ij}^{(2)} = \begin{pmatrix} \bar{\chi} & 0 \\ 0 & \bar{\eta} I_3\delta_{ij} + \left(\bar{\zeta} + \frac{1}{3}\bar{\eta}\right)(e_i \otimes e_j + e_j \otimes e_i) \end{pmatrix},$$

$$a^{(2)} = \begin{pmatrix} c_s^{-2} & 0 \\ 0 & I_3 \end{pmatrix}, \quad b_j = (e_j \otimes e_0 + e_0 \otimes e_j).$$

Finally, multiplying (3.2) by $(A^{(2)})^{-\frac{1}{2}}$ and writting it in variables $(A^{(2)})^{\frac{1}{2}}\psi$ gives

$$\psi_{tt} - \sum_{i,j=1}^{3} \bar{B}_{ij}\psi_{x_i x_j} + a\psi_t + \sum_{j=1}^{3} b_j\psi_{x_j} = 0, \tag{3.3}$$

where

$$\bar{B}_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \bar{\sigma}^{-1}\left(\bar{\eta} I_3\delta_{ij} + \left(\bar{\zeta} + \frac{1}{3}\bar{\eta}\right)(e_i \otimes e_j + e_j \otimes e_i)\right) \end{pmatrix},$$

$$a = \begin{pmatrix} c_s^{-2}\bar{\chi}^{-1} & 0 \\ 0 & \bar{\sigma}^{-1} I_3 \end{pmatrix}, \quad b_j = (\bar{\chi}\bar{\sigma})^{-\frac{1}{2}}(e_j \otimes e_0 + e_0 \otimes e_j).$$

The goal is to prove a decay estimate for the Cauchy problem associated with (3.3):

$$\psi_{tt} - \sum_{i,j=1}^{3} B_{ij}\psi_{x_i x_j} + a\psi_t + \sum_{j=1}^{3} b_j \psi_{x_j} = 0 \text{ on } (0,T] \times \mathbb{R}^3, \tag{3.4}$$

$$\psi(0) = {}^0\psi \text{ on } \mathbb{R}^3, \tag{3.5}$$

$$\psi_t(0) = {}^1\psi \text{ on } \mathbb{R}^3. \tag{3.6}$$

3.1 Proposition. *For some $s \in \mathbb{N}_0$ let $({}^0\psi, {}^1\psi) \in (H^{s+1} \times H^s) \cap (L^1)^2$ and $\psi(t) \in H^{s+1} \times H^s$ be a solution of* (3.4)-(3.6)*. Then there exist $c, C > 0$ such that for all integers $0 \le k \le s$ and all $t \in [0,T]$*

$$\|\partial_x^k \psi(t)\|_1 + \|\partial_x^k \psi_t(t)\| \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}} \left(\|{}^0\psi\|_{L^1} + \|{}^1\psi\|_{L^1}\right) + Ce^{-ct}\left(\|\partial_x^k({}^0\psi)\|_1 + \|\partial_x^k({}^1\psi)\|\right). \tag{3.7}$$

To prove Proposition 3.1 we consider (3.4)-(3.6) in Fourier space, i.e.

$$\hat{\psi}_{tt} + |\xi|^2 B(\check{\xi})\hat{\psi} + a\hat{\psi}_t - i|\xi| b(\check{\xi})\hat{\psi} = 0 \text{ on } (0,T] \times \mathbb{R}^3, \tag{3.8}$$

$$\hat{\psi}(0) = {}^0\hat{\psi}(\xi) \text{ on } \mathbb{R}^3, \tag{3.9}$$

$$\hat{\psi}_t(0) = {}^1\hat{\psi}(\xi) \text{ on } \mathbb{R}^3, \tag{3.10}$$

where $\check{\xi} = \xi/|\xi|$,

$$B(\omega) = \sum_{i,j=1}^{3} \omega_i \bar{B}_{ij}\omega_j = \begin{pmatrix} 1 & 0 \\ 0 & \bar{\sigma}^{-1}\left(\bar{\eta} I_3 + \left(\bar{\zeta} + \frac{1}{3}\bar{\eta}\right)(\omega \otimes \omega)\right) \end{pmatrix},$$

$$b(\omega) = \sum_{j=1}^{3} b_j \omega_j = \begin{pmatrix} 0 & \omega^t \\ \omega & 0 \end{pmatrix}, \ \omega \in \mathbb{S}^2.$$

We get the following pointwise decay estimate.

3.2 Lemma. *In the situation of Proposition 3.1 there exist $c, C > 0$ such that for $(t,\xi) \in [0,T] \times \mathbb{R}^n$*

$$(1+|\xi|^2)|\hat{\psi}(t,\xi)|^2 + |\hat{\psi}_t(t,\xi)|^2 \le C\exp(-c\rho(\xi)t)\left((1+|\xi|^2)|{}^0\hat{\psi}(\xi)|^2 + |{}^1\hat{\psi}(\xi)|^2\right), \tag{3.11}$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$.

Proof. Our goal is to arrive at an expression of the form

$$\frac{1}{2}\frac{d}{dt}E(t,\xi) + F(t,\xi) \le 0, \tag{3.12}$$

where $E(t,\xi)$ is uniformly equivalent to

$$E_0(t,\xi) = (1+|\xi|)^2|\hat{\psi}(t,\xi)|^2 + |\hat{\psi}_t(t,\xi)|^2$$

and $F \ge c\rho(\xi)E_0$. Then (3.11) follows by Gronwall's Lemma.

W.l.o.g. assume $\xi = (|\xi|, 0, 0)$ (otherwise rotate the coordinate system). Since $4/3\bar{\eta} + \bar{\zeta} = \bar{\sigma}$, (3.8) decomposes into the two uncoupled systems

$$w_{tt} + |\xi|^2 w + \tilde{a}w - i|\xi|\tilde{b}w = 0, \tag{3.13}$$

$$v_{tt} + \bar{\eta}\bar{\sigma}^{-1}|\xi|^2 v + \bar{\sigma}^{-1}v_t = 0, \tag{3.14}$$

where $w = (\hat{\psi}_0, \hat{\psi}_1)$, $v = (\hat{\psi}_2, \hat{\psi}_3)$,

$$\tilde{a} = \begin{pmatrix} \bar{\chi}^{-1}c_s^{-2} & 0 \\ 0 & \bar{\sigma}^{-1} \end{pmatrix}, \quad \tilde{b} = (\bar{\chi}\bar{\sigma})^{-\frac{1}{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.15}$$

Obviously, this allows us to prove estimate (3.11) for w and v independently.

First, consider (3.14), where the estimate is fairly easy to obtain. Take the scalar product (in $\mathbb{C}^2$) of this equations with $v_t + 1/(2\bar{\sigma})v$. The real part reads

$$\frac{1}{2}\frac{d}{dt}E^{(2)} + F^{(2)} = 0,$$

where

$$E^{(2)} = |v_t|^2 + \frac{\bar{\eta}}{\bar{\sigma}}|\xi|^2|v|^2 + \frac{1}{2\bar{\sigma}^2}|v|^2 + \frac{1}{\bar{\sigma}}\Re\langle v_t, v\rangle, \tag{3.16}$$

and

$$F^{(2)} = \frac{1}{2\bar{\sigma}}|v_t|^2 + \frac{\bar{\eta}}{2\bar{\sigma}^2}|\xi|^2|v|^2. \tag{3.17}$$

Since

$$|\bar{\sigma}^{-1}\Re\langle v_t, v\rangle| \le \frac{1}{3\bar{\sigma}^2}|v|^2 + \frac{3}{4}|v_t|^2$$

$E^{(2)}$ is uniformly equivalent to $E_0^{(2)} = |v_t|^2 + (1+|\xi|^2)|v|^2$ and as

$$|\xi|^2 \ge \frac{1}{2}\rho(\xi)(1+|\xi|^2)$$

$F^{(2)} \ge c_1\rho(\xi)E_0^{(2)}$ for some $c_1 > 0$.

Next, we study system (3.13). For notational purposes set $a_1 = \bar{\chi}^{-1} c_s^{-1}$, $a_2 = \bar{\sigma}^{-2}$ and $b_1 = (\bar{\chi}\bar{\sigma})^{-\frac{1}{2}}$. Now, take the scalar product of (3.13) with $\tilde{a} w_t$. The real part of the resulting equation reads

$$\frac{1}{2}\frac{d}{dt}\Big(\langle \tilde{a} w_t, w_t\rangle + |\xi|^2 \langle \tilde{a} w, w\rangle\Big) + |\tilde{a} w_t|^2 + \Re\langle -i|\xi|\tilde{b} w, \tilde{a} w_t\rangle = 0. \tag{3.18}$$

Taking the scalar product of (3.13) with $-i|\xi|\tilde{b}w$ and considering the real part gives

$$\frac{d}{dt}\Big(\Re\langle w_t, -i|\xi|\tilde{b}w\rangle\Big) + \Re\langle \tilde{a} w_t, -i|\xi|\tilde{b}w\rangle + |\xi|^2|\tilde{b}w|^2 = 0. \tag{3.19}$$

Then we take the scalar product of (3.13) with w. The real part is

$$\frac{1}{2}\frac{d}{dt}\left(\langle a w, w\rangle + 2\Re\langle w_t, w\rangle\right) - |w_t|^2 + |\xi|^2|w|^2 = 0. \tag{3.20}$$

Set

$$S = \frac{1}{2b_1}\begin{pmatrix} 0 & a_1 - a_2 \\ a_2 - a_1 & 0 \end{pmatrix}.$$

Since iS is hermitic

$$\Re\langle iSw, w_t\rangle = \frac{1}{2}\frac{d}{dt}\langle iSw, w\rangle$$

and we can write (3.20) as

$$\frac{1}{2}\frac{d}{dt}\left(\langle a w, w\rangle + 2\Re\langle w_t, w\rangle + 2|\xi|\langle iSw, w\rangle\right) - |w_t|^2 + |\xi|^2|w|^2 - 2\Re(|\xi|\langle iSw, w_t\rangle) = 0 \tag{3.21}$$

Now, add (3.18)+(3.19)+α(3.21) (for some $\alpha > 0$ to be determined later) to obtain

$$\frac{1}{2}\frac{d}{dt}E^{(1)} + F^{(1)} = 0, \tag{3.22}$$

where

$$E^{(1)} = \langle \tilde{a} w_t, w_t\rangle + |\xi|^2\langle \tilde{a} w, w\rangle + 2\Re(\langle w_t, -i|\xi|\tilde{b}w\rangle) + \alpha\left(\langle \tilde{a} w, w\rangle + 2\Re\langle w_t, w\rangle + 2|\xi|\langle iSw, w\rangle\right)$$

and

$$F^{(1)} = |\tilde{a} w_t|^2 - \alpha|w_t|^2 - 2\Re(i|\xi|\langle(\tilde{a}\tilde{b} - S)w, w_t\rangle) + |\xi|^2|\tilde{b}w|^2 + \alpha|w|^2.$$

First, show that $E^{(1)}$ is uniformly equivalent to $E_0^{(1)} = (1+|\xi|^2)|w|^2 + |w_t|^2$. Obviously, there exists $C_1 > 0$ such that

$$E^{(1)} \leq C_1 E_0^{(1)}.$$

For

$$M = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{a} \end{pmatrix}$$

and $W = (w_t, -i|\xi|w)$

$$\langle \tilde{a} w_t, w_t \rangle + |\xi|^2 \langle \tilde{a} w, w \rangle + 2\Re(\langle w_t, -i|\xi|\tilde{b} w \rangle) = \langle MW, W \rangle_{\mathbb{C}^4}.$$

It is easy to show that $\sigma(M) = \sigma(\tilde{a} + \tilde{b}) \cup \sigma(\tilde{a} - \tilde{b})$. Furthermore $c_s \in (0,1)$ yields $\tilde{a} + \tilde{b} > 0$, $\tilde{a} - \tilde{b} > 0$. Thus M is positive definite, i.e.

$$\langle \tilde{a} w_t, w_t \rangle + |\xi|^2 \langle \tilde{a} w, w \rangle + 2\Re(\langle w_t, -i|\xi|\tilde{b} w \rangle) \geq C_2(|w_t|^2 + |\xi|^2|w|^2)$$

for a $C_2 > 0$. Furthermore, by Young's inequality there exists $C_3 > 0$ such that

$$|2\Re\langle w_t, w \rangle + 2i|\xi|\langle Sw, w \rangle| \leq \frac{d}{2}|w|^2 + C_3(|\xi|^2|w|^2 + |w_t|^2),$$

where $d = \min\{a_1, a_2\}$. In conclusion

$$E^{(1)} \geq C_2(|w_t|^2 + |\xi|^2|w|^2) - \alpha C_3(|\xi|^2|w|^2 + |w_t|^2) + \alpha \frac{d}{2}|w|^2.$$

Hence, for α sufficiently small there exists $C_4 > 0$ such that

$$E^{(1)} \geq C_4 E_0^{(1)}.$$

Finally show $F^{(1)} \geq c\rho(\xi) E_0^{(1)}$ for α sufficiently small. To this end write $F^{(1)} = F_1^{(1)} + F_2^{(1)}$, where

$$\begin{aligned} F_1^{(1)} &= (a_1^2 - \alpha)|w_t^1|^2 + (b_1^2 + \alpha)|\xi|^2|w^2|^2 \\ &\quad - 2\Re\left(i|\xi|\left(a_1 b_1 + \alpha\frac{a_1 - a_2}{2b_1}\right) w^2 \bar{w}_t^1\right), \\ F_2^{(1)} &= (a_2^2 - \alpha)|w_t^2|^2 + (b_1^2 + \alpha)|\xi|^2|w^1|^2 \\ &\quad - 2\Re\left(i|\xi|\left(a_2 b_1 + \alpha\frac{a_2 - a_1}{2b_1}\right) w^1 \bar{w}_t^2\right). \end{aligned}$$

Since

$$(a_1^2 - \alpha)(b_1^2 + \alpha) - \left(a_1 b_1 + \alpha \frac{a_1 - a_2}{2b_1}\right)^2 = \alpha(a_1 a_2 - b_1^2) + O(\alpha^2)$$

and $a_1 a_2 > b_1^2$ there exist $c_2 > 0$ such that

$$F_1^{(1)} \geq \alpha c_2(|w_t^1|^2 + |\xi|^2 |w^2|^2)$$

for α sufficiently small. In the same way we get

$$F_2^{(1)} \geq \alpha c_2(|w_t^2|^2 + |\xi|^2 |w^1|^2).$$

Therefore

$$F^{(1)} \geq \alpha c_2(|w_t|^2 + |\xi|^2 |w|^2) \geq \alpha \frac{c_1}{2} \rho(\xi) E_0^{(1)},$$

which finishes the proof. □

Based on Lemma 3.2 the proof for Proposition 3.1 goes as [1, Proof of Theorem 3.1].

Next consider the inhomogeneous initial-value problem

$$\psi_{tt} - \sum_{i,j=1}^{3} B_{ij} \psi_{x_i x_j} + a\psi_t - \sum_{j=1}^{n} b_j \psi_{x_j} = h, \text{ on } (0,T] \times \mathbb{R}^3, \tag{3.23}$$

$$\psi(0) = \psi_0, \text{ on } \mathbb{R}^3, \tag{3.24}$$

$$\psi_t(0) = \psi_1, \text{on } \mathbb{R}^3. \tag{3.25}$$

for some $h : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^4$. We get the following results:

3.3 Proposition. *Let s be a non-negative integer, $({}^0\psi, {}^1\psi) \in (H^{s+1} \times H^s) \cap (L^1)^2$ and $h \in C([0,T], H^s \cap L^1)$. Then the solution ψ of (3.23)-(3.25) satisfies*

$$\begin{aligned} \|\partial_x^k \psi(t)\|_1 + \|\partial_x^k \psi_t(t)\| \leq{}& C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|{}^0\psi\|_{L^1} + \|{}^1\psi\|_{L^1}) \\ &+ Ce^{-ct}(\|\partial_x^k({}^0\psi)\|_1 + \|\partial_x^k({}^1\psi)\| \\ &+ C\int_0^t (1+t-\tau)^{-3/4-k/2}\|h(\tau)\|_{L^1} \\ &\quad + C\exp(-c(t-\tau))\|\partial_x^k h(\tau)\| d\tau \end{aligned} \tag{3.26}$$

for all $t \in [0,T]$ and $0 \leq k \leq s$.

Proof. For $t \in [0,T]$ let $T(t)$ be the linear operator which maps $({}^0\psi, {}^1\psi)$ to the solution $(\psi(t)), \psi_t(t))$ of the homogeneous IVP (3.4)-(3.6) at time t. By Duhamel's principle the solution of (3.23)-(3.25) is given by

$$(\psi(t), \psi_t(t)) = T(t)({}^0\psi, {}^1\psi) + \int_0^t T(t-\tau)(0, h(\tau))d\tau.$$

Hence the assertion is an immediate consequence of Proposition 3.1 □

3.4 Proposition. *Let s be a non-negative integer. There exist $C_1, C_2 > 0$ such that for all $({}^0\psi, {}^1\psi) \in H^{s+1} \times H^s$ and $h \in C([0,T], H^s)$ the solution ψ of* (3.23)-(3.25) *satisfies*

$$\begin{aligned} C_1 \left(\|\partial_x^\alpha \psi(t)\|_1^2 + \|\partial_x^\alpha \psi_t(t)\|^2 \right) + C_1 \int_0^t \|\partial_x^\alpha \partial_x \psi(\tau)\|^2 + \|\partial_x^\alpha \psi_t(\tau)\|^2 d\tau \\ \leq C_2 \left(\|\partial_x^\alpha ({}^0\psi)\|_1^2 + \|\partial_x^\alpha ({}^1\psi)\|^2 \right) \\ + \int_0^t C_2 \|\partial_x^\alpha \psi(\tau)\|^2 + \left(\partial_x^\alpha h(\tau), \frac{a}{2}\partial_x^\alpha \psi(\tau) + \partial_x^\alpha \psi_t(\tau) \right)_{L^2} d\tau \end{aligned} \tag{3.27}$$

for all $t \in [0,T]$ and $\alpha \in \mathbb{N}_0^3$, $|\alpha| = s$

Proof. Consider (3.23) in Fourier space, i.e.

$$\hat{\psi}_{tt} + |\xi|^2 B(\check{\xi})\hat{\psi} + a\hat{\psi}_t - i|\xi| b(\check{\xi})\hat{\psi} = \hat{h}$$

We proceed similar as in the proof of Lemma 3.2. Again w.l.o.g. assume $\xi = (|\xi|, 0, 0)$ then (3.23) reads

$$w_{tt} + |\xi|^2 w + \tilde{a} w - i|\xi|\tilde{b} w = (\hat{h}^0, \hat{h}^1)^t, \tag{3.28}$$

$$v_{tt} + \bar{\eta}\bar{\sigma}^{-1}|\xi|^2 v + \bar{\sigma}^{-1} v_t = (\hat{h}^2, \hat{h}^3)^t, \tag{3.29}$$

where $w = (\hat{\psi}_0, \hat{\psi}_1)$, $v = (\hat{\psi}_2, \hat{\psi}_3)$, $\tilde{a}, \tilde{b}$ are given by (3.15). First, take the scalar product of (3.29) with $v_t + 1/(2\bar{\sigma})v$ and consider the real part

$$\frac{1}{2}\frac{d}{dt}E^{(2)} + F^{(2)} = \Re \left\langle (\hat{h}^2, \hat{h}^3)^t, v_t + \frac{1}{2\bar{\sigma}} v \right\rangle \tag{3.30}$$

where $E^{(2)}, F^{(2)}$ are given by (3.16), (3.17). Since $E^{(2)}$ is uniformly equivalent to $|v_t|^2 + (1+|\xi|^2)|v|^2$ and $F^2 \geq c(|v_t|^2 + |\xi|^2|v|^2)$ integrating (3.30) leads to

$$\begin{aligned} C_1 \left(|v_t|^2 + (1+|\xi|^2)|v|^2 \right) + C_1 \int_0^t |v_t|^2 + |\xi|^2 |v|^2 d\tau \\ \leq C_2 \left(|v(0)|^2 + (1+|\xi|^2)|v_t(0)|^2 \right) + \int_0^t \Re \left\langle (\hat{h}^2, \hat{h}^3)^t, v_t + \frac{1}{2\bar{\sigma}} v \right\rangle d\tau. \end{aligned} \tag{3.31}$$

Next, take the scalar product of (3.29) with $w_t + \tilde{a}/2w$. The real part reads

$$\frac{1}{2}\frac{d}{dt}E^{(1)} + F^{(1)} = \Re\langle(\hat{h}^0, \hat{h}^1)^t, w_t + \frac{1}{2}\tilde{a}v\rangle, \tag{3.32}$$

where

$$E^{(1)} = |w_t|^2 + |\xi|^2|w|^2 + \frac{1}{2}|\tilde{a}w|^2 + 2\Re\langle\tilde{a}w_t, w\rangle$$

and

$$F^{(1)} = \frac{1}{2}\langle\tilde{a}w_t, w_t\rangle + \Re\langle -i|\xi|\tilde{b}w, w_t\rangle + \frac{1}{2}|\xi|^2\langle\tilde{a}w, w\rangle - \frac{1}{2}\Re\langle i|\xi|\tilde{b}w, \tilde{a}w\rangle.$$

Using Young's inequality it is easy to see that $E^{(1)}$ is uniformly equivalent to $|w_t|^2 + (1 + |\xi|^2)|w|^2$. Furthermore

$$F^{(1)} = \frac{1}{2}\langle MW, W\rangle_{\mathbb{C}^4} - \frac{1}{2}\Re\langle i|\xi|\tilde{b}w, \tilde{a}w\rangle,$$

where

$$M = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{a} \end{pmatrix}$$

and $W = (w_t, -i|\xi|w)$. As M is positive definite (see proof of Lemma 3.2) there exists $c_1, c_2 > 0$ such that

$$F^{(1)} \geq c_1(|w_t|^2 + |\xi|^2|w|^2) - c_2|\xi||w||w| \geq \frac{c_1}{2}(|w_t|^2 + |\xi|^2|w|^2) - \frac{c_2^2}{2c_1}|w|^2.$$

Thus integrating (3.32) leads to

$$\begin{aligned} &C_1\left(|w_t|^2 + (1 + |\xi|^2)|w|^2\right) + C_1\int_0^t |w_t|^2 + |\xi|^2|w|^2 d\tau \\ &\leq C_2\left(|w(0)|^2 + (1 + |\xi|^2)|w_t(0)|^2\right) + \int_0^t C_2|w|^2 + \Re\langle(\hat{h}^0, \hat{h}^1)^t, w_t + \frac{\tilde{a}}{2}w\rangle d\tau. \end{aligned} \tag{3.33}$$

Adding (3.31) and (3.33) gives

$$\begin{aligned} &C_1\left(|\hat{\psi}_t|^2 + (1 + |\xi|^2)|\hat{\psi}|^2\right) + C_1\int_0^t |\hat{\psi}_t|^2 + |\xi|^2|\hat{\psi}|^2 d\tau \\ &\quad\leq C_2\left(|{}^0\hat{\psi}|^2 + (1 + |\xi|^2)|{}^1\hat{\psi}|^2\right) + \int_0^t C_2|\hat{\psi}|^2 + \Re\langle\hat{h}, \hat{\psi}_t + \frac{a}{2}\hat{\psi}\rangle d\tau. \end{aligned} \tag{3.34}$$

Finally the assertion follows by multiplying (3.34) with $\xi^{2\alpha}$ for $\alpha \in \mathbb{N}_0^n$, $|\alpha| = s$, integrating with respect to ξ and using Plancherel's identity. $\square$

4. Global Existence and Asymptotic Decay of Small Solutions

The goal of this section is to prove Theorem 2.1. We will proceed as follows: First we show a decay estimate for all but the highest order derivatives of a solution, proposition 4.1, and then an energy estimate for the derivatives of highest order, proposition 4.3. Then Theorem 2.1 follows from combining the two, proposition 4.4.

As in section 3. fix $\theta > 0$, multiply (2.2) by $(ns)^{-1}\theta^{-2}(A^{(2)})^{-\frac{1}{2}}$ and change the variables to $(A^{(2)})^{\frac{1}{2}}\psi$ such that the linearisation at $(\theta^{-1}, 0, 0, 0)$ is given by (3.3). In addition consider $\psi - \bar{\psi}$ with $\bar{\psi} = (\theta^{-1}, 0, 0, 0)$ instead of ψ, $A(\cdot + \bar{\psi})$ instead of $A(\cdot)$ and so on, such that the rest state is shifted from $(\theta^{-1}, 0, 0, 0)$ to $(0, 0, 0, 0)$. Furthermore write $U = (\psi, \psi_t)$ and $U_0 = ({}^0\psi, {}^1\psi)$ for a solution to (2.3)-(2.5) and the initial values, respectively.

Throughout this section let $s \geq s_0 + 1$ ($s_0 = [3/2] + 1$), $T > 0$, $U_0 \in H^{s+1} \times H^s$, and ψ satisfy

$$\psi \in \bigcap_{j=0}^{s} C^j\left([0, T], H^{s+1-j}\right). \tag{4.1}$$

For $0 \leq t \leq t_1 \leq T$ define

$$N_s(t, t_1)^2 = \sup_{\tau \in [t, t_1]} \|U(\tau)\|_{s+1,s}^2 + \int_t^{t_1} \|U(\tau)\|_{s+1,s}^2 d\tau.$$

We write $N_s(t)$ instead of $N_s(0, t)$. Furthermore assume that $N_s(T) \leq a_0$ for an $a_0 > 0$. Since $s \geq s_0$, $H^s \hookrightarrow L^\infty$ is a continuous embedding. Hence $N_s(T) \leq a_0$ implies that $(\psi, \psi_t, \partial_x\psi)$ takes values in a closed ball $\overline{B(0, r)} \subset \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^{12}$ for some $r > 0$.

First we prove the decay estimate. To this end it is convenient to rewrite (the modified version of) (2.3) as

$$\psi_{tt} - \sum_{i,j=1}^{3} \bar{B}_{ij}\psi_{x_i x_j} + a\psi_t + \sum_{j=1}^{3} b_j \psi_{x_j} = h(\psi, \psi_t, \partial_x\psi, \partial_x^2\psi), \tag{4.2}$$

where

$$h(\psi, \psi_t, \partial_x\psi, \partial_x^2 u) = \sum_{i,j=1}^{3} \left(A(\psi)^{-1}B_{ij}(\psi) - \bar{B}_{ij}\right)\psi_{x_i x_j}$$
$$+ \sum_{j=1}^{3} A(\psi)^{-1}D_j(\psi)\psi_{tx_j}$$
$$- A(\psi)^{-1}f(\psi, \psi_t, \partial_x\psi) + \left(a\psi_t + \sum_{j=1}^{3} b_j\psi_{x_j}\right). \quad (4.3)$$

4.1 Proposition. *There exist constants* $a_1(\leq a_0), \delta_1 = \delta_1(a_1), C_1 = C_1(a_1, \delta_1) > 0$ *such that the following holds: If* $N_s(T)^2 \leq a_1$ *and* $\|U_0\|_{s,s-1,1}^2 \leq \delta_1$ *for a solution* ψ *of* (2.3)-(2.5) *satisfying* (4.1)*, then*

$$\|U(t)\|_{s,s-1} \leq C_1(1+t)^{-\frac{3}{4}}\|U_0\|_{s,s-1,1} \quad (t \in [0,T]). \quad (4.4)$$

Proof. Let $t \in [0,T]$ and ψ be a solution to (2.3)-(2.5). Since $B_{ij}(0) = \bar{B}_{ij}$, $D_j(0) = 0$ and

$$a\psi_t + \sum_{j=1}^{3} b_j\psi_{x_j} = Df(0)(\psi, \psi_t, \partial_x\psi),$$

Lemmas A.1, A.2 show that there exists a $c > 0$ ($c \leq a_0$) such that $h(t) \in H^{s-1} \cap L^1$ and

$$\begin{aligned}
\|h(t)\|_{s-1} &\leq C\|\psi(t)\|_{s-1}\left(\|\partial_x^2\psi(t)\|_{s-1} + \|\partial_x\psi_t(t)\|_{s-1}\right) \\
&\quad + C\|(\psi(t), \psi_t(t), \partial_x\psi(t)\|_{s-1}^2 \\
&\leq C\|U(t)\|_{s+1,s}\|U(t)\|_{s,s-1}, \\
\|h(t)\|_{L^1} &\leq C\|U(t)\|_{2,1}^2,
\end{aligned}$$

if $N_s(T) \leq c$, which we will assume throughout this proof. Then Proposition 3.3 yields

$$\|U(t)\|_{s,s-1} \leq C(1+t)^{-\frac{3}{4}}\|U_0\|_{s,s-1,1}$$
$$+ C\int_0^t \exp(-c(t-\tau))\|h(\tau)\|_{s-1} + (1+t-\tau)^{-\frac{3}{4}}\|h(\tau)\|_{L^1}d\tau.$$

This leads to

$$\|U(t)\|_{s-1,s} \leq C(1+t)^{-\frac{3}{4}}\|U_0\|_{s,s-1,1} + C \sup_{\tau\in[0,t]} \|U(\tau)\|_{s+1,s} \int_0^t \exp(-c(t-\tau))\|U(\tau)\|_{s,s-1} d\tau + C\int_0^t (1+t-\tau)^{-\frac{3}{4}}\|U(\tau)\|^2_{s,s-1} d\tau.$$

Multiplying with $(1+t)^{\frac{3}{4}}$ gives

$$(1+t)^{\frac{3}{4}}\|U(t)\|_{s,s-1} \leq C\|U_0\|_{s,s-1,1} + CN_s(t)\mu_1(t) \sup_{\tau\in[0,t]} (1+\tau)^{\frac{3}{4}}\|U(\tau)\|_{s,s-1} + C\mu_2(t) \sup_{\tau\in[0,t]} (1+\tau)^{\frac{3}{2}}\|U(\tau)\|^2_{s,s-1},$$

where

$$\mu_1(t) = (1+t)^{\frac{3}{4}} \int_0^t \exp(-c(t-\tau))(1+\tau)^{-\frac{3}{4}} d\tau$$
$$\mu_2(t) = (1+t)^{\frac{3}{4}} \int_0^t (1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{3}{2}} d\tau.$$

Since μ_1, μ_2 are bounded functions on $[0,\infty)$, we get

$$\sup_{\tau\in[0,t]} (1+\tau)^{\frac{3}{4}}\|U(\tau)\|_{s,s-1} \leq C\|U_0\|_{s,s-1,1} + CN_s(t) \sup_{\tau\in[0,t]} (1+\tau)^{\frac{3}{4}}\|U(\tau)\|_{s,s-1} + C \sup_{\tau\in[0,t]} (1+\tau)^{\frac{3}{2}}\|U(\tau)\|^2_{s,s-1}.$$

We can deduce from this equation that there in fact exists $a_1 > 0$ ($a_1 \leq c$), $\delta_1 > 0$ and $C_1 > 0$, such that

$$\sup_{\tau\in[0,t]} (1+\tau)^{\frac{3}{4}}\|U(\tau)\|_{s,s-1} \leq C_1\|U_0\|_{s,s-1,1},$$

whenever $N_s(T)^2 \leq a_1$ and $\|U_0\|^2_{s,s-1,1} \leq \delta_1$. □

4.2 Corollary. *In the situation of Proposition 4.1 there exists a* $C_2 = C_2(a_1, \delta_1) > 0$ *such that*

$$N_{s-1}(T)^2 \le C_2 \|U_0\|^2_{s,s-1,1} \tag{4.5}$$

whenever $N_s(T)^2 \le a_1$ *and* $\|U_0\|^2_{s,s-1,1} \le \delta_1$.

Proof. The function $t \mapsto (1+t)^{-\frac{3}{4}}$ is square-integrable on $[0,\infty)$. Therefore the assertion is a direct consequence of Proposition 4.1. $\square$

Now it is convenient to write (the modified version of) (2.3) as

$$\psi_{tt} - \sum_{i,j=1}^{3} \bar{B}_{ij}\psi_{x_i x_j} + a\psi_t + \sum_{j=1}^{3} b_j \psi_{x_j} = L(\psi)\psi + h_2(\psi, \psi_t, \partial_x \psi), \tag{4.6}$$

where

$$L(\psi)\psi = (I - A(\psi))\psi_{tt} + \sum_{i,j=1}^{3} (\bar{B}_{ij} - B_{ij}(\psi))\psi_{x_i x_j} - \sum_{j=1}^{3} D_j(\psi)\psi_{t x_j},$$

$$h_2(\psi, \psi_t, \partial_x \psi) = a\psi_t + \sum_{j=1}^{3} b_j \psi_{x_j} - f(\psi, \psi_t, \partial_x \psi).$$

4.3 Proposition. *There exist constants* $a_2 (\le a_0)$ *and* $c_3, C_3 = C_3(a_2) > 0$ *such that the following holds: If* $N_s(T)^2 \le a_2$ *for a solution* ψ *of* (2.3)-(2.5) *satisfying* (4.1), *then*

$$\begin{aligned} \|\partial_x^s \psi(t)\|_1^2 + \|\partial_x^s \psi_t(t)\|^2 + \int_0^t \|\partial_x^{s+1}\psi(\tau)\|^2 + \|\partial_x^s \psi_t(\tau)\|^2 d\tau \\ - c_3 \int_0^t \|\partial_x^s \psi(\tau)\|^2 d\tau \le C_3 \left(\|U_0\|^2_{s,s+1} + N_s(t)^3 \right) \quad (t \in [0,T]). \end{aligned} \tag{4.7}$$

Proof. We prove the result in two steps.

Step 1:

First let $U_0 = ({}^0\psi, {}^1\psi) \in H^{s+1} \times H^s$ and ψ be a solution to (2.3)-(2.5) such that

$$\psi \in \bigcap_{j=0}^{s} C^j \left([0,T], H^{s+2-j} \right). \tag{4.8}$$

By Lemma A.2 there exists a $c > 0$ such that $I - A(\psi), \bar{B}_{ij} - B_{ij}(\psi), D_j(\psi) \in H^{s+1}$ provided $N_s(T) \le c$. We will assume this throughout the proof. Then due to (4.8) and [6, Lemma 2.3] $L(\psi)\psi \in H^s$. Lemma A.2 yields $h_2 \in H^s$. Thus we can conlcude by Proposition 3.4 that

$$\begin{aligned} C_1\left(\|\partial_x^\alpha\psi(t)\|_1^2 + \|\partial_x^\alpha\psi_t(t)\|^2\right) + C_1\int_0^t \|\partial_x^\alpha\partial_x\psi(\tau)\|^2 + \|\partial_x^\alpha\psi_t(\tau)\|^2 d\tau \\ \le C_2\left(\|\partial_x^\alpha({}^0\psi)\|_1^2 + \|\partial_x^\alpha({}^1\psi)\|^2\right) \\ + C_2\int_0^t \|\partial_x^\alpha\psi(\tau)\|^2 d\tau \\ \int_0^t \left(\partial_x^\alpha(L(\psi(\tau))\psi(\tau) + h_2(\tau)), \partial_x^\alpha\psi_t(\tau) + \frac{a}{2}\partial_x^\alpha\psi(\tau)\right)_{L^2} d\tau \end{aligned} \tag{4.9}$$

for all $\alpha \in \mathbb{N}_0^3$, $|\alpha| = s$. First obviously

$$\left|\left(\partial_x^\alpha h_2, \partial_x^\alpha\psi_t + \frac{a}{2}\partial_x^\alpha\psi\right)_{L^2}\right| \le C\|h_2\|_s\|U\|_s \tag{4.10}$$

and integrating by parts gives

$$\begin{aligned} \left|\left(\partial_x^\alpha(L(\psi)\psi), \frac{a}{2}\partial_x^\alpha\psi\right)_{L^2}\right| &\le C\|L(\psi)\psi\|_{s-1}\|\psi\|_{s+1} \\ &\le C\|I - A(\psi)\|_s\|\psi_{tt}\|_{s-1}\|\psi\|_{s+1} \\ &+ C\sum_{i,j=1}^3 \|\bar{B}_{ij} - B_{ij}(\psi)\|_s\|\partial_x^2\psi\|_{s-1}\|\psi\|_{s+1} \\ &+ C\sum_{j=1}^3 \|D_j(\psi)\|_s\|\partial_x\psi\|_{s-1}\|\psi\|_{s+1}. \end{aligned} \tag{4.11}$$

Next write

$$\begin{aligned} \partial_x^\alpha\left(L(\psi)\psi\right) = L(\psi)\partial_x^\alpha\psi + [\partial_x^\alpha, (I - A(\psi))]\psi_{tt} \\ + \sum_{i,j=1}^n [\partial_x^\alpha, (\bar{B}_{ij} - B_{ij}(\psi))]\psi_{x_ix_j} + \sum_{j=1}^n [\partial_x^\alpha, D_j(\psi)]\psi_{tx_j}. \end{aligned}$$

Since $I - A(\psi), \bar{B}_{ij} - B_{ij}(\psi), D_j(\psi) \in H^s$, [6, Lemma 2.5(i)] yields

$$\begin{aligned} \|[\partial_x^\alpha, (-A(\psi))]\psi_{tt}\| &\le C\|\partial_x A(\psi)\|_{s-1}\|\psi_{tt}\|_{s-1} \\ \|[\partial_x^\alpha, (\bar{B}_{ij} - B_{ij}(\psi))]\psi_{x_ix_j}\| &\le C\|\partial_x B_{ij}(\psi)\|_{s-1}\|\psi_{x_ix_j}\|_{s-1} \\ \|[\partial_x^\alpha, D_j]\psi_{tx_j}\| &\le C\|\partial_x D_j(\psi)\|_{s-1}\|\psi_{tx_j}\|_{s-1}. \end{aligned} \tag{4.12}$$

Furthermore integration by parts and the symmetry of A, B_{ij} and D_j yield

$$
\begin{aligned}
\int_0^t (L(\psi)\partial_x^\alpha\psi, \partial_x^\alpha\psi_t)_{L^2}\, d\tau \\
&\le C\int_0^t \|\partial_t A\|\|\partial_x^\alpha(\partial_x\psi,\psi_t)\|^2 \\
&+ \left(\sum_{i,j=1}^n \|\partial_t B_{ij}\| + \|\partial_x B_{ij}\| + \sum_{j=1}^n \|\partial_x D_j\|\right)\|\partial_x^\alpha(\partial_x\psi,\psi_t)\|^2 d\tau \\
&+ C\left(\|I-A\| + \sum_{i,j=1}^n \|\bar{B}_{ij} - B_{ij}\|\right)\|\partial_x^\alpha(\partial_x\psi,\psi_t)\|^2 + C\|\partial_x^\alpha(\partial_x{}^0\psi, {}^1\psi)\|^2,
\end{aligned}
\tag{4.13}
$$

In conclusion, (4.9) and the estimates (4.10), (4.11), (4.12) and (4.13) lead to

$$
\begin{aligned}
\|\partial_x^\alpha\psi(t)\|_1^2 + \|\partial_x^\alpha\psi_t(t)\|^2 + \int_0^t \|\partial_x^\alpha\partial_x\psi(\tau)\|^2 + \|\partial_x^\alpha\psi_t(\tau)\|^2 d\tau \\
- c\int_0^t \|\partial_x^\alpha\psi(\tau)\|^2 d\tau \\
\le C\|U_0\|_{s+1,s}^2 + C\int_0^t (\|h_2\| + R_1(\psi))\|U\|_{s+1,s}^2 d\tau \\
+ C\int_0^t \|I - A(\psi)\|_s\|\psi_{tt}\|_{s-1}\|U\|_{s+1,s} d\tau \\
+ CR_2(\psi)\|U(t)\|_{s+1,s}^2,
\end{aligned}
\tag{4.14}
$$

where

$$
R_1(\psi) = \|\partial_t A(\psi)\| + \|I - A(\psi)\|_s + \sum_{i,j=1}^n \|\partial_t B_{ij}(\psi)\| + \|\bar{B}_{ij} - B_{ij}(\psi)\|_s + \sum_{j=1}^n \|D_j(\psi)\|_s
$$

and

$$
R_2(\psi) = \|I - A(\psi)\| + \sum_{i,j=1}^n \|\bar{B}_{ij} - B_{ij}(\psi)\|
$$

Step 2:

Now let ψ be a solution to (2.3)-(2.5) satisfying (4.1). For $\delta > 0$ set $\psi^\delta = \phi_\delta * \psi$. Applying $\phi_\delta *$ to (4.6) yields

$$\psi^\delta_{tt} - \sum_{i,j=1}^{3} \bar{B}_{ij}\psi^\delta + a\psi^\delta_t + \sum_{j=1}^{3} b_j \psi^\delta_{x_j} = L(\psi)\psi^\delta + R^\delta(\psi) + h^\delta_2$$

where $h^\delta = \phi_\delta * h_2$ and

$$R^\delta(\psi) = [\phi_\delta *, (I - A(\psi))]\psi_{tt} + \sum_{i,j=1}^{n} [\phi_\delta *, \bar{B}_{ij} - B_{ij}(\psi)]\psi_{x_i x_j} \\ + \sum_{j=1}^{3} [\phi_\delta, D_j(\psi)]\psi_{t x_j}.$$

Due to [6, Lemma 2.5 (ii)] $R^\delta \in H^s$. Hence $L(\psi)\psi^\delta + R^\delta + h^\delta_2 \in H^s$. Thus proceeding as in step 1 yields

$$\|\partial^\alpha_x \psi^\delta(t)\|^2_1 + \|\partial^\alpha_x \psi^\delta_t(t)\|^2 + \int_0^t \|\partial^\alpha_x \partial_x \psi^\delta(\tau)\|^2 + \|\partial^\alpha_x \psi^\delta_t(\tau)\|^2 d\tau \\ - c\int_0^t \|\partial^\alpha_x \psi^\delta(\tau)\|^2 d\tau \\ \leq C\|U^\delta_0\|^2_{s+1,s} \\ + C\int_0^t \left(\|h^\delta_2\|_s + R_1(\psi)\right)\|U^\delta\|^2_{s+1,s} + \|I - A(\psi)\|_s \|\psi^\delta_{tt}\|_{s-1}\|U^\delta\|_{s+1,s} d\tau \\ + C\int_0^t \|R^\delta(\psi)\|_s \|U^\delta\|_{s+1,s} d\tau + CR_2(\psi)\|U^\delta(t)\|^2_{s+1,s}$$

It is easy to see that $U^\delta \to U$ and $h^\delta_2 \to h_2$ in $L^\infty\left([0,T], H^{s+1} \times H^s\right)$ and in $L^2([0,T], H^s)$ respectively as $\delta \to 0$. Furthermore $R^\delta(\psi) \to 0$ in $L^2([0,T], H^s)$ as $\delta \to 0$ due to [6, Lemma 2.5(ii)]. Thus we get (4.14) for ψ satisfying (4.1).

Furthermore by Lemmas A.1 and A.2

$$\|h_2\|_s + R_1(\psi) + R_2(\psi) \leq C\|U\|^2_{s+1,s}.$$

for $N_s(T)$ sufficiently small. Finally, since ψ satisfies (2.3), we get

$$\|\psi_{tt}\|_{s-1} \leq C(\|\partial^2_x \psi\|_{s-1} + \|f(\psi, \psi_t, \partial_x \psi)\|_{s-1}) \leq \|U\|_{s+1,s}$$

for $N_s(T)$ sufficiently small. Hence we can deduce from (4.14) that

$$\|\partial_x^\alpha \psi(t)\|_1^2 + \|\partial_x^\alpha \psi_t(t)\|^2 + \int_0^t \|\partial_x^\alpha \partial_x \psi(\tau)\|^2 + \|\partial_x^\alpha \psi_t(\tau)\|^2 d\tau$$
$$- c \int_0^t \|\partial_x^\alpha \psi(\tau)\|^2 d\tau$$
$$\leq C\|U_0\|_{s+1,s}^2 + C\|U(t)\|_{s+1,s}^3 + C \int_0^t \|U(\tau)\|_{s+1,s}^3 d\tau.$$

The assertion is an immediate consequence of this inequality. □

4.4 Proposition. *In the situation of Proposition 4.1 there exist constants* $a_3(\leq \min\{a_2, a_1\}), C_4 = C_4(a_3, \delta_1) > 0$ *(δ_1 being the constant in Proposition 4.1) such that the the following holds: If* $N_s(T)^2 \leq a_3$ *and* $\|U_0\|_{s,s-1,1}^2 \leq \delta_1$ *for a solution* ψ *of* (2.3)-(2.5) *satisfying* (4.1)*, then*

$$N_s(t)^2 \leq C_4^2 \|U_0\|_{s+1,s,1}^2 \quad (t \in [0, T]). \tag{4.15}$$

Proof. This follows directly by adding (4.5)+ε(4.7) for ε sufficiently small. □

Finally we turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $T_1 > 0, \delta_2 > 0$ such that for all $U_0 = (\psi_0, \psi_1) \in H^{s+1} \times H^s$, where $\|U_0\|_{s+1,s}^2 < \delta_2$, there exists a solution $U = (\psi, \psi_t)$ of the Cauchy problem (2.3)-(2.5) with

$$\psi \in \bigcap_{j=1}^{s} C^j\left([0, T_1], H^{s+1-j}\right).$$

This is possible due to [5, Theorem III]. Furthermore let a_3, δ_1 and C_4 be the constants in Proposition 4.4. Choose $0 < \varepsilon < a_3/(2(1 + T_1))$. Due to [5, Ibid.] there exists $\delta_3 > 0, (\delta_3 \leq \delta_2)$ such that for all $U_0 = (\psi_0, \psi_1) \in H^{s+1} \times H^s$, where $\|U_0\|_{s+1,s}^2 < \delta_3$ the solution U of (2.3)-(2.5) satisfies

$$\sup_{t \in [0, T_1]} \|U(t)\|_{s+1,s}^2 < \varepsilon.$$

Now set $\delta_0 = \min\{\delta_1, \delta_3, \delta_3/C_4, a_3/(2C_4)\}$ and choose any $U_0 \in (H^{s+1} \times H^s) \cap (L^1 \times L^1)$ for which $\|U_0\|_{s+1,s,1}^2 < \delta_0$. Since $\delta_0 \leq \delta_3$

$$N_s(T_1)^2 < \varepsilon + T_1 \varepsilon < \frac{a_3}{2}.$$

Hence by Proposition 4.4 and $\|U_0\|^2_{s+1,s,1} < \delta_1$

$$N_s(T_1)^2 \le C_4 \|U_0\|^2_{s+1,s} < C_4\delta_0 \le \delta_3. \tag{4.16}$$

Furthermore due to Proposition 4.1, (2.7) holds for all $t \in [0, T_1]$. In particular (4.16) yields

$$\|U(T_1)\|^2_{s+1,s} < \delta_3. \tag{4.17}$$

Thus we can solve (2.3) on $[T_1, 2T_1]$ with initial values $(\psi(T_1), \psi_t(T_1))$ and get

$$N_s(T_1, 2T_1)^2 \le \varepsilon + T_1\varepsilon < \frac{a_3}{2}.$$

Now extend the solution (ψ, ψ_t) continuously on $[0, 2T_1]$. We can conclude

$$N_s(2T_1)^2 \le N_s(T_1)^2 + N_s(T_1, 2T_1)^2 < \frac{a_3}{2} + \frac{a_3}{2} = a_3.$$

Since we have already assumed $\|U_0\|^2_{s+1,s,1} < \delta_1$ Propositions 4.4 and 4.1 yield

$$N_s(2T_1) \le C_4\delta_0 \tag{4.18}$$

and (2.7) holds for all $t \in [0, 2T_1]$. Due to (4.18) we can repeat the former argument to obtain a solution on $[0, 3T_1]$ and further repetition proves the assertion. □

A. Appendix

A.1 Lemma. *Let $n, N \in \mathbb{N}$, $s \ge s_0 := [\frac{n}{2}]+1$ and $F \in C^\infty(\mathbb{R}^N)$, $F(0) = 0$. Then there exist $\delta, C = C(\delta) > 0$ such that for all $u \in H^s$ with $\|u\|_s \le \delta$, $F(u) - \partial_u F(0) \in H^s$ and*

$$\|F(u) - \partial_u F(0)u\|_s \le C\|u\|_s^2.$$

Proof. Since $s \ge s_0$ there exists a $C_1 > 0$ such that

$$\|u\|_{L^\infty} \le C_1\|u\|_s$$

for all $u \in H^s$. Furthermore due to $F(0) = 0$ there exist $\delta_1, C_2 = C_2(\delta) > 0$ such that

$$|F(y) - \partial_y F(0)y| \le C_2|y|^2.$$

for all $y \in \mathbb{R}^N$ with $|y| \le \delta_1$. Now let $u \in H^s$ such that $\|u\|_s \le \delta_1/C_1$ (i.e. $\|u\|_{L^\infty} \le \delta_1$). Then

$$\|F(u) - \partial_u F(0)u\| \le C_2\|u\|_{L^\infty}\|u\| \le C_1C_2\|u\|_s^2. \tag{A.1}$$

Furthermore for $\alpha \in \mathbb{N}_0^n$ with $1 \leq |\alpha| = j \leq s$ we get

$$\partial_x^\alpha F(u) = \partial_u F(u) \partial_x^\alpha u + R,$$

where

$$R = \sum_{1 \leq |\beta| < j} \binom{\alpha}{\beta} \partial_x^\beta u \; \partial_x^{\alpha-\beta} F(u).$$

Since $\partial_x u \in H^{s-1}$ and $\|u\|_{L^\infty} \leq \delta_1$, $\partial_x F(u) \in H^{s-1}$ and

$$\|\partial_x F(u)\|_{s-1} C_3 \|\partial_x u\|_{s-1}$$

for a $C_3 = C_3(\delta_2) > 0$ by [6, Lemma 2.4]. Therefore [6, Lemma 2.3] yields

$$\|R\| \leq C_4 \|\partial_x u\|_{s-1} \|\partial_x F(u)\|_{s-1} \leq C_3 C_4 \|\partial_x u\|_{s-1}^2$$

for a $C_4 > 0$. On the other hand there exist $\delta_2, C_5 = C_5(\delta_2) > 0$, such that

$$|\partial_y F(y) - \partial_y F(0)| \leq C_5 |y|$$

for all $y \in \mathbb{R}^N$ with $|y| \leq \delta_2$. Assuming $\|u\|_s \leq \delta_2 / C_1$ entails

$$\begin{aligned} \|\partial_x^\alpha (F(u) - \partial_u F(0))\| &\leq \|(\partial_u F(u) - \partial_u F(0)) \partial_x^\alpha u\| + \|R\| \\ &\leq \|\partial_u F(u) - \partial_u F(0)\|_{L^\infty} \|u\|_s + C_3 C_4 \|\partial_x u\|_{s-1} \\ &\leq \max\{C_3 C_4, C_5\} \|u\|_s^2. \end{aligned}$$

Since α was arbitrary this estimate together with (A.1) leads the assertion for $\delta = \min\{\delta_1, \delta_2\}/C_1$. □

A.2 Lemma. *Let $n, N \in \mathbb{N}$, $s \geq s_0$ and $F \in C^\infty(\mathbb{R}^N, \mathbb{R}^{N \times N})$. Then there exist $\delta, C = C(\delta) > 0$ such that for all $u \in H^s(\mathbb{R}^n, \mathbb{R}^N)$ with $\|u\|_s \leq \delta$, $(F(u) - F(0))u \in H^s$ and*

$$\|(F(u) - F(0))u\|_s \leq C \|u\|_s^2.$$

Proof. First note that there exist $\delta_1, C_1 = C_1(\delta_1) > 0$ such that

$$|F(y) - F(0)| \leq C_1 |y|$$

for all $y \in \mathbb{R}^N$, $|y| \leq \delta_1$ as well as $C_2 > 0$ such that

$$\|v\|_{L^\infty} \leq C_2 \|v\|_s$$

for all $v \in H^s$. Now let $u \in H^s$, $\|u\|_s \leq \delta_1/C_2$. Then

$$\|F(u) - F(0)\| \leq C_1\|u\|_s.$$

On the other hand by [6, Lemma 2.4] $\partial_x F(u) \in H^{s-1}$ and

$$\|\partial_x F(u)\|_{s-1} \leq C_3\|\partial_x u\|_{s-1}$$

for a $C_3 = C_3(\delta_1) > 0$. Hence $F(u) - F(0) \in H^s$ and

$$\|F(u) - F(0)\|_s \leq C_4\|u\|_s$$

for $\|u\|_s \leq \delta = \delta_1/C_2$. Now the assertion follows from [6, Lemma 2.4]. □

Conflict of interest: There is no conflict of interest.

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