Hilbert's 1888 Theorem for Cones along the Veronese Variety

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Abstract

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Hilbert's 1888 Theorem for Cones along the Veronese Variety

For $n, d \in \mathbb{N}$, the cone $\mathcal{P}_{n+1,2d}$ of positive semidefinite homogeneous polynomials with real coefficients in $n+1$ variables of degree $2d$ contains the cone $\Sigma_{n+1,2d}$ of those that are representable as finite sums of squares. Hilbert's 1888 theorem states that $\Sigma_{n+1,2d} = \mathcal{P}_{n+1,2d}$ exactly in the *Hilbert cases* $(2, 2d)_{d \geq 1}$, $(n+1, 2)_{n \geq 1}$ and $(3, 4)$.

In this thesis, we apply the Gram matrix method to construct and examine cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ along projective varieties containing the Veronese variety. Introducing a specific filtration $V_{k(n,d)-n} \subseteq \ldots \subseteq V_0$ of projective varieties containing the Veronese variety, we provide a specific cone filtration

$$\Sigma_{n+1,2d} = C_0 \subseteq \ldots \subseteq C_n \subseteq C_{n+1} \subseteq \ldots \subseteq C_{k(n,d)-n} = \mathcal{P}_{n+1,2d}.$$

Here, $k(n,d)+1$ denotes the dimension of the real vector space of homogeneous polynomials in $n+1$ variables of degree d. In a non-Hilbert case, at least one of the inclusions has to be strict, but it is not clear which one and how many. The identification of each strict inclusion is the goal of this thesis.

In order to achieve this, we examine the projective varieties $V_0, \ldots, V_{k(n,d)-n}$ in the non-Hilbert cases and prove that $V_0, \ldots, V_n$, and also V_{n+1} if $n = 2$, have minimal degree. Leveraging a result by Blekherman–Smith–Velasco, we thus demonstrate that $C_0, \ldots, C_n$, and also C_{n+1} if $n = 2$, coincide with $\Sigma_{n+1,2d}$. It remains to investigate the subfiltration $C_{n+1} \subseteq \ldots \subseteq C_{k(n,d)-n} = \mathcal{P}_{n+1,2d}$, and also $C_n \subseteq C_{n+1}$ if $n \geq 3$.

To this end, we show that all remaining inclusions are strict in the non-Hilbert cases $(n+1, 4)_{n \geq 3}$, $(n+1, 6)_{n \geq 2}$ by a reduction to the cases $(4, 4)$, $(3, 6)$ and the development of a first degree-jumping principle. We then generalize our findings to the non-Hilbert cases $(n+1, 2d)_{n \geq 2, d \geq 4}$ by establishing a second degree-jumping principle using circuit PSD-extremal forms. This allows us to give a refinement of Hilbert's 1888 theorem.

Lastly, we apply a method of Scheiderer to show that each identified strictly separating intermediate cone $\Sigma_{n+1,2d} \subsetneq C_i \subsetneq \mathcal{P}_{n+1,2d}$ fails to be a spectrahedral shadow. We therefore provide many counterexamples to the Helton–Nie conjecture that any convex semialgebraic set is a spectrahedral shadow.

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Contents

To my mother
Tanja Hess
who means the world to me.

Chapter 1

Introduction

1.1 Overview

A key question in real algebraic geometry is whether a given positive semidefinite (PSD) polynomial of even degree can be represented by a finite sum of squares (SOS) of half degree polynomials. Since the PSD and SOS properties are preserved under homogenization [Mar08, Proposition 1.2.4.], it suffices to examine $(n+1)$-ary $2d$-ics (i.e., homogeneous polynomials in $n+1$ variables of degree $2d$). The investigation of this (homogeneous) query goes back to Hilbert [Hil88], who in 1888 classified all cases $(n+1, 2d)$ in which every PSD $(n+1)$-ary $2d$-ic admits a SOS representation. These are the *Hilbert cases* $(2, 2d)_{d\geq 1}$, $(n+1, 2)_{n\geq 1}$ and $(3, 4)$. For the remaining *non-Hilbert cases*, Hilbert established the existence of PSD homogeneous polynomials that cannot be expressed as SOS. He achieved this by reducing his investigation to the *basic non-Hilbert cases* $(4, 4)$ and $(3, 6)$ using an argument that allowed him to increase the number of variables and the even degree while preserving the PSD-not-SOS property. The cone $\Sigma_{n+1,2d}$ of all SOS $(n+1)$-ary $2d$-ics is thus a proper subcone of the cone $\mathcal{P}_{n+1,2d}$ of all PSD $(n+1)$-ary $2d$-ics in any non-Hilbert case. Remarkably, Hilbert's initial proof was of an existential nature and did not produce explicit examples of homogeneous polynomials that are PSD but not SOS. His argumentation relied on the fact that $(n+1)$-ary d-ics satisfy linear Cayley–Bacharach relations which $(n+1)$-ary $2d$-ics fail [Ble12; Rez92; Rez00].

Almost eight decades after Hilbert, Motzkin [Mot65] was the first to give an explicit example of a PSD-not-SOS ternary sextic. Shortly thereafter and independent of Motzkin, Robinson [Rob73] successfully applied the methods of Hilbert in a simplified matter and constructed a PSD-not-SOS quaternary quartic and a PSD-not-SOS ternary sextic. A few years later, Choi and Lam [CL76; CL77] added to this list by presenting further examples of a PSD-not-SOS quaternary quartic and a PSD-not-SOS ternary sextic. Thus, explicit examples for PSD-not-SOS homogeneous polynomials were found in the basic non-Hilbert cases $(4, 4)$ and $(3, 6)$.

Building on his achievement from 1888, Hilbert further demonstrated that the PSD property of any ternary homogeneous polynomial can be certified by a sum of squares representation using rational functions [Hil90]. This led him to a broader question he presented at the International Congress of Mathematicians in Paris 1900 [Hil00]: Is it true that any PSD $(n+1)$-ary $2d$-ic is a sum of squares of rational functions? This inquiry, which is today known as Hilbert's 17th Problem, was affirmatively answered by Artin [Art27] in 1927 and deems the hour of birth of the Artin–Schreier theory of real closed fields.

The recent century now witnessed a significant interest in the cones $\mathcal{P}_{n+1,2d}$ and $\Sigma_{n+1,2d}$, which climaxed in a generalization of Hilbert's 1888 theorem along projective varieties by Blekherman–Smith–Velasco [BSV16] in 2016. In their work, they proclaimed that under specific conditions, any quadratic homogeneous polynomial that is PSD on a given irreducible projective variety admits a SOS representation in the respective homogeneous coordinate ring if and only if the considered projective variety has minimal degree. This result extends to homogeneous polynomials of any even degree via the Veronese embedding.

Turning our attention to applications, we see that it is inevitable for polynomial optimization, control theory, engineering and many more to have the ability to efficiently decide if a given homogeneous polynomial is PSD and to provide certificates for this property in case of an affirmative answer (see, e.g., [BPT13; Las10; Lau09]). However, up to date, there does not exist an algorithm to test whether a given $(n+1)$-ary $2d$-ic is PSD in polynomial time. Therefore, it is common practice to relax this task and focus on testing for the SOS property instead. Membership tests for the cone $\Sigma_{n+1,2d}$ can be reformulated into semidefinite programming problems, which can be solved efficiently in polynomial time [GLS93; Par03]. This efficiency stems from the fact that $\Sigma_{n+1,2d}$ is a spectrahedral shadow (i.e., the image of a spectrahedron under an affine linear map). Spectrahedral shadows are the feasible regions of positive semidefinite programs, which generalize the concept of linear programming, and find purpose in applied mathematics, computer vision and robotics (see, e.g. [AL12; SH21; VB99]).

Interestingly, there is a deeper connection between spectrahedral shadows and convex semialgebraic sets. For example, it is not too difficult to see that any spectrahedral shadow is a convex semialgebraic set. This led Nemirovski [Nem07] to ask if the converse is true as well in 2007. Helton and Nie [HN09] conjectured in 2009 that this is in fact the case. While several studies provided evidence for this conjecture in specific cases (see, e.g. [HN10; NS15; Sch18a]), a breakthrough came in 2018 when Scheiderer [Sch18b] disproved the Helton–Nie conjecture by generalizing the moment relaxation construction by Lasserre [Las09] and Parrilo [Par00] to establish a method to produce convex semialgebraic sets that are not spectrahedral shadows. In particular, he gave the first counterexample by showing that $\mathcal{P}_{n+1,2d}$ is not a spectrahedral shadow in the non-Hilbert cases. This clarified that the familiar semialgebraic optimization problem of minimizing a homogeneous polynomial in non-Hilbert cases does not allow an exact depiction as a semidefinite program. It is thus interesting to investigate at what stage between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ the property of being a spectrahedral shadow is lost.

To approach this question, we recall the Gram matrix method [CLR92] in Chapter 2 of this thesis and apply it as a tool in Chapter 3 to construct a specific cone filtration

$$\Sigma_{n+1,2d} = C_0 \subseteq \ldots \subseteq C_n \subseteq C_{n+1} \subseteq \ldots \subseteq C_{k(n,d)-n} = \mathcal{P}_{n+1,2d} \tag{1.1}$$

along $k(n,d) - n + 1$ projective varieties containing the Veronese variety. Here, the natural number $k(n,d)+1$ denotes the dimension of the real vector space of $(n+1)$-ary d-ics. Leveraging Blekherman–Smith–Velasco's result on varieties of minimal degree [BSV16], we demonstrate in Chapter 4 that the first $n+1$, respectively $n+2$ if $n = 2$, cones in (1.1) collapse to $\Sigma_{n+1,2d}$. In the next step, in Chapter 5, we identify all remaining cones to be strictly separating intermediate cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ (i.e., $\Sigma_{n+1,2d} \subsetneq C_i \subsetneq \mathcal{P}_{n+1,2d}$) in the non-Hilbert cases $(n+1,4)_{n\geq 3}$ and $(n+1,6)_{n\geq 2}$ by reconsidering the Motzkin and the Choi–Lam PSD-not-SOS homogeneous polynomials. Therafter, we extend our consideration to any non-Hilbert case by establishing a degree-jumping principle using circuit polynomials and the concept of extremality in the cone $\mathcal{P}_{n+1,2d}$ in Chapter 6. Hence, we identify all strictly separating intermediate cones in (1.1) in any non-Hilbert case. This allows us to state and prove a generalization of Hilbert's 1888 theorem. Moreover, we apply the method of Scheiderer [Sch18b] to demonstrate that each identified strictly separating intermediate cone in (1.1) fails to be a spectrahedral shadow in non-Hilbert cases in Chapter 7. We therefore provide many explicit examples of separating cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ which are convex semialgebraic sets but not spectrahedral shadows. In particular, the property of being a spectrahedral shadow is lost as soon as we move from $\Sigma_{n+1,2d}$ to the first strictly separating C_i in (1.1).

1.2 Structure of the Thesis

In Chapter 2, we lay the foundation for our investigation. To this end, Section 2.1 establishes essential notations and reviews core concepts from algebraic geometry and convexity. We assume that the reader is familiar with the basics of algebraic geometry and convexity. However, for the convenience of the reader, we also give a concise overview on these two theories in Appendix A.1 and Appendix A.3. In Section 2.2, we next introduce the key concept of positive semidefinite polynomials and sums of squares and lay out that it suffices to consider a homogeneous setting. This allows us to state Hilbert's celebrated 1888 theorem. Finally, Section 2.3 delves into the Gram matrix method by introducing the Gram map and the Veronese embedding. Using these maps, we reduce our examination of $\mathcal{P}_{n+1,2d}$, $\Sigma_{n+1,2d}$ and intermediate cones in between to a study of local non-negativity of quadratic homogeneous polynomials on sets of real points of some specific projective varieties.

In Chapter 3, we specify our consideration by establishing a construction method for cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ along sets of real points of projective varieties containing the Veronese variety in Section 3.1 (see Definition 3.1.1). We build on this method in Section 3.2 and give a specific cone filtration

$$\Sigma_{n+1,2d} = C_0 \subseteq \ldots \subseteq C_{k(n,d)-n} = \mathcal{P}_{n+1,2d}$$

(see ($\mathcal{CF}$)) along a filtration of projective varieties $V_{k(n,d)-n} \subsetneq \ldots \subsetneq V_0$ containing the Veronese variety (see (3.4)) with a corresponding filtration of sets of real points in which each inclusion is strict (see (3.5)). Hilbert's 1888 theorem guarantuees that at least one of the inclusions in the cone filtration has to be strict in a non-Hilbert case, but it is a priori not clear how many and which ones. This leads us to the main task of this thesis: the identification of each strict inclusion in the above specific cone filtration in non-Hilbert cases. To address this problem, we provide an explicit description of the projective varieties $V_0, \ldots, V_{k(n,d)-n}$ as the projective closure of certain affine varieties $K_0, \ldots, K_{k(n,d)-n}$ in Section 3.3 (see Construction 3.3.7 and Theorem 3.3.17). In Section 3.4, we leverage this result to show that it suffices to examine local non-negativity on the set of real points of the embedded affine variety K_i instead of on the set of real points of V_i in Corollary 3.4.5. This simplification proves to be useful in our later investigation.

Chapter 4 deepens our understanding of the projective varieties $V_0, \ldots, V_{k(n,d)-n}$ in non-Hilbert cases. We determine that each V_i is a non-degenerate irreducible totally-real projective variety of codimension i in Section 4.1. Moreover, we show that V_i has degree $i + 1$ for $i = 0, \ldots, n$, and also for $i = n + 1$ if $n = 2$, in Theorem 4.1.37. Hence, we conclude in Corollary 4.1.38 that $V_0, \ldots, V_n$, and also V_{n+1} if $n = 2$, are non-degenerate irreducible totally-real projective varieties of minimal degree. This finding, combined with a variation of Blekherman–Smith–Velasco's result on varieties of minimal degrees (see Theorem 4.2.1 and Theorem 4.2.2), allows us to demonstrate in Theorem 4.2.7 that the cones $C_0, \ldots, C_n$, and also C_{n+1} if $n = 2$, collapse to $\Sigma_{n+1,2d}$. The main task of this thesis therefore narrows down to an investigation of the cone filtration $C_{n+1} \subseteq \ldots \subseteq C_{k(n,d)-n} = \mathcal{P}_{n+1,2d}$, and also $C_n \subseteq C_{n+1}$ if $n \geq 3$.

In Chapter 5, we solve the main task of this thesis in the non-Hilbert cases of quartics and sextics by showing that each remaining cone inclusion is strict. To this end, Section 5.1 addresses the basic non-Hilbert cases where we provide answers in Theorem 5.1.2 and Theorem 5.1.6. Building on these findings, Section 5.2 follows a two step argument. Firstly, we provide a complete answer to the main query in the non-Hilbert cases $(n + 1, 4)_{n\geq 4}$ in Theorem 5.2.2. Secondly, we develope a first degree-jumping principle in Theorem 5.2.7 which, together with our results for ternary sextics, allows us to answer the query in the non-Hilbert cases $(n + 1, 6)_{n\geq 3}$ in Theorem 5.2.14.

Chapter 6 expands our findings beyond quartics and sextics to the non-Hilbert cases $(n+1, 2d)_{n\geq 2,d\geq 4}$. To achieve this, we introduce the concept of circuit polynomials and state related fundamental results in Section 6.1. In Section 6.2, we next determine the unique $i(f) \in \{0, \ldots, k(n, d) - n - 1\}$ such that $f \in C_{i(f)+1} \backslash C_{i(f)}$ for any given circuit $(n+1)$-ary $2d$-ic f that is extremal in the cone $\mathcal{P}_{n+1,2d}$ but not SOS in Theorem 6.2.12.

Leveraging this result, we develop a second degree-jumping principle in Section 6.3 (see Theorem 6.3.6) that allows us to solve the main task of this thesis by proving that all remaining cones C_i between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ are strictly separating for the non-Hilbert cases $(n+1, 2d)_{n\geq 2, d\geq 4}$ in Theorem 6.3.8.

Chapter 7 concludes our investigation by exploring some key properties of the strictly separating intermediate cones C_i between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$. In Section 7.1, we especially focus on topological properties and prove that our distinguished cones are closed (see Theorem 7.1.1). Moreover, we compute their interiors and boundaries (see Theorem 7.1.3 and Corollary 7.1.6). While a basic understanding of topology is helpful, we provide a concise overview on this theory in Appendix A.2 for the convenience of the reader. Section 7.2 lastly focuses on geometric properties and connects our findings to the Helton–Nie conjecture that any convex semialgebraic set is a spectrahedral shadow. We demonstrate that our identified strictly separating intermediate cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ serve as counterexamples by applying a method of Scheiderer. In order to understand the tools involved, we refer the reader to Appendix A.4 where we collect the relevant concepts and fundamental results from real algebraic geometry.

In Chapter 8, we summarize the results of this thesis and outline potential directions for further research. In Section 8.1, we highlight our key achievements by giving a refinement of Hilbert's 1888 theorem (see Theorem 8.1.1), which provides a more comprehensive understanding of the non-Hilbert cases, and collecting our findings on counterexamples to the Helton–Nie conjecture for convex semialgebraic sets in Theorem 8.1.2. Finally, in Section 8.2 to Section 8.4, we discuss possibilities for future research directions inspired by this thesis.

1.3 Published Contents in Advance

Parts of this thesis are based on papers with co-authors that are submitted for publication and were further worked out here. In [GHK24b], we firstly presented the constructions that are given in Section 2.3 and Section 3.2. Moreover, in this first paper, we also verified the consideration made in Section 3.3 and proved the main results from Chapter 5. In [GHK24a], we furthermore introduced a general construction method for intermediate cones along projective varieties in the sense of Section 3.1 and verified the statements given in Section 3.4. Moreover, we also established the main results from Chapter 6 and Chapter 7 in this second paper.

Chapter 2

Preliminaries

In this chapter, we provide the relevant background for this thesis. To this end, we introduce standard notations and basic concepts in Section 2.1. Moreover, in Section 2.2, we give an overview on positive semidefinite polynomials and sums of squares. Lastly, we introduce a method to systematically study positive semidefinite polynomials using Gram matrices and the Veronese embedding in Section 2.3.

2.1 Notations and Basic Concepts

Let us agree on some standard notations for this thesis. We denote the set of positive integers by $\mathbb{N}$ (the *natural numbers*), the set of non-zero integers by $\mathbb{N}_0$ and the set of integers by $\mathbb{Z}$. For $l \in \mathbb{N}$ and $a := (a_1, \ldots, a_l) \in \mathbb{N}_0^l$, we adopt a multi-index notation by setting $|a| := \sum_{i=1}^{l} |a_i|$ and $Y^a := \prod_{i=1}^{l} Y_i^{a_i}$ for a vector of indeterminants $Y = (Y_1, \ldots, Y_l)$. Moreover, we let $\mathbb{R}$ be the field of real numbers and set $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ to be the subsets of non-negative and positive real numbers, respectively. Furthermore, we denote the set of all non-zero real numbers by $\mathbb{R}^\times$ and the linear $\mathbb{R}$-span of a family of vectors $\mathfrak{V}$ by $\operatorname{span}_{\mathbb{R}}(\mathfrak{V})$. For $l \in \mathbb{N}$, we moreover set $\operatorname{Sym}_l(\mathbb{R})$ to be the $\mathbb{R}$-vector space of $l \times l$ symmetric matrices with real entries. The field of complex numbers is denoted by $\mathbb{C}$ and we set $\mathbb{C}^\times$ to be the subset of non-zero complex numbers. For any $a \in \mathbb{C}$, we furthermore appoint $\operatorname{Re}(a)$ to be the real part of a. For two arbitrary sets $\mathcal{A}$, $\mathcal{B}$ and a map $f\colon \mathcal{A} \to \mathcal{B}$, we moreover set $\ker(f)$ to be the kernel of f and denote the restriction of f to a subset A of $\mathcal{A}$ by $f|_A$. If $\mathcal{A}$ and $\mathcal{B}$ are vector spaces over a field $\mathbb{K}$, then $\operatorname{Hom}(\mathcal{A}, \mathcal{B})$ is the $\mathbb{K}$-vector space of linear maps $\mathcal{A} \to \mathcal{B}$. In the special case that $\mathcal{A} = \mathcal{B}$, we write $\operatorname{id}_{\mathcal{A}}$ for the identity map on $\mathcal{A}$.

In this thesis, we will often switch between an affine and a projective setting. Whenever we are in an affine setting, we will adapt bold notations. That is, we set $\boldsymbol{Y} := (Y_1, \ldots, Y_l)$ to be a vector of indeterminants and denote vectors in $\mathbb{K}^l$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ by $\boldsymbol{y} := (y_1, \ldots, y_l)$. Moreover, we let $\mathbb{K}[\boldsymbol{Y}]$ be the polynomial ring in the variables $Y_1, \ldots, Y_l$ with coefficients in $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and understand an affine variety in $\mathbb{C}^l$ to be the set of common zeros of a collection of polynomials $F \subseteq \mathbb{C}[\boldsymbol{Y}]$, which we abbreviate by $\boldsymbol{\mathcal{V}}(F)$. The vanishing ideal of an affine variety $\mathfrak{V}$ is denoted by $\boldsymbol{\mathcal{I}}(\mathfrak{V})$ and the corresponding coordinate ring is set to be given by $\mathbb{C}[\mathfrak{V}] := \mathbb{C}[\boldsymbol{Y}] \,/\, \boldsymbol{\mathcal{I}}(\mathfrak{V})$.

In order to draw a clear line to the affine setting, we drop any bold notations in a projective setting. We set $\mathbb{P}^l$ to be the l-dimensional projective space over $\mathbb{C}$ for $l \in \mathbb{N}$ and let $Y := (Y_0, \ldots, Y_l)$ be a vector of indeterminants. Hence, we denote vectors in $\mathbb{K}^l$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ by $y := (y_0, \ldots, y_l)$. Moreover, we set the homogeneous coordinates in $\mathbb{P}^l$ to be given by $[y] := [y_0 : \ldots : y_l]$ and introduce the embedding $\phi\colon \mathbb{C}^l \to \mathbb{P}^l, y \mapsto [y]$. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we furthermore let $\mathbb{K}[Y]$ be the polynomial ring in the variables $Y_0, \ldots, Y_l$ with coefficients in $\mathbb{K}$ and understand a projective variety in $\mathbb{P}^l$ to be the set of common zeros of a collection of homogeneous polynomials $F \subseteq \mathbb{C}[Y]$, which we abbreviate by $\mathcal{V}(F)$. The vanishing ideal of a projective variety $\mathfrak{V}$ is furthermore denoted by $\mathcal{I}(\mathfrak{V})$ and the corresponding graded homogeneous coordinate ring is set to be given by $\bigoplus_{t \geq 0} \mathbb{C}[\mathfrak{V}]_t = \mathbb{C}[\mathfrak{V}] := \mathbb{K}[Y] \,/\, \mathcal{I}(\mathfrak{V})$.

Throughout this thesis, $\boldsymbol{n}$ **denotes a natural number** that represents the number of variables considered. These are the variables $\boldsymbol{X} := (X_1, \ldots, X_n)$ in an affine setting and the variables $X := (X_0, \ldots, X_n)$ in a projective setting, respectively. Moreover, $\boldsymbol{d}$ **denotes a natural number** that provides information on the degree of the considered homogeneous polynomials, where the degree of a polynomial always refers to the total degree. We will furthermore later introduce another specific non-negative integer $k := k(n, d)$ that depends on n and d. When working with such a distinguished k, we will use the vector of indeterminants $\boldsymbol{Z} := (Z_1, \ldots, Z_k)$ in an affine setting and $Z := (Z_0, \ldots, Z_k)$ in a projective setting, respectively.

2.2 Positive Semidefinite Polynomials and Sums of Squares

We start our investigation by examining the fundamental objects of this thesis.

Definition 2.2.1. (i) A multivariate polynomial $g \in \mathbb{R}[\boldsymbol{X}]$ is *locally non-negative* or *locally positive semidefinite on* $K \subseteq \mathbb{R}^n$ if

$$g(x) \geq 0 \text{ for all } x \in K$$

and we write $g|_K \geq 0$. In the special case that $K = \mathbb{R}^n$, the polynomial g is *(globally) non-negative* or *(globally) positive semidefinite (PSD)*. We write $g \geq 0$.

(ii) A multivariate polynomial $g \in \mathbb{R}[\boldsymbol{X}]$ is a *sum of squares (SOS)* if there exist some $g_1, \ldots, g_s \in \mathbb{R}[\boldsymbol{X}]$ $(s \in \mathbb{N})$ such that

$$g = \sum_{i=1}^{s} g_i^2.$$

Lemma 2.2.2. *If $g \in \mathbb{R}[\boldsymbol{X}]$ is SOS, then g is PSD.*

Proof. Let $s \in \mathbb{N}$ and $g_1, \ldots, g_s \in \mathbb{R}[\boldsymbol{X}]$ be such that $g = \sum_{i=1}^{s} g_i^2$. For $\boldsymbol{x} \in \mathbb{R}^n$, we have

$$g(\boldsymbol{x}) = \sum_{i=1}^{s} g_i^2(\boldsymbol{x}) \geq 0. \qquad \blacksquare$$

Lemma 2.2.3. *For a permutation $\sigma \in S_n$ and a polynomial $g \in \mathbb{R}[\boldsymbol{X}]$ define $g^\sigma(\boldsymbol{X}) := g\left(X_{\sigma(1)}, \ldots, X_{\sigma(n)}\right) \in \mathbb{R}[\boldsymbol{X}]$, then the following are true:*

(i) If g is PSD, then g^σ is PSD.

(ii) If g is SOS, then g^σ is SOS.

Proof. (i) For $\boldsymbol{x} \in \mathbb{R}^n$, using that g is PSD, we compute

$$g^\sigma(\boldsymbol{x}) = g\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right) \geq 0.$$

(ii) Let $s \in \mathbb{N}$ and $g_1, \ldots, g_s \in \mathbb{R}[\boldsymbol{X}]$ be such that $g = \sum_{i=1}^{s} g_i^2$. For $i = 1, \ldots, s$, we set $g_i^\sigma(\boldsymbol{X}) := g_i\left(X_{\sigma(1)}, \ldots, X_{\sigma(n)}\right) \in \mathbb{R}[\boldsymbol{X}]$ and conclude that g^σ is SOS since

$$g^\sigma(\boldsymbol{X}) = \sum_{i=1}^{s} \left(g_i\left(X_{\sigma(1)}, \ldots, X_{\sigma(n)}\right)\right)^2 = \sum_{i=1}^{s} \left(g_i^\sigma(\boldsymbol{X})\right)^2. \qquad \blacksquare$$

Definition 2.2.4. A non-zero $f \in \mathbb{R}[X]$ of degree $d \in \mathbb{N}_0$ is called *homogeneous* or a *form (of degree d)* if all monomials in f have the same degree d. The zero polynomial is by convention a form which we call the *zero form*.

Remark 2.2.5. *The non-zero forms of degree zero are exactly the constant polynomials that are non-zero. Hence, the set of non-zero forms of degree zero together with the zero form can be identified with $\mathbb{R}$.*

If no further restrictions are made, we thus assume throughout the rest of this thesis that ***d* is a natural number**.

Notation 2.2.6. We set

$$\mathcal{F}_{n+1,d} := \{f \in \mathbb{R}[X] \mid f \text{ is a form of degree } d \text{ or the zero form}\}$$

and call any non-zero $f \in \mathcal{F}_{n+1,d}$ a $(n+1)$-*ary* d-*ic*.

Example 2.2.7. (i) The monomials in the cubic polynomial

$$f(X_0, X_1, X_2) := 5X_2^3 - 6X_1X_2^2 + X_0X_1X_2 + 2X_0^2X_1 \in \mathbb{R}[X_0, X_1, X_2]$$

are X_2^3, $X_1X_2^2$, $X_0X_1X_2$, $X_0^2X_1$ and each has degree three. It thus follows that f is a form in three indeterminants of degree three. We express this by writing $f \in \mathcal{F}_{3,3}$ and saying that f is a ternary cubic.

(ii) The monomials in the cubic polynomial

$$g(X_1, X_2) := 5X_2^3 - 6X_1X_2^2 + X_1X_2 + 2X_1 \in \mathbb{R}[X_1, X_2]$$

are X_2^3, $X_1X_2^2$, X_1X_2 and X_1. Each of the monomials X_2^3 and $X_1X_2^2$ has degree three, the monomial X_1X_2 has degree two and the monomial X_1 has degree one. Consequently, g is not a form.

Notation 2.2.8. We set

$$I_{n+1,d} := \{\alpha \in \mathbb{N}_0^{n+1} \mid |\alpha| = d\}.$$

Lemma 2.2.9. *$\mathcal{F}_{n+1,d}$ is a subspace of $\mathbb{R}[X]$ with basis $\{X^\alpha \mid \alpha \in I_{n+1,d}\}$. So,*

$$\dim(\mathcal{F}_{n+1,d}) = \binom{n+d}{n} = \binom{n+d}{d}. \tag{2.1}$$

Proof. A sum of two $(n+1)$-ary d-ics is a form of degree d. Likewise, a real multiple of a $(n+1)$-ary d-ic is a form of degree d. Hence, $\mathcal{F}_{n+1,d}$ is subspace of $\mathbb{R}[X]$.

The set $\{X^\alpha \mid \alpha \in I_{n+1,d}\}$ is linearly independent and any $(n+1)$-ary d-ic is a finite linear combination of monomials of degree d with scalars in $\mathbb{R}$. Consequently, $\{X^\alpha \mid \alpha \in I_{n+1,d}\}$ is a basis for $\mathcal{F}_{n+1,d}$ and we compute

$$\dim(\mathcal{F}_{n+1,d}) = |\{X^\alpha \mid \alpha \in I_{n+1,d}\}| = \binom{n+d}{d} = \binom{n+d}{n}. \qquad \blacksquare$$

Remark 2.2.10. *Collecting all terms of degree i of a given multivariate polynomial $g \in \mathbb{R}[X]$ of degree d in a form g_i for $i = 0, \dots, d$ gives us a unique decomposition $g = g_0 + \ldots + g_d$ into homogeneous components $g_i \in \mathcal{F}_{n+1,i}$ for $i = 0, \dots, d$.*

Lemma 2.2.11. *For $f \in \mathcal{F}_{n+1,d}$, $x \in \mathbb{R}^{n+1}$ and $\lambda \in \mathbb{R}^\times$, it holds $f(\lambda x) = \lambda^d f(x)$.*

Proof. Lemma 2.2.9 allows us to fix some coefficients $f_\alpha \in \mathbb{R}$ for $\alpha \in I_{n+1,d}$ such that $f(X) = \sum_{\alpha \in I_{n+1,d}} f_\alpha X^\alpha$ and we compute

$$f(\lambda x) = \sum_{\alpha \in I_{n+1,d}} f_\alpha (\lambda x)^\alpha = \sum_{\alpha \in I_{n+1,d}} f_\alpha \lambda^d x^\alpha = \lambda^d \cdot \sum_{\alpha \in I_{n+1,d}} f_\alpha x^\alpha = \lambda^d f(x). \qquad \blacksquare$$

Corollary 2.2.12. *If $f \in \mathcal{F}_{n+1,2d}$ is locally PSD on $\{1\} \times \mathbb{R}^n$, then f is PSD.*

Proof. For $x \in \mathbb{R}^{n+1}$ with $x_0 \neq 0$, we deduce from Lemma 2.2.11 that

$$f(x) = f\left(x_0\left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)\right) = (x_0)^{2d} f\left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right),$$

which allows us to conclude $f(x) \geq 0$ since $(x_0)^{2d} \geq 0$, $\left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in \{1\} \times \mathbb{R}^n$ and f is assumed to be locally PSD on $\{1\} \times \mathbb{R}^n$. Therefore, we know that f is locally PSD on $\mathbb{R}^\times \times \mathbb{R}^n$ and thus f is PSD by continuity. ∎

Proposition 2.2.13. *For $f \in \mathcal{F}_{n+1,d}$, the following are true:*

(i) If f is PSD, then d is even.

(ii) If f is SOS, then d is even and f is a finite sum of forms of degree $\frac{d}{2}$.

Proof. (i) Lemma 2.2.11 implies $f(-x) = (-1)^d f(x)$ for any $x \in \mathbb{R}^{n+1}$ and we know $f(-x), f(x) \geq 0$ for any $x \in \mathbb{R}^{n+1}$ since f is PSD. Hence, d is even.

(ii) See [Lau09, Lemma 3.1. and Lemma 3.2.]. ∎

Proposition 2.2.14. *For $f \in \mathcal{F}_{n+1,2d}$ and $i = 0, \dots, n$, the following are true:*

(i) $X_i^2 f(X) \in \mathcal{F}_{n+1,2d+2}$ is PSD if and only if f is PSD.

(ii) $X_i^2 f(X) \in \mathcal{F}_{n+1,2d+2}$ is SOS if and only if f is SOS.

Proof. (i) ($\Rightarrow$) For $x \in \mathbb{R}^{n+1}$ with $x_i \neq 0$, $x_i^2 f(x) \geq 0$ implies $f(x) \geq 0$. Hence, f is PSD by continuity.

($\Leftarrow$) For $x \in \mathbb{R}^{n+1}$, $f(x) \geq 0$ implies $x_i^2 f(x) \geq 0$. Hence, $X_i^2 f(X)$ is PSD.

(ii) ($\Rightarrow$) Recalling Proposition 2.2.13 (ii), we let $s \in \mathbb{N}$ and $h_1, \dots, h_s \in \mathcal{F}_{n+1,d+1}$ be such that $X_i^2 f(X) = \sum_{j=0}^{s} h_j^2(X) = \sum_{j=0}^{s} (h_j(X))^2$. Since $X_i^2 f(X)$ vanishes in $x_i = 0$, we conclude that $\sum_{j=0}^{s} (h_j(X))^2$ also vanishes in $x_i = 0$. This implies that $h_1, \dots, h_s$ vanish in $x_i = 0$ and thus X_i divides each of the forms $h_1, \dots, h_s$. Consequently, $f_j(X) := \frac{h_j(X)}{X_i}$ is a $(n+1)$-ary d-ic for $j = 1, \dots, s$ and

$$\sum_{j=1}^{s} f_j^2(X) = \sum_{j=1}^{s} (f_j(X))^2 = \sum_{j=1}^{s} \left(\frac{h_j(X)}{X_i}\right)^2 = \frac{1}{X_i^2} \sum_{j=0}^{s} (h_j(X))^2 = f(X).$$

($\Leftarrow$) Recalling Proposition 2.2.13 (ii), we let $s \in \mathbb{N}$ and $f_1, \dots, f_s \in \mathcal{F}_{n+1,d}$ be such that $f(X) = \sum_{j=0}^{s} f_j^2(X) = \sum_{j=0}^{s} (f_j(X))^2$. Consequently,

$$X_i^2 f(X) = X_i^2 \sum_{j=0}^{s} (f_j(X))^2 = \sum_{j=0}^{s} (X_i f_j(X))^2$$

and $X_i f_j(X)$ is a $(n+1)$-ary $(d+1)$-ic for $j = 0, \dots, s$. ∎

In general, not all monomials in a multivariate polynomial $g \in \mathbb{R}[\boldsymbol{X}]$ of degree d are of the same degree. However, if $g = g_0 + \ldots + g_d$ is the unique decomposition of g into homogeneous components $g_0, \ldots, g_d$ as in Remark 2.2.10, then introducing a new indeterminant X_0 to the polynomial ring $\mathbb{R}[\boldsymbol{X}]$ allows us to raise the degree of g_i to d by multiplying with the monomial X_0^{d-i} for $i = 0, \ldots, d$. This procedure extends g into the $(n+1)$-ary d-ic

$$X_0^d g_0 + X_0^{d-1} g_1 + \ldots + X_0 g_{d-1} + g_d = X_0^d g\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right).$$

Definition 2.2.15. The *homogenization* of a multivariate polynomial $g \in \mathbb{R}[\boldsymbol{X}]$ of degree d is given by

$$g^h(X_0, \ldots, X_n) := X_0^d g\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right) \in \mathcal{F}_{n+1,d}.$$

Example 2.2.16. We saw in Example 2.2.7 (ii) that the cubic polynomial

$$g(X_1, X_2) := 5X_2^3 - 6X_1X_2^2 + X_1X_2 + 2X_1 \in \mathbb{R}[X_1, X_2]$$

is not a form since each of the monomials X_2^3 and $X_1X_2^2$ has degree three, while each of the monomials X_1X_2 and X_1 has degree less than three. However, introducing the indeterminant X_0 to the polynomial ring $\mathbb{R}[X_1, X_2]$ allows us to raise the degree of X_1 and X_1X_2 to degree three by multiplying with X_0^2 and X_0, respectively. This gives us the ternary cubic

$$\begin{aligned} g^h(X_0, X_1, X_2) &= X_0^3 g\left(\frac{X_1}{X_0}, \frac{X_2}{X_0}\right) \\ &= X_0^3 \left(5\frac{X_2^3}{X_0^3} - 6\frac{X_1X_2^2}{X_0^3} + \frac{X_1X_2}{X_0^2} + 2\frac{X_1}{X_0}\right) \\ &= 5X_2^3 - 6X_1X_2^2 + X_0X_1X_2 + 2X_1X_0^2. \end{aligned}$$

The homogenization of g thus coincides with the ternary cubic from Example 2.2.7 (i).

Proposition 2.2.17. *For $g \in \mathbb{R}[\boldsymbol{X}]$ of degree $2d$, the following are true:*

(i) g is PSD if and only if g^h is PSD.

(ii) g is SOS if and only if g^h is SOS.

Proof. See [Mar08, 1.2.4 Proposition]. ■

Notation 2.2.18. We set

$$\begin{aligned} \mathcal{P}_{n+1,2d} &:= \{f \in \mathcal{F}_{n+1,2d} \mid f \text{ is PSD}\} \\ &= \{f \in \mathcal{F}_{n+1,2d} \mid \forall x \in \mathbb{R}^{n+1} \colon f(x) \geq 0\}, \end{aligned}$$

$$\begin{aligned}\Sigma_{n+1,2d} &:= \{f \in \mathcal{F}_{n+1,2d} \mid f \text{ is SOS}\} \\ &= \left\{f \in \mathcal{F}_{n+1,2d} \mid \exists s \in \mathbb{N}\ \exists f_1, \ldots, f_s \in \mathcal{F}_{n+1,d} : f = \sum_{i=1}^{s} f_i^2\right\}.\end{aligned}$$

Remark 2.2.19. *Lemma 2.2.2 states that a SOS representation certifies the PSD property. Consequently,* $\Sigma_{n+1,2d} \subseteq \mathcal{P}_{n+1,2d}$.

Notation 2.2.20. We set

$$\Delta_{n+1,2d} := \mathcal{P}_{n+1,2d} \backslash \Sigma_{n+1,2d}.$$

Lemma 2.2.21. $\mathcal{P}_{n+1,2d}$ *and* $\Sigma_{n+1,2d}$ *are pointed full-dimensional closed cones that are the convex hull of their extreme rays.*[1]

Proof. A straight forward computation verifies the cone property for $\mathcal{P}_{n+1,2d}$ and $\Sigma_{n+1,2d}$, respectively. Furthermore, we observe that only the zero form lies in $\mathcal{P}_{n+1,2d}$ and $-\mathcal{P}_{n+1,2d}$. Hence, $\mathcal{P}_{n+1,2d}$, and with that also its subcone $\Sigma_{n+1,2d}$, is pointed.

In order to show that $\Sigma_{n+1,2d}$ is full-dimensional, it suffices to prove that any monomial in $\mathcal{F}_{n+1,2d}$ is a $\mathbb{R}$-linear combination of sums of squares. Indeed, for $m \in \mathcal{F}_{n+1,2d}$, we let m_1 and m_2 be monomials in $\mathcal{F}_{n+1,d}$ such that $m_1 m_2 = m$ and compute $\frac{1}{2}(m_1+m_2)^2 - \frac{1}{2}m_1^2 - \frac{1}{2}m_2^2 = m_1 m_2$. Thus, the smallest affine subspace that contains $\Sigma_{n+1,2d}$ is $\mathcal{F}_{n+1,2d}$ itself. So, also the smallest affine subspace that contains the larger (w.r.t. $\subseteq$) cone $\mathcal{P}_{n+1,2d}$ is $\mathcal{F}_{n+1,2d}$. Hence, $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ are full-dimensional.

The cones $\mathcal{P}_{n+1,2d}$ and $\Sigma_{n+1,2d}$ are furthermore closed (see [Raj93, p. 82], [Rob73]) and for $f_1, f_2 \in \mathcal{F}_{n+1,2d}$ such that f_2 is not the zero form, we can always fix some $x \in \mathbb{R}^{n+1}$ such that $f_2(x) \neq 0$. Moreover, choosing $\lambda \in \mathbb{R}$ appropriately such that $f_1(x) < (-\lambda) f_2(x)$, it thus follows $f_1(x) + \lambda f_2(x) < 0$, which shows $f_1 + \lambda f_2 \notin \mathcal{P}_{n+1,2d}$. Therefore, $\mathcal{P}_{n+1,2d}$, and with that also its subcone $\Sigma_{n+1,2d}$, contains no straight lines. Corollary A.3.17 therefore yields that $\mathcal{P}_{n+1,2d}$ and $\Sigma_{n+1,2d}$ are the convex hull of their extreme rays, respectively. ■

A thorough investigation of the cases $(n+1, 2d)$ such that $\Sigma_{n+1,2d} = \mathcal{P}_{n+1,2d}$ was carried out by David Hilbert in [Hil88].

Theorem 2.2.22. Hilbert's 1888 Theorem
It holds $\Sigma_{n+1,2d} = \mathcal{P}_{n+1,2d}$ *if and only if* $n = 1$ *or* $d = 1$ *or* $(n+1, 2d) = (3, 4)$.

We refer to the cases $(2, 2d)_{d \geq 1}$, $(n+1, 2)_{n \geq 1}$ and $(3, 4)$ as *Hilbert cases*. All other cases, we call *non-Hilbert cases*. The simplest non-Hilbert cases are $(4, 4)$ and $(3, 6)$, which we call the *basic non-Hilbert cases*.

[1] See Appendix A.3 for an introduction to convex geometry.

Proof (Sketch). It was known even before Hilbert's investigation from 1888 that $\Sigma_{n+1,2d} = \mathcal{P}_{n+1,2d}$ in the cases $(2,2d)_{d\geq 1}$ and $(n+1,2)_{n\geq 1}$. As a consequence of the fundamental theorem of algebra, it indeed follows that any PSD binary $2d$-ic is a sum of at most two squares of binary d-ics. Moreover, as a consequence of the spectral theorem for Hermitian matrices, it also follows that any PSD $(n+1)$-ary quadratic is a sum of at most n squares of $(n+1)$-ary linear forms.

Hilbert's proof for showing $\Sigma_{3,4} = \mathcal{P}_{3,4}$ is more sophisticated and uses the theory of algebraic curves, which in particular allowed him to show that any PSD ternary quartic is a sum of at most three ternary quadratics. A more accessible proof for $\Sigma_{3,4} = \mathcal{P}_{3,4}$ was given by Choi–Lam in [CL77, Section 6] and uses extremal forms in $\mathcal{P}_{3,4}$. However, their proof does not provide any insight on how many forms suffice in an SOS representation of a PSD ternary quartic.

In order to show $\Sigma_{n+1,2d} \subsetneq \mathcal{P}_{n+1,2d}$ in any non-Hilbert case, Hilbert demonstrated that there exist PSD forms that are not SOS in the basic non-Hilbert cases using Cayley–Bacharach relations. He then extended his consideration to any non-Hilbert case by the two arguments below:

(i) Given $f \in \mathcal{P}_{4,4}\backslash\Sigma_{4,4}$, we see that f can be interpreted as a $(n+1)$-ary quadric $\hat{f}$ for $n \geq 4$ that is PSD but not SOS. Proposition 2.2.14 thus implies that $X_0^{2d-4}\hat{f}$ is a $(n+1)$-ary $2d$-ic that is PSD but not SOS for $d \geq 3$.

(ii) Given $f \in \mathcal{P}_{3,6}\backslash\Sigma_{3,6}$, Proposition 2.2.14 yields that $X_0^{2d-6}f$ is a ternary $2d$-ic that is PSD but not SOS for $d \geq 4$. ■

In principle, Hilbert's method for proving $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$ and $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$ is constructive but "rather complicated and did not lend itself to a really practical construction" (see [CL77, p. 1]). So, it is not surprising that for nearly 80 years no examples of PSD forms that are not SOS were known. Independent of Hilbert's method, Motzkin [Mot65] was the first to present an example of a ternary sextic that is PSD but not SOS. That is, the Motzkin form

$$\mathsf{M}(X_0,X_1,X_2) := X_0^4X_1^2 + X_0^2X_1^4 + X_2^6 - 3X_0^2X_1^2X_2^2.$$

Shortly thereafter and unrelated to Motzkin's discovery, Robinson found an unpublished example of a very complicated ternary sextic by W. J. Ellison that was constructed using Hilbert's method. After seeing this, Robinson was able to drastically simplify Hilbert's construction which allowed him to present the two examples below of PSD forms that are not SOS in the basic non-Hilbert cases in [Rob73]:

$$\begin{aligned}
\mathsf{R}_1(X_0,X_1,X_2,X_3) &:= X_0^2(X_0-X_3)^2 + X_1^2(X_1-X_3)^2 + X_2^2(X_2-X_3)^2 \\
&\quad +2X_0X_1X_2(X_0+X_1+X_2-2X_3), \\
\mathsf{R}_2(X_0,X_1,X_2) &:= X_0^6 + X_1^6 + X_2^6 - (X_0^4X_1^2 + X_0^2X_1^4 + X_0^4X_2^2 + X_0^2X_2^4 \\
&\quad +X_1^4X_2^2 + X_1^2X_2^4) + 3X_0^2X_1^2X_2^2.
\end{aligned}$$

A few years later, Choi and Lam [CL76; CL77] succeeded to develope two further examples of forms in $\mathcal{P}_{n+1,2d}\backslash\Sigma_{n+1,2d}$ in the basic non-Hilbert cases using a slight variation of Motzkin's construction. These are

$$\begin{aligned}\mathsf{C}(X_0,X_1,X_2,X_3) &:= X_0^2X_1^2+X_0^2X_2^2+X_1^2X_2^2+X_3^4-4X_0X_1X_2X_3,\\ \mathsf{L}(X_0,X_1,X_2) &:= X_0^4X_1^2+X_0^2X_2^4+X_1^4X_2^2-3X_0^2X_1^2X_2^2.\end{aligned}$$

2.3 The Gram Matrix Method

A method proven to be beneficial for a systematic investigation of the cone $\Sigma_{n+1,2d}$ is the Gram matrix method, which was introduced by Choi, Lam and Reznick in [CLR92]. The crux of their method is that a SOS representation $f=\sum_{i=1}^{s} f_i^2$ of a form $f\in\Sigma_{n+1,2d}$ corresponds to a real symmetric positive semidefinite matrix whose entries come from the coefficients of the $(n+1)$-ary d-ics $f_1,\dots,f_s$. Applying the Gram matrix method, Powers amd Wörmann [PW98] presented an algorithm for deciding if a given form is SOS which provides a particular SOS representation in case of an affirmative answer.

Notation 2.3.1. For $k\geq 0$, we introduce the map

$$\begin{aligned}Q\colon \mathrm{Sym}_{k+1}(\mathbb{R}) &\to \mathcal{F}_{k+1,2}\\ A &\mapsto q_A(Z):=ZAZ^t.\end{aligned}$$

Proposition 2.3.2. *For $k\geq 0$, the map $Q\colon \mathrm{Sym}_{k+1}(\mathbb{R})\to\mathcal{F}_{k+1,2}$ is a well-defined $\mathbb{R}$-vector space isomorphism.*

Proof. Q is a well-defined map between $\mathrm{Sym}_{k+1}(\mathbb{R})$ and $\mathcal{F}_{k+1,2}$ since

$$q_A(Z)=ZAZ^t=\sum_{s,t=0}^{k} a_{s,t}Z_sZ_t\in\mathcal{F}_{k+1,2}$$

for $A:=(a_{s,t})_{0\leq s,t\leq k}\in\mathrm{Sym}_{k+1}(\mathbb{R})$. It thus remains to show linearity and bijectivity.

Linearity: For $A,B\in\mathrm{Sym}_{k+1}(\mathbb{R})$ and $\lambda\in\mathbb{R}$, we compute

$$Q(A+\lambda B)(Z)=Z(A+\lambda B)Z^t=ZAZ^t+\lambda(ZBZ^t)=Q(A)(Z)+\lambda Q(B)(Z).$$

Injectivity: It suffices to show that the kernel of Q is trivial since Q is linear. To this end, we let $A=(a_{s,t})_{0\leq s,t\leq k}\in\mathrm{Sym}_{k+1}(\mathbb{R})$ be such that $Q(A)(Z)=\sum_{s,t=0}^{k} a_{s,t}Z_sZ_t$ is the zero form and observe that

(i) $0=a_{s,s}$ for $s=0,\dots,k$ and

(ii) $0=a_{s,t}+a_{t,s}$ for $s,t=0,\dots,k$, $s\neq t$.

Since (i) states that each diagonal entry of A is zero, it thus remains to show that $a_{s,t} = 0$ for $s, t = 0, \ldots, k$, $s \neq t$. This follows from (ii) and the symmetry of A. Indeed, $0 = a_{s,t} + a_{t,s} = 2a_{s,t}$ yields $0 = a_{s,t}$ for $s, t = 0, \ldots, k$, $s \neq t$.

Surjectivity: We compute

$$\dim(\mathrm{Sym}_{k+1}(\mathbb{R})) = \frac{(k+1)(k+2)}{2} = \binom{k+2}{2} \underset{(2.1)}{=} \dim(\mathcal{F}_{k+1,2})$$

and conclude that $Q\colon \mathrm{Sym}_{k+1}(\mathbb{R}) \to \mathcal{F}_{k+1,2}$ is surjective since Q is injective. ■

Remark 2.3.3. *For $A := (a_{s,t})_{0 \leq s,t \leq k} \in \mathrm{Sym}_{k+1}(\mathbb{R})$, we observed above that*

$$Q(A)(Z) = q_A(Z) = \sum_{s,t=0}^{k} a_{s,t} Z_s Z_t. \tag{2.2}$$

Notation 2.3.4. We introduce the well-defined binary map

$$\begin{array}{rcccl} k\colon \; \mathbb{N} \times \mathbb{N} & \to & \mathbb{N} & & \\ (n,d) & \mapsto & k(n,d) & := & \dim(\mathcal{F}_{n+1,d}) - 1 \\ & & & \underset{(2.1)}{=} & \binom{n+d}{n} - 1 = \binom{n+d}{d} - 1 \end{array}$$

and write k instead of $k(n,d)$ whenever n and d are clear from the context.

Lemma 2.3.5. *The following are true:*

(i) $k(1,d) = d$.

(ii) $k(n,1) = n$.

(iii) $k(n,d) < k(n,d+1)$.

(iv) $k(n,d) < k(n+1,d)$.

Proof. We compute

(i) $k(1,d) = \binom{1+d}{1} - 1 = (d+1) - 1 = d$,

(ii) $k(n,1) = \binom{n+1}{n} - 1 = (n+1) - 1 = n$,

(iii) $k(n,d) = \binom{n+d}{n} - 1 < \binom{n+(d+1)}{n} - 1 = k(n,d+1)$,

(iv) $k(n,d) = \binom{n+d}{d} - 1 < \binom{(n+1)+d}{d} - 1 = k(n+1,d)$. ■

Corollary 2.3.6. *If $d \geq 2$, then $k(n,d) > n$.*

Proof. Lemma 2.3.5 (ii) and (iii) together yield $k(n,d) > k(n,1) = n$. ∎

Construction 2.3.7. (1) We order $I_{n+1,d}$ lexicographically starting with the greatest element. This gives us the ordered set $\{\alpha_0, \ldots, \alpha_k\}$ and, for $i = 0, \ldots, k$, we write $\alpha_i = (\alpha_{i,0}, \ldots, \alpha_{i,n}) \in \mathbb{N}_0^{n+1}$.[2]

(2) For $i = 0, \ldots, k$, we set $m_i(X) := X^{\alpha_i}$. This gives us the ordered monomial basis $\{m_0, \ldots, m_k\}$ of $\mathcal{F}_{n+1,d}$.

Remark 2.3.8. *In the above construction, we chose to order $I_{n+1,d}$ lexicographically. This choice of a monomial order is crucial for many proofs of this thesis and may not be changed.*

Example 2.3.9. Let us follow Construction 2.3.7 in the Hilbert cases.

(i) Binary Forms
Let $n = 1$ and $d \geq 1$. Lemma 2.3.5 (i) gives $k = k(1,d) = d$ and, for $i = 0, \ldots, d$, we compute $\alpha_i = (d-i, i)$. Hence, $m_i(X_0, X_1) = X_0^{d-i} X_1^i$.

(ii) Quadratic Forms
Let $n \geq 1$ and $d = 1$. Lemma 2.3.5 (ii) gives $k = k(n,1) = n$ and we observe for $i = 0, \ldots, n$ that $\alpha_i = (\alpha_{i,0}, \ldots, \alpha_{i,n})$ with

$$\alpha_{i,s} := \begin{cases} 1, & \text{if } s = i \\ 0, & \text{else.} \end{cases}$$

Hence, $m_i(X_0, \ldots, X_n) = X_i$.

(iii) Ternary Quartics
Let $n = 2$ and $d = 2$. We compute $k = k(2,2) = \binom{2+2}{2} - 1 = 5$ and observe that the lexicographically ordered set $\{\alpha_0, \ldots, \alpha_5\}$ is given by

$$\begin{aligned} \alpha_0 &= (2,0,0), & \alpha_2 &= (1,0,1), & \alpha_4 &= (0,1,1), \\ \alpha_1 &= (1,1,0), & \alpha_3 &= (0,2,0), & \alpha_5 &= (0,0,2). \end{aligned}$$

Hence, the ordered monomial basis $\{m_0, \ldots, m_5\}$ of $\mathcal{F}_{3,2}$ is defined by

$$\begin{aligned} m_0(X) &= X_0^2, & m_2(X) &= X_0X_2, & m_4(X) &= X_1X_2, \\ m_1(X) &= X_0X_1, & m_3(X) &= X_1^2, & m_5(X) &= X_2^2. \end{aligned}$$

[2] That is, we set $\alpha <_{\text{lex}} \beta$ for $\alpha, \beta \in I_{n+1,d}$ if the first non-zero entry of $\beta - \alpha \in \mathbb{Z}^{n+1}$ is positive (cf. Definition A.1.61) and obtain $\alpha_0 >_{\text{lex}} \ldots >_{\text{lex}} \alpha_k$.

Example 2.3.10. Let us follow Construction 2.3.7 in the basic non-Hilbert cases.

(i) QUATERNARY QUARTICS
Let $n = 3$ and $d = 2$. We compute $k = k(3,2) = \binom{3+2}{3} - 1 = 9$ and observe that the lexicographically ordered set $\{\alpha_0, \ldots, \alpha_9\}$ is given by

$$\begin{array}{lll} \alpha_0 = (2,0,0,0), & \alpha_4 = (0,2,0,0), & \alpha_8 = (0,0,1,1), \\ \alpha_1 = (1,1,0,0), & \alpha_5 = (0,1,1,0), & \alpha_9 = (0,0,0,2). \\ \alpha_2 = (1,0,1,0), & \alpha_6 = (0,1,0,1), & \\ \alpha_3 = (1,0,0,1), & \alpha_7 = (0,0,2,0), & \end{array}$$

Hence, the ordered monomial basis $\{m_0, \ldots, m_9\}$ of $\mathcal{F}_{4,2}$ is defined by

$$\begin{array}{lll} m_0(X) = X_0^2, & m_4(X) = X_1^2, & m_8(X) = X_2X_3, \\ m_1(X) = X_0X_1, & m_5(X) = X_1X_2, & m_9(X) = X_3^2. \\ m_2(X) = X_0X_2, & m_6(X) = X_1X_3, & \\ m_3(X) = X_0X_3, & m_7(X) = X_2^2, & \end{array}$$

(ii) TERNARY SEXTICS
Let $n = 2$ and $d = 3$. We compute $k = k(2,3) = \binom{2+3}{2} - 1 = 9$ and observe that the lexicographically ordered set $\{\alpha_0, \ldots, \alpha_9\}$ is given by

$$\begin{array}{lll} \alpha_0 = (3,0,0), & \alpha_4 = (1,1,1), & \alpha_8 = (0,1,2), \\ \alpha_1 = (2,1,0), & \alpha_5 = (1,0,2), & \alpha_9 = (0,0,3). \\ \alpha_2 = (2,0,1), & \alpha_6 = (0,3,0), & \\ \alpha_3 = (1,2,0), & \alpha_7 = (0,2,1), & \end{array}$$

Hence, the ordered monomial basis $\{m_0, \ldots, m_9\}$ of $\mathcal{F}_{3,3}$ is defined by

$$\begin{array}{lll} m_0(X) = X_0^3, & m_4(X) = X_0X_1X_2, & m_8(X) = X_1X_2^2, \\ m_1(X) = X_0^2X_1, & m_5(X) = X_0X_2^2, & m_9(X) = X_2^3. \\ m_2(X) = X_0^2X_2, & m_6(X) = X_1^3, & \\ m_3(X) = X_0X_1^2, & m_7(X) = X_1^2X_2, & \end{array}$$

Definition 2.3.11. The *Gram map* is given by

$$\begin{array}{rcl} \mathcal{G}\colon \mathrm{Sym}_{k+1}(\mathbb{R}) & \to & \mathcal{F}_{n+1,2d} \\ A & \mapsto & f_A(X) := q_A(m_0(X), \ldots, m_k(X)). \end{array}$$

Lemma 2.3.12. *The Gram map is a well-defined surjective linear map.*

Proof. For $A := (a_{s,t})_{0\leq s,t\leq k} \in \mathrm{Sym}_{k+1}(\mathbb{R})$, we observe

$$f_A(X) = q_A(m_0(X), \dots, m_k(X)) \underset{(2.2)}{=} \sum_{s,t=0}^{k} a_{s,t} m_s(X) m_t(X) \in \mathcal{F}_{n+1,2d}.$$

Hence, $\mathcal{G}$ is well-defined and it remains to verify that $\mathcal{G}$ is linear and surjective.

Linearity: For $A, B \in \mathrm{Sym}_{k+1}(\mathbb{R})$ and $\lambda \in \mathbb{R}$, we recall from Proposition 2.3.2 that Q is linear and compute

$$\begin{aligned}
\mathcal{G}(A+\lambda B)(X) &= q_{A+\lambda B}(m_0(X), \dots, m_k(X)) \\
&= Q(A+\lambda B)(m_0(X), \dots, m_k(X)) \\
&= Q(A)(m_0(X), \dots, m_k(X)) + \lambda Q(B)(m_0(X), \dots, m_k(X)) \\
&= q_A(m_0(X), \dots, m_k(X)) + \lambda q_B(m_0(X), \dots, m_k(X)) \\
&= \mathcal{G}(A)(X) + \lambda \mathcal{G}(B)(X).
\end{aligned}$$

Surjectivity:[3] For $\beta \in I_{n+1,2d}$, we choose $s_\beta, t_\beta \in \{0, \dots, k\}$ such that

$$X^\beta = m_{s_\beta}(X) m_{t_\beta}(X)$$

and set $E_{s_\beta t_\beta} := (e_{s,t})_{0\leq s,t\leq k}$ to be the $(k+1)\times(k+1)$ matrix defined by

$$e_{s,t} = \begin{cases} 1, & \text{for } (s,t) = (s_\beta, t_\beta) \\ 0, & \text{else.} \end{cases}$$

The $(k+1)\times(k+1)$ symmetric matrix

$$A_{s_\beta t_\beta} := \frac{1}{2}\left(E_{s_\beta t_\beta} + E_{s_\beta t_\beta}^t\right) \tag{2.3}$$

with real entries thus satisfies

$$\begin{aligned}
\mathcal{G}(A_{s_\beta t_\beta})(X) &= q_{A_{s_\beta t_\beta}}(m_0(X), \dots, m_k(X)) \\
&= (m_0(X) \dots m_k(X)) A_{s_\beta t_\beta} (m_0(X) \dots m_k(X))^t \\
&= (m_0(X) \dots m_k(X)) \left(\frac{1}{2}\left(E_{s_\beta t_\beta} + E_{s_\beta t_\beta}^t\right)\right) (m_0(X) \dots m_k(X))^t \\
&= m_{s_\beta}(X) m_{t_\beta}(X) = X^\beta.
\end{aligned} \tag{2.4}$$

Moreover, for $f \in \mathcal{F}_{n+1,2d}$, we have $f(X) = \sum_{\beta \in I_{n+1,2d}} f_\beta X^\beta$ for appropriate $f_\beta \in \mathbb{R}$ $(\beta \in I_{n+1,2d})$ by Lemma 2.2.9. Hence, we set

$$A_f := \sum_{\beta \in I_{n+1,2d}} f_\beta A_{s_\beta t_\beta}$$

[3] For this proof, we apply the ideas of [Fid09, Theorem 2.4.3.] to forms.

and observe that A_f is a $\mathbb{R}$-linear combination of $(k+1)\times(k+1)$ symmetric matrices with real entries. So, $A_f \in \mathrm{Sym}_{k+1}(\mathbb{R})$ and, using the linearity of $\mathcal{G}$, we compute

$$\begin{aligned}\mathcal{G}(A_f)(X) &= \mathcal{G}\left(\sum_{\beta\in I_{n+1,2d}} f_\beta A_{s_\beta t_\beta}\right)(X)\\ &= \left(\sum_{\beta\in I_{n+1,2d}} f_\beta \mathcal{G}\left(A_{s_\beta t_\beta}\right)\right)(X)\\ &= \sum_{\beta\in I_{n+1,2d}} f_\beta \left(\mathcal{G}\left(A_{s_\beta t_\beta}\right)(X)\right)\\ &\underset{(2.4)}{=} \sum_{\beta\in I_{n+1,2d}} f_\beta X^\beta = f(X). \qquad \blacksquare\end{aligned}$$

The surjectivity of $\mathcal{G}$ in particular guarantees $\mathcal{G}^{-1}(f)\neq\emptyset$ for any $f\in\mathcal{F}_{n+1,2d}$.

Definition 2.3.13. For $f\in\mathcal{F}_{n+1,2d}$, $A\in\mathcal{G}^{-1}(f)$ is a *Gram matrix* associated to f.

Remark 2.3.14. *The preimage $\mathcal{G}^{-1}(f)$ is the coset of any a priori fixed Gram matrix A_f associated to f w.r.t. the kernel of the linear Gram map.*

Lemma 2.3.15. *For $f(X)=\sum_{\beta\in I_{n+1,2d}} f_\beta X^\beta\in\mathcal{F}_{n+1,2d}$, $A=(a_{s,t})_{0\le s,t\le k}\in\mathrm{Sym}_{k+1}(\mathbb{R})$ is a Gram matrix associated to f if and only if for any $\beta\in I_{n+1,2d}$, it holds*

$$f_\beta = \sum_{\alpha_s+\alpha_t=\beta} a_{s,t}. \tag{2.5}$$

Proof. We compute

$$\mathcal{G}(A)(X) = \sum_{s,t=0}^{k} a_{s,t} m_s(X) m_t(X). \tag{2.6}$$

($\Rightarrow$) If A is a Gram matrix associated to f, then

$$\begin{aligned}\sum_{\beta\in I_{n+1,2d}} f_\beta X^\beta &= \mathcal{G}(A)(X)\\ &\underset{(2.6)}{=} \sum_{s,t=0}^{k} a_{s,t} m_s(X) m_t(X)\\ &= \sum_{s,t=0}^{k} a_{s,t} X^{\alpha_s+\alpha_t}\\ &= \sum_{\beta\in I_{n+1,2d}} \left(\sum_{\alpha_s+\alpha_t=\beta} a_{s,t}\right) X^\beta.\end{aligned}$$

Hence, for $\beta\in I_{n+1,2d}$, we conclude $f_\beta=\sum_{\alpha_s+\alpha_t=\beta} a_{s,t}$ by comparision of coefficients.

($\Leftarrow$) If $f_\beta = \sum_{\alpha_s+\alpha_t=\beta} a_{s,t}$ for any $\beta \in I_{n+1,2d}$, then

$$\begin{aligned}\mathcal{G}(A)(X) &\underset{(2.6)}{=} \sum_{s,t=0}^{k} a_{s,t} m_s(X) m_t(X) \\ &= \sum_{s,t=0}^{k} a_{s,t} X^{\alpha_s+\alpha_t} \\ &= \sum_{\beta \in I_{n+1,2d}} \left(\sum_{\alpha_s+\alpha_t=\beta} a_{s,t} \right) X^\beta \\ &= \sum_{\beta \in I_{n+1,2d}} f_\beta X^\beta = f(X).\end{aligned}$$ ■

Example 2.3.16. Let us determine a Gram matrix associated to

(i) the Choi–Lam quaternary quartic

$$\mathsf{C}(X_0, X_1, X_2, X_3) = X_0^2 X_1^2 + X_0^2 X_2^2 + X_1^2 X_2^2 + X_3^4 - 4X_0X_1X_2X_3,$$

(ii) the Choi–Lam ternary sextic

$$\mathsf{L}(X_0, X_1, X_2) := X_0^4 X_1^2 + X_0^2 X_2^4 + X_1^4 X_2^2 - 3X_0^2 X_1^2 X_2^2,$$

(iii) the Motzkin ternary sextic

$$\mathsf{M}(X_0, X_1, X_2) = X_0^4 X_1^2 + X_0^2 X_1^4 + X_2^6 - 3X_0^2 X_1^2 X_2^2.$$

Recalling Example 2.3.10, we see that

(i) $\mathsf{C} = m_1^2 + m_2^2 + m_5^2 + m_9^2 - 4m_2m_6$,

(ii) $\mathsf{L} = m_1^2 + m_5^2 + m_7^2 - 3m_4^2$,

(iii) $\mathsf{M} = m_1^2 + m_3^2 + m_6^2 - 3m_4^2$.

Lemma 2.3.15 hence yields that associated Gram matrices are given by

(i) $A_{\mathsf{C}} = (a_{s,t})_{0\le s,t\le 9} \in \mathcal{G}^{-1}(\mathsf{C})$ with

$$a_{s,t} := \begin{cases} 1, & \text{for } (s,t) \in \{(1,1),(2,2),(5,5),(9,9)\} \\ -2, & \text{for } \{s,t\} = \{2,6\} \\ 0, & \text{else,} \end{cases}$$

(ii) $A_{\mathsf{L}} = (a_{s,t})_{0\leq s,t\leq 9} \in \mathcal{G}^{-1}(\mathsf{L})$ with

$$a_{s,t} := \begin{cases} 1, & \text{for } (s,t) \in \{(1,1),(5,5),(7,7)\} \\ -3, & \text{for } (s,t) = (4,4) \\ 0, & \text{else,} \end{cases}$$

(iii) $A_{\mathsf{M}} = (a_{s,t})_{0\leq s,t\leq 9} \in \mathcal{G}^{-1}(\mathsf{M})$ with

$$a_{s,t} := \begin{cases} 1, & \text{for } (s,t) \in \{(1,1),(3,3),(6,6)\} \\ -3, & \text{for } (s,t) = (4,4) \\ 0, & \text{else.} \end{cases}$$

We recall that any form in $(n+1)$ variables can be interpreted as a form in $(m+1)$ variables for $m \geq n$. A similar observation can be made for Gram matrices.

Lemma 2.3.17. *For $n \leq m$, $0 \leq i_0 < \ldots < i_n \leq m$ and $f \in \mathcal{F}_{n+1,2d}$, let*

$$g(X_0,\ldots,X_m) := f(X_{i_0},\ldots,X_{i_n}) \in \mathcal{F}_{m+1,2d}$$

and set $I := \{i \in \{0,\ldots,k(m,d)\} \mid \alpha_{i,s} = 0 \text{ for } s \neq i_0,\ldots,i_n\}$.

(i) For $A \in \mathcal{G}^{-1}(f)$ expand A to a matrix $B := (b_{i,j})_{0\leq i,j\leq k(m,d)} \in \mathrm{Sym}_{k(m,d)+1}(\mathbb{R})$ such that $B_I := (b_{i,j})_{i,j\in I} = A$ and all other entries are zero, then $B \in \mathcal{G}^{-1}(g)$.

(ii) For $B := (b_{i,j})_{0\leq i,j\leq k(m,d)} \in \mathcal{G}^{-1}(g)$ set $A := B_I \in \mathrm{Sym}_{k(n,d)+1}(\mathbb{R})$ to be the submatrix of B that is given by $B_I := (b_{i,j})_{i,j\in I}$, then $A \in \mathcal{G}^{-1}(f)$.

Proof. For the purpose of this proof, we write $X^{(N)}$ for the vector of indeterminants $(X_0,\ldots,X_N)$, set $k(N) := k(N,d)$ and, for $i = 0,\ldots,k(N)$, denote the monomial m_i in $\mathcal{F}_{N+1,d}$ by $m_i^{(N)}$ for $N \in \{n,m\}$.

(i) We set $X' := (X_{i_0},\ldots,X_{i_n})$ and, using $A \in \mathcal{G}^{-1}(f)$, compute

$$\begin{aligned}
&\mathcal{G}(B)\left(X^{(m)}\right) \\
&= q_B\left(m_0^{(m)}\left(X^{(m)}\right),\ldots,m_{k(m)}^{(m)}\left(X^{(m)}\right)\right) \\
&= \left(m_0^{(m)}\left(X^{(m)}\right),\ldots,m_{k(m)}^{(m)}\left(X^{(m)}\right)\right) B \left(m_0^{(m)}\left(X^{(m)}\right),\ldots,m_{k(m)}^{(m)}\left(X^{(m)}\right)\right)^t \\
&= \left(m_i^{(m)}\left(X^{(m)}\right)\right)_{i\in I} B_I \left(m_i^{(m)}\left(X^{(m)}\right)\right)_{i\in I}^t \\
&= \left(m_0^{(n)}\left(X'\right),\ldots,m_{k(n)}^{(n)}\left(X'\right)\right) A \left(m_0^{(n)}\left(X'\right),\ldots,m_{k(n)}^{(n)}\left(X'\right)\right)^t \\
&= q_A\left(m_0^{(n)}\left(X'\right),\ldots,m_{k(n)}^{(n)}\left(X'\right)\right) \\
&= \mathcal{G}(A)\left(X'\right) = f\left(X'\right) = g\left(X^{(m)}\right).
\end{aligned}$$

(ii) We set $X' := (X'_0, \dots, X'_m)$ to be given by

$$X'_i = \begin{cases} X_i, & \text{for } i \in I \\ 0, & \text{else} \end{cases}$$

and, using $B \in \mathcal{G}^{-1}(g)$, compute

$$\begin{aligned}
&\mathcal{G}(A)\left(X^{(n)}\right)\\
&= q_A\left(m_0^{(n)}\left(X^{(n)}\right), \dots, m_{k(n)}^{(n)}\left(X^{(n)}\right)\right)\\
&= \left(m_0^{(n)}\left(X^{(n)}\right), \dots, m_{k(n)}^{(n)}\left(X^{(n)}\right)\right) A \left(m_0^{(n)}\left(X^{(n)}\right), \dots, m_{k(n)}^{(n)}\left(X^{(n)}\right)\right)^t\\
&= \left(m_i^{(n)}\left(X^{(n)}\right)\right)_{i\in I} B_I \left(m_i^{(n)}\left(X^{(n)}\right)\right)_{i\in I}^t\\
&= \left(m_0^{(m)}\left(X'\right), \dots, m_{k(m)}^{(m)}\left(X'\right)\right) B \left(m_0^{(m)}\left(X'\right), \dots, m_{k(m)}^{(m)}\left(X'\right)\right)^t\\
&= q_B\left(m_0^{(m)}\left(X'\right), \dots, m_{k(m)}^{(m)}\left(X'\right)\right)\\
&= \mathcal{G}(B)\left(X'\right) = g\left(X'\right) = f\left(X^{(n)}\right). \qquad \blacksquare
\end{aligned}$$

Example 2.3.18. Let us consider the binary quartic

$$f(X_0, X_1) = X_0^4 + X_0^2X_1^2 - 2X_0X_1^3$$

and interpret f as the quaternary quartic

$$g(X_0, X_1, X_2, X_3) := f(X_1, X_3) = X_1^4 + X_1^2X_3^2 - 2X_1X_3^3.$$

Hence, $n = 1$, $m = 3$, $d = 2$ and $(i_0, i_1) = (1, 3)$ in the notations of Lemma 2.3.17.

(i) We compute $k(1, 2) = \binom{1+2}{1} - 1 = 2$ and determine the ordered monomial basis $\{m_0^{(1)}, m_1^{(1)}, m_2^{(1)}\}$ of $\mathcal{F}_{2,2}$ to consist of the monomials $m_0^{(1)}(X_0, X_1) = X_0^2$, $m_1^{(1)}(X_0, X_1) = X_0X_1$ and $m_2^{(1)}(X_0, X_1) = X_1^2$. Consequently,

$$f = \left(m_0^{(1)}\right)^2 + \left(m_1^{(1)}\right)^2 - 2m_1^{(1)}m_2^{(1)}$$

and thus, by Lemma 2.3.15, a Gram matrix associated to f is given by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Recalling Example 2.3.10 (i), we compute $I = \{4, 6, 9\}$ in the notation of Lemma 2.3.17 (i) and expand A to the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Sym}_{10}(\mathbb{R}).$$

For $i = 0, \ldots, 9$, we set $m_i^{(3)} := m_i \in \mathcal{F}_{4,2}$ (cf. Example 2.3.10 (i)) and compute

$$\begin{aligned} \mathcal{G}(B)(X_0, X_1, X_2, X_3) \underset{(2.6)}{=} & \left(m_4^{(3)}(X_0, X_1, X_2, X_3)\right)^2 + \left(m_6^{(3)}(X_0, X_1, X_2, X_3)\right)^2 \\ & -2m_6^{(3)}(X_0, X_1, X_2, X_3)m_9^{(3)}(X_0, X_1, X_2, X_3) \\ = & X_1^4 + X_1^2X_3^2 - 2X_1X_3^3 = g(X_0, X_1, X_2, X_3). \end{aligned}$$

(ii) Remaining in the notations of (i), we observe

$$g = \left(m_4^{(3)}\right)^2 + \left(m_6^{(3)}\right)^2 + 2m_5^{(3)}m_6^{(3)} - 2m_4^{(3)}m_8^{(3)} - 2m_6^{(3)}m_9^{(3)}$$

and thus, by Lemma 2.3.15, a Gram matrix associated to g is given by

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Sym}_{10}(\mathbb{R}).$$

Following Lemma 2.3.17 (ii), we consequently reduce B to the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \in \mathcal{G}^{-1}(f).$$

We recall that any form of degree $2d$ can be transformed into a form of degree 2δ for $\delta \geq d$ by multiplying with the square monomial $X_0^{2(\delta-d)}$. A similar observation can be made for Gram matrices.

Lemma 2.3.19. *For $\delta \geq d$, $f \in \mathcal{F}_{n+1,2d}$ and $A \in \mathcal{G}^{-1}(f)$, let*

$$g(X) := X_0^{2(\delta-d)} f(X) \in \mathcal{F}_{n+1,2\delta}$$

and set $B := (b_{s,t})_{0\leq s,t\leq k(n,\delta)} \in \mathrm{Sym}_{k(n,\delta)+1}(\mathbb{R})$ to be given by

$$b_{s,t} := \begin{cases} a_{s,t}, & \text{for } s,t = 0,\dots,k(n,d) \\ 0, & \text{else,} \end{cases}$$

then $B \in \mathcal{G}^{-1}(g)$.

Proof. For the purpose of this proof, we set $k(D) := k(n,D)$ and, for $i = 0,\dots,k(n,D)$, denote the monomial m_i in $\mathcal{F}_{n+1,D}$ by $m_i^{(D)}$for $D \in \{d,\delta\}$. We furthermore observe $m_i^{(\delta)}(X) = X_0^{\delta-d} m_i^{(d)}(X)$ for $i = 0,\dots,k(d)$ and, using $A \in \mathcal{G}^{-1}(f)$, conclude

$$\begin{aligned}
\mathcal{G}(B)(X) &= q_B\left(m_0^{(\delta)}(X),\dots,m_{k(\delta)}^{(\delta)}(X)\right) \\
&= \left(m_0^{(\delta)}(X),\dots,m_{k(\delta)}^{(\delta)}(X)\right) B \left(m_0^{(\delta)}(X),\dots,m_{k(\delta)}^{(\delta)}(X)\right)^t \\
&= \left(m_0^{(\delta)}(X),\dots,m_{k(d)}^{(\delta)}(X)\right) (b_{s,t})_{0\leq s,t\leq k(d)} \left(m_0^{(\delta)}(X),\dots,m_{k(d)}^{(\delta)}(X)\right)^t \\
&= X_0^{2(\delta-d)} \left(m_0^{(d)}(X),\dots,m_{k(d)}^{(d)}(X)\right) A \left(m_0^{(d)}(X),\dots,m_{k(d)}^{(d)}(X)\right)^t \\
&= X_0^{2(\delta-d)} q_A\left(m_0^{(d)}(X),\dots,m_{k(d)}^{(d)}(X)\right) = X_0^{2(\delta-d)} f(X) = g(X).
\end{aligned}$$

■

Example 2.3.20. Let us transform the binary quartic

$$f(X_0,X_1) = X_0^4 + X_0^2X_1^2 - 2X_0X_1^3$$

from Example 2.3.18 into a binary octic by setting

$$g(X_0,X_1) := X_0^4 f(X_0,X_1) = X_0^8 + X_0^6X_1^2 - 2X_0^5X_1^3.$$

Hence, $n = 1$, $d = 2$ and $\delta = 4$ in the notation of Lemma 2.3.19 and we compute $k(n,\delta) = k(1,4) = \binom{1+4}{1} - 1 = 4$. Moreover, we know that

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

is a Gram matrix associated to f by Example 2.3.18. Following Lemma 2.3.19, we thus expand A to the matrix

$$B := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Sym}_5(\mathbb{R}).$$

The ordered monomial basis $\{m_0, \ldots, m_4\}$ of $\mathcal{F}_{2,4}$ is furthermore given by

$$\begin{aligned} m_0(X_0, X_1) &= X_0^4, & m_2(X_0, X_1) &= X_0^2 X_1^2, & m_4(X_0, X_1) &= X_1^4. \\ m_1(X_0, X_1) &= X_0^3 X_1, & m_3(X_0, X_1) &= X_0 X_1^3, \end{aligned}$$

Therefore, we conclude

$$\begin{aligned} \mathcal{G}(B)(X_0, X_1) &\underset{(2.6)}{=} m_0(X_0, X_1)^2 + m_1(X_0, X_1)^2 - 2m_1(X_0, X_1)m_2(X_0, X_1) \\ &= X_0^8 + X_0^6 X_1^2 - 2X_0^5 X_1^3 = g(X_0, X_1). \end{aligned}$$

The surjectivity of the Gram map ensures that there exists a Gram matrix associated to any a priori fixed form. The uniqueness of associated Gram matrices however is guaranteed if and only if $\mathcal{G}\colon \mathrm{Sym}_{k+1}(\mathbb{R}) \to \mathcal{F}_{n+1,2d}$ is injective.

Lemma 2.3.21. *The Gram map is injective if and only if $d = 1$.*

Proof. ($\Rightarrow$) We assume $d \geq 2$ for a proof by contraposition and especially compute $\alpha_0 = (d, 0, \ldots, 0)$, $\alpha_1 = (d-1, 1, 0, \ldots, 0)$ and $\alpha_{k(n,d-2)+1} = (1, d-1, 0, \ldots, 0)$. For the two distinct matrices $A := (a_{s,t})_{0 \leq s,t \leq k}$, $B := (b_{s,t})_{0 \leq s,t \leq k} \in \mathrm{Sym}_{k+1}(\mathbb{R})$ given by

$$\begin{aligned} a_{s,t} &= \begin{cases} 1, & \text{for } (s,t) = (0,0) \\ 0, & \text{else,} \end{cases} \\ b_{s,t} &= \begin{cases} 1, & \text{for } \{s,t\} = \{1, k(n, d-2)+1\} \\ 0 & \text{else,} \end{cases} \end{aligned}$$

we thus obtain $\mathcal{G}(A)(X) = X_0^{2d} = \mathcal{G}(B)(X)$ by (2.6). Hence, $\mathcal{G}$ is not injectiv.

($\Leftarrow$) We compute $k = k(n,d) = k(n,1) = \binom{n+1}{n} - 1 = n$ and recall $m_i(X) = X_i$ for $i = 0, \ldots, n$ from Example 2.3.9 (ii). Therefore, the Gram map

$$\begin{aligned} \mathcal{G}\colon \mathrm{Sym}_{n+1}(\mathbb{R}) &\to \mathcal{F}_{n+1,2} \\ A &\mapsto f_A(X) = q_A(m_0(X), \ldots, m_n(X)) = q_A(X_0, \ldots, X_n) \end{aligned}$$

coincides with the $\mathbb{R}$-vector space isomorphism (cf. Proposition 2.3.2)

$$\begin{aligned} Q\colon \mathrm{Sym}_{n+1}(\mathbb{R}) &\to \mathcal{F}_{n+1,2} \\ A &\mapsto q_A(Z_0, \ldots, Z_n). \end{aligned}$$

∎

Example 2.3.22. In Example 2.3.16, we determined a Gram matrix

(i) A_{C} associated to $\mathsf{C} = m_1^2 + m_2^2 + m_5^2 + m_9^2 - 4m_2m_6$,

(ii) A_{L} associated to $\mathsf{L} = m_1^2 + m_5^2 + m_7^2 - 3m_4^2$,

(iii) A_{M} associated to $\mathsf{M} = m_1^2 + m_3^2 + m_6^2 - 3m_4^2$.

The above given expressions of the forms C, L and M via the monomials $m_0, \ldots, m_{k(n,d)}$ are not unique. Indeed, using our computations from Example 2.3.10, we also see

(i) $\mathsf{C} = m_0m_4 + m_0m_7 + m_4m_7 + m_9^2 - 4m_1m_8$,

(ii) $\mathsf{L} = m_0m_3 + m_2m_9 + m_6m_8 - 3m_1m_8$,

(iii) $\mathsf{M} = m_0m_3 + m_1m_6 + m_6^2 - 3m_1m_8$.

Applying Lemma 2.3.15, we hence obtain further associated Gram matrices

(i) $A = (a_{s,t})_{0\le s,t\le 9} \in \mathcal{G}^{-1}(\mathsf{C})$ with

$$a_{s,t} := \begin{cases} \frac{1}{2}, & \text{for } \{s,t\} \in \{\{0,4\},\{0,7\},\{4,7\}\} \\ 1, & \text{for } (s,t) = (9,9) \\ -2, & \text{for } \{s,t\} = \{1,8\} \\ 0, & \text{else,} \end{cases}$$

(ii) $A = (a_{s,t})_{0\le s,t\le 9} \in \mathcal{G}^{-1}(\mathsf{L})$ with

$$a_{s,t} := \begin{cases} \frac{1}{2}, & \text{for } \{s,t\} \in \{\{0,3\},\{2,9\},\{6,8\}\} \\ -\frac{3}{2}, & \text{for } \{s,t\} = \{1,8\} \\ 0, & \text{else,} \end{cases}$$

(iii) $A = (a_{s,t})_{0\le s,t\le 9} \in \mathcal{G}^{-1}(\mathsf{M})$ with

$$a_{s,t} := \begin{cases} \frac{1}{2}, & \text{for } \{s,t\} \in \{\{0,3\},\{1,6\}\} \\ 1, & \text{for } (s,t) = (6,6) \\ -\frac{3}{2}, & \text{for } \{s,t\} = \{1,8\} \\ 0, & \text{else.} \end{cases}$$

In a non-Hilbert case $(n+1, 2d)$, Hilbert's 1888 theorem implies $d \ge 2$. Therefore, the Gram map is not injective in non-Hilbert cases by Lemma 2.3.21. This is equivalent to the kernel of $\mathcal{G}$ being non-trivial in non-Hilbert cases. Using the Veronese embedding introduced below, we can characterize the kernel of $\mathcal{G}$.

Definition 2.3.23. The *(projective) Veronese embedding* is given by

$$\begin{aligned} V\colon \mathbb{P}^n &\to \mathbb{P}^k \\ [x] &\mapsto [m_0(x) : \ldots : m_k(x)]. \end{aligned}$$

Lemma 2.3.24. *The Veronese embedding is a well-defined embedding.*

Proof. For $[x], [x'] \in \mathbb{P}^n$ such that $[x] = [x']$, we fix $\lambda \in \mathbb{C}^\times$ such that $x = \lambda x'$ and observe $\lambda^d \in \mathbb{C}^\times$. Lemma 2.2.11 thus implies

$$(m_0(x), \ldots, m_k(x)) = (m_0(\lambda x'), \ldots, m_k(\lambda x')) = \lambda^d (m_0(x'), \ldots, m_k(x')).$$

So, $[m_0(x) : \ldots : m_k(x)] = [m_0(x') : \ldots : m_k(x')]$ and it remains to verify injectivity.

Injectivity: If $d = 1$, then V is the identity map on $\mathbb{P}^n$ (cf. Example 2.3.25 (ii) below). However, if $d \geq 2$, then we let $[x], [x'] \in \mathbb{P}^n$ be such that

$$[m_0(x) : \ldots : m_k(x)] = V([x]) = V([x']) = [m_0(x') : \ldots : m_k(x')]$$

and fix $\lambda \in \mathbb{C}^\times$ such that $(m_0(x), \ldots, m_k(x)) = \lambda(m_0(x'), \ldots, m_k(x'))$. We set

$$i := \min\{i \in \{0, \ldots, n\} \mid x_i \neq 0\}$$

and observe that $0 \neq x_i^d = \lambda(x_i')^d$ implies $x_i' \neq 0$. Moreover, if $i \geq 1$, then we also have $0 = x_j^d = \lambda(x_j')^d$ for $j = 0, \ldots, i-1$. In such a case, $x_0' = \ldots = x_{i-1}' = 0$ follows. We now set $\gamma := \frac{x_i}{x_i'} \in \mathbb{C}^\times$ and conclude $\lambda = \gamma^d$ from

$$\lambda(x_i')^d = x_i^d = \left(\frac{x_i}{x_i'}\right)^d (x_i')^d = \gamma^d (x_i')^d.$$

If $i = n$, then $\gamma x' = x$ and we are done. However, if $i < n$, then $\gamma x_j' = x_j$ for $j = 0, \ldots, i$ and we compute

$$x_i^{d-1} x_j = \lambda(x_i')^{d-1} x_j' = \gamma^d (x_i')^{d-1} x_j' = \left(\frac{x_i}{x_i'}\right)^d (x_i')^{d-1} x_j' = x_i^{d-1} \gamma x_j'$$

for $j = i+1, \ldots, n$. Consequently, $x = \gamma x'$ and we are again done. ■

Example 2.3.25. Let us explicitly give the Veronese embedding in the Hilbert cases.

(i) Binary Forms

Let $n = 1$ and $d \geq 1$. Using Example 2.3.9 (i), we see that

$$\begin{aligned} V\colon \mathbb{P}^1 &\to \mathbb{P}^d \\ [x] &\mapsto \left[x_0^d : x_0^{d-1} x_1 : \ldots : x_0 x_1^{d-1} : x_1^d\right]. \end{aligned}$$

(ii) Quadratic Forms
Let $n \geq 1$ and $d = 1$. Using Example 2.3.9 (ii), we see that

$$\begin{aligned} V\colon \mathbb{P}^n &\to \mathbb{P}^n \\ [x] &\mapsto [x_0 : x_1 : \ldots : x_{n-1} : x_n]. \end{aligned}$$

Hence, the (projective) Veronese embedding coincides with the identity map on $\mathbb{P}^n$ in the Hilbert cases of quadratic forms.

(iii) Ternary Quartics
Let $n = 2$ and $d = 2$. Using Example 2.3.9 (iii), we see that

$$\begin{aligned} V\colon \mathbb{P}^2 &\to \mathbb{P}^5 \\ [x] &\mapsto \left[x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2\right]. \end{aligned}$$

Example 2.3.26. Let us explicitly give V in the basic non-Hilbert cases.

(i) Quaternary Quartics
Let $n = 3$ and $d = 2$. Using Example 2.3.10 (i), we see that

$$\begin{aligned} V\colon \mathbb{P}^3 &\to \mathbb{P}^9 \\ [x] &\mapsto \left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3 : x_3^2\right]. \end{aligned}$$

(ii) Ternary Sextics
Let $n = 2$ and $d = 3$. Using Example 2.3.10 (ii), we see that

$$\begin{aligned} V\colon \mathbb{P}^2 &\to \mathbb{P}^9 \\ [x] &\mapsto \left[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1^2 : x_0x_1x_2 : x_0x_2^2 : x_1^3 : x_1^2x_2 : x_1x_2^2 : x_2^3\right]. \end{aligned}$$

Lemma 2.3.27. *The Veronese embedding is surjective if and only if $d = 1$.*

Proof. Lemma 2.3.24 states that V is an embedding of the finite-dimensional $\mathbb{C}$-vector space $\mathbb{P}^n$ in the finite-dimensional $\mathbb{C}$-vector space $\mathbb{P}^k$. Therefore, V is surjective if and only if $\dim(\mathbb{P}^n) = \dim(\mathbb{P}^k)$. That is, if and only if $n = k = k(n,d)$. Lemma 2.3.5 and Corollary 2.3.6 together show that $n = k(n,d)$ if and only if $d = 1$. ■

Notation 2.3.28. We set

$$\mathcal{S}_V := \{Z_iZ_j - Z_sZ_t \mid i,j,s,t \in \{0,\ldots,k\} : \alpha_i + \alpha_j = \alpha_s + \alpha_t\}. \tag{2.7}$$

Theorem 2.3.29. *It holds $V(\mathbb{P}^n) = \mathcal{V}(\mathcal{S}_V)$.*

Proof. See [Pla20, 3.5.6 Proposition]. ■

Definition 2.3.30. $V(\mathbb{P}^n)$ is called the *Veronese variety.*

Later in this thesis, we will construct projective varieties containing the Veronese variety using quadratic forms from $\mathcal{S}_V$. Therefore, we now prove two results related to $\mathcal{S}_V$ and its elements.

Lemma 2.3.31. *For $i = 1, \dots, k-n$, there exist $1 \leq s \leq n$ and $s \leq t < n+i$ such that $\alpha_s + \alpha_t = \alpha_0 + \alpha_{n+i}$. Consequently, $q(Z) := Z_0 Z_{n+i} - Z_s Z_t \in \mathcal{S}_V$.*

Proof. We compute $\alpha_{0,0} = d$ and $\alpha_{n+i,0} \leq d-2$. Hence, $d \leq (\alpha_0 + \alpha_{n+i})_0 \leq 2(d-1)$ from which it follows that there exists some $1 \leq s \leq t \leq k$ such that $\alpha_{s,0} = d-1$ and $\alpha_s + \alpha_t = \alpha_0 + \alpha_{n+i}$. Consequently, $\alpha_{t,0} = \alpha_{n+i,0} + 1$. We conclude $1 \leq s \leq n$ from $\alpha_{s,0} = d-1$ and $t < n+i$ from $\alpha_{t,0} > \alpha_{n+i,0}$. ∎

Lemma 2.3.32. *Any non-zero $q \in \mathcal{S}_V$ is irreducible.*

Proof. Without loss of generality, we assume that $q(Z) = Z_i Z_j - Z_s Z_t$ with $i \neq j$. Indeed, if $q(Z) = Z_i Z_j - Z_s Z_t$ with $i = j$, then either $s = t$ or $s \neq t$. If $s = t$, then

$$2\alpha_j = \alpha_i + \alpha_j \underset{(2.7)}{=} \alpha_s + \alpha_t = 2\alpha_s$$

implies $\alpha_i = \alpha_s$. Consequently, q would be the zero form which is not possible by the assumption that q is non-zero. Hence, $s \neq t$ and we replace q by $-q$.

For a proof by contradiction, we now assume that q is reducible which allows us to fix two linear forms $p_1(Z) = \sum_{l=0}^{k} a_l Z_l$, $p_2(Z) = \sum_{l=0}^{k} b_l Z_l \in \mathcal{F}_{k+1,1}$ such that

$$Z_i Z_j - Z_s Z_t = q(Z) = p_1(Z) p_2(Z) = \sum_{l,m=0}^{k} a_l b_m Z_l Z_m.$$

By comparision of coefficients, it follows

(i) $1 = a_i b_j + a_j b_i$,

(ii) $-1 = a_s b_t + a_t b_s$,

(iii) $0 = a_l b_m + a_m b_l$ for $l, m = 0, \dots, k$, $\{l, m\} \neq \{i, j\}, \{s, t\}$.

Without loss of generality, we conclude $a_i \neq 0$ and $b_i = 0$. Indeed, $a_i \neq 0$ or $b_i \neq 0$ by (i) and $i \neq j$ implies

$$2\alpha_i \neq \alpha_i + \alpha_j \underset{(2.7)}{=} \alpha_s + \alpha_t.$$

Hence, (iii) yields $0 = a_i b_i + a_i b_i = 2a_i b_i$ from which $a_i = 0$ or $b_i = 0$ follows. Altogether, $a_i \neq 0$ and $b_i = 0$ or $a_i = 0$ and $b_i \neq 0$.

The equality in (i) therefore reduces to $1 = a_i b_j$. Consequently, $b_j \neq 0$.

Claim 1: It holds $\{i,s\} \neq \{i,j\},\{s,t\}$.
Proof. If we assume $\{i,s\} = \{i,j\}$ for a proof by contradiction, then $s = j$ from which

$$\alpha_i + \alpha_j \underset{(2.7)}{=} \alpha_s + \alpha_t = \alpha_j + \alpha_t$$

follows. Consequently, $i = t$ and q is the zero form which is a contradiction to the assumption that q is non-zero.

Likewise, if $\{i,s\} = \{s,t\}$ for a proof by contradiction, then $i = t$ from which

$$\alpha_i + \alpha_j \underset{(2.7)}{=} \alpha_s + \alpha_t = \alpha_s + \alpha_i$$

follows. Consequently, $j = s$ and q is the zero form which is a contradiction to the assumption that q is non-zero. ∎

We thus conclude $0 = a_i b_s + a_s b_i = a_i b_s$ from (iii) since $b_i = 0$. Therefore, we also know $b_s = 0$ because $a_i \neq 0$.

Claim 2: It holds $\{i,t\} \neq \{i,j\},\{s,t\}$.
Proof. If we assume $\{i,t\} = \{i,j\}$ for a proof by contradiction, then $t = j$ from which

$$\alpha_i + \alpha_j \underset{(2.7)}{=} \alpha_s + \alpha_t = \alpha_s + \alpha_j$$

follows. Consequently, $i = s$ and q is the zero form which is a contradiction to the assumption that q is non-zero.

Likewise, if $\{i,t\} = \{s,t\}$ for a proof by contradiction, then $i = s$ from which

$$\alpha_i + \alpha_j \underset{(2.7)}{=} \alpha_s + \alpha_t = \alpha_i + \alpha_t$$

follows. Consequently, $j = t$ and q is the zero form which is a contradiction to the assumption that q is non-zero. ∎

We thus conclude $0 = a_i b_t + a_t b_i = a_i b_t$ from (iii) since $b_i = 0$. Therefore, we also know $b_t = 0$ because $a_i \neq 0$. Using $b_s = 0$, it hence follows $a_s b_t + a_t b_s = 0$ which contradicts (ii). The assumption that q is reducible must have been wrong. ∎

In Section 3.3, we will consider specific quadratic forms in $\mathcal{S}_V$ for which we will prove some further properties. However, for now we continue our introduction of the Gram matrix method by building a bridge between the real affine setting, that we worked in so far, and the complex projective setting in which we will primarily work throughout the rest of this thesis. We refer the reader to Appendix A.1 for an introduction to algebraic geometry.

Notation 2.3.33. For $l \geq 1$ and $W \subseteq \mathbb{P}^l$, we denote the set of real points of W by $W(\mathbb{R})$ and always assume y to be a real affine representative for $[y] \in W(\mathbb{R})$.

Proposition 2.3.34. *It holds* $V(\mathbb{P}^n(\mathbb{R})) = V(\mathbb{P}^n)(\mathbb{R})$.

Proof. ($\subseteq$) For $[z] \in V(\mathbb{P}^n(\mathbb{R}))$, we let $[x] \in \mathbb{P}^n(\mathbb{R})$ be such that $V([x]) = [z]$. Hence, $z = \lambda(m_0(x), \ldots, m_k(x))$ for some $\lambda \in \mathbb{C}^\times$ and $(m_0(x), \ldots, m_k(x)) \in \mathbb{R}^{k+1}$ since $x \in \mathbb{R}^{n+1}$. This shows that $[z]$ is a real point of the Veronese variety.
($\supseteq$) For $[z] \in V(\mathbb{P}^n)(\mathbb{R})$, we let $[x] \in \mathbb{P}^n$ be such that $V([x]) = [z]$. Hence,

$$z = \lambda(m_0(x), \ldots, m_k(x)) \tag{2.8}$$

for some $\lambda \in \mathbb{C}^\times$. We set $i := \min\{i \in \{0, \ldots, n\} \mid x_i \neq 0\}$ and fix $s \in \{0, \ldots, k\}$ such that $m_s(X) = X_i^d$. If $i > 0$, then $0 < s$ and so, for $t = 0, \ldots, s-1$, X_j divides $m_t(X)$ for some $0 \leq j \leq i-1$. If $i > 0$, we thus conclude for $t = 0, \ldots, s-1$ that

$$z_t \underset{(2.8)}{=} \lambda m_t(x) = 0. \tag{2.9}$$

Moreover, it is always true that

$$z_s \underset{(2.8)}{=} \lambda m_s(x) = \lambda x_i^d \neq 0 \tag{2.10}$$

and thus we assume $z_s > 0$ without loss of generality since $[z] = [-z]$. We furthermore fix $\gamma \in \mathbb{R}^\times$ such that $\gamma^d = z_s$ and distinguish two cases for i.

Case 1: If $i < n$, then, for $j = i+1, \ldots, n$, we have

$$z_{s+j} \underset{(2.8)}{=} \lambda m_{s+j}(x) = \lambda x_i^{d-1} x_j = \lambda x_i^d \left(\frac{x_j}{x_i}\right) \underset{(2.10)}{=} z_s \left(\frac{x_j}{x_i}\right) = \gamma^d \left(\frac{x_j}{x_i}\right).$$

From $z_{s+j} \in \mathbb{R}$ and $\gamma \in \mathbb{R}^\times$, it thus follows $y_j := \frac{x_j}{x_i} \in \mathbb{R}$ for $j = i+1, \ldots, n$. Hence, each entry of

$$y := \begin{cases} (1, y_1, \ldots, y_n), & \text{if } i = 0 \\ (0, \ldots, 0, 1, y_{i+1}, \ldots, y_n), & \text{else} \end{cases}$$

is real. Using Lemma 2.2.11, we conclude for $t = s, \ldots, k$ that

$$\begin{aligned} z_t &\underset{(2.8)}{=} \lambda m_t(x) \\ &= \begin{cases} \lambda m_t(x_0, \ldots, x_n), & \text{if } i = 0 \\ \lambda m_t(0, \ldots, 0, x_i, \ldots, x_k), & \text{else} \end{cases} \\ &= \begin{cases} \lambda m_t(x_0, y_1 x_0, \ldots, y_n x_0), & \text{if } i = 0 \\ \lambda m_t(0, \ldots, 0, x_i, y_{i+1} x_i, \ldots, y_k x_i), & \text{else} \end{cases} \\ &= \lambda x_i^d y^{\alpha_t} \underset{(2.10)}{=} z_s y^{\alpha_t} = \gamma^d y^{\alpha_t} = \gamma^d m_t(y) = m_t(\gamma y) \end{aligned}$$

and $\gamma y \in \mathbb{R}^{n+1}$ by construction. If $i = 0$, then $s = 0$ and we are done. However, if $i > 0$, then $0 < s$ and we are again done since

$$z_t \underset{(2.9)}{=} 0 = m_t(\gamma y) \text{ for } t = 0, \ldots, s-1.$$

In particular, $[z] = V([\gamma y]) \in V(\mathbb{P}^n(\mathbb{R}))$ independent from whether $i = 0$ or not.

Case 2: If $i = n$, then $s = k$ and

$$[z] = \left[0 : \ldots : 0 : \gamma^d\right] = V([0 : \ldots : 0 : \gamma]) \in V(\mathbb{P}^n(\mathbb{R}))). \qquad \blacksquare$$

The kernel of the Gram map can be described by the Veronese embedding since

$$\mathcal{G}(A)(X) = Q(A)(m_0(X), \ldots, m_k(X)) = (Q(A) \circ V)(X) \tag{2.11}$$

for any $A \in \mathrm{Sym}_{k+1}(\mathbb{R})$ by an abuse of notation. Consequently,

$$\ker(\mathcal{G}) = \{\mathfrak{A} \in \mathrm{Sym}_{k+1}(\mathbb{R}) \mid q_{\mathfrak{A}} \text{ vanishes on } V(\mathbb{P}^n)\} \tag{2.12}$$

and, for $f \in \mathcal{F}_{n+1,2d}$ with an a priori fixed associated Gram matrix A, it follows

$$\begin{aligned} \mathcal{G}^{-1}(f) &= A + \ker(\mathcal{G}) \\ &= \{A + \mathfrak{A} \mid \mathfrak{A} \in \mathrm{Sym}_{k+1}(\mathbb{R}) \wedge q_{\mathfrak{A}} \text{ vanishes on } V(\mathbb{P}^n)\} \\ &= \{\mathfrak{A} \in \mathrm{Sym}_{k+1}(\mathbb{R}) \mid q_{\mathfrak{A}-A} \text{ vanishes on } V(\mathbb{P}^n)\}. \end{aligned} \tag{2.13}$$

We conclude this section by characterizing $\mathcal{P}_{n+1,2d}$ and $\Sigma_{n+1,2d}$ using (local) positive semidefinite conditions for quadratic forms induced by Gram matrices via the $\mathbb{R}$-vector space isomorphism Q.

Definition 2.3.35. For $l \geq 1$ and $W \subseteq \mathbb{P}^l$, the quadratic form $q \in \mathcal{F}_{l+1,2}$ is *locally non-negative* or *locally positive semidefinite on* $W(\mathbb{R})$ if

$$q(y) \geq 0 \text{ for all } [y] \in W(\mathbb{R})$$

and we write $q|_{W(\mathbb{R})} \geq 0$. In the special case that $W(\mathbb{R}) = \mathbb{P}^l(\mathbb{R})$, the quadratic form q is *(globally) non-negative* or *(globally) positive semidefinite (PSD)*. We write $q \geq 0$.

Theorem 2.3.36. *For $f \in \mathcal{F}_{n+1,2d}$, the following are equivalent:*

(i) The form f is PSD.

(ii) For any $A \in \mathcal{G}^{-1}(f)$, the quadratic form q_A is locally PSD on $V(\mathbb{P}^n)(\mathbb{R})$.

(iii) There exists some $A \in \mathcal{G}^{-1}(f)$ such that q_A is locally PSD on $V(\mathbb{P}^n)(\mathbb{R})$.

Proof. (i) $\Rightarrow$ (ii) For $x \in \mathbb{R}^{n+1}$, using $A \in \mathcal{G}^{-1}(f)$, we see that

$$0 \leq f(x) = \mathcal{G}(A)(x) \underset{(2.11)}{=} (Q(A) \circ V)(x) = (q_A \circ V)(x) = q_A(V(x))$$

by an abuse of notation. Hence, the quadratic form q_A is locally PSD on $V(\mathbb{P}^n(\mathbb{R}))$ and $V(\mathbb{P}^n(\mathbb{R})) = V(\mathbb{P}^n)(\mathbb{R})$ by Proposition 2.3.34.

(ii) $\Rightarrow$ (iii) We recall that $\mathcal{G}^{-1}(f)$ is non-empty since $\mathcal{G}$ is surjective by Lemma 2.3.12. Hence, there exists some $A \in \mathcal{G}^{-1}(f)$ and the induced quadratic form q_A is thus locally PSD on $V(\mathbb{P}^n)(\mathbb{R})$ by assumption.

(iii) $\Rightarrow$ (i) For $x \in \mathbb{R}^{n+1}$, using $A \in \mathcal{G}^{-1}(f)$, we observe by an abuse of notation that

$$f(x) = \mathcal{G}(A)(x) \underset{(2.11)}{=} (Q(A) \circ V)(x) = (q_A(V(x)) \geq 0. \qquad \blacksquare$$

Theorem 2.3.37. *For $f \in \mathcal{F}_{n+1,2d}$, the following are equivalent:*

(i) The form f is SOS.

(ii) There exists some $A \in \mathcal{G}^{-1}(f)$ such that q_A is PSD.

Proof. See [CLR92, Theorem 2.4.]. $\blacksquare$

Theorem 2.3.36 and Theorem 2.3.37 yield

$$\begin{aligned}
\mathcal{P}_{n+1,2d} &= \left\{ f \in \mathcal{F}_{n+1,2d} \mid \forall\ A \in \mathcal{G}^{-1}(f) \colon q_A|_{\mathcal{V}(\mathbb{P}^n)(\mathbb{R})} \geq 0 \right\} \\
&= \left\{ f \in \mathcal{F}_{n+1,2d} \mid \exists\ A \in \mathcal{G}^{-1}(f) \colon q_A|_{\mathcal{V}(\mathbb{P}^n)(\mathbb{R})} \geq 0 \right\}, && (2.14) \\
\Sigma_{n+1,2d} &= \left\{ f \in \mathcal{F}_{n+1,2d} \mid \exists\ A \in \mathcal{G}^{-1}(f) \colon q_A|_{\mathbb{P}^k(\mathbb{R})} \geq 0 \right\}, && (2.15)
\end{aligned}$$

which sheds new light on Hilbert's 1888 theorem. Indeed, in order to verify that $\Sigma_{n+1,2d} \subsetneq \mathcal{P}_{n+1,2d}$, we realize that some $f \in \mathcal{F}_{n+1,2d}$ has to be found such that

(i) there exists some $A \in \mathcal{G}^{-1}(f)$ such that $q_A|_{V(\mathbb{P}^n)(\mathbb{R})} \geq 0$ and

(ii) for any $B \in \mathcal{G}^{-1}(f)$, there exists some $[z] \in \mathbb{P}^k(\mathbb{R})$ such that $q_B(z) < 0$.

Recalling from (2.12) that the kernel of the Gram map coincides with the set of all $\mathfrak{A} \in \mathrm{Sym}_{k+1}(\mathbb{R})$ such that $q_\mathfrak{A}$ vanishes on $V(\mathbb{P}^n)$, we therefore apply (2.13) to the Gram matrix A from (i) and replace (ii) by the equivalent condition that

(ii') for any $\mathfrak{A} \in \mathrm{Sym}_{k+1}(\mathbb{R})$ such that $q_\mathfrak{A}$ vanishes on $V(\mathbb{P}^n)$, there exists some $[z] \in \mathbb{P}^k(\mathbb{R})$ such that $q_{A+\mathfrak{A}}(z) < 0$.

Vice versa, we also see that $\mathcal{P}_{n+1,2d} = \Sigma_{n+1,2d}$ if and only if for any $f \in \mathcal{F}_{n+1,2d}$ with an associated Gram matrix A such that $q_A|_{V(\mathbb{P}^n)(\mathbb{R})} \geq 0$, there always exists some $B \in \mathcal{G}^{-1}(f)$ such that $q_B|_{\mathbb{P}^k(\mathbb{R})} \geq 0$. As before, using (2.12) and (2.13), we conclude that $\mathcal{P}_{n+1} = \Sigma_{n+1,2d}$ if and only if for any $f \in \mathcal{F}_{n+1,2d}$ with an associated Gram matrix A such that $q_A|_{V(\mathbb{P}^n)(\mathbb{R})} \geq 0$, there exists some $\mathfrak{A} \in \mathrm{Sym}_{k+1}(\mathbb{R})$ such that $q_{\mathfrak{A}}$ vanishes on $V(\mathbb{P}^n)$ and $q_{A+\mathfrak{A}}|_{\mathbb{P}^k(\mathbb{R})} \geq 0$.

We therefore see that the investigation of the cone inclusion $\Sigma_{n+1,2d} \subseteq \mathcal{P}_{n+1,2d}$ reduces to the question below.

Question 1. Let $q \in \mathcal{F}_{k+1,2}$ be locally PSD on $V(\mathbb{P}^n)(\mathbb{R})$. When does there exist a quadratic form $\mathfrak{q} \in \mathcal{F}_{k+1,2}$ such that $\mathfrak{q}$ vanishes on $V(\mathbb{P}^n)$ and $(q + \mathfrak{q})|_{\mathbb{P}^k(\mathbb{R})} \geq 0$?

Chapter 3

Intermediate Cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$

In this chapter, we start our investigation of intermediate cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$. Inspired by the Gram matrix method, we therefore introduce a general construction method for cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ along projective sets in Section 3.1. We moreover show that the investigation of such a cone filtration $\Sigma_{n+1,2d} \subseteq C \subseteq \mathcal{P}_{n+1,2d}$ for strict inclusions amounts to the question when local non-negativity of a given quadratic form can be extended to larger sets and we provide an answer in a special case.

In Section 3.2, we next establish a specific filtration of projective varieties containing the Veronese variety in which each inclusion is strict. Applying our construction method from the first section, we thus obtain a specific filtration of intermediate cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ which is the subject of interest for thesis. In particular, we wish to identify each strict inclusion in this specific cone filtration. That is the main query of this thesis.

In Section 3.3, we provide a further perspective on the distinguished projective varieties from the second section by introducing an algorithm to construct inducing forms for each of these projective varietiese. These explicit descriptions will be advantegeous at several points throughout this thesis.

Lastly, in Section 3.4, we apply our result from the first section on extending local non-negativity of a given quadratic form to larger sets in the special case of the projective varieties constructed in the second section.

3.1 A Construction Method

In Section 2.3, we applied the Gram matrix method to characterize $\mathcal{P}_{n+1,2d}$ and $\Sigma_{n+1,2d}$ by (local) PSD conditions for quadratic forms induced by Gram matrices on the sets of real points of the projective variety $V(\mathbb{P}^n)$ and the projective variety $\mathbb{P}^k$, respectively. In this section, we generalize this concept by introducing and examining cones in $\mathcal{F}_{n+1,2d}$ that come from local PSD restriction for quadratic froms induced by Gram matrices on arbitrary sets of real points. A priori, the sets of real points do not have to come from projective varieties. A similar construction to the one below was already proposed in [Goe14, Section 2.4].

Definition 3.1.1. For $W \subseteq \mathbb{P}^k$, we set

$$C_W := \{f \in \mathcal{F}_{n+1,2d} \mid \exists\, A \in \mathcal{G}^{-1}(f)\colon q_A|_{W(\mathbb{R})} \geq 0\}.$$

Lemma 3.1.2. *For $W \subseteq \mathbb{P}^k$, the following are true:*

(i) C_W is a cone.

(ii) $C_{V(\mathbb{P}^n)} = \mathcal{P}_{n+1,2d}$.

(iii) $C_{\mathbb{P}^k} = \Sigma_{n+1,2d}$.

Proof. (i) For $f, g \in C_W$, we fix $A \in \mathcal{G}^{-1}(f)$ and $B \in \mathcal{G}^{-1}(g)$ such that q_A and q_B are locally PSD on $W(\mathbb{R})$. For $\lambda \geq 0$, we set $C := A + \lambda B \in \mathrm{Sym}_{k+1}(\mathbb{R})$ and, using the linearity of the Gram map, compute

$$\mathcal{G}(C) = \mathcal{G}(A + \lambda B) = \mathcal{G}(A) + \lambda\mathcal{G}(B) = f + \lambda g.$$

Hence, $C \in \mathcal{G}^{-1}(f + \lambda g)$. Moreover, for $[z] \in W(\mathbb{R})$, the linearity of Q implies

$$q_C(z) = q_{A+\lambda B}(z) = Q(A + \lambda B)(z) = Q(A)(z) + \lambda Q(B)(z) = q_A(z) + \lambda q_B(z).$$

Since q_A and q_B are locally PSD on $W(\mathbb{R})$ and $\lambda \geq 0$ by choice, $q_C(z) \geq 0$ follows. Hence, for the Gram matrix C associated to $f + \lambda g$, the quadratic form q_C is locally PSD on $W(\mathbb{R})$. This shows $f + \lambda g \in C_W$.

(ii) Follows from (2.14).

(iii) Follows from (2.15). ■

Lemma 3.1.3. *Let $\mathfrak{W}_1, \mathfrak{W}_2 \subseteq \mathbb{P}^k$. If $\mathfrak{W}_1 \subseteq \mathfrak{W}_2$, then $C_{\mathfrak{W}_2} \subseteq C_{\mathfrak{W}_1}$.*

Proof. For $f \in C_{\mathfrak{W}_2}$, we fix $A \in \mathcal{G}^{-1}(f)$ such that q_A is locally PSD on $\mathfrak{W}_2(\mathbb{R})$. Since $\mathfrak{W}_1 \subseteq \mathfrak{W}_2$, it follows that q_A is locally PSD on the subset $\mathfrak{W}_1(\mathbb{R})$ of $\mathfrak{W}_2(\mathbb{R})$. This shows $f \in C_{\mathfrak{W}_1}$. ■

Corollary 3.1.4. *Let $W \subseteq \mathbb{P}^k$. If $V(\mathbb{P}^n) \subseteq W$, then C_W is an intermediate cone between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$.*

Proof. C_W is a cone by Lemma 3.1.2 (i). Moreover, Lemma 3.1.3 yields $C_{\mathbb{P}^k} \subseteq C_W$ by setting $\mathfrak{W}_1 := W$, $\mathfrak{W}_2 := \mathbb{P}^k$ and $C_{\mathbb{P}^k} = \Sigma_{n+1,2d}$ by Lemma 3.1.2 (iii). Likewise, Lemma 3.1.3 implies $C_W \subseteq C_{V(\mathbb{P}^n)}$ by setting $\mathfrak{W}_1 := V(\mathbb{P}^n)$, $\mathfrak{W}_2 := W$ and we have $C_{V(\mathbb{P}^n)} = \mathcal{P}_{n+1,2d}$ by Lemma 3.1.2 (ii). ■

Corollary 3.1.5. *Let $W \subseteq \mathbb{P}^k$. If $V(\mathbb{P}^n) \subseteq W$, then the cone C_W is pointed, full-dimensional and contains no straight lines.*

Proof. We recall from Lemma 2.2.21 that $\mathcal{P}_{n+1,2d}$ and $\Sigma_{n+1,2d}$ are pointed and full-dimensional. Corollary 3.1.4 thus yields that C_W is a subcone of the pointed cone $\mathcal{P}_{n+1,2d}$ and, therefore, it follows

$$\{0\} \subseteq C_W \cap -C_W \subseteq \mathcal{P}_{n+1,2d} \cap -\mathcal{P}_{n+1,2d} = \{0\},$$

where 0 denotes the zero form. Hence, C_W is pointed. Moreover, Corollary 3.1.4 yields that the full-dimensional cone $\Sigma_{n+1,2d}$ is a subcone of C_W and it thus follows that the smallest affine subspace that contains C_W, and with that especially the subcone $\Sigma_{n+1,2d}$, is $\mathcal{F}_{n+1,2d}$ itself. This shows that C_W is full-dimensional. Furthermore, we showed in the proof of Lemma 2.2.21 that $\mathcal{P}_{n+1,2d}$ contains no straight lines and thus also its subcone C_W contains no straight lines. ■

Unlike pointedness and full-dimensionality, the closure of an intermediate cone between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ does not follow from neither $\mathcal{P}_{n+1,2d}$ nor $\Sigma_{n+1,2d}$ being closed. Therefore, we postpone the discussion whether C_W is closed or not for particular choices of W to Section 7.1.

Observation 3.1.6. *If $(n+1, 2d)$ is a Hilbert case, then $\Sigma_{n+1,2d} = \mathcal{P}_{n+1,2d}$. Hence, for $W \subseteq \mathbb{P}^k$ such that $V(\mathbb{P}^n) \subseteq W$, Corollary 3.1.4 yields*

$$\mathcal{P}_{n+1,2d} = \Sigma_{n+1,2d} \subseteq C_W \subseteq \mathcal{P}_{n+1,2d}.$$

However, if $(n+1, 2d)$ is a non-Hilbert case, then $\Sigma_{n+1,2d} \subsetneq \mathcal{P}_{n+1,2d}$ and, for $W \subseteq \mathbb{P}^k$ such that $V(\mathbb{P}^n) \subseteq W$, Corollary 3.1.4 yields

$$\Sigma_{n+1,2d} \subseteq C_W \subseteq \mathcal{P}_{n+1,2d}. \tag{3.1}$$

Hence, at least one inclusion in (3.1) has to be strict, but it is a priori not clear how many of the inclusions are strict and which ones.

Following the train of thought of Observation 3.1.6, we intend to investigate inclusions of the type $C_{\mathfrak{W}_2} \subseteq C_{\mathfrak{W}_1}$ for projective sets $\mathfrak{W}_1$ and $\mathfrak{W}_2$ such that $\mathfrak{W}_1 \subseteq \mathfrak{W}_2$. In order to establish the strict inclusion $C_{\mathfrak{W}_2} \subsetneq C_{\mathfrak{W}_1}$, we realize that some $f \in \mathcal{F}_{n+1,2d}$ has to be found such that

(i) there exists some $A \in \mathcal{G}^{-1}(f)$ such that $q_A|_{\mathfrak{W}_1(\mathbb{R})} \geq 0$ and

(ii) for any $B \in \mathcal{G}^{-1}(f)$, there exists some $[z] \in \mathfrak{W}_2(\mathbb{R})$ such that $q_B(z) < 0$.

Recalling from (2.12) that the kernel of $\mathcal{G}$ coincides with the set of all $\mathfrak{A} \in \mathrm{Sym}_{k+1}(\mathbb{R})$ such that $q_{\mathfrak{A}}$ vanishes on $V(\mathbb{P}^n)$, applying (2.13) with the to-be-determined Gram matrix A from (i), we replace (ii) by the equivalent condition that (ii') for any $\mathfrak{A} \in \operatorname{Sym}_{k+1}(\mathbb{R})$ such that $q_{\mathfrak{A}}$ vanishes on $V(\mathbb{P}^n)$, there exists some $[z] \in \mathfrak{W}_2(\mathbb{R})$ such that $q_{A+\mathfrak{A}}(z) < 0$.

Vice versa, $C_{\mathfrak{W}_2} = C_{\mathfrak{W}_1}$ if and only if for any $f \in \mathcal{F}_{n+1,2d}$ with an associated Gram matrix A such that $q_A|_{\mathfrak{W}_1(\mathbb{R})} \geq 0$, there always exists some $B \in \mathcal{G}^{-1}(f)$ such that $q_B|_{\mathfrak{W}_2(\mathbb{R})} \geq 0$. As before, using (2.12) and (2.13), we conclude that $C_{\mathfrak{W}_2} = C_{\mathfrak{W}_1}$ if and only if for any $f \in \mathcal{F}_{n+1,2d}$ with an associated Gram matrix A such that $q_A|_{\mathfrak{W}_1(\mathbb{R})} \geq 0$, there exists some $\mathfrak{A} \in \operatorname{Sym}_{k+1}(\mathbb{R})$ such that $q_{\mathfrak{A}}$ vanishes on $V(\mathbb{P}^n)$ and $q_{A+\mathfrak{A}}|_{\mathfrak{W}_2(\mathbb{R})} \geq 0$. The investigation of $C_{\mathfrak{W}_2} \subseteq C_{\mathfrak{W}_1}$ thus reduce to the question below.

Question 2. For $\mathfrak{W}_1, \mathfrak{W}_2 \subseteq \mathbb{P}^k$ such that $\mathfrak{W}_1 \subseteq \mathfrak{W}_2$, let $q \in \mathcal{F}_{k+1,2}$ be locally PSD on $\mathfrak{W}_1(\mathbb{R})$. When does there exist a quadratic form $\mathfrak{q} \in \mathcal{F}_{k+1,2}$ such that $\mathfrak{q}$ vanishes on $V(\mathbb{P}^n)$ and $(q+\mathfrak{q})|_{\mathfrak{W}_2(\mathbb{R})} \geq 0$?

Remark 3.1.7. *This question was asked in [Goe14, Question 2.48] for $\mathfrak{W}_2 := \mathbb{P}^k$. If we moreover set $\mathfrak{W}_1 := V(\mathbb{P}^n)$, then we recover Question 1.*

Question 2 can be generalized even more.

Question 3. For $\mathfrak{W}_0, \mathfrak{W}_1, \mathfrak{W}_2 \subseteq \mathbb{P}^k$ such that $\mathfrak{W}_1 \subseteq \mathfrak{W}_2$, let $q \in \mathcal{F}_{k+1,2}$ be locally PSD on $\mathfrak{W}_1(\mathbb{R})$. When does there exist a quadratic form $\mathfrak{q} \in \mathcal{F}_{k+1,2}$ such that $\mathfrak{q}$ vanishes on $\mathfrak{W}_0$ and $(q+\mathfrak{q})|_{\mathfrak{W}_2(\mathbb{R})} \geq 0$?

Remark 3.1.8. *We recover Question 2 by setting $\mathfrak{W}_0 := V(\mathbb{P}^n)$.*

The Zariski closure $\mathfrak{W}_2 := \overline{\mathfrak{W}_1} \subseteq \mathbb{P}^k$ of a given set $\mathfrak{W}_1 := W \subseteq \mathbb{P}^k$ is the smallest projective variety in $\mathbb{P}^k$ w.r.t. $\subseteq$ containing $\mathfrak{W}_1$. In order to answer Question 2 for such choices of $\mathfrak{W}_1$ and $\mathfrak{W}_2$, we thus have to understand when for a given $q \in \mathcal{F}_{k+1,2}$ that is locally PSD on $W(\mathbb{R})$, there exists some quadratic form $\mathfrak{q} \in \mathcal{F}_{k+1,2}$ that vanishes on $V(\mathbb{P}^n)$ such that $q + \mathfrak{q}$ is locally PSD on $\overline{W}(\mathbb{R})$. Casually speaking, for quadratic forms, we have to understand when non-negativity over W can be extended (over the Veronese variety) to non-negativity on the Zariski closure $\overline{W}$. We refer an interested reader to Appendix A.2 for an overview on the Zariski topology.

If $\mathbb{P}^k$ is endowed with the Euclidean topology (instead of the Zariski topology), then the extension of non-negativity from a set to its Euclidean closure is always possible (even over the entire space) since quadratic forms are continuous w.r.t. the Euclidean topology. In the proof of Theorem 3.1.9 below, we use this fact to prove a similar result for the Zariski closure of an *embedded affine set* $W \subseteq \mathbb{P}^k$. That is, $W = \phi(K)$ for some $K \subseteq \mathbb{C}^k$ under the embedding

$$\begin{aligned} \phi\colon \mathbb{C}^k &\to \mathbb{P}^k \\ \boldsymbol{z} &\mapsto [1 : \boldsymbol{z}]. \end{aligned}$$

Theorem 3.1.9. *Let $\mathbb{P}^k$ be endowed with the Euclidean topology. For $K \subseteq \mathbb{C}^k$, let $q \in \mathcal{F}_{k+1,2}$ be such that $q|_{\phi(K)(\mathbb{R})} \geq 0$. If $\overline{\phi(K)(\mathbb{R})} = \overline{\phi(K)}(\mathbb{R})$, then $q|_{\overline{\phi(K)}(\mathbb{R})} \geq 0$.*

Proof. For $[z] \in \overline{\phi(K)(\mathbb{R})} \subseteq \mathbb{P}^k(\mathbb{R})$, Example A.2.24 allows us to fix $(\lambda_m)_{m\in\mathbb{N}} \subseteq \mathbb{C}^\times$ and $\left(\left[z^{(m)}\right]\right)_{m\in\mathbb{N}} \subseteq \phi(K)(\mathbb{R})$ such that $\lambda_m z^{(m)} \to z$ as $m \to \infty$. It thus follows $\mathrm{Re}(\lambda_m)z^{(m)} \to z$ as $m \to \infty$ and $\mathrm{Im}(\lambda_m)z^{(m)} \to 0$ as $m \to \infty$ since $\left(z^{(m)}\right)_{m\in\mathbb{N}} \subseteq \mathbb{R}^{k+1}$ and $z \in \mathbb{R}^{k+1}$. Moreover, the quadratic form q is continuous w.r.t. the Euclidean topology on $\mathbb{R}^{k+1}$. Hence, using Lemma 2.2.11 and $q|_{\phi(K)(\mathbb{R})} \geq 0$, we conclude

$$\begin{aligned} q(z) &= q\left(\lim_{m\to\infty} \mathrm{Re}(\lambda_m)z^{(m)}\right) \\ &= \lim_{m\to\infty} q\left(\mathrm{Re}(\lambda_m)z^{(m)}\right) \\ &= \lim_{m\to\infty} \mathrm{Re}(\lambda_m)^2 q\left(z^{(m)}\right) \geq 0. \end{aligned}$$ ■

Theorem 3.1.9 affirmatively answers Question 3 for any $\mathfrak{W}_0 \subseteq \mathbb{P}^k$ and $q \in \mathcal{F}_{k+1,2}$ if $\mathfrak{W}_1 := \phi(K)$, $\mathfrak{W}_2 := \overline{\phi(K)}$ (w.r.t. the Euclidean topology) and

$$\overline{\mathfrak{W}_1(\mathbb{R})} = \mathfrak{W}_2(\mathbb{R}) \text{ w.r.t. the Euclidean topology.}$$

Corollary 3.1.10. *Let $\mathbb{P}^k$ be endowed with the Euclidean topology and $K \subseteq \mathbb{C}^k$. If $\overline{\phi(K)(\mathbb{R})} = \overline{\phi(K)}(\mathbb{R})$, then $C_{\phi(K)} = C_{\overline{\phi(K)}}$.*

Proof. ($\subseteq$) For $f \in C_{\phi(K)}$, we fix $A \in \mathcal{G}^{-1}(f)$ such that q_A is locally PSD on $\phi(K)(\mathbb{R})$. Theorem 3.1.9 therefore implies $q_A|_{\overline{\phi(K)(\mathbb{R})}} \geq 0$ and it follows $f \in C_{\overline{\phi(K)}}$.
($\supseteq$) Lemma 3.1.3 with $W_1 := \phi(K)$ and $W_2 := \overline{\phi(K)}$ yields $C_{\overline{\phi(K)}} \subseteq C_{\phi(K)}$. ■

In the special case that $K \subseteq \mathbb{C}^k$ is not an arbitrary set but an affine variety, the Euclidean closure of the *embedded affine variety* $W := \phi(K) \subseteq \mathbb{P}^k$ can be related to its Zariski closure $\overline{W}$ (cf. Theorem 3.1.11 below). This allows us to give a sufficient criterion ensuring $C_W = C_{\overline{W}}$ in Corollary 3.1.12.

Theorem 3.1.11. *For $l \geq 1$, the Eulidean closure of an embedded affine variety $W \subseteq \mathbb{P}^l$ coincides with its Zariski closure.*

Proof. See [Mum76, (2.33) Theorem]. ■

For an embedded affine variety, the closure hence may be take w.r.t. the Euclidean or the Zariski topology and both lead to the same projective variety.

Corollary 3.1.12. *Let $W \subseteq \mathbb{P}^k$ be an embedded affine variety and $\mathfrak{W} \subseteq \mathbb{P}^k$ its Zariski closure. If $\overline{W(\mathbb{R})} = \overline{W}(\mathbb{R})$ w.r.t. the Euclidean topology on $\mathbb{P}^k$, then $C_W = C_{\mathfrak{W}}$.*

Proof. Follows from Corollary 3.1.10 and Theorem 3.1.11. ■

3.2 A Specific Cone Filtration

In this section, we introduce a specific filtration of cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ along a specific filtration of projective varieties containing the Veronese variety following our construction method from Section 3.1. Our idea for the construction of the filtration of projective varieties containing $V(\mathbb{P}^n)$ is to coordinatewise truncate the Veronese embedding.

Construction 3.2.1. For $i = 0, \ldots, k-n$,[1] we set

(1) $H_i := \{[z] \in \mathbb{P}^k \mid \exists\, x \in \mathbb{C}^{n+1} \colon (z_0, \ldots, z_{n+i}) = (m_0(x), \ldots, m_{n+i}(x))\}$ and

(2) let V_i be the Zariski closure of H_i in $\mathbb{P}^k$. That is, V_i is the smallest projective variety in $\mathbb{P}^k$ w.r.t. $\subseteq$ containing H_i.

Lemma 3.2.2. *For $i = 0, \ldots, k-n$, the projective set $H_i \subseteq \mathbb{P}^k$ is well-defined.*

Proof. Let $z \in \mathbb{C}^{k+1}$ be an arbitrary but fixed non-zero vector such that there exists some $x \in \mathbb{C}^{n+1}$ with $(z_0, \ldots, z_{n+i}) = (m_0(x), \ldots, m_{n+i}(x))$. For any non-zero vector $z' \in \mathbb{C}^{k+1}$ such that $z' \in [z]$, we fix $\lambda \in \mathbb{C}^\times$ such that $z' = \lambda z$ and let $\gamma \in \mathbb{C}^\times$ be such that $\gamma^d = \lambda$. For $x' := \gamma x \in \mathbb{C}^{n+1}$, using Lemma 2.2.11, we conclude

$$\begin{aligned}(z'_0, \ldots, z'_{n+i}) &= \lambda(z_0, \ldots, z_{n+i}) \\ &= \gamma^d(m_0(x), \ldots, m_{n+i}(x)) \\ &= \left(\gamma^d m_0(x), \ldots, \gamma^d m_{n+i}(x)\right) \\ &= (m_0(\gamma x), \ldots, m_{n+i}(\gamma x)) \\ &= (m_0(x'), \ldots, m_{n+i}(x')).\end{aligned}$$

■

Example 3.2.3. Let us determine $H_0, \ldots, H_{k-n}$ in the Hilbert cases.

(i) Binary Forms

Let $n = 1$ and we refer to Example 2.3.9 (i) for an explicit description of the monomials $m_0, \ldots, m_k$.

If $d = 1$, then $k - n = k(1,1) - 1 = 0$ and, using Example 2.3.25 (ii), we see that

$$H_0 = \{[x_0 : x_1] \in \mathbb{P}^1 \mid x_0, x_1 \text{ not both } 0\} = \mathbb{P}^1 = V(\mathbb{P}^1).$$

If $d \geq 2$, then $k - n = k(1,d) - 1 = d - 1 \geq 1$ and we compute

$$\begin{aligned} H_0 &= \left\{\left[x_0^d : x_0^{d-1}x_1 : z_2 : \ldots : z_d\right] \in \mathbb{P}^k \mid x_0, z_2, \ldots, z_d \text{ not all } 0\right\}, \\ &\vdots \\ H_{d-2} &= \left\{\left[x_0^d : x_0^{d-1}x_1 : \ldots : x_0 x_1^{d-1} : z_d\right] \in \mathbb{P}^k \mid x_0, z_d \text{ not both } 0\right\}, \\ H_{d-1} &= \left\{\left[x_0^d : x_0^{d-1}x_1 : \ldots : x_1^d\right] \in \mathbb{P}^1 \mid x_0, x_1 \text{ not both } 0\right\} = V(\mathbb{P}^1). \end{aligned}$$

[1] Lemma 2.3.5 ensures $k - n = k(n,d) - n \geq 0$.

(ii) Quadratic Forms

Let $n \geq 1$, $d = 1$ and we refer to Example 2.3.9 (ii) for an explicit description of the monomials $m_0, \ldots, m_k$. We compute $k - n = k(n,1) - n = n - n = 0$ and, using Example 2.3.25 (ii), we see that

$$H_0 = \{[x_0 : x_1 : \ldots : x_n] \in \mathbb{P}^n \mid x_0, \ldots, x_n \text{ not all } 0\} = \mathbb{P}^n = V(\mathbb{P}^n).$$

(iii) Ternary Quartics

Let $n = 2$, $d = 2$ and we refer to Example 2.3.9 (iii) for an explicit description of the monomials $m_0, \ldots, m_k$. We compute $k - n = k(2,2) - 2 = 3$ and conclude

$$\begin{aligned}
H_0 &= \left\{\left[x_0^2 : x_0x_1 : x_0x_2 : z_3 : z_4 : z_5\right] \in \mathbb{P}^5 \mid x_0, z_3, z_4, z_5 \text{ not all } 0\right\},\\
H_1 &= \left\{\left[x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : z_4 : z_5\right] \in \mathbb{P}^5 \mid x_0, x_1, z_4, z_5 \text{ not all } 0\right\},\\
H_2 &= \left\{\left[x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : z_5\right] \in \mathbb{P}^5 \mid x_0, x_1, z_5 \text{ not all } 0\right\},\\
H_3 &= \left\{\left[x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2\right] \in \mathbb{P}^5 \mid x_0, x_1, x_2 \text{ not all } 0\right\} = V(\mathbb{P}^2).
\end{aligned}$$

Example 3.2.4. Let us determine $H_0, \ldots, H_{k-n}$ in the basic non-Hilbert cases.

(i) Quaternary Quartics

Let $n = 3$, $d = 2$ and we refer to Example 2.3.10 (i) for an explicit description of the monomials $m_0, \ldots, m_k$. We compute $k - n = k(3,2) - 3 = 6$ and conclude

$$\begin{aligned}
H_0 &= \left\{\left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : z_4 : \ldots : z_9\right] \in \mathbb{P}^9 \mid x_0, z_4, \ldots, z_9 \text{ not all } 0\right\},\\
H_1 &= \left\{\left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : z_5 : \ldots : z_9\right] \in \mathbb{P}^9 \mid \right.\\
&\qquad\qquad \left. x_0, x_1, z_5, \ldots, z_9 \text{ not all } 0\right\},\\
H_2 &= \left\{\left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2 : z_6 : \ldots : z_9\right] \in \mathbb{P}^9 \mid \right.\\
&\qquad\qquad \left. x_0, x_1, z_6, \ldots, z_9 \text{ not all } 0\right\},\\
H_3 &= \left\{\left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2 : x_1x_3 : z_7 : z_8 : z_9\right] \in \mathbb{P}^9 \mid \right.\\
&\qquad\qquad \left. x_0, x_1, z_7, z_8, z_9 \text{ not all } 0\right\},\\
H_4 &= \left\{\left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : z_8 : z_9\right] \in \mathbb{P}^9 \mid \right.\\
&\qquad\qquad \left. x_0, x_1, x_2, z_8, z_9 \text{ not all } 0\right\},\\
H_5 &= \left\{\left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3 : z_9\right] \in \mathbb{P}^9 \mid \right.\\
&\qquad\qquad \left. x_0, x_1, x_2, z_9 \text{ not all } 0\right\},\\
H_6 &= \left\{\left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3 : x_3^2\right] \in \mathbb{P}^9 \mid \right.\\
&\qquad\qquad \left. x_0, x_1, x_2, x_3 \text{ not all } 0\right\} = V(\mathbb{P}^3).
\end{aligned}$$

(ii) Ternary Sextics

Let $n = 2$, $d = 3$ and we refer to Example 2.3.10 (ii) for an explicit description of the monomials $m_0, \ldots, m_k$. We compute $k - n = k(2,3) - 2 = 7$ and conclude

$$\begin{aligned}
H_0 &= \left\{\left[x_0^3 : x_0^2x_1 : x_0^2x_2 : z_3 : \ldots : z_9\right] \in \mathbb{P}^9 \mid x_0, z_3, \ldots, z_9 \text{ not all } 0\right\}, \\
H_1 &= \left\{\left[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1^2 : z_4 : \ldots : z_9\right] \in \mathbb{P}^9 \mid x_0, z_4, \ldots, z_9 \text{ not all } 0\right\}, \\
H_2 &= \left\{\left[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1^2 : x_0x_1x_2 : z_5 : \ldots : z_9\right] \in \mathbb{P}^9 \mid \right. \\
&\qquad \left. x_0, z_5, \ldots, z_9 \text{ not all } 0\right\}, \\
H_3 &= \left\{\left[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1^2 : x_0x_1x_2 : x_0x_2^2 : z_6 : \ldots : z_9\right] \in \mathbb{P}^9 \mid \right. \\
&\qquad \left. x_0, z_6, \ldots, z_9 \text{ not all } 0\right\}, \\
H_4 &= \left\{\left[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1^2 : x_0x_1x_2 : x_0x_2^2 : x_1^3 : z_7 : z_8 : z_9\right] \in \mathbb{P}^9 \mid \right. \\
&\qquad \left. x_0, x_1, z_7, z_8, z_9 \text{ not all } 0\right\}, \\
H_5 &= \left\{\left[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1^2 : x_0x_1x_2 : x_0x_2^2 : x_1^3 : x_1^2x_2 : z_8 : z_9\right] \in \mathbb{P}^9 \mid \right. \\
&\qquad \left. x_0, x_1, z_8, z_9 \text{ not all } 0\right\}, \\
H_6 &= \left\{\left[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1^2 : x_0x_1x_2 : x_0x_2^2 : x_1^3 : x_1^2x_2 : x_1x_2^2 : z_9\right] \in \mathbb{P}^9 \mid \right. \\
&\qquad \left. x_0, x_1, z_9 \text{ not all } 0\right\}, \\
H_7 &= \left\{\left[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1^2 : x_0x_1x_2 : x_0x_2^2 : x_1^3 : x_1^2x_2 : x_1x_2^2 : x_2^3\right] \in \mathbb{P}^9 \mid \right. \\
&\qquad \left. x_0, x_1, x_2 \text{ not all } 0\right\} = V(\mathbb{P}^2).
\end{aligned}$$

Lemma 3.2.5. *The following are true:*

(i) $V_0 = \mathbb{P}^k$.

(ii) $V_{k-n} = H_{k-n} = V(\mathbb{P}^n)$.

(iii) *If* $d \geq 2$, *then* $V_{i+1} \subseteq V_i$ *for* $i = 0, \ldots, k-n-1$.

Proof. Let $\mathbb{P}^k$ be endowed with the Zariski topology.

(i) ($\subseteq$) True by construction.
($\supseteq$) For $\boldsymbol{z} := (z_1, \ldots, z_k) \in \mathbb{C}^k$, we set $x_i := z_i$ for $i = 1, \ldots, n$. Hence, for $(1, \boldsymbol{x}) := (1, x_1, \ldots, x_n) \in \mathbb{C}^{n+1}$, we obtain

$$(1, z_1, \ldots, z_n) = (1, \boldsymbol{x}) = (m_0(1, \boldsymbol{x}), \ldots, m_n(1, \boldsymbol{x})).$$

This shows $\phi(\boldsymbol{z}) \in H_0$ and we conclude $\mathbb{P}^k = \overline{\phi(\mathbb{C}^k)} \subseteq \overline{H_0} = V_0$.

(ii) If $H_{k-n} = V(\mathbb{P}^n)$, then $V_{k-n} := \overline{H_{k-n}} = H_{k-n}$ since H_{k-n} is itself a projective variety. It thus suffices to show $H_{k-n} = V(\mathbb{P}^n)$.
($\subseteq$) For $[z] \in H_{k-n}$, we fix $x \in \mathbb{C}^{n+1}$ such that $z = (m_0(x), \ldots, m_k(x))$ and conclude $[z] = V([x]) \in V(\mathbb{P}^n)$.
($\supseteq$) For $[z] \in V(\mathbb{P}^n)$, we let $[x] \in \mathbb{P}^n$ be such that $[z] = V([x])$ and conclude $[z] = [m_0(x) : \ldots : m_k(x)] \in H_{k-n}$.

(iii) Corollary 2.3.6 states $k(n,d) > n$ if $d \geq 2$ and we know that $H_{i+1} \subseteq H_i$ for $i = 0, \ldots, k(n,d) - n - 1$ by Construction 3.2.1 (1). This allows us to conclude $V_{i+1} = \overline{H_{i+1}} \subseteq \overline{H_i} = V_i$ for $i = 0, \ldots, k(n,d) - n - 1$. ■

Remark 3.2.6. *In the special case that $d = 1$, we know by Lemma 2.3.5 (ii) that $k(n,d) - n = 0$. Hence, $V(\mathbb{P}^n) = V_{k-n} = V_0 = \mathbb{P}^n$ and V is indeed surjective if $d = 1$ by Lemma 2.3.27.*

Construction 3.2.1 in particular gives us a specific filtration of projective varieties

$$V(\mathbb{P}^n) = V_{k-n} \subseteq \ldots \subseteq V_0 = \mathbb{P}^k \tag{3.2}$$

with a corresponding specific filtration of sets of real points

$$V(\mathbb{P}^n)(\mathbb{R}) = V_{k-n}(\mathbb{R}) \subseteq \ldots \subseteq V_0(\mathbb{R}) = \mathbb{P}^k(\mathbb{R}). \tag{3.3}$$

Another equivalent way of thinking about the projective varieties $V_0, \ldots, V_{k-n}$ was pointed out to us by the anonymous referee of [GHK24b] and uses a standard construction from algebraic geometry. We refer an interested reader to Appendix A.1 for an overview.

Construction 3.2.7. For $i = 0, \ldots, k - n$, we let

(1) $\tilde{V}_i$ be the smallest projective variety in $\mathbb{P}^{n+i}$ w.r.t. $\subseteq$ containing the image of

$$\begin{aligned} \chi_i \colon \mathbb{P}^n &\dashrightarrow \mathbb{P}^{n+i} \\ [x] &\mapsto [m_0(x) : \ldots : m_{n+i}(x)]. \end{aligned}$$

(2) If $i < k - n$, then we set $V'_0 := \tilde{V}_i$ and, for $j = 1, \ldots, k - n - i$, let $V'_j \subseteq \mathbb{P}^{n+i+j}$ be the cone over $V'_{j-1} \subseteq \mathbb{P}^{n+i+j-1}$ with vertex $[0 : \ldots : 0 : 1] \in \mathbb{P}^{n+i+j}$ w.r.t. the embedding

$$\begin{aligned} \psi \colon \quad \mathbb{P}^{n+i+j-1} &\to \mathbb{P}^{n+i+j}, \\ [w_0 : \ldots : w_{n+i+j-1}] &\mapsto [w_0 : \ldots : w_{n+i+j-1} : 1]. \end{aligned}$$

(3) If $i < k - n$, then we set $V_i := V'_{k-n-i}$.

(4) If $i = k - n$, then we set $V_{k-n} := \tilde{V}_{k-n}$.

Remark 3.2.8. *Note that χ_{k-n} coincides with the Veronese embedding.*

Example 3.2.9. Let us determine the parametrizations $\chi_0, \ldots, \chi_{k-n}$ of $\tilde{V}_0, \ldots, \tilde{V}_{k-n}$ in the Hilbert cases.

(i) Binary Forms

Let $n = 1$ and we refer to Example 2.3.9 (i) for an explicit description of the monomials $m_0, \ldots, m_d$.

If $d = 1$, then $k - n = k(1,1) - 1 = 0$ and $\tilde{V}_0$ is parametrized by the map

$$\begin{aligned} \chi_0 \colon \mathbb{P}^1 &\dashrightarrow \mathbb{P}^1 \\ [x] &\mapsto [x]. \end{aligned}$$

If $d \geq 2$, then $k - n = k(1,d) - 1 = d - 1 \geq 1$ and we see that $\tilde{V}_0, \ldots, \tilde{V}_{d-1}$ are parametrized by the maps

$$\begin{aligned} \chi_0 \colon &\mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \\ &[x] \mapsto \left[x_0^d : x_0^{d-1} x_1\right], \\ &\vdots \\ \chi_{d-2} \colon &\mathbb{P}^1 \dashrightarrow \mathbb{P}^{d-1} \\ &[x] \mapsto \left[x_0^d : x_0^{d-1} x_1 : \ldots : x_0 x_1^{d-1}\right], \\ \chi_{d-1} \colon &\mathbb{P}^1 \dashrightarrow \mathbb{P}^d \\ &[x] \mapsto \left[x_0^d : x_0^{d-1} x_1 : \ldots : x_1^d\right]. \end{aligned}$$

(ii) QUADRATIC FORMS

Let $n \geq 1$, $d = 1$ and we refer to Example 2.3.9 (ii) for an explicit description of the monomials $m_0, \ldots, m_n$. We compute $k - n = k(n,1) - n = n - n = 0$ and see that $\tilde{V}_0$ is parametrized by the map

$$\begin{aligned} \chi_0 \colon \mathbb{P}^n &\dashrightarrow \mathbb{P}^n \\ [x] &\mapsto [x]. \end{aligned}$$

(iii) TERNARY QUARTICS

Let $n = 2$, $d = 2$ and we refer to Example 2.3.9 (iii) for an explicit description of the monomials $m_0, \ldots, m_5$. We compute $k - n = k(2,2) - 2 = 3$ and see that $\tilde{V}_0, \ldots, \tilde{V}_3$ are parametrized by the maps

$$\begin{aligned} \chi_0 \colon &\mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \\ &[x] \mapsto \left[x_0^2 : x_0 x_1 : x_0 x_2\right], \\ \chi_1 \colon &\mathbb{P}^2 \dashrightarrow \mathbb{P}^3 \\ &[x] \mapsto \left[x_0^2 : x_0 x_1 : x_0 x_2 : x_1^2\right], \\ \chi_2 \colon &\mathbb{P}^2 \dashrightarrow \mathbb{P}^4 \\ &[x] \mapsto \left[x_0^2 : x_0 x_1 : x_0 x_2 : x_1^2 : x_1 x_2\right], \\ \chi_3 \colon &\mathbb{P}^2 \dashrightarrow \mathbb{P}^5 \\ &[x] \mapsto \left[x_0^2 : x_0 x_1 : x_0 x_2 : x_1^2 : x_1 x_2 : x_2^2\right]. \end{aligned}$$

Example 3.2.10. Let us determine the parametrizations $\chi_0, \dots, \chi_{k-n}$ of $\tilde{V}_0, \dots, \tilde{V}_{k-n}$ in the basic non-Hilbert cases.

(i) Quaternary Quartics
Let $n = 3$, $d = 2$ and we refer to Example 2.3.10 (i) for an explicit description of the monomials $m_0, \dots, m_9$. We compute $k - n = k(3,2) - 3 = 6$ and see that $\tilde{V}_0, \dots, \tilde{V}_6$ are parametrized by the maps

$$\begin{aligned}
\chi_0 \colon \mathbb{P}^3 &\dashrightarrow \mathbb{P}^3 \\
[x] &\mapsto \left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3\right], \\
\chi_1 \colon \mathbb{P}^3 &\dashrightarrow \mathbb{P}^4 \\
[x] &\mapsto \left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2\right], \\
\chi_2 \colon \mathbb{P}^3 &\dashrightarrow \mathbb{P}^5 \\
[x] &\mapsto \left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2\right], \\
\chi_3 \colon \mathbb{P}^3 &\dashrightarrow \mathbb{P}^6 \\
[x] &\mapsto \left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2 : x_1x_3\right], \\
\chi_4 \colon \mathbb{P}^3 &\dashrightarrow \mathbb{P}^7 \\
[x] &\mapsto \left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2\right], \\
\chi_5 \colon \mathbb{P}^3 &\dashrightarrow \mathbb{P}^8 \\
[x] &\mapsto \left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3\right], \\
\chi_6 \colon \mathbb{P}^3 &\dashrightarrow \mathbb{P}^9 \\
[x] &\mapsto \left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3 : x_3^2\right].
\end{aligned}$$

(ii) Ternary Sextics
Let $n = 2$, $d = 3$ and we refer to Example 2.3.10 (ii) for an explicit description of the monomials $m_0, \dots, m_9$. We compute $k - n = k(2,3) - 2 = 7$ and see that $\tilde{V}_0, \dots, \tilde{V}_7$ are parametrized by the maps

$$\begin{aligned}
\chi_0 \colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^2 \\
[x] &\mapsto \left[x_0^3 : x_0^2x_1 : x_0^2x_2\right], \\
\chi_1 \colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^3 \\
[x] &\mapsto \left[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1^2\right], \\
\chi_2 \colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^4 \\
[x] &\mapsto \left[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1^2 : x_0x_1x_2\right], \\
\chi_3 \colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^5 \\
[x] &\mapsto \left[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1^2 : x_0x_1x_2 : x_0x_2^2\right],
\end{aligned}$$

$$\begin{aligned}
\chi_4\colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^6\\
[x] &\mapsto \left[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1^2 : x_0x_1x_2 : x_0x_2^2 : x_1^3\right],\\
\chi_5\colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^7\\
[x] &\mapsto \left[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1^2 : x_0x_1x_2 : x_0x_2^2 : x_1^3 : x_1^2x_2\right],\\
\chi_6\colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^8\\
[x] &\mapsto \left[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1^2 : x_0x_1x_2 : x_0x_2^2 : x_1^3 : x_1^2x_2 : x_1x_2^2\right],\\
\chi_7\colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^9\\
[x] &\mapsto \left[x_0^3 : x_0^2x_1 : x_0^2x_2 : x_0x_1^2 : x_0x_1x_2 : x_0x_2^2 : x_1^3 : x_1^2x_2 : x_1x_2^2 : x_2^3\right].
\end{aligned}$$

Before proceeding with the construction of a specific filtration of cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ along the filtration of projective varieties given in (3.2), we verify below that each inclusion in (3.3) is strict for $d \geq 2$. This will guarantee that no a priori obvious cone equality appears in the induced intermediate cone filtration.

Lemma 3.2.11. *For $i = 0, \ldots, k-n-1$, the inclusion $V_{i+1}(\mathbb{R}) \subsetneq V_i(\mathbb{R})$ is strict.*

Proof. We set $y := (y_0, \ldots, y_k) \in \mathbb{R}^{k+1}$ to be given by

$$y_s := \begin{cases} 1, & \text{for } s = 0, \ldots, n+i \\ 0, & \text{for } s = n+i+1, \ldots, k \end{cases}$$

and denote the all-one vector in $\mathbb{R}^{n+1}$ by e. Hence,

$$(y_0, \ldots, y_{n+i}) = (m_0(e), \ldots, m_{n+i}(e))$$

which shows $[y] \in H_i(\mathbb{R}) \subseteq V_i(\mathbb{R})$. It thus suffices to prove $[y] \notin V_{i+1}(\mathbb{R})$.

For $j \in \{i, i+1\}$, we set $G_j := \mathcal{S}_V \cap \mathbb{C}[Z_0, \ldots, Z_{n+j}]$ (cf. Notation 2.3.28) and, for $[z] \in H_j$, fix $x \in \mathbb{C}^{n+1}$ such that $(z_0, \ldots, z_{n+j}) = (m_0(x), \ldots, m_{n+j}(x))$. This allows us to observe for $g \in G_j$ that

$$0 = g(z) = g(m_0(x), \ldots, m_{n+j}(x), \ldots, m_k(x))$$

since $[m_0(x) : \ldots : m_k(x)] \in V(\mathbb{P}^n)$ and $V(\mathbb{P}^n) = \mathcal{V}(S_V)$ by Theorem 2.3.29. Hence, $[z] \in \mathcal{V}(G_j)$ and we conclude $H_j \subseteq \mathcal{V}(G_j)$ since $[z] \in H_j$ was arbitrarily chosen. Therefore, we have $V_j = \overline{H_j} \subseteq \mathcal{V}(G_j)$.

Lemma 2.3.31 moreover allows us to fix some $1 \leq s \leq n$ and $s \leq t < n + (i+1)$ such that $\alpha_s + \alpha_t = \alpha_0 + \alpha_{n+(i+1)}$. We set

$$q(Z) := Z_0 Z_{n+(i+1)} - Z_s Z_t \in G_{i+1}$$

and compute $q(y) = (-1) \neq 0$. This shows $[y] \notin \mathcal{V}(G_{i+1})(\mathbb{R})$. Using $V_{i+1} \subseteq \mathcal{V}(G_{i+1})$, we conclude $[y] \notin V_{i+1}(\mathbb{R})$. ■

Corollary 3.2.12. *For $i = 0, \ldots, k-n-1$, the inclusion $V_{i+1} \subsetneq V_i$ is strict.*

Proof. Immediate consequence of Lemma 3.2.11. ■

Lemma 3.2.11 and Corollary 3.2.12 together show the strictness of each inclusion in the specific filtration of projective varieties

$$V(\mathbb{P}^n) = V_{k-n} \subsetneq \ldots \subsetneq V_0 = \mathbb{P}^k \tag{3.4}$$

and also of each inclusion in the corresponding specific filtration of sets of real points

$$V(\mathbb{P}^n)(\mathbb{R}) = V_{k-n}(\mathbb{R}) \subsetneq \ldots \subsetneq V_0(\mathbb{R}) = \mathbb{P}^k(\mathbb{R}). \tag{3.5}$$

Construction 3.2.13. For $i = 0, \ldots, k-n$, we set $C_i := C_{V_i}$.

Lemma 3.1.2 (iii) and Lemma 3.2.5 (i) together imply

$$C_0 = C_{V_0} = C_{\mathbb{P}^k} = \Sigma_{n+1,2d}.$$

Likewise, Lemma 3.1.2 (ii) and Lemma 3.2.5 (ii) together show that

$$C_{k-n} = C_{V_{k-n}} = C_{V(\mathbb{P}^n)} = \mathcal{P}_{n+1,2d}.$$

Thus, we induced a specific cone filtration

$$\Sigma_{n+1,2d} = C_0 \subseteq \ldots \subseteq C_{k-n} = \mathcal{P}_{n+1,2d} \tag{$\mathcal{CF}$}$$

along the filtration of projective varieties given in (3.4) and each C_i in ($\mathcal{CF}$) is in particular an intermediate cone between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$.

Corollary 3.2.14. *For $i = 0, \ldots, k-n$, the cone C_i is pointed, full-dimensional and contains no straight lines.*

Proof. Follows from Corollary 3.1.5 by setting $W := V_i$. ■

Our consideration from Observation 3.1.6 extend to the cone filtration ($\mathcal{CF}$). That is, if $(n+1, 2d)$ denotes a Hilbert case, then ($\mathcal{CF}$) is given by

$$\Sigma_{n+1,2d} = C_0 = \ldots = C_{k-n} = \mathcal{P}_{n+1,2d}. \tag{3.6}$$

Hence, all inclusions in ($\mathcal{CF}$) are as a matter of fact equalities in Hilbert-cases.

However, if $(n+1, 2d)$ denotes a non-Hilbert case, then, by Hilbert's 1888 theorem (cf. Theorem 2.2.22), at least one of the inclusions in ($\mathcal{CF}$) has to be strict, but it is a priori not clear how many of the inclusions are strict and which ones. This brings us to the main query of this thesis whose answer will allow us to give a refinement of Hilbert's 1888 theorem.

Main Query. Which of the inclusions in ($\mathcal{CF}$) are strict in non-Hilbert cases?

Lemma 3.2.15. *If $(n+1, 2d)$ is a non-Hilbert case, then, for $f \in \Delta_{n+1,2d}$, there exists a unique $0 \leq i \leq k-n-1$ such that $f \in C_{i+1} \backslash C_i$.*

Proof. We set $I := \{i \in \{0, \ldots, k-n\} \mid f \in C_i\}$ and observe that I is non-empty since $f \in \mathcal{P}_{n+1,2d} = C_{k-n}$. Moreover, we deduce $\min(I) \geq 1$ from $f \notin \Sigma_{n+1,2d} = C_0$. For $i := \min(I) - 1 \in \{0, \ldots, k-n-1\}$, we furthermore have $i+1 = \min(I)$ by choice. Hence, $i+1 \in I$ from which it follows $f \in C_{i+1}$ and $i < \min(I)$ yields $i \notin I$. Consequently, $f \notin C_i$.

To prove the uniqueness of i, we let $0 \leq j \leq k-n-1$ be such that $f \in C_{j+1} \backslash C_j$. Hence, $f \in C_{j+1}$ implies $j+1 \in I$ and consequently $\min(I) \leq j+1$. Moreover, $f \notin C_j$ yields $j \leq \min(I) - 1$ since $C_0 \subseteq \ldots \subseteq C_{k-n}$ is a cone filtration. Altogether, $j = \min(I) - 1 = i$ follows. ■

Notation 3.2.16. Let $(n+1, 2d)$ be a non-Hilbert case. For $f \in \Delta_{n+1,2d}$, we denote the unique $0 \leq i \leq k-n-1$ such that $f \in C_{i+1} \backslash C_i$ by $i(f)$.

Remark 3.2.17. *In particular,*

$$\begin{aligned} i(f) &= \min\{i \in \{0, \ldots, k-n\} \mid f \in C_i\} - 1 \\ &= \max\{i \in \{0, \ldots, k-n-1\} \mid f \notin C_i\}. \end{aligned}$$

Definition 3.2.18. A cone $C \subseteq \mathcal{F}_{n+1,2d}$ is called a *strictly separating* intermediate cone between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ if $\Sigma_{n+1,2d} \subsetneq C \subsetneq \mathcal{P}_{n+1,2d}$.

Notation 3.2.19. We introduce the binary map

$$\begin{aligned} \mu \colon \mathbb{N} \times \mathbb{N} &\to \mathbb{N}_0 \\ (n, d) &\mapsto |\{i \in \{0, \ldots, k-n\} \mid \Sigma_{n+1,2d} \subsetneq C_i \subsetneq \mathcal{P}_{n+1,2d}\}|. \end{aligned}$$

The value $\mu(n, d)$ thus represents the number of strictly separating intermediate cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ in ($\mathcal{CF}$). In a Hilbert case, there are no strictly separating intermediate cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$. Hence,

$$\mu(n, d) = 0 \text{ if } n = 1 \text{ or } d = 1 \text{ or } (n+1, 2d) = (3, 4).$$

It therefore remains to determine $\mu(n, d)$ in any non-Hilbert case.

Lemma 3.2.20. *If $(n+1, 2d)$ is a non-Hilbert case, then exactly $\mu(n, d) + 1$ of the inclusions in ($\mathcal{CF}$) are strict.*

Proof. There are $k-n$ inclusions to be investigated in ($\mathcal{CF}$) of which at least one has to be strict since $(n+1, 2d)$ is a non-Hilbert case.

We set $1 \leq \mu \leq k-n$ to be the number of strict inclusions in ($\mathcal{CF}$). This allows us to reduce the specific cone filtration ($\mathcal{CF}$) to a cone subfiltration

$$\Sigma_{n+1,2d} = C'_0 \subsetneq \dots \subsetneq C'_\mu = \mathcal{P}_{n+1,2d} \tag{3.7}$$

in which each inclusion is strict. There are $\mu - 1$ pairwise distinct, strictly separating intermediate cones C'_i between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ in (3.7). Consequently, at least $\mu - 1$ pairwise distinct C_i in ($\mathcal{CF}$) are strictly separating intermediate cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$. This shows $\mu - 1 \leq \mu(n, d)$.

Vice versa, the fact that there are $\mu(n, d)$ strictly separating intermediate cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ in ($\mathcal{CF}$) yields that at least $\mu(n, d) + 1$ of the inclusions in ($\mathcal{CF}$) are strict. Consequently, $\mu(n+1, 2d) + 1 \leq \mu$. ■

Lemma 3.2.21. *If $(n+1, 2d)$ is a non-Hilbert case, then the following are true:*

(i) For $S \subseteq \Delta_{n+1,2d}$, it holds $|\{i(f) \mid f \in S\}| \leq \mu(n, d) + 1$.

(ii) There exists some $S \subseteq \Delta_{n+1,2d}$ such that $|\{i(f) \mid f \in S\}| = \mu(n, d) + 1$.

Proof. Lemma 3.2.20 states that $\mu(n, d) + 1$ of the inclusions in ($\mathcal{CF}$) are strict.

(i) Any form $f \in S$ testifies the strict inclusion $C_{i(f)} \subsetneq C_{i(f)+1}$ in ($\mathcal{CF}$).

(ii) For each strict inclusion $C_i \subsetneq C_{i+1}$ in ($\mathcal{CF}$), we fix some $f \in C_{i+1} \backslash C_i$ and conclude $i(f) = i$ from Lemma 3.2.15. The collection of all these separating forms gives us a set $S \subseteq \Delta_{n+1,2d}$ consisting of $\mu(n, d) + 1$ forms f with pairwise distinct $i(f)$. ■

Definition 3.2.22. Let $(n+1, 2d)$ be a non-Hilbert case. A set $S \subseteq \Delta_{n+1,2d}$ is called a *complete set of separating forms* for ($\mathcal{CF}$) if $|\{i(f) \mid f \in S\}| = \mu(n, d) + 1$.

3.3 An Explicit Description of $V_0, \dots, V_{k-n}$ in Non-Hilbert Cases

In a first step towards a complete answer to the main query, we give an explicit description of the projective varieties $V_0, \dots, V_{k-n}$ along the quadratic forms in the generating set $\mathcal{S}_V$ of the Veronese variety (cf. Notation 2.3.28 and Theorem 2.3.29) in the non-Hilbert cases below. To this end, we let **$(n+1, 2d)$ denote a non-Hilbert case** throughout this section.

Construction 3.3.1. For $i = 1, \dots, k - n$,[2]

(1) we set $s_i := \min\{s \in \{1, \dots, n\} \mid \exists\, t \in \{s, \dots, n+i-1\} \colon \alpha_s + \alpha_t = \alpha_0 + \alpha_{n+i}\}$ and fix $s_i \leq t_i \leq n+i-1$ correspondingly such that $\alpha_{s_i} + \alpha_{t_i} = \alpha_0 + \alpha_{n+i}$.[3]

(2) Moreover, we define

$$q_i(Z) := Z_0 Z_{n+i} - Z_{s_i} Z_{t_i} \in \mathcal{S}_V, \tag{3.8}$$

$$p_i(\boldsymbol{Z}) := q_i(1, \boldsymbol{Z}) \in \mathbb{R}[\boldsymbol{Z}]. \tag{3.9}$$

Remark 3.3.2. *By construction,* $p_i^h = q_i$.

Example 3.3.3. Quaternary Quartics
Let us explicitly carry out Construction 3.3.1 in the basic non-Hilbert case $(4,4)$. Hence, $n = 3$, $d = 2$ and we recall from Example 2.3.10 (i) that

$$\begin{aligned}
&\alpha_0 = (2,0,0,0), \quad \alpha_2 = (1,0,1,0), \quad \alpha_4 = (0,2,0,0), \quad \alpha_6 = (0,1,0,1), \quad \alpha_8 = (0,0,1,1),\\
&\alpha_1 = (1,1,0,0), \quad \alpha_3 = (1,0,0,1), \quad \alpha_5 = (0,1,1,0), \quad \alpha_7 = (0,0,2,0), \quad \alpha_9 = (0,0,0,2).
\end{aligned}$$

This allows us to compute

$$\begin{aligned}
s_1 &= \min\{s \in \{1,2,3\} \mid \exists\, t \in \{s,\dots,3\} \colon \alpha_s + \alpha_t = (2,2,0,0)\} = 1 \quad \text{and} \quad t_1 = 1,\\
s_2 &= \min\{s \in \{1,2,3\} \mid \exists\, t \in \{s,\dots,4\} \colon \alpha_s + \alpha_t = (2,1,1,0)\} = 1 \quad \text{and} \quad t_2 = 2,\\
s_3 &= \min\{s \in \{1,2,3\} \mid \exists\, t \in \{s,\dots,5\} \colon \alpha_s + \alpha_t = (2,1,0,1)\} = 1 \quad \text{and} \quad t_3 = 3,\\
s_4 &= \min\{s \in \{1,2,3\} \mid \exists\, t \in \{s,\dots,6\} \colon \alpha_s + \alpha_t = (2,0,2,0)\} = 2 \quad \text{and} \quad t_4 = 2,\\
s_5 &= \min\{s \in \{1,2,3\} \mid \exists\, t \in \{s,\dots,7\} \colon \alpha_s + \alpha_t = (2,0,1,1)\} = 2 \quad \text{and} \quad t_5 = 3,\\
s_6 &= \min\{s \in \{1,2,3\} \mid \exists\, t \in \{s,\dots,8\} \colon \alpha_s + \alpha_t = (2,0,0,2)\} = 3 \quad \text{and} \quad t_6 = 3.
\end{aligned}$$

Consequently,

$$\begin{aligned}
q_1(Z) &= Z_0 Z_4 - Z_1^2 &&\text{and}&& p_1(\boldsymbol{Z}) = Z_4 - Z_1^2,\\
q_2(Z) &= Z_0 Z_5 - Z_1 Z_2 &&\text{and}&& p_2(\boldsymbol{Z}) = Z_5 - Z_1 Z_2,\\
q_3(Z) &= Z_0 Z_6 - Z_1 Z_3 &&\text{and}&& p_3(\boldsymbol{Z}) = Z_6 - Z_1 Z_3,\\
q_4(Z) &= Z_0 Z_7 - Z_2^2 &&\text{and}&& p_4(\boldsymbol{Z}) = Z_7 - Z_2^2,\\
q_5(Z) &= Z_0 Z_8 - Z_2 Z_3 &&\text{and}&& p_5(\boldsymbol{Z}) = Z_8 - Z_2 Z_3,\\
q_6(Z) &= Z_0 Z_9 - Z_3^2 &&\text{and}&& p_6(\boldsymbol{Z}) = Z_9 - Z_3^2.
\end{aligned}$$

[2] Corollary 2.3.6 ensures $k - n = k(n,d) - n > 0$ in any non-Hilbert case.

[3] Lemma 2.3.31 ensures the existence of some $1 \leq s \leq n$ and some $s \leq t < n+i$ such that $\alpha_s + \alpha_t = \alpha_0 + \alpha_{n+i}$ for $i = 1, \dots, k-n$. Hence, s_i is well-defined.

Example 3.3.4. Ternary Sextics
Let us explicitly carry out Construction 3.3.1 in the basic non-Hilbert case $(3, 6)$. Hence, $n = 2$, $d = 3$ and we recall from Example 2.3.10 (ii) that

$$\begin{aligned} &\alpha_0 = (3,0,0), && \alpha_2 = (2,0,1), && \alpha_4 = (1,1,1), && \alpha_6 = (0,3,0), && \alpha_8 = (0,1,2), \\ &\alpha_1 = (2,1,0), && \alpha_3 = (1,2,0), && \alpha_5 = (1,0,2), && \alpha_7 = (0,2,1), && \alpha_9 = (0,0,3). \end{aligned}$$

This allows us to compute

$$\begin{aligned} s_1 &= \min\{s \in \{1,2\} \mid \exists\, t \in \{s,\dots,2\} \colon \alpha_s + \alpha_t = (4,2,0)\} = 1 \quad \text{and} \quad t_1 = 1, \\ s_2 &= \min\{s \in \{1,2\} \mid \exists\, t \in \{s,\dots,3\} \colon \alpha_s + \alpha_t = (4,1,1)\} = 1 \quad \text{and} \quad t_2 = 2, \\ s_3 &= \min\{s \in \{1,2\} \mid \exists\, t \in \{s,\dots,4\} \colon \alpha_s + \alpha_t = (4,0,2)\} = 2 \quad \text{and} \quad t_3 = 2, \\ s_4 &= \min\{s \in \{1,2\} \mid \exists\, t \in \{s,\dots,5\} \colon \alpha_s + \alpha_t = (3,3,0)\} = 1 \quad \text{and} \quad t_4 = 3, \\ s_5 &= \min\{s \in \{1,2\} \mid \exists\, t \in \{s,\dots,6\} \colon \alpha_s + \alpha_t = (3,2,1)\} = 1 \quad \text{and} \quad t_5 = 4, \\ s_6 &= \min\{s \in \{1,2\} \mid \exists\, t \in \{s,\dots,7\} \colon \alpha_s + \alpha_t = (3,1,2)\} = 1 \quad \text{and} \quad t_6 = 5, \\ s_7 &= \min\{s \in \{1,2\} \mid \exists\, t \in \{s,\dots,8\} \colon \alpha_s + \alpha_t = (3,0,3)\} = 2 \quad \text{and} \quad t_7 = 5. \end{aligned}$$

Consequently,

$$\begin{aligned} q_1(Z) &= Z_0Z_3 - Z_1^2 && \text{and} && p_1(\boldsymbol{Z}) = Z_3 - Z_1^2, \\ q_2(Z) &= Z_0Z_4 - Z_1Z_2 && \text{and} && p_2(\boldsymbol{Z}) = Z_4 - Z_1Z_2, \\ q_3(Z) &= Z_0Z_5 - Z_2^2 && \text{and} && p_3(\boldsymbol{Z}) = Z_5 - Z_2^2, \\ q_4(Z) &= Z_0Z_6 - Z_1Z_3 && \text{and} && p_4(\boldsymbol{Z}) = Z_6 - Z_1Z_3, \\ q_5(Z) &= Z_0Z_7 - Z_1Z_4 && \text{and} && p_5(\boldsymbol{Z}) = Z_7 - Z_1Z_4, \\ q_6(Z) &= Z_0Z_8 - Z_1Z_5 && \text{and} && p_6(\boldsymbol{Z}) = Z_8 - Z_1Z_5, \\ q_7(Z) &= Z_0Z_9 - Z_2Z_5 && \text{and} && p_7(\boldsymbol{Z}) = Z_9 - Z_2Z_5. \end{aligned}$$

Lemma 3.3.5. *For $i = 1, \dots, n$, it holds $q_i(Z) = Z_0Z_{n+i} - Z_1Z_i$.*

Proof. We set $e_i := (e_{i,0}, \dots, e_{i,n}) \in \mathbb{N}_0^{n+1}$ to be given by

$$e_{i,s} := \begin{cases} 1, & \text{if } s = i \\ 0, & \text{else} \end{cases}$$

and observe that $\alpha_0 = (d, 0, \dots, 0)$, $\alpha_1 = (d-1, 1, 0, \dots, 0)$, $\alpha_i = (d-1, 0, \dots, 0) + e_i$ and $\alpha_{n+i} = (d-2, 1, 0, \dots, 0) + e_i$. Hence,

$$\alpha_0 + \alpha_{n+i} = (2d-2, 1, 0, \dots, 0) + e_i = \alpha_1 + \alpha_i.$$

Consequently, $s_i = 1$, $t_i = i$ and we conclude from Construction 3.3.1 (2) that

$$q_i(Z) = Z_0 Z_{n+i} - Z_1 Z_i. \qquad \blacksquare$$

Corollary 3.3.6. *For $i = 1, \ldots, n$, it holds $p_i(\boldsymbol{Z}) = Z_{n+i} - Z_1 Z_i$.*

Proof. Construction 3.3.1 and Lemma 3.3.5 together imply

$$p_i(\boldsymbol{Z}) = q_i(1, \boldsymbol{Z}) = Z_{n+i} - Z_1 Z_i. \qquad \blacksquare$$

Construction 3.3.7. (1) We set $K_0 := \mathbb{C}^k$.

(2) For $i = 1, \ldots, k-n$, we set $K_i := \boldsymbol{\mathcal{V}}(p_1, \ldots, p_i) \subseteq \mathbb{C}^k$.

(3) For $i = 0, \ldots, k-n$, we let $W_i \subseteq \mathbb{P}^k$ be the projective closure of the affine variety $K_i \subseteq \mathbb{C}^k$.[4] That is, W_i is the Zariski closure of K_i under the embedding

$$\begin{array}{rccc} \phi\colon & \mathbb{C}^k & \to & \mathbb{P}^k \\ & \boldsymbol{z} & \mapsto & [1 : \boldsymbol{z}]. \end{array}$$

Remark 3.3.8. *$W_0 = \mathcal{V}(0) = \mathbb{P}^k$ and $W_i = \mathcal{V}(\langle p_1, \ldots, p_i \rangle^h) \subseteq \mathbb{P}^k$ by Theorem A.1.58.*

Lemma 3.3.9. *The following are true:*

(i) For $i = 0, \ldots, k-n-1$,

$$K_i = \{(m_1(1, \boldsymbol{x}), \ldots, m_{n+i}(1, \boldsymbol{x}), z_{n+i+1}, \ldots, z_k) \mid \boldsymbol{x} \in \mathbb{C}^n, z_{n+i+1}, \ldots, z_k \in \mathbb{C}\}.$$

(ii) $K_{k-n} = \{(m_1(1, \boldsymbol{x}), \ldots, m_k(1, \boldsymbol{x})) \mid \boldsymbol{x} \in \mathbb{C}^n\}$.

Proof. (i) ($\subseteq$) For $\boldsymbol{z} \in K_i$, we set $\boldsymbol{x} := (z_1, \ldots, z_n) \in \mathbb{C}^n$ and, for $j = 1, \ldots, n$, observe that $m_j(1, \boldsymbol{x}) = 1^{d-1} z_j = z_j$.

If $i = 0$, then we are done since $\boldsymbol{z} = (m_1(1, \boldsymbol{x}), \ldots, m_n(1, \boldsymbol{x}), z_{n+1}, \ldots, z_k)$.

However, if $i \geq 1$, then, for $j = n+1, \ldots, n+i$, we observe

$$\begin{aligned} 0 &= p_{j-n}(z_1, \ldots, z_k) \\ &= q_{j-n}(1, z_1, \ldots, z_k) \\ &= z_j - z_{s_{j-n}} z_{t_{j-n}} \end{aligned}$$

since $\boldsymbol{z} \in K_i = \boldsymbol{\mathcal{V}}(p_1, \ldots, p_i)$ by choice. Moreover, we recall that $1 \leq s_{j-n} \leq n$, $s_{j-n} \leq t_{j-n} \leq j-1$ and $\alpha_{s_{j-n}} + \alpha_{t_{j-n}} = \alpha_0 + \alpha_j$ by Construction 3.3.1 (1) and

[4] We refer to Appendix A.1.3 for an overview on the projective closure of an affine variety.

$m_0(1, \boldsymbol{x}) = 1^d = 1$. This allows us to iteratively conclude

$$z_j = z_{s_{j-n}} z_{t_{j-n}} = m_{s_{j-n}}(1, \boldsymbol{x}) m_{t_{j-n}}(1, \boldsymbol{x}) = m_0(1, \boldsymbol{x}) m_j(1, \boldsymbol{x}) = m_j(1, \boldsymbol{x})$$

for $j = n+1, \dots, n+i$ which shows $\boldsymbol{z} = (m_1(1, \boldsymbol{x}), \dots, m_{n+i}(1, \boldsymbol{x}), z_{n+i+1}, \dots, z_k)$.

($\supseteq$) If $i = 0$, then $K_i = \mathbb{C}^k$ by Construction 3.3.7 (1). Hence,

$$(m_1(1, \mathbf{x}), \dots, m_{n+i}(1, \mathbf{x}), z_{n+i+1}, \dots : z_k) \in K_i$$

follows for any $\boldsymbol{x} \in \mathbb{C}^n$ and any $z_{n+i+1}, \dots, z_k \in \mathbb{C}$.

However, if $i \geq 1$, then, for $\boldsymbol{x} \in \mathbb{C}^n$, $z_{n+i+1}, \dots, z_k \in \mathbb{C}$ and $j = 1, \dots, i$, using $m_0(1, \boldsymbol{x}) = 1^d = 1$, we compute

$$\begin{aligned} & p_j(m_1(1, \boldsymbol{x}), \dots, m_{n+i}(1, \boldsymbol{x}), z_{n+i+1}, \dots, z_k) \\ &= q_j(1, m_1(1, \boldsymbol{x}), \dots, m_{n+i}(1, \boldsymbol{x}), z_{n+i+1}, \dots, z_k) \\ &= q_j(m_0(1, \boldsymbol{x}), m_1(1, \boldsymbol{x}), \dots, m_{n+i}(1, \boldsymbol{x}), z_{n+i+1}, \dots, z_k) \\ &= m_0(1, \boldsymbol{x}) m_{n+j}(1, \boldsymbol{x}) - m_{s_j}(1, \boldsymbol{x}) m_{t_j}(1, \boldsymbol{x}) = 0 \end{aligned}$$

since $1 \leq s_j \leq n$, $s_j \leq t_j \leq n + i - 1$ and $\alpha_{s_j} + \alpha_{t_j} = \alpha_0 + \alpha_{n+j}$ according to Construction 3.3.1 (1). Hence,

$$(m_1(1, \mathbf{x}), \dots, m_{n+i}(1, \mathbf{x}), z_{n+i+1}, \dots, z_k) \in \boldsymbol{\mathcal{V}}(p_1, \dots, p_i) = K_i.$$

(ii) ($\subseteq$) Analogously as in ($\subseteq$) of (i), for $\boldsymbol{z} \in K_i$, we set $\boldsymbol{x} := (z_1, \dots, z_n) \in \mathbb{C}^n$ and compute $m_j(1, \boldsymbol{x}) = 1^{d-1} z_j = z_j$ for $j = 1, \dots, n$. Moreover, we observe

$$\begin{aligned} 0 &= p_{j-n}(z_1, \dots, z_k) \\ &= q_{j-n}(1, z_1, \dots, z_k) \\ &= z_j - z_{s_{j-n}} z_{t_{j-n}} \end{aligned}$$

for $j = n+1, \dots, k$ since $\boldsymbol{z} \in K_i = \boldsymbol{\mathcal{V}}(p_1, \dots, p_i)$ by choice. Using the fact that $1 \leq s_{j-n} \leq n$, $s_{j-n} \leq t_{j-n} \leq j - 1$ and $\alpha_{s_{j-n}} + \alpha_{t_{j-n}} = \alpha_0 + \alpha_j$ by Construction 3.3.1 (1) and $m_0(1, \boldsymbol{x}) = 1^d = 1$, we thus conclude iteratively for $j = n+1, \dots, k$ that

$$z_j = z_{s_{j-n}} z_{t_{j-n}} = m_{s_{j-n}}(1, \boldsymbol{x}) m_{t_{j-n}}(1, \boldsymbol{x}) = m_0(1, \boldsymbol{x}) m_j(1, \boldsymbol{x}) = m_j(1, \boldsymbol{x}).$$

This shows $\boldsymbol{z} = (m_1(1, \boldsymbol{x}), \dots, m_k(1, \boldsymbol{x}))$.

($\supseteq$) Analogously as in ($\supseteq$) of (i), for $\boldsymbol{x} \in \mathbb{C}^n$ and $j = 1, \dots, k-n$, using

$m_0(1,\boldsymbol{x}) = 1^d = 1$, we compute

$$\begin{aligned} p_j(m_1(1,\boldsymbol{x}),\ldots,m_k(1,\boldsymbol{x})) &= q_j(m_0(1,\boldsymbol{x}),m_1(1,\boldsymbol{x}),\ldots,m_k(1,\boldsymbol{x})) \\ &= m_0(1,\boldsymbol{x})m_{n+j}(1,\boldsymbol{x}) - m_{s_j}(1,\boldsymbol{x})m_{t_j}(1,\boldsymbol{x}) = 0 \end{aligned}$$

since $\alpha_{s_j} + \alpha_{t_j} = \alpha_0 + \alpha_{n+j}$ by Construction 3.3.1 (1). Hence,

$$(m_1(1,\mathbf{x}),\ldots,m_k(1,\mathbf{x})) \in \boldsymbol{\mathcal{V}}(p_1,\ldots,p_{k-n}) = K_{k-n}. \qquad \blacksquare$$

Example 3.3.10. Quaternary Quartics

Let us determine $K_0,\ldots,K_{k-n}$ in the basic non-Hilbert case $(4,4)$. Hence, $n=3$, $d=2$, $k-n = k(3,2)-3 = 6$ and we refer to Example 2.3.10 (i) for an explicit description of the monomials $m_0,\ldots,m_9$. Using Lemma 3.3.9, we compute

$$\begin{aligned}
K_0 &= \{(x_1,x_2,x_3,z_4,\ldots,z_9) \mid x_1,x_2,x_3,z_4,\ldots,z_9 \in \mathbb{C}\}, \\
K_1 &= \left\{\left(x_1,x_2,x_3,x_1^2,z_5,\ldots,z_9\right) \mid x_1,x_2,x_3,z_5,\ldots,z_9 \in \mathbb{C}\right\}, \\
K_2 &= \left\{\left(x_1,x_2,x_3,x_1^2,x_1x_2,z_6,\ldots,z_9\right) \mid x_1,x_2,x_3,z_6,\ldots,z_9 \in \mathbb{C}\right\}, \\
K_3 &= \left\{\left(x_1,x_2,x_3,x_1^2,x_1x_2,x_1x_3,z_7,z_8,z_9\right) \mid x_1,x_2,x_3,z_7,z_8,z_9 \in \mathbb{C}\right\}, \\
K_4 &= \left\{\left(x_1,x_2,x_3,x_1^2,x_1x_2,x_1x_3,x_2^2,z_8,z_9\right) \mid x_1,x_2,x_3,z_8,z_9 \in \mathbb{C}\right\}, \\
K_5 &= \left\{\left(x_1,x_2,x_3,x_1^2,x_1x_2,x_1x_3,x_2^2,x_2x_3,z_9\right) \mid x_1,x_2,x_3,z_9 \in \mathbb{C}\right\}, \\
K_6 &= \left\{\left(x_1,x_2,x_3,x_1^2,x_1x_2,x_1x_3,x_2^2,x_2x_3,x_3^2\right) \mid x_1,x_2,x_3 \in \mathbb{C}\right\}.
\end{aligned}$$

Example 3.3.11. Ternary Sextics

Let us determine $K_0,\ldots,K_{k-n}$ in the basic non-Hilbert case $(3,6)$. Hence, $n=2$, $d=3$, $k-n = k(2,3)-2 = 7$ and we refer to Example 2.3.10 (ii) for an explicit description of the monomials $m_0,\ldots,m_9$. Using Lemma 3.3.9, we compute

$$\begin{aligned}
K_0 &= \{(x_1,x_2,z_3,\ldots,z_9) \mid x_1,x_2,z_3,\ldots,z_9 \in \mathbb{C}\}, \\
K_1 &= \left\{\left(x_1,x_2,x_1^2,z_4,\ldots,z_9\right) \mid x_1,x_2,z_4,\ldots,z_9 \in \mathbb{C}\right\}, \\
K_2 &= \left\{\left(x_1,x_2,x_1^2,x_1x_2,z_5,\ldots,z_9\right) \mid x_1,x_2,z_5,\ldots,z_9 \in \mathbb{C}\right\}, \\
K_3 &= \left\{\left(x_1,x_2,x_1^2,x_1x_2,x_2^2,z_6,\ldots,z_9\right) \mid x_1,x_2,z_6,\ldots,z_9 \in \mathbb{C}\right\}, \\
K_4 &= \left\{\left(x_1,x_2,x_1^2,x_1x_2,x_2^2,x_1^3,z_7,z_8,z_9\right) \mid x_1,x_2,z_7,z_8,z_9 \in \mathbb{C}\right\}, \\
K_5 &= \left\{\left(x_1,x_2,x_1^2,x_1x_2,x_2^2,x_1^3,x_1^2x_2,z_8,z_9\right) \mid x_1,x_2,z_8,z_9 \in \mathbb{C}\right\}, \\
K_6 &= \left\{\left(x_1,x_2,x_1^2,x_1x_2,x_2^2,x_1^3,x_1^2x_2,x_1x_2^2,z_9\right) \mid x_1,x_2,z_9 \in \mathbb{C}\right\}, \\
K_7 &= \left\{\left(x_1,x_2,x_1^2,x_1x_2,x_2^2,x_1^3,x_1^2x_2,x_1x_2^2,x_2^3\right) \mid x_1,x_2 \in \mathbb{C}\right\}.
\end{aligned}$$

Lemma 3.3.12. *The following are true:*

(i) $W_{k-n} \subseteq V(\mathbb{P}^n)$.

(ii) For $i = 0, \ldots, k-n-1$, $W_{i+1} \subseteq W_i$.

Proof. (i) Lemma 3.3.9 (ii) states $K_{k-n} = \{(m_1(1, \mathbf{x}), \ldots, m_k(1, \mathbf{x})) \mid \boldsymbol{x} \in \mathbb{C}^n\}$ which implies $\phi(K_{k-n}) \subseteq V(\mathbb{P}^n)$ since $m_0(1, \boldsymbol{x}) = 1^d = 1$ for any $\boldsymbol{x} \in \mathbb{C}^n$. We conclude $W_{k-n} = \overline{\phi(K_{k-n})} \subseteq V(\mathbb{P}^n)$.

(ii) We have $K_{i+1} \subseteq K_i$ by Construction 3.3.7. Hence, $\phi(K_{i+1}) \subseteq \phi(K_i)$ from which $W_{i+1} = \overline{\phi(K_{i+1})} \subseteq \overline{\phi(K_i)} = W_i$ follows. ∎

Construction 3.3.7 gives us a specific filtration of projective varieties

$$W_{k-n} \subseteq \ldots \subseteq W_0 = \mathbb{P}^k \tag{3.10}$$

with a corresponding specific filtration of sets of real points

$$W_{k-n}(\mathbb{R}) \subseteq \ldots \subseteq W_0(\mathbb{R}) = \mathbb{P}^k(\mathbb{R}). \tag{3.11}$$

Lemma 3.3.13. *For* $i = 0, \ldots, k-n-1$*, the inclusion* $W_{i+1}(\mathbb{R}) \subseteq W_i(\mathbb{R})$ *is strict.*

Proof. We let e denote the all-one vector in $\mathbb{R}^{n+1}$ and set

$$\boldsymbol{z} := (m_1(e), \ldots, m_{n+i}(e), 0, \ldots, 0) \in \mathbb{R}^k.$$

In particular, $\phi(\boldsymbol{z}) \in \mathbb{P}^k(\mathbb{R})$ and we will show $\phi(\boldsymbol{z}) \in W_i \backslash W_{i+1}$ in two steps.

$\underline{\phi(\boldsymbol{z}) \in W_i}$: If $i = 0$, then $\boldsymbol{z} \in K_0$ is true by Construction 3.3.7 (1) according to which $K_0 = \mathbb{C}^k$. Consequently, we have $\phi(\boldsymbol{z}) \in \phi(K_0) \subseteq W_0$.

However, if $i \geq 1$, then $p_j(\boldsymbol{Z}) = Z_{n+i} - Z_{s_i} Z_{t_i}$ for $j = 1, \ldots, i$ and $s_i, t_i \leq n+i$ by Construction 3.3.1 (1). Therefore, we compute

$$p_j(\boldsymbol{z}) = m_{n+i}(e) - m_{s_i}(e) m_{t_i}(e) = 1 - 1 = 0$$

for $j = 1, \ldots, i$ and conclude $\boldsymbol{z} \in \boldsymbol{\mathcal{V}}(p_1, \ldots, p_i) = K_i$. It follows $\phi(\boldsymbol{z}) \in \phi(K_i) \subseteq W_i$.

$\underline{\phi(\boldsymbol{z}) \notin W_{i+1}}$: We have $p_{i+1}(\boldsymbol{Z}) = Z_{n+(i+1)} - Z_{s_{i+1}} Z_{t_{i+1}}$ and compute

$$p_{i+1}(\boldsymbol{z}) = 0 - m_{s_{i+1}}(e) m_{t_{i+1}}(e) = 0 - 1 = (-1)$$

since $s_i, t_i < n + (i+1)$ by Construction 3.3.1 (1). This shows

$$\boldsymbol{z} \notin \boldsymbol{\mathcal{V}}(p_1, \ldots, p_i, p_{i+1}) = K_{i+1}$$

which implies $\phi(\boldsymbol{z}) \notin \phi(K_{i+1})$. Moreover, W_{i+1} is the projective closure of K_{i+1} by Construction 3.3.7 (3). Hence, $W_{i+1} \cap \phi(\mathbb{C}^k) = \phi(K_{i+1})$ by Proposition A.1.53. Therefore, $\phi(\boldsymbol{z}) \in \phi(\mathbb{C}^k) \backslash \phi(K_{i+1})$ implies $\phi(\boldsymbol{z}) \notin W_{i+1}$. ∎

Corollary 3.3.14. *For $i = 0, \ldots, k-n-1$, the inclusion $W_{i+1} \subseteq W_i$ is strict.*

Proof. Immediate consequence of Lemma 3.3.13. ∎

Comparing Lemma 3.2.5 with Lemma 3.3.12, Lemma 3.2.11 with Lemma 3.3.13 and Corollary 3.2.12 with Corollary 3.3.14, we see that the projective varieties $V_0, \ldots, V_{k-n}$ from Construction 3.2.1 display a behaviour similar to the one of the projective varieties $W_0, \ldots, W_{k-n}$ introduced in Construction 3.3.7. We will show that both constructions indeed give the same projective varieties in the proof of Theorem 3.3.17 below for which we first have to acquire a deeper understanding of the underlying affine varieties $K_0, \ldots, K_{k-n}$.

Proposition 3.3.15. *For $i = 0, \ldots, k-n$, K_i and $\mathbb{C}^{k-i}$ are isomorphic as affine varieties.*[5]

Proof. If $i = 0$, then, by Construction 3.3.7 (1), $K_0 = \mathbb{C}^k$ and we are done. However, if $i \geq 1$, then we distinguish two cases and in each case propose candidates of polynomial maps for the isomorphy of K_i and $\mathbb{C}^{k-i}$.

<u>Case 1</u>: If $i < k - n$, then we consider the polynomial maps

$$\begin{array}{llll} \psi\colon & K_i & \to & \mathbb{C}^{k-i} \\ & (\boldsymbol{x}, z_{n+1}, \ldots, z_{n+i}, z_{n+i+1}, \ldots, z_k) & \mapsto & (\boldsymbol{x}, z_{n+i+1}, \ldots, z_k), \\ \varphi\colon & \mathbb{C}^{k-i} & \to & K_i \\ & (\boldsymbol{x}, z_{n+i+1}, \ldots, z_k) & \mapsto & (\boldsymbol{x}, m_{n+1}(1,\boldsymbol{x}), \ldots, m_{n+i}(1,\boldsymbol{x}), z_{n+i+1}, \ldots, z_k). \end{array}$$

Note that φ is well-defined: We have $\boldsymbol{x} = (m_1(1,\boldsymbol{x}), \ldots, m_n(1,\boldsymbol{x}))$ and, moreover, $m_0(1,\boldsymbol{x}) = 1^d = 1$. Hence, for $(\boldsymbol{x}, z_{n+i+1}, \ldots, z_k) \in \mathbb{C}^{k-i}$ and $j = 1, \ldots, i$,

$$\begin{aligned} p_j(\varphi(\boldsymbol{x}, z_{n+i+1}, \ldots, z_k)) &= p_j(\varphi(m_1(1,\boldsymbol{x}), \ldots, m_n(1,\boldsymbol{x}), z_{n+i+1}, \ldots, z_k)) \\ &= p_j(m_1(1,\boldsymbol{x}), \ldots, m_{n+i}(1,\boldsymbol{x}), z_{n+i+1}, \ldots, z_k) \\ &= m_{n+j}(1,\boldsymbol{x}) - m_{s_j}(1,\boldsymbol{x}) m_{t_j}(1,\boldsymbol{x}) \\ &= m_0(1,\boldsymbol{x}) m_{n+j}(1,\boldsymbol{x}) - m_{s_j}(1,\boldsymbol{x}) m_{t_j}(1,\boldsymbol{x}) = 0 \end{aligned}$$

since $\alpha_0 + \alpha_{n+j} = \alpha_{s_j} + \alpha_{t_j}$ by Construction 3.3.1 (1). This shows

$$\varphi(\boldsymbol{x}, z_{n+i+1}, \ldots, z_k) \in \boldsymbol{\mathcal{V}}(p_1, \ldots, p_i) = K_i.$$

[5] That is, there exist polynomial maps $\psi\colon K_i \to \mathbb{C}^{k-i}$ and $\varphi\colon \mathbb{C}^{k-i} \to K_i$ that are inverse to one another (cf. Definition A.1.9).

Case 2: If $i = k-n$, then we consider the polynomial maps

$$\begin{array}{rccc} \psi\colon & K_{k-n} & \to & \mathbb{C}^n \\ & (\boldsymbol{x}, z_{n+1}, \dots, z_k) & \mapsto & \boldsymbol{x}, \\ \varphi\colon & \mathbb{C}^n & \to & K_{k-n} \\ & \boldsymbol{x} & \mapsto & (\boldsymbol{x}, m_{n+1}(1,\boldsymbol{x}), \dots, m_k(1,\boldsymbol{x})). \end{array}$$

We now simultaneously prove for both cases that the polynomial maps ψ and φ are inverse to one another in two steps.

(1) For $y \in \mathbb{C}^{k-i}$, we compute $\psi(\varphi(y)) = y$. Hence, $\psi \circ \varphi = \mathrm{id}_{\mathbb{C}^{k-i}}$.

(2) For $(\boldsymbol{x}, z_{n+1}, \dots, z_k) \in K_i$, Lemma 3.3.9 yields

$$(\boldsymbol{x}, z_{n+1}, \dots, z_k) = \begin{cases} (m_1(1,\mathbf{y}), \dots, m_{n+i}(1,\mathbf{y}), z_{n+i+1}, \dots, z_k), & \text{if } i < k-n \\ (m_1(1,\mathbf{y}), \dots, m_k(1,\mathbf{y})), & \text{if } i = k-n \end{cases}$$

for some $\boldsymbol{y} \in \mathbb{C}^n$. Since $x_j = m_j(1,\boldsymbol{y}) = 1^{d-1} y_j$ for $j = 1, \dots, n$, we conclude $\boldsymbol{x} = \boldsymbol{y}$ and compute

$$\begin{aligned} &\varphi(\psi(\boldsymbol{x}, z_{n+1}, \dots, z_k)) \\ &= \begin{cases} (\boldsymbol{x}, m_{n+1}(1,\boldsymbol{x}), \dots, m_{n+i}(1,\boldsymbol{x}), z_{n+i+1}, \dots, z_k), & \text{if } i < k-n \\ (\boldsymbol{x}, m_{n+1}(1,\boldsymbol{x}), \dots, m_k(1,\boldsymbol{x})), & \text{if } i = k-n \end{cases} \\ &= \begin{cases} (m_1(1,\boldsymbol{y}), \dots, m_{n+i}(1,\boldsymbol{y}), z_{n+i+1}, \dots, z_k), & \text{if } i < k-n \\ (m_1(1,\boldsymbol{y}), \dots, m_k(1,\boldsymbol{y})), & \text{if } i = k-n \end{cases} \\ &= (\boldsymbol{x}, z_{n+1}, \dots, z_k). \end{aligned}$$

■

In the proof of the corollary below, we anticipate the concept of irreducibility and dimension of a projective variety from Section 4.1.2 and Section 4.1.3, respectively. We refer the reader to these section for an overview.

Corollary 3.3.16. *It holds $W_{k-n} = V(\mathbb{P}^n)$.*

Proof. K_{k-n} is isomorphic to $\mathbb{C}^n$ as affine varieties by Proposition 3.3.15 and $\mathbb{C}^n$ is irreducible of dimension n (cf. [Har77, Chapter 1, §1, Example 1.4.1, and Chapter 1, §1, Proposition 1.9]). Hence, K_{k-n} is irreducible of dimension n by Theorem A.1.13 and Theorem A.1.16. Thus, the projective closure W_{k-n} of the affine variety K_{k-n} is irreducible of dimension n by Theorem A.1.54 and Theorem A.1.55. Moreover, the projective variety $V(\mathbb{P}^n)$ is irreducible of dimension n (cf. [Pla20, 4.3.13 Satz][6]

[6] This result states that the Veronese embedding is an isomorphism between projective varieties and thus continuous w.r.t. the Zariski topology. Therefore, if $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are two projective varieties such that $\mathfrak{W}_1 \cup \mathfrak{W}_2 = V(\mathbb{P}^n)$, then $V^{-1}(\mathfrak{W}_1)$ and $V^{-1}(\mathfrak{W}_2)$ are two projective varieties such that $V^{-1}(\mathfrak{W}_1) \cup V^{-1}(\mathfrak{W}_2) = \mathbb{P}^n$. Hence, $V^{-1}(\mathfrak{W}_1) = \mathbb{P}^n$ or $V^{-1}(\mathfrak{W}_2) = \mathbb{P}^n$ since $\mathbb{P}^n$ is irreducible. Consequently, $\mathfrak{W}_1 = V(\mathbb{P}^n)$ or $\mathfrak{W}_2 = V(\mathbb{P}^n)$, which verifies the irreducibility of the Veronese variety.

and [Har92, Example 13.4][7]) and $W_{k-n} \subseteq V(\mathbb{P}^n)$ by Lemma 3.3.12 (i). We therefore conclude $W_{k-n} = V(\mathbb{P}^n)$ by Proposition A.1.41. ■

Theorem 3.3.17. *For $i = 0, \dots, k-n$, $W_i = V_i$.*

Proof. Remark 3.3.8 and Lemma 3.2.5 (i) together show $W_0 = \mathbb{P}^k = V_0$. Likewise, Corollary 3.3.16 and Lemma 3.2.5 (ii) together show $W_{k-n} = V(\mathbb{P}^n) = V_{k-n}$. Hence, if $k - n = 1$, then there is nothing left to prove. Therefore, we now assume $1 < k - n$ and set $\mathcal{O}_i := \{[z] \in H_i \mid z_0 = 0\}$ for $i = 1, \dots, k-n$. Moreover, we endow $\mathbb{P}^k$ with the Zariski topology.

Claim 1: $V_i = \overline{\mathcal{O}_i} \cup W_i$.
Proof. ($\subseteq$) On the one hand, for $[z] \in H_i$ with $z_0 = 0$, we have $[z] \in \mathcal{O}_i \subseteq \overline{\mathcal{O}_i}$. On the other hand, for $[z] \in H_i$ with $z_0 \neq 0$, we know that

$$(z_0, \dots, z_{n+i}) = \lambda(m_0(1, \boldsymbol{x}), \dots, m_{n+i}(1, \boldsymbol{x}))$$

for some $\lambda \in \mathbb{C}^\times$ and some $\boldsymbol{x} \in \mathbb{C}^n$. Recalling Lemma 3.3.9, we conclude

$$[z] \in \phi(K_i) \subseteq \overline{\phi(K_i)} = W_i.$$

Altogether, we thus know $H_i \subseteq \overline{\mathcal{O}_i} \cup W_i$ which implies $V_i = \overline{H_i} \subseteq \overline{\mathcal{O}_i} \cup W_i$.
($\supseteq$) $\mathcal{O}_i$ is contained in H_i by choice. Hence, $\overline{\mathcal{O}_i} \subseteq \overline{H_i} = V_i$ follows. It therefore remains to show $W_i \subseteq V_i$. For $\boldsymbol{z} \in K_i$, Lemma 3.3.9 allows us to fix some $\boldsymbol{x} \in \mathbb{C}^n$ such that $(z_1, \dots, z_{n+i}) = (m_1(1, \boldsymbol{x}), \dots, m_{n+i}(1, \boldsymbol{x}))$ and $m_0(1, \boldsymbol{x}) = 1^d = 1$. We conclude $\phi(\boldsymbol{z}) \in H_i$ and since $\boldsymbol{z} \in K_i$ was arbitrarily chosen, $\phi(K_i) \subseteq H_i \subseteq V_i$ follows. This implies $W_i = \overline{\phi(K_i)} \subseteq V_i$. ■

Construction 3.3.1 ensures $p_1(\boldsymbol{Z}), \dots, p_i(\boldsymbol{Z}) \subseteq \mathbb{C}[Z_1, \dots, Z_{n+i}]$. Hence, a Gröbner basis G of the ideal $\langle p_1, \dots, p_i \rangle \subseteq \mathbb{C}[\boldsymbol{Z}]$ w.r.t. some a priori fixed graded monomial order can be chosen such that $G \subseteq \mathbb{C}[Z_1, \dots, Z_{n+i}]$.[8] We denote the homogenization $G^h := \{g^h \mid g \in G\}$ of G by F and observe $F \subseteq \mathbb{C}[Z_0, \dots, Z_{n+i}]$. Moreover, we know $W_i = \mathcal{V}(F)$ by Theorem A.1.80.

Claim 2: $\langle F \rangle \subseteq \mathcal{I}(\mathcal{O}_i)$.
Proof. For $[z] \in \mathcal{O}_i$, we let $x \in \mathbb{C}^{n+1}$ be such that

$$(z_0, \dots, z_{n+i}) = (m_0(x), \dots, m_{n+i}(x))$$

and set $\tilde{z} := (m_0(x), \dots, m_k(x))$. In particular, $[\tilde{z}] \in V(\mathbb{P}^n)$ follows. Moreover,

$$V(\mathbb{P}^n) = W_{k-n} \subseteq W_i = \mathcal{V}(F)$$

[7]This result states that the Hilbert polynomial of the Veronese variety has degree n. Thus, the dimension of $V(\mathbb{P}^n)$ is n (cf. Theorem 4.1.14).

[8]Such a Gröbner basis can be constructed using Buchberger's algorithm (cf. Algorithm A.1.71). We refer an interested reader to Appendix A.1.3 for an introduction to Gröbner bases and the methods of Buchberger.

by Corollary 3.3.16 and Lemma 3.3.12 (ii) and thus we have $\mathcal{I}(\mathcal{V}(F)) \subseteq \mathcal{I}(V(\mathbb{P}^n))$. For $f \in F \subseteq \mathcal{I}(\mathcal{V}(F)) \subseteq \mathcal{I}(V(\mathbb{P}^n))$, we therefore conclude $0 = f(\tilde{z}) = f(z)$ using $F \subseteq \mathbb{C}[Z_0, \dots, Z_{n+i}]$. ■

Claim 2 and Hilbert's projective Nullstellensatz (cf. Theorem A.1.32) imply

$$\mathcal{I}(W_i) = \mathcal{I}(\mathcal{V}(F)) = \sqrt{\langle F \rangle} \subseteq \sqrt{\mathcal{I}(\mathcal{O}_i)} = \mathcal{I}(\mathcal{V}(\mathcal{I}(\mathcal{O}_i))).$$

Thus, we have $W_i = \mathcal{V}(F) \supseteq \mathcal{V}(\mathcal{I}(\mathcal{O}_i)) = \overline{\mathcal{O}_i}$ by Theorem A.1.31 and Example A.2.7. Recalling Claim 1, we conclude $V_i = \overline{O_i} \cup W_i = W_i$. ■

The above proof is constructive and allows us to compute a set of forms $F \subseteq \mathbb{C}[Z]$ (from $\{p_1, \dots, p_i\}$) such that $V_i = \mathcal{V}(F)$ for $i = 1, \dots, k-n$.

Algorithm 3.3.18. *Input*: $1 \leq i \leq k-n$

(1) Fix a graded monomial order $\leq_{\text{gr}}$ on $\mathbb{C}[\boldsymbol{Z}]$.

(2) Compute a Gröbner basis G of the ideal $\langle p_1, \dots, p_i \rangle \subseteq \mathbb{C}[\boldsymbol{Z}]$ w.r.t. $\leq_{\text{gr}}$.

(3) Set $F \subseteq \mathbb{C}[Z]$ to be the homogenization of G.

Output: $F \subseteq \mathbb{C}[Z]$ such that $V_i = \mathcal{V}(F)$.

Proof of Correct Output. Theorem 3.3.17 states $V_i = W_i$. Hence, V_i is the projective closure of the affine variety $K_i = \boldsymbol{\mathcal{V}}(p_1, \dots, p_i) = \boldsymbol{\mathcal{V}}(\langle p_1, \dots, p_i \rangle) \subseteq \mathbb{C}^k$ and we conclude $V_i = \mathcal{V}(\langle p_1, \dots, p_i \rangle^h) = \mathcal{V}(G^h) = \mathcal{V}(F)$ from Theorem A.1.80. ■

Remark 3.3.19. *Buchberger's algorithm (cf. Algorithm A.1.71) can be used for computing a Gröbner basis of the ideal $\langle p_1, \dots, p_i \rangle \subseteq \mathbb{C}[\boldsymbol{Z}]$ w.r.t. $\leq_{gr}$. In fact, using a modified version of Buchberger's algorithm (cf. Theorem A.1.74), one may compute the unique reduced Gröbner basis G of $\langle p_1, \dots, p_i \rangle \subseteq \mathbb{C}[\boldsymbol{Z}]$ w.r.t. $\leq_{gr}$ in Step (2) and thus obtain a unique F in Step (3).*

Example 3.3.20. Quaternary Quartics

Let us run Algorithm 3.3.18 to explicitly determine a set of forms $F_i \subseteq \mathbb{C}[Z]$ (from $\{p_1, \dots, p_i\}$) such that $V_i = \mathcal{V}(F_i)$ for $i = 1, \dots, k-n$ in the case of quaternary quartics. Hence, $n = 3$, $d = 2$, $k - n = 6$ and we recall from Example 3.3.3 that

$$\begin{aligned} p_1(1, \boldsymbol{Z}) &= Z_4 - Z_1^2, & p_4(1, \boldsymbol{Z}) &= Z_7 - Z_2^2, \\ p_2(1, \boldsymbol{Z}) &= Z_5 - Z_1 Z_2, & p_5(1, \boldsymbol{Z}) &= Z_8 - Z_2 Z_3, \\ p_3(1, \boldsymbol{Z}) &= Z_6 - Z_1 Z_3, & p_6(1, \boldsymbol{Z}) &= Z_9 - Z_3^2. \end{aligned}$$

For $i = 1, \dots, 6$, we compute the reduced Gröbner basis G_i of $\langle p_1, \dots, p_i \rangle$ w.r.t. the graded lexicographic ordering $\leq_{\text{grlex}}$ (cf. Definition A.1.77) using Matlab and set

$F_i := G_i^h$. This gives us

$$
\begin{aligned}
F_1 &= \left\{Z_1^2 - Z_0Z_4\right\},\\
F_2 &= \left\{Z_4Z_2^2 - Z_0Z_5^2, Z_1^2 - Z_0Z_4, Z_1Z_2 - Z_0Z_5, Z_1Z_5 - Z_2Z_4\right\},\\
F_3 &= F_2 \cup \left\{Z_2Z_3Z_4 - Z_0Z_5Z_6, Z_4Z_3^2 - Z_0Z_6^2, Z_1Z_3 - Z_0Z_6, Z_1Z_6 - Z_3Z_4,\right.\\
&\qquad \left. Z_2Z_6 - Z_3Z_5\right\},\\
F_4 &= \left\{Z_1^2 - Z_0Z_4, Z_1Z_2 - Z_0Z_5, Z_1Z_5 - Z_2Z_4, Z_2Z_3Z_4 - Z_0Z_5Z_6, Z_4Z_3^2 - Z_0Z_6^2,\right.\\
&\quad Z_1Z_3 - Z_0Z_6, Z_1Z_6 - Z_3Z_4, Z_2Z_6 - Z_3Z_5, Z_3^2Z_5^2 - Z_0Z_7Z_6^2,\\
&\quad \left. Z_2Z_3Z_5 - Z_0Z_6Z_7, Z_1Z_7 - Z_2Z_5, Z_2^2 - Z_0Z_7, Z_4Z_7 - Z_5^2\right\},\\
F_5 &= \left\{Z_4Z_3^2 - Z_0Z_6^2, Z_5Z_3^2 - Z_0Z_6Z_8, Z_7Z_3^2 - Z_0Z_8^2, Z_1^2 - Z_0Z_4, Z_1Z_2 - Z_0Z_5,\right.\\
&\quad Z_1Z_3 - Z_0Z_6, Z_1Z_5 - Z_2Z_4, Z_1Z_6 - Z_3Z_4, Z_1Z_7 - Z_2Z_5, Z_1Z_8 - Z_3Z_5,\\
&\quad Z_2^2 - Z_0Z_7, Z_2Z_3 - Z_0Z_8, Z_2Z_6 - Z_3Z_5, Z_2Z_8 - Z_3Z_7, Z_4Z_7 - Z_5^2,\\
&\quad \left. Z_4Z_8 - Z_5Z_6, Z_5Z_8 - Z_6Z_7\right\},\\
F_6 &= \left\{Z_1^2 - Z_0Z_4, Z_1Z_2 - Z_0Z_5, Z_1Z_3 - Z_0Z_6, Z_1Z_5 - Z_2Z_4, Z_1Z_6 - Z_3Z_4,\right.\\
&\quad Z_1Z_7 - Z_2Z_5, Z_1Z_8 - Z_3Z_5, Z_1Z_9 - Z_3Z_6, Z_2^2 - Z_0Z_7, Z_2Z_3 - Z_0Z_8,\\
&\quad Z_2Z_6 - Z_3Z_5, Z_2Z_8 - Z_3Z_7, Z_2Z_9 - Z_3Z_8, Z_3^2 - Z_0Z_9, Z_4Z_7 - Z_5^2,\\
&\quad \left. Z_4Z_8 - Z_5Z_6, Z_4Z_9 - Z_6^2, Z_5Z_8 - Z_6Z_7, Z_5Z_9 - Z_6Z_8, Z_7Z_9 - Z_8^2\right\}.
\end{aligned}
$$

Example 3.3.21. Ternary Sextics
Let us run Algorithm 3.3.18 to explicitly determine a set of forms $F_i \subseteq \mathbb{C}[Z]$ (from $\{p_1, \ldots, p_i\}$) such that $V_i = \mathcal{V}(F_i)$ for $i = 1, \ldots, k - n$ in the case of ternary sextics. Hence, $n = 2$, $d = 3$, $k - n = 7$ and we recall from Example 3.3.4 that

$$
\begin{aligned}
p_1(1, \boldsymbol{Z}) &= Z_3 - Z_1^2, & p_5(1, \boldsymbol{Z}) &= Z_7 - Z_1Z_4,\\
p_2(1, \boldsymbol{Z}) &= Z_4 - Z_1Z_2, & p_6(1, \boldsymbol{Z}) &= Z_8 - Z_1Z_5,\\
p_3(1, \boldsymbol{Z}) &= Z_5 - Z_2^2, & p_7(1, \boldsymbol{Z}) &= Z_9 - Z_2Z_5.\\
p_4(1, \boldsymbol{Z}) &= Z_6 - Z_1Z_3, & &
\end{aligned}
$$

For $i = 1, \ldots, 7$, we compute the reduced Gröbner basis G_i of $\langle p_1, \ldots, p_i\rangle$ w.r.t. the graded lexicographic ordering $\leq_{\text{grlex}}$ using Matlab and set $F_i := G_i^h$. This gives us

$$
\begin{aligned}
F_1 &= \left\{Z_1^2 - Z_0Z_3\right\},\\
F_2 &= \left\{Z_3Z_2^2 - Z_0Z_4^2, Z_1^2 - Z_0Z_3, Z_1Z_2 - Z_0Z_4, Z_1Z_4 - Z_2Z_3\right\},\\
F_3 &= \left\{Z_1^2 - Z_0Z_3, Z_1Z_2 - Z_0Z_4, Z_1Z_4 - Z_2Z_3, Z_1Z_5 - Z_2Z_4, Z_2^2 - Z_0Z_5,\right.\\
&\qquad \left. Z_3Z_5 - Z_4^2\right\},
\end{aligned}
$$

$$F_4 = \Big\{Z_4^6 - Z_0Z_5^3Z_6^2, Z_3Z_4^4 - Z_0Z_5^2Z_6^2, Z_2Z_4^3 - Z_0Z_6Z_5^2, Z_3^2Z_4^2 - Z_0Z_5Z_6^2,$$
$$Z_2Z_3^2 - Z_0Z_4Z_6, Z_2Z_3Z_4 - Z_0Z_5Z_6, Z_3^3 - Z_0Z_6^2, Z_1^2 - Z_0Z_3, Z_1Z_2 - Z_0Z_4,$$
$$Z_1Z_3 - Z_0Z_6, Z_1Z_4 - Z_2Z_3, Z_1Z_5 - Z_2Z_4, Z_1Z_6 - Z_3^2, Z_2^2 - Z_0Z_5,$$
$$Z_2Z_6 - Z_3Z_4, Z_3Z_5 - Z_4^2\Big\},$$
$$F_5 = \Big\{Z_5^3Z_6^4 - Z_0Z_7^6, Z_4Z_5^2Z_6^3 - Z_0Z_7^5, Z_5Z_4^2Z_6^2 - Z_0Z_7^4, Z_4^4 - Z_0Z_5Z_7^2,$$
$$Z_6Z_4^3 - Z_0Z_7^3, Z_2Z_4^2 - Z_0Z_5Z_7, Z_3^3 - Z_0Z_6^2, Z_4Z_3^2 - Z_0Z_6Z_7, Z_3Z_4^2 - Z_0Z_7^2,$$
$$Z_1^2 - Z_0Z_3, Z_1Z_2 - Z_0Z_4, Z_1Z_3 - Z_0Z_6, Z_1Z_4 - Z_0Z_7, Z_1Z_5 - Z_2Z_4,$$
$$Z_1Z_6 - Z_3^2, Z_1Z_7 - Z_3Z_4, Z_2^2 - Z_0Z_5, Z_2Z_3 - Z_0Z_7, Z_2Z_6 - Z_3Z_4,$$
$$Z_2Z_7 - Z_4^2, Z_3Z_5 - Z_4^2, Z_3Z_7 - Z_4Z_6, Z_4Z_7 - Z_5Z_6\Big\},$$
$$F_6 = \Big\{Z_5^3Z_7^2 - Z_0Z_8^4, Z_4Z_6Z_5^2 - Z_0Z_7Z_8^2, Z_6Z_5^3 - Z_0Z_8^3, Z_3^3 - Z_0Z_6^2, Z_4Z_3^2 - Z_0Z_6Z_7,$$
$$Z_3Z_4^2 - Z_0Z_7^2, Z_4^3 - Z_0Z_7Z_8, Z_5Z_4^2 - Z_0Z_8^2, Z_1^2 - Z_0Z_3, Z_1Z_2 - Z_0Z_4,$$
$$Z_1Z_3 - Z_0Z_6, Z_1Z_4 - Z_0Z_7, Z_1Z_5 - Z_0Z_8, Z_1Z_6 - Z_3^2, Z_1Z_7 - Z_3Z_4,$$
$$Z_1Z_8 - Z_4^2, Z_2^2 - Z_0Z_5, Z_2Z_3 - Z_0Z_7, Z_2Z_4 - Z_0Z_8, Z_2Z_6 - Z_3Z_4, Z_2Z_7 - Z_4^2,$$
$$Z_2Z_8 - Z_4Z_5, Z_3Z_5 - Z_4^2, Z_3Z_7 - Z_4Z_6, Z_3Z_8 - Z_5Z_6, Z_4Z_7 - Z_5Z_6,$$
$$Z_4Z_8 - Z_5Z_7, Z_6Z_8 - Z_7^2\Big\},$$
$$F_7 = \Big\{Z_3^3 - Z_0Z_6^2, Z_4Z_3^2 - Z_0Z_6Z_7, Z_3Z_4^2 - Z_0Z_7^2, Z_4^3 - Z_0Z_7Z_8, Z_5Z_4^2 - Z_0Z_8^2,$$
$$Z_4Z_5^2 - Z_0Z_8Z_9, Z_5^3 - Z_0Z_9^2, Z_1^2 - Z_0Z_3, Z_1Z_2 - Z_0Z_4, Z_1Z_3 - Z_0Z_6,$$
$$Z_1Z_4 - Z_0Z_7, Z_1Z_5 - Z_0Z_8, Z_1Z_6 - Z_3^2, Z_1Z_7 - Z_3Z_4, Z_1Z_8 - Z_4^2,$$
$$Z_1Z_9 - Z_4Z_5, Z_2^2 - Z_0Z_5, Z_2Z_3 - Z_0Z_7, Z_2Z_4 - Z_0Z_8, Z_2Z_5 - Z_0Z_9,$$
$$Z_2Z_6 - Z_3Z_4, Z_2Z_7 - Z_4^2, Z_2Z_8 - Z_4Z_5, Z_2Z_9 - Z_5^2, Z_3Z_5 - Z_4^2, Z_3Z_7 - Z_4Z_6,$$
$$Z_3Z_8 - Z_5Z_6, Z_3Z_9 - Z_5Z_7, Z_4Z_7 - Z_5Z_6, Z_4Z_8 - Z_5Z_7, Z_4Z_9 - Z_5Z_8,$$
$$Z_6Z_8 - Z_7^2, Z_6Z_9 - Z_7Z_8, Z_7Z_9 - Z_8^2\Big\}.$$

The above two examples illustrate the extensive nature of a to-be-determined Gröbner basis G of $\langle p_1, \ldots, p_i\rangle$ w.r.t. the graded lexicographic ordering for increasing i in the second step of Algorithm 3.3.18. However, for $i = 1, \ldots, n$, a specific simplified Gröbner basis G of $\langle p_1, \ldots, p_i\rangle$ w.r.t. the graded lexicographic ordering can be given.

Observation 3.3.22. *A Gröbner basis G of $\langle p_1\rangle \subseteq \mathbb{C}[\boldsymbol{Z}]$ w.r.t. $\leq_{grlex}$ is given by $G := \{p_1\}$. Consequently, $V_1 = \mathcal{V}(p_1^h) = \mathcal{V}(q_1)$.*

Theorem 3.3.23. *For $i = 2, \ldots, n$, a Gröbner basis of $\langle p_1, \ldots, p_i\rangle \subseteq \mathbb{C}[\boldsymbol{Z}]$ w.r.t. the graded monomial order $\leq_{grlex}$ is given by $G := \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3$ where*

$$\mathfrak{G}_1 := \{Z_1Z_s - Z_{n+s} \mid 1 \leq s \leq i\},$$
$$\mathfrak{G}_2 := \{Z_sZ_{n+t} - Z_tZ_{n+s} \mid 1 \leq s < t \leq i\},$$
$$\mathfrak{G}_3 := \{Z_{n+1}Z_sZ_t - Z_{n+s}Z_{n+t} \mid 2 \leq s \leq t \leq i\}.$$

Proof. For $l = 1, 2, 3$, we order the set $\mathfrak{G}_l$ by $\leq_{\text{grlex}}$ and denote it by $\mathfrak{G}_l^<$. We set $G^< := (\mathfrak{G}_1^<, \mathfrak{G}_2^<, \mathfrak{G}_3^<)$ and apply Buchberger's criterion (cf. Theorem A.1.70) in order to verify the assertion.

Claim: $\langle G^< \rangle = \langle p_1, \ldots, p_i \rangle$.
Proof. ($\subseteq$) For $j = 1, \ldots, n$, we know $p_j(\boldsymbol{Z}) = Z_{n+j} - Z_1 Z_j$ by Corollary 3.3.6. Consequently, $\mathfrak{G}_1^< = (-p_1, \ldots, -p_i)$ and we deduce $\mathfrak{G}_1^< \subseteq \langle p_1, \ldots, p_i \rangle$.

For $p_{st}(\boldsymbol{Z}) := Z_s Z_{n+t} - Z_t Z_{n+s} \in \mathfrak{G}_2^<$, we moreover observe

$$p_{st}(\boldsymbol{Z}) = Z_s p_t(\boldsymbol{Z}) - Z_t p_s(\boldsymbol{Z})$$

which allows us to conclude $\mathfrak{G}_2^< \subseteq \langle p_1, \ldots, p_i \rangle$.

For $Z_{n+1} Z_s Z_t - Z_{n+s} Z_{n+t} \in \mathfrak{G}_3^<$, we furthermore see that

$$Z_{n+1} Z_s Z_t - Z_{n+s} Z_{n+t} = -Z_s p_{1t}(\boldsymbol{Z}) - Z_{n+t} p_s(\boldsymbol{Z})$$

and thus $\mathfrak{G}_3^< \subseteq \langle p_1(\boldsymbol{Z}), \ldots, p_i(\boldsymbol{Z}) \rangle$.
($\supseteq$) We already observed $\mathfrak{G}_1^< = (-p_1, \ldots, -p_i)$ in ($\subseteq$) from which it follows that $\langle p_1, \ldots, p_i \rangle = \langle \mathfrak{G}_1^< \rangle \subseteq \langle G^< \rangle$. ■

For Buchberger's criterion, it remains to show that the remainder $\overline{S(f,g)}^{G^<, \leq_{\text{grlex}}}$ on division of the S-Polynomial $S(f,g)$ by $G^<$ w.r.t. the graded monomial order $\leq_{\text{grlex}}$ is zero for any distinct $f, g \in G^<$. This can be verified by a straight forward yet lengthy computation of $\overline{S(f,g)}^{G^<, \leq_{\text{grlex}}}$ for all choices of distinct $f, g \in G^<$. We consider the case of two distinct polynomials

$$\begin{aligned} f(\boldsymbol{Z}) &:= Z_s Z_{n+t} - Z_t Z_{n+s} \in \mathfrak{G}_2^<, \\ g(\boldsymbol{Z}) &:= Z_\sigma Z_{n+\tau} - Z_\tau Z_{n+\sigma} \in \mathfrak{G}_2^< \end{aligned}$$

with $1 < s < \sigma < \tau < t \leq i$ (if $i \geq 5$) and exemplarily compute $\overline{S(f,g)}^{G^<, \leq_{\text{grlex}}}$ below. For all other choices of distinct $f, g \in G^<$, the remainder $\overline{S(f,g)}^{G^<, \leq_{\text{grlex}}}$ can be similarly computed. This is left as an exercise to the reader. For all choices of distinct $f, g \in G^<$, the remainder $\overline{S(f,g)}^{G^<, \leq_{\text{grlex}}}$ is zero.

Exemplary Computation:

$$\begin{aligned} & S(f,g) \\ &= \frac{Z_s Z_{n+t} Z_\sigma Z_{n+\tau}}{Z_s Z_{n+t}} (Z_s Z_{n+t} - Z_t Z_{n+s}) - \frac{Z_s Z_{n+t} Z_\sigma Z_{n+\tau}}{Z_\sigma Z_{n+\tau}} (Z_\sigma Z_{n+\tau} - Z_\tau Z_{n+\sigma}) \\ &= Z_\sigma Z_{n+\tau}(Z_s Z_{n+t} - Z_t Z_{n+s}) - Z_s Z_{n+t}(Z_\sigma Z_{n+\tau} - Z_\tau Z_{n+\sigma}) \\ &= Z_s Z_\tau Z_{n+\sigma} Z_{n+t} - Z_\sigma Z_t Z_{n+s} Z_{n+\tau} \\ &= Z_\tau Z_{n+t}(Z_s Z_{n+\sigma} - Z_\sigma Z_{n+s}) + Z_\sigma Z_\tau Z_{n+s} Z_{n+t} - Z_\sigma Z_t Z_{n+s} Z_{n+\tau} \\ &= Z_\tau Z_{n+t}(Z_s Z_{n+\sigma} - Z_\sigma Z_{n+s}) + Z_\tau Z_{n+s}(Z_\sigma Z_{n+t} - Z_t Z_{n+\sigma}) \\ &\quad - Z_\sigma Z_t Z_{n+s} Z_{n+\tau} + Z_\tau Z_t Z_{n+s} Z_{n+\sigma} \end{aligned}$$

$$\begin{aligned}
&= Z_\tau Z_{n+t}(Z_s Z_{n+\sigma} - Z_\sigma Z_{n+s}) + Z_\tau Z_{n+s}(Z_\sigma Z_{n+t} - Z_t Z_{n+\sigma}) \\
&\quad - Z_t Z_{n+s}(Z_\sigma Z_{n+\tau} - Z_\tau Z_{n+\sigma}) - Z_\tau Z_t Z_{n+s} Z_{n+\sigma} + Z_\tau Z_t Z_{n+s} Z_{n+\sigma} \\
&= Z_\tau Z_{n+t}(Z_s Z_{n+\sigma} - Z_\sigma Z_{n+s}) + Z_\tau Z_{n+s}(Z_\sigma Z_{n+t} - Z_t Z_{n+\sigma}) \\
&\quad - Z_t Z_{n+s}(Z_\sigma Z_{n+\tau} - Z_\tau Z_{n+\sigma}).
\end{aligned}$$

Hence, the remainder $\overline{S(f,g)}^{G^<, \leq_{\mathrm{grlex}}}$ is zero. ■

Corollary 3.3.24. *For $i = 2, \dots, n$, it holds $V_i = \mathcal{V}(F)$ for $F := \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$,*

$$\begin{aligned}
\mathfrak{F}_1 &:= \{Z_1 Z_s - Z_0 Z_{n+s} \mid 1 \leq s \leq i\}, \\
\mathfrak{F}_2 &:= \{Z_s Z_{n+t} - Z_t Z_{n+s} \mid 1 \leq s < t \leq i\}, \\
\mathfrak{F}_3 &:= \{Z_{n+1} Z_s Z_t - Z_0 Z_{n+s} Z_{n+t} \mid 2 \leq s \leq t \leq i\}.
\end{aligned}$$

Proof. Theorem 3.3.23 states that $G := \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{G}_3$ with

$$\begin{aligned}
\mathfrak{G}_1 &:= \{Z_1 Z_s - Z_{n+s} \mid 1 \leq s \leq i\}, \\
\mathfrak{G}_2 &:= \{Z_s Z_{n+t} - Z_t Z_{n+s} \mid 1 \leq s < t \leq i\}, \\
\mathfrak{G}_3 &:= \{Z_{n+1} Z_s Z_t - Z_{n+s} Z_{n+t} \mid 2 \leq s \leq t \leq i\}
\end{aligned}$$

is a Gröbner basis of $\langle p_1, \dots, p_i \rangle \subseteq \mathbb{C}[\boldsymbol{Z}]$ w.r.t. the graded monomial order $\leq_{\mathrm{grlex}}$. Hence, $V_i = \mathcal{V}(F)$ by Algorithm 3.3.18. ■

Example 3.3.25. Quaternary Quartics
Let $n = 3$ and $d = 2$. Observation 3.3.22 yields $V_1 = \mathcal{V}(F_1)$ for

$$F_1 := \{q_1\} = \{Z_0 Z_4 - Z_1^2\}$$

and, for $i = 2, 3$, we compute F_i such that $V_i = \mathcal{V}(F_i)$ as in Corollary 3.3.24 below.

(i) For $i = 2$, we have $F_2 := \mathfrak{F}_1^{(2)} \cup \mathfrak{F}_2^{(2)} \cup \mathfrak{F}_3^{(2)}$ with

$$\begin{aligned}
\mathfrak{F}_1^{(2)} &= \{Z_1^2 - Z_0 Z_4, Z_1 Z_2 - Z_0 Z_5\}, \\
\mathfrak{F}_2^{(2)} &= \{Z_1 Z_5 - Z_2 Z_4\}, \\
\mathfrak{F}_3^{(2)} &= \{Z_4 Z_2^2 - Z_0 Z_5^2\}.
\end{aligned}$$

(ii) For $i = 3$, we have $F_3 := \mathfrak{F}_1^{(3)} \cup \mathfrak{F}_2^{(3)} \cup \mathfrak{F}_3^{(3)}$ with

$$\begin{aligned}
\mathfrak{F}_1^{(3)} &= \{Z_1^2 - Z_0 Z_4, Z_1 Z_2 - Z_0 Z_5, Z_1 Z_3 - Z_0 Z_6\}, \\
\mathfrak{F}_2^{(3)} &= \{Z_1 Z_5 - Z_2 Z_4, Z_1 Z_6 - Z_3 Z_4, Z_2 Z_6 - Z_3 Z_5\}, \\
\mathfrak{F}_3^{(3)} &= \{Z_4 Z_2^2 - Z_0 Z_5^2, Z_4 Z_2 Z_3 - Z_0 Z_5 Z_6, Z_4 Z_3^2 - Z_0 Z_6^2\}.
\end{aligned}$$

The sets F_1, F_2 and F_3 determined above coincide (up to signs) with the sets F_1, F_2 and F_3 computed in Example 3.3.20, respectively.

Example 3.3.26. Ternary Sextics
Let $n = 2$ and $d = 3$. Observation 3.3.22 yields $V_1 = \mathcal{V}(F_1)$ for

$$F_1 := \{q_1\} = \{Z_0Z_3 - Z_1^2\}.$$

Moreover, we compute F_2 such that $V_2 = \mathcal{V}(F_2)$ as in Corollary 3.3.24 to be given by $F_2 := \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$ where

$$\begin{aligned}
\mathfrak{F}_1 &= \{Z_1^2 - Z_0Z_3, Z_1Z_2 - Z_0Z_4\},\\
\mathfrak{F}_2 &= \{Z_1Z_4 - Z_2Z_3\},\\
\mathfrak{F}_3 &= \{Z_3Z_2^2 - Z_0Z_4^2\}.
\end{aligned}$$

The sets F_1 and F_2 determined above coincide (up to signs) with the sets F_1 and F_2 computed in Example 3.3.21, respectively.

3.4 Extending Non-Negativity on $\phi(K_i)(\mathbb{R})$ to $V_i(\mathbb{R})$ for Quadratic Forms

In Section 3.1, we showed for an embedded affine variety $\phi(K) \subseteq \mathbb{P}^k$ with Zariski closure $\mathfrak{K} \subseteq \mathbb{P}^k$ that if $\overline{\phi(K)(\mathbb{R})} = \overline{\phi(K)}(\mathbb{R})$ w.r.t. the Euclidean topology, then $C_{\phi(K)} = C_{\mathfrak{K}}$. In this section, we let $\boldsymbol{(n+1, 2d)}$ **denote a non-Hilbert case** and build on the above result to prove $C_i = C_{\phi(K_i)}$ for the embedded affine varieties $\phi(K_0), \ldots, \phi(K_{k-n})$. We argue in three steps.

(1) We prove $\overline{\phi(K_{k-n})(\mathbb{R})} = \overline{\phi(K_{k-n})}(\mathbb{R})$ w.r.t. the Euclidean topology.

(2) For $i = 0, \ldots, k-n-1$, we deduce $\overline{\phi(K_i)(\mathbb{R})} = \overline{\phi(K_i)}(\mathbb{R})$ w.r.t. the Euclidean topology from our observations made in (1).

(3) For $i = 0, \ldots, k-n$, we conclude $C_i = C_{\phi(K_i)}$.

The proofs of this section are of a topological nature and repeatedly use the fact that if $\mathbb{P}^k$ is endowed with the Euclidean topology, then, given $S \subseteq \mathbb{P}^k$, it holds $[z] \in \overline{S}$ if and only if there exists some $\left(\left[z^{(m)}\right]\right)_{m\in\mathbb{N}} \subseteq S$ and some $(\lambda_m)_{m\in\mathbb{N}} \subseteq \mathbb{C}^\times$ such that $\lambda_m z^{(m)} \to z$ as $m \to \infty$. We refer the reader to Appendix A.2 for a revision of this characterization of limit points (cf. Example A.2.24) and an overview of further relevant topological results.

Theorem 3.4.1. *If $\mathbb{P}^k$ is endowed with the Euclidean topology, then*

$$\overline{\phi(K_{k-n})(\mathbb{R})} = \overline{\phi(K_{k-n})}(\mathbb{R}).$$

Proof. ($\subseteq$) The inclusion $\phi(K_{k-n})(\mathbb{R}) \subseteq \phi(K_{k-n})$ implies $\overline{\phi(K_{k-n})(\mathbb{R})} \subseteq \overline{\phi(K_{k-n})}$. Likewise, $\phi(K_{k-n})(\mathbb{R}) \subseteq \mathbb{P}^k(\mathbb{R})$ yields $\overline{\phi(K_{k-n})(\mathbb{R})} \subseteq \overline{\mathbb{P}^k(\mathbb{R})} = \mathbb{P}^k(\mathbb{R})$. Altogether, it thus follows $\overline{\phi(K_{k-n})(\mathbb{R})} \subseteq \overline{\phi(K_{k-n})} \cap \mathbb{P}^k(\mathbb{R}) = \overline{\phi(K_{k-n})}(\mathbb{R})$.
($\supseteq$) Theorem 3.1.11 yields that the Euclidean closure $\overline{\phi(K_{k-n})}$ of $\phi(K_{k-n})$ coincides with its Zariski closure W_{k-n}. Corollary 3.3.16 and Proposition 2.3.34 therefore imply $\overline{\phi(K_{k-n})}(\mathbb{R}) = W_{k-n}(\mathbb{R}) = V(\mathbb{P}^n)(\mathbb{R}) = V(\mathbb{P}^n(\mathbb{R}))$. We now distinguish two cases for $[x] \in \mathbb{P}^n(\mathbb{R})$ and show $V([x]) \in \overline{\phi(K_{k-n})(\mathbb{R})}$ in each case.

Case 1: If $x_0 \neq 0$, then we set $\tilde{\boldsymbol{x}} := \frac{1}{x_0}(x_1, \ldots, x_n) \in \mathbb{R}^n$ and compute

$$\begin{aligned} V([x]) &= [m_0(x) : \ldots : m_k(x)] \\ &= [m_0(x_0(1,\tilde{\boldsymbol{x}})) : \ldots : m_k(x_0(1,\tilde{\boldsymbol{x}}))] \\ &= \left[x_0^d m_0(1,\tilde{\boldsymbol{x}}) : \ldots : x_0^d m_k(1,\tilde{\boldsymbol{x}})\right] \\ &= [m_0(1,\tilde{\boldsymbol{x}}) : \ldots : m_k(1,\tilde{\boldsymbol{x}})] \\ &= [1^d : m_1(1,\tilde{\boldsymbol{x}}) : \ldots : m_k(1,\tilde{\boldsymbol{x}})] \\ &= [1 : m_1(1,\tilde{\boldsymbol{x}}) : \ldots : m_k(1,\tilde{\boldsymbol{x}})] \end{aligned}$$

using Lemma 2.2.11. This shows $V([x]) \in \phi(K_{k-n})(\mathbb{R}) \subseteq \overline{\phi(K_{k-n})(\mathbb{R})}$.

Case 2: If $x_0 = 0$, then, for $m \in \mathbb{N}$, we set $\tilde{x}^{(m)} := \left(\frac{1}{m}, x_1, \ldots, x_n\right) \in \mathbb{R}^{n+1}$. For any $m \in \mathbb{N}$, we thus have $\tilde{x}_0^{(m)} = \frac{1}{m} \neq 0$ and, therefore, $V\left(\left[\tilde{x}^{(m)}\right]\right) \in \phi(K_{k-n})(\mathbb{R})$ follows from Case 1. Furthermore, we see that $\tilde{x}^{(m)} \to x$ as $m \to \infty$. Since the monomials $m_0, \ldots, m_k$ are continuous w.r.t. the Euclidean topology, we conclude

$$\left(m_0\left(\tilde{x}^{(m)}\right), \ldots, m_k\left(\tilde{x}^{(m)}\right)\right) \to (m_0(x), \ldots, m_k(x)) \text{ as } m \to \infty.$$

It follows $V([x]) \in \overline{\phi(K_{k-n})(\mathbb{R})}$ by Example A.2.24. ∎

Example 3.4.2. Quaternary Quartics
Let us illustrate the proof of Theorem 3.4.1 on the example of quaternary quartics. Hence, $n = 3$, $d = 2$, $k = 9$ and $k - n = 6$. We let $\mathbb{P}^9$ be endowed with the Euclidean topology and recall from Example 3.3.3 that

$$\begin{aligned} p_1(\boldsymbol{Z}) &= Z_4 - Z_1^2, & p_4(\boldsymbol{Z}) &= Z_7 - Z_2^2, \\ p_2(\boldsymbol{Z}) &= Z_5 - Z_1 Z_2, & p_5(\boldsymbol{Z}) &= Z_8 - Z_2 Z_3, \\ p_3(\boldsymbol{Z}) &= Z_6 - Z_1 Z_3, & p_6(\boldsymbol{Z}) &= Z_9 - Z_3^2. \end{aligned}$$

We refer to Example 2.3.10 (i) for an explicit description of the monomials $m_0, \ldots . m_9$ and apply Lemma 3.3.9 to compute

$$\begin{aligned} K_6 &= \left\{\left(x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2\right) \mid x_1, x_2, x_3 \in \mathbb{C}\right\}, \\ \phi(K_6) &= \left\{\left[1 : x_1 : x_2 : x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3 : x_3^2\right] \mid x_1, x_2, x_3 \in \mathbb{C}\right\}, \\ \phi(K_6)(\mathbb{R}) &= \left\{\left[1 : x_1 : x_2 : x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3 : x_3^2\right] \mid x_1, x_2, x_3 \in \mathbb{R}\right\}. \end{aligned}$$

Theorem 3.1.11, Construction 3.3.7 and Corollary 3.3.16 together yield

$$\overline{\phi(K_6)} = W_6 = V(\mathbb{P}^3)$$

and $V(\mathbb{P}^3)(\mathbb{R}) = V(\mathbb{P}^3(\mathbb{R}))$ by Proposition 2.3.34. Therefore, it suffices to show

$$V([x]) = \left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3 : x_3^2\right] \in \overline{\phi(K_6)}(\mathbb{R})$$

for $[x] \in \mathbb{P}^3(\mathbb{R})$. We do so by distinguishing two cases for $[x] \in \mathbb{P}^3(\mathbb{R})$.

Case 1: If $x_0 \neq 0$, then we set $\tilde{\boldsymbol{x}} := (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) := \frac{1}{x_0}(x_1, x_2, x_3) \in \mathbb{R}^3$ and compute

$$\begin{aligned} V([x]) &= \left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3 : x_3^2\right] \\ &= \left[x_0^2 : x_0^2\tilde{x}_1 : x_0^2\tilde{x}_2 : x_0^2\tilde{x}_3 : x_0^2\tilde{x}_1^2 : x_0^2\tilde{x}_1\tilde{x}_2 : x_0^2\tilde{x}_1\tilde{x}_3 : x_0^2\tilde{x}_2^2 : x_0^2\tilde{x}_2\tilde{x}_3 : x_0^2\tilde{x}_3^2\right] \\ &= \left[1 : \tilde{x}_1 : \tilde{x}_2 : \tilde{x}_3 : \tilde{x}_1^2 : \tilde{x}_1\tilde{x}_2 : \tilde{x}_1\tilde{x}_3 : \tilde{x}_2^2 : \tilde{x}_2\tilde{x}_3 : \tilde{x}_3^2\right]. \end{aligned}$$

Comparing with our above description of $\phi(K_6)(\mathbb{R})$, we conclude

$$V([x]) \in \phi(K_6)(\mathbb{R}) \subseteq \overline{\phi(K_6)(\mathbb{R})}.$$

Case 2: If $x_0 = 0$, then, for any $m \in \mathbb{N}$, we set $\tilde{x}^{(m)} := \left(\frac{1}{m}, x_1, x_2, x_3\right) \in \mathbb{R}^4$, $\tilde{\boldsymbol{y}}^{(\boldsymbol{m})} := \tilde{\boldsymbol{y}} := (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) := (x_1m, x_2m, x_3m) \in \mathbb{R}^3$ and compute

$$\begin{aligned} &V\left(\left[\tilde{x}^{(m)}\right]\right) \\ &= \left[\left(\frac{1}{m}\right)^2 : \left(\frac{1}{m}\right)x_1 : \left(\frac{1}{m}\right)x_2 : \left(\frac{1}{m}\right)x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3 : x_3^2\right] \\ &= m^2\left[\left(\frac{1}{m}\right)^2 : \left(\frac{1}{m}\right)x_1 : \left(\frac{1}{m}\right)x_2 : \left(\frac{1}{m}\right)x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3 : x_3^2\right] \\ &= \left[1 : \tilde{y}_1 : \tilde{y}_2 : \tilde{y}_3 : (\tilde{y}_1)^2 : (\tilde{y}_1)(\tilde{y}_2) : (\tilde{y}_1)(\tilde{y}_3) : (\tilde{y}_2)^2 : (\tilde{y}_2)(\tilde{y}_3) : (\tilde{y}_3)^2\right]. \end{aligned}$$

Comparing with our above description of $\phi(K_6)(\mathbb{R})$, we thus conclude

$$V\left(\left[\tilde{x}^{(m)}\right]\right) \in \phi(K_6)(\mathbb{R})$$

for any $m \in \mathbb{N}$. Moreover, we see that $(0, 0, 0, 0, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2)$ is the limit of the sequence $\left(\left(\left(\frac{1}{m}\right)^2, \left(\frac{1}{m}\right)x_1, \left(\frac{1}{m}\right)x_2, \left(\frac{1}{m}\right)x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2\right)\right)_{m\in\mathbb{N}}$ and $V([x]) = [0 : 0 : 0 : 0 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3 : x_3^2]$ since $x_0 = 0$ is assumed. Altogether, it thus follows $[z] \in \overline{\phi(K_6)(\mathbb{R})}$ by Example A.2.24.

Theorem 3.4.3. *If $\mathbb{P}^k$ is endowed with the Euclidean topology, then*

$$\overline{\phi(K_i)(\mathbb{R})} = \overline{\phi(K_i)}(\mathbb{R}) \text{ for } i = 0, \ldots, k-n-1.$$

Proof. ($\subseteq$) The inclusion $\phi(K_i)(\mathbb{R}) \subseteq \phi(K_i)$ implies $\overline{\phi(K_i)(\mathbb{R})} \subseteq \overline{\phi(K_i)}$. Likewise, $\phi(K_i)(\mathbb{R}) \subseteq \mathbb{P}^k(\mathbb{R})$ yields $\overline{\phi(K_i)(\mathbb{R})} \subseteq \overline{\mathbb{P}^k(\mathbb{R})} = \mathbb{P}^k(\mathbb{R})$. Altogether, it thus follows $\phi(K_i)(\mathbb{R}) \subseteq \overline{\phi(K_i)} \cap \mathbb{P}^k(\mathbb{R}) = \overline{\phi(K_i)}(\mathbb{R})$.
($\supseteq$) The set $H := \{[z] \in \mathbb{P}^k \mid (z_0, \ldots, z_{n+i}) \neq (0, \ldots, 0)\}$ is well-defined. Indeed, for $z \in \mathbb{C}^{k+1}$ with some $0 \le j \le n+i$ such that $z_j \neq 0$, we observe for any $z' \in [z]$ that also $z'_j \neq 0$ since $z' = \lambda z$ for some $\lambda \in \mathbb{C}^\times$.

Let $\mathbb{P}^{n+i}$ be endowed with the Euclidean topology and $\mathbb{P}^k \cap H$ with the subspace topology induced by the Euclidean topology on $\mathbb{P}^k$. We consider the continuous map

$$\begin{array}{cccc} \pi\colon & \mathbb{P}^k \cap H & \to & \mathbb{P}^{n+i} \\ & [z] & \mapsto & [z_0 : \ldots : z_{n+i}]. \end{array}$$

<u>Claim 1</u>: $\pi(V(\mathbb{P}^n) \cap H)(\mathbb{R}) \subseteq \pi((V(\mathbb{P}^n)(\mathbb{R}) \cap H)$.
Proof. For $[z] \in V(\mathbb{P}^n) \cap H$ with $(z_0, \ldots, z_{n+i}) \in \mathbb{R}^{n+i+1}$, we fix $x \in \mathbb{C}^{n+1}$ such that

$$(z_0, \ldots, z_{n+i}) = (m_0(x), \ldots, m_{n+i}(x)) \tag{3.12}$$

and conclude that there exists some $0 \le j \le n$ such that $x_j \neq 0$ and $X_j^d \ge_{\text{lex}} m_{n+i}(X)$. Therefore, we set

$$j := \min\{j \in \{0, \ldots, n\} \mid x_j \neq 0 \wedge X_j^d \ge_{\text{lex}} m_{n+i}(X)\}$$

and let $0 \le s \le n+i$ be such that $m_s(X) = X_j^d$. Hence,

$$z_s \underset{(3.12)}{=} m_s(x) = x_j^d \neq 0 \tag{3.13}$$

is the first non-zero entry of z and, after potentially replacing z by $-z$, we assume $z_s > 0$ without loss of generality. This allows us to fix some $\gamma \in \mathbb{R}^\times$ such that

$$z_s = \gamma^d. \tag{3.14}$$

We distinguish two cases for s and construct $[\tilde{x}] \in \mathbb{P}^n(\mathbb{R})$ such that $\pi([z]) = \pi(V([\tilde{x}]))$ in each case.

<u>Case 1</u>: If $s = n+i$, then $j \ge 1$ since $X_0^d = m_0(X)$. For $l = 0, \ldots, j-1$, we thus have $x_l = 0$ since $X_l^d >_{\text{lex}} X_j^d = m_{n+i}(X)$. This implies

$$z_t \underset{(3.12)}{=} m_t(x) = 0$$

for $t = 0, \ldots, n+i-1$. Therefore, we set $\tilde{x} := (\tilde{x}_0, \ldots, \tilde{x}_n) \in \mathbb{R}^{n+1}$ to be given by

$$\tilde{x}_\sigma := \begin{cases} \gamma, & \text{for } \sigma = s, \\ 0, & \text{else} \end{cases}$$

and conclude $\pi([z]) = \pi(V([\tilde{x}]))$ since

$$(z_0, \dots, z_{n+i}) = (0, \dots, 0, z_s) \underset{(3.14)}{=} (0, \dots, 0, \gamma^d) = (m_0(\tilde{x}), \dots, m_{n+i}(\tilde{x})).$$

Case 2: If $s < n+i$, then $j < n$ since $X_n^d = m_k(X)$. We set $r := \min\{n-j, n+i-s\}$ and conclude $r \geq 1$. For $l = 1, \dots, r$, we observe that

$$x_j^d \frac{x_{j+l}}{x_j} = x_j^{d-1} x_{j+l} = m_{s+l}(x) \underset{(3.12)}{=} z_{s+l} \in \mathbb{R}$$

and $x_j^d \underset{(3.13)}{=} z_s \in \mathbb{R}$. It thus follows $\lambda_l := \frac{x_{j+l}}{x_j} \in \mathbb{R}$ for $l = 1, \dots, r$. In the case that $r = n+i-s < n-j$, then we moreover set $\lambda_l := 0 \in \mathbb{R}$ for $l = r+1, \dots, n-j$.

If $j = 0$, then $s = 0$ and $n - j = n \leq n + i = n + i - s$. Therefore, we define $\tilde{x} := (\gamma, \lambda_1\gamma, \dots, \lambda_n\gamma) \in \mathbb{R}^{n+1}$ and, for $t = 0, \dots, n+i$, compute

$$\begin{aligned} m_t(\tilde{x}) &= m_t(\gamma, \lambda_1 y, \dots, \lambda_n\gamma) \\ &= \gamma^d m_t(1, \lambda_1, \dots, \lambda_n) \\ &\underset{(3.14)}{=} z_s m_t(1, \lambda_1, \dots, \lambda_n) \\ &\underset{(3.13)}{=} x_0^d m_t(1, \lambda_1, \dots, \lambda_n) \\ &= x_0^d m_t\left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \\ &= m_t(x_0, x_1, \dots, x_n) \\ &\underset{(3.12)}{=} z_t. \end{aligned}$$

We thus conclude $\pi([z]) = \pi(V([\tilde{x}]))$ since

$$(z_0, \dots, z_{n+i}) \underset{(3.13)}{=} (m_0(x), \dots, m_{n+i}(x)) = (m_0(\tilde{x}), \dots, m_{n+i}(\tilde{x})). \tag{3.15}$$

However, if $j \geq 1$, then $s \geq 1$ and for $l = 0, \dots, j-1$, we have $x_l = 0$ since $X_l^d >_{\text{lex}} X_j^d >_{\text{lex}} m_{n+i}(X)$. This implies

$$z_t \underset{(3.12)}{=} m_t(x) = 0$$

for $t = 0, \dots, s-1$. We thus set $\tilde{x} := (0, \dots, 0, \tilde{x}_j, \dots, \tilde{x}_n) \in \mathbb{R}^{n+1}$ to be given by

$$\tilde{x}_\sigma := \begin{cases} \gamma, & \text{for } \sigma = j \\ \lambda_{\sigma-j}\gamma, & \text{for } \sigma = j+1, \dots, n \end{cases}$$

and, for $t = 0, \dots, s-1$, observe $m_t(\tilde{x}) = 0 = m_t(x) = z_t$. For $t = s, \dots, n+i$, using Lemma 2.2.11 and the fact that $X_{j+r+1}, \dots, X_n$ does not appear in $m_t(X)$ if

$r = n+i-s < n-j$, we moreover compute

$$\begin{aligned} m_t(\tilde{x}) &= m_t(0,\dots,0,\gamma,\lambda_1 y,\dots,\lambda_{n-j}\gamma) \\ &= \gamma^d m_t(0,\dots,0,1,\lambda_1,\dots,\lambda_{n-j}) \\ &\underset{(3.14)}{=} z_s m_t(0,\dots,0,1,\lambda_1,\dots,\lambda_{n-j}) \\ &\underset{(3.13)}{=} x_j^d m_t(0,\dots,0,1,\lambda_1,\dots,\lambda_{n-j}) \\ &= \begin{cases} x_j^d m_t\left(0,\dots,0,1,\frac{x_{j+1}}{x_j},\dots,\frac{x_n}{x_j}\right), & \text{if } n-j \le n+i-s \\ x_j^d m_t\left(0,\dots,0,1,\frac{x_{j+1}}{x_j},\dots,\frac{x_{j+r}}{x_j},0,\dots,0\right), & \text{if } n+i-s < n-j \end{cases} \\ &= x_j^d m_t\left(0,\dots,0,1,\frac{x_{j+1}}{x_j},\dots,\frac{x_n}{x_j}\right) \\ &= m_t(0,\dots,0,x_j,x_{j+1},\dots,x_n) \\ &= m_t(x) \\ &\underset{(3.12)}{=} z_t. \end{aligned}$$

Therefore, we conclude $\pi([z]) = \pi(V([\tilde{x}]))$ since

$$(z_0,\dots,z_{n+i}) \underset{(3.13)}{=} (m_0(x),\dots,m_{n+i}(x)) = (m_0(\tilde{x}),\dots,m_{n+i}(\tilde{x})). \tag{3.16}$$

In Case 1 and Case 2, we each found some $[\tilde{x}] \in \mathbb{P}^n(\mathbb{R})$ such that $\pi([z]) = \pi(V([\tilde{x}]))$. This shows $\pi([z]) = \pi([\tilde{x}]) \in \pi(V(\mathbb{P}^n(\mathbb{R})) \cap H)$ and we know $V(\mathbb{P}^n(\mathbb{R})) = V(\mathbb{P}^n)(\mathbb{R})$ by Proposition 2.3.34. Therefore, $\pi([z]) \in \pi(V(\mathbb{P}^n)(\mathbb{R}) \cap H)$ was proven. ■

<u>Claim 2</u>: $\overline{\pi(\phi(K_{k-n}))}(\mathbb{R}) \subseteq \overline{\pi(\phi(K_{k-n})(\mathbb{R}))}$.

Proof. Let $\overline{(\cdot)}^H$ denote the closure of the indicated subset in $\mathbb{P}^k \cap H$ and deduce $\overline{\phi(K_{k-n})}^H = \overline{\phi(K_{k-n})} \cap H = V(\mathbb{P}^n) \cap H$ from Theorem 3.1.11, Construction 3.3.7 and Corollary 3.3.16. Hence, $\overline{\phi(K_{k-n})}^H = V(\mathbb{P}^n) \cap H$ and the continuity of π implies

$$\overline{\pi(\phi(K_{k-n}))}(\mathbb{R}) \subseteq \pi\left(\overline{\phi(K_{k-n})}^H\right)(\mathbb{R}) = \pi(V(\mathbb{P}^n) \cap H)(\mathbb{R}). \tag{3.17}$$

Claim 1 thus yields

$$\overline{\pi(\phi(K_{k-n}))}(\mathbb{R}) \underset{(3.17)}{\subseteq} \pi(V(\mathbb{P}^n) \cap H)(\mathbb{R}) \subseteq \pi(V(\mathbb{P}^n)(\mathbb{R}) \cap H). \tag{3.18}$$

Since $V(\mathbb{P}^n) = W_{k-n} = \overline{\phi(K_{k-n})}$, it therefore follows

$$\overline{\pi(\phi(K_{k-n}))}(\mathbb{R}) \underset{(3.18)}{\subseteq} \pi(V(\mathbb{P}^n)(\mathbb{R}) \cap H) = \pi(\overline{\phi(K_{k-n})}(\mathbb{R}) \cap H) \tag{3.19}$$

and $\overline{\phi(K_{k-n})}(\mathbb{R}) = \overline{\phi(K_{k-n})(\mathbb{R})}$ by Theorem 3.4.1. Consequently,

$$\overline{\phi(K_{k-n})}(\mathbb{R}) \cap H = \overline{\phi(K_{k-n})(\mathbb{R})} \cap H = \overline{\phi(K_{k-n})(\mathbb{R})}^H$$

which allows us to conclude

$$\overline{\pi(\phi(K_{k-n}))}(\mathbb{R}) \underset{(3.19)}{\subseteq} \pi(\overline{\phi(K_{k-n})}(\mathbb{R}) \cap H) = \pi\left(\overline{\phi(K_{k-n})(\mathbb{R})}^H\right). \tag{3.20}$$

Recalling that π is continuous, we hence see

$$\overline{\pi(\phi(K_{k-n}))}(\mathbb{R}) \underset{(3.20)}{\subseteq} \pi\left(\overline{\phi(K_{k-n})(\mathbb{R})}^H\right) \subseteq \overline{\pi(\phi(K_{k-n})(\mathbb{R}))}.$$ ■

Claim 3: $\pi(\phi(K_i)) \subseteq \pi(\phi(K_{k-n}))$.
Proof. If $i = 0$, then $K_i = \mathbb{C}^k$ by Construction 3.3.7 (1) and

$$\begin{array}{rccc} \pi\colon & \mathbb{P}^k \cap H & \to & \mathbb{P}^n \\ & [z] & \mapsto & [z_0 : \ldots : z_n]. \end{array}$$

Hence, using Lemma 3.3.9 (ii), we conclude

$$\pi(\phi(K_i)) = \pi(\phi(\mathbb{C}^k)) = \{[z] \in \mathbb{P}^n \mid z_0 \neq 0\} = \pi(\phi(K_{k-n})).$$

However, if $i \geq 1$, then Lemma 3.3.9 (i) yields for any $\boldsymbol{z} \in K_i$ that

$$(z_1, \ldots, z_{n+i}) = (m_1(1, \boldsymbol{x}), \ldots, m_{n+i}(1, \boldsymbol{x}))$$

for some $\boldsymbol{x} \in \mathbb{C}^n$. For such choices, using Lemma 3.3.9 (ii), we obtain

$$\pi(\phi(\boldsymbol{z})) \quad = \quad \pi([1 : m_1(1, \boldsymbol{x}) : \ldots : m_k(1, \boldsymbol{x})]) \in \pi(\phi(K_{k-n})).$$

Altogether, we thus conclude $\pi(\phi(K_i)) \subseteq \pi(\phi(K_{k-n}))$. ■

From Claim 2, it follows

$$\overline{\pi(\phi(K_i))}(\mathbb{R}) \subseteq \overline{\pi(\phi(K_{k-n}))}(\mathbb{R}) \subseteq \overline{\pi(\phi(K_{k-n})(\mathbb{R}))}. \tag{3.21}$$

Moreover, $K_{k-n} \subseteq K_i$ by Construction 3.3.7. Hence,

$$\overline{\pi(\phi(K_i))}(\mathbb{R}) \underset{(3.21)}{\subseteq} \overline{\pi(\phi(K_{k-n})(\mathbb{R}))} \subseteq \overline{\pi(\phi(K_i)(\mathbb{R}))} \subseteq \overline{\pi(\phi(K_i))}(\mathbb{R}). \tag{3.22}$$

To conclude the proof, we now let $[z] \in \overline{\phi(K_i)}(\mathbb{R})$ be arbitrary but fixed and use (3.22) to show $[z] \in \overline{\phi(K_i)(\mathbb{R})}$ by a case distinction.

Case 1: If $[z] \notin H$, then $z_0 = \ldots = z_{n+i} = 0$ and we set $x^{(m)} := \left(\frac{1}{m}, \ldots, \frac{1}{m}\right) \in \mathbb{R}^{n+1}$, $y^{(m)} := \left(m_0\left(x^{(m)}\right), \ldots, m_{n+i}\left(x^{(m)}\right), z_{n+i+1}, \ldots, z_k\right) \in \mathbb{R}^{k+1}$ for $m \in \mathbb{N}$. Hence,

$$y^{(m)} \to z \text{ as } m \to \infty.$$

Using Lemma 2.2.11, we moreover see for $m \in \mathbb{N}$ that

$$\begin{aligned} \left[y^{(m)}\right] &= \left[m^d y^{(m)}\right] \\ &= \left[m^d m_0\left(x^{(m)}\right) : \ldots : m^d m_{n+i}\left(x^{(m)}\right) : m^d z_{n+i+1} : \ldots : m^d z_k\right] \\ &= \left[m_0\left(m x^{(m)}\right) : \ldots : m_{n+i}\left(m x^{(m)}\right) : m^d z_{n+i+1} : \ldots : m^d z_k\right] \\ &= \left[m_0(1,\ldots,1) : \ldots : m_{n+i}(1,\ldots,1) : m^d z_{n+i+1} : \ldots : m^d z_k\right] \\ &= \left[1 : m_1(1,\ldots,1) : \ldots : m_{n+i}(1,\ldots,1) : m^d z_{n+i+1} : \ldots : m^d z_k\right] \end{aligned}$$

which shows $\left[y^{(m)}\right] \in \phi(K_i)(\mathbb{R})$ by Lemma 3.3.9 (i). We conclude $[z] \in \overline{\phi(K_i)(\mathbb{R})}$.

Case 2: If $[z] \in H$, then $[z] \in \overline{\phi(K_i)}(\mathbb{R}) \subseteq \overline{\phi(K_i)}$ allows us to fix $\left(\left[y^{(m)}\right]\right)_{m\in\mathbb{N}} \subseteq \phi(K_i)$ and $(\lambda_m)_{m\in\mathbb{N}} \subseteq \mathbb{C}^\times$ such that

$$\lambda_m y^{(m)} \to z \text{ as } m \to \infty.$$

For $(\pi([y^{(m)}]))_{m\in\mathbb{N}} \subseteq \pi(\phi(K_i))$, we hence know that

$$\lambda_m\left(y_0^{(m)},\ldots,y_{n+i}^{(m)}\right) \to (z_0,\ldots,z_{n+i}) \text{ as } m \to \infty.$$

Therefore, we conclude $\pi([z]) \in \overline{\pi(\phi(K_i))}$ and recall $[z] \in \mathbb{P}^k(\mathbb{R})$ by choice. Thus, $\pi([z]) \in \overline{\pi(\phi(K_i))}(\mathbb{R})$ follows and $\overline{\pi(\phi(K_i))}(\mathbb{R}) \subseteq \overline{\pi(\phi(K_i)(\mathbb{R}))}$ by (3.22). This observation allows us to fix $\left(\left[\tilde{y}^{(m)}\right]\right)_{m\in\mathbb{N}} \subseteq \phi(K_i)(\mathbb{R})$ and $(\gamma_m)_{m\in\mathbb{N}} \subseteq \mathbb{C}^\times$ such that

$$\gamma_m\left(\tilde{y}_0^{(m)},\ldots,\tilde{y}_{n+i}^{(m)}\right) \to (z_0,\ldots,z_{n+i}) \text{ as } m \to \infty.$$

Since $\left(\left(\tilde{y}_0^{(m)},\ldots,\tilde{y}_{n+i}^{(m)}\right)\right)_{m\in\mathbb{N}} \subseteq \mathbb{R}^{n+i+1}$ and $(z_0,\ldots,z_{n+i}) \in \mathbb{R}^{n+i+1}$, we hence know

$$\operatorname{Re}(\gamma_m)\left(\tilde{y}_0^{(m)},\ldots,\tilde{y}_{n+i}^{(m)}\right) \to (z_0,\ldots,z_{n+i}) \text{ as } m \to \infty. \tag{3.23}$$

Recalling $[z] \in H$ by assumption, we assume $\operatorname{Re}(\gamma_m) \neq 0$ for $m \in \mathbb{N}$ without loss of generality after potentially going over to subsequences. This allows us to set

$$\overline{y}^{(m)} := \left(\tilde{y}_0^{(m)},\ldots,\tilde{y}_{n+i}^{(m)}, \frac{1}{\operatorname{Re}(\gamma_m)} z_{n+i+1},\ldots,\frac{1}{\operatorname{Re}(\gamma_m)} z_k\right) \in \mathbb{R}^{k+1}$$

and we have $\left(\left[\overline{y}^{(m)}\right]\right)_{m\in\mathbb{N}} \subseteq \phi(K_i)(\mathbb{R})$ since $\left(\left[\tilde{y}^{(m)}\right]\right)_{m\in\mathbb{N}} \subseteq \phi(K_i)(\mathbb{R})$ by choice. Moreover, by (3.23), we know

$$\operatorname{Re}(\gamma_m)\overline{y}^{(m)} \to z \text{ as } m \to \infty.$$

Altogether, we conclude $[z] \in \overline{\phi(K_i)(\mathbb{R})}$. ■

Example 3.4.4. Quaternary Quartics
Let $n = 3$, $d = 2$ and compute $k = 9$, $k - n = 6$. In this example, we illustrate the proof of Theorem 3.4.3 for $i = 3$. To this end, we let $\mathbb{P}^9$ be endowed with the Euclidean topology and recall from Example 3.3.3 that

$$p_1(\mathbf{Z}) = Z_4 - Z_1^2, \qquad p_2(\mathbf{Z}) = Z_5 - Z_1Z_2, \qquad p_3(\mathbf{Z}) = Z_6 - Z_1Z_3.$$

We refer to Example 2.3.10 (i) for an explicit description of the monomials $m_0, \dots, m_9$ and apply Lemma 3.3.9 to compute

$$\begin{aligned} K_3 &= \Big\{\Big(x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, z_7, z_8, z_9\Big) \mid x_1, x_2, x_3, z_7, z_8, z_9 \in \mathbb{C}\Big\},\\ \phi(K_3) &= \Big\{\Big[1 : x_1 : x_2 : x_3 : x_1^2 : x_1x_2 : x_1x_3 : z_7 : z_8 : z_9\Big] \mid x_1, x_2, x_3, z_7, z_8, z_9 \in \mathbb{C}\Big\},\\ \phi(K_3)(\mathbb{R}) &= \Big\{\Big[1 : x_1 : x_2 : x_3 : x_1^2 : x_1x_2 : x_1x_3 : z_7 : z_8 : z_9\Big] \mid x_1, x_2, x_3, z_7, z_8, z_9 \in \mathbb{R}\Big\}. \end{aligned}$$

Theorem 3.1.11, Construction 3.3.7, Theorem 3.3.17 and Corollary 3.3.24 together yield $\overline{\phi(K_3)} = W_3 = V_3 = \mathcal{V}(F)$ for $F := \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$ with

$$\begin{aligned} \mathfrak{F}_1 &:= \{Z_1^2 - Z_0Z_4, Z_1Z_2 - Z_0Z_5, Z_1Z_3 - Z_0Z_6\},\\ \mathfrak{F}_2 &:= \{Z_1Z_5 - Z_2Z_4, Z_1Z_6 - Z_3Z_4, Z_2Z_6 - Z_3Z_5\},\\ \mathfrak{F}_3 &:= \{Z_4Z_2^2 - Z_0Z_5^2, Z_4Z_2Z_3 - Z_0Z_5Z_6, Z_4Z_3^2 - Z_0Z_6^2\}. \end{aligned}$$

Therefore, we know that

$$\begin{aligned} \overline{\phi(K_3)}(\mathbb{R}) = &\Big\{\Big[1 : x_1 : x_2 : x_3 : x_1^2 : x_1x_2 : x_1x_3 : z_7 : z_8 : z_9\Big] \mid x_1, x_2, x_3, z_7, z_8, z_9 \in \mathbb{R}\Big\}\\ &\cup\ \{[0 : 0 : 1 : z_3 : 0 : z_5 : z_3z_5 : z_7 : z_8 : z_9] \mid z_3, z_5, z_7, z_8, z_9 \in \mathbb{R}\}\\ &\cup\ \{[0 : 0 : 0 : 1 : 0 : 0 : z_6 : z_7 : z_8 : z_9] \mid z_6, z_7, z_8, z_9 \in \mathbb{R}\}\\ &\cup\ \{[0 : 0 : 0 : 0 : z_4 : \dots : z_9] \mid z_4, \dots, z_9 \in \mathbb{R} \text{ not all zero}\}. \end{aligned}$$

For $[z] \in \overline{\phi(K_3)}(\mathbb{R})$, we thus prove $[z] \in \overline{\phi(K_3)(\mathbb{R})}$ by a case distinction.

Case 1: If $[z] = [1 : x_1 : x_2 : x_3 : x_1^2 : x_1x_2 : x_1x_3 : z_7 : z_8 : z_9]$ for some $x_1, x_2, x_3, z_7, z_8, z_9 \in \mathbb{R}$, then, comparing with our above description of $\phi(K_3)(\mathbb{R})$, we conclude $[z] \in \phi(K_3)(\mathbb{R}) \subseteq \overline{\phi(K_3)(\mathbb{R})}$.

Case 2: If $[z] = [0 : 0 : 1 : z_3 : 0 : z_5 : z_3z_5 : z_7 : z_8 : z_9]$ for some $z_3, z_5, z_7, z_8, z_9 \in \mathbb{R}$, then, for $m \in \mathbb{N}$, we set

$$\overline{y}^{(m)} \quad := \quad \left(\left(\frac{1}{m}\right)^2, \frac{z_5}{m^2}, 1, z_3, \left(\frac{z_5}{m}\right)^2, z_5, z_3z_5, z_7, z_8, z_9\right) \in \mathbb{R}^{10}$$

and observe that $\overline{y}^{(m)} \to (0, 0, 1, z_3, 0, z_5, z_3z_5, z_7, z_8, z_9)$ as $m \to \infty$. We compute

$$\left[\overline{y}^{(m)}\right] \quad = \quad \left[\left(\frac{1}{m}\right)^2 : \frac{z_5}{m^2} : 1 : z_3 : \left(\frac{z_5}{m}\right)^2 : z_5 : z_3z_5 : z_7 : z_8 : z_9\right]$$

$$
\begin{aligned}
&= \left(\frac{1}{m}\right)^2 \left[1 : z_5 : m^2 : z_3 m^2 : z_5^2 : z_5 m^2 : z_5(z_3 m^2) : z_7 m^2 : z_8 m^2 : z_9 m^2\right] \\
&= \left[1 : z_5 : m^2 : z_3 m^2 : z_5^2 : z_5 m^2 : z_5(z_3 m^2) : z_7 m^2 : z_8 m^2 : z_9 m^2\right].
\end{aligned}
$$

Comparing with our above description of $\phi(K_3)(\mathbb{R})$, we conclude $\left[\overline{y}^{(m)}\right] \in \phi(K_3)(\mathbb{R})$ for $m \in \mathbb{N}$. Altogether, it follows $[z] \in \overline{\phi(K_3)(\mathbb{R})}$.

Case 3: If $[z] = [0 : 0 : 0 : 1 : 0 : 0 : z_6 : z_7 : z_8 : z_9]$ for some $z_6, z_7, z_8, z_9 \in \mathbb{R}$, then, for $m \in \mathbb{N}$, we set

$$
\overline{y}^{(m)} := \left(\left(\frac{1}{m}\right)^2, \frac{z_6}{m^2}, \left(\frac{1}{m}\right)^2, 1, \left(\frac{z_6}{m}\right)^2, \frac{z_6}{m^2}, z_6, z_7, z_8, z_9\right) \in \mathbb{R}^{10}
$$

and observe that $\overline{y}^{(m)} \to (0, 0, 0, 1, 0, 0, z_6, z_7, z_8, z_9)$ as $m \to \infty$. We compute

$$
\begin{aligned}
\left[\overline{y}^{(m)}\right] &= \left[\left(\frac{1}{m}\right)^2 : \frac{z_6}{m^2} : \left(\frac{1}{m}\right)^2 : 1 : \left(\frac{z_6}{m}\right)^2 : \frac{z_6}{m^2} : z_6 : z_7 : z_8 : z_9\right] \\
&= \left(\frac{1}{m}\right)^2 \left[1 : z_6 : 1 : m^2 : z_6^2 : z_6 : z_6 m^2 : z_7 m^2 : z_8 m^2 : z_9 m^2\right] \\
&= \left[1 : z_6 : 1 : m^2 : z_6^2 : z_6 : z_6 m^2 : z_7 m^2 : z_8 m^2 : z_9 m^2\right].
\end{aligned}
$$

Comparing with our above description of $\phi(K_3)(\mathbb{R})$, we conclude $\left[\overline{y}^{(m)}\right] \in \phi(K_3)(\mathbb{R})$ for $m \in \mathbb{N}$. Altogether, it follows $[z] \in \overline{\phi(K_3)(\mathbb{R})}$.

Case 4: If $[z] = [0 : 0 : 0 : 0 : z_4 : \ldots : z_9]$ for some $z_4, \ldots, z_9 \in \mathbb{R}$, then we have to examine further case distinctions.

Case 4.1: If $[z] = [0 : 0 : 0 : 0 : 1 : z_5 : \ldots : z_9]$ for some $z_5, \ldots, z_9 \in \mathbb{R}$, then, for $m \in \mathbb{N}$, we set

$$
\overline{y}^{(m)} := \left(\left(\frac{1}{m}\right)^2, \frac{1}{m}, \frac{z_5}{m}, \frac{z_6}{m}, 1, z_5, z_6, z_7, z_8, z_9\right) \in \mathbb{R}^{10}
$$

and observe that $\overline{y}^{(m)} \to (0, 0, 0, 0, 1, z_5, z_6, z_7, z_8, z_9)$ as $m \to \infty$. We compute

$$
\begin{aligned}
\left[\overline{y}^{(m)}\right] &= \left[\left(\frac{1}{m}\right)^2 : \frac{1}{m} : \frac{z_5}{m} : \frac{z_6}{m} : 1 : z_5 : z_6 : z_7 : z_8 : z_9\right] \\
&= \left(\frac{1}{m}\right)^2 \left[1 : m : z_5 m : z_6 m : m^2 : z_5 m^2 : z_6 m^2 : z_7 m^2 : z_8 m^2 : z_9 m^2\right] \\
&= \left[1 : m : z_5 m : z_6 m : m^2 : z_5 m^2 : z_6 m^2 : z_7 m^2 : z_8 m^2 : z_9 m^2\right].
\end{aligned}
$$

Comparing with our above description of $\phi(K_3)(\mathbb{R})$, we conclude $\left[\overline{y}^{(m)}\right] \in \phi(K_3)(\mathbb{R})$ for $m \in \mathbb{N}$. Altogether, it follows $[z] \in \overline{\phi(K_3)(\mathbb{R})}$.

Case 4.2: If $[z] = [0:0:0:0:0:1:z_6:\ldots:z_9]$ for some $z_6,\ldots,z_9 \in \mathbb{R}$, then, for $m \in \mathbb{N}$, we set

$$\overline{y}^{(m)} := \left(\left(\frac{1}{m^2}\right)^2, \frac{1}{m^3}, \frac{1}{m}, \frac{z_6}{m}, \frac{1}{m^2}, 1, z_6, z_7, z_8, z_9\right) \in \mathbb{R}^{10}$$

and observe that $\overline{y}^{(m)} \to (0,0,0,0,0,1,z_6,z_7,z_8,z_9)$ as $m \to \infty$. We compute

$$\begin{aligned} \left[\overline{y}^{(m)}\right] &= \left[\left(\frac{1}{m^2}\right)^2 : \frac{1}{m^3} : \frac{1}{m} : \frac{z_6}{m} : \frac{1}{m^2} : 1 : z_6 : z_7 : z_8 : z_9\right] \\ &= \left(\frac{1}{m^2}\right)^2 \left[1 : m : m^3 : z_6m^3 : m^2 : m^4 : z_6m^4 : z_7m^4 : z_8m^4 : z_9m^4\right] \\ &= \left[1 : m : m^3 : z_6m^3 : m^2 : m^4 : z_6m^4 : z_7m^4 : z_8m^4 : z_9m^4\right]. \end{aligned}$$

Comparing with our above description of $\phi(K_3)(\mathbb{R})$, we conclude $\left[\overline{y}^{(m)}\right] \in \phi(K_3)(\mathbb{R})$ for $m \in \mathbb{N}$. Altogether, it follows $[z] \in \overline{\phi(K_3)(\mathbb{R})}$.

Case 4.3: If $[z] = [0:0:0:0:0:0:1:z_7:z_8:z_9]$ for some $z_7,z_8,z_9 \in \mathbb{R}$, then, for $m \in \mathbb{N}$, we set

$$\overline{y}^{(m)} := \left(\left(\frac{1}{m^2}\right)^2, \frac{1}{m^3}, \frac{1}{m^3}, \frac{1}{m}, \frac{1}{m^2}, \frac{1}{m^2}, 1, z_7, z_8, z_9\right) \in \mathbb{R}^{10}$$

and observe that $\overline{y}^{(m)} \to (0,0,0,0,0,0,1,z_7,z_8,z_9)$ as $m \to \infty$. We compute

$$\begin{aligned} \left[\overline{y}^{(m)}\right] &= \left[\left(\frac{1}{m^2}\right)^2 : \frac{1}{m^3} : \frac{1}{m^3} : \frac{1}{m} : \frac{1}{m^2} : \frac{1}{m^2} : 1 : z_7 : z_8 : z_9\right] \\ &= \left(\frac{1}{m^2}\right)^2 \left[1 : m : m : m^3 : m^2 : m^2 : m^4 : z_7m^4 : z_8m^4 : z_9m^4\right] \\ &= \left[1 : m : m : m^3 : m^2 : m^2 : m^4 : z_7m^4 : z_8m^4 : z_9m^4\right]. \end{aligned}$$

Comparing with our above description of $\phi(K_3)(\mathbb{R})$, we conclude $\left[\overline{y}^{(m)}\right] \in \phi(K_3)(\mathbb{R})$ for $m \in \mathbb{N}$. Altogether, it follows $[z] \in \overline{\phi(K_3)(\mathbb{R})}$.

Case 4.4: If $[z] = [0:0:0:0:0:0:0:z_7:z_8:z_9]$ for some $z_7,z_8,z_9 \in \mathbb{R}$, then, for $m \in \mathbb{N}$, we set

$$y^{(m)} := \left(\left(\frac{1}{m}\right)^2, \left(\frac{1}{m}\right)^2, \left(\frac{1}{m}\right)^2, \left(\frac{1}{m}\right)^2, \left(\frac{1}{m}\right)^2, \left(\frac{1}{m}\right)^2, \left(\frac{1}{m}\right)^2, z_7, z_8, z_9\right) \in \mathbb{R}^{10}$$

and observe that $y^{(m)} \to (0,0,0,0,0,0,0,z_7,z_8,z_9)$ as $m \to \infty$. We compute

$$\begin{aligned} \left[y^{(m)}\right] &= \left[\left(\frac{1}{m}\right)^2 : \left(\frac{1}{m}\right)^2 : \left(\frac{1}{m}\right)^2 : \left(\frac{1}{m}\right)^2 : \left(\frac{1}{m}\right)^2 : \left(\frac{1}{m}\right)^2 : \left(\frac{1}{m}\right)^2 : z_7 : z_8 : z_9\right] \\ &= \left(\frac{1}{m}\right)^2 \left[1:1:1:1:1:1:1 : z_7m^2 : z_8m^2 : z_9m^2\right] \\ &= \left[1:1:1:1:1:1:1 : z_7m^2 : z_8m^2 : z_9m^2\right]. \end{aligned}$$

Comparing with our above description of $\phi(K_3)(\mathbb{R})$, we conclude $\left[\overline{y}^{(m)}\right] \in \phi(K_3)(\mathbb{R})$ for $m \in \mathbb{N}$. Altogether, it follows $[z] \in \overline{\phi(K_3)(\mathbb{R})}$.

Corollary 3.4.5. *For $i = 0, \ldots, k-n$, it holds $C_i = C_{\phi(K_i)}$.*

Proof. Theorem 3.4.1 and Theorem 3.4.3 together imply $\overline{\phi(K_i)(\mathbb{R})} = \overline{\phi(K_i)}(\mathbb{R})$ w.r.t. the Euclidean topology on $\mathbb{P}^k$. Moreover, Theorem 3.3.17 states $V_i = W_i$ and W_i is the Zariski closure of $\phi(K_i)$ by Construction 3.3.7 (3). Corollary 3.1.12 thus yields $C_{\phi(K_i)} = C_{W_i} = C_{V_i} = C_i$. ∎

Chapter 4

Cones Coinciding with $\Sigma_{n+1,2d}$

In 2016, Blekherman, Smith and Valsco [BSV16] showed that any quadratic form which is locally non-negative on a given non-degenerate irreducible totally-real projective variety is a sum of squares modulo the vanishing ideal of this variety if and only if the degree of the variety equals its codimension increased by one.

In Section 4.1, we validate that the above result is applicable in our setting in non-Hilbert cases by showing that the specific projective varieties from Construction 3.2.1 are non-degenerate, irreducible and totally-real. Moreover, we verify for the first $n+1$, respectively $n+2$ if $n=2$, of these distinguished projective varieties that the degree equals the codimension increased by one.

In Section 4.2, we apply the result of Blekherman–Smith–Velasco to deduce a sufficient criterion for intermediate cones along non-degenerate irreducible totally-real projective varieties to coincide with $\Sigma_{n+1,2d}$. Using the consideration of the first section, we furthermore see that this criterion is applicable to the first $n+1$, respectively $n+2$ if $n=2$, cones in our specific cone filtration in non-Hilbert cases.

4.1 Properties of the Varieties $V_0, \dots, V_{k-n}$

Throughout this section, $(n+1, 2d)$ **denotes a non-Hilbert case** if not explicitly mentioned otherwise.

4.1.1 Non-Degeneracy

Definition 4.1.1. For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a projective variety. If there exists a hyperplane $H \subseteq \mathbb{P}^l$ such that $W \subseteq H$, then W is called *degenerate*. Otherwise, W is called *non-degenerate*.

Proposition 4.1.2. *For $l \in \mathbb{N}$, let $\mathfrak{W}_1, \mathfrak{W}_2 \subseteq \mathbb{P}^l$ be two projective varieties such that $\mathfrak{W}_1 \subseteq \mathfrak{W}_2$. If $\mathfrak{W}_2$ is degenerate, then $\mathfrak{W}_1$ is degenerate.*

Proof. Let $H \subseteq \mathbb{P}^l$ be a hyperplane such that $\mathfrak{W}_2 \subseteq H$. Since $\mathfrak{W}_1$ is a subvariety of $\mathfrak{W}_2$, we conclude $\mathfrak{W}_1 \subseteq \mathfrak{W}_2 \subseteq H$. Therefore, $\mathfrak{W}_1$ is contained in the hyperplane H, which shows that $\mathfrak{W}_1$ is degenerate. ∎

Lemma 4.1.3. *$V(\mathbb{P}^n)$ is non-degenerate.*

Proof. Let us assume for a proof by contradiction that there exists some non-zero linear form $f \in \mathbb{C}[Z]$ such that $V(\mathbb{P}^n) \subseteq \mathcal{V}(f)$. Consequently, the non-zero $(n+1)$-ary d-ic $g(X) := f(m_0(X), \dots, m_k(X)) \in \mathbb{C}[X]$ vanishes on $\mathbb{C}^{n+1}$. This is impossible. ∎

Corollary 4.1.4. *For $i = 0, \dots, k-n$, V_i is non-degenerate.*

Proof. The Veronese variety is a non-degenerate subvariety of V_i by Lemma 4.1.3 and Lemma 3.2.5. The contraposition of Proposition 4.1.2 with $\mathfrak{W}_1 := V(\mathbb{P}^n)$ and $\mathfrak{W}_2 := V_i$ therefore yields that V_i is non-degenerate. ∎

4.1.2 Irreducibility

Definition 4.1.5. For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a projective variety. If there exist projective subvarieties $\mathfrak{W}_1, \mathfrak{W}_2 \subsetneq W$ such that $\mathfrak{W}_1 \cup \mathfrak{W}_2 = W$, then W is called *reducible*. Otherwise, W is called *irreducible*.

Proposition 4.1.6. *For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a projective variety, then there exists a unique $m \in \mathbb{N}$ and unique irreducible projective subvarieties $\mathfrak{W}_1, \dots, \mathfrak{W}_m \subseteq \mathfrak{W}$ such that $\mathfrak{W}_i \not\subseteq \mathfrak{W}_j$ for $i \neq j$ and $W = \bigcup_{i=1}^{m} \mathfrak{W}_i$.*

Proof. See [Har92, Theorem 5.7.]. ∎

Definition 4.1.7. For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a projective variety and $\mathfrak{W}_1, \dots, \mathfrak{W}_m$ ($m \in \mathbb{N}$) be the unique irreducible projective subvarieties of W such that $\mathfrak{W}_i \not\subseteq \mathfrak{W}_j$ for $i \neq j$ and $W = \bigcup_{i=1}^{m} \mathfrak{W}_i$. The *irreducible decomposition* of W is given by $\bigcup_{i=1}^{m} \mathfrak{W}_i$ and $\mathfrak{W}_1, \dots, \mathfrak{W}_m$ are the *irreducible components* of W.

Proposition 4.1.8. *For $i = 0, \dots, k-n$, V_i is irreducible.*

Proof. Proposition 3.3.15 states that K_i and $\mathbb{C}^{k-i}$ are isomorphic as affine varieties and $\mathbb{C}^{k-i}$ is irreducible as an affine variety (cf. [Har77, Chapter 1, §1, Example 1.4.1]). Hence, K_i is irreducible by Theorem A.1.13 and consequently W_i (the projective closure of the affine variety K_i) is irreducible by Theorem A.1.54. Furthermore, we have $W_i = V_i$ by Theorem 3.3.17. ∎

4.1.3 Dimension

Definition 4.1.9. Fix $l \in \mathbb{N}$.

(i) Let $W \subseteq \mathbb{P}^l$ be an irreducible projective variety. The *dimension* of W is the greatest integer $\delta \geq 0$ for which there exist irreducible projective subvarieties $\mathfrak{W}_0, \dots, \mathfrak{W}_\delta \subseteq W$ such that $\emptyset \subsetneq \mathfrak{W}_0 \subsetneq \dots \subsetneq \mathfrak{W}_\delta = W$.

(ii) Let $W \subseteq \mathbb{P}^l$ be a projective variety. The *dimension* of W is the maximal dimension of its irreducible components.

Notation 4.1.10. For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a projective variety. We denote the dimension of W by $\dim(W)$.

We now introduce a useful tool for computing the dimension of a projective variety.

Definition 4.1.11. For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a projective variety and denote the graded homogeneous coordinate ring of W by $\mathbb{C}[W] = \bigoplus_{t \geq 0} \mathbb{C}[W]_t$. The *Hilbert function* of W is given by

$$\begin{aligned} h_W \colon \mathbb{N}_0 &\to \mathbb{N}_0 \\ t &\mapsto \dim(\mathbb{C}[W]_t). \end{aligned}$$

Proposition 4.1.12. *For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a projective variety with Hilbert function h_W, then there exists a unique polynomial p_W such that $h_W(t) = p_W(t)$ for all sufficiently large $t \in \mathbb{N}_0$.*

Proof. See [Har92, Proposition 13.2.]. ∎

Definition 4.1.13. For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a projective variety with Hilbert function h_W. The unique polynomial p_W such that $h_W(t) = p_W(t)$ for all sufficiently large $t \in \mathbb{N}_0$ is called the *Hilbert polynomial* of W.

Theorem 4.1.14. *For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a projective variety with Hilbert polynomial p_W, then $\dim(W) = \deg(p_W)$.*

Proof. See [Pla20, 3.7.5 Satz]. ∎

Corollary 4.1.15. *For $l \in \mathbb{N}$, let $\mathfrak{W}_1, \mathfrak{W}_2 \subseteq \mathbb{P}^l$ be two projective varieties with Hilbert polynomials $p_{\mathfrak{W}_1}$ and $p_{\mathfrak{W}_2}$, respectively. If $p_{\mathfrak{W}_1} = p_{\mathfrak{W}_2}$, then $\dim(\mathfrak{W}_1) = \dim(\mathfrak{W}_2)$.*

Proof. Theorem 4.1.14 implies $\dim(\mathfrak{W}_1) = \deg(p_{\mathfrak{W}_1}) = \deg(p_{\mathfrak{W}_2}) = \dim(\mathfrak{W}_2)$. ∎

Lemma 4.1.16. *For $l \in \mathbb{N}$ and $i = 0, \dots, l-1$, the dimension of $\mathcal{V}(Z_0, \dots, Z_i) \subseteq \mathbb{P}^l$ is $l - (i+1)$.*

Proof. The vanishing ideal of $\mathcal{V}(Z_0, \dots, Z_i)$ coincide with $\langle Z_0, \dots, Z_i \rangle$. Therefore, we have $\mathbb{C}[\mathcal{V}(Z_0, \dots, Z_i)] \simeq \mathbb{C}[Z_{i+1}, \dots, Z_l]$. The Hilbert function is thus given by

$$\begin{aligned} h_{\mathcal{V}(Z_0,\dots,Z_i)} \colon \mathbb{N}_0 &\to \mathbb{N}_0 \\ t &\mapsto \dim(\mathbb{C}[Z_{i+1}, \dots, Z_l]_t) = k(l-(i+1), t) \end{aligned}$$

and can be interpreted as a polynomial. Hence, $p_{\mathcal{V}(Z_0,\dots,Z_i)} = h_{\mathcal{V}(Z_0,\dots,Z_i)} \in \mathbb{R}[T]$ is a polyonimal of degree $l - (i+1)$. Theorem 4.1.14 hence implies

$$\dim(\mathcal{V}(Z_0, \dots, Z_i)) = l - (i+1).$$ ∎

Lemma 4.1.17. *For $l \in \mathbb{N}$, the projective variety $\mathbb{P}^l$ is l-dimensional and its Hilbert polynomial is given by*

$$\begin{aligned} p_{\mathbb{P}^l} : \mathbb{N}_0 &\to \mathbb{N}_0 \\ t &\mapsto k(l,t)+1. \end{aligned}$$

Proof. The vanishing ideal of $\mathbb{P}^l$ is the zero ideal. Therefore, we see $\mathbb{C}[\mathbb{P}^l] \simeq \mathbb{C}[Z_0, \ldots, Z_l]$. The Hilbert function

$$\begin{aligned} h_{\mathbb{P}^l} : \mathbb{N}_0 &\to \mathbb{N}_0 \\ t &\mapsto \dim\left(\mathbb{C}[\mathbb{P}^l]_t\right) = \dim(\mathbb{C}[Z_0, \ldots, Z_l]_t) = k(l,t)+1 \end{aligned}$$

of $\mathbb{P}^l$ can therefore be interpreted as a polynomial. Hence, $p_{\mathbb{P}^l} = h_{\mathbb{P}^l} \in \mathbb{C}[T]$, which is a polyonimal of degree l. Theorem 4.1.14 thus yields $\dim(\mathbb{P}^l) = l$. ∎

Notation 4.1.18. For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a projective variety. The *codimension* of W is given by $\operatorname{codim}(W) := \dim(\mathbb{P}^l) - \dim(W) = l - \dim(W)$.

Lemma 4.1.19. *For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be an irreducible projective variety of dimension δ. If there exists some integer $\delta' \geq 0$ and irreducible varieties $\mathfrak{W}_0, \ldots, \mathfrak{W}_{\delta'} \subseteq \mathbb{P}^l$ such that $W = \mathfrak{W}_0 \subsetneq \ldots \subsetneq \mathfrak{W}_{\delta'} = \mathbb{P}^l$, then $\delta' \leq \operatorname{codim}(W)$.*

Proof. Since $\dim(W) = \delta$, we know that there exist irreducible projective subvarieties $\mathfrak{V}_0, \ldots, \mathfrak{V}_\delta \subseteq W$ such that $\emptyset \subsetneq \mathfrak{V}_0 \subsetneq \ldots \subsetneq \mathfrak{V}_\delta = W$. Hence,

$$\emptyset \subsetneq \mathfrak{V}_0 \subsetneq \ldots \subsetneq \mathfrak{V}_\delta = W = \mathfrak{W}_0 \subsetneq \ldots \subsetneq \mathfrak{W}_{\delta'} = \mathbb{P}^l$$

is a filtration of irreducible varieties in which each inclusion is strict. We conclude $\delta + \delta' \leq \dim(\mathbb{P}^l)$. Thus, $\delta' \leq \dim(\mathbb{P}^l) - \delta = \dim(\mathbb{P}^l) - \dim(W) = \operatorname{codim}(W)$ follows. ∎

Proposition 4.1.20. *For $i = 0, \ldots, k-n$, V_i has codimension i.*

Proof. Proposition 3.3.15 states that K_i and $\mathbb{C}^{k-i}$ are isomorphic as affine varieties and $\mathbb{C}^{k-i}$ has dimension $k-i$ (cf. [Har77, Chapter 1, §1, Proposition 1.9]). Hence, K_i has dimension $k-i$ by Theorem A.1.16 and consequently W_i (the projective closure of the affine variety K_i) has dimension $k-i$ by Theorem A.1.55. Furthermore, $W_i = V_i$ by Theorem 3.3.17 and thus $\operatorname{codim}(V_i) = k - \dim(V_i) = i$ follows. ∎

Proposition 4.1.20 and Lemma 3.2.5 (ii) together imply $\operatorname{codim}(V(\mathbb{P}^n)) = k - n$ and $V_0, \ldots, V_{k-n}$ are irreducible projective varieties such that

$$V(\mathbb{P}^n) = V_{k-n} \subsetneq \ldots \subsetneq V_0 = \mathbb{P}^k$$

by (3.4) and Proposition 4.1.8. The above filtration of projective varieties can therefore not be extended by any further, i.e., to $V_0, \dots, V_{k-n}$ distinct, irreducible intermediate varieties according to Lemma 4.1.19. Hence, (3.4) is a filtration of irreducible projective varieties between $V(\mathbb{P}^n)$ and $\mathbb{P}^k$ that has maximal length w.r.t. $\subseteq$ such that each inclusion is strict.

4.1.4 Totally-Realness

Definition 4.1.21. For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a projective variety that is induced by forms with real coefficients.[1] If the Zariski closure of $W(\mathbb{R})$ coincides with W, then W is called *totally-real*.

The idea behind totally-realness is that we want to ensure that an in $\mathbb{P}^k$ embedded real variety has enough real points. An accessible characterization of totally-real projective varieties is therefore based on the existence of non-singular real points. We recall that a point $[x]$ of an irreducible projective variety $W \subseteq \mathbb{P}^l$ with vanishing ideal $\mathcal{I}(W) = \langle f_1, \dots, f_s \rangle$ for some homogeneous polynomials $f_1, \dots, f_s \in \mathbb{C}[X_0, \dots, X_l]$ ($s \in \mathbb{N}$) is *non-singular* if and only if the rank of the Jacobian matrix J of $f_1, \dots, f_s$ in $[x]$ equals the codimension of W (cf. [Per08, Chapter V, Proposition 2.6]).

Theorem 4.1.22. *For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a projective variety that is induced by forms with real coefficients, then the Zariski closure of $W(\mathbb{R})$ coincides with W if and only if there exists a non-singular real point in each irreducible component of W.*

Proof. See [Man20, Theorem 2.2.9 3.]. ■

Proposition 4.1.23. *For $i = 0, \dots, k-n$, V_i is totally-real.*

Proof. We verify the assertion in two steps.

Claim 1: V_i is the set of common zeros of some forms with real coefficients.
Proof. If $i = 0$, then $V_0 = \mathbb{P}^k = \mathcal{V}(0)$ by Lemma 3.2.5 (i) and we are done. However, for $i = 1, \dots, k-n$, Theorem 3.3.17 states $V_i = W_i$ and W_i is the projective closure of the affine variety $K_i \subseteq \mathbb{C}^k$ in $\mathbb{P}^k$. The affine variety K_i is moreover the set of common zeros of the polynomials $p_1, \dots, p_i$ that only have real coefficient. Buchberger's algorithm (cf. Algorithm A.1.71) thus allows us to construct a Gröbner basis $G \subseteq \mathbb{R}[\boldsymbol{Z}]$ for the ideal $\langle p_1, \dots, p_i \rangle \subseteq \mathbb{C}[\boldsymbol{Z}]$ w.r.t. the graded monomial order $\leq_{\text{grlex}}$ from $\{p_1, \dots, p_i\} \subseteq \mathbb{R}[\boldsymbol{Z}]$. We conclude by Theorem A.1.58 and Theorem A.1.79 that

$$V_i = W_i = \overline{\phi(K_i)} = \overline{\phi(\boldsymbol{\mathcal{V}}(G))} = \mathcal{V}\left(G^h\right).$$

Hence, V_i is the set of common zeros of $G^h \subseteq \mathbb{R}[Z]$. ■

[1] The base field of the projective space $\mathbb{P}^l$ is $\mathbb{C}$. Therefore, a projective variety is a priori the set of common zeros of some polynomials that might have complex coefficients. In the particular case of this definition, we now especially require that there exist forms with only real coefficients which induce W. In light of our definition of forms (cf. Definition 2.2.4), the coefficients are always real. However, in the literature, it is common to define forms with coefficients in arbitrary fields. Therefore, we added the information "with real coefficients" to avoid any confusion.

Claim 2: $\overline{V_i(\mathbb{R})} = V_i$ w.r.t. the Zariski topology.

Proof. V_i is an irreducible projective variety by Proposition 4.1.8 and $V_i = W_i$ by Theorem 3.3.17. Thus, it suffices to show that $W_i(\mathbb{R})$ contains a non-singular point by Theorem 4.1.22. To this end, we let $U := \phi(\mathbb{C}^k) = \left\{[1 : \boldsymbol{z}] \mid \boldsymbol{z} \in \mathbb{C}^k\right\} \subseteq \mathbb{P}^k$ be the open 0^{th} affine patch of $\mathbb{P}^k$ with $Z_0 \neq 0$ and recall from Proposition A.1.53 that $U \cap W_i = U \cap V_i = \phi(K_i)$. Moreover, we set $\boldsymbol{e} := (1, \ldots, 1) \in \mathbb{C}^k$ and observe $\phi(\boldsymbol{e}) \in V_i \cap U = W_i \cap U = \phi(K_i)$. Hence, $\phi(\boldsymbol{e})$ is a non-singular point of V_i if and only if $\boldsymbol{e}$ is non-singular for K_i by [Per08, Chapter V, Remark 1.11, Definition 2.1] and [CLO15, Chapter 9, §5, Corollary 3].

It thus suffices to show that $\boldsymbol{e}$ is a non-singular point of K_i. We recall that $K_i \subseteq \mathbb{C}^k$ is an irreducible $(k-i)$-dimensional affine variety that is the set of common zeros of the polynomials $p_1, \ldots, p_i \in \mathbb{R}[\boldsymbol{X}]$, where $p_j(\boldsymbol{Z}) := Z_{n+j} - Z_{s_j} Z_{t_j}$ for $j = 1, \ldots, i$ as in Construction 3.3.1. Moreover, we let I_i denote the $i \times i$ identity matrix, O the $i \times (k - (n+i))$ matrix with only zero entries and observe that the Jacobian matrix of $p_1, \ldots, p_i$ in $\boldsymbol{e}$ is given by

$$J := (J_{s,t})_{1 \leq s \leq i, 1 \leq t \leq k} := \left(\frac{\partial p_s}{\partial Z_t}\right)_{1 \leq s \leq i, 1 \leq t \leq k} (\boldsymbol{e}) = (A \mid I_i \mid O)$$

for some real $i \times n$ matrix A. Therefore, we see that the rank of J is $i = k - \dim(K_i)$ and thus $\boldsymbol{e}$ is a non-singular point of K_i by [Per08, Chapter V, Remark 2.4]. ∎

4.1.5 Degree

Definition 4.1.24. Fix $l \in \mathbb{N}$.

(i) Let $W \subseteq \mathbb{P}^l$ be an irreducible projective variety and p_W its Hilbert polynomial. The *degree* of W is $\dim(W)!$ times the leading coefficient of p_W.

(ii) Let $W \subseteq \mathbb{P}^l$ be a projective variety. The *degree* of W is the sum of the degrees of its irreducible components with dimension $\dim(W)$.

Notation 4.1.25. For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a projective variety. We denote the degree of W by $\deg(W)$.

Lemma 4.1.26. *For $l \in \mathbb{N}$, let $\mathfrak{W}_1, \mathfrak{W}_2 \subseteq \mathbb{P}^l$ be two irreducible projective varieties with Hilbert polynomials $p_{\mathfrak{W}_1}$ and $p_{\mathfrak{W}_2}$, respectively. If $p_{\mathfrak{W}_1} = p_{\mathfrak{W}_2}$, then*

$$\deg(\mathfrak{W}_1) = \deg(\mathfrak{W}_2).$$

Proof. Corollary 4.1.15 yields $\dim(\mathfrak{W}_1) = \dim(\mathfrak{W}_2)$ and thus it follows

$$\deg(\mathfrak{W}_1) = (\dim(\mathfrak{W}_1)!)\mathrm{LC}(p_{\mathfrak{W}_1}) = (\dim(\mathfrak{W}_2)!)\mathrm{LC}(p_{\mathfrak{W}_2}) = \deg(\mathfrak{W}_2),$$

where $\mathrm{LC}(\cdot)$ denotes the leading coefficient of the indicated polynomial. ∎

Lemma 4.1.27. *It holds* $\deg(V_{k-n}) = \deg(V(\mathbb{P}^n)) = d^n$.

Proof. Lemma 3.2.5 (ii), Proposition 4.1.8 and Proposition 4.1.20 together imply that $V_{k-n} = V(\mathbb{P}^n)$ is an n-dimensional irreducible projective variety. Moreover, the Hilbert polynomial of $V(\mathbb{P}^n)$ is given by

$$p_{V(\mathbb{P}^n)}(T) = \binom{dT+n}{n} \in \mathbb{C}[T]$$

(cf. [Har92, Example 13.4.]) and has leading coefficient $\frac{d^n}{n!}$. Recalling Theorem 4.1.14, we thus conclude $\deg(V(\mathbb{P}^n)) = n!\frac{d^n}{n!} = d^n$. ∎

Lemma 4.1.28. *For* $l \in \mathbb{N}$ *and* $i = 0, \dots, l-1$, *the degree of* $\mathcal{V}(Z_0, \dots, Z_i) \subseteq \mathbb{P}^l$ *is* 1.

Proof. In the proof of Lemma 4.1.16, we determined that the Hilbert polynomial of the irreducible $(l-(i+1))$-dimensional projective $\mathcal{V}(Z_0, \dots, Z_i) \subseteq \mathbb{P}^l$ is given by

$$p_{\mathcal{V}(Z_0,\dots,Z_i)}(T) = k(l-(i+1), T) \in \mathbb{C}[T].$$

The leading coefficient of $p_{\mathcal{V}(Z_0,\dots,Z_i)}$ is $\frac{1}{(l-(i+1))!}$ and thus we conclude

$$\deg(\mathcal{V}(Z_0, \dots, Z_i)) = (l-(i+1))!\frac{1}{(l-(i+1))!} = 1.$$ ∎

Lemma 4.1.29. *For* $l \in \mathbb{N}$, *it holds* $\deg(V_0) = \deg(\mathbb{P}^l) = 1$.

Proof. Lemma 4.1.17 states that the irreducible projective variety $\mathbb{P}^l$ is l-dimensional and has Hilbert polynomial

$$p_{\mathbb{P}^l}(T) = \binom{l+T}{l} \in \mathbb{R}[T].$$

The leading coefficient of $p_{\mathbb{P}^l}$ is $\frac{1}{l!}$ and thus we conclude $\deg(\mathbb{P}^l) = l!\frac{1}{l!} = 1$. ∎

Lemma 4.1.30. *For* $q \in \mathcal{S}_V$, *it holds* $\deg(\mathcal{V}(q)) = 2$.

Proof. Since q is an irreducible quadratic form by Lemma 2.3.32, we know that $\mathcal{V}(q)$ has degree 2 by [Har92, pp.224]. ∎

Corollary 4.1.31. *It holds* $\deg(V_1) = 2$.

Proof. Theorem 3.3.17, Theorem A.1.58 and Remark 3.3.2 together imply

$$V_1 = W_1 = \overline{\phi(K_1)} = \overline{\phi(\boldsymbol{\mathcal{V}}(p_1))} = \mathcal{V}\left(p_1^h\right) = \mathcal{V}(q_1).$$

Lemma 4.1.30 with $q := q_1 \in \mathcal{S}_V$ thus yields $\deg(V_1) = \deg(\mathcal{V}(q_1)) = 2$. ∎

In order to determine the degrees of the varieties $V_2, \dots, V_n$ and also V_{n+1} if $n = 2$, further results on the dimension and the degree of an a priori given projective variety are needed. We refer the reader to Appendix A.1 for an overview. In the special case that the given projective variety is non-degenerate and irreducible, an elementary lower bound for the degree is given by its codimension.

Proposition 4.1.32. *For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a non-degenerate irreducible projective variety, then* $\deg(W) \geq \operatorname{codim}(W) + 1$.

Proof. See [Har92, Corollary 18.12.]. ∎

Definition 4.1.33. For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a non-degenerate irreducible projective variety. If $\deg(W) = \operatorname{codim}(W) + 1$, then W is a projective variety of *minimal degree*.

Proposition 4.1.34. *It holds* $\deg(V_2) = 3$.

Proof. Theorem 3.3.17, Theorem A.1.58 and Remark 3.3.2 together imply

$$V_1 = W_1 = \overline{\phi(K_1)} = \overline{\phi(\boldsymbol{\mathcal{V}}(p_1))} = \mathcal{V}\left(p_1^h\right) = \mathcal{V}(q_1)$$

and we recall $q_1(Z) = Z_0 Z_{n+1} - Z_1^2$, $q_2(Z) = Z_0 Z_{n+2} - Z_1 Z_2$ from Lemma 3.3.5. Hence, $\mathfrak{W}_1 := \mathcal{V}(Z_0, Z_1)$ is an irreducible projective subvariety of

$$W := V_1 \cap \mathcal{V}(q_2) = \mathcal{V}(q_1, q_2) = \mathcal{V}\left(Z_0 Z_{n+1} - Z_1^2, Z_0 Z_{n+2} - Z_1 Z_2\right). \tag{4.1}$$

Likewise, we also deduce

$$\mathfrak{W}_2 := V_2 = W_2 = \overline{\phi(K_2)} = \overline{\phi(\boldsymbol{\mathcal{V}}(p_1, p_2))} = \mathcal{V}\left(\langle p_1, p_2\rangle^h\right) \subseteq \mathcal{V}\left(p_1^h, p_2^h\right) \underset{(4.1)}{=} W.$$

Claim: $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are two distinct irreducible components of W.
Proof. The projective variety V_1 is irreducible of dimension $k-1$ by Proposition 4.1.8, Proposition 4.1.20 and $V_1 \not\subseteq \mathcal{V}(q_2)$. Hence, Proposition A.1.43 yields

$$\dim(W) = \dim(V_1 \cap \mathcal{V}(q_2)) = \dim(V_1) - 1 = k - 2.$$

Moreover, the dimension of the irreducible projective subvariety $\mathfrak{W}_1$ of W is $k-2$ by Lemma 4.1.16 and $\mathfrak{W}_2 = V_2 \subseteq W$ is irreducible with $\dim(\mathfrak{W}_2) = k - 2$ by Proposition 4.1.8 and Proposition 4.1.20. Therefore, $\dim(\mathfrak{W}_1) = \dim(\mathfrak{W}_2) = \dim(W)$ and we conclude that $\mathfrak{W}_1$, $\mathfrak{W}_2$ are irreducible components of W by Theorem A.1.42.

The projective variety $\mathfrak{W}_1$ is furthermore degenerate by construction while the projective variety $\mathfrak{W}_2 = V_2$ is non-degenerate by Corollary 4.1.4. Therefore, the irreducible components $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are distinct. ∎

Lemma 4.1.30 and Corollary 4.1.31 together yield $\deg(V_1) = \deg(\mathcal{V}(q_2)) = 2$. Moreover, $\deg(\mathfrak{W}_1) = 1$ by Lemma 4.1.28 and

$$\deg(\mathfrak{W}_2) = \deg(V_2) \geq \operatorname{codim}(V_2) + 1 = 3$$

by Proposition 4.1.20 and Proposition 4.1.32. An application of Bézout's theorem (cf. Theorem A.1.49) thus allows us to conclude for some $c_1, c_2 \in \mathbb{N}$, $c \in \mathbb{N}_0$ that

$$\begin{aligned} 4 &= \deg(V_1) \cdot \deg(\mathcal{V}(q_2)) \\ &= c_1 \cdot \deg(\mathfrak{W}_1) + c_2 \cdot \deg(\mathfrak{W}_2) + c \\ &\geq c_1 + 3c_2 + c \\ &\geq 4. \end{aligned}$$

Hence, $c_1 = c_2 = 1$, $c = 0$ and $\deg(V_2) = \deg(\mathfrak{W}_2) = 3$. ∎

Remark 4.1.35. *We in particular showed that the irreducible decomposition of the projective variety $\mathcal{V}(q_1, q_2)$ is given by $\mathcal{V}(Z_0, Z_1) \cup V_2$ and each irreducible component has dimension $k-2$. Hence, $\deg(\mathcal{V}(q_1, q_2)) = \deg(\mathcal{V}(Z_0, Z_1)) + \deg(V_2) = 4$.*

Theorem 4.1.36. *For $n \geq 3$ and $i = 3, \dots, n$, set*

$$B := \{Z_0, Z_1, Z_{n+1}, Z_{n+s} - Z_s Z_{n+2} \mid s = 3, \dots, i\} \subseteq \mathbb{R}[Z_0, Z_1, Z_3, \dots, Z_k]$$

and let $W \subseteq \mathbb{P}^k$ be the projective closure of the affine variety $\boldsymbol{\mathcal{V}}(B) \subseteq \mathbb{C}^k$ under the embedding $\psi \colon \mathbb{C}^k \mapsto \mathbb{P}^k, (z_0, z_1, z_3, \dots, z_k) \mapsto [z_0 : z_1 : 1 : z_3 : \dots : z_k]$, then W is an irreducible, $(k-(i+1))$-dimensional projective variety with $\deg(W) \geq i-1$.

Proof. For the convenience of the reader, this proof is organized in five claims.

<u>Claim 1</u>: $\boldsymbol{\mathcal{V}}(B)$ and $\mathbb{C}^{k-(i+1)}$ are isomorphic as affine varieties.
Proof. For $y := (y_3, \dots, y_n, y_{n+2}, y_{n+i+1}, \dots, y_k) \in \mathbb{C}^{k-(i+1)}$, we set

$$\boldsymbol{y_\varphi} := (0, 0, y_3, \dots, y_n, 0, y_{n+2}, y_3 y_{n+2}, \dots, y_i y_{n+2}, y_{n+i+1}, \dots, z_k) \in \boldsymbol{\mathcal{V}}(B)$$

and consider the well-defined polynomial maps

$$\begin{array}{rccc} \zeta \colon & \boldsymbol{\mathcal{V}}(B) & \to & \mathbb{C}^{k-(i+1)} \\ & (z_0, z_1, z_3, \dots, z_k) & \mapsto & (z_3, \dots, z_n, z_{n+2}, z_{n+i+1}, \dots, z_k), \\ \varphi \colon & \mathbb{C}^{k-(i+1)} & \to & \boldsymbol{\mathcal{V}}(B) \\ & y & \mapsto & \boldsymbol{y_\varphi}. \end{array}$$

We now prove that ζ and φ are inverse to one another in two steps.

(1) For $y := (y_3, \dots, y_n, y_{n+2}, y_{n+i+1}, \dots, y_k) \in \mathbb{C}^{k-(i+1)}$, we compute

$$\zeta(\varphi(y)) = (y_3, \dots, y_n, y_{n+2}, y_{n+i+1}, \dots, y_k) = y.$$

(2) For $(z_0, z_1, z_3, \dots, z_k) \in \boldsymbol{\mathcal{V}}(B)$, we observe $z_0 = z_1 = z_{n+1} = 0$ and $z_{n+s} = z_s z_{n+2}$ for $s = 3, \dots, i$. Therefore, we have

$$\begin{aligned}&\varphi(\zeta(z_0, z_1, z_3, \dots, z_k))\\&\quad= (0, 0, z_3, \dots, z_n, 0, z_{n+2}, z_3 z_{n+2}, \dots, z_i z_{n+2}, z_{n+i+1}, \dots, z_k)\\&\quad= (z_0, z_1, z_3, \dots, z_k).\end{aligned}$$

Hence, $\varphi \circ \zeta = \mathrm{id}_{\boldsymbol{\mathcal{V}}(B)}$. ■

Since $\mathbb{C}^{k-(i+1)}$ is irreducible of dimension $k-(i+1)$ (cf. [Har77, Chapter 1, §1, Example 1.4.1 and Proposition 1.9]), we conclude that also $\boldsymbol{\mathcal{V}}(B)$ is irreducible of dimension $k-(i+1)$ by Theorem A.1.13 and Theorem A.1.16. So, W (the projective closure of the affine variety $\boldsymbol{\mathcal{V}}(B)$) is irreducible of dimension $k-(i+1)$ by Theorem A.1.54 and Theorem A.1.55.

It hence remains to show $\deg(W) \geq i - 1$. To this end, we set

$$B' := \{Z_{n+s} - Z_s Z_{n+2} \mid s = 3, \dots, i\} \subseteq \mathbb{R}[Z_3, \dots, Z_n, Z_{n+2}, \dots, Z_k]$$

and let $W' \subseteq \mathbb{P}^{k-3}$ be the projective closure of the affine variety $\boldsymbol{\mathcal{V}}(B') \subseteq \mathbb{C}^{k-3}$ under the embedding

$$\begin{aligned}\psi' :\ & \mathbb{C}^{k-3} && \to && \mathbb{P}^{k-3}\\ & (z_3, \dots, z_n, z_{n+2}, \dots, z_k) && \mapsto && [1 : z_3 : \ldots : z_n : z_{n+2} : \ldots : z_k].\end{aligned}$$

<u>Claim 2</u>: $\boldsymbol{\mathcal{V}}(B')$ and $\mathbb{C}^{k-(i+1)}$ are isomorphic as affine varieties.
Proof. For $y := (y_3, \dots, y_n, y_{n+2}, y_{n+i+1}, \dots, y_k) \in \mathbb{C}^{k-(i+1)}$, we set

$$\boldsymbol{y}_{\varphi'} := (y_3, \dots, y_n, y_{n+2}, y_3 y_{n+2}, \dots, y_i y_{n+2}, y_{n+i+1}, \dots, y_k) \in \boldsymbol{\mathcal{V}}(B')$$

and consider the well-defined polynomial maps

$$\begin{aligned}\zeta' :\ & \boldsymbol{\mathcal{V}}(B') && \to && \mathbb{C}^{k-(i+1)}\\ & (z_3, \dots, z_n, z_{n+2}, \dots, z_k) && \mapsto && (z_3, \dots, z_n, z_{n+2}, z_{n+i+1}, \dots, z_k),\end{aligned}$$

$$\begin{aligned}\varphi' :\ & \mathbb{C}^{k-(i+1)} && \to && \boldsymbol{\mathcal{V}}(B')\\ & y && \mapsto && \boldsymbol{y}_{\varphi'}.\end{aligned}$$

We now prove that ζ' and φ' are inverse to one another in two steps.

(1) For $y := (y_3, \dots, y_n, y_{n+2}, y_{n+i+1}, \dots, y_k) \in \mathbb{C}^{k-(i+1)}$, we compute

$$\zeta'(\varphi'(y) = (y_3, \dots, y_n, y_{n+2}, y_{n+i+1}, \dots, y_k) = y.$$

(2) For $(z_3, \dots, z_n, z_{n+2}, \dots, z_k) \in \boldsymbol{\mathcal{V}}(B')$, we observe $z_{n+s} = z_s z_{n+2}$ for $s = 3, \dots, i$. Therefore, we have

$$\begin{aligned}
&\varphi'(\zeta'(z_3, \dots, z_n, z_{n+2}, \dots, z_k)) \\
&\quad = (z_3, \dots, z_n, z_{n+2}, z_3 z_{n+2}, \dots, z_i z_{n+2}, z_{n+i+1}, \dots, z_k) \\
&\quad = (z_3, \dots, z_n, z_{n+2}, \dots, z_k).
\end{aligned}$$

Hence, $\varphi' \circ \zeta' = \mathrm{id}_{\boldsymbol{\mathcal{V}}(B')}$. ∎

Since $\mathbb{C}^{k-(i+1)}$ is irreducible of dimension $k-(i+1)$ (cf. [Har77, Chapter 1, §1, Example 1.4.1 and Proposition 1.9]), we conclude that $\boldsymbol{\mathcal{V}}(B')$ is irreducible of dimension $k-(i+1)$ by Theorem A.1.13 and Theorem A.1.16. So, W' (the projective closure of the affine variety $\boldsymbol{\mathcal{V}}(B')$) is irreducible of dimension $k-(i+1)$ by Theorem A.1.54 and Theorem A.1.55.

Claim 3: W' is non-degenerate.
Proof. We set the vector of indeterminants $\boldsymbol{Z}' := (Z_3, \dots, Z_n, Z_{n+2}, \dots, Z_k)$ and let

$$f(Z_2, \boldsymbol{Z}') := \sum_{j=2}^{n} f_j Z_j + \sum_{j=n+2}^{k} f_j Z_j \subseteq \mathbb{C}[Z']$$

be such that $W' \subseteq \mathcal{V}(f)$. It thus suffices to show $f_2 = \ldots = f_n = f_{n+2} = \ldots = f_k = 0$. We do so in three steps.

(1) For $j = n+i+1, \dots, k$, we set $\left(\boldsymbol{z}^{(j)}\right)'(t) := (z_3, \dots, z_n, z_{n+2}, \dots, z_k) \in \boldsymbol{\mathcal{V}}(B')$ to be given by

$$z_l := \begin{cases} t, & \text{if } l = j \\ 0, & \text{else} \end{cases}$$

for $t \in \mathbb{C}$ and observe $\psi'\left(\left(\boldsymbol{z}^{(j)}\right)'(t)\right) \in \psi'(\boldsymbol{\mathcal{V}}(B')) \subseteq W$ for all $t \in \mathbb{C}$. This allows us to conclude

$$0 = f\left(1, \left(\boldsymbol{z}^{(j)}\right)'(t)\right) = f_2 + f_j t$$

for all $t \in \mathbb{C}$. Therefore, we know that $f_2 + f_j T \in \mathbb{C}[T]$ is the zero form and thus $f_2 = f_j = 0$ follows.

(2) For $z_3, \dots, z_n \in \mathbb{C}$, we observe $\boldsymbol{z}' := (z_3, \dots, z_n, 0, \dots, 0) \in \boldsymbol{\mathcal{V}}(B')$ and conclude $\psi'(\boldsymbol{z}') \in \psi'(\boldsymbol{\mathcal{V}}(B')) \subseteq W$. Therefore, we see that

$$0 = f(1, \boldsymbol{z}') = \sum_{j=3}^{n} f_j z_j$$

for any $z_3, \dots, z_n \in \mathbb{C}$ which shows that $\sum_{j=3}^{n} f_j Z_j \in \mathbb{C}[Z_3, \dots, Z_n]$ is the zero form. It thus follows $f_3 = \ldots = f_n = 0$.

(3) For $z_3, \ldots, z_i \in \mathbb{C}$, we observe

$$\boldsymbol{z}' := (z_3, \ldots, z_i, \underbrace{0, \ldots, 0,}_{(n-i)-\text{many}} 1, z_3, \ldots, z_i, 0, \ldots, 0) \in \boldsymbol{\mathcal{V}}(B')$$

and conclude $\psi'(\boldsymbol{z}') \in \psi'(\boldsymbol{\mathcal{V}}(B')) \subseteq W$. Therefore, we see that

$$0 = f(1, \boldsymbol{z}') = f_{n+2} + \sum_{j=n+3}^{n+i} f_j z_{j-n}$$

for any $z_3, \ldots, z_i \in \mathbb{C}$ which shows that $f_{n+2} + \sum_{j=n+3}^{n+i} f_j Z_{j-n} \in \mathbb{C}[Z_3, \ldots, Z_i]$ is the zero form. It thus follows $f_{n+2} = \ldots = f_{n+i} = 0$. ■

Putting it all together, $W' \subseteq \mathbb{P}^{k-3}$ is a non-degenerate irreducible projective variety of dimension $k - (i + 1)$. Therefore, by Proposition 4.1.32, we see that

$$\deg(W') \geq \operatorname{codim}(W') + 1 = (k - 3) - (k - (i + 1)) + 1 = i - 1. \tag{4.2}$$

To finish the proof, we let G' be a Gröbner basis of $\langle B' \rangle \subseteq \mathbb{C}[Z_3, \ldots, Z_n, Z_{n+2}, \ldots, Z_k]$ w.r.t. to the graded monomial order $\leq_{\text{grlex}}$ and consider the set

$$G := \{Z_0, Z_1, Z_{n+1}\} \cup G' \subseteq \mathbb{C}[Z_0, Z_1, Z_3, \ldots, Z_k].$$

<u>Claim 4</u>: G is a Gröbner basis of $\langle B \rangle$ w.r.t. the graded monomial order $\leq_{\text{grlex}}$.
Proof. Since G' is assumed to be a (Gröbner) basis of $\langle B' \rangle$, we know that any $g \in G'$ lies in $\langle B' \rangle \subseteq \langle B \rangle$ by an abuse of notation. Recalling $Z_0, Z_1, Z_{n+1} \in B$ by construction, it thus follows $G = \{Z_0, Z_1, Z_{n+1}\} \cup G' \subseteq \langle B \rangle$. We deduce

$$\langle \operatorname{LT}(g) \mid g \in G \rangle \subseteq \langle \operatorname{LT}(f) \mid f \in B \rangle,$$

where $\operatorname{LT}(\cdot)$ denotes the leading term of the indicated form w.r.t. $\leq_{\text{grlex}}$.

For the reverse inclusion, we let $f \in \langle B \rangle$ be arbitrary but fixed and choose appropriate $f_1 \in \langle Z_0, Z_1, Z_{n+1} \rangle$, $f_2 \in \langle B' \rangle \subseteq \mathbb{C}[Z_3, \ldots, Z_n, Z_{n+2}, \ldots, Z_k]$ such that $f = f_1 + f_2$ by an abuse of notation. Consquently, we see that $\operatorname{LT}(f) = \operatorname{LT}(f_1)$ or $\operatorname{LT}(f) = \operatorname{LT}(f_2)$. If $\operatorname{LT}(f) = \operatorname{LT}(f_1)$, then

$$\operatorname{LT}(f) \in \langle Z_0, Z_1, Z_{n+1} \rangle \subseteq \langle \operatorname{LT}(g) \mid g \in G \rangle$$

follows. Otherwise, if $\operatorname{LT}(f) = \operatorname{LT}(f_2)$, then, using that G' is a Gröbner basis of $\langle B' \rangle$ w.r.t. $\leq_{\text{grlex}}$, we see by an abuse of notation that

$$\operatorname{LT}(f) \in \langle \operatorname{LT}(g) \mid g \in \langle B' \rangle \rangle = \langle \operatorname{LT}(g) \mid g \in G' \rangle \subseteq \langle \operatorname{LT}(g) \mid g \in G \rangle. \quad ■$$

Theorem A.1.58 and Theorem A.1.79 therefore together imply

$$W = \mathcal{V}\left(G^h\right) = \mathcal{V}\left(Z_0, Z_1, Z_{n+1}, (G')^h\right)$$

and we set I' to be the vanishing ideal of $\mathcal{V}\left((G')^h\right) = W'$ in $\mathbb{C}[Z_2, \dots, Z_n, Z_{n+2}, \dots, Z_k]$. By an abuse of notation, we moreover denote the interpretation of I' as a subset of $\mathbb{C}[Z_0, \dots, Z_k]$ again by I'.

Claim 5: $\mathcal{I}(W) = \langle Z_0, Z_1, Z_{n+1}\rangle + I'$.
Proof. ($\subseteq$) For $f \in \mathcal{I}(W)$, we are able to choose some appropriate $f_1 \in \langle Z_0, Z_1, Z_{n+1}\rangle$ and $f_2 \in \mathbb{C}[Z_2, \dots, Z_n, Z_{n+2}, \dots, Z_k]$ such that $f = f_1 + f_2$ by an abuse of notation. It thus suffices to show $f_2 \in I'$. To this end, we observe that there exists some $m \in \mathbb{N}$ such that $f^m \in \langle (G')^h\rangle$ by Hilbert's projective Nullstellensatz (cf. Theorem A.1.32) and $f^m = (f_1 + f_2)^m = g + f_2^m$ for some appropriate $g \in \langle Z_0, Z_1, Z_{n+1}\rangle$ that is a multiple of f_1. It therefore follows

$$f_2^m = f^m - g \in \langle Z_0, Z_1, Z_{n+1}, (G')^h\rangle = \langle G^h\rangle.$$

Moreover, $f_2 \in \mathbb{C}[Z_2, \dots, Z_n, Z_{n+2}, \dots, Z_k]$ implies $f_2^m \in \mathbb{C}[Z_2, \dots, Z_n, Z_{n+2}, \dots, Z_k]$. Hence, by an abuse of notation, we have

$$f_2^m \in \langle G^h\rangle \cap \mathbb{C}[Z_2, \dots, Z_n, Z_{n+2}, \dots, Z_k]$$

which implies $f_2^m \in \langle (G')^h\rangle \subseteq \mathbb{C}[Z_2, \dots, Z_n, Z_{n+2}, \dots, Z_k]$. Again applying Hilbert's projective Nullstellensatz, we thus conclude that f_2 is an element of the vanishing ideal I' of $W' \subseteq \mathbb{P}^{k-3}$ in $\mathbb{C}[Z_2, \dots, Z_n, Z_{n+2}, \dots, Z_k]$.

($\supseteq$) For $f_1 \in \langle Z_0, Z_1, Z_{n+1}\rangle$, $f_2 \in I'$ and $[z] \in W = \mathcal{V}(Z_0, Z_1, Z_{n+1}, (G')^h)$, we observe $z_0 = z_1 = z_{n+1} = 0$, $[z_2 : \dots : z_n : z_{n+2} : \dots : z_k] \in \mathcal{V}\left((G')^h\right)$ and thus compute $(f_1 + f_2)(z) = f_1(z) + f_2(z) = 0 + 0 = 0$. ■

The observation of Claim 5 implies

$$\begin{aligned} \mathbb{C}[W] &= \mathbb{C}[Z_0, \dots, Z_k]/\mathcal{I}(W) \\ &= \mathbb{C}[Z_0, \dots, Z_k]/(\langle Z_0, Z_1, Z_{n+1}\rangle + I') \\ &\simeq \mathbb{C}[Z_2, \dots, Z_n, Z_{n+2}, \dots, Z_k]/I' = \mathbb{C}[W']. \end{aligned}$$

Hence, for any $t \in \mathbb{N}_0$, we observe for the corresponding Hilbert functions that

$$h_W(t) = \dim(\mathbb{C}[W]_t) = \dim(\mathbb{C}[W']_t) = h_{W'}(t).$$

This shows that the Hilbert polynomials of W and W' coincide. Since W and W' are both irreducible, Lemma 4.1.26 thus yields $\deg(W) = \deg(W') \geq i - 1$. ■

Theorem 4.1.37. *For* $i = 0, \ldots, n$, *it holds* $\deg(V_i) = i + 1$. *Moreover, if* $n = 2$, *then also* $\deg(V_{n+1}) = n + 2$.

Proof. Lemma 4.1.29, Corollary 4.1.31 and Proposition 4.1.34 yield $\deg(V_0) = 1$, $\deg(V_1) = 2$ and $\deg(V_2) = 3$, respectively. If $n \geq 3$, it thus remains to show $\deg(V_i) = i+1$ for $i = 3, \ldots, n$. Moreover, if $n = 2$, then we have to show $\deg(V_3) = 4$.

(1) If $n \geq 3$, then we verify $\deg(V_i) = i + 1$ by an induction on i. The base case $i = 2$ is solved by the above consideration since $\deg(V_2) = 2+1$. Therefore, we now assume for the inductive step that the assertion was already verified up to some $2 \leq i < n$ and investigate the situation for $i + 1$ using the same strategy as applied in the proof of Proposition 4.1.34. To this end, we set $W := V_i \cap \mathcal{V}(q_{i+1})$, $\mathfrak{W}_1 := \mathcal{V}(Z_0, \ldots, Z_i)$, $\mathfrak{W}_2 := V_{i+1} \subseteq \mathbb{P}^k$,

$$B^{(2)} := \{Z_0, Z_1, Z_{n+1}, Z_{n+s} - Z_s Z_{n+2} \mid s = 3, \ldots, i\} \subseteq \mathbb{R}[Z_0, Z_1, Z_3, \ldots, Z_k]$$

and let $\mathfrak{W}_3 \subseteq \mathbb{P}^k$ be the projective closure of $\boldsymbol{\mathcal{V}}\left(B^{[2]}\right) \subseteq \mathbb{C}^k$ under the embedding

$$\begin{array}{rccc} \phi^{(2)}: & \mathbb{C}^k & \rightarrow & \mathbb{P}^k \\ & (z_0, z_1, z_3, \ldots, z_k) & \mapsto & [z_0 : z_1 : 1 : z_3 : \ldots : z_n]. \end{array}$$

Moreover, Corollary 3.3.24 yields that $V_i = \mathcal{V}(F)$ for $F := \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$,

$$\begin{aligned} \mathfrak{F}_1 &:= \{Z_1 Z_s - Z_0 Z_{n+s} \mid 1 \leq s \leq i\}, \\ \mathfrak{F}_2 &:= \{Z_s Z_{n+t} - Z_t Z_{n+s} \mid 1 \leq s < t \leq i\}, \\ \mathfrak{F}_3 &:= \{Z_{n+1} Z_s Z_t - Z_0 Z_{n+s} Z_{n+t} \mid 2 \leq s \leq t \leq i\}. \end{aligned}$$

and $q_{i+1}(Z) = Z_0 Z_{n+i+1} - Z_1 Z_{i+1}$ by Lemma 3.3.5. Hence,

$$W = \mathcal{V}(F, Z_0 Z_{n+i+1} - Z_1 Z_{i+1})$$

and we conclude $\mathfrak{W}_1 \subseteq W$. Furthermore, we see $\phi^{(2)}(B^{(2)}) \subseteq W$ from which it follows that $\mathfrak{W}_3 \subseteq W$. Lemma 3.2.5 (iii) moreover implies $V_{i+1} \subseteq V_i$ and

$$V_{i+1} = W_{i+1} = \mathcal{V}\left(\langle p_1, \ldots, p_{i+1}\rangle^h\right) \subseteq \mathcal{V}\left(p_1^h, \ldots, p_{i+1}^h\right) = \mathcal{V}(q_1, \ldots, q_{i+1}) \subseteq \mathcal{V}(q_{i+1})$$

by Theorem 3.3.17, Theorem A.1.58 and Remark 3.3.2. We thus conclude

$$\mathfrak{W}_2 = V_{i+1} \subseteq V_i \cap \mathcal{V}(q_{i+1}) = W.$$

Claim: $\mathfrak{W}_1$, $\mathfrak{W}_2$ and $\mathfrak{W}_3$ are three pairwise distinct irreducible components of W.
Proof. The projective variety V_i is irreducible with $\dim(V_i) = k-i$ by Proposition 4.1.8 and Proposition 4.1.20 and $V_i \not\subseteq \mathcal{V}(q_{i+1})$. Hence, Proposition A.1.43 yields

$$\dim(W) = \dim(V_i \cap \mathcal{V}(q_{i+1})) = \dim(V_i) - 1 = k - (i + 1).$$

Moreover, the dimension of each of the irreducible subvarieties $\mathfrak{W}_1$, $\mathfrak{W}_2$, $\mathfrak{W}_3$ of W is $k-(i+1)$ by Lemma 4.1.16, Proposition 4.1.8, Proposition 4.1.20 and Theorem 4.1.36, respectively. Thus, $\dim(\mathfrak{W}_1) = \dim(\mathfrak{W}_2) = \dim(\mathfrak{W}_3) = \dim(W)$ and we conclude that $\mathfrak{W}_1$, $\mathfrak{W}_2$ and $\mathfrak{W}_3$ are irreducible components of W by Theorem A.1.42.

It therefore remains to show that $\mathfrak{W}_1$, $\mathfrak{W}_2$ and $\mathfrak{W}_3$ are pairwise distinct. To see this, we observe $\mathfrak{W}_1, \mathfrak{W}_3 \subseteq \mathcal{V}(Z_0)$ and $\mathfrak{W}_2 \not\subseteq \mathcal{V}(Z_0)$. Thus, $\mathfrak{W}_1 \neq \mathfrak{W}_2$ and $\mathfrak{W}_2 \neq \mathfrak{W}_3$. Likewise, $\mathfrak{W}_1 \subseteq \mathcal{V}(Z_2)$ and $\mathfrak{W}_3 \not\subseteq \mathcal{V}(Z_2)$ implies $\mathfrak{W}_1 \neq \mathfrak{W}_3$. ■

The inductive assumption implies $\deg(V_i) = i+1$ and Lemma 4.1.30 furthermore yields $\deg(\mathcal{V}(q_{i+1})) = 2$. Moreover, $\deg(\mathfrak{W}_1) = 1$ by Lemma 4.1.28,

$$\deg(\mathfrak{W}_2) = \deg(V_{i+1}) \geq \operatorname{codim}(V_{i+1}) + 1 = i+2$$

by Proposition 4.1.20, Proposition 4.1.32 and $\deg(\mathfrak{W}_3) \geq i-1$ by Theorem 4.1.36. Applying Bézout's theorem (cf. Theorem A.1.49), we thus observe for some appropriate $c_1, c_2, c_3 \in \mathbb{N}$, $c \in \mathbb{N}_0$ that

$$\begin{aligned} 2(i+1) &= \deg(V_i) \cdot \deg(\mathcal{V}(q_{i+1})) \\ &= c_1 \cdot \deg(\mathfrak{W}_1) + c_2 \cdot \deg(\mathfrak{W}_2) + c_3 \cdot \deg(\mathfrak{W}_3) + c \\ &\geq c_1 + c_2(i+2) + c_3(i-1) + c \\ &\geq 1 + (i+2) + (i-1) \\ &= 2(i+1). \end{aligned}$$

Hence, $c_1 = c_2 = c_3 = 1$, $c = 0$, $\deg(V_{i+1}) = \deg(\mathfrak{W}_2) = i+2$ and the irreducible decomposition of W is given by $\mathfrak{W}_1 \cup \mathfrak{W}_2 \cup \mathfrak{W}_3$.

(2) If $n = 2$, then we set

$$G := \left\{ Z_3 - Z_1^2, Z_4 - Z_1Z_2, Z_5 - Z_2^2, Z_1Z_4 - Z_2Z_3, Z_1Z_5 - Z_2Z_4, Z_3Z_5 - Z_4^2 \right\},$$

$\mathfrak{V} := \mathcal{V}\left(G^h\right) \subseteq \mathbb{P}^{k(2,d)}$ and think of the Veronese surface. A straight forward verification using Buchberger's criterion (cf. Theorem A.1.70) shows that G is a Gröbner basis of $\langle p_1, p_2, p_3 \rangle \subseteq \mathbb{C}[\boldsymbol{X}]$ w.r.t. the graded monomial order $\leq_{\text{grlex}}$. Theorem 3.3.17, Theorem A.1.58 and Theorem A.1.79 thus together imply

$$V_3 = W_3 = \mathcal{V}\left(\langle p_1, p_2, p_3 \rangle^h\right) = \mathcal{V}\left(\langle G \rangle^h\right) = \mathcal{V}\left(\langle G^h \rangle\right) = \mathcal{V}\left(G^h\right) = \mathfrak{V}. \tag{4.3}$$

Therefore, we know that $\mathfrak{V}$ is a non-degenerate irreducible projective variety with $\operatorname{codim}(\mathfrak{V}) = 3$ by Corollary 4.1.4, Proposition 4.1.8 and Proposition 4.1.20, respectively. Proposition 4.1.32 hence gives

$$\deg(\mathfrak{V}) \geq 1 + \operatorname{codim}(\mathfrak{V}) = 4. \tag{4.4}$$

Claim 1: $\deg(\mathfrak{V}) \leq 5$.

Proof. We define $W := V_2 \cap \mathcal{V}(q_3)$, $\mathfrak{W}_1 := \mathcal{V}(Z_0, Z_1, Z_2)$, $\mathfrak{W}_2 := V_3 \underset{(4.3)}{=} \mathfrak{V}$ and deduce from a straight forward calculation based on Buchberger's criterion that $\mathfrak{G} := \{Z_3 - Z_1^2, Z_4 - Z_1Z_2, Z_1Z_4 - Z_2Z_3, Z_3Z_2^2 - Z_4^2\}$ is a Gröbner basis of the ideal $\langle p_1, p_2\rangle \subseteq \mathbb{C}[\boldsymbol{X}]$ w.r.t. the graded monomial order $\leq_{\text{grlex}}$. Hence,

$$\begin{aligned} V_2 &= W_3 = \mathcal{V}\left(\langle p_1, p_2\rangle^h\right) = \mathcal{V}\left(\langle G\rangle^h\right) = \mathcal{V}\left(\langle G^h\rangle\right) \\ &= \mathcal{V}\left(Z_0Z_3 - Z_1^2, Z_0Z_4 - Z_1Z_2, Z_1Z_4 - Z_2Z_3, Z_3Z_2^2 - Z_4^2 Z_0\right) \end{aligned}$$

by Theorem 3.3.17, Theorem A.1.58, Theorem A.1.79 and $q_3(Z) = Z_0Z_5 - Z_2^2$ by Construction 3.3.1. It thus follows $\mathfrak{W}_1 \subseteq V_2 \cap \mathcal{V}(q_3) = W$. Moreover, we see that $\mathfrak{W}_2 = V_3 \subseteq V_2$ by Lemma 3.2.5 (iii) and

$$\mathfrak{W}_2 = V_3 = W_3 = \mathcal{V}\left(\langle p_1, p_2\rangle^h\right) \subseteq \mathcal{V}\left(p_1^h, p_2^h, p_3^h\right) \subseteq \mathcal{V}(q_3)$$

by Theorem 3.3.17 and Theorem A.1.58. Consequently, $\mathfrak{W}_2 \subseteq \mathfrak{W}$.

Subclaim: $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are two distinct irreducible components of $\mathfrak{W}$.

Proof. Proposition 4.1.8 and Proposition 4.1.20 imply that the projective variety V_2 is irreducible with dimension $k-2$ and $V_2 \not\subseteq \mathcal{V}(q_3)$. Hence, Proposition A.1.43 yields

$$\dim(W) = \dim(V_2 \cap \mathcal{V}(q_3)) = \dim(V_2) - 1 = k - 3.$$

Moreover, the irreducible subvariety $\mathfrak{W}_1$ of W has dimension $k-3$ by Lemma 4.1.16 and $\mathfrak{W}_2 = V_3 \subseteq W$ is irreducible with dimension $k-3$ by Proposition 4.1.8 and Proposition 4.1.20. Thus, $\dim(\mathfrak{W}_1) = \dim(\mathfrak{W}_2) = \dim(W)$ and we conclude that $\mathfrak{W}_1$, $\mathfrak{W}_2$ are irreducible components of W by Theorem A.1.42. We furthermore observe $\mathfrak{W}_1 \subseteq \mathcal{V}(Z_0)$, $\mathfrak{W}_2 \not\subseteq \mathcal{V}(Z_0)$ and, therefore, know $\mathfrak{W}_1 \neq \mathfrak{W}_2$. ■

Proposition 4.1.34 yields $\deg(V_2) = 3$ and Lemma 4.1.30 gives $\deg(\mathcal{V}(q_3)) = 2$. Moreover, we have $\deg(\mathfrak{W}_1) = 1$ by Lemma 4.1.28. An application of Bézout's theorem (cf. Theorem A.1.49) thus allows us to conclude

$$\begin{aligned} 6 &= \deg(V_2) \cdot \deg(\mathcal{V}(q_3)) \\ &\geq \deg(\mathfrak{W}_1) + \deg(\mathfrak{W}_2) \\ &\geq 1 + \deg(\mathfrak{V}), \end{aligned}$$

which shows $\deg(\mathfrak{V}) \leq 5$ as claimed. ■

Claim 2: The degree of $\mathfrak{V}$ is divisible by 4.

Proof. We compute $k(2,2) = 5$, set $\mathfrak{U} := \mathfrak{V} \cap \mathcal{V}(Z_6, \ldots, Z_k)$ and let $I \subseteq \mathbb{C}[Z_0, \ldots, Z_5]$ be the vanishing ideal of the projective variety $\mathfrak{V}' \subseteq \mathbb{P}^5$ that is given by

$$\mathcal{V}\left(Z_0Z_3 - Z_1^2, Z_0Z_4 - Z_1Z_2, Z_0Z_5 - Z_2^2, Z_1Z_4 - Z_2Z_3, Z_1Z_5 - Z_2Z_4, Z_3Z_5 - Z_4^2\right).$$

By an abuse of notation, we moreover denote the interpretation of $I \subseteq \mathbb{C}[Z_0, \dots, Z_5]$ as a subset of $\mathbb{C}[Z_0, \dots, Z_k]$ again by I.

Subclaim 2.1 $\mathfrak{V}' = V(\mathbb{P}^2)$.
Proof. ($\subseteq$) We let $[z] \in \mathfrak{V}'$ be arbitrary but fixed. If $z_0 = 1$, then

$$[z] = \left[1 : z_1 : z_2 : z_1^2 : z_1 z_2 : z_2^2\right] = V([1 : z_1 : z_2]).$$

Otherwise, if $z_0 = 0$ and $z_3 = 1$, then $[z] = [0 : 0 : 0 : 1 : z_4 : z_4^2] = V_2([0 : 1 : z_4])$. Lastly, if $z_1 = z_3 = 0$, then $[z] = [0 : 0 : 0 : 0 : 0 : z_5] = V_2([0 : 0 : \sqrt{z_5}])$.

($\supseteq$) We observe $Z_0Z_3 - Z_1^2$, $Z_0Z_4 - Z_1Z_2$, $Z_0Z_5 - Z_2^2$, $Z_1Z_4 - Z_2Z_3$, $Z_1Z_5 - Z_2Z_4$, $Z_3Z_5 - Z_4^2 \in \mathcal{S}_V$. Theorem 2.3.29 thus implies $V(\mathbb{P}^2) = \mathcal{V}(\mathcal{S}_V) \subseteq \mathfrak{V}'$. ■

Subclaim 2.2: $\mathcal{I}(\mathfrak{U}) = I + \langle Z_6, \dots, Z_k\rangle$.
Proof. ($\subseteq$) For $f \in \mathcal{I}(\mathfrak{U})$, we are able to choose some appropriate $f_1 \in \mathbb{C}[Z_0, \dots, Z_5]$ and $f_2 \in \langle Z_6, \dots, Z_k\rangle$ such that $f = f_1 + f_2$. It thus suffices to show $f_1 \in I$. To this end, we observe that there exists some $m \in \mathbb{N}$ such that f^m lies in

$$\langle Z_0Z_3 - Z_1^2, Z_0Z_4 - Z_1Z_2, Z_0Z_5 - Z_2^2, Z_1Z_4 - Z_2Z_3, Z_1Z_5 - Z_2Z_4, Z_3Z_5 - Z_4^2, Z_6, \dots, Z_k\rangle$$

by Hilbert's projective Nullstellensatz (cf.Theorem A.1.32) and

$$f^m = (f_1 + f_2)^m = f_0^m + g$$

for some appropriate $g \in \langle Z_6, \dots, Z_k\rangle$ that is a multiple of f_2. It therefore follows that $f_1^m = f^m - g$ lies in

$$\langle Z_0Z_3 - Z_1^2, Z_0Z_4 - Z_1Z_2, Z_0Z_5 - Z_2^2, Z_1Z_4 - Z_2Z_3, Z_1Z_5 - Z_2Z_4, Z_3Z_5 - Z_4^2, Z_6, \dots, Z_k\rangle.$$

Moreover, $f_1 \in \mathbb{C}[Z_0, \dots, Z_5]$ implies $f_1^m \in \mathbb{C}[Z_0, \dots, Z_5]$. Hence, by an abuse of notation, $f_1^m \in I \subseteq \mathbb{C}[Z_0, \dots, Z_5]$ follows. Again applying Hilbert's projective Nullstellensatz, we thus conclude that f_1 lies in the vanishing ideal of $\mathfrak{V}' \subseteq \mathbb{P}^5$.

($\supseteq$) For $f_1 \in I$, $f_2 \in \langle Z_6, \dots, Z_k\rangle$ and $[z] \in \mathfrak{U}$, we observe $z_6 = \dots = z_k = 0$, $[z_0 : \dots : z_5] \in \mathfrak{V}'$ and thus compute $(f_1 + f_2)(z) = f_1(z) + f_2(z) = 0 + 0 = 0$. ■

Our observations from Subclaim 2.1 allow us to deduce

$$\begin{aligned}
\mathbb{C}[\mathfrak{U}] &= \mathbb{C}[Z_0, \dots, Z_k]/\mathcal{I}(\mathfrak{U}) \\
&= \mathbb{C}[Z_0, \dots, Z_k]/(I + \langle Z_6, \dots, Z_k\rangle) \\
&\simeq \mathbb{C}[Z_0, \dots, Z_5]/I \\
&= \mathbb{C}\left[\mathfrak{V}'\right] \\
&= \mathbb{C}\left[V\left(\mathbb{P}^2\right)\right].
\end{aligned}$$

Hence, for any $t \in \mathbb{N}_0$, we observe for the corresponding Hilbert functions that

$$h_{\mathfrak{U}}(t) = \dim(\mathbb{C}[\mathfrak{U}]_t) = \dim(\mathbb{C}[V(\mathbb{P}^2)]_t) = h_{V(\mathbb{P}^2)}(t).$$

Therefore, the Hilbert polynomials of $\mathfrak{U}$, $V(\mathbb{P}^2)$ coincide and we deduce that $\mathfrak{U} \subseteq \mathbb{P}^k$ is irreducible from Corollary A.1.36 since $V(\mathbb{P}^2) \subseteq \mathbb{P}^5$ is irreducible by Proposition 4.1.8. Recalling $\deg(V(\mathbb{P}^2)) = 4$ by Lemma 4.1.27, we conclude

$$\deg(\mathfrak{U}) = \deg(V(\mathbb{P}^2)) = 4$$

from Lemma 4.1.26. Moreover, we know that $\mathfrak{V}$ is irreducible of dimension $k-3$ and $\mathcal{V}(Z_6, \ldots, Z_k)$ is irreducible of dimension 5 by Lemma 4.1.16. Hence, it follows

$$\dim(\mathfrak{V}) + \dim(\mathcal{V}(Z_6, \ldots, Z_k)) - k = (k-3) + 5 - k = 2 = \dim(\mathfrak{U}).$$

Recalling that the projective variety $\mathfrak{U} = \mathfrak{V} \cap \mathcal{V}(Z_6, \ldots, Z_k)$ is irreducible of degree 4 and that the degree of $\mathcal{V}(Z_6, \ldots, Z_k)$ is 1 by Lemma 4.1.28, we thus conclude

$$\deg(\mathfrak{V}) = \deg(\mathfrak{V}) \cdot \deg(\mathcal{V}(Z_6, \ldots, Z_k)) = \deg(\mathfrak{U}) \cdot c = 4c$$

for some $c \in \mathbb{N}$ by Bézout's theorem (cf. Theorem A.1.49). ■

Altogether, the degree of $\mathfrak{V}$ therefore is an integer between 4 and 5 (cf. (4.4) and Claim 1) that is divisible by 4 (cf. Claim 2). Hence,

$$\deg(V_3) \underset{(4.3)}{=} \deg(\mathfrak{V}) = 4. \quad ■$$

Corollary 4.1.38. *For $i = 0, \ldots, n$, V_i is a projective variety of minimal degree. Moreover, if $n = 2$, then also V_{n+1} is a projective variety of minimal degree.*

Proof. The projective variety V_i is non-degenerate and irreducible of codimension i by Corollary 4.1.4, Proposition 4.1.8 and Proposition 4.1.20, respectively. Using Theorem 4.1.37, we thus see $\deg(V_{i+1}) = i + 1 = \operatorname{codim}(V_i) + 1$. Therefore, V_i is a projective variety of minimal degree and the same is true for V_{n+1} if $n = 2$. ■

4.2 Hilbert's 1888 Theorem in the Light of Varieties of Minimal Degree

In 2016, Blekherman, Smith and Velsco [BSV16] presented a generalization of Hilbert's 1888 theorem by providing a full answer to Question 3 when $\mathfrak{W}_0 := \mathfrak{W}_1 := W$ is a non-degenerate irreducible totally-real projective variety and $\mathfrak{W}_2 := \mathbb{P}^k$.

Theorem 4.2.1. *For $l \in \mathbb{N}$, let $W \subseteq \mathbb{P}^l$ be a non-degenerate irreducible totally-real projective variety, then the following are equivalent:*

(i) For every $q \in \mathcal{F}_{l+1,2}$ that is locally PSD on $W(\mathbb{R})$, there exist $h_1, \ldots, h_s \in \mathcal{F}_{l+1,1}$ $(s \in \mathbb{N})$ such that $q - \sum_{i=0}^{s} h_i^2 \in \mathcal{I}(W)$.

(ii) W is a projective variety of minimal degree.

Proof. See [BSV16, Theorem 1.1]. ■

The beauty of this result lies in the fact that projective varieties of minimal degree are well understood. In 1886, Del Pezzo [Pez86] classified the surfaces of minimal degree. Approximately 20 years later, Bertini [Ber07] extended Del Pezzo's consideration to varieties of any dimension. Their results were collected by Eisenbud and Harris in [EH85, Theorem 1]. We here include their united catalogue of projective varieties of minimal degree for the convenience of the reader. That is, there are three types of projective varieties of minimal degree:

- quadratic hypersurfaces
- rational normal scrolls
- cones over the Veronese surface $V(\mathbb{P}^2)$

In Corollary 4.1.38, we observed that $V_0, \ldots, V_n$ and also V_{n+1} if $n = 2$ are projective varieties of minimal degree in non-Hilbert cases. Therefore, they fit in the above catalogue as follows:

- The projective variety $V_1 = \mathcal{V}(q_1)$, which is defined by the irreducible quadratic form q_1, is a quadratic hypersurface.
- The projective variety $V_0 = \mathbb{P}^k$ and also the projective varieties $V_2, \ldots, V_n$ are of minimal degree, but they are neither quadratic hypersurfaces nor cones over the Veronese surface. Therefore, they are rational normal scrolls.
- If $n = 2$, then Construction 3.2.7 yields that $V_{n+1} = V_3$ is a cone over the Veronese surface.

Keeping this classification in mind, let us now illuminate how Blekherman–Smith–Velasco's generalization of Hilbert's 1888 theorem can be applied in our setting.

Theorem 4.2.2. *Let $W \subseteq \mathbb{P}^k$ be a non-degenerate irreducible totally-real projective variety such that $V(\mathbb{P}^n)(\mathbb{R}) \subseteq W(\mathbb{R})$. If W is a projective variety of minimal degree, then $\Sigma_{n+1,2d} = C_W$.*

Proof. $(\subseteq)$ We deduce $\overline{V(\mathbb{P}^n)(\mathbb{R})} \subseteq \overline{W(\mathbb{R})}$ w.r.t. the Zariski topology from the assumption $V(\mathbb{P}^n)(\mathbb{R}) \subseteq \mathfrak{W}(\mathbb{R})$. Proposition 4.1.23 furthermore yields that $V(\mathbb{P}^n) = V_{k-n}$ is totally-real and the same is true for W by choice. Hence, we conclude

$$V(\mathbb{P}^n) = \overline{V(\mathbb{P}^n)(\mathbb{R})} \subseteq \overline{W(\mathbb{R})} = W \tag{4.5}$$

and thus $\Sigma_{n+1,2d} \subseteq C_W$ follows by Corollary 3.1.4.

($\supseteq$) For $f \in \mathcal{F}_{n+1,2d}$, we let $A \in \mathcal{G}^{-1}(f)$ be such that q_A is locally PSD on $W(\mathbb{R})$. Since W is assumed to be a projective variety of minimal degree, Theorem 4.2.1 yields that there exist some $h_1, \ldots, h_s \in \mathcal{F}_{k+1,1}$ ($s \in \mathbb{N}$) such that $q_A - \sum_{i=1}^{s} h_i^2 \in \mathcal{I}(W)$. This allows us to conclude

$$q_A - \sum_{i=1}^{s} h_i^2 \in \mathcal{I}(V(\mathbb{P}^n))$$

since $V(\mathbb{P}^n)$ is a subvariety of W by (4.5). We define $q := \sum_{i=1}^{s} h_i^2 \in \Sigma_{k+1,2}$, fix some $B \in Q^{-1}(q)$ and conclude $q_{A-B} = q_A - q_B = q_A - q \in \mathcal{I}(V(\mathbb{P}^n))$. Thus, $B \in \mathcal{G}^{-1}(f)$ follows from (2.13). Since q_B is PSD as a sum of squares, we hence know $f \in C_{\mathbb{P}^k} = \Sigma_{n+1,2d}$ by Lemma 3.1.2 (iii). ∎

Theorem 4.2.3. *Let $W \subseteq \mathbb{P}^k$ be a non-degenerate irreducible totally-real projective variety such that $W(\mathbb{R}) \subseteq V(\mathbb{P}^n)(\mathbb{R})$. If $\Sigma_{n+1,2d} = C_W$, then W is a projective variety of minimal degree.*

Proof. Lemma 3.1.3 implies $\Sigma_{n+1,2d} \subseteq C_W$ since $W \subseteq \mathbb{P}^k$ and $C_{\mathbb{P}^k} = \Sigma_{n+1,2d}$ by Lemma 3.1.2 (iii). For a proof by contraposition, we moreover assume that W is not a projective variety of minimal degree and show that the inclusion $\Sigma_{n+1,2d} \subsetneq C_W$ is strict. To this end, it suffices to find some $f \in C_W \backslash C_{\mathbb{P}^k}$.

Since $W \subseteq \mathbb{P}^k$ is assumed to be a non-degenerate irreducible totally-real projective variety that is not of minimal degree, we know that there exists some $q \in \mathcal{F}_{k+1,2}$ which is locally PSD on $W(\mathbb{R})$ such that $q - \sum_{i=1}^{s} h_i^2 \notin \mathcal{I}(W)$ for any $s \in \mathbb{N}$ and $h_1, \ldots, h_s \in \mathcal{F}_{k+1,1}$ by Theorem 4.2.1. We fix such $q \in \mathcal{F}_{k+1,2}$ and choose some $A \in Q^{-1}(q)$ for which we set $f := \mathcal{G}(A) = f_A \in \mathcal{F}_{n+1,2d}$. It follows $f \in C_W$.

We moreover assume $f \in C_{\mathbb{P}^k}$ for a contradiction. This allows us to fix some $B \in \mathcal{G}^{-1}(f)$ such that q_B is PSD. Therefore, we know that $q_B = \sum_{i=0}^{s} h_i^2$ for some $h_1, \ldots, h_s \in \mathcal{F}_{n+1,1}$ ($s \in \mathbb{N}$) since $\mathcal{P}_{k+1,2} = \Sigma_{k+1,2}$ by Hilbert's 1888 theorem. Since A and B are both Gram matrices associated to f, we conclude

$$q - \sum_{i=0}^{s} h_i^2 = q_A - q_B = q_{A-B} \in \mathcal{I}(V(\mathbb{P}^n)) \tag{4.6}$$

from (2.13). Furthermore, we deduce that $\overline{W(\mathbb{R})} \subseteq \overline{V(\mathbb{P}^n)(\mathbb{R})}$ w.r.t. the Zariski topology from the assumption $W(\mathbb{R}) \subseteq V(\mathbb{P}^n)(\mathbb{R})$. Since $V(\mathbb{P}^n) = V_{k-n}$ and W are both totally-real, we hence see

$$W = \overline{W(\mathbb{R})} \subseteq \overline{V(\mathbb{P}^n)(\mathbb{R})} = V(\mathbb{P}^n)$$

which implies $\mathcal{I}(W) \subseteq \mathcal{I}(V(\mathbb{P}^n))$. Therefore, $q - \sum_{i=1}^{s} h_i^2 \in \mathcal{I}(W)$ follows from (4.6) which is a contradiction to the choice of q. ■

Remark 4.2.4. *In the above proof, we argued $\Sigma_{k+1,2} = \mathcal{P}_{k+1,2}$ by Hilbert's 1888 theorem. However, as we already explained in Section 2.2, $\Sigma_{k+1,2} = \mathcal{P}_{k+1,2}$ also follows as a consequence to the spectral theorem for Hermitian matrices.*

We are now in possession of all the tools to recover Hilbert's 1888 theorem from Blekherman–Smith–Velasco's result on varieties of minimal degree as it was done in [BSV16, Example 5.5]. We illustrate their argument in our framework below.

Theorem 4.2.5. *$\Sigma_{n+1,2d} = \mathcal{P}_{n+1,2d}$ if and only if $V(\mathbb{P}^n)$ is a projective variety of minimal degree.*

Proof. $V(\mathbb{P}^n) = V_{k-n}$ is a non-degenerate irreducible totally-real projective variety by Corollary 4.1.4, Proposition 4.1.8 and Proposition 4.1.23 and $C_{V(\mathbb{P}^n)} = \mathcal{P}_{n+1,2d}$ by Lemma 3.1.2 (ii). If $\Sigma_{n+1,2d} = \mathcal{P}_{n+1,2d} = C_{V(\mathbb{P}^n)}$, then Theorem 4.2.3 yields that $V(\mathbb{P}^n)$ has minimal degree by setting $W := V(\mathbb{P}^n)$. Vice versa, if $V(\mathbb{P}^n)$ is a projective variety of minimal degree, then Theorem 4.2.2 implies $\Sigma_{n+1,2d} = C_{V(\mathbb{P}^n)} = \mathcal{P}_{n+1,2d}$ by setting $W := V(\mathbb{P}^n)$. ■

Lemma 4.2.6. *$V(\mathbb{P}^n)$ is a projective variety of minimal degee if and only if $n = 1$ or $d = 1$ or $(n+1, 2d) = (3, 4)$.*

Proof. Proposition 4.1.20 yields $\operatorname{codim}(V(\mathbb{P}^n)) = \operatorname{codim}(V_{k-n}) = k-n$ and, moreover, Lemma 4.1.27 states $\deg(V(\mathbb{P}^n)) = d^n$. Therefore, we see that $V(\mathbb{P}^n)$ is a projective variety of minimal degree if and only if

$$d^n = k(n,d) - n + 1 := \binom{n+d}{n} - n.$$

The latter is true if and only if $n = 1$ or $d = 1$ or $(n+1, 2d) = (3, 4)$. ■

Hilbert's 1888 theorem thus follows as a corollary to Theorem 4.2.5 and Lemma 4.2.6. In this spirit, we conclude this section by showing that the cones $C_0, \ldots, C_n$ and also C_{n+1} if $n \leq 2$ collapse to $\Sigma_{n+1,2d}$ using Theorem 4.2.2.

Theorem 4.2.7. *For $i = 0, \ldots, n-1$, it holds $C_i = C_{i+1}$. Moreover, if $n \leq 2$, then also $C_n = C_{n+1}$.*

Proof. If $(n+1, 2d)$ is a Hilbert case, then $\Sigma_{n+1,2d} = C_0 = \ldots = C_{k-n} = \mathcal{P}_{n+1,2d}$ by Hilbert's 1888 theorem. However, if $(n+1, 2d)$ denotes a non-Hilbert case, then

V_i is a non-degenerate irreducible totally-real projective variety of minimal degree by Corollary 4.1.4, Proposition 4.1.8, Proposition 4.1.23 and Corollary 4.1.38. Thus, Theorem 4.2.2 yields $\Sigma_{n+1,2d} = C_{V_i}$ by setting $W := V_i$ and the same is true for V_{n+1} if $n = 2$. Altogether, $\Sigma_{n+1,2d} = C_0 = \ldots = C_n$ and also $C_n = C_{n+1}$ if $n \leq 2$. ■

The main query reduces to an investigation of the inclusions in $C_{n+1} \subseteq \ldots \subseteq C_{k-n}$ and also the inclusion $C_n \subseteq C_{n+1}$ if $n \geq 3$.

Chapter 5

Quartics and Sextics

In this chapter, we answer the main query of this thesis for the non-Hilbert cases of $(n+1)$-ary quartics ($n \geq 3$) and $(n+1)$-ary sextics ($n \geq 2$).

In Section 5.1, we determine specific complete sets of separating forms for the cone filtration ($\mathcal{CF}$) in the basic non-Hilbert cases. This allows us to identify each strict inclusion in the distinguished cone filtration and to compute the exact number of strictly separating intermediate cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ for quaternary quartics and ternary sextics, respectively.

In Section 5.2, we firstly generalize our consideration of the first section from quaternary quartics to $(n+1)$-ary quartics ($n \geq 4$). This allows us to answer the main query for the non-Hilbert cases of quartics. Secondly, we introduce a first degree-jumping principle that allows us to extend our consideration from quartics to sextics. We therefore obtain a partial answer to the main query for the non-Hilbert cases of sextics. To give a complete answer, we thirdly extend our consideration of the first section from ternary sextics to $(n+1)$-ary sextics ($n \geq 3$). Altogether, we thus also answer the main query for the non-Hilbert cases of sextics.

5.1 Quaternary Quartics and Ternary Sextics

Observation 5.1.1. *Quaternary Quartics*
Let $n = 3$ and $d = 2$. We compute $k = 9$, $k - n = 6$ and observe that the specific cone filtration ($\mathcal{CF}$) for quaternary quartics is given by

$$\Sigma_{4,4} = C_0 \subseteq C_1 \subseteq C_2 \subseteq C_3 \subseteq C_4 \subseteq C_5 \subseteq C_6 = \mathcal{P}_{4,4}. \tag{5.1}$$

Theorem 4.2.7 yields $C_0 = C_1 = C_2 = C_3$ and, therefore, it remains to determine each strict inclusion in $C_3 \subseteq C_4 \subseteq C_5 \subseteq C_6$.

Theorem 5.1.2. *If $(n+1, 2d) = (4, 4)$, then $C_3 \subsetneq C_4 \subsetneq C_5 \subsetneq C_6$.*

Proof. Let us consider the Choi–Lam quaternary quartic

$$\mathsf{C}(X_0, X_1, X_2, X_3) := X_0^2X_1^2 + X_0^2X_2^2 + X_1^2X_2^2 + X_3^4 - 4X_0X_1X_2X_3$$

and its permuted forms

$$\begin{aligned}\mathsf{C}^{\sigma}(X_0,X_1,X_2,X_3) &:= \mathsf{C}(X_0,X_3,X_1,X_2),\\ \mathsf{C}^{\tau}(X_0,X_1,X_2,X_3) &:= \mathsf{C}(X_3,X_1,X_2,X_0).\end{aligned}$$

Since C is PSD but not SOS by [CL76; CL77], also the permuted forms C^{σ} and $\mathsf{C}^{\tau} \in \mathcal{P}_{4,4}\backslash\Sigma_{4,4}$ are PSD but not SOS. As a matter of fact, we claim

(i) $\mathsf{C}^{\sigma} \in C_4\backslash C_3$,

(ii) $\mathsf{C}^{\tau} \in C_5\backslash C_4$,

(iii) $\mathsf{C} \in C_6\backslash C_5$.

To prove this, for $i = 3, 4, 5$, we argue in two steps (i.e., (1) and (2) below) that the proposed quaternary quartic lies in $C_{i+1}\backslash C_i$. Let us firstly sketch the idea of this two-step-argumentation and then, secondly, make our consideration precise for the special case when $i = 3$. For $i = 4$ and 5, similar arguments apply.

<u>Outline of the Proof</u>:

(1) We fix a Gram matrix $A := (a_{s,t})_{0\leq s,t\leq 9}$ associated to the proposed quaternary quartic such that $a_{s,t} = 0$ for $s \geq i+5$ or $t \geq i+5$ and conclude that q_A is locally PSD on $\phi(K_{i+1})(\mathbb{R})$. By Definition 3.1.1 and Corollary 3.4.5, the proposed quaternary quartic thus lies in C_i.

(2) We assume for a proof by contradiction that there exists some Gram matrix $B := (b_{s,t})_{0\leq s,t\leq 9}$ associated to the proposed quaternary quartic such that q_B is locally PSD on $V_i(\mathbb{R})$. We then show that necessarily $b_{i+4,i+4} = 1$ and $b_{s,t} = 0$ for $s \geq i+4$ or $t \geq i+4$ with $(s,t) \neq (i+4, i+4)$. Consequently, $q_B(z) < 0$ for $[z] \in \phi(K_i)(\mathbb{R}) \subseteq V_i(\mathbb{R})$ given by

$$z_s := \begin{cases} 1, & \text{for } s = 0,\ldots,n+i,\\ 0, & \text{else}\end{cases}$$

which contradicts the assumption that q_B is locally PSD on $V_i(\mathbb{R})$.

<u>Detailed Proof</u>: For $i = 3$, we compute

$$\begin{aligned}\mathsf{C}^{\sigma}(X_0,X_1,X_2,X_3) = X_0^2X_1^2 + X_0^2X_3^2 + X_1^2X_3^2 + X_2^4\\ - 4(X_0X_2)(X_1X_3)\end{aligned} \tag{5.2}$$

and recall from Example 2.3.10 (i) that

$$\begin{aligned}
m_1(X_0,X_1,X_2,X_3) &= X_0X_1, & m_6(X_0,X_1,X_2,X_3) &= X_1X_3,\\
m_2(X_0,X_1,X_2,X_3) &= X_0X_2, & m_7(X_0,X_1,X_2,X_3) &= X_2^2.\\
m_3(X_0,X_1,X_2,X_3) &= X_0X_3, & &
\end{aligned}$$

Hence, $\mathsf{C}^\sigma \underset{(5.2)}{=} m_1^2 + m_3^2 + m_6^2 + m_7^2 - 4m_2m_6$ and, recalling Example 3.2.10 (i), we know that $\tilde{V}_4$ is parametrized by

$$\begin{array}{rcl} \chi_4 \colon \mathbb{P}^3 & \dashrightarrow & \mathbb{P}^7 \\ [x] & \mapsto & \left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2\right]. \end{array}$$

Moreover, Example 3.3.10 showed that

$$\begin{array}{rcl} K_3 & = & \left\{\left(x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, z_7, z_8, z_9\right) \in \mathbb{C}^9 \mid x_1, x_2, x_3, z_7, z_8, z_9 \in \mathbb{C}\right\}, \\ K_4 & = & \left\{\left(x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, z_8, z_9\right) \in \mathbb{C}^9 \mid x_1, x_2, x_3, z_8, z_9 \in \mathbb{C}\right\}. \end{array}$$

These consideration, combined with the above sketched two-step-argument, allows us to give an extensive proof for (i).

(1) We set $A := (a_{s,t})_{0\le s,t\le 9} \in \mathrm{Sym}_{10}(\mathbb{R})$ to be given by

$$a_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{1,3,6,7\} \\ -2, & \text{if } \{s,t\} = \{2,6\} \\ 0, & \text{else} \end{cases}$$

and deduce from Lemma 2.3.15 that A is a Gram matrix associated to C^σ. For $[z] := [1 : x_1 : x_2 : x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : z_8 : z_9] \in \phi(K_4)(\mathbb{R})$, we compute

$$q_A(z) = x_1^2 + x_3^2 + (x_1x_3)^2 + x_2^4 - 4x_1x_2x_3 = \mathsf{C}^\sigma(1, x_1, x_2, x_3) \ge 0.$$

This shows that q_A is locally PSD on $\phi(K_4)(\mathbb{R})$. Hence, $\mathsf{C}^\sigma \in C_{\phi(K_4)} = C_4$ follows by Definition 3.1.1 and Corollary 3.4.5.

(2) For a proof by contradiction, we assume that it is possible to fix some Gram matrix $B = (b_{s,t})_{0\le s,t\le 9}$ associated to C^σ such that q_B is locally PSD on $V_3(\mathbb{R})$ and observe that $b_{9,9}$ is the coefficient of X_3^4 in C^σ by Lemma 2.3.15. That is, $b_{9,9} = 0$. Hence, for $X := (X_0, X_1, X_2, X_3)$ as usual, we conclude that

$$\begin{aligned} q(X,Y) &:= q_B(m_0(X), \dots, m_8(X), Y) \\ &= q_B(m_0(X), \dots, m_8(X), m_9(X)) - 2\sum_{s=0}^{8} b_{s,9} m_s(X) m_9(X) \\ &\quad + \left(2\sum_{s=0}^{8} b_{s,9} m_s(X)\right) Y \\ &= \mathsf{C}^\sigma(X) - 2\sum_{s=0}^{8} b_{s,9} m_s(X) m_9(X) + \left(2\sum_{s=0}^{8} b_{s,9} m_s(X)\right) Y \end{aligned}$$

is PSD. Consequently, $b_{s,9} = 0$ for $s = 0, \dots, 8$ to avoid the potential linearity of q in Y. We therefore know that any entry in the ninth column (and by the symmetry of B also in the ninth row) of B is zero.

We now iterate this argument for the eighth column (and by the symmetry of B also for the eighth row) and observe that $b_{8,8}$ is the coefficient of $X_2^2X_3^2$ in C^σ by Lemma 2.3.15. That is, $b_{8,8}=0$. Hence, we conclude that

$$\begin{aligned}\tilde{q}(X,Y) &:= q_B(m_0(X),\ldots,m_7(X),Y,m_9(X))\\ &= q_B(m_0(X),\ldots,m_7(X),m_8(X),m_9(X))\\ &\quad -2\sum_{s=0}^{7} b_{s,8}m_s(X)m_8(X)+\left(2\sum_{s=0}^{7} b_{s,8}m_s(X)\right)Y\\ &= \mathsf{C}^\sigma(X)-2\sum_{s=0}^{7} b_{s,8}m_s(X)m_8(X)+\left(2\sum_{s=0}^{7} b_{s,8}m_s(X)\right)Y\end{aligned}$$

is PSD. Consequently, $b_{s,8}=0$ for $s=0,\ldots,7$ to avoid the potential linearity of $\tilde{q}$ in Y. We therefore know that any entry in the eighth column (and by the symmetry of B also in the eighth row) of B is zero.

For the seventh column (and by the symmetry of B also for the seventh row), we observe that $b_{7,7}$ is the coefficient of X_2^4 in C^σ by Lemma 2.3.15. That is, $b_{7,7}=1$. Hence, for $\boldsymbol{X} := (X_1,X_2,X_3)$ as usual, we conclude that

$$\begin{aligned}\hat{q}(\boldsymbol{X},Y) &:= q_B(m_0(1,\boldsymbol{X}),\ldots,m_6(1,\boldsymbol{X}),Y,m_8(1,\boldsymbol{X}),m_9(1,\boldsymbol{X}))\\ &= X_1^2+X_3^2+X_1^2X_3^2-4X_1X_2X_3+2(b_{0,7}+b_{1,7}X_1+b_{2,7}X_2\\ &\quad +b_{3,7}X_3+b_{4,7}X_1^2+b_{5,7}X_1X_2+b_{6,7}X_1X_3)(Y-X_2^2)+Y^2\end{aligned}$$

is PSD. Consequently, the greatest degree of X_2 appearing in $\hat{q}$ cannot be odd and it follows $b_{2,7}=b_{5,7}=0$. Moreover, evaluating $\hat{q}$ in $\boldsymbol{x}=(0,0,0)$ yields that

$$\hat{q}(0,0,0,Y)=2b_{0,7}Y+Y^2=(Y+b_{0,7})^2-b_{0,7}^2$$

is PSD. Hence, $b_{0,7}=0$. Evaluating $\hat{q}$ in $\boldsymbol{x}=(x_1,t,t)$ for any $x_1,t\in\mathbb{R}$ likewise also shows that

$$\begin{aligned}\hat{q}(X_1,T,T,Y) &= X_1^2+T^2+X_1^2T^2-4X_1T^2\\ &\quad +2(b_{1,7}X_1+b_{3,7}T+b_{4,7}X_1^2+b_{6,7}X_1T)(Y-T^2)+Y^2\end{aligned}$$

is PSD. Hence, the greatest degree of T appearing cannot be odd and we conclude $b_{3,7}=b_{6,7}=0$. Similarly, we observe that

$$\hat{q}(T,X_2,X_3,T)=T^2+X_3^2+T^2X_3^2-4TX_2X_3+2(b_{1,7}T+b_{4,7}T^2)(T-X_2^2)+T^2$$

is PSD and deduce $b_{4,7}=0$. Lastly, also

$$\hat{q}(T,T,X_3,Y)=T^2+X_3^2+T^2X_3^2-4T^2X_3+2b_{1,7}T(Y-T^2)+Y^2$$

is PSD and, hence, $b_{1,7} = 0$. It thus altogether follows that $b_{7,7} = 1$ and $b_{s,t} = 0$ for $s \geq 7$ or $t \geq 7$ with $(s,t) \neq (7,7)$ by the symmetry of B.

For $[z] := [1 : \ldots : 1 : 0 : 0 : 0] \in \phi(K_3)(\mathbb{R}) \subseteq V_3(\mathbb{R})$, we conclude

$$q_B(z) = q_B(1, \ldots, 1, 0, 0, 0) = q_B(1, \ldots, 1) - 1^2 = \mathsf{C}^\sigma(1, 1, 1, 1) - 1 = (-1)$$

which is a contradiction to the assumption that q_B is locally PSD on $V_3(\mathbb{R})$. ∎

Remark 5.1.3. *Similar arguments as in the detailed proof for $i = 3$ apply for $i = 4, 5$.*

(i) If $i = 4$, then we compute

$$\begin{aligned} \mathsf{C}^\tau(X_0, X_1, X_2, X_3) &= X_1^2X_2^2 + X_1^2X_3^2 + X_2^2X_3^2 + X_0^4 - 4X_0X_1X_2X_3 \\ &= (X_1X_2)^2 + (X_1X_3)^2 + (X_2X_3)^2 + \left(X_0^2\right)^2 \\ &\quad - 4(X_0X_2)(X_1X_3) \end{aligned} \tag{5.3}$$

and recall from Example 2.3.10 (i) that

$$\begin{aligned} m_0(X_0, X_1, X_2, X_3) &= X_0^2, & m_6(X_0, X_1, X_2, X_3) &= X_1X_3, \\ m_2(X_0, X_1, X_2, X_3) &= X_0X_2, & m_8(X_0, X_1, X_2, X_3) &= X_2X_3. \\ m_5(X_0, X_1, X_2, X_3) &= X_1X_2, \end{aligned}$$

Hence, $\mathsf{C}^\tau = m_5^2 + m_6^2 + m_8^2 + m_0^2 - 4m_2m_6$ and, recalling Example 3.2.10 (i), we know that $\tilde{V}_5$ is parametrized by

$$\begin{aligned} \chi_5 \colon \mathbb{P}^3 &\dashrightarrow \mathbb{P}^8 \\ [x] &\mapsto \left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3\right]. \end{aligned}$$

Lemma 2.3.15 implies that an adequate choice for $A := (a_{s,t})_{0 \leq s,t \leq 9} \in \mathcal{G}^{-1}(\mathsf{C}^\tau)$ such that q_A is locally PSD on $\phi(K_5)(\mathbb{R})$ in the first step is given by

$$a_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{0, 5, 6, 8\} \\ -2, & \text{if } \{s, t\} = \{2, 6\} \\ 0, & \text{else.} \end{cases}$$

(ii) If $i = 5$, then we compute

$$\begin{aligned} \mathsf{C}(X_0, X_1, X_2, X_3) =& X_0^2X_1^2 + X_0^2X_2^2 + X_1^2X_2^2 + X_3^4 - 4X_0X_1X_2X_3 \\ &= (X_0X_1)^2 + (X_0X_2)^2 + (X_1X_2)^2 + \left(X_3^2\right)^2 \\ &\quad - 4(X_0X_2)(X_1X_3) \end{aligned} \tag{5.4}$$

and recall from Example 2.3.10 (i) that

$$\begin{aligned} m_1(X_0,X_1,X_2,X_3) &= X_0X_1, & m_6(X_0,X_1,X_2,X_3) &= X_1X_3,\\ m_2(X_0,X_1,X_2,X_3) &= X_0X_2, & m_9(X_0,X_1,X_2,X_3) &= X_3^2.\\ m_5(X_0,X_1,X_2,X_3) &= X_1X_2, \end{aligned}$$

Hence, $\mathsf{C} = m_1^2 + m_2^2 + m_5^2 + m_9^2 - 4m_2m_6$ *and, recalling Example 3.2.10 (i), we know that* $\tilde{V}_6$ *is parametrized by*

$$\begin{aligned} \chi_6\colon \mathbb{P}^3 &\dashrightarrow \mathbb{P}^9\\ [x] &\mapsto \left[x_0^2 : x_0x_1 : x_0x_2 : x_0x_3 : x_1^2 : x_1x_2 : x_1x_3 : x_2^2 : x_2x_3 : x_3^2\right]. \end{aligned}$$

Lemma 2.3.15 implies that an adequate choice for $A := (a_{s,t})_{0\le s,t\le 9} \in \mathcal{G}^{-1}(\mathsf{C})$ *such that* q_A *is locally PSD on* $\phi(K_6)(\mathbb{R})$ *in the first step is given by*

$$a_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{1,2,5,9\}\\ -2, & \text{if } \{s,t\} = \{2,6\}\\ 0, & \text{else.} \end{cases}$$

Corollary 5.1.4. *If* $(n+1, 2d) = (4,4)$*, then* $\mu(3,2) = 2$ *and* $\{\mathsf{C}^\sigma, \mathsf{C}^\tau, \mathsf{C}\}$ *is a complete set of separating forms for* ($\mathcal{CF}$)*.*

Proof. Observation 5.1.1 and 5.1.2 together yield

$$\Sigma_{4,4} = C_0 = C_1 = C_2 = C_3 \subsetneq C_4 \subsetneq C_5 \subsetneq C_6 = \mathcal{P}_{4,4}. \tag{5.5}$$

Hence, there are two strictly separating intermediate cones between $\Sigma_{4,4}$ and $\mathcal{P}_{4,4}$ in (5.5), namely, C_4 and C_5. This gives us $\mu(3,2) = 2$. Moreover, we know

$$|\{i(f) \mid f \in \{\mathsf{C}^\sigma, \mathsf{C}^\tau, \mathsf{C}\}\}| = |\{3,4,5\}| = 3 = \mu(3,2) + 1$$

by the proof of Theorem 5.1.2. Thus, $\{\mathsf{C}^\sigma, \mathsf{C}^\tau, \mathsf{C}\}$ is a complete set of separating forms for ($\mathcal{CF}$) in the case of quaternary quartics. ■

Observation 5.1.5. *Ternary Sextics*
Let $n = 2$ *and* $d = 3$*. We compute* $k = 9$*,* $k - n = 7$ *and observe that the specific cone filtration* ($\mathcal{CF}$) *for ternary sextics is given by*

$$\Sigma_{3,6} = C_0 \subseteq C_1 \subseteq C_2 \subseteq C_3 \subseteq C_4 \subseteq C_5 \subseteq C_6 \subseteq C_7 = \mathcal{P}_{3,6}. \tag{5.6}$$

Theorem 4.2.7 yields $C_0 = C_1 = C_2 = C_3$ *and, therefore, it remains to determine each strict inclusion in* $C_3 \subseteq C_4 \subseteq C_5 \subseteq C_6 \subseteq C_7$*.*

Theorem 5.1.6. *If* $(n+1, 2d) = (3, 6)$*, then* $C_3 \subsetneq C_4 \subsetneq C_5 \subsetneq C_6 \subsetneq C_7$*.*

Proof. We consider the Motzkin ternary sextic M and the Choi–Lam ternary sextic L

$$\begin{aligned} \mathsf{M}(X_0, X_1, X_2) &:= X_0^4 X_1^2 + X_0^2 X_1^4 + X_2^6 - 3X_0^2 X_1^2 X_2^2, \\ \mathsf{L}(X_0, X_1, X_2) &:= X_0^4 X_1^2 + X_0^2 X_2^4 + X_1^4 X_2^2 - 3X_0^2 X_1^2 X_2^2 \end{aligned}$$

and their permuted forms

$$\begin{aligned} \mathsf{M}^\sigma(X_0, X_1, X_2) &:= \mathsf{M}(X_0, X_2, X_1), \\ \mathsf{L}^\sigma(X_0, X_1, X_2) &:= \mathsf{L}(X_0, X_2, X_1). \end{aligned}$$

Since M and L are PSD but not SOS by [Mot65] and [CL76; CL77], respectively, also the permuted forms M^σ and L^σ are PSD but not SOS. As a matter of fact, analogously as in the proof of Theorem 5.1.2, we can show

(i) $\mathsf{M}^\sigma \in C_4 \backslash C_3$,

(ii) $\mathsf{L} \in C_5 \backslash C_4$,

(iii) $\mathsf{L}^\sigma \in C_6 \backslash C_5$,

(iv) $\mathsf{M} \in C_7 \backslash C_6$. ■

Remark 5.1.7. *(i) We compute*

$$\begin{aligned} \mathsf{M}^\sigma(X_0, X_1, X_2) &:= X_0^4 X_2^2 + X_0^2 X_2^4 + X_1^6 - 3X_0^2 X_1^2 X_2^2, \\ &= (X_0^2 X_2)^2 + (X_0 X_2^2)^2 + (X_1^3)^2 - 3(X_0 X_1 X_2)^2 \end{aligned} \tag{5.7}$$

and recall from Example 2.3.10 (ii) that

$$\begin{aligned} m_2(X_0, X_1, X_2) &= X_0^2 X_2, & m_5(X_0, X_1, X_2) &= X_0 X_2^2, \\ m_4(X_0, X_1, X_2) &= X_0 X_1 X_2, & m_6(X_0, X_1, X_2) &= X_1^3. \end{aligned}$$

Hence, $\mathsf{M}^\sigma \underset{(5.7)}{=} m_2^2 + m_5^2 + m_6^2 - 3m_4^2$ *and, recalling Example 3.2.10 (ii), we know that* $\tilde{V}_4$ *is parametrized by*

$$\begin{aligned} \chi_4 \colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^6 \\ [x] &\mapsto \left[x_0^3 : x_0^2 x_1 : x_0^2 x_2 : x_0 x_1^2 : x_0 x_1 x_2^2 : x_0 x_2^2 : x_1^3\right]. \end{aligned}$$

Lemma 2.3.15 implies that an adequate choice for $A := (a_{s,t})_{0 \le s,t \le 9} \in \mathcal{G}^{-1}(\mathsf{M}^\sigma)$ *such that* q_A *is locally PSD on* $\phi(K_4)(\mathbb{R})$ *in the first step is given by*

$$a_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{2,5,6\} \\ -3, & \text{if } s = t = 4 \\ 0, & \text{else.} \end{cases}$$

(ii) We compute

$$\begin{aligned} \mathsf{L}(X_0, X_1, X_2) &:= X_0^4 X_1^2 + X_0^2 X_2^4 + X_1^4 X_2^2 - 3X_0^2 X_1^2 X_2^2, \\ &= (X_0^2 X_1)^2 + (X_0 X_2^2)^2 + (X_1^2 X_2)^2 - 3(X_0 X_1 X_2)^2 \end{aligned} \tag{5.8}$$

and recall from Example 2.3.10 (ii) that

$$\begin{aligned} m_1(X_0, X_1, X_2) &= X_0^2 X_1, & m_5(X_0, X_1, X_2) &= X_0 X_2^2, \\ m_4(X_0, X_1, X_2) &= X_0 X_1 X_2, & m_7(X_0, X_1, X_2) &= X_1^2 X_2. \end{aligned}$$

Hence, $\mathsf{L} \underset{(5.8)}{=} m_1^2 + m_5^2 + m_7^2 - 3m_4^2$ *and, recalling Example 3.2.4 (ii), we know that* $\tilde{V}_5$ *is parametrized by*

$$\begin{aligned} \chi_5 \colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^7 \\ [x] &\mapsto \left[x_0^3 : x_0^2 x_1 : x_0^2 x_2 : x_0 x_1^2 : x_0 x_1 x_2^2 : x_0 x_2^2 : x_1^3 : x_1^2 x_2\right]. \end{aligned}$$

Lemma 2.3.15 implies that an adequate choice for $A := (a_{s,t})_{0 \le s,t \le 9} \in \mathcal{G}^{-1}(\mathsf{L})$ *such that* q_A *is locally PSD on* $\phi(K_5)(\mathbb{R})$ *in the first step is given by*

$$a_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{1,5,7\} \\ -3, & \text{if } s = t = 4 \\ 0, & \text{else.} \end{cases}$$

(iii) We compute

$$\begin{aligned} \mathsf{L}^\sigma(X_0, X_1, X_2) &:= X_0^4 X_2^2 + X_0^2 X_1^4 + X_1^2 X_2^4 - 3X_0^2 X_1^2 X_2^2, \\ &= (X_0^2 X_2)^2 + (X_0 X_1^2)^2 + (X_1 X_2^2)^2 - 3(X_0 X_1 X_2)^2 \end{aligned} \tag{5.9}$$

and recall from Example 2.3.10 (ii) that

$$\begin{aligned} m_2(X_0, X_1, X_2) &= X_0^2 X_2, & m_4(X_0, X_1, X_2) &= X_0 X_1 X_2, \\ m_3(X_0, X_1, X_2) &= X_0 X_1^2, & m_8(X_0, X_1, X_2) &= X_1 X_2^2. \end{aligned}$$

Hence, $\mathsf{L}^\sigma \underset{(5.9)}{=} m_2^2 + m_3^2 + m_8^2 - 3m_4^2$ *and, recalling Example 3.2.4 (ii), we know that* $\tilde{V}_6$ *is parametrized by*

$$\begin{aligned} \chi_6 \colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^8 \\ [x] &\mapsto \left[x_0^3 : x_0^2 x_1 : x_0^2 x_2 : x_0 x_1^2 : x_0 x_1 x_2^2 : x_0 x_2^2 : x_1^3 : x_1^2 x_2 : x_1 x_2^2\right]. \end{aligned}$$

Lemma 2.3.15 implies that an adequate choice for $A := (a_{s,t})_{0 \le s,t \le 9} \in \mathcal{G}^{-1}(\mathsf{L}^\sigma)$ *such that* q_A *is locally PSD on* $\phi(K_6)(\mathbb{R})$ *in the first step is given by*

$$a_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{2,3,8\} \\ -3, & \text{if } s = t = 4 \\ 0, & \text{else.} \end{cases}$$

(iv) We compute

$$\begin{aligned} \mathsf{M}(X_0, X_1, X_2) &:= X_0^4 X_1^2 + X_0^2 X_1^4 + X_2^6 - 3 X_0^2 X_1^2 X_2^2, \\ &= (X_0^2 X_1)^2 + (X_0 X_1^2)^2 + (X_2^3)^2 - 3(X_0 X_1 X_2)^2 \end{aligned} \tag{5.10}$$

and recall from Example 2.3.10 (ii),

$$\begin{aligned} m_1(X_0, X_1, X_2) &= X_0^2 X_1, & m_4(X_0, X_1, X_2) &= X_0 X_1 X_2, \\ m_3(X_0, X_1, X_2) &= X_0 X_1^2, & m_9(X_0, X_1, X_2) &= X_2^3. \end{aligned}$$

Hence, $\mathsf{M} \underset{(5.10)}{=} m_1^2 + m_3^2 + m_9^2 - 3m_4^2$ *and, recalling Example 3.2.4 (ii), we know that* $\tilde{V}_7$ *is parametrized by*

$$\begin{aligned} \chi_7 \colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^9 \\ [x] &\mapsto \left[x_0^3 : x_0^2 x_1 : x_0^2 x_2 : x_0 x_1^2 : x_0 x_1 x_2^2 : x_0 x_2^2 : x_1^3 : x_1^2 x_2 : x_1 x_2^2 : x_2^3\right]. \end{aligned}$$

Lemma 2.3.15 implies that an adequate choice for $A := (a_{s,t})_{0 \le s,t \le 9} \in \mathcal{G}^{-1}(\mathsf{M})$ *such that* q_A *is locally PSD on* $\phi(K_7)(\mathbb{R})$ *in the first step is given by*

$$a_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{1,3,9\} \\ -3, & \text{if } s = t = 4 \\ 0, & \text{else.} \end{cases}$$

Corollary 5.1.8. *If* $(n+1, 2d) = (3,6)$*, then* $\mu(2,3) = 3$ *and* $\{\mathsf{M}^\sigma, \mathsf{L}, \mathsf{L}^\sigma, \mathsf{M}\}$ *is a complete set of separating forms for (*$\mathcal{CF}$*).*

Proof. Observation 5.1.5 and Theorem 5.1.6 together yield

$$\Sigma_{3,6} = C_0 = C_1 = C_2 = C_3 \subsetneq C_4 \subsetneq C_5 \subsetneq C_6 \subsetneq C_7 = \mathcal{P}_{3,6}. \tag{5.11}$$

Hence, there are three strictly separating intermediate cones between $\Sigma_{3,6}$ and $\mathcal{P}_{3,6}$ in (5.11), namely, C_4, C_5 and C_6. This gives us $\mu(2,3) = 3$. Moreover, we know

$$|\{i(f) \mid f \in \{\mathsf{M}^\sigma, \mathsf{L}, \mathsf{L}^\sigma, \mathsf{M}\}\}| = |\{3,4,5,6\}| = 4 = \mu(3,2) + 1$$

by the proof of Theorem 5.1.6. Thus, $\{\mathsf{M}^\sigma, \mathsf{L}, \mathsf{L}^\sigma, \mathsf{M}\}$ is a complete set of separating forms for ($\mathcal{CF}$) in the case of ternary sextics. ■

Concluding Remark. *Altogether, we answered the main query in the basic non-Hilbert cases by specifying the cone filtration (5.1) and (5.6) to be given by*

$$\Sigma_{4,4} = C_0 = C_1 = C_2 = C_3 \subsetneq C_4 \subsetneq C_5 \subsetneq C_6 = \mathcal{P}_{4,4} \text{ and}$$
$$\Sigma_{3,6} = C_0 = C_1 = C_2 = C_3 \subsetneq C_4 \subsetneq C_5 \subsetneq C_6 \subsetneq C_7 = \mathcal{P}_{3,6}, \text{ respectively.}$$

This especially allowed us to deduce $\mu(3,2) = 2$ and $\mu(3,2) = 3$.

5.2 $(n+1)$-ary Quartics $(n \geq 4)$ and $(n+1)$-ary Sextics $(n \geq 3)$

Observation 5.2.1. $(n+1)$-ARY QUARTICS $(n \geq 4)$
For $n \geq 4$, let $d = 2$ and observe that the specific cone filtration ($\mathcal{CF}$) is given by

$$\Sigma_{n+1,4} = C_0 \subseteq \ldots \subseteq C_n \subseteq \ldots \subseteq C_{k-n} = \mathcal{P}_{n+1,4}. \tag{5.12}$$

Theorem 4.2.7 yields $C_0 = \ldots = C_n$ and, therefore, it remains to determine each strict inclusion in $C_n \subseteq \ldots \subseteq C_{k-n}$.

The proof below generalizes the consideration of Theorem 5.1.2 from quaternary quartics to $(n+1)$-ary quartics for $n \geq 4$.

Theorem 5.2.2. *For $(n+1,4)_{n\geq 4}$ and $i = n, \ldots, k-n-1$, the inclusion $C_i \subsetneq C_{i+1}$ is strict.*

Proof. For the purpose of this proof, for $j = 3, \ldots, 6$, we denote the subcone C_j of $\mathcal{P}_{4,4}$ by C_j^3 and the subcones C_i and C_{i+1} of $\mathcal{P}_{n+1,4}$ by C_i^n and C_{i+1}^n, respectively. Likewise, we set K_3^3 to be the affine variety $K_3 \subseteq \mathbb{C}^{k(3,2)}$, K_i^n to be the affine variety $K_i \subseteq \mathbb{C}^{k(n,2)}$ and let α_j^n be the exponent $\alpha_j \in I_{n+1,2}$ for $j = 0, \ldots, k(n,2)$.

For $i = n, \ldots, k-n-1$, we distinguish three cases, which can be treated similarly:

(i) $m_{n+i}(X) = X_j X_n$ for some $1 \leq j \leq n-2$

(ii) $m_{n+i}(X) = X_j X_l$ for some $2 \leq j \leq l \leq n-1$

(iii) $m_{n+i}(X) = m_{k-1}(X)$

In each case, it suffices to find some $g \in \mathcal{P}_{n+1,4}$ with the following two properties:

(1) There exists some $B \in \mathcal{G}^{-1}(g)$ such that q_B is locally PSD on $\phi(K_{i+1}^n)(\mathbb{R})$.

(2) For any $B \in \mathcal{G}^{-1}(g)$, there exists a $[y] \in \phi(K_i^n)$ with $q_B(y) < 0$.

Indeed, (1) shows $g \in C_{\phi(K_{i+1}^n)}$ and $C_{\phi(K_{i+1}^n)} = C_{i+1}^n$ by Corollary 3.4.5. Moreover, (2) yields $g \notin C_i^n$. Altogether, such a g hence testifies $C_i^n \subsetneq C_{i+1}^n$.

We therefore now give some $g \in \mathcal{P}_{n+1,4}$ with property (1) and (2) for Case (i). To this end, we fix $f \in \mathcal{C}_4^3 \backslash C_3^3$, set $g(X_0, \ldots, X_n) := f(X_0, X_j, X_{j+1}, X_n) \in \mathcal{F}_{n+1,4}$ and, recalling Lemma 2.3.17, define

$$I := \left\{ i \in \{0, \ldots, k(n,2)\} \mid \alpha_{i,s}^n = 0 \text{ for } s \neq 0, j, j+1, n \right\}.$$

(1) We let $A \in \mathcal{G}^{-1}(f)$ be such that q_A is locally PSD on $\phi(K_4^3)(\mathbb{R})$. Applying Lemma 2.3.17 (i), we expand A to a Gram matrix B associated to g such that $B_I := A$. Moreover, we define

$$\begin{array}{rccc} \pi\colon & \mathbb{C}^{k(n,2)+1} & \to & \mathbb{C}^{k(3,2)+1} \\ & z & \mapsto & y_z := (z_i)_{i \in I} \end{array}$$

and point out that $[y_z] \in \mathbb{P}^{k(3,2)}$ is well-defined for $[z] \in \phi(K_{i+1}^n)(\mathbb{R})$. Indeed, for $[z] \in \phi(K_{i+1}^n)(\mathbb{R})$, we have $z_0 \neq 0$ and $0 \in I$ since $\alpha_0 = (2, 0, \ldots, 0) \in I_{n+1,2}$. Consequently, $y_z = \pi(z) \neq (0, \ldots, 0)$. Moreover, for $z' \in [z]$, we let $\lambda \in \mathbb{C}^\times$ be such that $z = \lambda z'$. Hence, $y_z = \pi(z) = \pi(\lambda z') = \lambda \pi(z') = \lambda y_{z'}$ and so $[y_z] = [\pi(z)] = [\pi(z')] = [y_{z'}]$ follows.

For $[z] \in \phi(K_{i+1}^n)(\mathbb{R})$, we in particular see $[y_z] \in \phi(K_4^3)(\mathbb{R})$ by construction. Using that q_A is locally PSD on $\phi(K_4^3)(\mathbb{R})$, we deduce for $[z] \in \phi(K_{i+1}^n)(\mathbb{R})$ that

$$q_B(z) = zBz^t = (z_i)_{i \in I} B_I (z_i)_{i \in I}^t = \pi(z) A \pi(z)^t = y_z A y_z^t = q_A(y_z) \geq 0.$$

This proves that q_B is locally PSD on $\phi(K_{i+1}^n(\mathbb{R}))$.

(2) We let $B \in \mathcal{G}^{-1}(g)$ be arbitrary but fixed. Applying Lemma 2.3.17 (ii), we know that the submatrix $A := B_I$ of B is a Gram matrix associated to f. Since $f \notin C_3^3$ by choice, we can fix some $[y] \in \phi(K_3^3)(\mathbb{R})$ such that $q_A(y) < 0$. We expand $[y]$ to $[z_y] \in \phi(K_i^n)(\mathbb{R})$ by zero entries such that $(z_i)_{i \in I} = y$ and conclude

$$q_A(z_y) = z_y A z_y^t = yBy^t = q_B(y) < 0.$$

Case (ii) and (iii) can be treated similarly to Case (i). That is, for Case (ii), we fix $f \in C_5^3 \backslash C_4^3$ and set $g(X_0, \ldots, X_n) := f(X_0, X_1, X_j, X_{l+1})$. Likewise, for Case (iii), we fix $f \in C_6^3 \backslash C_5^3$ and set $g(X_0, \ldots, X_n) := f(X_0, X_1, X_{n-1}, X_n)$. ∎

Corollary 5.2.3. *For* $(n+1,4)_{n\geq 4}$*, it holds*

$$\mu(n,2) = k(n,2) - 2n - 1 = \frac{n(n-1)}{2} - 1.$$

Proof. Observation 5.2.1 and Theorem 5.2.2 together yield

$$\Sigma_{n+1,4} = C_0 = \ldots = C_n \subsetneq C_{n+1} \subsetneq \ldots \subsetneq C_{k(n,2)-n} = \mathcal{P}_{n+1,4}. \tag{5.13}$$

Hence, there are $k(n,2)-2n-1$ strictly separating intermediate cones between $\Sigma_{n+1,4}$ and $\mathcal{P}_{n+1,4}$ in (5.13), namely $C_{n+1},\ldots,C_{k(n,2)-n-1}$. This gives

$$\mu(n,2) = k(n,2) - 2n - 1 = \binom{n+2}{2} - 2n - 2 = \frac{n(n-1)}{2} - 1. \qquad \blacksquare$$

Corollary 5.2.4. *Let* $\{f_1,f_2,f_3\}$ *be a complete set of separating forms for* ($\mathcal{CF}$) *such that* $i(f_j) = j+2$ *for* $j = 1,2,3$ *in the basic non-Hilbert case* $(4,4)$*. For* $n \geq 4$*, a complete set of separating forms for* ($\mathcal{CF}$) *in the non-Hilbert case* $(n+1,4)$ *is given by* $F_1 \cup F_2 \cup F_3$ *where*

$$\begin{aligned}
F_1 &:= \{f_1(X_0,X_j,X_{j+1},X_n) \mid j = 1,\ldots,n-2\},\\
F_2 &:= \{f_2(X_0,X_1,X_j,X_{l+1}) \mid 2 \leq j \leq l \leq n-1\},\\
F_3 &:= \{f_3(X_0,X_1,X_{n-1},X_n)\}.
\end{aligned}$$

Proof. The proof of Theorem 5.2.2 is constructive and can be carried out using the forms f_1, f_2 and f_3. This gives us the sets

$$\begin{aligned}
F_1 &:= \{f_1(X_0,X_j,X_{j+1},X_n) \mid j = 1,\ldots,n-2\},\\
F_2 &:= \{f_2(X_0,X_1,X_j,X_{l+1}) \mid 2 \leq j \leq l \leq n-1\},\\
F_3 &:= \{f_3(X_0,X_1,X_{n-1},X_n)\}
\end{aligned}$$

and $i(f) \neq i(g)$ for distinct $f,g \in F_1 \cup F_2 \cup F_3$. Recalling Corollary 5.2.3, we conclude

$$\begin{aligned}
|\{i(f) \mid f \in F_1 \cup F_2 \cup F_3\}| &= |F_1| + |F_2| + |F_3|\\
&= (n-2) + \frac{(n-2)(n-1)}{2} + 1\\
&= \frac{n(n-1)}{2}\\
&= \mu(n,2) + 1.
\end{aligned}$$

Hence, $F_1 \cup F_2 \cup F_3$ is a complete set of separating forms for ($\mathcal{CF}$) in the non-Hilbert case $(n+1,4)$. $\blacksquare$

Corollary 5.2.5. *In the non-Hilbert case $(n+1,4)$ for $n \geq 4$, a complete set of separating forms for $(\mathcal{CF})$ is given by $F_1 \cup F_2 \cup F_3$ where*

$$\begin{aligned} F_1 &:= \{\mathsf{C}^\sigma(X_0, X_j, X_{j+1}, X_n) \mid j = 1, \dots, n-2\}, \\ F_2 &:= \{\mathsf{C}^\tau(X_0, X_1, X_j, X_{l+1}) \mid 2 \leq j \leq l \leq n-1\}, \\ F_3 &:= \{\mathsf{C}(X_0, X_1, X_{n-1}, X_n)\}. \end{aligned}$$

Proof. Corollary 5.1.4 states that $\{\mathsf{C}^\sigma, \mathsf{C}^\tau, \mathsf{C}\}$ is a complete set of separating forms for $(\mathcal{CF})$ in the basic non-Hilbert case $(4,4)$ and $i(\mathsf{C}^\sigma) = 3$, $i(\mathsf{C}^\tau) = 4$, $i(\mathsf{C}) = 5$ by the proof of Theorem 5.1.2. Hence, the assertion follows from Corollary 5.2.4 by setting $f_1 := \mathsf{C}^\sigma$, $f_2 := \mathsf{C}^\tau$ and $f_3 := \mathsf{C}$. ∎

Example 5.2.6. Quinary Quartics

Let $n = 4$, $d = 2$ and we compute $k = k(4,2) = 14$, $k - n = 10$. Following the proof of Theorem 5.2.2, we now construct a separating form $g \in C_8 \backslash C_7$. In the notations of that proof, we have $i = 7$ and determine $m_{n+i}(X) = m_{11}(X) = X_2X_3$. We are consequently in the situation of Case (ii) with $j = 2$ and $l = 3$. Hence, we reconsider the separating quaternary quartic

$$\mathsf{C}^\tau(X_0, X_1, X_2, X_3) = X_1^2X_2^2 + X_1^2X_3^2 + X_2^2X_3^2 + X_0^4 - 4X_0X_1X_2X_3 \in C_5^3 \backslash C_4^3$$

from the proof of Theorem 5.1.2 and set

$$\begin{aligned} g(X_0, X_1, X_2, X_3, X_4) &:= \mathsf{C}^\tau(X_0, X_1, X_2, X_4) \\ &= X_1^2X_2^2 + X_1^2X_4^2 + X_2^2X_4^2 + X_0^4 - 4X_0X_1X_2X_4. \end{aligned}$$

According to the proof of Theorem 5.2.2, $g \in \mathcal{F}_{5,4}$ thus testifies $C_7 \subsetneq C_8$. We now indicate why that is by reconsidering our arguments given in (1) and (2) in the proof of Theorem 5.2.2. To this end, we compute

$$\begin{array}{llll} \alpha_0^4 = (2,0,0,0,0), & \alpha_4^4 = (1,0,0,0,1), & \alpha_8^4 = (0,1,0,0,1), & \alpha_{12}^4 = (0,0,0,2,0), \\ \alpha_1^4 = (1,1,0,0,0), & \alpha_5^4 = (0,2,0,0,0), & \alpha_9^4 = (0,0,2,0,0), & \alpha_{13}^4 = (0,0,0,1,1), \\ \alpha_2^4 = (1,0,1,0,0), & \alpha_6^4 = (0,1,1,0,0), & \alpha_{10}^4 = (0,0,1,1,0), & \alpha_{14}^4 = (0,0,0,0,2). \\ \alpha_3^4 = (1,0,0,1,0), & \alpha_7^4 = (0,1,0,1,0), & \alpha_{11}^4 = (0,0,1,0,1), & \end{array}$$

Hence, $I := \{i \in \{0, \dots, 14\} \mid \alpha_{i,s}^4 = 0 \text{ for } s \neq 0,1,2,4\} = \{0,1,2,4,5,6,8,9,11,14\}$.

(1) Using Remark 5.1.3 (i), we observe that for $A := (a_{s,t})_{0 \leq s,t \leq 9} \in \mathcal{G}^{-1}(\mathsf{C}^\tau)$ with

$$a_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{0,5,6,8\} \\ -2, & \text{if } \{s,t\} = \{2,6\} \\ 0, & \text{else,} \end{cases}$$

q_A is locally PSD on $\phi(K_5^3)(\mathbb{R})$. Applying Lemma 2.3.17 (i), we therefore expand A to the Gram matrix $B := (b_{s,t})_{0\le s,t\le 14}$ associated to g that is given by

$$b_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{0, 6, 8, 11\} \\ -2, & \text{if } \{s,t\} = \{2, 8\} \\ 0, & \text{else.} \end{cases}$$

Example 3.3.10 and Lemma 3.3.9 moreover give

$$\begin{aligned} K_5^3 &= \left\{ \left(x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, z_9\right) \mid x_1, x_2, x_3, z_9 \in \mathbb{C} \right\}, \\ K_8^4 &= \left\{ \left(x_1, x_2, x_3, x_4, x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2^2, x_2x_3, x_2x_4, x_3^2, z_{13}, z_{14}\right) \mid \right. \\ &\qquad \left. x_1, x_2, x_3, x_4, z_{13}, z_{14} \in \mathbb{C} \right\}. \end{aligned}$$

We set

$$\begin{aligned} \pi\colon \quad \mathbb{C}^{15} &\to \mathbb{C}^{10} \\ z &\mapsto y_z := (z_0, z_1, z_2, z_4, z_5, z_6, z_8, z_9, z_{11}, z_{14}) \end{aligned}$$

and thus observe for $[z] \in \phi(K_8^4)(\mathbb{R})$ that

$$\begin{aligned} [y_z] &= \left[\pi\left(1, x_1, x_2, x_3, x_4, x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2^2, x_2x_3, x_2x_4, x_3^2, z_{13}, z_{14}\right)\right] \\ &= \left[1 : x_1 : x_2 : x_4 : x_1^2 : x_1x_2 : x_1x_4 : x_2^2 : x_2x_4 : z_{14}\right] \in \phi(K_5^3)(\mathbb{R}). \end{aligned}$$

Moreover, using that C^τ is PSD, we compute for $[z] \in \phi(K_8^4)(\mathbb{R})$ that

$$\begin{aligned} q_B(z) &= 1^2 + x_1^2x_2^2 + x_1^2x_4^2 + x_2^2x_4^2 - 4x_1x_2x_4 \\ &= g(1, x_1, x_2, x_3, x_4) \\ &= \mathsf{C}^\tau(1, x_1, x_2, x_4) \ge 0. \end{aligned}$$

Let us point out that the local PSD property of q_A on $\phi(K_5^3)(\mathbb{R})$ instead of the PSD property of C^τ also yields

$$\begin{aligned} q_B(z) &= 1^2 + x_1^2x_2^2 + x_1^2x_4^2 + x_2^2x_4^2 - 4x_1x_2x_4 \\ &= q_A\left(1, x_1, x_2, x_4, x_1^2, x_1x_2, x_1x_4, x_2^2, x_2x_4, z_{14}\right) \ge 0 \end{aligned}$$

for $[z] \in \phi(K_8^4)(\mathbb{R})$. Either way, $g \in C_8$ follows.

(2) Using the matrix $B := (b_{s,t})_{0 \leq s,t \leq 14} \in \mathrm{Sym}_{15}(\mathbb{R})$ that is given by

$$b_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{0,6,8\} \\ \frac{1}{2}, & \text{if } \{s,t\} = \{9,14\} \\ -2, & \text{if } \{s,t\} = \{1,11\} \\ -6, & \text{if } s = t = 4 \\ 3, & \text{if } \{s,t\} = \{0,14\} \\ 10, & \text{if } s = t = 7 \\ -5, & \text{if } \{s,t\} = \{5,12\} \\ 0, & \text{else,} \end{cases}$$

we now illustrate the idea of argument (2) from the proof of Theorem 5.2.2. To this end, we observe that B is a Gram matrix associated to g by Lemma 2.3.15. Hence, by Lemma 2.3.17 (ii), the matrix $A := (a_{s,t})_{0 \leq s,t \leq 9} \in \mathrm{Sym}_{10}(\mathbb{R})$ given by

$$a_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{0,5,6\} \\ \frac{1}{2}, & \text{if } \{s,t\} = \{7,9\} \\ -2, & \text{if } \{s,t\} = \{1,8\} \\ -6, & \text{if } s = t = 3 \\ 3, & \text{if } \{s,t\} = \{0,9\} \\ 0, & \text{else} \end{cases}$$

is a Gram matrix associated to C^τ. Comparing with Lemma 2.3.15, we see that this is indeed true. From the proof of Theorem 5.1.2, we furthermore know $\mathsf{C}^\tau \notin C_4^3 = C_{\phi(K_4^3)}$. Hence, there exists some $[y] \in \phi(K_4^3)(\mathbb{R})$ such that $q_A(z) < 0$. Indeed, recalling Example 3.3.10 and applying Lemma 3.3.9, we see

$$\begin{aligned} K_4^3 &= \left\{ \left(x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, z_8, z_9 \right) \mid x_1, x_2, x_3, z_8, z_9 \in \mathbb{C} \right\}, \\ K_7^4 &= \left\{ \left(x_1, x_2, x_3, x_4, x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2^2, x_2x_3, x_2x_4, z_{12}, z_{13}, z_{14} \right) \mid \right. \\ &\qquad \left. x_1, x_2, x_3, x_4, z_{12}, z_{13}, z_{14} \in \mathbb{C} \right\} \end{aligned}$$

and compute $q_A(y) = (-3) < 0$ for

$$[y] := [1:1:1:1:1:1:1:1:0:0] \in \phi(K_4^3)(\mathbb{R}).$$

Expanding $[y]$ by zero entries into

$$[z_y] = [1:1:1:0:1:1:1:0:1:1:0:0:0:0:0] \in \phi(K_7^4)(\mathbb{R}),$$

we thus find $q_B(z_y) = (-3) < 0$.

Theorem 5.2.7. Degree-Jumping Principle I
Let $(n+1,2d)$ be a non-Hilbert case and $i=0,\dots,k(n,d)-n-1$. If $C_i \subsetneq C_{i+1}$ as subcones of $\mathcal{P}_{n+1,2d}$, then $C_i \subsetneq C_{i+1}$ as subcones of $\mathcal{P}_{n+1,2\delta}$ for $\delta \geq d$.

Proof. For the purpose of this proof, we denote the subcones C_i and C_{i+1} of $\mathcal{P}_{n+1,2\delta}$ by C_i^δ and C_{i+1}^δ, respectively. Likewise, we denote the subvarieties V_i and V_{i+1} of $\mathbb{P}^{k(n,\delta)}$ by V_i^δ and V_{i+1}^δ, respectively. Moreover, for $j=0,\dots,k(n,\delta)$, we let m_j^δ be the monomial $m_j \in \mathcal{F}_{n+1,\delta}$.

We now argue by an induction on $\delta \geq d$ where the inductive base case $d=\delta$ is true by assumption. Thus, we assume that the assertion was already verified up to some $\delta \geq d$ and investigate the situation for $\delta+1$. To this end, we fix $f(X) \in C_{i+1}^\delta \backslash C_i^\delta$, set $g(X) := X_0^2 f(X) \in \mathcal{F}_{n+1,2(\delta+1)}$ and claim $g \in C_{i+1}^{\delta+1} \backslash C_i^{\delta+1}$.

Indeed, let $A \in \mathcal{G}^{-1}(f)$ be such that $q_A(y) \geq 0$ for any $[y] \in V_{i+1}^\delta(\mathbb{R})$. We set $B := (b_{s,t})_{0 \leq s,t \leq k(n,\delta+1)} \in \mathrm{Sym}_{k(n,\delta+1)+1}(\mathbb{R})$ to be given by

$$b_{s,t} := \begin{cases} a_{s,t}, & \text{for } s,t = 0,\dots,k(n,\delta) \\ 0, & \text{else} \end{cases}$$

and conclude from Lemma 2.3.19 that B is a Gram matrix associated to g. For any $[z] \in V_{i+1}^{\delta+1}(\mathbb{R})$, we moreover define $[y_z] := \left[z_0 : \dots : z_{k(n,\delta)}\right] \in \mathbb{P}^{k(n,\delta)}(\mathbb{R})$ and observe $[y_z] \in V_{i+1}^\delta(\mathbb{R})$ by Construction 3.2.1. This allows us to conclude

$$q_B(z) = zBz^t = \left(z_0,\dots,z_{k(n,\delta)}\right) A \left(z_0,\dots,z_{k(n,\delta)}\right)^t = y_z A y_z^t = q_A(y_z) \geq 0.$$

It thus remains to show $g \notin C_i^{\delta+1}$. To this end, we assume for a proof by contradiction that there exists some $D := (d_{s,t})_{0 \leq s,t \leq k(n,\delta+1)} \in \mathcal{G}^{-1}(g)$ such that $q_D(z) \geq 0$ for any $[z] \in V_i^{\delta+1}(\mathbb{R})$. We deduce from Lemma 2.3.15 that $d_{k(n,\delta+1),k(n,\delta+1)}$ is the coefficient of $X_n^{2(\delta+1)}$ in g. That is, $d_{k(n,\delta+1),k(n,\delta+1)} = 0$. Hence, for $X := (X_0,\dots,X_n)$ as usual, we conclude that

$$\begin{aligned} q(X,Z) &:= q_D(m_0(X),\dots,m_{k(n,\delta+1)-1}(X),Z) \\ &= q_D(m_0(X),\dots,m_{k(n,\delta+1)-1}(X),m_{k(n,\delta+1)}(X)) \\ &\quad -2\sum_{s=0}^{k(n,\delta+1)-1} d_{s,k(n,\delta+1)} m_s(X) m_{k(n,\delta+1)}(X) \\ &\quad + \left(2\sum_{s=0}^{k(n,\delta+1)-1} d_{s,k(n,\delta+1)} m_s(X)\right) Z \\ &= g(X) - 2\sum_{s=0}^{k(n,\delta+1)-1} d_{s,k(n,\delta+1)} m_s(X) m_{k(n,\delta+1)}(X) \\ &\quad + \left(2\sum_{s=0}^{k(n,\delta+1)-1} d_{s,k(n,\delta+1)} m_s(X)\right) Z \end{aligned}$$

is PSD. Consequently, $d_{s,k(n,\delta+1)} = 0$ for $s = 0, \ldots, k(n, \delta+1) - 1$ to avoid the potential linearity of q in Z. We therefore know that any entry in the $k(n, \delta+1)^{th}$ column (and by the symmetry of D also in the $k(n, \delta+1)^{th}$ row) of D is zero. Iterating this argument allows us to conclude that $d_{s,t} = d_{t,s} = 0$ for $t = k(n, \delta)+1, \ldots, k(n, \delta+1)$ and $s = 0, \ldots, t$. Hence, for $I := \{0, \ldots, k(n, \delta)\}$ and $D_I := (d_{s,t})_{s,t \in I} \in \mathrm{Sym}_{k(n,\delta)+1}(\mathbb{R})$,

$$\begin{aligned} X_0^2(\mathcal{G}(D_I)(X)) &\overset{(2.6)}{=} X_0^2 \left(\sum_{s,t=0}^{k(n,\delta)} d_{s,t} m_s^\delta(X) m_t^\delta(X) \right) \\ &= \sum_{s,t=0}^{k(n,\delta)} d_{s,t} m_s^{\delta+1}(X) m_t^{\delta+1}(X) \\ &= \sum_{s,t=0}^{k(n,\delta)+1} d_{s,t} m_s^{\delta+1}(X) m_t^{\delta+1}(X) \\ &\overset{(2.6)}{=} \mathcal{G}(D)(X) = f(X) = X_0^2 g(X). \end{aligned}$$

Consequently, D_I is a Gram matrix associated to g. We recall $f \notin C_i^\delta$ which allows us to fix $[y] \in V_i^\delta(\mathbb{R})$ such that $q_{D_I}(y) < 0$. We set $[z_y] := [y : 0 : \ldots : 0] \in \mathbb{P}^{k(n,\delta+1)}(\mathbb{R})$ and observe $[z_y] \in V_i^{\delta+1}(\mathbb{R})$ by Construction 3.2.1. Thus, we compute

$$q_D(z_y) = z_y D z_y^t = y D_I y^t = q_{D_I}(y) < 0$$

which contradicts q_D being locally PSD on $V_i^{\delta+1}(\mathbb{R})$. It follows $g \notin C_i^{\delta+1}$. ■

Remark 5.2.8. *Given a form $f \in \mathcal{F}_{n+1,2d}$ that testifies $C_i \subsetneq C_{i+1}$ as subcones of $\mathcal{P}_{n+1,2d}$, the above proof showed that the form $g(X) := X_0^{2(\delta-d)} f(X) \in \mathcal{F}_{n+1,2\delta}$ testifies $C_i \subsetneq C_{i+1}$ as subcones of $\mathcal{P}_{n+1,2\delta}$ for $\delta \geq d$.*

Corollary 5.2.9. *For $(n+1, 6)_{n \geq 3}$ and $i = n, \ldots, k(n,2) - n - 1$, the inclusion $C_i \subsetneq C_{i+1}$ is strict.*

Proof. Theorem 5.2.2 states $C_i \subsetneq C_{i+1}$ as subcones of $\mathcal{P}_{n+1,4}$. Therefore, applying Theorem 5.2.7 with $\delta = 3$, we also know that $C_i \subsetneq C_{i+1}$ as subcones of $\mathcal{P}_{n+1,6}$. ■

Example 5.2.10. Quaternary Sextics

Let $n = 3$, $\delta = 3$ and $d = 2$. We compute $k(3,3) = 19$, $k(3,3) - 3 = 16$ and $k(3,2) = 9$, $k(3,2) - 3 = 6$. Setting $i := 5$, Corollary 5.2.9 implies that the inclusion $C_5 \subsetneq C_6$ as subcones of $\mathcal{P}_{4,6}$ is strict. To illustrate why that is, we adapt the notations from the proof of Theorem 5.2.7 and recall that C testifies $C_5 \subsetneq C_6$ as subcones of $\mathcal{P}_{4,4}$ by Theorem 5.1.2. Following the proof of Theorem 5.2.7, we set

$$g(X) := X_0^2 \mathsf{C}(X) = X_0^4 X_1^2 + X_0^4 X_2^2 + X_0^2 X_1^2 X_2^2 + X_0^2 X_3^4 - 4 X_0^3 X_1 X_2 X_3 \in \mathcal{F}_{4,6}.$$

Remark 5.1.3 (ii) states that a Gram matrix $A := (a_{s,t})_{0 \leq s,t \leq 9}$ associated to C that is locally PSD on $V_6^2(\mathbb{R})$ is given by

$$a_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{1,2,5,9\} \\ -2, & \text{if } \{s,t\} = \{2,6\} \\ 0, & \text{else.} \end{cases}$$

We therefore set $B := (b_{s,t})_{0 \leq s,t \leq 19} \in \mathrm{Sym}_{19}(\mathbb{R})$ to be given by

$$b_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{1,2,5,9\} \\ -2, & \text{if } \{s,t\} = \{2,6\} \\ 0, & \text{else} \end{cases}$$

and observe, using Lemma 2.3.19, that B is a Gram matrix associated to g. Let us point out that Lemma 2.3.15 also yields $B \in \mathcal{G}^{-1}(\mathsf{C})$. Recalling Construction 3.3.7, Lemma 3.3.9 and Example 3.3.11, we also know $V_6^2 = \overline{\phi(K_6^2)}$ and $V_6^3 = \overline{\phi(K_6^3)}$ for

$$\begin{aligned} K_6^2 &= \left\{\left(x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2\right) \mid x_1, x_2, x_3 \in \mathbb{C}\right\}, \\ K_6^3 &= \left\{\left(x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, z_{10}, \ldots, z_{19}\right) \mid \right. \\ &\qquad\qquad \left. x_1, x_2, x_3, z_{10}, \ldots, z_{19} \in \mathbb{C}\right\}, \end{aligned}$$

respectively. Using that C is PSD, we hence deduce for $[z] \in \phi(K_6^3)(\mathbb{R})$ that

$$\begin{aligned} q_B(z) &= x_1^2 + x_2^2 + x_1^2x_2^2 + x_3^4 - 4x_1x_2x_3 \\ &= g(1, x_1, x_2, x_3) \\ &= \mathsf{C}(1, x_1, x_2, x_3) \geq 0. \end{aligned}$$

We point out that the local PSD property of q_A on $V_6^2(\mathbb{R})$ instead of the PSD property of C also yields

$$q_B(z) = x_1^2 + x_2^2 + x_1^2x_2^2 + x_3^4 - 4x_1x_2x_3 = q_A(y_z) \geq 0$$

for $[z] \in \phi(K_6^3)(\mathbb{R})$ with $[y_z] \in \phi(K_6^2)(\mathbb{R})$ Either way, $g \in C_{\phi(K_6^3)} = C_6^3$ follows.

To give an intuition for $g \notin C_5^3$, we let $D := (d_{s,t})_{0 \leq s,t \leq 19} \in \mathrm{Sym}_{20}(\mathbb{R})$ be given by

$$d_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{1,5,9\} \\ \frac{1}{2}, & \text{if } \{s,t\} = \{0,7\} \\ -2, & \text{if } \{s,t\} = \{2,6\} \\ -6, & \text{if } s = t = 3 \\ 3, & \text{if } \{s,t\} = \{0,9\} \\ 0, & \text{else.} \end{cases}$$

Lemma 2.3.15 thus implies $D \in \mathcal{G}^1(g)$. We set $I := \{0, \ldots, 9\}$ and observe that $A_D := (a^D_{s,t})_{0 \leq s,t \leq 9} := D_I \in \mathrm{Sym}_{10}(\mathbb{R})$ is given by

$$a^D_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{1,5,9\} \\ \frac{1}{2}, & \text{if } \{s,t\} = \{0,7\} \\ -2, & \text{if } \{s,t\} = \{2,6\} \\ -6, & \text{if } s = t = 3 \\ 3, & \text{if } \{s,t\} = \{0,9\} \\ 0, & \text{else.} \end{cases}$$

Lemma 2.3.15 yields $A_D \in \mathcal{G}^{-1}(\mathsf{C})$ and, recalling Construction 3.3.7, Lemma 3.3.9 and Example 3.3.11, we also know $V_5^2 = \overline{\phi(K_5^2)}$ and $V_5^3 = \overline{\phi(K_5^3)}$ for

$$\begin{aligned} K_5^2 &= \left\{ \left(x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, z_9 \right) \mid x_1, x_2, x_3, z_9 \in \mathbb{C} \right\}, \\ K_5^3 &= \left\{ \left(x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, z_9, \ldots, z_{19} \right) \mid \right. \\ &\qquad\qquad\qquad\qquad \left. x_1, x_2, x_3, z_9, \ldots, z_{19} \in \mathbb{C} \right\}, \end{aligned}$$

respectively. The proof of Theorem 5.1.2 moreover showed $\mathsf{C} \notin C_5^2$. Hence, there exists some $[y] \in V_5^2(\mathbb{R})$ such that $q_{A_D}(y) < 0$. For example, $q_{A_D}(y) = (-1) < 0$ for

$$[y] := [1:1:1:1:1:1:1:1:1:0] \in \phi(K_5^2)(\mathbb{R})$$

and, consequently, also $q_D(z_y) = (-1) < 0$ for

$$[z_y] := [y:0:\ldots:0] = [1:1:1:1:1:1:1:1:1:0:0:\ldots:0] \in \phi(K_5^3)(\mathbb{R}).$$

Observation 5.2.11. $(n+1)$-ary Sextics $(n \geq 3)$
For $n \geq 3$, let $d = 3$ and observe that the specific cone filtration ($\mathcal{CF}$) is given by

$$\Sigma_{n+1,6} = C_0 \subseteq \ldots \subseteq C_n \subseteq \ldots \subseteq C_{k(n,2)-n} \subseteq \ldots \subseteq C_{k(n,3)-n} = \mathcal{P}_{n+1,6}. \tag{5.14}$$

Theorem 4.2.7 and Corollary 5.2.9 together imply $C_0 = \ldots = C_n \subsetneq \ldots \subsetneq C_{k(n,2)-n}$ and thus it remains to determine each strict inclusion in $C_{k(n,2)-n} \subseteq \ldots \subseteq C_{k(n,3)-n}$.

The proof below generalizes the consideration of Theorem 5.1.6 from ternary sextics to $(n+1)$-ary sextics for $n \geq 3$.

Theorem 5.2.12. *For $(n+1, 6)_{n \geq 3}$ and $i = k(n,2) - n, \ldots, k(n,3) - n - 1$, the inclusion $C_i \subsetneq C_{i+1}$ is strict.*

Proof. For $i = k(n,2) - n, \ldots, k(n,3) - n - 1$, we distinguish the five cases:

(i) $m_{n+i}(X) = m_{k(n,2)}(X)$

(ii) $m_{n+i}(X) = X_j^2 X_l$ for some $1 \leq j \leq l \leq n-1$

(iii) $m_{n+i}(X) = X_j X_l X_n$ for some $1 \leq j \leq l \leq n-1$

(iv) $m_{n+i}(X) = X_j X_n^2$ for some $1 \leq j \leq n-1$

(v) $m_{n+i}(X) = X_j X_l X_r$ for some $1 \leq j < l \leq r \leq n-1$

These cases can be treated similarly as the ones in the proof of Theorem 5.2.2 by a reduction to ternary (for Case (i) – (iv)), respectively, quaternary (for Case (v)) sextics and subsequent extensions to $\mathcal{F}_{n+1,6}$. These are:

(i) $g(X_0, \ldots, X_n) := f(X_0, X_1, X_n)$ for some ternary sextic $f \in C_4 \backslash C_3$

(ii) $g(X_0, \ldots, X_n) := f(X_0, X_j, X_{l+1})$ for some ternary sextic $f \in C_5 \backslash C_4$

(iii) $g(X_0, \ldots, X_n) := f(X_0, X_j, X_{l+1})$ for some ternary sextic $f \in C_6 \backslash C_5$

(iv) $g(X_0, \ldots, X_n) := f(X_0, X_j, X_{j+1})$ for some ternary sextic $f \in C_7 \backslash C_6$

(v) $g(X_0, \ldots, X_n) := f(X_0, X_j, X_l, X_{r+1})$ for some quaternary sextic $f \in C_{11} \backslash C_{10}$

An example of a quaternary sextic in $C_{11} \backslash C_{10}$ is given by

$$f(X) := X_1^2 \mathsf{C}^\tau(X) = X_1^4 X_2^2 + X_1^4 X_3^2 + X_1^2 X_2^2 X_3^2 + X_0^4 X_1^2 - 4X_0 X_1^3 X_2 X_3. \tag{5.15}$$

The separating property of $f \in C_{11} \backslash C_{10}$ may be verified by an argument similar to the one given in the detailed proof of Theorem 5.1.2, which involves plenty straight forward yet lengthy computations. For a more elegant proof of $f \in C_{11} \backslash C_{10}$, using a more advanced method that will be developed in Chapter 6, we point a curious reader to Example 6.2.13. ∎

Remark 5.2.13. *Case (v) treats the inclusion $C_i \subseteq C_{i+1}$ with $m_{n+i}(X) = X_j X_l X_r$ and $m_{n+i+1}(X) = X_j X_l X_{r+1}$ for some $1 \leq j < l \leq r \leq n-1$. Hence, in order to handle up to four pairwise distinct variables X_j, X_l, X_r and X_{r+1}, we need to reduce to quaternary sextics, instead of ternary sextics, in the above proof.*

Theorem 5.2.14. *For $(n+1,6)_{n\geq 3}$ and $i = n, \ldots, k-n-1$, the inclusion $C_i \subsetneq C_{i+1}$ is strict.*

Proof. Corollary 5.2.9 and Theorem 5.2.12 together yield

$$C_n \subsetneq \ldots \subsetneq C_{k(n,2)-n} \subsetneq \ldots \subsetneq C_{k(n,3)-n} = \mathcal{P}_{n+1,6}.$$ ∎

Corollary 5.2.15. *For $(n+1,6)_{n\geq 3}$, it holds*

$$\mu(n,3) = k(n,3) - 2n - 1 = \frac{n(n^2+6n-1)}{6} - 1.$$

Proof. Observation 5.2.11 and Theorem 5.2.14 together yield

$$\Sigma_{n+1,6} = C_0 = \ldots = C_n \subsetneq \ldots \subsetneq C_{k(n,2)-n} \subsetneq \ldots \subsetneq C_{k(n,3)-n} = \mathcal{P}_{n+1,6}. \tag{5.16}$$

Hence, there are $k(n,3)-2n-1$ strictly separating intermediate cones between $\Sigma_{n+1,6}$ and $\mathcal{P}_{n+1,6}$ in (5.16), namely, $C_{n+1},\ldots,C_{k(n,3)-n-1}$. This gives

$$\mu(n,3) = k(n,3) - 2n - 1 = \binom{n+3}{3} - 2n - 2 = \frac{n(n^2+6n-1)}{6} - 1. \qquad \blacksquare$$

Corollary 5.2.16. *Let $\{f_1,f_2,f_3,f_4\}$ be a complete set of separating forms for $(\mathcal{CF})$ such that $i(f_j) = j+2$ for $j=1,\ldots,4$ in the basic non-Hilbert case $(3,6)$. Moreover, let $g \in \Delta_{4,6}$ be such that $g \in C_{11}\backslash C_{10}$ in the non-Hilbert case $(4,6)$. For $n \geq 3$, let $\{h_1,\ldots,h_{\mu(n,2)+1}\}$ be a complete set of separating forms for $(\mathcal{CF})$ in the non-Hilbert case $(n+1,4)$. Then a complete set of separating forms for $(\mathcal{CF})$ in the non-Hilbert case $(n+1,6)$ is given by $\bigcup_{j=1}^{4} F_j \cup G \cup H$ where*

$$\begin{aligned}
F_1 &:= \{f_1(X_0,X_1,X_n)\},\\
F_2 &:= \{f_2(X_0,X_j,X_{l+1}) \mid 1 \leq j \leq l \leq n-1\},\\
F_3 &:= \{f_3(X_0,X_j,X_{l+1}) \mid 1 \leq j \leq l \leq n-1\},\\
F_4 &:= \{f_4(X_0,X_j,X_{j+1}) \mid j = 1,\ldots,n-1\},\\
G &:= \{g(X_0,X_jX_l,X_{r+1}) \mid 1 \leq j \leq l \leq r \leq n-1\},\\
H &:= \{X_0^2 h_j(X) \mid j = 1,\ldots,\mu(n,2)+1\}.
\end{aligned}$$

Proof. The proof of Theorem 5.2.12 is constructive and can be carried out using the forms f_1,f_2,f_3,f_4 and g. This gives us the sets

$$\begin{aligned}
F_1 &:= \{f_1(X_0,X_1,X_n)\},\\
F_2 &:= \{f_2(X_0,X_j,X_{l+1}) \mid 1 \leq j \leq l \leq n-1\},\\
F_3 &:= \{f_3(X_0,X_j,X_{l+1}) \mid 1 \leq j \leq l \leq n-1\},\\
F_4 &:= \{f_4(X_0,X_j,X_{j+1}) \mid j = 1,\ldots,n-1\},\\
G &:= \{g(X_0,X_jX_l,X_{r+1}) \mid 1 \leq j \leq l \leq r \leq n-1\}.
\end{aligned}$$

Remark 5.2.8 moreover implies that the forms in H testify the strictness of the inclusions in $C_n \subsetneq \ldots \subsetneq C_{k(n,2)-n}$. By construction, we conclude $i(f) \neq i(g)$ for distinct

$f, g \in \bigcup_{j=1}^{4} F_j \cup G \cup H$. Recalling Corollary 5.2.15, we therefore have

$$\begin{aligned} |\{i(f) \mid f \in \bigcup_{j=1}^{4} F_j \cup G \cup H\}| &= |F_1| + |F_2| + |F_3| + |F_4| + |G| + |H| \\ &= 1 + \frac{n(n-1)}{2} + \frac{n(n-1)}{2} + (n-1) \\ &\quad + \frac{n(n^2-3n+2)}{6} + (\mu(n,2)+1) \\ &= \frac{n(n^2+6n-1)}{6} \\ &= \mu(n,3) + 1. \end{aligned}$$

Hence, $\bigcup_{j=1}^{4} F_j \cup G \cup H$ is a complete set of separating forms for ($\mathcal{CF}$) in the non-Hilbert case $(n+1, 6)$. ■

Corollary 5.2.17. *In the non-Hilbert case $(n+1, 6)$ for $n \geq 3$, a complete set of separating forms for ($\mathcal{CF}$) is given by $\bigcup_{j=1}^{4} F_j \cup G \cup (H_1 \cup H_2 \cup H_3)$ where*

$$\begin{aligned} F_1 &:= \{\mathsf{M}^\sigma(X_0, X_1, X_n)\}, \\ F_2 &:= \{\mathsf{L}(X_0, X_j, X_{l+1}) \mid 1 \leq j \leq l \leq n-1\}, \\ F_3 &:= \{\mathsf{L}^\sigma(X_0, X_j, X_{l+1}) \mid 1 \leq j \leq l \leq n-1\}, \\ F_4 &:= \{\mathsf{M}(X_0, X_j, X_{j+1}) \mid j = 1, \ldots, n-1\}, \\ G &:= \{X_1^2 \mathsf{C}^\tau(X_0, X_j, X_l, X_{r+1}) \mid 1 \leq j \leq l \leq r \leq n-1\}, \\ H_1 &:= \{X_0^2 \mathsf{C}^\sigma(X_0, X_j, X_{j+1}, X_n) \mid j = 1, \ldots, n-2\}, \\ H_2 &:= \{X_0^2 \mathsf{C}^\tau(X_0, X_1, X_j, X_{l+1}) \mid 2 \leq j \leq l \leq n-1\}, \\ H_3 &:= \{X_0^2 \mathsf{C}(X_0, X_1, X_{n-1}, X_n)\}. \end{aligned}$$

Proof. Corollary 5.1.8 states that $\{\mathsf{M}^\sigma, \mathsf{L}, \mathsf{L}^\sigma, \mathsf{M}\}$ is a complete set of separating forms for ($\mathcal{CF}$) in the basic non-Hilbert case $(3, 6)$ and $i(\mathsf{M}^\sigma) = 3$, $i(\mathsf{L}) = 4$, $i(\mathsf{L}^\sigma) = 5$, $i(\mathsf{M}) = 6$ by the proof of Theorem 5.1.6. Moreover, Corollary 5.2.5 yields that a complete set of separating forms for ($\mathcal{CF}$) in the non-Hilbert case $(n+1, 4)$ is given by $\mathfrak{H}_1 \cup \mathfrak{H}_2 \cup \mathfrak{H}_3$ where

$$\begin{aligned} \mathfrak{H}_1 &:= \{\mathsf{C}^\sigma(X_0, X_j, X_{j+1}, X_n) \mid j = 1, \ldots, n-2\}, \\ \mathfrak{H}_2 &:= \{\mathsf{C}^\tau(X_0, X_1, X_j, X_{l+1}) \mid 2 \leq j \leq l \leq n-1\}, \\ \mathfrak{H}_3 &:= \{\mathsf{C}(X_0, X_1, X_{n-1}, X_n)\}. \end{aligned}$$

The assertion follows from Corollary 5.2.16 by setting $f_1 := \mathsf{M}^\sigma$, $f_2 := \mathsf{L}$, $f_3 := \mathsf{L}^\sigma$, $f_4 := \mathsf{M}$, $g(X) := X_1^2 \mathsf{C}^\tau(X)$ (cf. (5.15)) and $\{h_1, \ldots, h_{\mu(n,2)+1}\} := \mathfrak{H}_1 \cup \mathfrak{H}_2 \cup \mathfrak{H}_3$. ■

Example 5.2.18. Quaternary Sextics
Let $n = 3$, $d = 3$ and we compute $k(3,3) = 19$, $k(3,3) - 3 = 16$ and $k(3,2) = 9$, $k(3,2) - 3 = 6$. Following the proof of Theorem 5.2.12, we now construct a separating form $g \in C_{12} \backslash C_{11}$. In the notations of that proof, we have $i = 11$ and determine $m_{n+i}(X) = m_{14}(X) = X_1X_2X_3$. We are consequently in the situation of Case (iii) with $j = 1$ and $l = 2$. Hence, we reconsider the separating ternary sextic

$$\mathsf{L}^\sigma(X_0, X_1, X_2) = X_0^4X_2^2 + X_0^2X_1^4 + X_1^2X_2^4 - 3X_0^2X_1^2X_2^2 \in C_6 \backslash C_5$$

from the proof of Theorem 5.1.6 and set

$$\begin{aligned} g(X_0, X_1, X_2, X_3) &:= \mathsf{L}^\sigma(X_0, X_1, X_3) \\ &= X_0^4X_3^2 + X_0^2X_1^4 + X_1^2X_3^4 - 3X_0^2X_1^2X_3^2. \end{aligned}$$

According to the proof of Theorem 5.2.12, $g \in \mathcal{F}_{4,6}$ thus testifies $C_{11} \subsetneq C_{12}$. We now indicate why that is by following the proof of Theorem 5.2.12, which is based on the two-step-proof of Theorem 5.2.2. For this purpose, we denote the subcone C_5 of $\mathcal{P}_{4,6}$ by C_5^3. Likewise, we set K_5^2, K_6^2 to be the affine subvariety K_5, K_6 of $\mathbb{C}^{k(2,3)}$ and K_{11}^3, K_{12}^3 to be the affine subvariety K_{11}, K_{12} of $\mathbb{C}^{k(3,3)}$, respectively. Moreover, for $j = 0, \ldots, 19$, we let α_j^3 be the exponent $\alpha_j \in I_{4,3}$ and compute

$$\begin{array}{llll} \alpha_0^3 = (3,0,0,0), & \alpha_5^3 = (1,1,1,0), & \alpha_{10}^3 = (0,3,0,0), & \alpha_{15}^3 = (0,1,0,2), \\ \alpha_1^3 = (2,1,0,0), & \alpha_6^3 = (1,1,0,1), & \alpha_{11}^3 = (0,2,1,0), & \alpha_{16}^3 = (0,0,3,0), \\ \alpha_2^3 = (2,0,1,0), & \alpha_7^3 = (1,0,2,0), & \alpha_{12}^3 = (0,2,0,1), & \alpha_{17}^3 = (0,0,2,1), \\ \alpha_3^3 = (2,0,0,1), & \alpha_8^3 = (1,0,1,1), & \alpha_{13}^3 = (0,1,2,0), & \alpha_{18}^3 = (0,0,1,2), \\ \alpha_4^3 = (1,2,0,0), & \alpha_9^3 = (1,0,0,2), & \alpha_{14}^3 = (0,1,1,1), & \alpha_{19}^3 = (0,0,0,3). \end{array}$$

Hence, $I := \{i \in \{0, \ldots, 19\} \mid \alpha_{i,s}^3 = 0 \text{ for } s \neq 0,1,3\} = \{0,1,3,4,6,9,10,12,15,19\}$.

(1) Using Remark 5.1.7 (iii), we observe that for $A := (a_{s,t})_{0 \leq s,t \leq 9} \in \mathcal{G}^{-1}(\mathsf{L}^\sigma)$ with

$$a_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{2,3,8\} \\ -3, & \text{if } s = t = 4 \\ 0, & \text{else,} \end{cases}$$

q_A is locally PSD on $\phi(K_6^2)(\mathbb{R})$. Applying Lemma 2.3.17 (i), we therefore expand A to the Gram matrix $B := (b_{s,t})_{0 \leq s,t \leq 19}$ associated to g that is given by

$$b_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{3,4,15\} \\ -3, & \text{if } s = t = 6 \\ 0, & \text{else.} \end{cases}$$

Example 3.3.11 and Lemma 3.3.9 moreover give

$$\begin{aligned} K_6^2 &= \left\{\left(x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, z_9\right) \mid x_1, x_2, z_9 \in \mathbb{C}\right\}, \\ K_{12}^3 &= \left\{\left(x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2,\right.\right. \\ &\qquad \left.\left. x_1x_2x_3, x_1x_3^2, z_{16}, \dots, z_{19}\right) \mid x_1, x_2, x_3, z_{16}, \dots, z_{19} \in \mathbb{C}\right\}. \end{aligned}$$

We set

$$\begin{aligned} \pi\colon \quad \mathbb{C}^{20} &\to \mathbb{C}^{10} \\ z &\mapsto y_z := (z_0, z_1, z_3, z_4, z_6, z_9, z_{10}, z_{12}, z_{15}, z_{19}) \end{aligned}$$

and thus observe for $[z] \in \phi(K_{12}^3)(\mathbb{R})$ that

$$\begin{aligned} [y_z] &= \left[\pi\left(1, x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2,\right.\right. \\ &\qquad \left.\left. x_1x_2x_3, x_1x_3^2, z_{16}, \dots, z_{19}\right)\right] \\ &= \left[1 : x_1 : x_3 : x_1^2 : x_1x_3 : x_3^2 : x_1^3 : x_1^2x_3 : x_1x_3^2 : z_{19}\right] \in \phi(K_6^2)(\mathbb{R}). \end{aligned}$$

Hence, using that L^σ is PSD, we compute for $[z] \in \phi(K_{12}^3)(\mathbb{R})$ that

$$q_B(z) = x_3^2 + x_1^4 + x_1^2x_3^4 - 3x_1^2x_3^2 = g(1, x_1, x_2, x_3) = \mathsf{L}^\sigma(1, x_1, x_3) \geq 0.$$

We point out that the local PSD property of q_A on $\phi(K_6^2)(\mathbb{R})$ instead of the PSD property of L^σ also yields

$$\begin{aligned} q_B(z) &= x_3^2 + x_1^4 + x_1^2x_3^4 - 3x_1^2x_3^2 \\ &= q_A\left(1, x_1, x_3, x_1^2, x_1x_3, x_3^2, x_1^3, x_1^2x_3, x_1x_3^2, z_{19}\right) \geq 0 \end{aligned}$$

for $[z] \in \phi(K_{12}^3)(\mathbb{R})$. Either way, $g \in C_{12}$ follows.

(2) To motivate $g \notin C_{11}$, we now consider the matrix $B := (b_{s,t})_{0 \leq s,t \leq 19} \in \mathrm{Sym}_{15}(\mathbb{R})$ that is given by

$$b_{s,t} := \begin{cases} 1, & \text{if } s = t \in \{3, 15\} \\ \frac{1}{2}, & \text{if } \{s,t\} = \{1, 10\} \\ -\frac{3}{2}, & \text{if } \{s,t\} = \{4, 9\} \\ -2, & \text{if } s = t = 12 \\ 1, & \text{if } \{s,t\} = \{10, 15\} \\ 4, & \text{if } s = t = 5 \\ -2, & \text{if } \{s,t\} = \{1, 13\} \\ 0, & \text{else} \end{cases}$$

and observe that B is a Gram matrix associated to g by Lemma 2.3.15. Hence, by Lemma 2.3.17 (ii), the matrix $A := (a_{s,t})_{0\leq s,t\leq 9} \in \mathrm{Sym}_{10}(\mathbb{R})$ given by

$$a_{s,t} := \begin{cases} 1, & \text{if } s=t \in \{2,8\} \\ \frac{1}{2}, & \text{if } \{s,t\}=\{1,6\} \\ -\frac{3}{2}, & \text{if } \{s,t\}=\{3,5\} \\ -2, & \text{if } s=t=7 \\ 1, & \text{if } \{s,t\}=\{6,8\} \\ 0, & \text{else} \end{cases}$$

is a Gram matrix associated to L^σ. Comparing with Lemma 2.3.15, we see that this is indeed true. From the proof of Theorem 5.1.6, we furthermore know $\mathsf{L}^\sigma \notin C_5^2 = C_{\phi(K_5^2)}$. Hence, there exists some $[y] \in \phi(K_5^2)(\mathbb{R})$ such that $q_A(z) < 0$. Indeed, recalling Example 3.3.11 and applying Lemma 3.3.9, we see

$$\begin{aligned} K_5^2 &= \Big\{\Big(x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, z_8, z_9\Big) \mid x_1, x_2, z_8, z_9 \in \mathbb{C}\Big\}, \\ K_{11}^3 &= \Big\{\Big(x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2, \\ &\qquad x_1x_2x_3, z_{15}, \ldots, z_{19}) \mid x_1, x_2, x_3, z_{15}, \ldots, z_{19} \in \mathbb{C}\} \end{aligned}$$

and compute $q_A(y) = (-3) < 0$ for

$$[y] := [1:1:1:1:1:1:1:1:0:0] \in \phi(K_5^2)(\mathbb{R}).$$

Expanding $[y]$ by zero entries into

$$[z_y] = [1:1:0:1:1:0:1:0:0:1:1:0:1:0:0:0:0:0:0:0] \in \phi(K_{11}^3)(\mathbb{R}),$$

we thus find $q_B(z_y) = (-3) < 0$.

Concluding Remark. *Altogether, we answered the main query in the non-Hilbert cases $(n+1,4)_{n\geq 4}$ and $(n+1,6)_{n\geq 3}$ by specifying the cone filtrations (5.12) and (5.14) to be given by*

$$\Sigma_{n+1,2d} = C_0 = \ldots = C_n \subsetneq \ldots \subsetneq C_{k(n,2)-n} = \mathcal{P}_{n+1,2d}. \tag{5.17}$$

This allowed us to deduce $\mu(n,d) = k(n,d) - 2n - 1$ for $(n+1,4)_{n\geq 4}$ and $(n+1,6)_{n\geq 3}$.

Chapter 6

The Non-Hilbert Cases $(n+1,2d)_{n\geq 2,d\geq 4}$

In this chapter, we answer the main query in the non-Hilbert cases $(n+1,2d)_{n\geq 2,d\geq 4}$. To this end, we firstly introduce circuit forms in Section 6.1 and examine them for their basic properties. Moreover, we put them into the context of this thesis.

Secondly, in Section 6.2, we establish a sufficient criterion that allows us to determine the greatest cone w.r.t. $\subseteq$ in our specific cone filtration to which an a priori fixed circuit form, that spans an extreme ray of $\mathcal{P}_{n+1,2d}$ but which is not SOS, belongs.

We thirdly apply this criterion in Section 6.3 to establish a second degree-jumping principle that allows us to algorithmically construct complete sets of separating forms for ($\mathcal{CF}$) in the non-Hilbert cases $(n+1,2d)_{n\geq 2,d\geq 4}$. We thereby also determine the number of strictly separating intermediate cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ for the non-Hilbert cases $(n+1,2d)_{n\geq 2,d\geq 4}$.

6.1 Preliminaries: Circuit Forms

The PSD property of the Choi–Lam quaternary quartic, the Choi–Lam ternary sextic, respectively, the Motzkin ternary sextic follows from the *arithmetic–geometric mean inequality* which states for $m\in\mathbb{N}$ and $a_1,\ldots,a_m\in\mathbb{R}_{\geq 0}$ that

$$\frac{1}{m}\left(\sum_{j=1}^{m} a_j\right) \geq \left(\prod_{j=1}^{m} a_j\right)^{\frac{1}{m}}.$$

Example 6.1.1. Let us examine the Choi–Lam quaternary quartic C, the Choi–Lam ternary sextic L and the Motzkin ternary sextic M for the PSD property.

(i) For $\mathsf{C}(X_0,X_1,X_2,X_3) = X_0^2X_1^2 + X_0^2X_2^2 + X_1^2X_2^2 + X_3^4 - 4X_0X_1X_2X_3$, we set $m:=4$ and substitute $a_1 := x_0^2x_1^2$, $a_2 := x_0^2x_2^2$, $a_3 := x_1^2x_2^2$, $a_4 := x_3^4$ for $x\in\mathbb{R}^4$

in the arithmetic–geometric mean inequality. This gives us

$$\frac{1}{4}\left(x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2 + x_3^4\right) \geq \left(x_0^4x_1^4x_2^4x_3^4\right)^{\frac{1}{4}} = x_0x_1x_2x_3.$$

(ii) For $\mathsf{L}(X_0, X_1, X_2) := X_0^4X_1^2 + X_0^2X_2^4 + X_1^4X_2^2 - 3X_0^2X_1^2X_2^2$, we set $m := 3$ and substitute $a_1 := x_0^4x_1^2$, $a_2 := x_0^2x_2^4$, $a_3 := x_1^4x_2^2$ for $x \in \mathbb{R}^3$ in the arithmetic–geometric mean inequality. This gives us

$$\frac{1}{3}\left(x_0^4x_1^2 + x_0^2x_2^4 + x_1^4x_2^2\right) \geq \left(x_0^6x_1^6x_2^6\right)^{\frac{1}{3}} = x_0^2x_1^2x_2^2.$$

(iii) For $\mathsf{M}(X_0, X_1, X_2) := X_0^4X_1^2 + X_0^2X_1^4 + X_2^6 - 3X_0^2X_1^2X_2^2$, we set $m := 3$ and substitute $a_1 := x_0^4x_1^2$, $a_2 := x_0^2x_1^4$, $a_3 := x_2^6$ for $x \in \mathbb{R}^3$ in the arithmetic–geometric mean inequality. This gives us

$$\frac{1}{3}\left(x_0^4x_1^2 + x_0^2x_1^4 + x_2^6\right) \geq \left(x_0^6x_1^6x_2^6\right)^{\frac{1}{3}} = x_0^2x_1^2x_2^2.$$

Casually speaking, we see that C, L and M are forms that come from substituting adequate monomials in the arithmetic–geometric mean inequality. Forms of this kind are called *agiforms*. More precisely, $f \in \mathcal{F}_{n+1,2d}$ is an *agiform* if there exists some $A := \{a_1, \ldots, a_r\} \subseteq I_{n+1,2d}$ $(r \in \mathbb{N})$ such that any $a \in A$ is even[1] and there exists some $b \in \operatorname{conv}(A) \cap \mathbb{Z}^{n+1}$ with barycentric coordinates[2] $\lambda_0, \ldots, \lambda_r$ such that

$$f(X) = \sum_{j=0}^{r} \lambda_j X^{a_j} - X^b.$$

Recalling that the *support* of a form $f(X) = \sum\limits_{\beta \in I_{n+1,2d}} f_\beta X^\beta \in \mathcal{F}_{n+1,2d}$ is given by

$$\operatorname{supp}(f) := \{\beta \in I_{n+1,2d} \mid f_\beta \neq 0\},$$

we thus see that the PSD property of agiforms follows from the simple structure of their supports. Bruce Reznick [Rez89] extensively studied agiforms by an investigation of their supports, which allowed him to provide a necessary condition for an agiform to be SOS and a necessary and sufficient condition for an agiform to span an extreme ray of $\mathcal{P}_{n+1,2d}$. The idea behind agiforms can be generalized to a wider class of sparse forms, which brings us to the definition of circuit forms below. We refer an interested reader to [IW16] for a concise overview on circuit polynomials.

Definition 6.1.2. Let $f \in \mathcal{F}_{n+1,2d}$ and $A \subseteq I_{n+1,2d}$ be such that $\operatorname{supp}(f) \subseteq A$. If all elements in the set $\operatorname{vert}(A)$ of vertices of the convex hull of A are even and if

[1] We call $a \in \mathbb{Z}^{n+1}$ *even* if there exists some $b \in \mathbb{Z}^{n+1}$ such that $a = 2b$.

[2] That is, non-negative real numbers $\lambda_0, \ldots, \lambda_r$ such that $\sum\limits_{j=0}^{r} \lambda_j = 1$ and $b = \sum\limits_{j=0}^{r} \lambda_j a_j$.

$$f(X) = \sum_{j=0}^{r} f_{a(j)} X^{a(j)} + f_b X^b$$

for some $r \leq n+1$, pairwise distinct exponents $a(0), \ldots, a(r) \in A$ with corresponding $f_{a(0)}, \ldots, f_{a(r)} \in \mathbb{R}_{>0}$ and exponent $b \in A$ with corresponding $f_b \in \mathbb{R}$ such that

(i) $\mathrm{vert}(A) = \{a(0), \ldots, a(r)\}$,

(ii) $a(0), \ldots, a(r)$ are affinely independent and

(iii) there exist unique $\lambda_0, \ldots, \lambda_r > 0$ with $\sum_{j=0}^{r} \lambda_j = 1$ and $b = \sum_{j=0}^{r} \lambda_j a(j)$,

then f is a *circuit form* with *outer exponents* $a(0), \ldots, a(r)$ and *inner exponent* b. The corresponding terms $f_{a(0)} X^{a(0)}, \ldots, f_{a(r)} X^{a(r)}$ and $f_b X^b$ are called *outer terms* and *inner term* of f, respectively. The *circuit number* of f is given by

$$\Theta_f := \prod_{j=0}^{r} \left(\frac{f_{a(j)}}{\lambda_j} \right)^{\lambda_j}.$$

Remark 6.1.3. *Setting $f_{a(j)} := \lambda_j$ for $j = 0, \ldots, r$ and $f_b := (-1)$, we recover an agiform as a special case of a circuit form.*

Notation 6.1.4. We denote the set of circuit forms in $\mathcal{F}_{n+1,2d}$ by $\mathfrak{C}_{n+1,2d}$ and write

$$\begin{aligned} \mathcal{P}^{\mathfrak{C}}_{n+1,2d} &:= \mathcal{P}_{n+1,2d} \cap \mathfrak{C}_{n+1,2d}, \\ \Sigma^{\mathfrak{C}}_{n+1,2d} &:= \Sigma_{n+1,2d} \cap \mathfrak{C}_{n+1,2d}, \\ \Delta^{\mathfrak{C}}_{n+1,2d} &:= \Delta_{n+1,2d} \cap \mathfrak{C}_{n+1,2d}. \end{aligned}$$

Example 6.1.5. Let us examine the Choi–Lam quaternary quartic C, the Choi–Lam ternary sextic L and the Motzkin ternary sextic M for the circuit property.

(i) We recall that the Choi–Lam quaternary quartic is given by

$$\mathsf{C}(X_0, X_1, X_2, X_3) = X_0^2 X_1^2 + X_0^2 X_2^2 + X_1^2 X_2^2 + X_3^4 - 4 X_0 X_1 X_2 X_3.$$

In the notations of Definition 6.1.2, we therefore set $r := 3$, $a(0) := (2,2,0,0)$, $a(1) := (2,0,2,0)$, $a(2) := (0,2,2,0)$, $a(3) := (0,0,0,4)$ and $b := (1,1,1,1)$ with coefficients $\mathsf{C}_{a(0)} := \mathsf{C}_{a(1)} := \mathsf{C}_{a(2)} := \mathsf{C}_{a(3)} = 1$, $\mathsf{C}_b := (-4)$ and conclude that

$$\mathsf{C}(X) = \sum_{j=0}^{r} \mathsf{C}_{a(j)} X^{a(j)} + \mathsf{C}_b X^b$$

is of the appropriate structure. Moreover, we determine

$$\mathrm{supp}(\mathsf{C}) = \{(2,2,0,0), (2,0,2,0), (0,2,2,0), (0,0,0,4), (1,1,1,1)\}$$

and thus $\mathrm{vert}(\mathrm{supp}(\mathsf{C})) = \{(2,2,0,0),(2,0,2,0),(0,2,2,0),(0,0,0,4)\}$. Hence, any element in $\mathrm{vert}(\mathrm{supp}(\mathsf{C}))$ is even and $\mathrm{vert}(\mathrm{supp}(\mathsf{C})) = \{a(0),a(1),a(2),a(3)\}$. A straight forward computation furthermore shows that $a(0),\ldots,a(3)$ are affinely independent and that the unique $\lambda_0,\ldots,\lambda_3 > 0$ such that $\sum_{j=0}^{3}\lambda_j = 1$ and $b = \sum_{j=0}^{3}\lambda_j a(j)$ are given by $\lambda_0 := \lambda_1 := \lambda_2 := \lambda_3 := \frac{1}{4}$. Therefore, C is a circuit form with outer exponents $(2,2,0,0)$, $(2,0,2,0)$, $(0,2,2,0)$, $(0,0,0,4)$, inner exponent $(1,1,1,1)$ and circuit number $\Theta_{\mathsf{C}} = 4$.

(ii) We recall that the Choi–Lam ternary sextic is given by

$$\mathsf{L}(X_0,X_1,X_2) := X_0^4X_1^2 + X_0^2X_2^4 + X_1^4X_2^2 - 3X_0^2X_1^2X_2^2.$$

In the notations of Definition 6.1.2, we therefore set $r := 2$, $a(0) := (4,2,0)$, $a(1) := (2,0,4)$, $a(2) := (0,4,2)$, $b := (2,2,2)$, $\mathsf{L}_{a(0)} := \mathsf{L}_{a(1)} := \mathsf{L}_{a(2)} := 1$, $\mathsf{L}_b := (-3)$ and conclude that

$$\mathsf{L}(X) = \sum_{j=0}^{r}\mathsf{L}_{a(j)}X^{a(j)} + \mathsf{L}_bX^b$$

is of the appropriate structure. Moreover, we determine

$$\mathrm{supp}(\mathsf{L}) = \{(4,2,0),(2,0,4),(0,4,2),(2,2,2)\}$$

and thus $\mathrm{vert}(\mathrm{supp}(\mathsf{L})) = \{(4,2,0),(2,0,4),(0,4,2)\}$. Hence, any element in $\mathrm{vert}(\mathrm{supp}(\mathsf{L}))$ is even and $\mathrm{vert}(\mathrm{supp}(\mathsf{L})) = \{a(0),a(1),a(2)\}$. A straight forward computation shows that $a(0),a(1),a(2)$ are affinely independent and that the unique $\lambda_0,\lambda_1,\lambda_2 > 0$ such that $\sum_{j=0}^{2}\lambda_j = 1$ and $b = \sum_{j=0}^{2}\lambda_j a(j)$ are given by $\lambda_0 := \lambda_1 := \lambda_2 := \frac{1}{3}$. Therefore, L is a circuit form with outer exponents $(4,2,0)$, $(2,0,4)$, $(0,4,2)$, inner exponent $(2,2,2)$ and circuit number $\Theta_{\mathsf{L}} = 3$.

(iii) We recall that the Motzkin ternary sextic is given by

$$\mathsf{M}(X_0,X_1,X_2) := X_0^4X_1^2 + X_0^2X_1^4 + X_2^6 - 3X_0^2X_1^2X_2^2.$$

In the notations of Definition 6.1.2, we therefore set $r := 2$, $a(0) := (4,2,0)$, $a(1) := (2,4,0)$, $a(2) := (0,0,6)$, $b := (2,2,2)$, $\mathsf{M}_{a(0)} := \mathsf{M}_{a(1)} := \mathsf{M}_{a(2)} := 1$, $\mathsf{M}_b := (-3)$ and conclude that

$$\mathsf{M}(X) = \sum_{j=0}^{r}\mathsf{M}_{a(j)}X^{a(j)} + \mathsf{M}_bX^b$$

is of the appropriate structure. Moreover, we determine

$$\mathrm{supp}(\mathsf{M}) = \{(4,2,0),(2,4,0),(0,0,6),(2,2,2)\}$$

and thus $\mathrm{vert}(\mathrm{supp}(\mathsf{M})) = \{(4,2,0),(2,4,0),(0,0,6)\}$. Hence, any element in $\mathrm{vert}(\mathrm{supp}(\mathsf{M}))$ is even and $\mathrm{vert}(\mathrm{supp}(\mathsf{M})) = \{a(0),a(1),a(2)\}$. A straight forward computation shows that $a(0),a(1),a(2)$ are affinely independent and that the unique $\lambda_0,\lambda_1,\lambda_2 > 0$ such that $\sum_{j=0}^{2}\lambda_j = 1$ and $b = \sum_{j=0}^{2}\lambda_j a(j)$ are given by $\lambda_0 := \lambda_1 := \lambda_2 := \frac{1}{3}$. Therefore, M is a circuit form with outer exponents $(4,2,0)$, $(2,4,0)$, $(0,0,6)$, inner exponent $(2,2,2)$ and circuit number $\Theta_{\mathsf{M}} = 3$.

According to Remark 6.1.3, we in particular verified that C, L and M are agiforms.

Lemma 6.1.6. *Let $\sigma \in S_{n+1}$ be a permutation and $f \in \mathcal{F}_{n+1,2d}$. If $f \in \mathfrak{C}_{n+1,2d}$, then $f^\sigma(X) := f\left(X_{\sigma(0)},\ldots,X_{\sigma(n)}\right) \in \mathfrak{C}_{n+1,2d}$.*

Proof. In the scope of Definition 6.1.2, we assume that f is supported on an adequate $A \subseteq I_{n+1,2d}$ and write

$$f(X) = \sum_{j=0}^{r} f_{a(j)}X^{a(j)} + f_bX^b.$$

Moreover, we set $X^\sigma := \left(X_{\sigma(0)},\ldots,X_{\sigma(n)}\right)$ and define $a^\sigma := \left(a_{\sigma^{-1}(0)},\ldots,a_{\sigma^{-1}(n)}\right)$ for $a := (a_0,\ldots,a_n) \in I_{n+1,2d}$. Since f is supported on A and any element in $\mathrm{vert}(A)$ is even by the choice of A, we know that f^σ is supported on $A^\sigma := \{a^\sigma \in I_{n+1,2d} \mid a \in A\}$ and also any element in $\mathrm{vert}(A^\sigma) = \{a^\sigma \mid a \in \mathrm{vert}(A)\}$ is even. We furthermore set $f_{a(0)^\sigma} := f_{a(0)},\ldots,f_{a(r)^\sigma} := f_{a(r)} \in \mathbb{R}_{>0}$, $f_{b^\sigma} := f_b \in \mathbb{R}$ and conclude that

$$\begin{aligned} f^\sigma(X) &= f(X^\sigma) \\ &= \sum_{j=0}^{r} f_{a(j)}(X^\sigma)^{a(j)} + f_b(X^\sigma)^b \\ &= \sum_{j=0}^{r} f_{a(j)}X^{a(j)^\sigma} + f_bX^{b^\sigma} \\ &= \sum_{j=0}^{r} f_{a(j)^\sigma}X^{a(j)^\sigma} + f_{b^\sigma}X^{b^\sigma} \end{aligned}$$

with $r \le n+1$, pairwise distinct $a(0)^\sigma,\ldots,a(r)^\sigma \in A^\sigma$, $f_{a(0)^\sigma},\ldots,f_{a(r)^\sigma} \in \mathbb{R}_{>0}$, $b^\sigma \in A^\sigma$ and $f_b^\sigma \in \mathbb{R}$. Our choice of A especially secures $\mathrm{vert}(A) = \{a(0),\ldots,a(r)\}$ and thus we deduce $\mathrm{vert}(A^\sigma) = \{a^\sigma \mid a \in \mathrm{vert}(A)\} = \{a(0)^\sigma,\ldots,a(r)^\sigma\}$. Moreover, we know that $a(0)^\sigma,\ldots,a(r)^\sigma$ are affinely independent since $a(0),\ldots,a(r)$ are affinely independent and we conclude for the unique $\lambda_0,\ldots,\lambda_r > 0$ such that $\sum_{j=0}^{r}\lambda_j = 1$ and $b = \sum_{j=0}^{r}\lambda_j a(j)$ that $b^\sigma = \sum_{j=0}^{r}\lambda_j a(j)^\sigma$. As a matter of fact, for any $\lambda_0^\sigma,\ldots,\lambda_r^\sigma > 0$ such that $\sum_{j=0}^{r}\lambda_j^\sigma = 1$ and $b^\sigma = \sum_{j=0}^{r}\lambda_j^\sigma a(j)^\sigma$, we see that $b = \sum_{j=0}^{r}\lambda_j^\sigma a(j)$ which implies $\lambda_j = \lambda_j^\sigma$ for $j = 0,\ldots,r$ by the uniqueness of $\lambda_0,\ldots,\lambda_r$. ■

Lemma 6.1.7. *For $i = 0,\ldots,n$, $f \in \mathcal{F}_{n+1,2d}$ and $g(X) := X_i^2 f(X) \in \mathcal{F}_{n+1,2(d+1)}$, it holds $f \in \mathfrak{C}_{n+1,2d}$ if and only if $g \in \mathfrak{C}_{n+1,2(d+1)}$.*

Proof. ($\subseteq$) In the scope of Definition 6.1.2, we assume that f is supported on an adequate $A \subseteq I_{n+1,2d}$, write

$$f(X) = \sum_{j=0}^{r} f_{a(j)} X^{a(j)} + f_b X^b$$

and set $e_i := (e_{i,0}, \dots, e_{i,n})$ to be given by

$$e_{i,j} = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{else.} \end{cases}$$

Since f is supported on A and any element in $\operatorname{vert}(A)$ is even by the choice of A, we know that g is supported on $B := \{a + 2e_i \mid a \in A\} \subseteq I_{n+1,2(d+1)}$ and also any element in $\operatorname{vert}(B) = \{a + 2e_i \mid a \in \operatorname{vert}(A)\}$ is even. We furthermore set $g_{a(0)+2e_i} := f_{a(0)}, \dots, g_{a(r)+2e_i} := f_{a(r)} \in \mathbb{R}_{>0}$, $g_{b+2e_i} := f_b \in \mathbb{R}$ and conclude that

$$\begin{aligned} g(X) &= X_i^2 f(X) \\ &= X_i^2 \sum_{j=0}^{r} f_{a(j)} X^{a(j)} + f_b X^b \\ &= \sum_{j=0}^{r} f_{a(j)} X^{a(j)+2e_i} + f_b X^{b+2e_i} \\ &= \sum_{j=0}^{r} g_{a(j)+2e_i} X^{a(j)+2e_i} + g_{b+2e_i} X^{b+2e_i} \end{aligned}$$

with $r \leq n+1$, pairwise distinct $a(0)+2e_i, \dots, a(r)+2e_i \in B$, $g_{a(0)+2e_i}, \dots, g_{a(r)+2e_i}$ in $\mathbb{R}_{>0}$, $b+2e_i \in B$ and $g_{b+2e_i} \in \mathbb{R}$. Our choice of A secures $\operatorname{vert}(A) = \{a(0), \dots, a(r)\}$ and thus we deduce

$$\operatorname{vert}(B) = \{a + 2e_i \mid a \in \operatorname{vert}(A)\} = \{a(0) + 2e_i, \dots, a(r) + 2e_i\}.$$

Moreover, we know that $a(0) + 2e_i, \dots, a(r) + 2e_i$ are affinely independent since $a(0), \dots, a(r)$ are affinely independent and we conclude for the unique $\lambda_0, \dots, \lambda_r > 0$ such that $\sum_{j=0}^{r} \lambda_j = 1$ and $b = \sum_{j=0}^{r} \lambda_j a(j)$ that

$$b + 2e_i = \sum_{j=0}^{r} \lambda_j a(j) + 2\left(\sum_{j=0}^{r} \lambda_j\right) e_i = \sum_{j=0}^{r} \lambda_j (a(j) + 2e_i).$$

In fact, for $\lambda'_0, \dots, \lambda'_r > 0$ such that $\sum_{j=0}^{r} \lambda'_j = 1$, $b + 2e_i = \sum_{j=0}^{r} \lambda'_j (a(j) + 2e_i)$, we have

$$b = (b + 2e_i) - 2e_i = \sum_{j=0}^{r} \lambda'_j (a(j) + 2e_i) - 2\left(\sum_{j=0}^{r} \lambda'_j\right) e_i = \sum_{j=0}^{r} \lambda'_j a(j)$$

which implies $\lambda_j = \lambda'_j$ for $j = 0, \dots, r$ by the uniqueness of $\lambda_0, \dots, \lambda_r$.

($\supseteq$) Vice versa, again in the notation of Definition 6.1.2, we assume that g is supported on an adequate $B \subseteq I_{n+1,2(d+1)}$, write

$$g(X) = \sum_{j=0}^{r} g_{a(j)} X^{a(j)} + g_b X^b$$

and set $e_i := (e_{i,0}, \ldots, e_{i,n})$ to be given by

$$e_{i,j} = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{else.} \end{cases}$$

Since $g(X) = X_i^2 f(X)$, we know that any monomial in g is divisible by X_i^2. This allows us to conclude $c - 2e_i \in I_{n+1,2d}$ for any $c \in B$. Moreover, g is supported on B and any element in $\text{vert}(B)$ is even by the choice of B. Hence, f is supported on $A := \{c - 2e_i \mid c \in B\} \subseteq I_{n+1,2d}$ and any element in $\text{vert}(A) = \{c - 2e_i \mid c \in \text{vert}(B)\}$ is even. We set $f_{a(0)-2e_i} := g_{a(0)}, \ldots, f_{a(r)-2e_i} := g_{a(r)} \in \mathbb{R}_{>0}$, $f_{b-2e_i} := g_b \in \mathbb{R}$ and conclude from $g(X) = X_i^2 f(X)$ that

$$\begin{aligned} f(X) &= \sum_{j=0}^{r} g_{a(j)} X^{a(j)-2e_i} + g_b X^{b-2e_i} \\ &= \sum_{j=0}^{r} f_{a(j)-2e_i} X^{a(j)-2e_i} + f_{b-2e_i} X^{b-2e_i} \end{aligned}$$

with $r \leq n+1$, pairwise distinct $a(0)-2e_i, \ldots, a(r)-2e_i \in A$, $f_{a(0)-2e_i}, \ldots, f_{a(r)-2e_i}$ in $\mathbb{R}_{>0}$, $b - 2e_i \in A$ and $f_{b-2e_i} \in \mathbb{R}$. Our choice of B secures $\text{vert}(B) = \{a(0), \ldots, a(r)\}$ and thus we deduce

$$\text{vert}(A) = \{c - 2e_i \mid c \in \text{vert}(B)\} = \{a(0) - 2e_i, \ldots, a(r) - 2e_i\}.$$

Moreover, we know that $a(0) - 2e_i, \ldots, a(r) - 2e_i$ are affinely independent since $a(0), \ldots, a(r)$ are affinely independent and we conclude for the unique $\lambda_0, \ldots, \lambda_r > 0$ such that $\sum_{j=0}^{r} \lambda_j = 1$ and $b = \sum_{j=0}^{r} \lambda_j a(j)$ that

$$b - 2e_i = \sum_{j=0}^{r} \lambda_j a(j) - 2\left(\sum_{j=0}^{r} \lambda_j\right) e_i = \sum_{j=0}^{r} \lambda_j (a(j) - 2e_i).$$

In fact, for $\lambda'_0, \ldots, \lambda'_r > 0$ such that $\sum_{j=0}^{r} \lambda'_j = 1$, $b - 2e_i = \sum_{j=0}^{r} \lambda'_j (a(j) - 2e_i)$, we have

$$b = (b - 2e_i) + 2e_i = \sum_{j=0}^{r} \lambda'_j (a(j) - 2e_i) + 2\left(\sum_{j=0}^{r} \lambda'_j\right) e_i = \sum_{j=0}^{r} \lambda'_j a(j)$$

which implies $\lambda_j = \lambda'_j$ for $j = 0, \ldots, r$ by the uniqueness of $\lambda_0, \ldots, \lambda_r$. ∎

The advantage of agiforms over arbitrary forms is that the PSD property always follows as an immediate consequence of the arithmetic–geometric mean inequality. In the more general context of circuit forms, the circuit number offers an easy-to-be-checked criterion for the PSD property that acts as a counterpart.

Theorem 6.1.8. *Let $f \in \mathcal{F}_{n+1,2d}$ be a circuit form with inner term $f_b X^b$ and circuit number Θ_f, then f is PSD if and only if*

(i) $|f_b| \leq \Theta_f$ and b is not even or

(ii) $f_b \geq -\Theta_f$ and b is even.

Proof. See [IW16, Theorem 3.8]. ∎

Example 6.1.9. Let us reconsider Example 6.1.5.

(i) We know that the Choi–Lam quaternary quartic C is a circuit form with inner term $\mathsf{C}_b X^b = (-4)X_0X_1X_2X_3$ and circuit number $\Theta_{\mathsf{C}} = 4$. Hence, we have $|\mathsf{C}_b| = |-4| = 4 = \Theta_{\mathsf{C}}$ and $b = (1,1,1,1)$ is not even. Condition (i) of Theorem 6.1.8 is therefore satisfied and it follows $\mathsf{C} \in \mathcal{P}^{\mathfrak{C}}_{4,4}$.

(ii) We know that the Choi–Lam ternary sextic L is a circuit form with inner term $\mathsf{L}_b X^b = (-3)X_0^2X_1^2X_2^2$ and circuit number $\Theta_{\mathsf{L}} = 3$. Hence, $\mathsf{L}_b = (-3) = (-\Theta_{\mathsf{L}})$ and $b = (2,2,2)$ is even. Condition (ii) of Theorem 6.1.8 is therefore satisfied and it follows $\mathsf{L} \in \mathcal{P}^{\mathfrak{C}}_{3,6}$.

(iii) We know that the Motzkin ternary sextic M is a circuit form with inner term $\mathsf{M}_b X^b = (-3)X_0^2X_1^2X_2^2$ and circuit number $\Theta_{\mathsf{M}} = 3$. Therefore, we see that $\mathsf{M}_b = (-3) = (-\Theta_{\mathsf{M}})$ and $b = (2,2,2)$ is even. Condition (ii) of Theorem 6.1.8 is therefore satisfied and it follows $\mathsf{M} \in \mathcal{P}^{\mathfrak{C}}_{3,6}$.

In the following three sections, we will use circuit forms to answer the main query for the non-Hilbert cases $(n+1,2d)_{n\geq 2,d\geq 4}$. We thus conclude this section by putting circuit forms into the framework of this thesis.

Observation 6.1.10. *Let $f(X) = \sum_{j=0}^{r} f_{a(j)} X^{a(j)} + f_b X^b$ be a circuit form in $\mathcal{F}_{n+1,2d}$. We recall from Construction 2.3.7 (1) that $\{\alpha_0, \ldots, \alpha_k\}$ is the lexicographically ordered set $I_{n+1,d}$. The even outer exponents $a(0), \ldots, a(r) \in I_{n+1,2d}$ of f are therefore of the type $2\alpha_{j_0}, \ldots, 2\alpha_{j_r}$ for some $j_0, \ldots, j_r \in \{0, \ldots, k\}$. Thus,*

$$f(X) = \sum_{s=0}^{r} f_{2\alpha_{j_s}} X^{2\alpha_{j_s}} + f_b X^b.$$

Notation 6.1.11. For $f(X) = \sum_{s=0}^{r} f_{2\alpha_{j_s}} X^{2\alpha_{j_s}} + f_b X^b \in \mathfrak{C}_{n+1,2d}$, we set

$$j(f) := \max\{j_0, \ldots, j_r\}.$$

Example 6.1.12. Let us reconsider Example 6.1.5.

(i) The Choi–Lam quaternary quartic C is a circuit form with outer exponents $a(0) := (2,2,0,0)$, $a(1) := (2,0,2,0)$, $a(2) := (0,2,2,0)$ and $a(3) := (0,0,0,4)$. Recalling Example 2.3.10 (i), we also know that $\alpha_1 = (1,1,0,0)$, $\alpha_2 = (1,0,1,0)$, $\alpha_5 = (0,1,1,0)$ and $\alpha_9 = (0,0,0,2)$. Hence,

$$\begin{aligned} a(0) &= (2,2,0,0) = 2(1,1,0,0) = 2\alpha_1, \\ a(1) &= (2,0,2,0) = 2(1,0,1,0) = 2\alpha_2, \\ a(2) &= (0,2,2,0) = 2(0,1,1,0) = 2\alpha_5, \\ a(3) &= (0,0,0,4) = 2(0,0,0,2) = 2\alpha_9 \end{aligned}$$

and we compute $j(\mathsf{C}) = \max\{1,2,5,9\} = 9$.

(ii) The Choi–Lam ternary sextic L is a circuit form with outer exponents given by $a(0) := (4,2,0)$, $a(1) := (2,0,4)$ and $a(2) := (0,4,2)$. By Example 2.3.10 (ii), we furthermore know $\alpha_1 = (2,1,0)$, $\alpha_5 = (1,0,2)$ and $\alpha_7 = (0,2,1)$. Hence,

$$\begin{aligned} a(0) &= (4,2,0) = 2(2,1,0) = 2\alpha_1, \\ a(1) &= (2,0,4) = 2(1,0,2) = 2\alpha_5, \\ a(2) &= (0,4,2) = 2(0,2,1) = 2\alpha_7 \end{aligned}$$

and we compute $j(\mathsf{L}) = \max\{1,5,7\} = 7$.

(iii) The Motzkin ternary sextic M is a circuit form with outer exponents given by $a(0) := (4,2,0)$, $a(1) := (2,4,0)$ and $a(2) := (0,0,6)$. By Example 2.3.10 (ii), we furthermore know $\alpha_1 = (2,1,0)$, $\alpha_3 = (1,2,0)$ and $\alpha_9 = (0,0,3)$. Hence,

$$\begin{aligned} a(0) &= (4,2,0) = 2(2,1,0) = 2\alpha_1, \\ a(1) &= (2,4,0) = 2(1,2,0) = 2\alpha_3, \\ a(2) &= (0,0,6) = 2(0,0,3) = 2\alpha_9 \end{aligned}$$

and we compute $j(\mathsf{M}) = \max\{1,3,9\} = 9$.

Lemma 6.1.13. *For $f \in \mathfrak{C}_{n+1,2d}$, any monomial with non-zero coefficient in f is at least as great as $m^2_{j(f)}$ w.r.t. $\leq_{lex}$.*

Proof. In the notations of Observation 6.1.10, we write $f(X) = \sum_{s=0}^{r} f_{2\alpha_{j_s}} X^{2\alpha_{j_s}} + f_b X^b$ and recall from Construction 2.3.7 (1) that $\{\alpha_0, \ldots, \alpha_k\}$ is lexicographically ordered starting with the greatest element. Hence, $\alpha_j \geq_{\text{lex}} \alpha_{j(f)}$ for $j \leq j(f)$ from which it follows that

$$2\alpha_{j_0}, \ldots, 2\alpha_{j_r} \geq_{\text{lex}} 2\alpha_{j(f)}. \tag{6.1}$$

Moreover, we have $b=2\sum_{s=0}^{r}\lambda_s\alpha_{j_s}$ for some $\lambda_0,\dots,\lambda_r>0$ such that $\sum_{s=0}^{r}\lambda_s=1$ by Definition 6.1.2 (iii). Applying (6.1), we conclude

$$b=2\sum_{s=0}^{r}\lambda_s\alpha_{j_s}\geq_{\text{lex}}2\sum_{s=0}^{r}\lambda_s\alpha_{j(f)}=2\left(\sum_{s=0}^{r}\lambda_s\right)\alpha_{j(f)}=2\alpha_{j(f)}.$$

■

Lemma 6.1.14. *For $f\in\mathfrak{C}_{n+1,2d}$ and $g(X):=X_0^2f(X)\in\mathfrak{C}_{n+1,2(d+1)}$, it holds $j(f)=j(g(X))$.*

Proof. For the purpose of this proof, we denote $\alpha\in\{\alpha_0,\dots,\alpha_{k(n,D)}\}\subseteq I_{n+1,D}$ by $\alpha^{(D)}$ for $D\in\{d,d+1\}$. In the scope of Observation 6.1.10, we write

$$f(X)=\sum_{s=0}^{r}f_{2\alpha_{j_s}^{(d)}}X^{2\alpha_{j_s}^{(d)}}+f_bX^b,$$

set $e_0:=(1,0,\dots,0)\in I_{n+1,1}$ and recall from the proof of Lemma 6.1.7 that

$$g(X)=X_0^2f(X)=\sum_{s=0}^{r}f_{2\alpha_{j_s}^{(d)}}X^{2\alpha_{j_s}^{(d)}+2e_0}+f_bX^{b+2e_0}=\sum_{s=0}^{r}f_{2\alpha_{j_s}^{(d)}}X^{2(\alpha_{j_s}^{(d)}+e_0)}+f_bX^{b+2e_0}.$$

Using that $\alpha_j^{(d)}+e_0=\alpha_j^{(d+1)}$ for $j=0,\dots,k(n,d)$, we deduce

$$g(X)=\sum_{s=0}^{r}f_{2\alpha_{j_s}^{(d)}}X^{2\alpha_{j_s}^{(d+1)}}+f_bX^{b+2e_0}.$$

Therefore, we conclude $j(f)=\max\{j_0,\dots,j_r\}=j(g)$. ■

6.2 Computing $i(f)$ for PSD-extremal Circuit Forms

Throughout this section, we let **$(n+1,2d)$ denote a non-Hilbert case** if not explicitly mentioned otherwise. Moreover, we recall from Lemma 3.2.15 that for every $f\in\Delta_{n+1,2d}$, there exists a unique $i(f)\in\{0,\dots,k-n-1\}$ such that $f\in C_{i(f)+1}\setminus C_{i(f)}$. Striving to determine $i(f)$ for any a priori fixed $f\in\Delta_{n+1,2d}^{\mathfrak{C}}$, that spans an extreme ray of $\mathcal{P}_{n+1,2d}$,[3] we state and verify a sufficient condition for the membership of a PSD form to C_i in Theorem 6.2.2 below that uses the linear space introduced next.

Notation 6.2.1. For $n,d\geq 1$ and $i=0,\dots,k$, we set

$$\mathfrak{S}_i:=\operatorname{span}_{\mathbb{R}}\{m_sm_t\mid 0\leq s,t\leq i\}.$$

Theorem 6.2.2. *For $i=0,\dots,k-n$, it holds*

$$C_i\cap\mathfrak{S}_{n+i}=\mathcal{P}_{n+1,2d}\cap\mathfrak{S}_{n+i}.\qquad(6.2)$$

[3] We refer to Appendix A.3 for an introduction to convex geometry.

Proof. ($\subseteq$) We have $C_i \subseteq \mathcal{P}_{n+1,2d}$ by construction and thus

$$C_i \cap \mathfrak{S}_{n+i} \subseteq \mathcal{P}_{n+1,2d} \cap \mathfrak{S}_{n+i}.$$

($\supseteq$) For $f \in \mathcal{P}_{n+1,2d} \cap \mathfrak{S}_{n+i}$, we fix $A := (a_{s,t})_{0\le s,t\le k} \in \mathcal{G}^{-1}(f)$ such that $a_{s,t} = 0$ for all $0 \le s,t \le k$ with $n+i < \max\{s,t\}$. Moreover, for $[z] \in \phi(K_i)(\mathbb{R})$, Lemma 3.3.9 allows us to choose $\boldsymbol{x} \in \mathbb{R}^n$ such that $(z_0,\dots,z_{n+i}) = z_0(m_0(1,\boldsymbol{x}),\dots,m_{n+i}(1,\boldsymbol{x}))$. Altogether, using that f is PSD, it follows

$$\begin{aligned} q_A(z) &= q_A(z_0(m_0(1,\mathbf{x}),\dots,m_k(1,\mathbf{x}))) \\ &= z_0^2 q_A(m_0(1,\mathbf{x}),\dots,m_k(1,\mathbf{x})) \\ &= z_0^2 f(1,\boldsymbol{x}) \ge 0. \end{aligned}$$

We thus conclude $f \in C_i \cap \mathfrak{S}_{n+i}$ since $C_{\phi(K_i)} = C_i$ by Corollary 3.4.5. ■

Remark 6.2.3. *If $i = k-n$, then $C_i = C_{k-n} = \mathcal{P}_{n+1,2d}$ and $\mathfrak{S}_{n+i} = \mathfrak{S}_k = \mathcal{F}_{n+1,2d}$. Hence, $C_i \subseteq \mathfrak{S}_{n+i}$ and we see that (6.2) reduces to the tautology $\mathcal{P}_{n+1,2d} = \mathcal{P}_{n+1,2d}$.*

Lemma 6.2.4. *For $i = 0,\dots,k-n-1$, it holds $C_i \not\subseteq \mathfrak{S}_{n+i}$.*

Proof. We set $f(X) := X_n^{2d} \in \Sigma_{n+1,2d} \subseteq C_i$ and observe that the unique representation of f as a linear combination of two-factor-products of $m_0,\dots,m_k$ is $f(X) = m_k(X)^2$. Since $n+i<k$, we thus conclude $f \notin \mathfrak{S}_{n+i}$. ■

Theorem 6.2.5. *Let $n,d \ge 1$. If $f \in \mathfrak{C}_{n+1,2d}$, then $f \in \mathfrak{S}_{j(f)}$.*

Proof. In the notations of Observation 6.1.10, we write

$$f(X) = \sum_{s=0}^{r} f_{2\alpha_{j_s}} X^{2\alpha_{j_s}} + f_b X^b.$$

The outer terms of f are thus given by $f_{2\alpha_{j_s}} X^{2\alpha_{j_s}} = f_{2\alpha_{j_s}} m_{j_s}(X)^2$ for $s = 0,\dots,r$ and $j_0,\dots,j_r \le \max\{j_0,\dots,j_r\} = j(f)$. Therefore, all outer terms of f lie in $\mathfrak{S}_{j(f)}$ and it remains to verify $f_b X^b \in \mathfrak{S}_{j(f)}$, which we accomplish by an induction on $d \ge 1$.

For the inductive base case $d = 1$, we compute $k = k(n,1) = n$ and recall from Example 2.3.9 (ii) that $m_j(X) = X_j$ for $j = 0,\dots,n$. Moreover, Definition 6.1.2 (iii) states that the inner exponent of f is given by $b = 2\sum_{s=0}^{r} \lambda_{j_s}\alpha_{j_s}$ for some appropriate $\lambda_{j_0},\dots,\lambda_{j_r} > 0$ such that $\sum_{s=0}^{r} \lambda_{j_s} = 1$. Therefore, the inner term $f_b X^b$ of f is given by $f_b X_{j_s}^2$ for some $0 \le s \le r$ or by $f_b X_{j_{s_1}} X_{s_2}$ for some $0 \le s_1, s_2 \le r$. In both cases, $f_b X^b \in \mathfrak{S}_{j(f)}$ follows.

For the inductive step, we now assume that the assertion was already verified up to some $d \ge 1$ and investigate the situation for $d+1$ by a case distinction for $j(f)$.

Case 1: If $j(f) \leq k(n,d)$, then Lemma 6.1.13 and Lemma 6.1.7 imply $f(X) = X_0^2 g(X)$ for some $g \in \mathfrak{C}_{n+1,2d}$. The inductive assumption moreover yields $g \in \mathfrak{S}_{j(g)} \subseteq \mathcal{F}_{n+1,2d}$ and thus $f(X) \in \mathfrak{S}_{j(f)} \subseteq \mathcal{F}_{n+1,2(d+1)}$ follows since $j(g) = j(f)$ by Lemma 6.1.14.

Case 2: If $j(f) \geq k(n,d)+1$, then we set

$$l(f) := \max\left\{l \in \{1,\dots,n\} \mid X_l^{d+1} \geq_{\text{lex}} m_{j(f)}(X)\right\}$$

and argue by an induction on $n \geq 1$ that $f_b X^b$ lies in $\mathfrak{S}_{j(f)}$.

For the inductive base case $n = 1$, we compute $k(1,d) = d$, $k(1,d+1) = d+1$ and, therefore, conclude $j(f) = k(1,d+1)$. It thus follows

$$\mathfrak{S}_{j(f)} = \mathfrak{S}_{k(1,d+1)} = \operatorname{span}_{\mathbb{R}}\{m_s m_t \mid 0 \leq s,t \leq k(1,d+1)\} = \mathcal{F}_{2,2d}$$

and so, the assertion $f_b X^b \in \mathfrak{S}_{j(f)}$ is true.

For the inductive step, let us now assume that the assertion was already verified up to some $n \geq 1$ (i.e., for circuit forms in up to $n+1$ variables of degree $2(d+1)$) and investigate the situation for $n+1$.

If $j_0, \dots, j_r \geq k(n+1,d)+1$, then we are done by the inductive assumption since f can be interpreted as a circuit form in the $n+1$ variables $X_1, \dots, X_{n+1}$ of degree $2(d+1)$. We therefore now assume that there exists some $0 \leq \sigma \leq r$ with $0 \leq j_\sigma \leq k(n+1,d)$ and recall from Definition 6.1.2 (iii) that there exist some unique $\lambda_0, \dots, \lambda_r > 0$ such that $\sum\limits_{s=0}^{r} \lambda_s = 1$ and

$$b = 2\sum_{s=0}^{r} \lambda_s \alpha_{j_s}. \tag{6.3}$$

Since $0 \leq j_\sigma \leq k(n+1,d)$ is assumed, we consequently know that X_0 divides X^b.

If X_0^2 divides X^b, then we are done because in this case $X^b = m_s(X) m_t(X)$ for some $0 \leq s,t \leq k$ such that X_0 divides $m_s(X)$ and $m_t(X)$, respectively. In particular, $s,t \leq j(f)$ and $f_b X^b \in \mathfrak{S}_{j(f)}$ follows.

We thus restrict to the case when $X^b = X_0 m(X_1, \dots, X_{n+1})$ for some monomial m of degree $2d+1$ in the $n+1$ variables $X_1, \dots, X_{n+1}$.

If $l(f) \geq 2$ and X_1 divides X^b, then we are done because in this special case we have $X^b = m_s(X) m_t(X)$ for some $0 \leq s,t \leq k$ such that X_0 divides $m_s(X)$ and X_1 divides $m_t(X)$. In particular, $s,t \leq j(f)$ and $f_b X^b \in \mathfrak{S}_{j(f)}$ follows.

If $l(f) \geq 2$ and X_1 does not divide X^b, then, by (6.3), X_1 does not appear in f. Thus, f can be interpreted as a circuit form in the $n+1$ variables $X_0, X_2, \dots, X_{n+1}$ of degree $2(d+1)$ and the assertion follows from the inductive assumption.

It hence remains to investigate the case when $l(f) = 1$. To this end, we set

$$I := \{s \in \{0,\dots,r\} \mid \alpha_{j_s,0} \geq 1\}$$

and observe, using (6.3), that I is non-empty since X_0 divides X^b. Moreover, we set

$$I' := \{0,\dots,r\}\backslash I = \{s \in \{0,\dots,r\} \mid \alpha_{j_s,0} = 0\}.$$

Since $X^b = X_0 m(X_1,\dots,X_{n+1})$ by assumption, we conclude

$$1 = b_0 \underset{(6.3)}{=} 2\sum_{s\in I}\lambda_s\alpha_{j_s,0} + 2\sum_{s\in I'}\lambda_s\alpha_{j_s,0} = 2\sum_{s\in I}\lambda_s\alpha_{j_s,0} \geq 2\sum_{s\in I}\lambda_s. \tag{6.4}$$

Recalling $\sum_{s=0}^{r}\lambda_s = 1$, it therefore follows

$$\sum_{s\in I'}\lambda_s \geq \frac{1}{2} \geq \sum_{s\in I}\lambda_l. \tag{6.5}$$

Case 2.1: If $\alpha_{j_s,1} \geq 1$ for some $s \in I$, then we are done. Indeed, for any $t \in I'$, we have $\alpha_{j_t,1} \geq \alpha_{j(f),1}$ since $2\alpha_{j_t} \geq_{\text{lex}} 2\alpha_{j(f)}$ by Lemma 6.1.13 and $\alpha_{j_t,0} = 0 = \alpha_{j(f),0}$ by $t \in I'$ and $l(f) = 1$. Consequently, using (6.3), we know

$$b_1 > 2\sum_{s\in I'}\lambda_s\alpha_{j_s,1} \geq 2\left(\sum_{s\in I'}\lambda_s\right)\alpha_{j(f),1} \underset{(6.5)}{\geq} \alpha_{j(f),1}.$$

Hence, $X^b = m_s(X)m_t(X)$ for some $0 \leq s,t \leq k$ such that X_0 divides $m_s(X)$ and X_1^τ divides $m_t(X)$ for $\tau := \min\{\alpha_{j(f),1}+1, d+1\}$. In particular, $s,t \leq j(f)$ and it follows $f_b X^b \in \mathfrak{S}_{j(f)}$.

Case 2.2: If $\alpha_{j_s,1} > \alpha_{j(f),1}$ for some $s \in I'$, then we are done. Indeed, recalling $\alpha_{j_t,1} \geq \alpha_{j(f),1}$ for any $t \in I'$ from Case 2.1 and knowing $\alpha_{j_s,1} > \alpha_{j(f),1}$ in particular, we deduce, using (6.3), that

$$b_1 \geq 2\sum_{s\in I'}\lambda_s\alpha_{j_s,1} > 2\left(\sum_{s\in I'}\lambda_s\right)\alpha_{j(f),1} \underset{(6.5)}{\geq} \alpha_{j(f),1}.$$

Hence, $X^b = m_s(X)m_t(X)$ for some $0 \leq s,t \leq k$ such that X_0 divides $m_s(X)$ and X_1^τ divides $m_t(X)$ for $\tau := \min\{\alpha_{j(f),1}+1, d+1\}$. In particular, $s,t \leq j(f)$ and it follows $f_b X^b \in \mathfrak{S}_{j(f)}$.

Case 2.3: If we are neither in Case 2.1 nor in Case 2.2, then $\alpha_{j_s,1} = 0$ for any $s \in I$ and $\alpha_{j_s,1} = \alpha_{j(f),1}$ for any $s \in I'$. Hence,

$$\begin{aligned} b_1 &\underset{(6.3)}{=} 2\sum_{s\in I}\lambda_s\alpha_{j_s,1} + 2\sum_{s\in I'}\lambda_s\alpha_{j_s,1} \\ &= 2\sum_{s\in I'}\lambda_s\alpha_{j_s,1} \\ &= 2\left(\sum_{s\in I'}\lambda_s\right)\alpha_{j(f),1} \underset{(6.5)}{\geq} \alpha_{j(f),1}. \end{aligned} \tag{6.6}$$

If the inequality in (6.6) is strict, then we are done because then $X^b = m_s(X)m_t(X)$ for some $0 \leq s, t \leq k$ such that X_0 divides $m_s(X)$ and also X_1^τ divides $m_t(X)$ for $\tau := \min\{\alpha_{j(f),1} + 1, d+1\}$. In particular, $s, t \leq j(f)$ and $f_b X^b \in \mathfrak{S}_{j(f)}$ follows.

However, if we have equality in (6.6), then $\sum_{s=0}^{r} \lambda_s = 1$ yields

$$\sum_{s\in I'} \lambda_s = \frac{1}{2} = \sum_{s\in I} \lambda_s,$$

which implies equality in (6.4), and we conclude $\alpha_{j_s,0} = 1$ for $s \in I$. Consequently, we have $(\alpha_{j_s,0}, \alpha_{j_s,1}) = (1, 0)$ for $s \in I$ and $(\alpha_{j_s,0}, \alpha_{j_s,1}) = (0, \alpha_{j(f),1})$ for $s \in I'$. An iteration of the arguments given in Case 2.1, Case 2.2 and Case 2.3 for $b_2, \ldots, b_{n+1}$ (instead of b_1) therefore necessarily terminates in the solved Case 2.1 or the solved Case 2.2. Otherwise, we would have $\alpha_{j_s} = (1, 0, \ldots, 0)$ for any $s \in I$ which is impossible since $|\alpha_{j_s}| = d + 1 \geq 2$ for any $s \in I$. ■

Remark 6.2.6. *If $j(f) = k$, then $\mathfrak{S}_{j(f)} = \mathcal{F}_{n+1,2d}$.*

Example 6.2.7. The above proof of Theorem 6.2.5 is constructive as we now illustrate on the example of the Choi–Lam quaternary quartic C, the Choi–Lam ternary sextic L and the Motzkin ternary sextic M.

(i) In Example 6.1.5 (i) and Example 6.1.12 (i), we observed that the outer terms of the Choi–Lam quaternary quartic

$$\mathsf{C}(X_0, X_1, X_2, X_3) = X_0^2X_1^2 + X_0^2X_2^2 + X_1^2X_2^2 + X_3^4 - 4X_0X_1X_2X_3$$

are given by m_1^2, m_2^2, m_5^2, m_9^2 and deduced $j(\mathsf{C}) = 9$. Hence, all outer terms of C lie in $\mathfrak{S}_{j(\mathsf{C})}$ and it remains to examine the inner term $(-4)X_0X_1X_2X_3$.

Since $2 > 1$ and $j(\mathsf{C}) = 9 \geq 4 = k(3, 1) + 1$, we are in Case 2 of the inductive step of the first induction on the degree with $d = 1$ and $d + 1 = 2$. Hence,

$$l(\mathsf{C}) := \max\left\{l \in \{1, 2, 3\} \mid X_l^2 \geq_{\text{lex}} m_9(X) = X_3^2\right\} = 3$$

and we observe that we are in the inductive step of the second induction on the number of variables with $n = 2$ and $n + 1 = 3$. Moreover, for the index of the outer term m_1^2, we have $0 \leq 1 \leq 3 = k(3, 1)$. Since $l(\mathsf{C}) = 3 > 2$, we are therefore done by chosing $0 \leq s, t \leq 9$ such that X_0 divides $m_s(X)$, X_1 divides $m_t(X)$ and $X_0X_1X_2X_3 = m_s(X)m_t(X)$. For example, $s := 2$ and $t := 6$ suffice since $m_2(X) = X_0X_2$ and $m_6(X) = X_1X_3$ by Example 2.3.10 (i). Altogether, we conclude $\mathsf{C} = m_1^2 + m_2^2 + m_5^2 + m_9^2 - 4m_2m_6 \in \mathfrak{S}_9 = \mathfrak{S}_{j(\mathsf{C})}$.

However, let us point out that since $j(\mathsf{C}) = 9 = k(3, 2)$, we are in the trivial case that $\mathfrak{S}_{j(\mathsf{C})} = \mathcal{F}_{4,4}$ by Remark 6.2.6. Thus, $\mathsf{C} \in \mathfrak{S}_{j(\mathsf{C})}$ actually follows without any further argument.

(ii) In Example 6.1.5 (ii) and Example 6.1.12 (ii), we observed that the outer terms of the Choi–Lam ternary sextic

$$\mathsf{L}(X_0, X_1, X_2) := X_0^4 X_1^2 + X_0^2 X_2^4 + X_1^4 X_2^2 - 3X_0^2 X_1^2 X_2^2$$

are given by m_1^2, m_5^2, m_7^2 and deduced $j(\mathsf{L}) = 7$. Hence, all outer terms of L lie in $\mathfrak{S}_{j(\mathsf{L})}$ and it remains to examine the inner term $(-3)X_0^2 X_1^2 X_2^2$.

Since $3 > 1$ and $j(\mathsf{L}) = 7 \geq 6 = k(2,2) + 1$, we are in Case 2 of the inductive step of the first induction on the degree with $d = 2$ and $d + 1 = 3$. Hence,

$$l(\mathsf{L}) := \max\left\{l \in \{1,2\} \mid X_l^3 \geq_{\text{lex}} m_7(X) = X_1^2 X_2\right\} = 1$$

and we observe that we are in the inductive step of the second induction on the number of variables with $n = 1$ and $n + 1 = 2$. Moreover, for the index of the outer term m_1^2, we have $0 \leq 1 \leq 5 = k(2,2)$. Since X_0^2 divides $X_0^2 X_1^2 X_2^2$, we are therefore done by chosing $0 \leq s, t \leq 9$ such that $X_0^2 X_1^2 X_2^2 = m_s(X) m_t(X)$ and such that X_0 divides both $m_s(X)$ and $m_t(X)$. For example, $s := 3$ and $t := 5$ suffice since $m_3(X) = X_0 X_1^2$ and $m_5(X) = X_0 X_2^2$ by Example 2.3.10 (ii).

Altogether, we conclude $\mathsf{L} = m_1^2 + m_5^2 + m_7^2 - 3m_3 m_5 \in \mathfrak{S}_7 = \mathfrak{S}_{j(\mathsf{L})}$.

(iii) In Example 6.1.5 (iii) and Example 6.1.12 (iii), we showed that the outer terms of the Motzkin ternary sextic

$$\mathsf{M}(X_0, X_1, X_2) := X_0^4 X_1^2 + X_0^2 X_1^4 + X_2^6 - 3X_0^2 X_1^2 X_2^2$$

are given by m_1^2, m_3^2, m_9^2 and deduced $j(\mathsf{M}) = 9$. Hence, all outer terms of M lie in $\mathfrak{S}_{j(\mathsf{M})}$ and it remains to examine the inner term $(-3)X_0^2 X_1^2 X_2^2$.

Since $3 > 1$ and $j(\mathsf{M}) = 9 \geq 6 = k(2,2) + 1$, we are in Case 2 of the inductive step of the first induction on the degree with $d = 2$ and $d + 1 = 3$. Hence,

$$l(\mathsf{M}) := \max\left\{l \in \{1,2\} \mid X_l^3 \geq_{\text{lex}} m_9(X) = X_2^3\right\} = 2$$

and we observe that we are in the inductive step of the second induction on the number of variables with $n = 1$ and $n + 1 = 2$. Moreover, for the index of the outer term m_1^2, we have $0 \leq 1 \leq 5 = k(2,2)$. Since X_0^2 divides $X_0^2 X_1^2 X_2^2$, we are therefore done by chosing $0 \leq s, t \leq 9$ such that $X_0^2 X_1^2 X_2^2 = m_s(X) m_t(X)$ and such that X_0 divides both $m_s(X)$ and $m_t(X)$. For example, $s := 3$ and $t := 5$ suffice since $m_3(X) = X_0 X_1^2$ and $m_5(X) = X_0 X_2^2$ by Example 2.3.10 (ii).

Altogether, we conclude $\mathsf{M} = m_1^2 + m_3^2 + m_9^2 - 3m_3 m_5 \in \mathfrak{S}_9 = \mathfrak{S}_{j(\mathsf{M})}$.

However, let us point out that since $j(\mathsf{M}) = 9 = k(2,3)$, we are in the trivial case that $\mathfrak{S}_{j(\mathsf{M})} = \mathcal{F}_{3,6}$ by Remark 6.2.6. Thus, $\mathsf{M} \in \mathfrak{S}_{j(\mathsf{M})}$ actually follows without any further argument.

Theorem 6.2.8. *For $n, d \geq 1$ and $f \in \mathcal{P}^{\mathfrak{C}}_{n+1,2d}$, the following are true:*

(i) If $0 \leq j(f) \leq n-1$, then $f \in \Sigma^{\mathfrak{C}}_{n+1,2d}$.

(ii) If $n \leq j(f) \leq k(n,d)$, then $f \in C_{j(f)-n}$.

Proof. Theorem 6.2.5 states $f \in \mathfrak{S}_{j(f)}$ and thus we have $f \in \mathcal{P}_{n+1,2d} \cap \mathfrak{S}_{j(f)}$ since f is assumed to be PSD.

(i) If $0 \leq j(f) \leq n-1$, then $\mathfrak{S}_{j(f)} \subseteq \mathfrak{S}_{n+1}$ and $\alpha_{0,0}, \ldots, \alpha_{n+1,0} \geq d+1$. Therefore, $f \in \mathfrak{S}_{j(f)} \subseteq \mathfrak{S}_{n+1}$ implies $f(X) = X_0^{2d-2} g(X)$ for some $g \in \mathcal{F}_{n+1,2}$. Moreover, $f \in \mathcal{P}_{n+1,2d}$ and thus $g \in \mathcal{P}_{n+1,2} = \Sigma_{n+1,2}$ follows from Proposition 2.2.14 (i) and Hilbert's 1888 theorem (cf. Theorem 2.2.22). We conclude $f \in \Sigma_{n+1,2d}$ by Proposition 2.2.14 (ii).

(ii) If $n \leq j(f) \leq k(n,d)$, then applying Theorem 6.2.2 with $i := j(f) - n$ yields $f \in C_{j(f)-n} \cap \mathfrak{S}_{j(f)}$. ■

Corollary 6.2.9. *For $f \in \Delta^{\mathfrak{C}}_{n+1,2d}$, the following are true:*

(i) If $n = 2$, then $j(f) \geq 6$.

(ii) If $n \geq 3$, then $j(f) \geq 2n+1$.

Proof. Theorem 6.2.8 yields $n \leq j(f)$ since $f \notin \Sigma_{n+1,2d}$ is assumed and thus also $f \in C_{j(f)-n}$ holds.

(i) If $n = 2$, then Theorem 4.2.7 yields $\Sigma_{3,2d} = C_3$ and we conclude $j(f) - 2 \geq 4$.

(ii) If $n \geq 3$, then Theorem 4.2.7 yields $\Sigma_{n+1,2d} = C_n$ and we conclude

$$j(f) - n \geq n+1.$$ ■

Corollary 6.2.10. *For $f \in \Delta^{\mathfrak{C}}_{n+1,2d}$, it holds $i(f) \leq j(f) - n - 1$.*

Proof. Theorem 6.2.8 gives $f \in C_{j(f)-n}$ and Remark 3.2.17 states that $i(f)+1$ is minimal in $\{1, \ldots, k-n\}$ such that $f \in C_{i(f)}$. Corollary 6.2.9 moreover secures $j(f) \geq 2n+1$ and thus $1 < n < n+1 \leq j(f) - n \leq k - n$ follows. Altogether, we have $i(f) + 1 \leq j(f) - n$. ■

We conclude this section by computing $i(f)$ for any a priori given circuit form $f \in \mathcal{F}_{n+1,2d}$, that spans an extreme ray of $\mathcal{P}_{n+1,2d}$, in Theorem 6.2.12 below using Theorem 6.2.11. To this end, we recall that any $f \in \mathcal{F}_{n+1,2d}$ that spans an extreme ray of $\mathcal{P}_{n+1,2d}$ is called *PSD-extremal*. Moreover, we observe that $f \in \mathcal{F}_{n+1,2d}$ is PSD-extremal if and only if for any $f_1, f_2 \in \mathcal{P}_{n+1,2d}$ such that $f = f_1 + f_2$, there exist some $\lambda_1, \lambda_2 \geq 0$ such that $f_1 = \lambda_1 f$ and $f_2 = \lambda_2 f$. This characterization especially illuminates why PSD-extremality is maintained under permutation of variables and under multiplication with monomial squares.

Theorem 6.2.11. *Let $f \in \Delta^{\mathfrak{C}}_{n+1,2d}$ be PSD-extremal, then $i(f) \geq j(f) - n - 1$.*

Proof. In the notations of Observation 6.1.10, we assume after a possible scaling of f that the coefficient $f_{2\alpha_{j(f)}}$ of the outer term $f_{2\alpha_{j(f)}} X^{2\alpha_{j(f)}}$ of f is 1 and recall $j(f) \geq 2n+1$ from Corollary 6.2.9. For a proof by contradiction, we moreover assume $i(f) < j(f) - n - 1$. Hence, we have $f \in C_{i(f)+1} \subseteq C_{j(f)-n-1}$, which allows us to fix some $A := (a_{i,j})_{0\leq i,j\leq k} \in \mathcal{G}^{-1}(f)$ such that q_A is locally PSD on $V_{j(f)-n-1}(\mathbb{R})$.

Claim: For $j \geq j(f)+1$, all entries of $(a_{i,j})_{0\leq i\leq j}$ are zero.
Proof. If $j(f) = k$, then there is nothing to prove. Hence, we assume $j(f) < k$. Lemma 2.3.15 yields that $a_{k,k}$ is the coefficient of X_n^{2d} in f. Since $j(f) < k$ and X_n^{2d} is uniquely expressible as $m_k(X)^2$, we conclude $a_{k,k} = 0$ from Lemma 6.1.13. Moreover, since $[m_0(x) : \ldots : m_{k-1}(x) : y] \in V_{k-n-1}(\mathbb{R}) \subseteq V_{j(f)-n-1}(\mathbb{R})$ for any $x \in \mathbb{R}^{n+1}$ and any non-zero $y \in \mathbb{R}$, we know that

$$\begin{aligned} q(X,Y) &:= q_A(m_0(X),\ldots,m_{k-1}(X),Y) \\ &= q_A(m_0(X),\ldots,m_{k-1}(X),m_k(X)) - 2\sum_{i=0}^{k-1} a_{i,k} m_i(X) m_k(X) \\ &\quad + \left(2\sum_{i=0}^{k-1} a_{i,k} m_i(X)\right) Y \\ &= f(X) - 2\sum_{i=0}^{k-1} a_{i,k} m_i(X) m_k(X) + \left(2\sum_{i=0}^{k-1} a_{i,k} m_i(X)\right) Y \end{aligned}$$

is PSD. Consequently, $a_{i,k} = 0$ for $i = 0,\ldots,k-1$ to avoid the potential linearity of q in Y. Iterating this argument for $j = k-1,\ldots,j(f)+1$ yields the assertion. ∎

Since $a_{j(f),j(f)}$ is the coefficient of $m_{j(f)}(X)^2 = X^{2\alpha_{j(f)}}$ in f by Lemma 2.3.15 and all entries of $(a_{i,j})_{0\leq i\leq j}$ are zero for $j \geq j(f)+1$ by the above consideration, we thus know $a_{j(f),j(f)} = 1$. Therefore, we set

$$g(X,Y) := f(X) + \left(Y + \sum_{i=0}^{j(f)-1} a_{i,j(f)} m_i(X)\right)^2 - \left(\sum_{i=0}^{j(f)-1} a_{i,j(f)} m_i(X) + m_{j(f)}(X)\right)^2$$

and show that g is PSD by a case distinction for $j(f)$.

Case 1: If $j(f) = k$, then $[m_0(x) : \ldots : m_{j(f)-1}(x) : y] \in V_{j(f)-n-1}(\mathbb{R})$ for any $x \in \mathbb{R}^{n+1}$ and any non-zero $y \in \mathbb{R}$ yields

$$\begin{aligned} 0 &\leq q_A(m_0(X),\ldots,m_{j(f)-1}(X),Y) \\ &= q_A(m_0(X),\ldots,m_k(X)) - 2\left(\sum_{i=0}^{k-1} a_{i,k} m_i(X)\right)(m_k(X) - Y) - m_k(X)^2 + Y^2 \\ &= f(X) + \left(Y + \sum_{i=0}^{k-1} a_{i,k} m_i(X)\right)^2 - \left(\sum_{i=0}^{k-1} a_{i,k} m_i(X) + m_k(X)\right)^2 \\ &= g(X,Y). \end{aligned}$$

Case 2: If $j(f) < k$, then

$$[m_0(x) : \ldots : m_{j(f)-1}(x) : y : m_{j(f)+1}(x) : \ldots : m_k(x)] \in V_{j(f)-n-1}(\mathbb{R})$$

for any $x \in \mathbb{R}^{n+1}$ and any non-zero $y \in \mathbb{R}$ yields

$$\begin{aligned} 0 &\leq q_A(m_0(X), \ldots, m_{j(f)-1}(X), Y, m_{j(f)+1}(X), \ldots, m_k(X)) \\ &= q_A(m_0(X), \ldots, m_k(X)) - 2\left(\sum_{i=0}^{j(f)-1} a_{i,j(f)} m_i(X)\right)(m_{j(f)}(X) - Y) \\ &\quad -m_{j(f)}(X)^2 + Y^2 \\ &= f(X) + \left(Y + \sum_{i=0}^{j(f)-1} a_{i,j(f)} m_i(X)\right)^2 - \left(\sum_{i=0}^{j(f)-1} a_{i,j(f)} m_i(X) + m_{j(f)}(X)\right)^2 \\ &= g(X,Y). \end{aligned}$$

Therefore, g is PSD in both cases and we also know that

$$h(X) := g\left(X, -\sum_{i=0}^{j(f)-1} a_{i,j(f)} m_i(X)\right) = f(X) - \left(\sum_{i=0}^{j(f)-1} a_{i,j(f)} m_i(X) + m_{j(f)}(X)\right)^2$$

is PSD. Consequently, $f = f_1 + f_2$ for the PSD forms

$$\begin{aligned} f_1(X) &:= \frac{1}{2}h(X) = \frac{1}{2}\left(f(X) - \left(\sum_{i=0}^{j(f)-1} a_{i,j(f)} m_i(X) + m_{j(f)}(X)\right)^2\right), \\ f_2(X) &:= \frac{1}{2}\left(f(X) + \left(\sum_{i=0}^{j(f)-1} a_{i,j(f)} m_i(X) + m_{j(f)}(X)\right)^2\right). \end{aligned} \tag{6.7}$$

Since f is PSD-extremal, this allows us to fix some $\lambda_1 \geq 0$ such that $f_1 = \lambda_1 f$. We now observe that the monomial square $m_{j(f)}(X)^2 = X^{2\alpha_{j(f)}}$ has coefficient one in f and coefficient zero in f_1. Thus, $\lambda_1 = 0$ follows. Hence, f_1 is the zero form and we conclude $f = f_2$. Recalling (6.7), we conclude

$$f(X) = \left(\sum_{i=0}^{j(f)-1} a_{i,j(f)} m_i(X) + m_{j(f)}(X)\right)^2$$

which is in contradiction to $f \notin \Sigma_{n+1,2d}$. ■

Theorem 6.2.12. *Let $f \in \Delta^{\mathfrak{C}}_{n+1,2d}$ be PSD-extremal, then $i(f) = j(f) - n - 1$.*

Proof. Corollary 6.2.9 and Theorem 6.2.11 together imply $i(f) = j(f) - n - 1$ for any PSD-extremal $f \in \Delta^{\mathfrak{C}}_{n+1,2d}$. ■

Example 6.2.13. Quaternary Sextics
We let $n = d = 3$ and reconsider the quaterneray sextic

$$X_1^2\mathsf{C}^\tau(X) = X_1^4X_2^2 + X_1^4X_3^2 + X_1^2X_2^2X_3^2 + X_0^4X_1^2 - 4X_0X_1^3X_2X_3$$

from (5.15). Since C^τ is a PSD-extremal circuit form that is not SOS and X_1^2 is a monomial square, we conclude that also $X_1^2\mathsf{C}^\tau(X)$ is a PSD-extremal circuit form that is not SOS. Moreover, we compute $\alpha_1 = (2,1,0,0)$, $\alpha_4 = (1,2,0,0)$, $\alpha_{11} = (0,2,1,0)$, $\alpha_{12} = (0,2,0,1)$, $\alpha_{14} = (0,1,1,1)$ and thus

$$\begin{aligned} X_1^2\mathsf{C}^\tau(X) &= \left(X_1^2X_2\right)^2 + \left(X_1^2X_3\right)^2 + (X_1X_2X_3)^2 + \left(X_0^2X_1\right)^2 \\ &\quad - 4\left(X_0X_1^2\right)(X_1X_2X_3) \\ &= (m_{11}(X))^2 + (m_{12}(X))^2 + (m_{14}(X))^2 + (m_1(X))^2 \\ &\quad - 4m_4(X)m_{14}(X). \end{aligned} \tag{6.8}$$

In light of Observation 6.1.10, the outer exponents of $X_1^2\mathsf{C}^\tau(X)$ are given by $2\alpha_1$, $2\alpha_{11}$, $2\alpha_{12}$, $2\alpha_{14}$. Hence, we have $j\left(X_1^2\mathsf{C}^\tau(X)\right) = 14$. Applying Theorem 6.2.12, we therefore conclude $i\left(X_1^2\mathsf{C}^\tau(X)\right) = j\left(X_1^2\mathsf{C}^\tau(X)\right) - n - 1 = 14 - 4 = 10$ which yields $X_1\mathsf{C}^\tau(X) \in C_{11}\backslash C_{10}$ as claimed in the proof of Theorem 5.2.12.

Theorem 6.2.12 only provides a sufficient condition to compute $i(f)$ for $f \in \Delta_{n+1,2d}^{\mathfrak{C}}$. That is, the PSD-extremality of f. Yet, the PSD-extremality is not a necessary condition as there exists $f \in \Delta_{n+1,2d}^{\mathfrak{C}}$ such that $i(f) = j(f) - n - 1$ which is not PSD-extremal. For example, the form below.

Example 6.2.14. In the non-Hilbert case $(4,6)$, let us consider the quaternary sextic

$$\mathsf{D}(X_0,X_1,X_2,X_3) := 2X_0^4X_3^2 + 2X_0^2X_3^4 + X_1^4X_2^2 + X_1^2X_2^4 - 6X_0^2X_1X_2X_3^2.$$

We compute $\operatorname{supp}(\mathsf{D}) = \{(4,0,0,2),(2,0,0,4),(0,4,2,0),(0,2,4,0),(2,1,1,2)\}$ and set $r := 3$, $a(0) := (4,0,0,2)$, $a(1) := (2,0,0,4)$, $a(2) := (0,4,2,0)$, $a(3) := (0,2,4,0)$, $b := (2,1,1,2)$, $\mathsf{D}_{a(0)} := \mathsf{D}_{a(1)} := 2$, $\mathsf{D}_{a(2)} := \mathsf{D}_{a(3)} := 1$ and $\mathsf{D}_b := (-6)$. Hence, for $X := (X_0,X_1,X_2,X_3)$ as usual, we have

$$\mathsf{D}(X) = \sum_{j=0}^{r} \mathsf{D}_{a(j)}X^{a(j)} + \mathsf{D}_bX^b.$$

Moreover, we see that $\operatorname{vert}(\operatorname{supp}(\mathsf{D})) = \{a(0),a(1),a(2),a(3)\}$ and any element in $\operatorname{vert}(\operatorname{supp}(\mathsf{D}))$ is even. A straight forward computation furthermore shows that $a(0),\ldots,a(3)$ are affinely independent. Lastly, another straight forward computation reveals that the unique $\lambda_0,\ldots,\lambda_3 > 0$ such that $\sum_{j=0}^{3}\lambda_j = 1$ and $b = \sum_{j=0}^{3}\lambda_j a(j)$ are given by $\lambda_0 := \lambda_1 := \frac{1}{3}$, $\lambda_2 := \lambda_3 := \frac{1}{6}$. Altogether, D is a circuit form with outer exponents $(4,0,0,2)$, $(2,0,0,4)$, $(0,4,2,0)$, $(0,2,4,0)$, inner exponents $(2,1,1,2)$ and circuit number $\Theta_{\mathsf{D}} = 6$. Hence, we have $|\mathsf{D}_b| = 6 = \Theta_{\mathsf{D}}$ and, therefore, Condition (i) of Theorem 6.1.8 is satisfied. It follows $\mathsf{D} \in \mathcal{P}^{\mathfrak{C}}_{4,6}$.

Moreover, we compute

$$\begin{array}{llll}
\alpha_0 = (3,0,0,0), & \alpha_4 = (1,2,0,0), & \alpha_8 = (1,0,1,1), & \alpha_{12} = (0,2,0,1), \\
\alpha_1 = (2,1,0,0), & \alpha_5 = (1,1,1,0), & \alpha_9 = (1,0,0,2), & \alpha_{13} = (0,1,2,0) \\
\alpha_2 = (2,0,1,0), & \alpha_6 = (1,1,0,1), & \alpha_{10} = (0,3,0,0), & \\
\alpha_3 = (2,0,0,1), & \alpha_7 = (1,0,2,0), & \alpha_{11} = (0,2,1,0), &
\end{array}$$

and deduce $a(0) = 2\alpha_3$, $a(1) = 2\alpha_9$, $a(2) = 2\alpha_{11}$, $a(3) = 2\alpha_{13}$. This implies $j(\mathfrak{D}) = 13$ and thus, by Theorem 6.2.8, we know $\mathsf{D} \in C_{10}$. However, D is not PSD-extremal by [Rez89, (9.9) Example]. Hence, Theorem 6.2.12, respectively Theorem 6.2.11, cannot be applied to conclude $\mathsf{D} \notin C_9$. Yet, this is true.

Claim: $\mathsf{D} \notin C_9$.

Proof. We compute $k = k(3,3) = 19$ and assume for a proof by contradiction that it is possible to fix some $A := (a_{s,t})_{0\leq s,t\leq 19} \in \mathcal{G}^{-1}(\mathsf{D})$ such that q_A is locally PSD on $V_9(\mathbb{R})$. Arguing as in the proof of Theorem 6.2.11 above, we see that $a_{s,t} = 0$ for $s \geq 14$ or $t \geq 14$. Since $a_{13,13}$ is the coefficient of $m_{13}(X)^2 = X_1^2X_2^4$ in f by Lemma 2.3.15, we moreover know $a_{13,13} = 1$. For $\boldsymbol{X} := (X_1, X_2, X_3)$ as usual, we thus deduce that

$$\begin{aligned}
\hat{q}(\boldsymbol{X}, Y) \;&:=\; q_A(m_0(1,\boldsymbol{X}), \ldots, m_{12}(1,\boldsymbol{X}), Y, m_{14}(1,\boldsymbol{X}), \ldots, m_{19}(1,\boldsymbol{X})) \\
&=\; 2X_3^2 + 2X_3^4 + X_1^4X_2^2 + -6X_1X_2X_3^2 + 2(a_{0,13} + a_{1,13}X_1 + a_{2,13}X_2 \\
&\quad +a_{3,13}X_3 + a_{4,13}X_1^2 + a_{5,13}X_1X_2 + a_{6,13}X_1X_3 + a_{7,13}X_2^2 + a_{8,13}X_2X_3 \\
&\quad +a_{9,13}X_3^2 + a_{10,13}X_1^3 + a_{11,13}X_1^2X_2 + a_{12,13}X_1^2X_3)(Y - X_1X_2^2) + Y^2
\end{aligned}$$

is PSD. Evaluating $\hat{q}$ in $\boldsymbol{x} = (0,0,0)$ now yields that

$$\hat{q}(0,0,0,Y) = 2a_{0,13}Y + Y^2 = (Y + a_{0,13})^2 - a_{0,13}^2$$

is PSD. Consequently, $a_{0,13} = 0$. Likewise, evaluating $\hat{q}$ in $(\boldsymbol{x}, y) = (0,t,0,t)$ for any $t \in \mathbb{R}$ implies that

$$\hat{q}(0,T,0,T) = 2(a_{2,13}T + a_{7,13}T^2)T + T^2$$

is PSD. Hence, the greatest degree of T appearing cannot be odd and we conclude $a_{7,13} = 0$. Similarly, evaluating $\hat{q}$ in $(\boldsymbol{x}, y) = (t,0,0,t^2)$ for any $t \in \mathbb{R}$, we observe that

$$\hat{q}(T,0,0,T^2) = 2(a_{1,13}T + a_{4,13}T^2 + a_{10,13}T^3)T^2 + T^4$$

is PSD and deduce $a_{10,13} = 0$. Moreover, evaluating $\hat{q}$ in $(\boldsymbol{x}, y) = (t, 0, 0, t)$ for any $t \in \mathbb{R}$ gives us that

$$\hat{q}(T, 0, 0, T) = 2(a_{1,13}T + a_{4,13}T^2)T + T^2$$

is PSD. We conclude $a_{4,13} = 0$. Likewise, evaluating $\hat{q}$ in $(\boldsymbol{x}, y) = (t, 0, 1, t)$ for any $t \in \mathbb{R}$ implies that

$$\hat{q}(T, 0, 1, T) = 4 + 2(a_{1,13}T + a_{3,13} + a_{6,13}T + a_{9,13} + a_{12,13}T^2)T + T^2$$

is PSD. So, $a_{12,13} = 0$. Furthermore, evaluating $\hat{q}$ in $(\boldsymbol{x}, y) = (1, t^2, t, 0)$ for any $t \in \mathbb{R}$, we observe that

$$\begin{aligned} \hat{q}(1, T^2, T, 0) \quad = \quad & 2T^2 - 3T^4 + 2(a_{1,13} + a_{1,13}T^2 + a_{3,13}T + a_{5,13}T^2 + a_{6,13}T \\ & + a_{8,13}T^3 + a_{9,13}T^2 + a_{11,13}T^2)(-T^4) \end{aligned}$$

is PSD and deduce $a_{8,13} = 0$. Moreover, evaluating $\hat{q}$ in $(\boldsymbol{x}, y) = (1, t, 0, 0)$ for any $t \in \mathbb{R}$ gives us that

$$\hat{q}(1, T, 0, 0) = T^2 + 2(a_{1,13} + a_{2,13}T + a_{5,13}T + a_{11,13}T)(-T^2)$$

is PSD and we conclude

$$a_{2,13} + a_{5,13} + a_{11,13} = 0. \tag{6.9}$$

Likewise, evaluating $\hat{q}$ in $(\boldsymbol{x}, y) = (-1, t, 0, 0)$ for any $t \in \mathbb{R}$ implies that

$$\hat{q}(-1, T, 0, 0) = T^2 + 2(-a_{1,13} + a_{2,13}T - a_{5,13}T + a_{11,13}T)(T^2)$$

is PSD. So, $a_{2,13} - a_{5,13} + a_{11,13} = 0$. Recalling (6.9), we deduce $a_{5,13} = 0$ and

$$a_{2,13} = (-a_{11,13}). \tag{6.10}$$

Furthermore, evaluating $\hat{q}$ in $(\boldsymbol{x}, y) = (2, t, 0, 0)$ for $t \in \mathbb{R}$ yields that

$$\hat{q}(2, T, 0, 0) = 16T^2 + 2(2a_{1,13} + a_{2,13}T + 4a_{11,13}T)(-2T^2)$$

is PSD. Consequently, $a_{2,13} = (-4a_{11,13})$. Recalling (6.10), we therefore see that $a_{11,13} = 0 = a_{2,13}$. Moreover, evaluating $\hat{q}$ in $(\boldsymbol{x}, y) = (0, t, 0, 1)$ for $t \in \mathbb{R}$ shows that

$$\hat{q}(0, T, 0, 1) = 2a_{1,13}T + 1$$

is PSD. We conclude $a_{1,13} = 0$. Similarly, evaluating $\hat{q}$ in $(\boldsymbol{x}, y) = (t, 0, 1, 1)$ for any $t \in \mathbb{R}$ ensures that

$$\hat{q}(T, 0, 1, 1) = 5 + 2(a_{3,13} + a_{6,13}T + a_{9,13})$$

is PSD. We hence obtain $a_{6,13} = 0$. Moreover, evaluating $\hat{q}$ in $(\boldsymbol{x}, y) = (s, t, t, 0)$ for any $s, t \in \mathbb{R}$ gives us that

$$\hat{q}(S, T, T, 0) = 2T^2 + 2T^4 + S^4T^2 - 6ST^3 + 2(a_{3,13}T + a_{9,13}T^2)(-ST^2)$$

is PSD. Consequently, when interpreted as a monomial in T, the coefficient $2 - a_{9,13}S$ of T^4 must be PSD as a polynomial in S. Hence, $a_{9,13} = 0$. Lastly, evaluating $\hat{q}$ in $(\boldsymbol{x}, y) = (1, t, s, 0)$ for any $s, t \in \mathbb{R}$ yields that

$$\hat{q}(1, T, S, 0) = 2S^2 + 2S^4 + T^2 - 6TS^2 - 2a_{3,13}ST^2$$

is PSD. Therefore, when interpreted as a monomial in T, the coefficient $1 - 2a_{3,13}S$ of T^2 must be PSD as a polynomial in S and it thus follows $a_{3,13} = 0$.

Altogether, we conclude $a_{13,13} = 1$ and $a_{s,t} = 0$ for $s \geq 13$ or $t \geq 13$ such that $(s, t) \neq (13, 13)$ by the symmetry of A. For

$$[z] := [1 : \ldots : 1 : 0 : 0 : 0 : 0 : 0 : 0] \in H_9(\mathbb{R}) \subseteq V_9(\mathbb{R}),$$

we therefore obtain

$$q_A(z) = q_A(1, \ldots, 1, 0, 0, 0, 0, 0, 0) = q_A(1, \ldots, 1) - 1^2 = \mathsf{D}(1, 1, 1, 1) - 1 = (-1)$$

which is a contradiction to the assumption that q_A is locally PSD on $V_9(\mathbb{R})$. ■

Hence, the PSD quaternary sextic D is a circuit form that is not PSD-extremal but for which $i(\mathsf{D}) = 9 = j(\mathsf{D}) - 3 - 1$ holds anyway since $\mathsf{D} \in C_{10} \backslash C_9$.

6.3 Separation of $C_{n+1}, \ldots, C_{k-n}$ and also C_n if $n \geq 3$

In Chapter 5, we showed $C_n \subsetneq \ldots \subsetneq C_{k-n}$, and also $C_n \subsetneq C_{n+1}$ if $n \geq 3$, in the non-Hilbert cases $(n+1, 4)_{n\geq 3}$ and $(n+1, 6)_{n\geq 2}$ by arguing as follows.

- Firstly, in the basic non-Hilbert cases $(4, 4)$ and $(3, 6)$, we established explicit complete sets $S_{3,2}$ and $S_{2,3}$ of separating forms for $(\mathcal{CF})$ (cf. Corollary 5.1.4 and Corollary 5.1.8), respectively, and verified the separating property of each form via several point evaluations. This gave us the two main results of Section 5.1.

  Theorem 5.1.2. *If $(n+1, 2d) = (4, 4)$, then $C_3 \subsetneq C_4 \subsetneq C_5 \subsetneq C_6$.*

  Theorem 5.1.6. *If $(n+1, 2d) = (3, 6)$, then $C_3 \subsetneq C_4 \subsetneq C_5 \subsetneq C_6 \subsetneq C_7$.*

- Secondly, we generalized our findings of Theorem 5.1.2 from quaternary quartics to $(n+1)$-ary quartics $(n \geq 4)$, which led to the first main result of Section 5.2.

  Theorem 5.2.2. *For $(n+1, 4)_{n\geq 4}$ and $i = n, \ldots, k-n-1$, the inclusion $C_i \subsetneq C_{i+1}$ is strict.*

The proof of the above theorem was constructive and allowed us to establish an explicit complete set $S_{n,2}$ of separating forms for ($\mathcal{CF}$) for $n \geq 4$ and $d = 2$ in Corollary 5.2.5.

- Thirdly, we developed a first degree-jumping principle in Theorem 5.2.7 that allowed us to transfer our result of Theorem 5.2.2 from $(n+1)$-ary quartics to $(n+1)$-ary sextics for $n \geq 3$. Moreover, we generalized our findings of Theorem 5.1.6 from ternary sextics to $(n+1)$-ary sextics ($n \geq 3$). Combining both considerations, we showed the second main result of Section 5.2.

  Theorem 5.2.14. *For $(n+1, 6)_{n \geq 3}$ and $i = n, \ldots, k-n-1$, the inclusion $C_i \subsetneq C_{i+1}$ is strict.*

  The proof of the above theorem was constructive and allowed us to establish an explicit complete set $S_{n,3}$ of separating forms for ($\mathcal{CF}$) for $n \geq 3$ and $d = 3$ in Corollary 5.2.17.

We now give alternative proofs for the above four main results of Chapter 5. To this end, we show that each form in the constructed explicit complete sets of separating forms for ($\mathcal{CF}$) is a PSD-extremal circuit form and applying Theorem 6.2.12.

Alternative Proof of Theorem 5.1.2. Observation 5.1.1 states

$$\Sigma_{4,4} = C_0 = C_1 = C_2 = C_3 \subseteq C_4 \subseteq C_5 \subseteq C_6 = \mathcal{P}_{4,4}.$$

Hence, $\mu(3,2) \leq 2$ and $\mu(3,2) = 2$ if and only if each remaining inclusion is strict. Let us consider the set $S_{3,2} := \{\mathsf{C}^\sigma, \mathsf{C}^\tau, \mathsf{C}\}$ (cf. Corollary 5.1.4) and we recall that $\mathsf{C} \in \Delta_{4,4}$ is a circuit form by Example 6.1.5 (i). Moreover, we know that C is PSD-extremal by [CL77, Theorem 3.5] and, therefore, also the permuted forms $\mathsf{C}^\sigma, \mathsf{C}^\tau \in \Delta_{4,4}$ are PSD-extremal circuit forms. We compute $j(\mathsf{C}^\sigma) = 7$, $j(\mathsf{C}^\tau) = 8$ and $j(\mathsf{C}) = 9$. Theorem 6.2.12 thus implies $i(\mathsf{C}^\sigma) = 3$, $i(\mathsf{C}^\tau) = 4$ and $i(\mathsf{C}) = 5$. Therefore, we conclude $|\{i(f) \mid f \in S_{3,2}\}| = 3 \geq \mu(3,2) + 1$. Hence,

$$|\{i(f) \mid f \in S_{3,2}\}| = 3 = \mu(3,2) + 1$$

by Lemma 3.2.21 (i). This shows that $S_{3,2}$ is a complete set of separating forms for ($\mathcal{CF}$) in the basic non-Hilbert case $(4,4)$, $\mu(3,2) = 2$ and $C_3 \subsetneq C_4 \subsetneq C_5 \subsetneq C_6$. ∎

Remark 6.3.1. *In the above proof, we in particular gave an alternative proof for Corollary 5.1.4 by computing $i(\mathsf{C}^\sigma) = 3$, $i(\mathsf{C}^\tau) = 4$ and $i(\mathsf{C}) = 5$ using Theorem 6.2.12 and the PSD-extremality of these distinguished circuit quaternary quartics.*

Alternative Proof of Theorem 5.1.6. Observation 5.1.5 states

$$\Sigma_{3,6} = C_0 = C_1 = C_2 = C_3 \subseteq C_4 \subseteq C_5 \subseteq C_6 \subseteq C_7 = \mathcal{P}_{3,6}.$$

Hence, $\mu(2,3) \leq 3$ and $\mu(2,3) = 3$ if and only if each remaining inclusion is strict. Let us consider $S_{2,3} := \{\mathsf{M}^\sigma, \mathsf{L}, \mathsf{L}^\sigma, \mathsf{M}\}$ (cf. Corollary 5.1.8) and we recall that $\mathsf{L}, \mathsf{M} \in \Delta_{3,6}$ are circuit forms from Example 6.1.5 (ii) and (iii), respectively. Moreover, L and M are PSD-extremal by [CL77, Theorem 3.4 and Corollary 3.3], respectively, and thus also the permuted forms $\mathsf{M}^\sigma, \mathsf{L}, \mathsf{L}^\sigma, \mathsf{M} \in \Delta_{3,6}$ are PSD-extremal circuit forms. Moreover, we compute $j(\mathsf{M}^\sigma) = 6$, $j(\mathsf{L}) = 7$, $j(\mathsf{L}^\sigma) = 8$, $j(\mathsf{M}) = 9$ and deduce $i(\mathsf{M}^\sigma) = 3$, $i(\mathsf{L}) = 4$, $i(\mathsf{L}^\sigma) = 5$, $i(\mathsf{M}) = 6$ from Theorem 6.2.12. It therefore follows $|\{i(f) \mid f \in S_{2,3}\}| = 4 \geq \mu(2,3) + 1$. Hence, using Lemma 3.2.21 (i), we conclude $|\{i(f) \mid f \in S_{2,3}\}| = 4 = \mu(2,3) + 1$. This shows that $S_{2,3}$ is a complete set of separating forms for ($\mathcal{CF}$) in the basic non-Hilbert case $(3,6)$, $\mu(2,3) = 3$ and

$$C_3 \subsetneq C_4 \subsetneq C_5 \subsetneq C_6 \subsetneq C_7. \qquad \blacksquare$$

Remark 6.3.2. *In the above proof, we in particular gave an alternative proof for Corollary 5.1.8 by computing* $i(\mathsf{M}^\sigma) = 3$, $i(\mathsf{L}) = 4$, $i(\mathsf{L}^\sigma) = 5$ *and* $i(\mathsf{M}) = 6$ *using Theorem 6.2.12 and the PSD-extremality of these circuit ternary sextics.*

Alternative Proof of Theorem 5.2.2. Observation 5.2.1 states

$$\Sigma_{n+1,4} = C_0 = \ldots = C_n \subseteq C_{n+1} \subseteq \ldots \subseteq C_{k-n} = \mathcal{P}_{n+1,4}.$$

Hence, $\mu(n,2) \leq k(n,2) - 2n - 1$ and $\mu(n,2) = k(n,2) - 2n - 1$ if and only if each remaining inclusion is strict. Let us consider the set $S_{n,2} := S_1 \cup S_2 \cup S_3$ given by

$$\begin{aligned} F_1 &:= \{\mathsf{C}^\sigma(X_0, X_j, X_{j+1}, X_n) \mid j = 1, \ldots, n-2\}, \\ F_2 &:= \{\mathsf{C}^\tau(X_0, X_1, X_j, X_{l+1}) \mid 2 \leq j \leq l \leq n-1\}, \\ F_3 &:= \{\mathsf{C}(X_0, X_1, X_{n-1}, X_n)\} \end{aligned}$$

(cf. Corollary 5.2.5) and, similarly as in the above alternative proof of Theorem 5.1.2, observe that each $f \in S_{n,2}$ is a PSD-extremal circuit form that is not SOS. Moreover, we also compute

(i) $m_{j(\mathsf{C}^\sigma(X_0,X_j,X_{j+1},X_n))}(X) = X_{j+1}^2$ for $j = 1, \ldots, n-2$,

(ii) $m_{j(\mathsf{C}^\tau(X_0,X_1,X_j,X_{l+1}))}(X) = X_j X_{l+1}$ for $2 \leq j \leq l \leq n-1$,

(iii) $m_{j(\mathsf{C}(X_0,X_1,X_{n-1},X_n))}(X) = X_n^2$.

Theorem 6.2.12 thus implies $i(f) \neq i(g)$ for any distinct $f, g \in S_{n,2}$ and we conclude $|\{i(f) \mid f \in S_{n,2}\}| = |S_{n,2}| = k(n,2) - 2n \geq \mu(n,2) + 1$. Hence, using Lemma 3.2.21 (i), $|\{i(f) \mid f \in S_{n,2}\}| = k(n,2) - 2n = \mu(n,2) + 1$ follows. Thus, for $n \geq 4$, we see that $S_{n,2}$ is a complete set of separating forms for ($\mathcal{CF}$) in the non-Hilbert case $(n+1, 4)$, $\mu(n, 2) = k(n, 2) - 2n - 1$ and the inclusion $C_i \subsetneq C_{i+1}$ is strict for $i = n, \dots, k-n-1$. ∎

Remark 6.3.3. *In the original proof of Theorem 5.2.2 from Chapter 5, we distinguished three cases for $i = n, \dots, k-n-1$. These were*

(i) $m_{n+i}(X) = X_j X_n$ *for some* $1 \leq j \leq n-2$,

(ii) $m_{n+i}(X) = X_j X_l$ *for some* $2 \leq j \leq l \leq n-1$,

(iii) $m_{n+i}(X) = m_{k-1}(X)$

and we verified the existence of separating forms for $C_i \subsetneq C_{i+1}$ in each case. In the above alternative proof, corresponding to the cases (i) – (iii), we gave an alternative argument for the separating property of the PSD-extremal circuit $(n+1)$-ary quartics

(i) $\mathsf{C}^\sigma(X_0, X_j, X_{j+1}, X_n)$ *for* $j = 1, \dots, n-2$,

(ii) $\mathsf{C}^\tau(X_0, X_1, X_j, X_{l+1})$ *for* $2 \leq j \leq l \leq n-1$,

(iii) $\mathsf{C}(X_0, X_1, X_{n-1}, X_n)$

using Theorem 6.2.12. We hence also provided an alternative proof for Corollary 5.2.5.

Alternative Proof of Theorem 5.2.14. Observation 5.2.11 states

$$\Sigma_{n+1,6} = C_0 = \ldots = C_n \subseteq C_{n+1} \subseteq \ldots \subseteq C_{k-n} = \mathcal{P}_{n+1,6}.$$

Hence, $\mu(n, 3) \leq k(n, 3) - 2n - 1$ and $\mu(n, 3) = k(n, 3) - 2n - 1$ if and only if each remaining inclusion is strict. Let us consider the set

$$S_{n,3} := \bigcup_{j=1}^{4} F_j \cup G \cup (H_1 \cup H_2 \cup H_3)$$

that is given by

$$\begin{aligned}
F_1 &:= \{\mathsf{M}^\sigma(X_0, X_1, X_n)\}, \\
F_2 &:= \{\mathsf{L}(X_0, X_j, X_{l+1}) \mid 1 \leq j \leq l \leq n-1\}, \\
F_3 &:= \{\mathsf{L}^\sigma(X_0, X_j, X_{l+1}) \mid 1 \leq j \leq l \leq n-1\}, \\
F_4 &:= \{\mathsf{M}(X_0, X_j, X_{j+1}) \mid j = 1, \dots, n-1\}, \\
G &:= \{X_1^2 \mathsf{C}^\tau(X_0, X_j, X_l, X_{r+1}) \mid 1 \leq j \leq l \leq r \leq n-1\}, \\
H_1 &:= \{X_0^2 \mathsf{C}^\sigma(X_0, X_j, X_{j+1}, X_n) \mid j = 1, \dots, n-2\}, \\
H_2 &:= \{X_0^2 \mathsf{C}^\tau(X_0, X_1, X_j, X_{l+1}) \mid 2 \leq j \leq l \leq n-1\}, \\
H_3 &:= \{X_0^2 \mathsf{C}(X_0, X_1, X_{n-1}, X_n)\}
\end{aligned}$$

(cf. Corollary 5.2.17) and, similarly as in the alternative proof of Theorem 5.1.6 above, observe that each $f \in F_1 \cup F_2 \cup F_3 \cup F_4$ is a PSD-extremal circuit form that is not SOS. Moreover, each $f \in G \cup H_1 \cup H_2 \cup H_3 \cup H_4$ is a square monomial multiple of a PSD-extremal circuit form that is not SOS (cf. alternative proof of Theorem 5.2.2 above) and, therefore, also any $f \in G \cup H_1 \cup H_2 \cup H_3 \cup H_4$ is a PSD-extremal circuit form that is not SOS. We compute

(i) $m_{j(\mathsf{M}^\sigma(X_0,X_1,X_n))}(X) = X_1^3$,

(ii) $m_{j(\mathsf{L}(X_0,X_j,X_{l+1}))}(X) = X_j^2 X_{l+1}$ for $1 \leq j \leq l \leq n-1$,

(iii) $m_{j(\mathsf{L}^\sigma(X_0,X_j,X_{l+1}))}(X) = X_j X_{k+1}^2$ for $1 \leq j \leq l \leq n-1$,

(iv) $m_{j(\mathsf{M}(X_0,X_j,X_{l+1}))}(X) = X_{j+1}^3$ for $j = 1, \ldots, n-1$,

(v) $m_{j(X_1^2\mathsf{C}^\tau(X_0,X_j,X_l,X_{r+1}))}(X) = X_j X_l X_{r+1}$ for $1 \leq j \leq l \leq r \leq n-1$,

(vi) $m_{j(X_0^2\mathsf{C}^\sigma(X_0,X_j,X_{j+1},X_n))}(X) = X_0 X_{j+1}^2$ for $j = 1, \ldots, n-2$,

(vii) $m_{j(X_0^2\mathsf{C}^\tau(X_0,X_1,X_j,X_{l+1}))}(X) = X_0 X_j X_{l+1}$ for $2 \leq j \leq l \leq n-1$,

(viii) $m_{j(X_0^2\mathsf{C}(X_0,X_1,X_{n-1},X_n))}(X) = X_0 X_n^2$.

Theorem 6.2.12 thus implies $i(f) \neq i(g)$ for any distinct $f, g \in S_{n,3}$ and we conclude $|\{i(f) \mid f \in S_{n,3}\}| = |S_{n,3}| = k(n,3) - 2n \geq \mu(n,3) + 1$. Hence, using Lemma 3.2.21 (i), $|\{i(f) \mid f \in S_{n,3}\}| = k(n,3) - 2n = \mu(n,3) + 1$ follows. Thus, for $n \geq 3$, we see that $S_{n,3}$ is a complete set of separating forms for $(\mathcal{CF})$ in the non-Hilbert case $(n+1,6)$, $\mu(n,3) = k(n,3) - 2n - 1$ and the inclusion $C_i \subsetneq C_{i+1}$ is strict for $i = n, \ldots, k-n-1$. ■

Remark 6.3.4. *In the original proof of Theorem 5.2.14, applying Theorem 5.2.7 to Theorem 5.2.2, it sufficed to show $C_i \subsetneq C_{i+1}$ for $i = k(n,2)-n, \ldots, k(n,3)-n-1$. The latter, we verified in the proof of Theorem 5.2.12 by distinguishing five cases for $i = k(n,2)-n, \ldots, k(n,3)-n-1$. These were*

(i) $m_{n+i}(X) = m_{k(n,2)}(X)$,

(ii) $m_{n+i}(X) = X_j^2 X_l$ *for some* $1 \leq j \leq l \leq n-1$,

(iii) $m_{n+i}(X) = X_j X_l X_n$ *for some* $1 \leq j \leq l \leq n-1$,

(iv) $m_{n+i}(X) = X_j X_n^2$ *for some* $1 \leq j \leq n-1$,

(v) $m_{n+i}(X) = X_j X_l X_r$ *for some* $1 \leq j < l \leq r \leq n-1$

and we proved the existence of a separating form for $C_i \subsetneq C_{i+1}$ in each case. In the above alternative proof, corresponding to the cases (i)–(v), we gave an alternative argument for the separating property of the PSD-extremal circuit $(n+1)$-ary sextics

(i) $\mathsf{M}^\sigma(X_0, X_1, X_n)$,

(ii) $\mathsf{L}(X_0, X_j, X_{l+1})$ *for* $1 \leq j \leq l \leq n-1$,

(iii) $\mathsf{L}^\sigma(X_0, X_j, X_{l+1})$ *for* $1 \leq j \leq l \leq n-1$,

(iv) $\mathsf{M}(X_0, X_j, X_{j+1})$ *for* $1 \leq j \leq n-1$,

(v) $X_1^2\mathsf{C}^\tau(X_0, X_j, X_l, X_{r+1})$ *for* $1 \leq j < l \leq r \leq n-1$

using Theorem 6.2.12. We hence also provided an alternative proof for Corollary 5.2.17. Moreover, we identified $F_1 \cup F_2 \cup F_3 \cup F_4 \subseteq \Delta^{\mathfrak{C}}_{n+1,6}$ *with*

$$\begin{aligned} F_1 &:= \{\mathsf{M}^\sigma(X_0, X_1, X_n)\}, \\ F_2 &:= \{\mathsf{L}(X_0, X_j, X_{l+1}) \mid 1 \leq j \leq l \leq n-1\}, \\ F_3 &:= \{\mathsf{L}^\sigma(X_0, X_j, X_{l+1}) \mid 1 \leq j \leq l \leq n-1\}, \\ F_4 &:= \{\mathsf{M}(X_0, X_j, X_{j+1}) \mid j = 1, \ldots, n-1\} \end{aligned}$$

to be a set of PSD-extremal circuit $(n+1)$*-ary sextics such that*

$$\{i(f) \mid f \in F_1 \cup F_2 \cup F_3 \cup F_4\} = \{k(n,2) - n, \ldots, k(n,3) - n - 1\}.$$

We will now generalize our findings of the previous chapter from the non-Hilbert cases $(n+1, 6)_{n\geq 2}$ to the non-Hilbert cases $(n+1, 2d)_{n\geq 2, d\geq 4}$ by an inductive argument on the degree $2d$ with inductive base case $2d = 6$. To this end, we observe for $n \geq 2$ and $d \geq 3$ that the specific cone filtration ($\mathcal{CF}$) is given by

$$\Sigma_{n+1,2d} = C_0 \subseteq \ldots \subseteq C_n \subseteq \ldots \subseteq C_{k-n} = \mathcal{P}_{n+1,2d}. \tag{6.11}$$

Theorem 4.2.7 yields $C_0 = \ldots = C_n$ and also $C_n = C_{n+1}$ if $n = 2$. Therefore,

$$\mu(n,d) \leq \begin{cases} k(n,d) - 2n - 2, & \text{if } n = 2 \\ k(n,d) - 2n - 1, & \text{else.} \end{cases}$$

If for some particular $n \geq 2$ and $d \geq 3$, we have

$$\mu(n,d) = \begin{cases} k(n,d) - 2n - 2, & \text{if } n = 2 \\ k(n,d) - 2n - 1, & \text{else,} \end{cases}$$

then the proof of our first degree-jumping principle (cf. Theorem 5.2.7) implies that for any complete set $S_{n,d}$ of separating forms for ($\mathcal{CF}$), it holds

$$\left\{ i(X_0^2 f(X)) \mid f \in S_{n,d} \right\} = \begin{cases} \{n+1, \ldots, k(n,d) - n - 1\}, & \text{if } n = 2 \\ \{n, \ldots, k(n,d) - n - 1\}, & \text{else.} \end{cases}$$

Thus, in order to show that also

$$\mu(n,d+1) = \begin{cases} k(n,d+1) - 2n - 2, & \text{if } n = 2 \\ k(n,d+1) - 2n - 1, & \text{else} \end{cases}$$

in the inductive step $2(d+1)$, it suffices to construct $J_{n,d+1} \subseteq \Delta_{n+1,2(d+1)}$ such that

$$\{i(f) \mid f \in J_{n,d+1}\} = \{k(n,d)-n,\ldots,k(n,d+1)-n-1\}.$$

Indeed, a complete set $S_{n,d+1}$ of separating forms for $(\mathcal{CF})$ in the non-Hilbert case $(n+1,2(d+1))$ is then given by $S_{n,d+1} := \{X_0^2 f(X) \mid f \in S_{n,d}\} \cup J_{n,d+1}$.

Observation 6.3.5. *For $n \geq 2$ and $d = 3$, Corollary 5.1.8 and Corollary 5.2.15 together imply*

$$\mu(n,3) = \begin{cases} k(n,3)-2n-2, & \text{if } n = 2 \\ k(n,3)-2n-1, & \text{else} \end{cases}$$

and we identified a set $J_{n,3} \subseteq \Delta^{\mathfrak{C}}_{n+1,3}$ of PSD-extremal forms such that

$$\{i(f) \mid f \in J_{n,3}\} = \{k(n,2)-n,\ldots,k(n,3)-n-1\}$$

in the above alternative proofs of Theorem 5.1.6 and Theorem 5.2.12.

For an arbitrary but fixed $n \geq 2$, the target of our second degree-jumping principle below is to extend a given set $J_{n,3} \subseteq \Delta^{\mathfrak{C}}_{n+1,6}$, that is of the same type as the set in Observation 6.3.5 above, for $d \geq 4$, to a set $J_{n,d} \subseteq \Delta^{\mathfrak{C}}_{n+1,2d}$ such that

$$\{i(f) \mid f \in J_{n,d}\} = \{k(n,d-1)-n,\ldots,k(n,d)-n-1\}.$$

Theorem 6.3.6. Degree-Jumping Principle II
Let $n \geq 2$ and assume that there exists a set $J_{n,3} \subseteq \Delta^{\mathfrak{C}}_{n+1,6}$ of PSD-extremal forms such that $\{i(f) \mid f \in J_{n,3}\} = \{k(n,2)-n,\ldots,k(n,3)-n-1\}$, then, for $d \geq 4$, there exists a set $J_{n,d} \subseteq \Delta^{\mathfrak{C}}_{n+1,2d}$ of PSD-extremal forms such that

$$\{i(f) \mid f \in J_{n,d}\} = \{k(n,d-1)-n,\ldots,k(n,d)-n-1\}.$$

Proof. We verify the assertion by an induction on $d \geq 3$ and note that the inductive base case $d = 3$ is solved by assumption. Thus, we assume that the assertion was already verified up to some $d \geq 3$ and investigate the situation for $d+1$.

For the purpose of this proof, we denote the lexicographically ordered monomial basis of $\mathcal{F}_{n+1,\delta}$ by $\left\{m_0^{(\delta)},\ldots,m_{k(n,\delta)}^{(\delta)}\right\}$ and the corresponding exponents by $\alpha_0^{(\delta)},\ldots,\alpha_{k(n,\delta)}^{(\delta)}$ for $\delta \in \{d,d+1\}$.

For $i = k(n,d)-n,\ldots,k(n,d+1)-n-1$, we moreover set

$$l := \max\left\{l \in \{1,\ldots,n\} \mid X_l^{d+1} \geq_{\text{lex}} m_{n+i+1}^{(d+1)}(X)\right\}$$

and observe that $m_{n+i+1}^{(d+1)}$ is divisible by X_l. We let $s \in \{0,\ldots,k(n,d)\}$ be such that

$$X_l m_s^{(d)}(X) = m_{n+i+1}^{(d+1)}(X) \tag{6.12}$$

and deduce $X_l^d \geq_{\text{lex}} m_s^{(d)}(X)$ from the fact that $X_l^{d+1} \geq_{\text{lex}} m_{n+i+1}^{(d+1)}(X)$. It follows $s \geq k(n,d) - k(n-l,d)$ and we conclude for $j := s - n - 1$ that

$$j \in \{k(n,d) - k(n-l,d) - n - 1, \ldots, k(n,d) - n - 1\}.$$

Furthermore, we see that

$$X_l m_{n+j+1}^{(d)}(X) = X_l m_s^{(d)}(X) = m_{n+i+1}^{(d+1)}(X).$$

Since $j \geq k(n,d) - k(n-l,d) - n - 1 \geq k(n,d-1) - n$, the inductive assumption secures the existence of a PSD-extremal form $f \in \Delta_{n+1,2d}^{\mathfrak{C}}$ such that $i(f) = j$ and Theorem 6.2.12 gives $j(f) = i(f) + n + 1$. Hence, $j(f) = j + n + 1 = s$ and thus $m_{j(f)}^{(d)}(X) = m_s^{(d)}(X)$ follows. Consequently, $g(X) := X_l^2 f(X) \in \Delta_{n+1,2(d+1)}^{\mathfrak{C}}$ is a PSD-extremal form such that

$$m_{j(g)}^{(d+1)}(X) = X_l m_s^{(d)}(X) \underset{(6.12)}{=} m_{n+i+1}^{(d+1)}(X).$$

We conclude $i(g) = j(g) - n - 1 = i$ using Theorem 6.2.12. ■

Corollary 6.3.7. *For $n \geq 2$ and $d \geq 4$, there exists a set $J_{n,d} \subseteq \Delta_{n+1,2d}^{\mathfrak{C}}$ of PSD-extremal forms such that $\{i(f) \mid f \in J_{n,d}\} = \{k(n,d-1) - n, \ldots, k(n,d) - n - 1\}$.*

Proof. We set $J_{2,3} := \{\mathsf{M}^\sigma, \mathsf{L}, \mathsf{L}^\sigma, \mathsf{M}\}$ and $J_{n,3} := F_1 \cup F_2 \cup F_3 \cup F_4$ with

$$\begin{aligned}
F_1 &:= \{\mathsf{M}^\sigma(X_0, X_1, X_n)\}, \\
F_2 &:= \{\mathsf{L}(X_0, X_j, X_{l+1}) \mid 1 \leq j \leq l \leq n-1\}, \\
F_3 &:= \{\mathsf{L}^\sigma(X_0, X_j, X_{l+1}) \mid 1 \leq j \leq l \leq n-1\}, \\
F_4 &:= \{\mathsf{M}(X_0, X_j, X_{j+1}) \mid j = 1, \ldots, n-1\}
\end{aligned}$$

for $n \geq 3$. Hence, Remark 6.3.2 and Remark 6.3.4 together yield that $J_{n,3} \subseteq \Delta_{n+1,6}^{\mathfrak{C}}$ is a set of PSD-extremal forms such that

$$\{i(f) \mid f \in J_{n,3}\} = \{k(n,2) - n, \ldots, k(n,3) - n - 1\}$$

for $n \geq 2$. The assertion therefore follows from Theorem 6.3.6. ■

Theorem 6.3.8. *For $(n+1, 2d)_{n \geq 2, d \geq 4}$ and $i = n+1, \ldots, k-n-1$, the inclusion $C_i \subsetneq C_{i+1}$ is strict. Moreover, if $n \geq 3$, then also the inclusion $C_n \subsetneq C_{n+1}$ is strict.*

Proof. We fix $n \geq 2$ and recall from Theorem 5.1.6 and Theorem 5.2.14 that in the non-Hilbert cases of $(n+1)$-ary sextics, the inclusion $C_i \subsetneq C_{i+1}$ is strict for $i = n+1, \ldots, k-n-1$. Moreover, if $n \geq 3$, then also the inclusion $C_n \subsetneq C_{n+1}$ is strict. In an inductive argument, we therefore assume that the assertion was already verified up to some $d \geq 3$ and investigate the situation for $d+1$.

By the inductive assumption, each inclusion in $C_{n+1} \subsetneq \ldots \subsetneq C_{k(n,d)-n}$ as subcones of $\mathcal{P}_{n+1,2d}$ is strict. Thus, also each inclusion in $C_{n+1} \subsetneq \ldots \subsetneq C_{k(n,d)-n}$ as subcones of $\mathcal{P}_{n+1,2d+2}$ is strict by the first degree-jumping principle (cf. Theorem 5.2.7). The same is true for the inclusion $C_n \subsetneq C_{n+1}$ if $n \geq 3$. Moreover, Corollary 6.3.7, as a consequence of the second degree-jumping principle (cf. Theorem 6.3.6), yields that each inclusion in the cone filtration $C_{k(n,d)-n} \subsetneq \ldots \subsetneq C_{k(n,d+1)-n}$ is strict. ■

Corollary 6.3.9. *For $(n+1, 2d)_{n\geq 2, d\geq 4}$, it holds*

$$\mu(n,d) = \begin{cases} k(n,d) - 2n - 2, & \text{if } n = 2 \\ k(n,d) - 2n - 1, & \text{else.} \end{cases}$$

Proof. Theorem 4.2.7 and Theorem 6.3.8 together imply

$$\begin{cases} \Sigma_{n+1,2d} = C_0 = \ldots = C_n = C_{n+1} \subsetneq \ldots \subsetneq C_{k(n,d)-n} = \mathcal{P}_{n+1,2d}, & \text{if } n = 2 \\ \Sigma_{n+1,2d} = C_0 = \ldots = C_n \subsetneq C_{n+1} \subsetneq \ldots \subsetneq C_{k(n,d)-n} = \mathcal{P}_{n+1,2d}, & \text{else.} \end{cases}$$

Hence, there are

$$\begin{cases} (k(n,d) - n) - (n+1) - 1, & \text{if } n = 2 \\ (k(n,d) - n) - (n+1), & \text{else} \end{cases}$$

strictly separating cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ in the above filtrations. ■

The above proof of Theorem 6.3.8 makes use of our two degree-jumping principles given in Theorem 5.2.7 and Theorem 6.3.6, respectively. The verification of these principles was constructive and thus provides an iterative method to establish complete sets $S_{n,d} \subseteq \Delta^{\mathfrak{C}}_{n+1,2d}$ of separating forms for $(\mathcal{CF})$ such that any $f \in S_{n,d}$ is PSD-extremal in any non-Hilbert case $(n+1, 2d)_{n\geq 2, d\geq 4}$.

Algorithm 6.3.10. *Input*: Natural numbers $n \geq 2$, $d \geq 4$ and a complete set of separating forms $S_{n,d-1} \subseteq \Delta^{\mathfrak{C}}_{n+1,2(d-1)}$ for $(\mathcal{CF})$ in the non-Hilbert case $(n+1, 2(d-1))$ that consists of PSD-extremal forms.

(1) Set $\mathfrak{M}_{n,d-1} := \{X_0^2 f(X) \mid f \in S_{n,d-1}\}$.

(2) For $i = k(n, d-1) - n, \ldots, k(n,d) - n - 1$,

 (a) set $l := \max\left\{l \in \{1, \ldots, n\} \mid X_l^d \geq_{\text{lex}} m^{(d)}_{n+i+1}(X)\right\}$ and

 (b) let $j \in \{k(n,d-1) - k(n-l, d-1) - n - 1, \ldots, k(n, d-1) - n - 1\}$ be such that $X_l m^{(d-1)}_{n+j+1}(X) = m^{(d)}_{n+i+1}(X)$.

 (c) Choose $f \in S_{n,d-1}$ such that $i(f) = j$ and

 (d) set $g_i(X) := X_l^2 f(X)$.

(3) Set $J_{n,d} := \left\{g_{k(n,d-1)-n}, \ldots, g_{k(n,d)-n-1}\right\}$.

(4) Set $S_{n,d} := \mathfrak{M}_{n,d-1} \cup J_{n,d}$.

Output: A complete set of separating forms $S_{n,d}\subseteq\Delta^{\mathfrak{C}}_{n+1,2d}$ for ($\mathcal{CF}$) in the non-Hilbert case $(n+1,2d)$ that consists of PSD-extremal forms.

Proof of Correct Output. The set $\mathfrak{M}_{n,d-1}$ consist of PSD-extremal circuit forms that are not SOS since any $f\in S_{n,d-1}$ is assumed to be a PSD-extremal circuit form that is not SOS and X_0^2 is a monomial square. Moreover, Lemma 6.1.14 and Theorem 6.2.12 together imply $i(X_0^2f(X)) = j(X_0^2f(X))-n-1 = j(f)-n-1 = i(f)$ for any $f\in S_{n,d-1}$. Hence,

$$\begin{aligned}\{i(g)\mid g\in\mathfrak{M}_{n,d-1}\} &= \{i(f)\mid f\in S_{n,d-1}\}\\ &= \begin{cases}\{n+1,\dots,k(n,d-1)-n-1\}, & \text{if } n=2\\ \{n,\dots,k(n,d-1)-n-1\}, & \text{else.}\end{cases}\end{aligned}$$

The proof of Theorem 6.3.6 moreover justifies that Step (2) of the above algorithm can be carried out for $i=k(n,d-1)-n,\dots,k(n,d)-n-1$ and yields a PSD-extremal $g_i\in\Delta^{\mathfrak{C}}_{n+1,2d}$ such that $i(g_i)=i$. Hence, $J_{n,d}\subseteq\Delta^{\mathfrak{C}}_{n+1,2d}$ from Step (3) is a set of PSD-extremal forms such that $\{i(g)\mid f\in J_{n,d}\}=\{k(n,d-1)-n,\dots,k(n,d)-n-1\}$. Therefore, $S_{n,d}\subseteq\Delta^{\mathfrak{C}}_{n+1,2d}$ from Step (4) is a set of PSD-extremal forms such that

$$\{i(g)\mid g\in S_{n,d}\} = \begin{cases}\{n+1,\dots,k(n,d)-n-1\}, & \text{if } n=2\\ \{n,\dots,k(n,d)-n-1\} & \text{else.}\end{cases}$$

We conclude $|\{i(g)\mid g\in S_{n,d}\}| = \mu(n,d)+1$ from Corollary 6.3.9. Thus, $S_{n,d}$ is a complete set of separating forms for ($\mathcal{CF}$) in the non-Hilbert case $(n+1,2d)$. ■

Example 6.3.11. Ternary Octics
Let us run Algorithm 6.3.10 to establish a complete set of separating forms $S_{2,4}\subseteq\Delta^{\mathfrak{C}}_{3,8}$ for ($\mathcal{CF}$) in the non-Hilbert case $(3,8)$ that consists of PSD-extremal forms. Hence, $n=2$, $d=4$ and, recalling Remark 6.3.2, we choose $S_{2,3}:=\{\mathsf{M}^\sigma,\mathsf{L},\mathsf{L}^\sigma,\mathsf{M}\}$ as a complete set of separating forms for ($\mathcal{CF}$) in the non-Hilbert case $(3,6)$ that consists of PSD-extremal forms for the input of the algorithm. In Step (1), we set

$$\mathfrak{M}_{2,3}:=\left\{X_0^2f(X)\mid f\in S_{2,3}\right\}=\left\{X_0^2\mathsf{M}^\sigma(X),X_0^2\mathsf{L}(X),X_0^2\mathsf{L}^\sigma(X),X_0^2\mathsf{M}(X)\right\}.$$

Step (2) now runs for $i=7,\dots,11$ since $k(n,d-1)-n=k(2,3)-2=7$ and $k(n,d)-n-1=k(2,4)-3=11$.

For $i=7$, $l_7:=\max\left\{l\in\{1,2\}\mid X_l^4\geq_{\text{lex}} m_{10}^{(4)}(X)=X_1^4\right\}=1$ and we compute

$$m_{n+i+1}^{(d)}=m_{10}^{(4)}=X_1^4=X_1\cdot\left(X_1^3\right)=X_1m_6^{(3)}(X).$$

Therefore, we set $j_7:=6-n-1=3$ and observe

$$k(n,d-1)-k(n-l,d-1)-n-1=3\leq j_7\leq 6=k(n,d-1)-n-1.$$

Step (2c) thus requires us to choose some $f \in S_{2,3}$ such that $i(f) = 3$. Remark 6.3.2 yields that $f := \mathsf{M}^\sigma$ is an appropriate choice and we set $g_7(X) := X_1^2\mathsf{M}^\sigma(X)$ as proposed by Step (2d). Running Step (2) for $i = 8, \ldots, 11$, we similarly obtain

$$\begin{array}{llll} l_8 = 1, & j_8 = 4 & \text{and} & g_8(X) = X_1^2\mathsf{L}(X), \\ l_9 = 1, & j_9 = 5 & \text{and} & g_9(X) = X_1^2\mathsf{L}^\sigma(X), \\ l_{10} = 1, & j_{10} = 6 & \text{and} & g_{10}(X) = X_1^2\mathsf{M}(X), \\ l_{11} = 2, & j_{11} = 6 & \text{and} & g_{11}(X) = X_2^2\mathsf{M}(X). \end{array}$$

Following Step (3), we set $J_{2,4} := \{g_7, \ldots, g_{11}\}$. Hence, $S_{2,4} := \mathfrak{M}_{2,3} \cup J_{2,4} \subseteq \Delta^{\mathfrak{C}}_{3,8}$ is a complete set of separating forms for $(\mathcal{CF})$ in the non-Hilbert case $(3,8)$ that consists of PSD-extremal forms by Step (4). To be explicit,

$$\begin{aligned} S_{2,4} \;=\; & \Big\{X_0^2\mathsf{M}^\sigma(X), X_0^2\mathsf{L}(X), X_0^2\mathsf{L}^\sigma(X), X_0^2\mathsf{M}(X), X_1^2\mathsf{M}^\sigma(X), X_1^2\mathsf{L}(X), \\ & X_1^2\mathsf{L}^\sigma(X), X_1^2\mathsf{M}(X), X_2^2\mathsf{M}(X)\Big\}. \end{aligned}$$

The above example illuminates that Algorithm 6.3.10 allows us to keep track of the monomials appearing in the forms of our algorithmically established complete sets of separating forms for $(\mathcal{CF})$. This observation will be crucial for our investigation in Section 7.2 and is therefore made precise in the theorem below.

Theorem 6.3.12. *Let $(n+1, 2d)$ be a non-Hilbert case and $i = n+1, \ldots, k-n-1$, then there exists a PSD-extremal circuit form $f_i \in (C_{i+1} \backslash C_i) \cap \mathfrak{S}_{n+i+1}$. Moreover, if $n \geq 3$, then there also exists a PSD-extremal circuit form $f_n \in (C_{n+1} \backslash C_n) \cap \mathfrak{S}_{2n+1}$.*

Proof. In this proof, we combine plenty of the observations made in Chapter 5 and this chapter. Therefore, the proof is lengthy and we thus firstly sketch the to-be-taken steps for the convenience of the reader before secondly going into detail.

Outline of the Proof:

(1) We verify the assertion for the basic non-Hilbert case $(4,4)$ using the circuit quaternary quartics $\mathsf{C}^\sigma, \mathsf{C}^\tau$ and C identified in the proof of Theorem 5.1.2.

(2) Similarly as done in the proof of Theorem 5.2.2, we extend our consideration of (1) from the basic non-Hilbert case $(4,4)$ to the non-Hilbert cases $(n+1,4)_{n\geq 4}$.

(3) We verify the assertion for the basic non-Hilbert case $(3,6)$ using the ternary sextics $\mathsf{M}^\sigma, \mathsf{L}, \mathsf{L}^\sigma$ and M identified in the proof of Theorem 5.1.6.

(4) Similarly as done in the proof of Theorem 5.2.14, we firstly use the construction made in the proof of Theorem 5.2.7 to partly verify the assertion by extending our findings of (2) from the non-Hilbert cases $(n+1,4)_{n\geq 3}$ to the non-Hilbert cases $(n+1,6)_{n\geq 2}$. Secondly, we fully prove the assertion by extending our consideration of (3) from the basic non-Hilbert case $(3,6)$ to the non-Hilbert cases $(n+1,6)_{n\geq 3}$.

(5) Similarly as done in the proof of Theorem 6.3.8, we fix $n\geq 2$ and generalize our findings of (3) and (4) from the non-Hilbert case $(n+1,6)$ to the non-Hilbert cases $(n+1,2d)_{d\geq 4}$ by an induction on the degree $2d$ using the construction made in the proof of Theorem 6.3.6.

Detailed Proof:

(1) In the basic non-Hilbert case $(4,4)$, Remark 6.3.1 especially implies $\mathsf{C}^\sigma\in C_4\backslash C_3$, $\mathsf{C}^\tau\in C_5\backslash C_4$, $\mathsf{C}\in C_6\backslash C_5$ and states that each of these distinguished circuit quaternary quartics is PSD-extremal. Moreover, recalling the proof of Theorem 5.1.2 and Remark 5.1.3, we know

$$\begin{aligned}\mathsf{C}^\sigma &= m_1^2+m_3^2+m_6^2+m_7^2-4m_2m_6\in\mathfrak{S}_7,\\ \mathsf{C}^\tau &= m_5^2+m_6^2+m_8^2+m_0^2-4m_2m_6\in\mathfrak{S}_8,\\ \mathsf{C} &= m_1^2+m_2^2+m_5^2+m_9^2-4m_2m_6\in\mathfrak{S}_9.\end{aligned}$$

(2) In the non-Hilbert case $(n+1,4)$ for $n\geq 4$, we now distinguish three cases for $i=n,\dots,k-n-1$:

(i) $m_{n+i}(X)=X_jX_n$ for some $1\leq j\leq n-2$

(ii) $m_{n+i}(X)=X_jX_l$ for some $2\leq j\leq l\leq n-1$

(iii) $m_{n+i}(X)=m_{k-1}(X)$

Remark 6.3.3 states that in each case a PSD-extremal circuit $(n+1)$-ary quartic separating $C_i\subsetneq C_{i+1}$ is given by

(i) $\mathsf{C}^\sigma(X_0,X_j,X_{j+1},X_n)$ for $j=1,\dots,n-2$,

(ii) $\mathsf{C}^\tau(X_0,X_1,X_j,X_{l+1})$ for $2\leq j\leq l\leq n-1$,

(iii) $\mathsf{C}(X_0,X_1,X_{n-1},X_n)$

and we compute the following:

(i) For $j=1,\dots,n-2$, we see that

$$\begin{aligned}\mathsf{C}^\sigma(X_0,X_j,X_{j+1},X_n) &\underset{(5.2)}{=} (X_0X_j)^2+(X_0X_n)^2+(X_jX_n)^2+\left(X_{j+1}^2\right)^2\\ &\quad -4\,(X_0X_{j+1})\,(X_jX_n)\\ &\in \operatorname{span}_{\mathbb{R}}\left\{m_sm_t\mid m_s(X),m_t(X)\geq_{\text{lex}}X_{j+1}^2\right\}\end{aligned}$$

and $X_{j+1}^2=m_{n+i+1}(X)$. Hence, $\mathsf{C}^\sigma\in\mathfrak{S}_{n+i+1}$.

(ii) For $2 \leq j \leq l \leq n-1$, we see that

$$\begin{aligned} \mathsf{C}^{\tau}(X_0,X_1,X_j,X_{l+1}) &\underset{(5.3)}{=} (X_1X_j)^2+(X_1X_{l+1})^2+(X_jX_{l+1})^2+\left(X_0^2\right)^2 \\ &\quad -4(X_0X_j)(X_1X_{l+1}) \\ &\in \operatorname{span}_{\mathbb{R}}\{m_sm_t \mid m_s(X),m_t(X) \geq_{\text{lex}} X_jX_{l+1}\} \end{aligned}$$

and $X_jX_{l+1}=m_{n+i+1}(X)$. Hence, $\mathsf{C}^{\tau} \in \mathfrak{S}_{n+i+1}$.

(iii) Moreover, we see that

$$\begin{aligned} \mathsf{C}(X_0,X_1,X_{n-1},X_n) &\underset{(5.4)}{=} (X_0X_1)^2+(X_0X_{n-1})^2+(X_1X_{n-1})^2+\left(X_n^2\right)^2 \\ &\quad -4(X_0X_{n-1})(X_1X_n) \\ &\in \operatorname{span}_{\mathbb{R}}\left\{m_sm_t \mid m_s(X),m_t(X) \geq_{\text{lex}} X_n^2\right\} \end{aligned}$$

and $X_n^2=m_{n+i+1}(X)$. Hence, $\mathsf{C} \in \mathfrak{S}_{n+i+1}$.

(3) In the basic non-Hilbert case $(3,6)$, Remark 6.3.2 especially implies $\mathsf{M}^{\sigma} \in C_4 \backslash C_3$, $\mathsf{L} \in C_5\backslash C_4$, $\mathsf{L}^{\sigma} \in C_6\backslash C_5$, $\mathsf{M} \in C_7\backslash C_6$ and states that each of these distinguished circuit ternary sextics is PSD-extremal. Recalling Remark 5.1.7, we also know

$$\begin{aligned} \mathsf{M}^{\sigma} &= m_2^2+m_5^2+m_6^2-3m_4^2 \in \mathfrak{S}_6, \\ \mathsf{L} &= m_1^2+m_5^2+m_7^2-3m_4^2 \in \mathfrak{S}_7, \\ \mathsf{L}^{\sigma} &= m_2^2+m_3^2+m_8^2-3m_4^2 \in \mathfrak{S}_8, \\ \mathsf{M} &= m_1^2+m_3^2+m_9^2-3m_4^2 \in \mathfrak{S}_9. \end{aligned}$$

(4) We let $n \geq 3$ be arbitrary but fixed and examine the non-Hilbert case $(n+1,6)$.

For the purpose of this step, for $i=n,\ldots,k(n,2)-n$ and $\delta \in \{2,3\}$, we denote the subcones C_i and C_{i+1} of $\mathcal{F}_{n+1,2\delta}$ by $C_i^{(\delta)}$ and $C_{i+1}^{(\delta)}$, respectively. Moreover, we denote $\mathfrak{S}_{n+i+1} \subseteq \mathcal{F}_{n+1,2\delta}$ by $\mathfrak{S}_{n+i+1}^{(\delta)}$ and set the lexicographically ordered monomial basis of $\mathcal{F}_{n+1,\delta}$ to be given by $\left\{m_0^{(\delta)},\ldots,m_{k(n,\delta)}^{(\delta)}\right\}$.

Step (1) and (2) together allow us to fix a PSD-extremal circuit $(n+1)$-ary quartic $f \in \left(C_{i+1}^{(2)}\backslash C_i^{(2)}\right) \cap \mathfrak{S}_{n+i+1}^{(2)}$ for $i=n,\ldots,k(n,2)-n$. Similarly as done in the proof of Theorem 5.2.7, we set $g(X):=X_0^2f(X) \in \Delta_{n+1,2d}^{\mathfrak{C}}$ and observe that g is a PSD-extremal circuit $(n+1)$-ary sextic separating $C_{i+1}^{(3)} \subsetneq C_i^{(3)}$. Moreover, $f \in \mathfrak{S}_{n+i+1}^{(2)} = \operatorname{span}_{\mathbb{R}}\left\{m_s^{(2)}m_t^{(2)} \mid m_s^{(2)},m_t^{(2)} \geq_{\text{lex}} m_{n+i+1}^{(2)}\right\}$ yields

$$\begin{aligned} g(X) &\in \operatorname{span}_{\mathbb{R}}\left\{\left(X_0m_s^{(2)}(X)\right)\left(X_0m_t^{(2)}(X)\right) \mid m_s^{(2)},m_t^{(2)} \geq_{\text{lex}} m_{n+i+1}^{(2)}\right\} \\ &= \operatorname{span}_{\mathbb{R}}\left\{m_s^{(3)}m_t^{(3)} \mid m_s^{(3)},m_t^{(3)} \geq_{\text{lex}} m_{n+i+1}^{(3)}\right\} = \mathfrak{S}_{n+i+1}^{(3)}. \end{aligned}$$

For $i=k(n,2)-n,\dots,k-n-1$, we now distinguish five cases:

(i) $m_{n+i}(X)=m_{k(n,2)}(X)$

(ii) $m_{n+i}(X)=X_j^2X_l$ for some $1\leq j\leq l\leq n-1$

(iii) $m_{n+i}(X)=X_jX_lX_n$ for some $1\leq j\leq l\leq n-1$

(iv) $m_{n+i}(X)=X_jX_n^2$ for some $1\leq j\leq n-1$

(v) $m_{n+i}(X)=X_jX_lX_r$ for some $1\leq j<l\leq r\leq n-1$

Remark 6.3.4 states that a PSD-extremal circuit $(n+1)$-ary sextic separating $C_i\subsetneq C_{i+1}$ as subcones of $\mathcal{F}_{n+1,6}$ is given by

(i) $\mathsf{M}^\sigma(X_0,X_1,X_n)$,

(ii) $\mathsf{L}(X_0,X_j,X_{l+1})$ for $1\leq j\leq l\leq n-1$,

(iii) $\mathsf{L}^\sigma(X_0,X_j,X_{l+1})$ for $1\leq j\leq l\leq n-1$,

(iv) $\mathsf{M}(X_0,X_j,X_{j+1})$ for $1\leq j\leq n-1$,

(v) $X_1^2\mathsf{C}^\tau(X_0,X_j,X_l,X_{r+1})$ for $1\leq j<l\leq r\leq n-1$

and we compute the following:

(i) We see that

$$\begin{aligned}\mathsf{M}^\sigma(X_0,X_1,X_n) &\underset{(5.7)}{=} \left(X_0^2X_n\right)^2+\left(X_0X_n^2\right)^2+\left(X_1^3\right)^2-3\left(X_0X_1X_n\right)^2\\ &\in \operatorname{span}_{\mathbb{R}}\left\{m_sm_t \mid m_s(X),m_t(X)\geq_{\text{lex}} X_1^3\right\}\end{aligned}$$

and $X_1^3=m_{n+i+1}(X)$. Hence, $\mathsf{M}^\sigma\in\mathfrak{S}_{n+i+1}$.

(ii) For $1\leq j\leq l\leq n-1$, we see that

$$\begin{aligned}\mathsf{L}(X_0,X_j,X_{l+1}) &\underset{(5.8)}{=} \left(X_0^2X_j\right)^2+\left(X_0X_{l+1}^2\right)^2+\left(X_j^2X_{l+1}\right)^2\\ &\quad -3\left(X_0X_jX_{l+1}\right)^2\\ &\in \operatorname{span}_{\mathbb{R}}\left\{m_sm_t \mid m_s(X),m_t(X)\geq_{\text{lex}} X_j^2X_{l+1}\right\}\end{aligned}$$

and $X_j^2X_{l+1}=m_{n+i+1}(X)$. Hence, $\mathsf{L}\in\mathfrak{S}_{n+i+1}$.

(iii) For $1\leq j\leq l\leq n-1$, we see that

$$\begin{aligned}\mathsf{L}^\sigma(X_0,X_j,X_{l+1}) &\underset{(5.9)}{=} \left(X_0^2X_{l+1}\right)^2+\left(X_0X_j^2\right)^2+\left(X_jX_{l+1}^2\right)^2\\ &\quad -3(X_0X_jX_{l+1})^2\\ &\in \operatorname{span}_{\mathbb{R}}\left\{m_sm_t \mid m_s(X),m_t(X)\geq_{\text{lex}} X_jX_{l+1}^2\right\}\end{aligned}$$

and $X_jX_{l+1}^2=m_{n+i+1}(X)$. Hence, $\mathsf{L}^\sigma\in\mathfrak{S}_{n+i+1}$.

(iv) For $1 \leq j \leq n-1$, we see that

$$\begin{aligned} \mathsf{M}(X_0, X_j, X_{l+1}) &\underset{(5.10)}{=} \left(X_0^2 X_j\right)^2 + \left(X_0 X_j^2\right)^2 + \left(X_{j+1}^3\right)^2 \\ &\quad -3(X_0 X_j X_{j+1})^2 \\ &\in \operatorname{span}_{\mathbb{R}}\left\{m_s m_t \mid m_s(X), m_t(X) \geq_{\text{lex}} X_{j+1}^3\right\} \end{aligned}$$

and $X_{j+1}^3 = m_{n+i+1}(X)$. Hence, $\mathsf{M} \in \mathfrak{S}_{n+i+1}$.

(v) For $1 \leq j < l \leq r \leq n-1$, we see that

$$\begin{aligned} X_1^2 \mathsf{C}^\tau(X_0, X_j, X_l, X_{r+1}) &\underset{(6.8)}{=} \left(X_j^2 X_l\right)^2 + \left(X_j^2 X_{r+1}\right)^2 + (X_j X_l X_{r+1})^2 \\ &\quad + \left(X_0^2 X_j\right)^2 - 4\left(X_0 X_j^2\right)(X_j X_l X_{r+1}) \\ &\in \operatorname{span}_{\mathbb{R}}\{m_s m_t \mid m_s(X), m_t(X) \geq_{\text{lex}} X_{j+1}^3\} \end{aligned}$$

and $X_j X_l X_{r+1} = m_{n+i+1}(X)$. Hence, $X_1^2 \mathsf{C}^\tau \in \mathfrak{S}_{n+i+1}$.

(5) For the non-Hilbert cases $(n+1, 2d)_{n\geq 2, d\geq 4}$, we let $n \geq 2$ be arbitrary but fixed and argue by an induction on the degree $2d$. Recalling our findings from Step (3) and Step (4), we see that the assertion was already verified for the non-Hilbert case $(n+1, 6)$. The non-Hilbert case $(n+1, 6)$ thus acts as the base cases for the induction. Therefore, we now assume that the assertion was already verified up to some $d \geq 3$ and investigate the situation for $d+1$.

For the purpose of the inductive step, for $i = n+1, \ldots, k(n,d) - n - 1$ and $\delta \in \{d, d+1\}$, we denote the subcones C_i and C_{i+1} of $\mathcal{F}_{n+1,2\delta}$ by $C_i^{(\delta)}$ and $C_{i+1}^{(\delta)}$, respectively. Moreover, we denote $\mathfrak{S}_{n+i+1} \subseteq \mathcal{F}_{n+1,2\delta}$ and also $\mathfrak{S}_{2n+1}$ if $n \geq 3$ by $\mathfrak{S}_{n+i+1}^{(\delta)}$ and $\mathfrak{S}_{2n+1}^{(\delta)}$, respectively. Furthermore, we set the lexicographically ordered monomial basis of $\mathcal{F}_{n+1,\delta}$ to be given by $\left\{m_0^{(\delta)}, \ldots, m_{k(n,\delta)}^{(\delta)}\right\}$.

The inductive assumption allows us to fix a complete set $S_{n,d} \subseteq \Delta_{n+1,2d}^{\mathfrak{C}}$ of separating forms for $(\mathcal{CF})$ in the non-Hilbert case $(n+1, 2d)$ that consists of PSD-extremal forms such that for $i = n+1, \ldots, k(n,d) - n - 1$, there exists some $f \in S_{n,d}$ with $f \in \left(C_{i+1}^{(d)} \backslash C_i^{(d)}\right) \cap \mathfrak{S}_{n+i+1}^{(d)}$. If $n \geq 3$, then there also exists some $f \in S_{n,d}$ with $f \in \left(C_{n+1}^{(d)} \backslash C_n^{(d)}\right) \cap \mathfrak{S}_{2n+1}^{(d)}$.

We set $S_{n,d+1} \subseteq \Delta_{n+1,2(d+1)}^{\mathfrak{C}}$ to be the complete set of separating forms for $(\mathcal{CF})$ in the non-Hilbert case $(n+1, 2(d+1))$ that is generated from the set $S_{n,d}$ by Algorithm 6.3.10. Hence, any form in $S_{n,d+1}$ is PSD-extremal and it remains to show that, for $i = n+1, \ldots, k(n, d+1) - n - 1$, there exists some $g \in S_{n,d+1}$ with $g \in \left(C_{i+1}^{(d+1)} \backslash C_i^{(d+1)}\right) \cap \mathfrak{S}_{n+i+1}^{(d+1)}$ and, moreover, that there also exists some $g \in S_{n,d+1}$ with $g \in \left(C_{n+1}^{(d+1)} \backslash C_n^{(d+1)}\right) \cap \mathfrak{S}_{2n+1}^{(d+1)}$ if $n \geq 3$.

To see this, for $i = n+1, \ldots, k(n,d) - n - 1$, we fix some $f \in S_{n,d}$ that satisfies $f \in \left(C_{i+1}^{(d)} \backslash C_i^{(d)}\right) \cap \mathfrak{S}_{n+i+1}^{(d)}$ and, similarly as done in the proof of Theorem 5.2.7, conclude for $g(X) := X_0^2 f(X) \in S_{n,d+1}$ that $g \in C_{i+1}^{(d+1)} \backslash C_i^{(d+1)}$. Furthermore, $f \in \mathfrak{S}_{n+i+1}^{(d)} = \operatorname{span}_{\mathbb{R}} \left\{ m_s^{(d)} m_t^{(d)} \mid m_s^{(d)}, m_t^{(d)} \geq_{\text{lex}} m_{n+i+1}^{(d)} \right\}$ yields

$$\begin{aligned} g(X) &\in \operatorname{span}_{\mathbb{R}} \left\{ \left(X_0 m_s^{(d)}(X) \right) \left(X_0 m_t^{(d)}(X) \right) \mid m_s^{(d)}, m_t^{(d)} \geq_{\text{lex}} m_{n+i+1}^{(d)} \right\} \\ &= \operatorname{span}_{\mathbb{R}} \left\{ m_s^{(d+1)} m_t^{(d+1)} \mid m_s^{(d+1)}, m_t^{(d+1)} \geq_{\text{lex}} m_{n+i+1}^{(d+1)} \right\} = \mathfrak{S}_{n+i+1}^{(d+1)}. \end{aligned}$$

Moreover, if $n \geq 3$, then we let $f \in S_{n,d}$ be such that $f \in \left(C_{n+1}^{(d)} \backslash C_n^{(d)} \right) \cap \mathfrak{S}_{2n+1}^{(d)}$ and conclude for $g(X) := X_0^2 f(X) \in S_{n,d+1}$ that $g \in \left(C_{n+1}^{(d+1)} \backslash C_n^{(d+1)} \right) \cap \mathfrak{S}_{2n+1}$ by the same argument.

For $i = k(n,d) - n, \ldots, k(n,d+1) - n - 1$, we lastly set

$$l := \max \left\{ l \in \{1, \ldots, n\} \mid X_l^{d+1} \geq_{\text{lex}} m_{n+i+1}^{(d+1)}(X) \right\}$$

and let $j \in \{k(n,d) - k(n-l,d) - n - 1, \ldots, k(n,d) - n - 1\}$ be such that $X_l m_{n+j+1}^{(d)}(X) = m_{n+i+1}^{(d+1)}(X)$. We moreover fix some $f \in S_{n,d}$ that satisfies $f \in \left(C_{j+1}^{(d)} \backslash C_j^{(d)} \right) \cap \mathfrak{S}_{n+j+1}^{(d)}$ and set $g(X) := X_l^2 f(X) \in S_{n,d+1}$. Recalling the proof of Theorem 6.3.6, we thus have $g \in C_{i+1} \backslash C_i$ and we furthermore see that $f \in \operatorname{span}_{\mathbb{R}} \left\{ m_s^{(d)} m_t^{(d)} \mid m_s^{(d)}, m_t^{(d)} \geq_{\text{lex}} m_{n+j+1}^{(d)} \right\}$ implies

$$\begin{aligned} g(X) &\in \operatorname{span}_{\mathbb{R}} \left\{ \left(X_l m_s^{(d)}(X) \right) \left(X_l m_t^{(d)}(X) \right) \mid m_s^{(d)}, m_t^{(d)} \geq_{\text{lex}} m_{n+j+1}^{(d)} \right\} \\ &= \operatorname{span}_{\mathbb{R}} \left\{ m_s^{(d+1)} m_t^{(d+1)} \mid m_s^{(d+1)}, m_t^{(d+1)} \geq_{\text{lex}} X_l m_{n+j+1}^{(d)} \right\} \\ &= \operatorname{span}_{\mathbb{R}} \left\{ m_s^{(d+1)} m_t^{(d+1)} \mid m_s^{(d+1)}, m_t^{(d+1)} \geq_{\text{lex}} m_{n+i+1}^{(d+1)} \right\} = \mathfrak{S}_{n+i+1}^{(d+1)}. \end{aligned}$$ ■

Concluding Remark. *Altogether, we answered the main query in the non-Hilbert cases $(n+1, 2d)_{n \geq 2, d \geq 4}$ by specifying the cone filtration ($\mathcal{CF}$) to be given by*

$$\begin{cases} \Sigma_{n+1,2d} = C_0 = \ldots = C_n = C_{n+1} \subsetneq \ldots \subsetneq C_{k(n,d)-n} = \mathcal{P}_{n+1,2d}, & \text{if } n = 2 \\ \Sigma_{n+1,2d} = C_0 = \ldots = C_n \subsetneq C_{n+1} \subsetneq \ldots \subsetneq C_{k(n,d)-n} = \mathcal{P}_{n+1,2d}, & \text{else.} \end{cases} \tag{6.13}$$

This especially allowed us to deduce

$$\mu(n,d) = \begin{cases} k(n,d) - 2n - 2, & \text{if } n = 2 \\ k(n,d) - 2n - 1, & \text{else.} \end{cases}$$

Chapter 7

Properties of the Cones $C_0, \ldots, C_{k-n}$

In this chapter, we close our investigation and determine properties of our specific intermediate cones in ($\mathcal{CF}$). To this end, we examine the distinguished cones for topological properties and show that they are in particular closed in Section 7.1. Furthermore, we determine their interiors and describe their boundaries.

In Section 7.2, we moreover introduce the geometric concept of spectrahedra and spectrahedral shadows. We investigate our distinguished cones for these properties and show that each identified strictly spearating intermediate cone in ($\mathcal{CF}$) is not a spectrahedral shadow in non-Hilbert cases.

7.1 Closure, Interior and Boundary

Throughout this section, we let **$(n+1, 2d)$ denote a non-Hilbert case**. The $\mathbb{R}$-vector spaces $\mathrm{Sym}_{k+1}(\mathbb{R})$, $\mathcal{F}_{n+1,2d}$ and $\mathcal{F}_{k+1,2}$ are finite-dimensional. Hence, all norms on $\mathrm{Sym}_{k+1}(\mathbb{R})$, $\mathcal{F}_{n+1,2d}$, respectively, $\mathcal{F}_{k+1,2}$ are equivalent and induce the same topology (cf. Example A.2.31). Therefore, we may endow each of the finite-dimensional $\mathbb{R}$-vector spaces $\mathrm{Sym}_{k+1}(\mathbb{R})$, $\mathcal{F}_{n+1,2d}$ and $\mathcal{F}_{k+1,2}$ with some norm $||\cdot||$ (no matter which one) and treat them as normed vector spaces. In the proofs of this section, several topological results are used. We refer the reader to Appendix A.2 for an overview.

Theorem 7.1.1. *For $i = 0, \ldots, k-n$, C_i is closed.*

Proof. The Gram map is a linear map between the finite-dimensional normed vector spaces $\mathrm{Sym}_{k+1}(\mathbb{R})$ and $\mathcal{F}_{n+1,2d}$. Therefore, the Gram map is continuous and bounded by Proposition A.2.40. The normed vector spaces $\mathrm{Sym}_{k+1}(\mathbb{R})$ and $\mathcal{F}_{n+1,2d}$ are furthermore Banach spaces by Corollary A.2.43 and thus $\mathcal{G}$ is open by the open mapping theorem (cf. Theorem A.2.44). Let us now consider the map

$$\begin{aligned} \overline{\mathcal{G}}\colon {}^{\textstyle \mathrm{Sym}_{k+1}(\mathbb{R})}\big/_{\textstyle \ker(\mathcal{G})} &\to \mathcal{F}_{n+1,2d} \\ [A] &\mapsto \overline{\mathcal{G}}([A]) := \mathcal{G}(A), \end{aligned}$$

where $[A]$ denotes the equivalence class of $A \in \mathrm{Sym}_{k+1}(\mathbb{R})$ w.r.t. $\sim_{\ker(\mathcal{G})}$.[1]

Claim 1: $\overline{\mathcal{G}}$ is a well-defined bijective linear map.

Proof. We recall from Lemma 2.3.12 that $\mathcal{G}$ is a well-defined surjective linear map.

Well-definedness: For $A, B \in \mathrm{Sym}_{k+1}(\mathbb{R})$ such that $A \sim_{\ker(\mathcal{G})} B$, we see that

$$\overline{\mathcal{G}}([A]) = \mathcal{G}(A) = \mathcal{G}(B + (A - B)) = \mathcal{G}(B) + \mathcal{G}(A - B) = \mathcal{G}(B) = \overline{\mathcal{G}}([B].$$

Linearity: For $[A], [B] \in \mathrm{Sym}_{k+1}(\mathbb{R}) \,/\, \ker(\mathcal{G})$ and $\lambda \in \mathbb{R}$, the linearity of $\mathcal{G}$ yields

$$\overline{\mathcal{G}}([A] + \lambda[B]) = \overline{\mathcal{G}}([A + \lambda B]) = \mathcal{G}(A + \lambda B) = \mathcal{G}(A) + \lambda\mathcal{G}(B) = \overline{\mathcal{G}}([A]) + \lambda\overline{\mathcal{G}}([B]).$$

Surjectivity: For $f \in \mathcal{F}_{n+1,2d}$, we fix $A \in \mathcal{G}^{-1}(f)$, which is possible by the surjectivity of $\mathcal{G}$, and conclude for $[A] \in \mathrm{Sym}_{k+1}(\mathbb{R}) \,/\, \ker(\mathcal{G})$ that $\overline{\mathcal{G}}([A]) = \mathcal{G}(A) = f$.

Injectivity: For $[A] \in \mathrm{Sym}_{k+1}(\mathbb{R}) \,/\, \ker(\mathcal{G})$ such that $\overline{\mathcal{G}}([A]) = \mathcal{G}(A)$ is the zero form, we deduce $A \in \ker(\mathcal{G})$. Therefore, A is equivalent to the $(k+1) \times (k+1)$ zero matrix O w.r.t. the kernel of $\mathcal{G}$. This shows $[A] = [O]$ in $\mathrm{Sym}_{k+1}(\mathbb{R}) \,/\, \ker(\mathcal{G})$. ■

We now interpret the quotient space $\mathrm{Sym}_{k+1}(\mathbb{R}) \,/\, \ker(\mathcal{G})$ as the topological space that is obtained by endowing the quotient set $\mathrm{Sym}_{k+1}(\mathbb{R}) \,/\, \ker(\mathcal{G})$ with the quotient topology induced by the normed vector spaces $\mathrm{Sym}_{k+1}(\mathbb{R})$.

Claim 2: $\overline{\mathcal{G}}$ is open.

Proof. Let $\pi\colon \mathrm{Sym}_{k+1}(\mathbb{R}) \to \mathrm{Sym}_{k+1}(\mathbb{R}) \,/\, \ker(\mathcal{G}), A \mapsto [A]$ be the canonical quotient map and $\mathcal{U} \subseteq \mathrm{Sym}_{k+1}(\mathbb{R}) \,/\, \ker(\mathcal{G})$ open. Thus, $U := \pi^{-1}(\mathcal{U}) \in \mathrm{Sym}_{k+1}(\mathbb{R})$ is open. Therefore, also $\mathcal{G}(U)$ is open since $\mathcal{G}$ is an open map. Hence, it suffices to show $\overline{\mathcal{G}}(\mathcal{U}) = \mathcal{G}(U)$.

($\subseteq$) For $[A] \in \mathcal{U}$, we observe $A \in \pi^{-1}([A]) \subseteq \pi^{-1}(\mathcal{U}) = U$. Therefore, it follows $\overline{\mathcal{G}}([A]) = \mathcal{G}(A) \in \mathcal{G}(U)$.

($\supseteq$) For $A \in U = \pi^{-1}(\mathcal{U})$, we have $[A] = \pi(A) \in \mathcal{U}$. Thus, $\mathcal{G}(A) = \overline{\mathcal{G}}([A]) \in \overline{\mathcal{G}}(\mathcal{U})$. ■

Claim 1 and Claim 2 together imply that $\overline{\mathcal{G}}$ is a bijective open map between two topological spaces. Thus, $\overline{\mathcal{G}}$ is closed by Lemma A.2.26. Keeping this in mind, we interpret $\mathcal{F}_{k+1,2}$ as a metric space by endowing $\mathcal{F}_{k+1,2}$ with the metric induced by the norm of $\mathcal{F}_{k+1,2}$. We define $W := \phi(K_i) \subseteq \mathbb{P}^k$ and set

$$\mathrm{PSD}(W) := \left\{ q \in \mathcal{F}_{k+1,2} \mid q|_{W(\mathbb{R})} \geq 0 \right\} \subseteq \mathcal{F}_{k+1,2}.$$

[1] For $A, B \in \mathrm{Sym}_{k+1}(\mathbb{R})$, we set $A \sim_{\ker(\mathcal{G})} B$ if and only if $A - B \in \ker(\mathcal{G})$ (cf. Example A.2.23).

Claim 3: $\mathrm{PSD}(W)$ is closed.

Proof. According to Proposition A.2.15, it suffices to show that the limit of any converging sequence in $\mathrm{PSD}(W)$ lies in $\mathrm{PSD}(W)$. To this end, we let $g \in \mathcal{F}_{k+1,2}$ be the limit of a converging sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathrm{PSD}(W)$. This implies that the non-negative sequence $(g_n(x))_{n\in\mathbb{N}} \subseteq \mathbb{R}_{\geq 0}$ converges to $g(x) \in \mathbb{R}$ for any $[x] \in W(\mathbb{R})$ and thus $g(x) \geq 0$ for any $[x] \in W(\mathbb{R})$. ■

We recall from Proposition 2.3.2 that Q is a linear map between the two finite-dimensional normed vector spaces $\mathrm{Sym}_{k+1}(\mathbb{R})$ and $\mathcal{F}_{k+1,2}$. Therefore, Q is continuous by Proposition A.2.40. Thus, the preimage

$$\mathfrak{W} := Q^{-1}(\mathrm{PSD}(W)) \subseteq \mathrm{Sym}_{k+1}(\mathbb{R})$$

of the closed set $\mathrm{PSD}(W)$ under the continuous map Q is closed. We set

$$\mathcal{W} := \left\{ [A] \in {}^{\mathrm{Sym}_{k+1}(\mathbb{R})}\big/_{\ker(\mathcal{G})} \mid \exists B \in \ker(\mathcal{G})\colon\, A + B \in \mathfrak{W} \right\}.$$

Claim 4: $\mathcal{W} \subseteq \mathrm{Sym}_{k+1}(\mathbb{R}) \,/\, \ker(\mathcal{G})$ is a well-defined closed set.

Proof. For $A \in \mathrm{Sym}_{k+1}(\mathbb{R})$ with $B \in \ker(\mathcal{G})$ such that $A + B \in \mathfrak{W}$, we choose $C \in \mathrm{Sym}_{k+1}(\mathbb{R})$ such that $A \sim_{\ker(\mathcal{G})} C$ and set $D := (A - C) + B \in \mathrm{Sym}_{k+1}(\mathbb{R})$. Moreover, we denote the zero form in $\mathcal{F}_{n+1,2d}$ by 0 and deduce

$$\mathcal{G}(D) = \mathcal{G}((A - C) + B) = \mathcal{G}(A - C) + \mathcal{G}(B) = 0 + 0 = 0$$

from the linearity of $\mathcal{G}$. This shows $D \in \ker(\mathcal{G})$. Furthermore, we see

$$C + D = C + ((A - C) + B) = A + B \in \mathfrak{W}.$$

Therefore, it remains to show that $\mathcal{W}$ is closed which, by the definition of the quotient topology, is the case if and only if $\{A \in \mathrm{Sym}_{k+1}(\mathbb{R}) \mid [A] \in \mathcal{W}\} \subseteq \mathrm{Sym}_{k+1}(\mathbb{R})$ is closed.

Subclaim 1: $\{A \in \mathrm{Sym}_{k+1}(\mathbb{R}) \mid [A] \in \mathcal{W}\} = \mathfrak{W} + \ker(\mathcal{G})$.

Proof. ($\subseteq$) For $A \in \mathrm{Sym}_{k+1}(\mathbb{R})$ with $[A] \in \mathcal{W}$, we let $B \in \ker(\mathcal{G})$ be such that $C := A + B \in \mathfrak{W}$ and conclude $A = C - B \in \mathfrak{W} - \ker(\mathcal{G}) = \mathfrak{W} + \ker(\mathcal{G})$.

($\supseteq$) For $A \in \mathfrak{W}$ and $B \in \ker(\mathcal{G})$, we observe $[A + B] = [A] + [B] = [A] \in \mathcal{W}$. ■

It hence remains to show that $\mathfrak{W} + \ker(\mathcal{G})$ is closed. To this end, we firstly observe that $\mathfrak{W}$ and $\ker(\mathcal{G})$ are non-empty cones in $\mathrm{Sym}_{k+1}(\mathbb{R})$.

Indeed, the $(k+1) \times (k+1)$ zero matrix is contained in both $\mathfrak{W}$ and $\ker(\mathcal{G})$. Moreover, for $A, B \in \mathfrak{W}$ and $\lambda \geq 0$, we have $Q(A + \lambda B) = Q(A) + \lambda Q(B) = q_A + \lambda q_B$ by the linearity of Q and we know that q_A and q_B are locally PSD on $W(\mathbb{R})$. Since λ is assumed to be non-negative, we thus obtain that $q_A + \lambda q_B$ is locally PSD on $W(\mathbb{R})$. This shows $A + \lambda B \in \mathfrak{W}$. Likewise, for $A, B \in \ker(\mathcal{G})$ and $\lambda \geq 0$, we denote the zero form in $\mathcal{F}_{n+1,2d}$ by 0 and deduce $\mathcal{G}(A + \lambda B) = \mathcal{G}(A) + \lambda \mathcal{G}(B) = 0 + \lambda \cdot 0 = 0$ from the linearity of $\mathcal{G}$. This shows $A + \lambda B \in \ker(\mathcal{G})$.

Secondly, we recall from Example A.2.36, Example A.2.34 and Example A.2.51 that $\mathrm{Sym}_{k+1}(\mathbb{R})$ is a locally convex Hausdorff space that is locally compact. Since we know that $\mathfrak{W}$, $\ker(\mathcal{G}) \subseteq \mathrm{Sym}_{k+1}(\mathbb{R})$ are closed, we thus deduce that $\mathfrak{W}$ and $\ker(\mathcal{G})$ are locally compact by Theorem A.2.52. Moreover, Theorem A.3.19 gives that the recession cones of the cones $\mathfrak{W}$ and $\ker(\mathcal{G})$ coincide with $\mathfrak{W}$ and $\ker(\mathcal{G})$, respectively. By an application of Dieudonné's theorem (cf. Theorem A.3.20), it suffices to show that $\mathfrak{W} \cap \ker(\mathcal{G})$ is a linear space to conclude that $\mathfrak{W} - \ker(\mathcal{G}) = \mathfrak{W} + \ker(\mathcal{G})$ is closed.

Subclaim 2: The following are true:

(i) If $i = k - n$, then $\mathfrak{W} \cap \ker(\mathcal{G}) = \ker(\mathcal{G})$.

(ii) If $i < k - n$, then $\mathfrak{W} \cap \ker(\mathcal{G})$ coincides with the set

$$\{A := (a_{s,t})_{0 \le s,t \le k} \in \ker(\mathcal{G}) \mid \forall t \ge n + i + 1 \colon a_{s,t} = 0 \text{ for } s = 0, \dots, t\}.$$

Proof. (i) It suffices to show $\ker(\mathcal{G}) \subseteq \mathfrak{W}$ and

$$\mathfrak{W} = Q^{-1}(\mathrm{PSD}(W)) = \left\{A \in \mathrm{Sym}_{k+1}(\mathbb{R}) \mid q_A|_{W(\mathbb{R})} \ge 0\right\}$$

by construction. We observe that Theorem 3.1.9, Theorem 3.4.1 and Theorem 3.1.11 together imply $\mathfrak{W} = \left\{A \in \mathrm{Sym}_{k+1}(\mathbb{R}) \mid q_A|_{V_{k-n}(\mathbb{R})} \ge 0\right\}$ since $V_{k-n} = W_{k-n}$ is the Zariski closure of $W = \phi(K_{k-n})$ by Construction 3.3.7 (3). Moreover, we know by Lemma 3.2.5 (ii) that $V_{k-n} = V(\mathbb{P}^n)$ and, therefore, we see that

$$\mathfrak{W} = \left\{A \in \mathrm{Sym}_{k+1}(\mathbb{R}) \mid q_A|_{V(\mathbb{P}^n)(\mathbb{R})} \ge 0\right\}$$

which contains the set

$$\ker(\mathcal{G}) \underset{(2.12)}{=} \{A \in \mathrm{Sym}_{k+1}(\mathbb{R}) \mid q_A \text{ vanishes on } V(\mathbb{P}^n)\}.$$

(ii) ($\subseteq$) For $A := (a_{s,t})_{0 \le s,t \le k} \in \mathfrak{W} \cap \ker(\mathcal{G})$, Lemma 2.3.15 yields that $a_{k,k}$ is the coefficient of X_n^{2d} in $\mathcal{G}(A)$. Therefore, we deduce $a_{k,k} = 0$ from $A \in \ker(\mathcal{G})$. Moreover, since $[m_0(1, \boldsymbol{x}) : \dots : m_{k-1}(1, \boldsymbol{x}) : y] \in \phi(K_{k-n-1})(\mathbb{R}) \subseteq \phi(K_i)(\mathbb{R})$ for any $\boldsymbol{x} \in \mathbb{R}^n$ and any non-zero $y \in \mathbb{R}$, we conclude from $A \in \mathfrak{W}$ that

$$\begin{aligned} q(\boldsymbol{X}, Y) &:= q_A(m_0(1, \boldsymbol{X}), \dots, m_{k-1}(1, \boldsymbol{X}), Y) \\ &= q_A(m_0(1, \boldsymbol{X}), \dots, m_{k-1}(1, \boldsymbol{X}), m_k(1, \boldsymbol{X})) - 2\sum_{s=0}^{k-1} a_{s,k} m_s(1, \boldsymbol{X}) m_k(1, \boldsymbol{X}) \\ &\quad + \left(2\sum_{s=0}^{k-1} a_{s,k} m_s(1, \boldsymbol{X})\right) Y \end{aligned}$$

is PSD. Consequently, we see that $a_{i,k} = 0$ for $i = 0, \dots, k-1$ in order to avoid the potential linearity of q in Y. If $i = k-n-1$, then we are done. Otherwise, we iterate the above argument for $t = k-1, \dots, n+i+1$ and conclude $a_{s,t} = 0$ for $t = n+i+1, \dots, k-n$ and $s = 0, \dots, t$.

($\supseteq$) For $A := (a_{s,t})_{0 \le s,t \le k} \in \ker(\mathcal{G})$ such that $a_{s,t} = 0$ for $t = n+i+1, \dots, k$ and $s = 0, \dots, t$, we observe for $\mathbf{x} \in \mathbb{R}^n$ and $z_{n+i+1}, \dots, z_k \in \mathbb{R}$ that

$$\begin{aligned} q_A(m_0(1,\boldsymbol{x}), \dots, m_{n+i}(1,\boldsymbol{x}), z_{n+i+1}, \dots, z_k) &= q_A(m_0(1,\boldsymbol{x}), \dots, m_k(1,\boldsymbol{x})) \\ &= \mathcal{G}(A)(1,\boldsymbol{x}) = 0. \end{aligned}$$

Recalling Lemma 3.3.9, we thus conclude $A \in \mathfrak{W}$ since $W = \phi(K_i)$. ■

If $i = k-n$, then $\mathfrak{W} \cap \ker(\mathcal{G}) = \ker(\mathcal{G})$ is a linear space. Otherwise, if $i < k-n$, then we compute for $\lambda \in \mathbb{R}$ and $A := (a_{s,t})_{0 \le s,t \le k}, B := (b_{s,t})_{0 \le s,t \le k} \in \ker(\mathcal{G})$ such that $a_{s,t} = b_{s,t} = 0$ for $t = n+i+1, \dots, k$ and $s = 0, \dots, t$ that

$$(A + \lambda B)_{s,t} = a_{s,t} + \lambda b_{s,t} = 0 + \lambda \cdot 0 = 0$$

for $t = n+i+1, \dots, k$ and $s = 0, \dots, t$. Hence, recalling Subclaim 2 (ii), we deduce that $\mathfrak{W} \cap \ker(\mathcal{G})$ also is a linear space if $i < k-n$. ■

Claim 5: $\overline{\mathcal{G}}(\mathcal{W}) = C_i$.
Proof. ($\subseteq$) For $[A] \in \mathcal{W}$, we fix $B \in \ker(\mathcal{G})$ such that $C := A + B \in \mathfrak{W}$ and set $f := \mathcal{G}(A) = \overline{\mathcal{G}}([A])$. Hence, C is a Gram matrix associated to f and $q_C = Q(C)$ is locally PSD on $W(\mathbb{R})$ since $C \in \mathfrak{W}$. We have $W = \phi(K_i)$ by construction and thus $\overline{\mathcal{G}}(A) = f \in C_W = C_{\phi(K_i)} = C_i$ follows using Corollary 3.4.5.
($\supseteq$) For $f \in C_i$, we fix $A \in \mathcal{G}^{-1}(f)$ such that q_A is locally PSD on $V_i(\mathbb{R})$ and observe that $Q(A) = q_A$ is especially locally PSD on the subset $\phi(K_i)(\mathbb{R})$ of $V_i(\mathbb{R})$. This shows $A \in \mathfrak{W}$ which implies $[A] \in \mathcal{W}$. We conclude $f = \mathcal{G}(A) = \overline{\mathcal{G}}([A]) \in \overline{\mathcal{G}}(\mathcal{W})$. ■

Since $\overline{\mathcal{G}}(\mathcal{W})$ is the image of the closed set $\mathcal{W}$ under the closed map $\overline{\mathcal{G}}$, we thus deduce that $\overline{\mathcal{G}}(\mathcal{W}) = C_i$ is closed. ■

Remark 7.1.2. *It follows that the cones $\mathcal{P}_{n+1,2d} = C_{k-n}$ and $\Sigma_{n+1,2d} = C_0$ are closed, which agrees with common knowledge (cf. Lemma 2.2.21).*

In order to describe the interior of the cone C_i, we need to introduce a non-negativity notion that is stronger than the (local) positive semidefinite property of a given quadratic form. That is, $q \in \mathcal{F}_{k+1,2}$ is *(locally) positive* or *(locally) positive definite* on $W(\mathbb{R}) \subseteq \mathbb{P}^k$, if $q(z) > 0$ for all $[z] \in W(\mathbb{R})$ and we write $q|_{W(\mathbb{R})} > 0$. In the special case that $W(\mathbb{R}) = \mathbb{P}^k(\mathbb{R})$, we say that q is *(globally) positive* or *(globally) positive definite (PD)* and write $q > 0$.

Theorem 7.1.3. *For $i = 0, \dots, k-n$, the interior of C_i is given by*

$$\mathring{C}_i = \left\{ f \in \mathcal{F}_{n+1,2d} \;\middle|\; \exists A \in \mathcal{G}^{-1}(f) \colon q_A|_{V_i(\mathbb{R})} > 0 \right\}.$$

Proof. At the beginning of the proof of Theorem 7.1.1, we observed that the Gram map $\mathcal{G}$ is a homeomorphism between $\mathrm{Sym}_{k+1}(\mathbb{R})$ and $\mathcal{F}_{n+1,2d}$.[2] Thus, we set

$$\mathfrak{W} := \left\{ A \in \mathrm{Sym}_{k+1} \mid q_A|_{V_i(\mathbb{R})} \geq 0 \right\}$$

and deduce $C_i = \mathcal{G}(\mathfrak{W})$. Lemma A.2.28 hence yields $\mathring{C}_i = (\mathcal{G}(\mathring{\mathfrak{W}})) = \mathcal{G}(\mathring{\mathfrak{W}})$ and it consequently suffices to show $\mathring{\mathfrak{W}} = \{A \in \mathrm{Sym}_{k+1}(\mathbb{R}) \mid q_A|_{V_i(\mathbb{R})} > 0\}$.

($\subseteq$) For $A \in \mathring{\mathfrak{W}}$, there exists some $\varepsilon > 0$ such that the open ball $B_\varepsilon(A)$ of radius ε with center A in $\mathrm{Sym}_{k+1}(\mathbb{R})$ is contained in $\mathfrak{W}$. We fix such $\varepsilon > 0$ and let $\lambda > 0$ be sufficiently small such that $|| - \lambda I_{k+1}|| = | - \lambda| \cdot ||I_{k+1}|| = \lambda ||I_{k+1}|| < \varepsilon$, where I_{k+1} denotes the $(k+1)\times(k+1)$ identity matrix. Hence, $||(A - \lambda I_{k+1}) - A|| = ||-\lambda I_{k+1}|| < \varepsilon$ implies $A - \lambda I_{k+1} \in B_\varepsilon(A)$. We thus deduce $A - \lambda I_{k+1} \in \mathfrak{W}$ since $B_\varepsilon(A) \subseteq \mathfrak{W}$ by choice. This implies that

$$q_{A-\lambda I_{k+1}}(Z) = q_A(Z) - \lambda q_{I_{k+1}}(Z) = q_A(Z) - \lambda \sum_{j=0}^{k} Z_j^2$$

is locally PSD on $V_i(\mathbb{R})$. Since $\lambda \sum\limits_{j=0}^{k} z_j^2 > 0$ for any $[z] \in V_i(\mathbb{R})$, we conclude $q_A(z) > 0$ for any $[z] \in V_i(\mathbb{R})$.

($\supseteq$) We set $\mathcal{B}$ to be the closed unit ball in $\mathbb{R}^{k+1}$ and recall from Example A.2.49 that $\mathcal{B}$ is compact. Moreover, we specify the norm on $\mathrm{Sym}_{k+1}(\mathbb{R})$ to be given by

$$\begin{aligned} || \cdot ||\colon \mathrm{Sym}_{k+1}(\mathbb{R}) &\to \mathbb{R}_{\geq 0} \\ A &\mapsto \max_{z \in \mathcal{B}} |q_A(z)|. \end{aligned}$$

<u>Claim 1</u>: $|| \cdot ||$ is a well-defined norm on $\mathrm{Sym}_{k+1}(\mathbb{R})$.
Proof. For $A \in \mathrm{Sym}_{k+1}(\mathbb{R})$, the quadratic form q_A attains both a maximum and a minimum on the compact set $\mathcal{B}$ by Proposition A.2.47. Hence,

$$\max_{z \in \mathcal{B}} |q_A(z)| = \max \left\{ |\max_{z \in \mathcal{B}} q_A(z)|, |\min_{z \in \mathcal{B}} q_A(z)| \right\} \geq 0$$

is a well-defined expression for any $A \in \mathrm{Sym}_{k+1}(\mathbb{R})$.

<u>Positive definiteness</u>: For $A \in \mathrm{Sym}_{k+1}(A)$ such that $0 = ||A|| = \max\limits_{z \in \mathcal{B}} |q_A(z)|$, we deduce $q_A(Z) = 0$ for all $z \in \mathcal{B}$. Moreover, for arbitrary but fixed $z \in \mathbb{R}^{k+1}$, we let $\lambda > 0$ be such that $|\lambda z| = \lambda |z| \leq 1$ and observe $\lambda z \in \mathcal{B}$. Therefore, we know $0 = q_A(\lambda z) = \lambda^2 q_A(z)$ using Lemma 2.2.11. Since $\lambda^2 \neq 0$, we conclude $q_A(z) = 0$. Therefore, we see that $q_A = Q(A)$ is the zero form and thus, by the injectivity of Q, A has to be the $(k+1) \times (k+1)$ zero matrix.

[2] That is, $\mathcal{G}$ is a bijective continuous open map (cf. Definition A.2.27).

Subadditivity: For $A, B \in \mathrm{Sym}_{k+1}(\mathbb{R})$, using the subadditivity of $|\cdot|$ and the linearity of Q, we compute

$$\begin{aligned} ||A+B|| &= \max_{z\in\mathcal{B}} |q_{A+B}(z)| \\ &= \max_{z\in\mathcal{B}} |q_A(z)+q_B(z)| \\ &\leq \max_{z\in\mathcal{B}} (|q_A(z)|+|q_B(z)|) \\ &\leq \max_{z\in\mathcal{B}} |q_A(z)| + \max_{z\in\mathcal{B}} |q_B(z)| \\ &= ||A|| + ||B||. \end{aligned}$$

Absolut homogeneity: For $A \in \mathrm{Sym}_{k+1}(\mathbb{R})$ and $\lambda \in \mathbb{R}$, using the absolute homogeneity of $|\cdot|$ and the linearity of Q, we compute

$$\begin{aligned} ||\lambda A|| &= \max_{z\in\mathcal{B}} |q_{\lambda A}(z)| \\ &= \max_{z\in\mathcal{B}} |\lambda q_A(z)| \\ &= \max_{z\in\mathcal{B}} (|\lambda|\cdot|q_A(z)|) \\ &= |\lambda| \cdot \max_{z\in\mathcal{B}} |q_A(z)| \\ &= |\lambda|\cdot||A||. \end{aligned}$$ ■

We now interpret $V_i(\mathbb{R})$ as an affine set by setting $W := \{z \in \mathbb{R}^{k+1} \mid [z] \in V_i(\mathbb{R})\}$.

Claim 2: W is closed.

Proof. According to Proposition A.2.15, it suffices to show that the limit of any converging sequence in W lies in W. To this end, we let $\left(z^{(n)}\right)_{n\in\mathbb{N}} \subseteq W$ be a converging sequence with limit $z \in \mathbb{R}^{k+1}$ and choose $S \subseteq \mathbb{C}[Z]$ such that $V_i = \mathcal{V}(S)$. For $n \in \mathbb{N}$, $z^{(n)} \in W$ implies $\left[z^{(n)}\right] \in V_i(\mathbb{R}) = \mathcal{V}(S)(\mathbb{R})$ and thus we conclude $f\left(z^{(n)}\right) = 0$ for any $f \in S$ and any $n \in \mathbb{N}$. Therefore,

$$f(z) = f\left(\lim_{n\to\infty} z^{(n)}\right) = \lim_{n\to\infty} f\left(z^{(n)}\right) = 0$$

follows for any $f \in S$. This implies $[z] \in \mathcal{V}(S)(\mathbb{R}) = V_i(\mathbb{R})$, respectively, $z \in W$. ■

Hence, $\mathcal{B} \cap W$ is compact by Lemma A.2.46 and we deduce for arbitrary but fixed $A \in \mathrm{Sym}_{k+1}(\mathbb{R})$ with $q_A|_{V_i(\mathbb{R})} > 0$ that q_A attains a minimum on $\mathcal{B} \cap W$ according to Proposition A.2.47. This allows us to fix some $\varepsilon > 0$ such that

$$\min_{z\in\mathcal{B}\cap W} q_A(z) > \varepsilon > 0$$

and we let $B_\varepsilon(A)$ be the open ball of radius ε with center A in $\mathrm{Sym}_{k+1}(\mathbb{R})$.

Claim 3: $B_\varepsilon(A) \subseteq \mathfrak{W}$.

Proof. For $B \in B_\varepsilon(A)$, it holds $\varepsilon > ||B - A|| = \max_{z\in\mathcal{B}} |q_{B-A}(z)|$ and we have to show that q_B is locally PSD on $V_i(\mathbb{R})$. To this end, we let $[z] \in V_i(\mathbb{R})$ be arbitrary but fixed and deduce $z \in W$. Moreover, we choose a sufficiently small $\lambda > 0$ such that $|\lambda z| = \lambda|z| \le 1$ and observe $\lambda z \in \mathcal{B} \cap W$ since $[\lambda z] = [z] \in V_i(\mathbb{R})$. Recalling that $\min_{y\in\mathcal{B}\cap W} q_A(y) > \varepsilon > 0$ by our choice of ε, we conclude

$$\begin{aligned} q_B(\lambda z) &= q_{A+(B-A)}(\lambda z) \\ &= q_A(\lambda z) + q_{B-A}(\lambda z) \\ &\ge \min_{y\in\mathcal{B}\cap W} q_A(y) + q_{B-A}(\lambda z) \\ &> \varepsilon + q_{B-A}(\lambda z) \\ &= \varepsilon - (-q_{B-A}(\lambda z)). \end{aligned}$$

If we assume $q_B(\lambda z) < 0$ for a contradiction, then the above computation implies $-q_{B-A}(\lambda z) > \varepsilon > 0$ and we thus have

$$||B - A|| = \max_{z\in\mathcal{B}} |q_{B-A}(z)| = \max_{z\in\mathcal{B}} | - q_{B-A}(z)| \ge -q_{B-A}(\lambda z) > \varepsilon.$$

This contradicts $B \in B_\varepsilon(A)$. Hence, $q_B(\lambda z) < 0$ cannot be the case and it follows $0 \le q_B(\lambda z) = \lambda^2 q_B(z)$ by Lemma 2.2.11. Thus, $q_B(z) > 0$ follows. ■

Altogether, we showed that any $A \in \mathrm{Sym}_{k+1}(\mathbb{R})$ such that $q_A|_{V_i(\mathbb{R})} > 0$ lies in the interior of $\mathfrak{W}$. ■

Example 7.1.4. The Interior of $\mathcal{P}_{n+1,2d}$

It is known that the interior of the cone $\mathcal{P}_{n+1,2d}$ is given by

$$\mathring{\mathcal{P}}_{n+1,2d} = \left\{ f \in \mathcal{F}_{n+1,2d} \mid \forall x \in \mathbb{R}^{n+1} \text{ non-zero } : f(x) > 0 \right\}$$

(cf. [Rez92, Theorem 3.14]). Let us relate this result to Theorem 7.1.3 according to which the interior of $\mathcal{P}_{n+1,2d} = C_{k-n}$ is given by

$$\mathring{\mathcal{P}}_{n+1,2d} = \mathring{C}_{k-n} = \left\{ f \in \mathcal{F}_{n+1,2d} \mid \exists A \in \mathcal{G}^{-1}(f) \colon q_A|_{V(\mathbb{P}^n(\mathbb{R}))} > 0 \right\}$$

since $V_{k-n}(\mathbb{R}) = V(\mathbb{P}^n)(\mathbb{R}) = V(\mathbb{P}^n(\mathbb{R}))$ by Lemma 3.2.5 (ii) and Proposition 2.3.34.

To this end, we let $f \in \mathcal{F}_{n+1,2d}$ be such that $f(x) > 0$ for all non-zero $x \in \mathbb{R}^{n+1}$ and fix $A \in \mathcal{G}^{-1}(f)$. Hence, we see $q_A(m_0(x), \dots, m_k(x)) = f(x) > 0$ for all non-zero $x \in \mathbb{R}^{n+1}$ which shows that q_A is locally PD on $V((\mathbb{P}^n(\mathbb{R}))$.

Vice versa, we let $f \in \mathcal{F}_{n+1,2d}$ be such that there exists some $A \in \mathcal{G}^{-1}(f)$ such that q_A is locally PD on $V(\mathbb{P}^n(\mathbb{R}))$. For such a choice of $A \in \mathcal{G}^{-1}(f)$, we conclude $f(x) = q_A(m_0(x), \dots, m_k(x)) > 0$ for any non-zero $x \in \mathbb{R}^{n+1}$ since we know that $[m_0(x) : \dots : m_k(x)] = V([x]) \in V(\mathbb{P}^n(\mathbb{R}))$ for any non-zero $x \in \mathbb{R}^{n+1}$.

Example 7.1.5. The Interior of $\Sigma_{n+1,2d}$
It is known that the interior of the cone $\Sigma_{n+1,2d}$ is given by

$$\mathring{\Sigma}_{n+1,2d} = \left\{ f \in \mathcal{F}_{n+1,2d} \mid \exists A \in \mathcal{G}^{-1}(f) \text{ such that } A \text{ is non-singular and } q_A \geq 0 \right\}$$

(cf. [CLR92, Proposition 5.5.]). Let us relate this result to Theorem 7.1.3 according to which the interior of $\Sigma_{n+1,2d} = C_0$ is given by

$$\mathring{\Sigma}_{n+1,2d} = \mathring{C}_0 = \left\{ f \in \mathcal{F}_{n+1,2d} \mid \exists A \in \mathcal{G}^{-1}(f) \colon q_A > 0 \right\}$$

since $V_0(\mathbb{R}) = \mathbb{P}^k(\mathbb{R})$ by Lemma 3.2.5 (i).

To this end, we let $f \in \mathcal{F}_{n+1,2d}$ be such that there exists some non-singular $A \in \mathcal{G}^{-1}(f)$ for which q_A is PSD. For such a choice of $A \in \mathcal{G}^{-1}(f)$, we deduce that all eigenvalues of A are non-negative real numbers since $q_A \geq 0$ implies that the matrix $A \in \mathrm{Sym}_{k+1}(\mathbb{R})$ is positive semidefinite. Moreover, 0 is not an eigenvalue of A since A is assumed to be non-singular. Altogether, we thus know that any eigenvalue of A is a positive real number. Therefore, the matrix A is positive definite which implies that q_A is PD.

Vice versa, we let $f \in \mathcal{F}_{n+1,2d}$ be such that there exists some $A \in \mathcal{G}^{-1}(f)$ for which q_A is PD. For such a choice of $A \in \mathcal{G}^{-1}(f)$, we especially know that q_A is PSD and we moreover deduce that the matrix $A \in \mathrm{Sym}_{k+1}(\mathbb{R})$ is positive definite since $q_A > 0$. Therefore, we know that any eigenvalue of A is a positive real number and thus especially non-zero. We conclude that A is non-singular.

Corollary 7.1.6. *For $i = 0, \ldots, k-n$, the boundary of C_i is given by*

$$\partial C_i = \left\{ f \in C_i \mid \forall A \in \mathcal{G}^{-1}(f) \ \exists [z] \in V_i(\mathbb{R}) \colon q_A(z) \leq 0 \right\}.$$

Proof. Theorem 7.1.1 states that C_i is closed and thus $\partial C_i = C_i \backslash \mathring{C}_i$ (cf. Lemma A.2.6). The assertion therefore follows from Theorem 7.1.3 according to which

$$\mathring{C}_i = \left\{ f \in \mathcal{F}_{n+1,2d} \mid \exists A \in \mathcal{G}^{-1}(f) \colon q_A|_{V_i(\mathbb{R})} > 0 \right\}. \qquad \blacksquare$$

7.2 (Non-)Spectrahedral Shadows

Positive semidefinite forms and sum of square representations find application in polynomial optimization for which it is crucial to be able to determine whether a given $(n+1)$-ary $2d$-ic is PSD. Unfortunately, this is a very challenging task to solve. However, we know that a sum of square representation certifies the PSD property of a given $(n+1)$-ary $2d$-ic. Therefore, tests for the PSD property are often replaced by tests for the SOS property, which can be solved in polynomial time. The geometric reason for the efficient decidability of membership to $\Sigma_{n+1,2d}$ is that the latter cone represents a feasible region of a semidefinite programming problem, which brings us to the definition below. We refer an interested reader to [Kle02], [Las10] and [VB96] for an introduction to semidefinite programming.

Notation 7.2.1. For $s \in \mathbb{N}$, we denote the cone of positive semidefinite matrices in $\mathrm{Sym}_s(\mathbb{R})$ by $\mathrm{Sym}_s^+(\mathbb{R})$.

Definition 7.2.2. Fix $l \in \mathbb{N}$.

(i) $C \subseteq \mathbb{R}^l$ is a *spectrahedron* if there exists some $s \in \mathbb{N}$ and an affine-linear map $\tau\colon \mathbb{R}^l \to \mathrm{Sym}_s(\mathbb{R})$ such that $C = \tau^{-1}(\mathrm{Sym}_s^+(\mathbb{R}))$.

(ii) $C \subseteq \mathbb{R}^l$ is a *spectrahedral shadow* if there exists some $s \in \mathbb{N}$ and an affine-linear map $\tau\colon \mathbb{R}^s \to \mathbb{R}^l$ such that $C = \tau(D)$ for some spectrahedron $D \subseteq \mathbb{R}^s$.

Remark 7.2.3. *Any finite-dimensional $\mathbb{R}$-vector space $\mathcal{X}$ can be identified with $\mathbb{R}^l$ for $l := \dim(\mathcal{X})$. In this sense, the notion of spectrahedra and spectrahedral shadows can be extended to finite-dimensional $\mathbb{R}$-vector spaces.*

Spectrahedral shadows are the feasible regions of semidefinite programming problems and are therefore also often called *semidefinitely representable* or *SDP representable* sets. We refer an interested reader to [GN11], [HN09] and [NP23] for an overview on spectrahedra and spectrahedral shadows. Here, we only include a brief list of properties of spectrahedral shadows that will be used in the proofs below for the convenience of the reader.

Proposition 7.2.4. *The following are true:*

(i) Finite intersections of spectrahedral shadows are spectrahedral shadows.

(ii) The affine-linear image of a spectrahedral shadow is a spectrahedral shadow.

(iii) The dual cone of a spectrahedral shadow is a spectrahedral shadow.[3]

Proof. See [NP23, Theorem 3.5 1., 3. and 9.]. ■

A famous example of a spectrahedral shadow is the cone $\Sigma_{n+1,2d}$, which is the image of the spectrahedron $\mathrm{Sym}_{k+1}^+(\mathbb{R})$ under the linear Gram map by (2.15). If we are thus in a Hilbert case, then $\Sigma_{n+1,2d} = C_0 = \ldots = C_{k-n} = \mathcal{P}_{n+1,2d}$ implies that also $C_0, \ldots, C_{k-n} = \mathcal{P}_{n+1,2d}$ are spectrahedral shadows. In any non-Hilbert case, a similar argument is valid only for the first $n+1$, respectively $n+2$ if $n = 2$, cones in ($\mathcal{CF}$).

Lemma 7.2.5. *If $(n+1, 2d)$ is a non-Hilbert case, then C_i is a spectrahedral shadow for $i = 0, \ldots, n$. Moreover, if $n = 2$, then also C_{n+1} is a spectrahedral shadow.*

Proof. Theorem 4.2.7 yields $\Sigma_{n+1,2d} = C_0 = \ldots = C_n$ and also $C_n = C_{n+1}$ if $n = 2$. The assertion thus follows since $\Sigma_{n+1,2d}$ is a spectrahedral shadow. ■

[3] The *dual cone* of a subset K of a $\mathbb{R}$-vector space X is $K^\vee := \{l \in \mathrm{Hom}(X, \mathbb{R}) \mid \forall x \in K\colon l(x) \geq 0\}$.

For the remaining C_i's in a non-Hilbert case, the situation is opposite. To see this, we use a technique of Scheiderer [Sch18b] that is based on methods from real algebraic geometry in the proofs below. We refer the reader to [BCR98] for a general introduction to real algebraic geometry and to Appendix A.4 for a concise overview on the here used results. Moreover, we let **$(n+1, 2d)$ denote a non-Hilbert case** throughout the rest of this section if not explicitly mentioned otherwise.

Theorem 7.2.6. *The following are true:*

(i) If $d = 2$, then, for $i = n+1, \ldots, k-n$, the cone C_i is not a spectrahedral shadow.

(ii) If $d \geq 3$, then, for $i = k(n, d-1) - n + 1, \ldots, k(n, d) - n$, the cone C_i is not a spectrahedral shadow.

Proof. We set $N := |\{m_s m_t \mid 0 \leq s, t \leq n+i\}| \in \mathbb{N}$ and enumerate the set

$$\{m_s m_t \mid 0 \leq s, t \leq n+i\} =: \{e_0, \ldots, e_N\}$$

such that $e_0(X) = m_0(X)^2 = X_0^{2d}$. Morever, we set $\boldsymbol{X} := (X_1, \ldots, X_n)$ as usual and define $\boldsymbol{e_j}(\boldsymbol{X}) := e_j(1, \boldsymbol{X})$ for $j = 1, \ldots, N$. We furthermore consider the maps

$$\begin{array}{rclcc} v & : & \mathbb{C}^{n+1} & \to & \mathbb{C}^{N+1} \\ & & x & \mapsto & (e_0(x), \ldots, e_N(x)), \\ \boldsymbol{v} & : & \mathbb{C}^n & \to & \mathbb{C}^N \\ & & \boldsymbol{x} & \mapsto & (\boldsymbol{e_1}(\boldsymbol{x}), \ldots, \boldsymbol{e_N}(\boldsymbol{x})). \end{array}$$

<u>Claim 1</u>: $\overline{\operatorname{conv}(\boldsymbol{v}(\mathbb{R}^n))}$ is not a spectrahedral shadow.
Proof. We set $j := i - 1$ and observe, using Lemma 2.3.5, that

$$\begin{cases} j \geq n, & \text{if } d = 2, n \geq 3 \\ j \geq k(n, d-1) - n + 1 \geq k(n, 2) - n + 1 = \frac{n(n+1)}{2} \geq n + 1, & \text{if } d \geq 3, n \geq 2. \end{cases}$$

Theorem 6.3.12 allows us to fix some $f \in \Delta_{n+1,2d}$ such that $f \in \mathfrak{S}_{n+j+1}$. Moreover, we set L to be the finite-dimensional linear space spanned by $\{\boldsymbol{e_1}, \ldots, \boldsymbol{e_n}\}$ in $\mathbb{R}[\boldsymbol{X}]$ and let $R \supsetneq \mathbb{R}$ be a real closed field extension with canonical valuation ring B and maximal ideal $\mathfrak{m}_B$.

For an infinitesimal $\varepsilon > 0$ in R and $\eta \in \mathbb{R}^n$, [Sch18b, Proposition 4.18.] implies that $f_\varepsilon(\boldsymbol{X}) := f(\varepsilon, \boldsymbol{X}) \in B[\boldsymbol{X}]$ is not a sum of squares in $B[\boldsymbol{X}] \,/\, \langle X_1, \ldots, X_n \rangle^{2d+1} B[\boldsymbol{X}]$. We let $M_{\mathbb{C}^n,\eta} \subseteq B[\mathbb{C}^n]$ denote the kernel of the evaluation map $B[\mathbb{C}^n] \to B, g \mapsto g(\eta)$ and interpret $f_\eta(\boldsymbol{X}) := f_\varepsilon(\boldsymbol{X} - \eta)$ as an element in $B[\mathbb{C}^n]$. Hence, we deduce that f_η is not a sum of squares in $B[\mathbb{C}^n] \,/\, (M_{\mathbb{C}^n,\eta})^{2d+1}$ by [Sch18b, Lemma 4.17.].

However, since $f \in \mathcal{F}_{n+1,2d}$ is PSD, we see that $f(X_0, \boldsymbol{X} - \eta)$ is PSD. The Tarski-Transfer principle (cf. Theorem A.4.4) yields $f(x_0, \boldsymbol{x} - \eta) \geq 0$ for any $(x_0, \boldsymbol{x}) \in R^{n+1}$.

Moreover, we set $L_B := L \otimes B$ and observe that L_B can be identified with the linear space spanned by $\{\boldsymbol{e_1}, \dots, \boldsymbol{e_N}\}$ in $B[\boldsymbol{X}]$. In order to verify the assertion, it thus remains to show $f_\eta \in L_B + B1$ according to [Sch18b, Proposition 4.19.]. To this end, we observe that all non-constant monomials in the variables $X_1, \dots, X_n$ of degree at most $2d-2$ are among $\boldsymbol{e_1}, \dots, \boldsymbol{e_N}$ since $i \geq k(n, d-1) - n + 1$ in both cases. Moreover,

$$f \in \mathfrak{S}_{n+j+1} = \mathfrak{S}_{n+i} = \operatorname{span}_{\mathbb{R}}\{m_s m_t \mid 0 \leq s, t \leq n+i\} = \operatorname{span}_{\mathbb{R}}\{e_0, \dots, e_N\}$$

yields $f_\eta \in \operatorname{span}_{\mathbb{R}} \{e_0(\varepsilon, \boldsymbol{X} - \eta), \dots, e_N(\varepsilon, \boldsymbol{X} - \eta)\}$. It hence suffices to show

$$e_l(\varepsilon, \boldsymbol{X} - \eta) \in L_B + B1 \text{ for } l = 1, \dots, N$$

since $e_0(\varepsilon, \boldsymbol{X} - \eta) = \varepsilon^{2d} \in L_B + B1$. We distinguish three cases for $l = 1, \dots, N$.

Case 1: If X_0^2 divides e_l, then $e_l(\varepsilon, \boldsymbol{X} - \eta)$ is a linear combination of monomials in the variables $X_1, \dots, X_n$ of degree at most $2d - 2$ with scalars in B that are products of some of the elements $\varepsilon, -\eta_1, \dots, -\eta_n \in B$. By our general observation above that all non-constant monomials in the variables $X_1, \dots, X_n$ of degree at most $2d - 2$ are among $\boldsymbol{e_1}, \dots, \boldsymbol{e_N}$ and the identification of L_B with the linear space spanned by $\{\boldsymbol{e_1}, \dots, \boldsymbol{e_N}\}$ in $B[\boldsymbol{X}]$, it thus follows $e_l(\varepsilon, \boldsymbol{X} - \eta) \in L_B + B1$.

Case 2: If $e_l(X) = X_0 \boldsymbol{X}^\beta$ for some $\beta \in \mathbb{N}_0^n$ such that $|\beta| = 2d - 1$, then we see that $e_l(\varepsilon, \boldsymbol{X} - \eta) = \varepsilon(\boldsymbol{X} - \eta)^\beta$ is a linear combination of

(i) monomials in the variables $X_1, \dots, X_n$ of degree at most $2d - 2$ with scalars in B that are products of some of the elements $\varepsilon, -\eta_1, \dots, -\eta_n \in B$ and

(ii) $\boldsymbol{X}^\beta = e_l(1, \boldsymbol{X}) = \boldsymbol{e_l}(\boldsymbol{X})$ with scalar $\varepsilon \in B$.

By our general observation above that all non-constant monomials in the variables $X_1, \dots, X_n$ of degree at most $2d - 2$ are among $\boldsymbol{e_1}, \dots, \boldsymbol{e_N}$ and the identification of L_B with the linear space spanned by $\{\boldsymbol{e_1}, \dots, \boldsymbol{e_N}\}$ in $B[\boldsymbol{X}]$, we therefore conclude $e_l(\varepsilon, \boldsymbol{X} - \eta) \in L_B + B1$.

Case 3: If X_0 does not divide e_l, then $e_l = m_s m_t$ for some $0 < s, t \leq n + i$ such that $\alpha_{s,0} = \alpha_{t,0} = 0$. Consequently, $e_l(\varepsilon, \boldsymbol{X} - \eta) = e_l(1, \boldsymbol{X} - \eta) = \boldsymbol{e_l}(\boldsymbol{X} - \eta)$ is a linear combination of

(i) monomials in the variables $X_1, \dots, X_n$ of degree at most $2d - 2$ with scalars in B that are products of some of the elements $-\eta_1, \dots, -\eta_n \in B$,

(ii) monomials in the variables $X_1, \dots, X_n$ of degree $2d - 1$ with scalars in the set $\{-\eta_1, \dots, -\eta_n\} \subseteq B$ and

(iii) $\boldsymbol{e_l}(\boldsymbol{X})$ with scalar 1.

For Case (i) and Case (iii), we recall our general observation above that all non-constant monomials in the variables $X_1, \dots, X_n$ of degree at most $2d - 2$ are among $\boldsymbol{e_1}, \dots, \boldsymbol{e_N}$ and our identification of L_B with the linear space spanned by $\{\boldsymbol{e_1}, \dots, \boldsymbol{e_N}\}$

in $B[\boldsymbol{X}]$. Moreover, for Case (ii), we observe that the monomials to be considered are of the structure $\boldsymbol{X}^{\boldsymbol{\alpha_s}+\boldsymbol{\alpha_t}-\chi_r}$ with scalar $-\eta_r \in B$, r^{th} unit vector χ_r in $\mathbb{Z}^n$, exponents $\boldsymbol{\alpha_s} := (\alpha_{s,1}, \dots, \alpha_{s,n})$, $\boldsymbol{\alpha_t} := (\alpha_{t,1}, \dots, \alpha_{t,n})$ and $1 \leq r \leq n$ such that $(\boldsymbol{\alpha_s} + \boldsymbol{\alpha_t})_r \geq 1$. Without loss of generality, we furthermore assume $\alpha_{s,r} \geq 1$ and set $m(X) := X_0 \boldsymbol{X}^{\boldsymbol{\alpha_s}-\chi_r}$. It thus follows

$$m_0(X) >_{\text{lex}} m(X) \geq_{\text{lex}} X_0 X_n^{d-1} = m_{k(n,d-1)} >_{\text{lex}} m_{n+i}(X)$$

since $|\boldsymbol{\alpha_s} - \chi_r| = |\boldsymbol{\alpha_s}| - 1 = d - 1 \geq 1$ and $i \geq k(n, d-1) - n + 1$. Moreover, we have $m_0 >_{\text{lex}} m_t \geq_{\text{lex}} m_{n+i}$ since $0 < t \leq n + i$. Hence, recalling $\alpha_{t,0} = 0$, we see that

$$X_0 \boldsymbol{X}^{\boldsymbol{\alpha_s}+\boldsymbol{\alpha_t}-\chi_r} = (X_0 \boldsymbol{X}^{\boldsymbol{\alpha_s}-\chi_r}) \boldsymbol{X}^{\boldsymbol{\alpha_t}} = (X_0 \boldsymbol{X}^{\boldsymbol{\alpha_s}-\chi_r}) X^{\alpha_t} = m(X) m_t(X) \tag{7.1}$$

and also that $m(X)m_t(X)$ lies in

$$\{m_\sigma m_\tau \mid m_0 >_{\text{lex}} m_\sigma, m_\tau \geq_{\text{lex}} m_{n+i}\} = \{m_s m_t \mid 0 < \sigma, \tau \leq n + i\} \subseteq \{e_1, \dots, e_N\}.$$

This shows

$$\boldsymbol{X}^{\boldsymbol{\alpha_s}+\boldsymbol{\alpha_t}-\chi_r} \underset{(7.1)}{=} m(1, \boldsymbol{X}) m_t(1, \boldsymbol{X}) \in \{e_1(1, \boldsymbol{X}), \dots, e_N(1, \boldsymbol{X})\} = \{\boldsymbol{e_1}, \dots, \boldsymbol{e_N}\}.$$

Altogether, it thus follows $e_l(\varepsilon, \boldsymbol{X} - \eta) \in L_B + B1$. ■

Claim 2: $\overline{\operatorname{conv}(v(\{1\} \times \mathbb{R}^n))}$ is not a spectrahedral shadow.
Proof. We assume for a proof by contradiction that $\overline{\operatorname{conv}(v(\{1\} \times \mathbb{R}^n))}$ is a spectrahedral shadow and deduce from Proposition 7.2.4 (ii) that $\pi(\overline{\operatorname{conv}(v(\{1\} \times \mathbb{R}^n))})$ is a spectrahedral shadow for the linear projection $\pi \colon \mathbb{R}^{N+1} \to \mathbb{R}^N, (z_0, \boldsymbol{z}) \mapsto \boldsymbol{z}$.

Subclaim: $\pi(\overline{\operatorname{conv}(v(\{1\} \times \mathbb{R}^n))}) = \overline{\operatorname{conv}(\boldsymbol{v}(\mathbb{R}^n))}$.
Proof of Subclaim. ($\subseteq$) The continuity of π yields

$$\pi(\overline{\operatorname{conv}(v(\{1\} \times \mathbb{R}^n))}) \subseteq \overline{\pi(\operatorname{conv}(v(\{1\} \times \mathbb{R}^n)))} \tag{7.2}$$

and $\pi(\operatorname{conv}(v(\{1\} \times \mathbb{R}^n))) = \operatorname{conv}(\boldsymbol{v}(\mathbb{R}^n))$ by the linearity of π. Hence,

$$\pi(\overline{\operatorname{conv}(v(\{1\} \times \mathbb{R}^n))}) \underset{(7.2)}{\subseteq} \overline{\operatorname{conv}(\boldsymbol{v}(\mathbb{R}^n))}.$$

($\supseteq$) For $z \in \overline{\operatorname{conv}(\boldsymbol{v}(\mathbb{R}^n))}$, we choose $\left(z^{(j)}\right)_{j \in \mathbb{N}} \subseteq \operatorname{conv}(\boldsymbol{v}(\mathbb{R}^n))$ such that $z^{(j)} \to z$ as $j \to \infty$. For any $j \in \mathbb{N}$, it is moreover possible to fix some $l \in \mathbb{N}$, $\lambda_1, \dots, \lambda_l \geq 0$ and $x^{(1)}, \dots, x^{(l)} \in \mathbb{R}^n$ such that $\sum_{r=1}^{l} \lambda_r = 1$ and $\sum_{r=1}^{l} \lambda_r \left(\boldsymbol{e_1}\left(x^{(r)}\right), \dots, \boldsymbol{e_N}\left(x^{(r)}\right)\right) = z^{(j)}$.
Hence, we have

$$\begin{aligned}\sum_{r=1}^{l} \lambda_r \left(e_0\left(1, x^{(r)}\right), \ldots, e_N\left(1, x^{(r)}\right)\right) &= \sum_{r=1}^{l} \lambda_r \left(e_0\left(1, x^{(r)}\right), \boldsymbol{e_1}\left(x^{(r)}\right), \ldots, \boldsymbol{e_N}\left(x^{(r)}\right)\right) \\ &= \sum_{r=1}^{l} \lambda_r \left(1, \boldsymbol{e_1}\left(x^{(r)}\right), \ldots, \boldsymbol{e_N}\left(x^{(r)}\right)\right) \\ &= \left(\sum_{r=1}^{l} \lambda_r, \sum_{r=1}^{l} \lambda_r \left(\boldsymbol{e_1}\left(x^{(r)}\right), \ldots, \boldsymbol{e_N}\left(x^{(r)}\right)\right)\right) \\ &= \left(1, z^{(j)}\right)\end{aligned}$$

which shows $\left(1, z^{(j)}\right) \in \operatorname{conv}(v(\{1\} \times \mathbb{R}^n))$. We conclude $z \in \pi(\overline{\operatorname{conv}(v(\{1\} \times \mathbb{R}^n))})$ since $\left(1, z^{(j)}\right) \to (1, z)$ as $j \to \infty$ and $\pi(1, z) = z$. ■

Hence, $\overline{\operatorname{conv}(\boldsymbol{v}(\mathbb{R}^n))}$ is a spectrahedral shadow which contradicts Claim 1. The assumption that $\overline{\operatorname{conv}(v(\{1\} \times \mathbb{R}^n))}$ is a spectrahedral shadow must have been wrong and the assertion follows. ■

We now lastly assume for a proof by contradiction that C_i is a spectrahedral shadow and note that $\mathfrak{S}_{n+i}$ is a spectrahedral shadow as well. Recalling Theorem 6.2.2, we thus deduce from Proposition 7.2.4 (i) that $\mathcal{P}_{n+1,2d} \cap \mathfrak{S}_{n+i} = C_i \cap \mathfrak{S}_{n+i}$ is a spectrahedral shadow. Moreover, we let $\mathbb{R}[\boldsymbol{X}]_{\leq 2d}$ denote the $\mathbb{R}$-vector space of polynomials in $\boldsymbol{X}$ of degree at most $2d$ and consider the linear map

$$\begin{aligned}\pi\colon \mathcal{F}_{n+1,2d} &\to \mathbb{R}[\boldsymbol{X}]_{\leq 2d} \\ f(X) &\mapsto f(1, \boldsymbol{X}).\end{aligned}$$

This allows us to conclude from Proposition 7.2.4 (ii) that also $\pi(\mathcal{P}_{n+1,2d} \cap \mathfrak{S}_{n+i})$ is a spectrahedral shadow.

Claim 3: $\pi(\mathcal{P}_{n+1,2d} \cap \mathfrak{S}_{n+i}) = \{g \in \operatorname{span}_{\mathbb{R}}(1, \boldsymbol{e_1}, \ldots, \boldsymbol{e_N}) \mid \forall \boldsymbol{x} \in \mathbb{R}^n \colon g(\boldsymbol{x}) \geq 0\}$.
Proof. ($\subseteq$) For $f \in \mathcal{P}_{n+1,2d} \cap \mathfrak{S}_{n+i}$, we see that

$$\pi(f)(\boldsymbol{X}) = f(1, \boldsymbol{X}) \in \operatorname{span}_{\mathbb{R}}\{e_0(1, \boldsymbol{X}), \ldots, e_N(1, \boldsymbol{X})\} = \operatorname{span}_{\mathbb{R}}\{1, \boldsymbol{e_1}(\boldsymbol{X}), \ldots, \boldsymbol{e_N}(\boldsymbol{X})\}$$

since $f \in \mathfrak{S}_{n+i} = \operatorname{span}_{\mathbb{R}}\{e_0, \ldots, e_N\}$. Furthermore, we deduce $\pi(f)(\boldsymbol{x}) = f(1, \boldsymbol{x}) \geq 0$ for $\boldsymbol{x} \in \mathbb{R}^n$ using that f is PSD.

($\supseteq$) For $g \in \operatorname{span}_{\mathbb{R}}(1, \boldsymbol{e_1}, \ldots, \boldsymbol{e_N})$ such that $g(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$, we complete each monomial of g by powers of X_0 to a monomial of degree $2d$ in X. This gives us a form $f \in \mathcal{F}_{n+1,2d}$ such that $\pi(f)(\boldsymbol{X}) = f(1, \boldsymbol{X}) = g(\boldsymbol{X})$ and, since $g \in \operatorname{span}_{\mathbb{R}}\{\boldsymbol{e_1}, \ldots, \boldsymbol{e_N}\}$, especially $f \in \operatorname{span}_{\mathbb{R}}\{e_0, \ldots, e_N\} = \mathfrak{S}_{n+i}$ follows. Moreover, we compute

$$f(1, \boldsymbol{x}) = \pi(f)(\boldsymbol{x}) = g(\boldsymbol{x}) \geq 0$$

for $\boldsymbol{x} \in \mathbb{R}^n$ using that g is PSD. Hence, $f \in \mathcal{P}_{n+1,2d}$ by Corollary 2.2.12. ■

We therefore conclude that

$$\begin{aligned}
&\{g \in \operatorname{span}_{\mathbb{R}}(1, \boldsymbol{e_1}, \dots, \boldsymbol{e_N}) \mid \forall \boldsymbol{x} \in \mathbb{R}^n \colon g(\boldsymbol{x}) \geq 0\} \\
&\quad \simeq \{l \in \operatorname{Hom}(\mathbb{R}^{N+1}, \mathbb{R}) \mid \forall \boldsymbol{x} \in \mathbb{R}^n \colon l(1, \boldsymbol{e_1}(\boldsymbol{x}), \dots, \boldsymbol{e_N}(\boldsymbol{x})) \geq 0\} \\
&\quad = \{l \in \operatorname{Hom}(\mathbb{R}^{N+1}, \mathbb{R}) \mid \forall \boldsymbol{x} \in \mathbb{R}^n \colon l(e_0(1, \boldsymbol{x}), \dots, e_N(1, \boldsymbol{x})) \geq 0\} \\
&\quad = \{l \in \operatorname{Hom}(\mathbb{R}^{N+1}, \mathbb{R}) \mid l(v(\{1\} \times \mathbb{R}^n)) \geq 0\} \\
&\quad = \{l \in \operatorname{Hom}(\mathbb{R}^{N+1}, \mathbb{R}) \mid l(\operatorname{conv}(v(\{1\} \times \mathbb{R}^n))) \geq 0\} \\
&\quad = \{l \in \operatorname{Hom}(\mathbb{R}^{N+1}, \mathbb{R}) \mid l(\overline{\operatorname{conv}(v(\{1\} \times \mathbb{R}^n))}) \geq 0\} \\
&\quad = (\overline{\operatorname{conv}(v(\{1\} \times \mathbb{R}^n))})^\vee
\end{aligned}$$

is a spectrahedral shadow. Thus, also the bidual cone

$$((\overline{\operatorname{conv}(v(\{1\} \times \mathbb{R}^n))})^\vee)^\vee = \overline{\operatorname{conv}(v(\{1\} \times \mathbb{R}^n))}$$

is a spectrahedral shadow by Proposition 7.2.4 (iii). This contradicts Claim 2. The assumption that C_i is a spectrahedral shadow must have been wrong. ■

Theorem 7.2.7. *Let $n, d \geq 1$ and $i = 0, \dots, k-n$. If C_i is not a spectrahedral shadow as a subcone of $\mathcal{P}_{n+1,2d}$, then C_i is not a spectrahedral shadow as a subcone of $\mathcal{P}_{n+1,2\delta}$ for $\delta \geq d$.*

Proof. For the purpose of this proof, for $D \in \{d, \delta\}$, we denote the ordered monomial basis vectors of $\mathcal{F}_{n+1,D}$ by $m_0^{(D)}, \dots, m_{k(n,D)}^{(D)}$ and write $K_i^{(D)}$ for $K_i \subseteq \mathbb{C}^{k(n,D)}$. We furthermore recall from Corollary 3.4.5 that $C_i = C_{\phi(K_i^{(D)})}$ as a subcone of $\mathcal{P}_{n+1,2D}$.

Moreover, we assume for a proof by contraposition that $C_{\phi(K_i^{(\delta)})}$ is a spectrahedral shadow as a subcone of $\mathcal{P}_{n+1,2\delta}$ and set

$$\mathfrak{S} := \operatorname{span}_{\mathbb{R}}\{X^\beta \mid \beta \in \mathbb{Z}_0^{n+1} \colon |\beta| = 2\delta \wedge \beta_0 \geq 2(\delta - d)\} \subseteq \mathcal{F}_{n+1,2\delta}$$

which we note to be a spectrahedral shadow. This allows us to deduce that $C_{\phi(K_i^{(\delta)})} \cap \mathfrak{S}$ is a spectrahedral shadow from Proposition 7.2.4 (i). Furthermore, we observe that any $f \in \mathfrak{S}$ is divisible by $X_0^{2(\delta-d)}$ and thus there exists some $g \in \mathcal{F}_{n+1,2d}$ such that $f(X) = X_0^{2(\delta-d)} g(X)$. Therefore, the linear map

$$\begin{array}{rccl}
\pi & : & \mathfrak{S} & \to \mathcal{F}_{n+1,2d} \\
& & X_0^{2(\delta-d)} g(X) & \mapsto g(X)
\end{array}$$

is well-defined and we obtain that $\pi\left(C_{\phi(K_i^{(\delta)})} \cap \mathfrak{S}\right)$ is a spectrahedral shadow by Proposition 7.2.4 (ii).

Claim: $\pi\left(C_{\phi(K_i^{(\delta)})} \cap \mathfrak{S}\right) = C_{\phi(K_i^{(d)})}$.

Proof. ($\subseteq$) For $g \in \mathcal{F}_{n+1,2d}$ such that $f(X) := X_0^{2(\delta-d)} g(X) \in C_{\phi(K_i^{(\delta)})} \cap \mathfrak{S}$, we fix

$A \in \mathcal{G}^{-1}(f)$ such that q_A is locally PSD on $\phi\left(K_i^{(\delta)}\right)(\mathbb{R})$. In particular,

$$A = \begin{pmatrix} B & O \\ O & O \end{pmatrix}$$

for some $B \in \mathrm{Sym}_{k(n,d)+1}(\mathbb{R})$ and zero matrices O of appropriate size. We compute

$$\begin{aligned} X_0^{2(\delta-d)} q_B\left(m_0^{(d)}(X), \ldots, m_{k(n,d)}^{(d)}(X)\right) &= q_B\left(X_0^{\delta-d} m_0^{(d)}(X), \ldots, X_0^{\delta-d} m_{k(n,d)}^{(d)}(X)\right) \\ &= q_B\left(m_0^{(\delta)}(X), \ldots, m_{k(n,d)}^{(\delta)}(X)\right) \\ &= q_A\left(m_0^{(\delta)}(X), \ldots, m_{k(n,\delta)}^{(\delta)}(X)\right) \\ &= f(X) \\ &= X_0^{2(\delta-d)} \cdot g(X) \end{aligned}$$

using Lemma 2.2.11. It thus follows $B \in \mathcal{G}^{-1}(g)$ by continuity.

If $i < k(n,d) - n$, then, recalling Lemma 3.3.9, we moreover observe

$$\begin{aligned} & q_B\left(m_0^{(d)}(1,\boldsymbol{x}), \ldots, m_{n+i}^{(d)}(1,\boldsymbol{x}), z_{n+i+1}, \ldots, z_{k(n,d)}\right) \\ &\quad = q_A\left(m_0^{(\delta)}(1,\boldsymbol{x}), \ldots, m_{n+i}^{(\delta)}(1,\boldsymbol{x}), z_{n+i+1}, \ldots, z_{k(n,d)}, 0, \ldots, 0\right) \geq 0 \end{aligned}$$

for $\boldsymbol{x} \in \mathbb{R}^n$ and $z_{n+i+1}, \ldots, z_{k(n,d)} \in \mathbb{R}$ since q_A is locally PSD on $\phi\left(K_i^{(\delta)}\right)(\mathbb{R})$.

However, $i = k(n,d) - n$, then, recalling Lemma 3.3.9, we likewise observe

$$q_B\left(m_0^{(d)}(1,\boldsymbol{x}), \ldots, m_{k(n,d)}^{(d)}(1,\boldsymbol{x})\right) = q_A\left(m_0^{(\delta)}(1,\boldsymbol{x}), \ldots, m_{k(n,d)}^{(\delta)}(1,\boldsymbol{x}), 0, \ldots, 0\right) \geq 0$$

for $\boldsymbol{x} \in \mathbb{R}^n$ since q_A is locally PSD on $\phi\left(K_{k-n}^{(\delta)}\right)(\mathbb{R})$.

For both cases, we conclude $g \in C_{\phi(K_i^{(d)})}$.

($\supseteq$) For $g \in C_{\phi(K_i^{(d)})}$, we let $B \in \mathcal{G}^{-1}(g)$ be such that q_B is locally PSD on $\phi\left(K_i^{(d)}\right)(\mathbb{R})$ and expand B by zero rows and columns to a matrix $A \in \mathrm{Sym}_{k(n,\delta)+1}(\mathbb{R})$ such that

$$A = \begin{pmatrix} B & O \\ O & O \end{pmatrix}$$

for zero matrices O of appropriate size. For $f(X) := X_0^{2(\delta-d)} g(X) \in \mathfrak{S}$, we compute

$$\begin{aligned} q_A\left(m_0^{(\delta)}(X), \ldots, m_{k(n,\delta)}^{(\delta)}(X)\right) &= q_B\left(m_0^{(\delta)}(X), \ldots, m_{k(n,d)}^{(\delta)}(X)\right) \\ &= q_B\left(X_0^{\delta-d} m_0^{(d)}(X), \ldots, X_0^{\delta-d} m_{k(n,d)}^{(d)}(X)\right) \\ &= X_0^{2(\delta-d)} q_B\left(m_0^{(d)}(X), \ldots, m_{k(n,d)}^{(d)}(X)\right) \\ &= X_0^{2(\delta-d)} g(X) = f(X) \end{aligned}$$

using Lemma 2.2.11. This shows $A \in \mathcal{G}^{-1}(f)$.

If $i < k(n,d) - n$, then, recalling Lemma 3.3.9, we moreover observe

$$\begin{aligned} &q_A\left(m_0^{(\delta)}(1,\boldsymbol{x}), \ldots, m_{n+i}^{(\delta)}(1,\boldsymbol{x}), z_{n+i+1}, \ldots, z_{k(n,\delta)}\right) \\ &\qquad = q_B\left(m_0^{(d)}(1,\boldsymbol{x}), \ldots, m_{n+i}^{(d)}(1,\boldsymbol{x}), z_{n+i+1}, \ldots, z_{k(n,d)}\right) \geq 0 \end{aligned}$$

for $\boldsymbol{x} \in \mathbb{R}^n$ and $z_{n+i+1}, \ldots, z_{k(n,\delta)} \in \mathbb{R}$ since q_B is locally PSD on $\phi\left(K_i^{(d)}\right)(\mathbb{R})$.

However, if $i = k(n,d) - n$, then, recalling Lemma 3.3.9, we likewise observe

$$\begin{aligned} &q_A\left(m_0^{(\delta)}(1,\boldsymbol{x}), \ldots, m_{k(n,d)}^{(\delta)}(1,\boldsymbol{x}), z_{k(n,d)+1}, \ldots, z_{k(n,\delta)}\right) \\ &\qquad\qquad = q_B\left(m_0^{(d)}(1,\boldsymbol{x}), \ldots, m_{k(n,d)}^{(d)}(1,\boldsymbol{x})\right) \geq 0 \end{aligned}$$

for $\boldsymbol{x} \in \mathbb{R}^n$ and $z_{k(n,d)+1}, \ldots, z_{k(n,\delta)} \in \mathbb{R}$ since q_B is locally PSD on $\phi\left(K_{k-n}^{(d)}\right)(\mathbb{R})$.

For both cases, we conclude $f \in C_{\phi(K_i^{(\delta)})}$ and $\pi(f) = g$. ■

Altogether, $C_{\phi(K_i^{(d)})}$ is a spectrahedral shadow as a subcone of $\mathcal{P}_{n+1,2d}$. ■

Theorem 7.2.8. *For $i = n+2, \ldots, k-n$, C_i is not a spectrahedral shadow. Moreover, if $n \geq 3$, then also C_{n+1} is not a spectrahedral shadow.*

Proof. For fixed n, we verify the assertion by an induction on d. For the start of induction, we thus have to consider the non-Hilbert cases $(n+1,4)_{n\geq 3}$ and $(3,6)$. Theorem 7.2.6 (i) validates the assertion for the non-Hilbert cases $(n+1,4)_{n\geq 3}$. Moreover, Theorem 7.2.6 (ii) with $n := 2$, $d := 3$ validates the assertion for the basic non-Hilbert case $(3,6)$ since $k(2,2) - 2 + 1 = 4 = 2 + 2$. Hence, we now assume that the assertion was already verified up to some d and investigate the situation for $d+1$.

For $i = n+2, \ldots, k(n,d)-n$, C_i is not a spectrahedral shadow as a subcone of $\mathcal{P}_{n+1,2d}$ by the inductive assumption. Therefore, for $i = n+2, \ldots, k(n,d) - n$, C_i is not a spectrahedral shadow as a subcone of $\mathcal{P}_{n+1,2(d+1)}$ by Theorem 7.2.7. Moreover, if $n \geq 3$, then the same is also true for $i = n+1$. It thus remains to show that C_i is not a spectrahedral shadow for $i = k(n,d) - n + 1, \ldots, k(n,d+1) - n$. This, however, is true by Theorem 7.2.6 (ii). ■

Chapter 8

Conclusion and Future Work

In the first section of this chapter, we draw conclusions from the results of this thesis. In the other three sections, we outline possible directions for future investigations.

8.1 Concluding Remarks

Our results from Chapter 5 and Chapter 6 together allow us to answer the main query of this thesis and to give a refinement of Hilbert's 1888 theorem.

Theorem 8.1.1. A Refinement of Hilbert's 1888 Theorem
In a non-Hilbert case $(n+1, 2d)$, *it holds*

$$\begin{cases} \Sigma_{n+1,2d} = C_0 = \ldots = C_n = C_{n+1} \subsetneq \ldots \subsetneq C_{k(n,d)-n} = \mathcal{P}_{n+1,2d}, & \textit{if } n = 2 \\ \Sigma_{n+1,2d} = C_0 = \ldots = C_n \subsetneq C_{n+1} \subsetneq \ldots \subsetneq C_{k(n,d)-n} = \mathcal{P}_{n+1,2d}, & \textit{else} \end{cases}$$

and there are

$$\mu(n,d) = \begin{cases} k(n,d) - 2n - 2, & \textit{if } n = 2 \\ k(n,d) - 2n - 1, & \textit{else} \end{cases}$$

pairwise distinct strictly separating intermediate cones between $\Sigma_{n+1,2d}$ *and* $\mathcal{P}_{n+1,2d}$ *in the above cone filtration.*

In Chapter 7, we examined the intermediate cones $C_0, \ldots, C_{k-n}$ for some topological and geometric properties and showed that each identified strictly separating intermediate cone between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ fails to be a spectrahedral shadow in a non-Hilbert case. The study of spectrahedral shadows goes back to Nemirovski [Nem07] who in 2006 pointed out that any spectrahedral shadow is a convex semialgebraic set and asked if the converse is true as well.[1] Helton and Nie [HN09] took up this question and conjectured that this is in fact the case in 2009. It was nearly a decade later that Scheiderer [Sch18b] developed a method to produce convex semialgebraic sets that are not spectrahedral shadows and thus disproved the conjecture of Helton and Nie. In particular, he gave the first counterexample by showing that

[1] A set in a finite-dimensional vector space over a real closed field is *semialgebraic* if it can be described by finitely many polynomial equalities and inequalities.

$\mathcal{P}_{n+1,2d}$, interpreted as a convex semialgebraic set, is not a spectrahedral shadow in non-Hilbert cases. A similar observation can be made for our strictly separating intermediate cones between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ in non-Hilbert cases using our results from Chapter 7.

Theorem 8.1.2. *In a non-Hilbert case, the convex semialgebraic sets $C_{n+2}, \dots, C_{k-n}$ are not spectrahedral shadows. Moreover, if $n \geq 3$, then also the convex semialgebraic set C_{n+1} is not a spectrahedral shadow.*

Proof. For $i = n+2, \dots, k-n$, we know that C_i is convex by Lemma 3.1.2 (i). Moreover, we observe that $\{q \in \mathcal{F}_{k+1,2} \mid q|_{V_i(\mathbb{R})} \geq 0\}$ is semialgebraic since the set $\{z \in \mathbb{R}^{k+1} \mid [z] \in V_i(\mathbb{R})\}$ is semialgebraic. Hence, we conclude that the preimage $\{A \in \mathrm{Sym}_{k+1}(\mathbb{R}) \mid q_A|_{V_i(\mathbb{R})} \geq 0\}$ of the semialgebraic set $\{q \in \mathcal{F}_{k+1,2} \mid q|_{V_i(\mathbb{R})} \geq 0\}$ under the linear map Q is semialgebraic by [BCR98, Proposition 2.2.7.]. Thus, also the image of the semialgebraic set $\{A \in \mathrm{Sym}_{k+1}(\mathbb{R}) \mid q_A|_{V_i(\mathbb{R})} \geq 0\}$ under the linear Gram map is semialgebraic by [BCR98, Proposition 2.2.7.]. It therefore follows that

$$C_i = \mathcal{G}\left(\left\{A \in \mathrm{Sym}_{k+1}(\mathbb{R}) \mid q_A|_{V_i(\mathbb{R})} \geq 0\right\}\right)$$

is a convex semialgebraic set which is not a spectrahedral shadow by Theorem 7.2.8. The same is also true for C_{n+1} if $n \geq 3$. ∎

We thus constructed many counterexamples to the Helton–Nie conjecture on convex semialgebraic sets and spectrahedral shadows. In particular, the spectrahedral shadow property is lost as soon as we move from $\Sigma_{n+1,2d}$ to the first strictly separating intermediate cone in ($\mathcal{CF}$) in non-Hilbert cases.

8.2 Further Properties of the Intermediate Cones

In Section 7.1, we investigated the cones $C_0, \dots, C_{k-n}$ for some topological properties in the non-Hilbert cases and showed that these specific cones are in particular closed. Moreover, we know that none of $C_0, \dots, C_{k-n}$ contains a straight line by Corollary 3.1.5. The result below thus follows as a consequence to Krein–Milman's theorem for cones (cf. Corollary A.3.17).

Lemma 8.2.1. *If $(n+1, 2d)$ is a non-Hilbert case, then C_i is the convex hull of its extreme rays for $i = 0, \dots, k-n$.*

It is therefore compelling to gain a better understanding of the extreme rays of the cones $C_0, \dots, C_{k-n}$ in non-Hilbert cases as this will lead to a profound comprehension of these distinguished cones. In this context, it is furthermore interesting to determine (exposed) faces of our intermediate cones in general.

From an applied viewpoint, we explained in Section 7.2, that the benefit of $\Sigma_{n+1,2d}$ over $\mathcal{P}_{n+1,2d}$ is that membership can be efficiently tested. The reason for this is that a $(n+1)$-ary $2d$-ic lies in $\Sigma_{n+1,2d}$ if and only if there exists an associated Gram matrix with a corresponding quadratic form that is locally PSD on $\mathbb{P}^k(\mathbb{R})$. The latter is equivalent to the associated Gram matrix being positive semidefinite. It is well-understood when a given symmetric matrix with real entries is positive semidefinite. An extract of characterization of this property is provided in the list below, which is taken from [Mar08, 0.2.1 Proposition].

Proposition 8.2.2. *For a symmetric $l \times l$ matrix A with real entries ($l \in \mathbb{N}$), the following are equivalent:*

(i) A is positive semidefinite.

(ii) Every eigenvalue of A is non-negative.

(iii) Each principal minor of A is non-negative.

(iv) There exists some $l \times l$ matrix U with real entries such that $A = U^t U$.

(v) There exist $s \in \mathbb{N}$, $y^{(1)}, \ldots, y^{(s)} \in \mathbb{R}^l$ and $\lambda_1, \ldots, \lambda_s \geq 0$ such that

$$A = \sum_{i=1}^{s} \lambda_i \left(y^{(i)}\right)^t \left(y^{(i)}\right).$$

In the setting of this thesis, a brutal force verification of whether a given matrix $A \in \mathrm{Sym}_{k+1}(\mathbb{R})$ is positive semidefinite by evaluating q_A in each $x \in \mathbb{R}^{k+1}$ can thus be replaced by a test for one of the intrinsic properties (ii) – (v) from Proposition 8.2.2.

The downside of replacing membership tests to $\mathcal{P}_{n+1,2d}$ by membership tests to $\Sigma_{n+1,2d}$ is that not every PSD $(n+1)$-ary $2d$-ic is SOS and thus membership tests in $\Sigma_{n+1,2d}$ do not suffices to decide whether a given form is PSD in all cases. In an attempt to provide certificates of the PSD property for more $(n+1)$-ary $2d$-ics than those that are SOS, we therefore propose to test membership in the larger (w.r.t. $\subseteq$) strictly separating intermediate cones in (6.13). This brings us to the following task.

Open Problem 1. Let C_i be a strictly separating intermediate cone in (6.13) in a non-Hilbert case. Provide a (preferably intrinsic) membership test for C_i.

The above problem has to be addressed with tools other than the ones coming from semidefinite programming since no strictly separating intermediate cone in (6.13) is a spectrahedral shadow by Theorem 7.2.8 and with that not a feasible region of a semidefinite programming problem. However, other methods might provide numerically efficient (partial) membership tests for our distinguished intermediate cones.

8.3 Dual Cones of the Intermediate Cones

Throughout this section, we let $\boldsymbol{(n+1, 2d)}$ **denote a non-Hilbert case** and we recall that $C_0, \ldots, C_{k-n}$ are closed. Thus, the bidual cone $(C_i^{\vee})^{\vee}$ coincides with C_i

for $i = 0, \dots, k-n$ by Theorem A.3.24. Hence, an examination of the cone filtration ($\mathcal{CF}$) can equivalently be replaced by an investigation of the dual cone filtration

$$\Sigma_{n+1,2d}^{\vee} \supseteq C_0^{\vee} \supseteq \ldots \supseteq C_{k-n}^{\vee} = \mathcal{P}_{n+1,2d}^{\vee}$$

since $C_i \subsetneq C_{i+1}$ if and only if $C_i^{\vee} \supsetneq C_{i+1}^{\vee}$ for $i = 0, \dots, k-n-1$.

This brings us to a classical problem of functional analysis. That is, given a linear functional $L\colon \mathbb{R}[\boldsymbol{X}] \to \mathbb{R}$, the *classical n-dimensional moment problem* asks if there exists a non-negative Borel measure μ on $\mathbb{R}^n$ such that $L(g) = \int g \mathrm{d}\mu$ for all $g \in \mathbb{R}[\boldsymbol{X}]$.

However, in this thesis, we focused on forms of a fixed even degree or, equivalently in a dehomogenized setting, on polynomials up to a fixed even degree. We are therefore in the context of the *truncated moment problem* which asks if for a given sequence $(y_a)_{a \in I_{n+1,d}} \subseteq \mathbb{R}$, there exists a non-negative Borel measure μ on $\mathbb{R}^n$ such that $y_a = \int \boldsymbol{X}^{(a_1,\dots,a_n)} \mathrm{d}\mu(\boldsymbol{X})$ for all $a \in I_{n+1,d}$. We call $(y_a)_{a \in I_{n+1,d}}$ a *truncated moment sequence* and say that this sequence *admits a measure* if the truncated moment problem can be affirmatively answered with a Borel measure μ. Any such μ is then called a *representing measure* of $(y_a)_{a \in I_{n+1,d}}$ and we denote the set of all truncated moment sequences $(y_a)_{a \in I_{n+1,d}}$ that admit a measure by $\mathcal{R}_{n,d}$. Moreover, we set $\mathbb{R}[\boldsymbol{X}]_{\leq d}$ to be the $\mathbb{R}$-vector space of polynomials in $\boldsymbol{X}$ up to degree d with coefficients in $\mathbb{R}$ and observe that a truncated moment sequence $y := (y_a)_{a \in I_{n+1,d}} \subseteq \mathbb{R}$ induces the *Riesz functional*

$$\mathcal{L}_y\colon \begin{array}{ccc} \mathbb{R}[\boldsymbol{X}]_{\leq d} & \to & \mathbb{R} \\ \sum\limits_{a \in I_{n+1,d}} g_a \boldsymbol{X}^{(a_1,\dots,a_n)} & \mapsto & \sum\limits_{a \in I_{n+1,d}} g_a y_a. \end{array}$$

Therefore, the truncated moment problem in particular asks if for a given linear functional $L\colon \mathbb{R}[\boldsymbol{X}]_{\leq d} \to \mathbb{R}$, there exists a non-negative Borel measure μ on $\mathbb{R}^n$ such that $L(g) = \int g \mathrm{d}\mu$ for all $g \in \mathbb{R}[\boldsymbol{X}]_{\leq d}$.

Over the years, the truncated moment problem has been extensively studied by Curto–Fialkow [CF96; CF05; CF08], Fialkow–Nie [FN12] and many more in the classical setting and by Curto et al. [Cur+23] for unital commutative $\mathbb{R}$-algebras.

A condition that is necessary for a truncated moment sequence $y := (y_a)_{a \in I_{n+1,d}}$ to admit a measure is, for example, that the Riesz functional $\mathcal{L}_y$ is non-negative in any positive semidefinite $g \in \mathbb{R}[\boldsymbol{X}]_{\leq d}$. That is, $\mathcal{L}_y$ must lie in the dual cone of the cone $\mathcal{P}_{n,\leq d}$ of positive semidefinite $g \in \mathbb{R}[\boldsymbol{X}]_{\leq d}$. In fact, by [FN10, Theorem 2.2.], we have

$$\overline{\mathcal{R}_{n,d}} = \left\{ y := (y_a)_{a \in I_{n+1,2d}} \subseteq \mathbb{R} \mid \mathcal{L}_y \in \mathcal{P}_{n,\leq d}^{\vee} \right\} \simeq \mathcal{P}_{n,\leq d}^{\vee}.$$

In order to transfer this consideration into a homogeneous setting, we observe that a truncated moment sequence $y := (y_a)_{a \in I_{n+1,d}} \subseteq \mathbb{R}$ admits a measure if and only if there exists a non-negative Borel measure μ supported on the n-dimensional unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ such that $y_a = \int_{\mathbb{S}^n} X^{\alpha} \mathrm{d}\mu(X)$ for all $a \in I_{n+1,d}$. In this sense, y is a *homogeneous* truncated moment sequence and we denote the set of homogeneous truncated moment sequences admitting a representing measure supported on $\mathbb{S}^n$ by $\mathcal{R}^h_{n+1,d}$. We now observe for even degrees that

$$\overline{\mathcal{R}_{n,2d}} = \mathcal{R}^h_{n+1,2d} = \left\{ y := (y_a)_{a \in I_{n+1,d}} \subseteq \mathbb{R} \mid \mathcal{L}_y \in \mathcal{P}^\vee_{n+1,2d} \right\} \simeq \mathcal{P}^\vee_{n+1,2d}$$

by [FN10, Corollary 2.4. and Theorem 3.1.]. The dual cone $\mathcal{P}^\vee_{n+1,2d}$ can thus be interpreted as the cone of homogeneous truncated moment sequences that admit a representing measure supported on $\mathbb{S}^n$. Similarly as in our primal setting, testing membership in $\mathcal{P}^\vee_{n+1,2d}$ is a challenging task. Therefore, this task is often replaced by membership tests to the greater (w.r.t. $\subseteq$) well-understood dual cone $\Sigma^\vee_{n+1,2d}$. Yet, membership in $\Sigma^\vee_{n+1,2d}$ is only necessary but not sufficient for membership in the subcone $\mathcal{P}^\vee_{n+1,2d}$.

Coming back to the setting of this thesis, we observe that Theorem 8.1.1 yields

$$\begin{cases} \Sigma^\vee_{n+1,2d} = C^\vee_0 = \ldots = C^\vee_n = C^\vee_{n+1} \supsetneq \ldots \supsetneq C^\vee_{k(n,d)-n} = \mathcal{P}^\vee_{n+1,2d}, & \text{if } n = 2 \\ \Sigma^\vee_{n+1,2d} = C^\vee_0 = \ldots = C^\vee_n \supsetneq C^\vee_{n+1} \supsetneq \ldots \supsetneq C^\vee_{k(n,d)-n} = \mathcal{P}^\vee_{n+1,2d}, & \text{else.} \end{cases}$$

Therefore, we propose to establish (partial) membership tests for the dual cone $C^\vee_i$ of a strictly separating intermediate cone C_i of $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$. The benefit of such tests is that they provide an affirmative answer for fewer linear functionals $L\colon \mathcal{F}_{n+1,2d} \to \mathbb{R}$ than membership tests for $\Sigma^\vee_{n+1,2d}$. Indeed, $\mathcal{P}^\vee_{n+1,2d} \subsetneq C^\vee_i \subsetneq \Sigma^\vee_{n+1,2d}$ shows that $C^\vee_i$ approximate $\mathcal{P}^\vee_{n+1,2d}$ better than $\Sigma^\vee_{n+1,2d}$. These consideration bring us to the two problems below.

Open Problem 2. Give an explicit description of the dual cone $C^\vee_i$ for each strictly separating intermediate cone C_i between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ in (6.13).

Open Problem 3. Put $C^\vee_i$ into the context of the truncated moment problem for each strictly separating intermediate cone C_i between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ in (6.13).

In Chapter 6, we moreover illuminated a connection between the cone C_i and the $\mathbb{R}$-vector space $\mathfrak{S}_{n+i} := \operatorname{span}_{\mathbb{R}}\{m_s m_t \mid 0 \le s,t \le n+i\}$ by showing

$$C_i \cap \mathfrak{S}_{n+i} = \mathcal{P}_{n+1,2d} \cap \mathfrak{S}_{n+i} \tag{8.1}$$

for $i = 0, \ldots, k-n$ in Theorem 6.2.2. Casually speaking, we are thus not only interested in $(n+1)$-ary $2d$-ics but especially in those that are $\mathbb{R}$-linear combinations of the monomials $m_s m_t$ for $0 \le s,t \le n+i$ for an a priori fixed $i \in \{0, \ldots, k-n\}$.

This brings us to the $\mathcal{A}$-*truncated moment problem* for a given finite set $\mathcal{A} \subseteq \mathbb{N}^n_0$ which asks if for a given *truncated multisequence* $(y_{\boldsymbol{a}})_{\boldsymbol{a} \in \mathcal{A}} \subseteq \mathbb{R}$, there exists a non-negative Borel measure μ such that $y_{\boldsymbol{a}} = \int \boldsymbol{X}^{\boldsymbol{a}} \mathrm{d}\mu(\boldsymbol{X})$ for all $\boldsymbol{a} \in \mathcal{A}$. The $\mathcal{A}$-truncated moment problem finds application in sparse polynomial optimization (cf. [Las06]) and was studied by Nie [Nie14]. However, up to date, only few is known. In the light of (8.1), we therefore pose the following question.

Open Problem 4. For $i = 0, \dots, k-n$, consider the finite set

$$\mathcal{A}_i := \{\alpha_s + \alpha_t \mid 0 \leq s, t \leq n+i\} \subseteq I_{n+1,2d} \subseteq \mathbb{N}_0^{n+1}.$$

How does the dual cone $C_i^\vee$ relate to the $\mathcal{A}_i$-truncated moment problem?

8.4 Cones along Toric Varieties

At the beginning of our investigation, we stressed in Remark 2.3.8 that our choice of ordering $I_{n+1,d}$ lexicographically cannot be neglected. We now illustrate what may happens if $I_{n+1,d}$ is ordered by a monomial order other than the lexicographical.

Example 8.4.1. Ternary Decics
We consider the non-Hilbert case of ternary decics. Hence, $n = 2$, $d = 5$ and we compute $k = 20$. Instead of ordering $I_{3,5}$ lexicographically, we now propose another monomial order $\leq$. That is, for $\alpha, \beta \in \mathbb{N}_0^3$, we set $\alpha < \beta$ if

- $\alpha_0 + \alpha_1 < \beta_0 + \beta_1$ or
- $\alpha_0 + \alpha_1 = \beta_0 + \beta_1$ and $\alpha_0 < \beta_0$ or
- $(\alpha_0, \alpha_1) = (\beta_0, \beta_1)$ and $\alpha_2 < \beta_2$.

A straight forward computation verifies that $\leq$ is a monomial order. Therefore, ordering $I_{3,5}$ by $\leq$ starting with the greatest element, we obtain the ordered monomial basis $\{m_0, \dots, m_{20}\}$ and we see that $m_0(X) = X_0^5$, $m_1(X) = X_0^4X_1$, $m_2(X) = X_0^3X_1^2$, $m_3(X) = X_0^2X_1^3$, $m_4(X) = X_0X_1^4$, $m_5(X) = X_1^5$.

In order to retrieve a counterpart to Construction 3.2.1, we furthermore compute

$$\begin{aligned}
T_0 &:= \left\{[z] \in \mathbb{P}^{20} \mid \exists x \in \mathbb{C}^3 \colon z_0 = x_0^5\right\} = \mathbb{P}^{20},\\
T_1 &:= \left\{[z] \in \mathbb{P}^{20} \mid \exists x \in \mathbb{C}^3 \colon (z_0, z_1) = \left(x_0^5, x_0^4x_1\right)\right\} = T_0,\\
T_2 &:= \left\{[z] \in \mathbb{P}^{20} \mid \exists x \in \mathbb{C}^3 \colon (z_0, z_1, z_2) = \left(x_0^5, x_0^4x_1, x_0^3x_1^2\right)\right\} \subsetneq T_1,\\
T_3 &:= \left\{[z] \in \mathbb{P}^{20} \mid \exists x \in \mathbb{C}^3 \colon (z_0, z_1, z_2, z_3) = \left(x_0^5, x_0^4x_1, x_0^3x_1^2, x_0^2x_1^3\right)\right\} \subsetneq T_2,\\
T_4 &:= \left\{[z] \in \mathbb{P}^{20} \mid \exists x \in \mathbb{C}^3 \colon (z_0, z_1, z_2, z_3, z_4) = \left(x_0^5, x_0^4x_1, x_0^3x_1^2, x_0^2x_1^3, x_0x_1^4\right)\right\} \subsetneq T_3,\\
T_5 &:= \left\{[z] \in \mathbb{P}^{20} \mid \exists x \in \mathbb{C}^3 \colon (z_0, \dots, z_5) = \left(x_0^5, x_0^4x_1, x_0^3x_1^2, x_0^2x_1^3, x_0x_1^4, x_1^5\right)\right\} \subsetneq T_4,\\
T_6 &:= \left\{[z] \in \mathbb{P}^{20} \mid \exists x \in \mathbb{C}^3 \colon (z_0, \dots, z_6) = \left(x_0^5, \dots, x_1^5, x_0^4x_2\right)\right\} = T_5,\\
T_7 &:= \left\{[z] \in \mathbb{P}^{20} \mid \exists x \in \mathbb{C}^3 \colon (z_0, \dots, z_7) = \left(x_0^5, \dots, x_0^4x_2, x_0^3x_1x_2\right)\right\} \subsetneq T_6,\\
&\vdots\\
T_{20} &:= \left\{[z] \in \mathbb{P}^{20} \mid \exists x \in \mathbb{C}^3 \colon (z_0, \dots, z_{20}) = \left(x_0^5, \dots, x_1x_2^4, x_2^5\right)\right\} \subsetneq T_{19}.
\end{aligned}$$

Thus, we set $H_0 := T_0$, $H_1 := T_2$, $H_2 := T_3$, $H_3 := T_4$, $H_4 := T_5$, $H_5 := T_7$, $H_i := T_{i+2}$ for $i = 6, \ldots, 18$ and let V_i be the Zariski closure of H_i in $\mathbb{P}^{20}$ for $i = 0, \ldots, 18$. Arguing as in Section 3.2, this construction gives us a specific filtration of projective varieties

$$V(\mathbb{P}^2) = V_{18} \subsetneq \ldots \subsetneq V_0 = \mathbb{P}^{20}$$

with a corresponding specific filtration of sets of real points

$$V(\mathbb{P}^2)(\mathbb{R}) = V_{18}(\mathbb{R}) \subsetneq \ldots \subsetneq V_0(\mathbb{R}) = \mathbb{P}^{20}(\mathbb{R})$$

such that each inclusion appearing is strict. For $i = 0, \ldots, 18$, we lastly set $C_i := C_{V_i}$ and obtain a specific cone filtration

$$\Sigma_{3,10} = C_0 \subseteq \ldots \subseteq C_{18} = \mathcal{P}_{3,10} \tag{8.2}$$

in which at least one inclusion has to be strict by Hilbert's 1888 theorem but it is not clear how many of the inclusions are strict and which ones.

We hence now examine the subfiltration

$$\Sigma_{3,10} = C_0 \subseteq C_1 \subseteq C_2 \subseteq C_3 \subseteq C_4 \tag{8.3}$$

for strict inclusions. To this end, as a counterpart to Construction 3.3.1, we set

$$\begin{aligned}
q_1(Z) &:= Z_0Z_2 - Z_1^2 &\quad\text{and}\quad&& p_1(\boldsymbol{Z}) &:= Z_2 - Z_1^2,\\
q_2(Z) &:= Z_0Z_3 - Z_1Z_2 &\quad\text{and}\quad&& p_2(\boldsymbol{Z}) &:= Z_3 - Z_1Z_2,\\
q_3(Z) &= Z_0Z_4 - Z_1Z_3 &\quad\text{and}\quad&& p_3(\boldsymbol{Z}) &= Z_4 - Z_1Z_3,\\
q_4(Z) &= Z_0Z_5 - Z_1Z_4 &\quad\text{and}\quad&& p_4(\boldsymbol{Z}) &= Z_5 - Z_1Z_4.
\end{aligned}$$

Following Construction 3.3.7, we furthermore consider the affine varieties $K_0 := \mathbb{C}^{20}$ and $K_i := \boldsymbol{\mathcal{V}}(p_1, \ldots, p_i) \subseteq \mathbb{C}^{20}$ for $i = 1, \ldots, 4$ and let $W_i \subseteq \mathbb{P}^{20}$ be the projective closure of $K_i \subseteq \mathbb{C}^{20}$ for $i = 0, \ldots, 4$. Repeating our arguments from Section 3.3, we conclude $V_i = W_i$ for $i = 0, \ldots, 4$. Hence, applying our methods from Section 4.1, we see that V_i is a non-degenerate irreducible totally-real projective variety of codimension i for $i = 0, \ldots, 4$. Moreover, in the spirit of Construction 3.2.7, we see that $V_0, \ldots, V_4$ are cones over the varieties $\tilde{V}_0, \ldots, \tilde{V}_4$ that are parametrized by

$$\begin{aligned}
\chi_0 \colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1\\
[x] &\mapsto \left[x_0^5 : x_0^4x_1\right],\\
\chi_1 \colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^2\\
[x] &\mapsto \left[x_0^5 : x_0^4x_1 : x_0^3x_1^2\right],\\
\chi_2 \colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^3\\
[x] &\mapsto \left[x_0^5 : x_0^4x_1 : x_0^3x_1^2 : x_0^2x_1^3\right],
\end{aligned}$$

$$\begin{aligned}\chi_3\colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^4\\ [x] &\mapsto \left[x_0^5 : x_0^4x_1 : x_0^3x_1^2 : x_0^2x_1^3 : x_0x_1^4\right],\\ \chi_4\colon \mathbb{P}^2 &\dashrightarrow \mathbb{P}^5\\ [x] &\mapsto \left[x_0^5 : x_0^4x_1 : x_0^3x_1^2 : x_0^2x_1^3 : x_0x_1^4 : x_1^5\right].\end{aligned}$$

In particular, the map χ_4 is the Veronese embedding (of degree 5)

$$\begin{aligned}V^{(5)}\colon \mathbb{P}^1 &\to \mathbb{P}^5,\\ [x_0 : x_1] &\mapsto \left[x_0^5 : x_0^4x_1 : x_0^3x_1^2 : x_0^2x_1^3 : x_0x_1^4 : x_1^5\right]\end{aligned}$$

in disguise and thus V_4 is a multiple cone over the Veronese variety $V^{(5)}(\mathbb{P}^1)$ in $\mathbb{P}^{k(1,5)}$. The Hilbert polynomial of the irreducible projective variety $V^{(5)}(\mathbb{P}^1)$ is given by

$$p_{V^{(5)}(\mathbb{P}^1)}(T) = \binom{5T+1}{1} = 5T + 1 \in \mathbb{C}[T]$$

according to [Har92, Example 13.4.] and, therefore, we conclude $\deg\left(V^{(5)}(\mathbb{P}^1)\right) = 5$. Consequently, the multiple cone V_4 over $V^{(5)}(\mathbb{P}^1)$ has degree 5 by Proposition A.1.50.

Altogether, we thus know that V_4 is a non-degenerate irreducible totally-real projective variety of minimal degree such that $V(\mathbb{P}^n)(\mathbb{R}) \subseteq V_4(\mathbb{R})$. Theorem 4.2.2 therefore yields $\Sigma_{3,10} = C_4$ which shows that the subfiltration (8.3) of (8.2) collapses. Hence, we conclude that there are at most 13 strictly separating intermediate cones between $\Sigma_{3,10}$ and $\mathcal{P}_{3,10}$ in (8.2) in comparision to $\mu(2,5) = k(2,5) - 4 - 2 = 14$ strictly separating intermediate cones between $\Sigma_{3,10}$ and $\mathcal{P}_{3,10}$ in (6.13).

The example above illustrates that different monomial orders lead to potentially different behaviours of the inclusions in cone filtrations similar to ($\mathcal{CF}$) that are constructed in the spirit of this thesis. It would therefore be interesting to investigate what properties of a monomial order influence the occurence of cone equalities, respectively cone inequalities, and how.

The specific projective varieties of this thesis, and more generally also the projective varieties along a monomial order in the light of Construction 3.2.1, are projective varieties that are described by monomials. This brings us to so called *toric varieties* and we refer an interested reader to [CLS11] for an introduction to this concept. We here only include the definition of a projective toric variety in the framework of this thesis for the convenience of the reader.

Definition 8.4.2. For a finite set $\mathcal{B} := \{\beta_0, \ldots, \beta_s\} \subseteq I_{n+1,d} \subseteq \mathbb{Z}^{n+1}$ ($s \in \mathbb{N}$), we set

$$\begin{aligned}\phi_{\mathcal{B}}\colon (\mathbb{C}^{\times})^{n+1} &\to (\mathbb{C}^{\times})^{s+1}\\ x &\mapsto \left(x^{\beta_0}, \ldots, x^{\beta_s}\right).\end{aligned}$$

and consider the canonical map $\pi\colon (\mathbb{C}^\times)^{s+1} \to \mathbb{P}^s, y \mapsto [y]$. The Zariski closure of the image of $\pi \circ \phi_\mathcal{B}$ in $\mathbb{P}^s$ is a *projective toric variety.*

The projective varieties $\tilde{V}_0, \ldots, \tilde{V}_{k-n}$ from Construction 3.2.7 are thus projective toric varieties and, in this sense, also $V_0, \ldots, V_{k-n} \subseteq \mathbb{P}^k$ can be interpreted as projective toric varieties. The same remains true when $I_{n+1,d}$ is ordered by a monomial order other than the lexicographical.

Since toric varieties are well-understood, they might serve as a good starting point for further investigations that attempt to establish a general criterion for an intermediate cone C_W between $\Sigma_{n+1,2d}$ and $\mathcal{P}_{n+1,2d}$ along a projective variety W to be strictly separating in non-Hilbert cases. We therefore propose the following question.

Open Problem 5. For projective toric varieties $\mathfrak{W}_1, \mathfrak{W}_2 \subseteq \mathbb{P}^k$ such that $\mathfrak{W}_1 \subseteq \mathfrak{W}_2$, let $q \in \mathcal{F}_{k+1,2}$ be locally PSD on $\mathfrak{W}_1(\mathbb{R})$. When does there exist a quadratic form $\mathfrak{q} \in \mathcal{F}_{k+1,2}$ such that $\mathfrak{q}$ vanishes on $V(\mathbb{P}^n)$ and $(q + \mathfrak{q})|_{\mathfrak{W}_2(\mathbb{R})} \geq 0$?

Replacing the projective toric variety $V(\mathbb{P}^n)$ by some arbitrary projective toric variety $\mathfrak{W}_0 \subseteq \mathbb{P}^k$ in Open Problem 5, we obtain the more general problem below whose answer will be a first step towards a complete answer for Question 3.

Open Problem 6. For three projective toric varieties $\mathfrak{W}_0, \mathfrak{W}_1, \mathfrak{W}_2 \subseteq \mathbb{P}^k$ such that $\mathfrak{W}_1 \subseteq \mathfrak{W}_2$, let $q \in \mathcal{F}_{k+1,2}$ be locally PSD on $\mathfrak{W}_1(\mathbb{R})$. When does there exist a quadratic form $\mathfrak{q} \in \mathcal{F}_{k+1,2}$ such that $\mathfrak{q}$ vanishes on $\mathfrak{W}_0$ and $(q + \mathfrak{q})|_{\mathfrak{W}_2(\mathbb{R})} \geq 0$?

Our consideration of Section 8.2 and Section 8.3 can be repeated for intermediate cones that come from arbitrary projective toric varieties.

Chapter 9

Deutsche Zusammenfassung

9.1 Überblick

Eines der zentralen Probleme der reell-algebraischen Geometrie beschäftigt sich mit der Frage, wann ein beliebiges gegebenes positiv semidefinites (PSD) Polynom in n Variablen des Grades $2d$ als Summe von Polynomquadraten (SOS) dargestellt werden kann. Da Nicht-Negativität und die SOS-Eigenschaft unter Homogenisierung stabil sind [Mar08, Proposition 1.2.4.], reicht es aus, homogene Polynome in $n+1$ Variablen vom Grad $2d$ zu betrachten. Die Untersuchung dieser Fragestellung geht auf Hilbert [Hil88] zurück, welcher bereits 1888 sämtliche Fälle $(n+1, 2d)$ klassifizierte, in denen jedes PSD homogene Polynom in $n+1$ Variablen vom Grad $2d$ als Summe von Formenquadraten halben Grades d dargestellt werden kann. Dies sind die *Hilbert-Fälle* $(2, 2d)_{d\geq 1}$, $(n+1, 2)_{n\geq 1}$ und $(3, 4)$. Für die verbleibenden *Nicht-Hilbert-Fälle* belegte Hilbert die Existenz homogener PSD Polynome welche nicht SOS sind. Zunächst zeigte er dies für die *grundlegenden Nicht-Hilbert-Fälle* $(4, 4)$ und $(3, 6)$. Anschließend argumentierte er, dass die PSD-nicht-SOS-Eigenschaft solcher homogener Polynome stabil unter Interpretation in mehr Variablen und Multiplikation mit Monomenquadraten ist. Daraus schloss Hilbert, dass es sich in den Nicht-Hilbert-Fällen bei dem konvexen Kegel $\Sigma_{n+1,2d}$ aller SOS homogener Polynome in $n+1$ Variablen vom Grad $2d$ um einen echten Unterkegel des konvexen Kegels $\mathcal{P}_{n+1,2d}$ aller PSD homogener Polynome in $n+1$ Variablen vom Grad $2d$ handelt. Hilberts Beweisführung war jedoch nicht konstruktiv und lieferte daher keine expliziten Beispiele für PSD-nicht-SOS homogene Polynome.

Erst knapp 80 Jahre nach Hilberts Entdeckung gelang es Motzkin [Mot65] ein erstes explizites Beispiel eines PSD-nicht-SOS homogenen Polynoms in dem grundlegenden Nicht-Hilbert-Fall $(3, 6)$ zu finden. Unabhängig dessen gelang es Robinson [Rob73] kurze Zeit später PSD-nicht-SOS homogene Polynome in den Fällen $(4, 4)$ und $(3, 6)$ zu konstruieren, indem er Hilberts Argumente wesentlich vereinfachte. In den folgenden Jahren erweiterten Choi und Lam [CL76; CL77] diese Liste um weitere PSD-nicht-SOS homogene Polynome in den grundlegenden Nicht-Hilbert-Fällen.

Aufbauend auf seiner Entdeckung von 1888 bewies Hilbert [Hil90] ferner, dass jedes PSD homogene Polynom in drei Variablen als Summe von Quadraten rationaler Funktionen geschrieben werden kann. Diese Beobachtung veranlasste ihn dazu auf dem internationalem Kongress der Mathematiker in Paris 1900 [Hil00] die Frage zu stellen, ob dies für PSD homogene Polynome in beliebig vielen Variablen verallgemeinert werden kann. Diese, heute als 17. Problem von Hilbert bekannte Fragestellung, konnte 1927 durch Artin [Art27] gelöst werden und markiert die Geburtsstunde der Artin–Schreier-Theorie reell-abgeschlossener Körper.

Im Laufe des letzten Jahrhunderts rückten die konvexen Kegel $\mathcal{P}_{n+1,2d}$ und $\Sigma_{n+1,2d}$ immer weiter in den Fokus aktueller Forschungsarbeiten. Die Untersuchungen gipfelten schlussendlich 2016 in einer Verallgemeinerung von Hilberts Satz von 1888 entlang projektiver Varietäten durch Blekherman–Smith–Velasco [Ble12]. Den Autoren gelang es zu zeigen, dass unter bestimmten Voraussetzungen genau dann jedes lokal auf einer irreduziblen projektiven Varietät positiv semidefinite homogene Polynom vom Grad 2 eine Summe von Formenquadraten in dem zugehörigen homogenen Koordinatenring ist, wenn die entsprechende Varietät minimalen Grad hat. Unter Verwendung der Veronese-Einbettung kann dieses Resultat ferner auf homogene Polynome beliebigen geraden Grades $2d$ übertragen werden.

Die Untersuchung positiv semidefiniter homogener Polynome spielt eine entscheidende Rolle in diversen anwendungsorientierten Gebieten wie der polynomiellen Optimierung, der Kontrolltheorie und dem Ingenieurwesen (vgl. [BPT13; Las10; Lau09]). Bis zum heutigen Tag existiert jedoch kein Algorithmus, der in polynomieller Zeit entscheiden kann, ob ein beliebig gegebenes homogenes Polynom PSD ist. Daher wird in der Praxis häufig stattdessen auf die SOS-Eigenschaft getestet, da diese effizient in polynomieller Zeit überprüfbar ist. Dies beruht auf der Tatsache, dass die Zugehörigkeit zu $\Sigma_{n+1,2d}$ als semidefinites Optimierungsproblem formuliert werden kann [GLS93; Par03]. Spektralschatten repräsentieren dabei die realisierbaren Bereiche semidefiniter Optimierungsprobleme. Eben zu solchem zählt $\Sigma_{n+1,2d}$.

Bei einem Spektralschatten handelt es sich insbesondere stets um eine konvexe semialgebraische Menge. Diese Beobachtung motivierte Nemirovski [Nem07] im Jahr 2007 die Frage zu stellen, ob bereits jede konvexe semialgebraische Menge auch ein Spektralschatten ist. Helton und Nie [HN09] stellten 2009 die Vermutung auf, dass dies tatsächlich der Fall sei. Diverse Ergebnisse in Spezialfällen von unterschiedlichen Autoren schienen diese Vermutung zunächst zu bestätigen (vgl. [HN10; NS15; Sch18a]). 2018 gelang es Scheider [Sch18b] dann jedoch die Vermutung von Helton und Nie zu widerlegen, indem er eine Methodik entwickelte, mit Hilfe derer konvexe semialgebraische Mengen erzeugt werden können, die keine Spektralschatten sind. Der konvexe Kegel $\mathcal{P}_{n+1,2d}$ stellt einen Spezialfall einer solchen Menge dar. In Anbetracht der Tatsache, dass $\Sigma_{n+1,2d}$ ein Spektralschatten ist, während $\mathcal{P}_{n+1,2d}$ diese Eigenschaft nicht aufweist, ist es daher von großem Interesse zu verstehen, in welchem Schritt beim Übergang von dem kleinen konvexen Kegel $\Sigma_{n+1,2d}$ zu dem größeren konvexen Kegel $\mathcal{P}_{n+1,2d}$ die Spektralschatten-Eigenschaft verloren geht.

Um diese Frage zu untersuchen, wird in Kapitel 2 der vorliegenden Arbeit zunächst die Gram-Matrix-Methode [CLR92] vorgestellt und sodann für die Konstruktion einer Filtrierung konvexer Kegel

$$\Sigma_{n+1,2d} = C_0 \subseteq \ldots \subseteq C_n \subseteq C_{n+1} \subseteq \ldots \subseteq C_{k(n,d)-n} = \mathcal{P}_{n+1,2d} \tag{9.1}$$

entlang $k(n, d) - n + 1$ projektiver Varietäten, welche die Veronese-Varietät enthalten, in Kapitel 3 herangezogen. Hierbei steht die natürliche Zahl $k(n, d) + 1$ für die Dimension des $\mathbb{R}$-Vektorraums homogener Polynome in $n + 1$ Variablen vom Grad d. Unter Verwendung der Verallgemeinerung von Hilberts Satz von 1888 durch Blekherman–Smith–Velasco wird in Kapitel 4 weiter aufgezeigt, dass die ersten $n+1$, bzw. die ersten $n + 2$ wenn $n = 2$, konvexen Kegel in (9.1) mit $\Sigma_{n+1,2d}$ übereinstimmen. Bei den verbleibenden Inklusionen handelt es sich in den Nicht-Hilbert-Fällen dagegen um strikte Inklusionen, wie zunächst für homogene Polynome vom Grad 4 und 6 unter Einbezug der PSD-nicht-SOS-Beispiele nach Motzkin und Choi–Lam in Kapitel 5 bewiesen wird. Diese Beobachtung wird durch die Entwicklung eines Grad-Sprung-Verfahrens für PSD-extremale Kreispolynome auf beliebige Nicht-Hilbert-Fälle in Kapitel 6 verallgemeinert. Hieraus ergibt sich in Kapitel 7 eine Verfeinerung von Hilberts Satz von 1888 und gleichzeitig die Widerlegung der Helton–Nie-Vermutung über konvexe semialgebraische Mengen, da die identifizierten echten semialgebraischen Zwischenkegel keine Spektralschatten sind. Insbesondere geht die Spektralschatten-Eigenschaft in (9.1) daher unmittelbar im ersten strikten Inklusionsschritt verloren.

9.2 Detailierte Inhaltsangabe

Kapitel 2 legt das Fundament dieser Arbeit. Hierfür werden in Unterkapitel 2.1 zunächst sämtliche verwendeten Notationen eingeführt und relevante Konzepte kurz wiederholt. Es wird vorausgesetzt, dass die Leserin oder der Leser mit den Grundlagen der algebraischen Geometrie und der Konvexitätslehre vertraut ist. Bei Fragen kann jedoch Anhang A.1 und Anhang A.3 hinzugezogen werden. Das Schlüsselkonzept positiv semidefiniter Polynome und Summen von Formenquadraten wird in Unterkapitel 2.2 erläutert. Dabei wird insbesondere aufgezeigt, dass die Betrachtung homogener Polynome genügt und Hilberts Satz von 1888 präsentiert. Abschließend wird in Unterkapitel 2.3 die Gram-Matrix-Methode erläutert, mit derer Hilfe die Untersuchung der konvexen Kegel $\mathcal{P}_{n+1,2d}$, $\Sigma_{n+1,2d}$ und ihrer konvexen Zwischenkegel auf die Analyse von lokalen Nicht-Negativitätseigenschaften homogener Polynome vom Grad 2 auf reellen Punktemengen projektiver Varietäten reduziert werden kann.

Kapitel 3 vertieft die Überlegungen aus Kapitel 2. Unterkapitel 3.1 etabliert hierfür eine Konstruktionsmethode konvexer Kegel zwischen $\Sigma_{n+1,2d}$ und $\mathcal{P}_{n+1,2d}$ entlang reeller Punktemengen projektiver Varietäten (vgl. Definition 3.1.1). Dies führt in

Unterkapitel 3.2 zu einer Filtrierung konvexer Kegel

$$\Sigma_{n+1,2d} = C_0 \subseteq \ldots \subseteq C_{k(n,d)-n} = \mathcal{P}_{n+1,2d}$$

(vgl. ($\mathcal{CF}$)) entlang einer Filtrierung projektiver Varietäten $V_{k(n,d)-n} \subsetneq \ldots \subsetneq V_0$ (vgl. (3.4)) mit einer zugehörigen Filtrierung reeller Punktemengen, in der jede auftretende Inklusion strikt ist (vgl. (3.5)). Hilberts Satz von 1888 impliziert, dass in jedem Nicht-Hilbert-Fall mindestens eine der Inklusionen in obiger Filtrierung konvexer Kegel strikt sein muss. Es ist a priori jedoch weder klar welche Inklusion diese ist, noch wie viele Inklusionen strikt sind. Dies liefert die zentralen Thematik dieser Doktorarbeit: die Identifikation jeder strikten Inklusion in obiger konvexen Kegelfiltrierung in allen Nicht-Hilbert-Fällen. Um sich dieser Problematik zu nähern, wird in Unterkapitel 3.3 zunächst eine explizite Beschreibung der projektiven Varietäten $V_0, \ldots, V_{k(n,d)-n}$ als projektiver Abschluss gewisser affiner Varietäten $K_0, \ldots, K_{k(n,d)-n}$ etabliert (vgl. Konstruktion 3.3.7 und Satz 3.3.17). Diese Beobachtung wird in Unterkapitel 3.4 genutzt, um aufzuzeigen, dass eine Analyse der lokalen Nicht-Negativitätseigenschaft über die reelle Punktemenge der eingebetteten affinen Varietät K_i gegenüber der projektiven Varietät V_i genügt (vgl. Korollar 3.4.5). Diese Erkenntnis ist zentral für weiterführende Untersuchungen.

In Kapitel 4 werden die projektiven Varietäten $V_0, \ldots, V_{k(n,d)-n}$ in den Nicht-Hilbert-Fällen einer tiefgreifenden Betrachtung unterzogen. Hierfür zeigt Unterkapitel 4.1 zunächst auf, dass es sich bei V_i um eine nicht-ausgeartete irreduzible total-reelle projektive Varietät der Kodimension i handelt. Ferner beweist Satz 4.1.37, dass die Varietät V_i Grad $i+1$ für $i = 0, \ldots, n$, bzw. $i = 0, \ldots, n+1$ falls $n = 2$, hat. Daraus folgt in Korollar 4.1.38, dass es sich bei $V_0, \ldots, V_n$, und auch bei V_{n+1} falls $n = 2$, um nicht-ausgeartete irreduzible total-reelle projektive Varietäten minimalen Grades handelt. Eine Anwendung der Verallgemeinerung von Hilberts Satz von 1888 nach Blekherman, Smith und Velasco (vgl. Satz 4.2.1 und Satz 4.2.2) führt in Satz 4.2.7 zu der Beobachtung, dass die konvexen Kegel $C_0, \ldots, C_n$, und auch der konvexe Kegel C_{n+1} falls $n = 2$, mit dem konvexen Kegel $\Sigma_{n+1,2d}$ übereinstimmen. Die zentrale Fragestellung dieser Doktorarbeit reduziert sich somit auf die Untersuchung der konvexen Kegelfiltrierung $C_{n+1} \subseteq \ldots \subseteq C_{k(n,d)-n} = \mathcal{P}_{n+1,2d}$ und auch der Inklusion $C_n \subseteq C_{n+1}$ falls $n \geq 3$.

In Kapitel 5 wird die zentrale Frage dieser Doktorarbeit sodann in den Nicht-Hilbert-Fällen homogener Polynome vom Grad 4 und 6 beantwortet, indem gezeigt wird, dass alle verbleibende Inklusionen strikt sind. Zu diesem Zweck werden zunächst die grundlegenden Nicht-Hilbert-Fälle in Unterkapitel 5.1 analysiert. Dies ermöglicht in Satz 5.1.2 und Satz 5.1.6 die Beantwortung der Forschungsfrage für die grundlegenden Nicht-Hilbert-Fälle. In Unterkapitel 5.2 werden aufbauend auf diesen Beobachtungen weiter in Satz 5.2.2 die Nicht-Hilbert-Fälle homogener Polynome vom Grad 4 gelöst. Im nächsten Schritt wird in Satz 5.2.7 ein erstes Grad-Sprung-Verfahren entwickelt, das eine Verallgemeinerung der Erkenntnisse dieses Kapitels für homogene Polynome vom Grad 6 erlaubt (vgl. Satz 5.2.14).

Das Ziel von Kapitel 6 ist es, die Ergebnisse aus Kapitel 5 auf beliebige Nicht-Hilbert-Fälle zu übertragen. Hierfür werden in Unterkapitel 6.1 zunächst Kreispolynome und ihre grundlegenden Eigenschaften eingeführt. Unterkapitel 6.2 beschäftigt sich weiterführend mit der Berechnung des eindeutigen $i(f) \in \{0, \ldots, k(n, d) - n - 1\}$ sodass $f \in C_{i(f)+1} \backslash C_{i(f)}$ für ein gegebenes PSD-extremales homogenes Kreispolynom f, das nicht SOS ist, gilt (vgl. Satz 6.2.12). Diese Formel wird in Unterkapitel 6.3 für die Entwicklung eines zweiten Grad-Sprung-Verfahrens in Satz 6.3.6 aufgegriffen, welches wiederum eine Antwort auf die zentrale Frage dieser Doktorarbeit für die verbleibenden Nicht-Hilbert-Fälle $(n+1, 2d)_{n\geq 2, d\geq 4}$ in Satz 6.3.8 ermöglicht. Insbesondere wird aufgezeigt, dass sämtliche verbleibenden konvexen Zwischenkegel von $\Sigma_{n+1,2d}$ und $\mathcal{P}_{n+1,2d}$ strikt separierend sind.

Kapitel 7 schließt die Untersuchungen dieser Doktorarbeit ab und widmet sich der Analyse der identifizierten strikt separierenden Zwischenkegel von $\Sigma_{n+1,2d}$ und $\mathcal{P}_{n+1,2d}$ auf geometrische und topologische Eigenschaften in den Nicht-Hilbert-Fällen. Insbesondere wird dabei die Abgeschlossenheit jener konvexer Kegel in Unterkapitel 7.1 belegt (vgl. Satz 7.1.1) und deren Inneres sowie deren Rand (vgl. Satz 7.1.3 und Korollar 7.1.6) bestimmt. Grundlegende topologische Kentnisse sind für das Verständnis dieses Unterkapitels hilfreich, jedoch kann bei Bedarf auch auf Anhang A.2 zurückgegriffen werden. In Unterkapitel 7.2 wird ferner ein Bezug zu der Vermutung von Helton–Nie über konvexe semialgebraische Mengen und Spektralschatten hergestellt. Unter Verwendung einer Methode nach Scheiderer wird insbesondere bewiesen, dass die in dieser Arbeit identifizierten strikt separierende Zwischenkegel von $\Sigma_{n+1,2d}$ und $\mathcal{P}_{n+1,2d}$ keine Spektralschatten sind und somit Gegenbeispiele zu der Vermutung von Helton–Nie darstellen. Für das Verständnis der in diesem Unterkapitel verwendeten Methoden wird die Leserin oder der Leser an Anhang A.4 verwiesen, in welchem grundlegende Konzepte und zentrale Resultate der reell-algebraischen Geometrie nachzulesen sind.

Kapitel 8 fasst die zentralen Ergebnisse dieser Arbeit zusammen und zeigt mögliche Ansatzpunkte für weiterführende Forschungen auf. In Unterkapitel 8.1 wird insbesondere eine Verfeinerung von Hilberts Satz von 1888 (vgl. Satz 8.1.1) präsentiert und eine Liste an Gegenbeispielen für die Vermutung von Helton–Nie über konvexe semialgebraische Mengen (vgl. Satz 8.1.2) etabliert. Ferner werden in Unterkapitel 8.2 bis Unterkapitel 8.4 mögliche, auf dieser Doktorarbeit aufbauende, weiterführende Forschungsrichtungen erläutert.

Appendix A

Appendix

In this appendix, we provide an overview on concepts and fundamental results from algebraic geometry, topology, convex geometry and real algebraic geometry that were used in this thesis.

A.1 Algebraic Geometry

An interested reader may consult [CLO15], [Har92], [Har77] and [Pla20] for a comprehensive overview on algebraic geometry. We here only give a brief introduction to this topic. To this end, we let **$\mathbb{K}$ denote a field** and **l, m be natural numbers** throughout this section. Moreover, we define the vectors of indeterminants $\boldsymbol{Y} := (Y_1, \ldots, Y_l)$, $Y := (Y_0, \ldots, Y_l)$ and denote the corresponding polynomial rings in the variables $\boldsymbol{Y}$ and Y with coefficients in $\mathbb{K}$ by $\mathbb{K}[\boldsymbol{Y}]$ and $\mathbb{K}[Y]$, respectively.

A.1.1 Affine Varieties

Definition A.1.1. An *affine variety* $\mathfrak{V} \subseteq \mathbb{K}^l$ is the set of common zeros of finitely many polynomials $g_1, \ldots, g_s \in \mathbb{K}[\boldsymbol{Y}]$ $(s \in \mathbb{N})$. That is,

$$\mathfrak{V} := \boldsymbol{\mathcal{V}}(g_1, \ldots, g_s) := \left\{ \boldsymbol{y} \in \mathbb{K}^l \mid g_i(\boldsymbol{y}) = 0 \text{ for } i = 1, \ldots, s \right\}.$$

Lemma A.1.2. *Arbitrary intersections of affine varieties are affine varieties. Moreover, finite unions of affine varieties are affine varieties.*

Proof. See [Pla20, 2.1.2 Proposition]. ■

Theorem A.1.3. Hilbert's Basis Theorem
For every ideal $I \subseteq \mathbb{K}[\boldsymbol{Y}]$, there exists some $s \in \mathbb{N}$ and $g_1, \ldots, g_s \in \mathbb{K}[\boldsymbol{Y}]$ such that

$$I = \langle g_1, \ldots, g_s \rangle.$$

Proof. See [CLO15, Chapter 2, §5, Theorem 4] for a modern interpretation of Hilbert's result [Hil90, Theorem I]. ■

Proposition A.1.4. *Let* $I = \langle g_1, \ldots, g_s \rangle \subseteq \mathbb{K}[\boldsymbol{Y}]$ $(s \in \mathbb{N})$ *be an ideal, then*

$$\boldsymbol{\mathcal{V}}(g_1, \ldots, g_s) = \left\{ \boldsymbol{y} \in \mathbb{K}^l \mid \forall g \in I \colon g(\boldsymbol{y}) = 0 \right\}.$$

Proof. See [CLO15, Chapter 2, §5, Proposition 9]. ∎

Notation A.1.5. For an ideal $I = \langle g_1, \ldots, g_s \rangle \subseteq \mathbb{K}[\boldsymbol{Y}]$ ($s \in \mathbb{N}$), we denote the affine variety that is induced by I as above by $\boldsymbol{\mathcal{V}}(I)$. That is, $\boldsymbol{\mathcal{V}}(I) := \boldsymbol{\mathcal{V}}(g_1, \ldots, g_s)$.

Definition A.1.6. The *vanishing ideal* of an affine variety $\mathfrak{V} \subseteq \mathbb{K}^l$ is given by

$$\boldsymbol{\mathcal{I}}(\mathfrak{V}) := \{ g \in \mathbb{K}[\boldsymbol{Y}] \mid \forall \boldsymbol{y} \in \mathfrak{V} \colon g(\boldsymbol{y}) = 0 \}.$$

Lemma A.1.7. *The vanishing ideal of an affine variety is an ideal.*

Proof. See [CLO15, Chapter 1, §4, Lemma 6]. ∎

Definition A.1.8. The *coordinate ring* of an affine variety $\mathfrak{V} \subseteq \mathbb{K}^l$ is given by

$$\mathbb{K}[\mathfrak{V}] := {}^{\mathbb{K}[\boldsymbol{Y}]}\big/_{\boldsymbol{\mathcal{I}}(\mathfrak{V})}.$$

Definition A.1.9. Two affine varieties $\mathfrak{V} \subseteq \mathbb{K}^l$ and $\mathfrak{W} \subseteq \mathbb{K}^m$ are *isomorphic* if there exist two polynomial maps $\psi \colon \mathfrak{V} \to \mathfrak{W}$ and $\varphi \colon \mathfrak{W} \to \mathfrak{V}$ such that $\varphi \circ \psi = \mathrm{id}_{\mathfrak{V}}$ and $\psi \circ \varphi = \mathrm{id}_{\mathfrak{W}}$.

Theorem A.1.10. *Two affine varieties* $\mathfrak{V} \subseteq \mathbb{K}^l$ *and* $\mathfrak{W} \subseteq \mathbb{K}^m$ *are isomorphic if and only if there exists a ring isomorphism between* $\mathbb{K}[\mathfrak{V}]$ *and* $\mathbb{K}[\mathfrak{W}]$ *that is the identity on the constant functions.*

Proof. See [CLO15, Chapter 5, §4, Theorem 9]. ∎

Definition A.1.11. An affine variety $\mathfrak{V} \subseteq \mathbb{K}^l$ is *reducible* if there exist affine varieties $\mathfrak{V}_1, \mathfrak{V}_2 \subsetneq \mathfrak{V}$ such that $\mathfrak{V}_1 \cup \mathfrak{V}_2 = \mathfrak{V}$. Otherwise, $\mathfrak{V}$ is called *irreducible*.

Proposition A.1.12. *For an affine variety* $\mathfrak{V} \subseteq \mathbb{K}^l$*, the following are equivalent:*

(i) $\mathfrak{V}$ *is irreducible.*

(ii) $\boldsymbol{\mathcal{I}}(\mathfrak{V})$ *is a prime ideal.*

(iii) $\mathbb{K}[\mathfrak{V}]$ *is an integral domain.*

Proof. See [CLO15, Chapter 5, §1, Proposition 4]. ∎

Theorem A.1.13. *Let* $\mathfrak{V} \subseteq \mathbb{K}^l$ *and* $\mathfrak{W} \subseteq \mathbb{K}^m$ *be two isomorphic affine varieties. If* $\mathfrak{V}$ *is irreducible, then* $\mathfrak{W}$ *is irreducible.*

Proof. Theorem A.1.10 implies that $\mathbb{K}[\mathfrak{V}]$ and $\mathbb{K}[\mathfrak{W}]$ are isomorphic as rings. Moreover, we know that $\mathbb{K}[\mathfrak{V}]$ is an integral domain by Proposition A.1.12 since $\mathfrak{V}$ is irreducible. Therefore, also $\mathbb{K}[\mathfrak{W}]$ is an integral domain and thus Proposition A.1.12 yields that $\mathfrak{W}$ is irreducible. ■

Definition A.1.14. The *dimension* of an affine variety $\mathfrak{V} \subseteq \mathbb{K}^l$ is the greatest integer $\delta \geq 0$ for which there exist irreducible affine varieties $\mathfrak{V}_0, \dots, \mathfrak{V}_\delta \subseteq \mathfrak{V}$ such that

$$\emptyset \subsetneq \mathfrak{V}_0 \subsetneq \dots \subsetneq \mathfrak{V}_\delta \subseteq \mathfrak{V}.$$

Notation A.1.15. We denote the dimension of an affine variety $\mathfrak{V} \subseteq \mathbb{K}^l$ by $\dim(\mathfrak{V})$.

Theorem A.1.16. *If $\mathfrak{V} \subseteq \mathbb{K}^l$ and $\mathfrak{W} \subseteq \mathbb{K}^m$ are two isomorphic affine varieties, then* $\dim(\mathfrak{V}) = \dim(\mathfrak{W})$.

Proof. See [CLO15, Chapter 9, §5, Corollary 3]. ■

A.1.2 Projective Varieties

Notation A.1.17. For $y, y' \in \mathbb{K}^{l+1}$, we set

$$y \sim y' \text{ if and only if there exists some } \lambda \in \mathbb{K}^\times \text{such that } y = \lambda y'.$$

This defines an equivalence relation on $\mathbb{K}^{l+1}$.

Definition A.1.18. Let 0 denote the origin of $\mathbb{K}^{l+1}$. The *l-dimensional projective space* over $\mathbb{K}$ is given by

$$\mathbb{P}^l_{\mathbb{K}} := \mathbb{K}^{l+1} \backslash \{0\} \big/ \sim.$$

The equivalence class of $y := (y_0, \dots, y_l) \in \mathbb{K}^{l+1}$ in $\mathbb{P}^l_{\mathbb{K}}$ is denoted by $[y] := [y_0 : \dots : y_l]$ and called the *homogeneous coordinate* of y.

We recall that a polynomial $f \in \mathbb{K}[Y]$ is called a *form* or *homogeneous* if all monomials in f have the same degree.

Proposition A.1.19. *Let $f \in \mathbb{K}[Y]$ be a form and $y \in \mathbb{K}^{l+1}$. If $f(y) = 0$, then $f(\lambda y) = 0$ for all $\lambda \in \mathbb{K}^\times$.*

Proof. See [CLO15, Chapter 8, §2, Proposition 4]. ■

Definition A.1.20. A *projective variety* $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ is the set of homogeneous coordinates of common zeros of finitely many forms $f_1, \dots, f_s \in \mathbb{K}[Y]$ $(s \in \mathbb{N})$. That is,

$$\mathfrak{V} := \mathcal{V}(f_1, \dots, f_s) := \left\{ [y] \in \mathbb{P}^l_{\mathbb{K}} \mid f_i(y) = 0 \text{ for } i = 1, \dots, s \right\}.$$

Lemma A.1.21. *Arbitrary intersections of projective varieties are projective varieties. Moreover, finite unions of projective varieties are projective varieties.*

Proof. See [Pla20, pp. 72-73]. ■

We recall that collecting all terms of the same degree of a given polynomial g in a form gives us a unique decomposition of g into its *homogeneous components*.

Definition A.1.22. An ideal $I \subseteq \mathbb{K}[Y]$ is *homogeneous* if each homogeneous component of each $g \in I$ lies in I.

Theorem A.1.23. *An ideal $I \subseteq \mathbb{K}[Y]$ is homogeneous if and only if there exists some $s \in \mathbb{N}$ and forms $f_1, \dots, f_s \in \mathbb{K}[Y]$ such that $I = \langle f_1, \dots, f_s \rangle$.*

Proof. See [CLO15, Chapter 8, §3, Theorem 2]. ■

Proposition A.1.24. *Let $I = \langle f_1, \dots, f_s \rangle \subseteq \mathbb{K}[Y]$ be a homogeneous ideal with forms $f_1, \dots, f_s \in \mathbb{K}[Y]$ $(s \in \mathbb{N})$, then*

$$\mathcal{V}(f_1, \dots, f_s) = \left\{ [y] \in \mathbb{P}^l_{\mathbb{K}} \mid \forall f \in I \colon f(y) = 0 \right\}.$$

Proof. See [CLO15, Chapter 8, §3, Proposition 3]. ■

Notation A.1.25. For a homogeneous ideal $I = \langle f_1, \dots, f_s \rangle \subseteq \mathbb{K}[Y]$ with forms $f_1, \dots, f_s \in \mathbb{K}[Y]$ $(s \in \mathbb{N})$, we denote the projective variety that is induced by I as above by $\mathcal{V}(I)$. That is, $\mathcal{V}(I) := \mathcal{V}(f_1, \dots, f_s)$.

Definition A.1.26. For $l \geq 2$, let $\mathfrak{V} \subseteq \mathbb{P}^{l-1}_{\mathbb{K}}$ be a projective variety over an algebraically closed field $\mathbb{K}$ and ψ an embedding of $\mathbb{P}^{l-1}_{\mathbb{K}}$ in $\mathbb{P}^l_{\mathbb{K}}$. The *cone over $\mathfrak{V}$ with vertex* $[y] \in \mathbb{P}^l_{\mathbb{K}} \backslash \psi(\mathfrak{V})$ is the union of lines joining $[y]$ to points of $\psi(\mathfrak{V})$ in $\mathbb{P}^l_{\mathbb{K}}$.

Lemma A.1.27. *For $l \geq 2$, let $\mathfrak{V} = \mathcal{V}(f_1, \dots, f_s) \subseteq \mathbb{P}^{l-1}_{\mathbb{K}}$ be a projective variety over an algebraically closed field $\mathbb{K}$ for some forms $f_1, \dots, f_s \in \mathbb{K}[Y_0, \dots, Y_{l-1}]$ $(s \in \mathbb{N})$ and ψ the embedding $\psi \colon \mathbb{P}^{l-1}_{\mathbb{K}} \to \mathbb{P}^l_{\mathbb{K}}, [w_0 : \dots : w_{l-1}] \mapsto [w_0 : \dots : w_{l-1} : 1]$. The cone over $\mathfrak{V}$ with vertex $[0 : \dots : 0 : 1] \in \mathbb{P}^l_{\mathbb{K}}$ is given by $\mathcal{V}(f_1, \dots, f_s) \subseteq \mathbb{P}^l_{\mathbb{K}}$, where $f_1, \dots, f_s$ are interpreted as forms in $\mathbb{K}[Y_0, \dots, Y_l]$.*

Proof. See [CLO15, pp. 32]. ■

Corollary A.1.28. *For $l \geq 2$, let $\mathfrak{V} \subseteq \mathbb{P}^{l-1}_{\mathbb{K}}$ be a projective variety over an algebraically closed field $\mathbb{K}$ and ψ an embedding of $\mathbb{P}^{l-1}_{\mathbb{K}}$ in $\mathbb{P}^l_{\mathbb{K}}$. The cone over $\mathfrak{V}$ with vertex $[y] \in \mathbb{P}^l_{\mathbb{K}} \backslash \psi(\mathfrak{V})$ is a projective variety in $\mathbb{P}^l_{\mathbb{K}}$.*

Proof. See [CLO15, pp. 32]. ■

Definition A.1.29. The *vanishing ideal* of a projective variety $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ is given by

$$\mathcal{I}(\mathfrak{V}) := \langle \{ f \in \mathbb{K}[Y] \mid f \text{ is a form and } f(y) = 0 \text{ for all } [y] \in \mathfrak{V} \} \rangle.$$

Lemma A.1.30. *The vanishing ideal of a projective variety over an infinite field $\mathbb{K}$ is a homogeneous ideal.*

Proof. See [CLO15, Chapter 8, §3, Proposition 4]. ∎

Theorem A.1.31. *Let $\mathfrak{V}, \mathfrak{W} \subseteq \mathbb{P}^l_{\mathbb{K}}$ be two projective varieties over an infinite field $\mathbb{K}$. Then $\mathfrak{V} \subseteq \mathfrak{W}$ if and only if $\mathcal{I}(\mathfrak{W}) \subseteq \mathcal{I}(\mathfrak{V})$.*

Proof. See [CLO15, Chapter 8, §3, Theorem 5]. ∎

We recall that the *radical* of an ideal I in a commutative ring R is given by

$$\sqrt{I} := \{ r \in R \mid \exists m \in \mathbb{N} \colon r^m \in I \}.$$

Theorem A.1.32. Hilbert's Projective Nullstellensatz
Let $I \subseteq \mathbb{K}[Y]$ be a homogeneous ideal over an algebraically closed field $\mathbb{K}$. If $\mathcal{V}(I) \subseteq \mathbb{P}^l_{\mathbb{K}}$ is non-empty, then $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$.

Proof. See [CLO15, Chapter 8, §3, Theorem 9]) for a modern interpretation of Hilbert's result [Hil00, II, §3]. ∎

Definition A.1.33. The *homogeneous coordinate ring* of a projective variety $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ is given by

$$\mathbb{K}[\mathfrak{V}] := {}^{\mathbb{K}[Y]}\big/_{\mathcal{I}(\mathfrak{V})}.$$

Definition A.1.34. A projective variety $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ is *reducible* if there exist projective varieties $\mathfrak{V}_1, \mathfrak{V}_2 \subsetneq \mathfrak{V}$ such that $\mathfrak{V}_1 \cup \mathfrak{V}_2 = \mathfrak{V}$. Otherwise, $\mathfrak{V}$ is called *irreducible*.

Proposition A.1.35. *For a projective variety $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ over an algebraically closed field $\mathbb{K}$, the following are equivalent:*

(i) *$\mathfrak{V}$ is irreducible.*

(ii) *$\mathcal{I}(\mathfrak{V})$ is a prime ideal.*

(iii) *$\mathbb{K}[\mathfrak{V}]$ is an integral domain.*

Proof. See [Pla20, 3.2.5 Proposition]. ∎

Corollary A.1.36. *Let $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ and $\mathfrak{W} \subseteq \mathbb{P}^m_{\mathbb{K}}$ be two projective varieties over an algebraically closed field $\mathbb{K}$ such that $\mathbb{K}[\mathfrak{V}]$ and $\mathbb{K}[\mathfrak{W}]$ are isomorphic as rings. If $\mathfrak{V}$ is irreducible, then also $\mathfrak{W}$ is irreducible.*

Proof. Proposition A.1.35 yields that $\mathbb{K}[\mathfrak{V}]$ is an integral domain. Hence, also the to $\mathbb{K}[\mathfrak{V}]$ isomorphic ring $\mathbb{K}[\mathfrak{W}]$ is an integral domain. Therefore, $\mathfrak{W}$ is irreducible by Proposition A.1.35. ∎

Proposition A.1.37. *Let $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ be a projective variety, then there exists a unique $s \in \mathbb{N}$ and unique irreducible projective varieties $\mathfrak{V}_1, \ldots, \mathfrak{V}_s \subseteq \mathfrak{V}$ such that $\mathfrak{V}_i \not\subseteq \mathfrak{V}_j$ for $i \neq j$ and $\mathfrak{V} = \bigcup_{i=1}^{s} \mathfrak{V}_i$.*

Proof. See [Har92, Theorem 5.7]. ∎

Definition A.1.38. Let $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ be a projective variety and $\mathfrak{V}_1, \ldots, \mathfrak{V}_s$ ($s \in \mathbb{N}$) the unique irreducible projective subvarieties of $\mathfrak{V}$ such that $\mathfrak{V}_i \not\subseteq \mathfrak{V}_j$ for $i \neq j$ and $\mathfrak{V} = \bigcup_{i=1}^{s} \mathfrak{V}_i$. The *irreducible decomposition* of $\mathfrak{V}$ is $\bigcup_{i=1}^{s} \mathfrak{V}_i$ and $\mathfrak{V}_1, \ldots, \mathfrak{V}_s$ are called the *irreducible components* of $\mathfrak{V}$.

Definition A.1.39.
(i) Let $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ be an irreducible projective variety. The *dimension* of $\mathfrak{V}$ is the greatest integer $\delta \geq 0$ for which there exist irreducible projective subvarieties $\mathfrak{V}_0, \ldots, \mathfrak{V}_\delta \subseteq \mathfrak{V}$ such that $\emptyset \subsetneq \mathfrak{V}_0 \subsetneq \ldots \subsetneq \mathfrak{V}_\delta = \mathfrak{V}$.

(ii) The *dimension* of a (not necessarily irreducible) projective variety $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ is the maximum of the dimensions of the irreducible components of $\mathfrak{V}$.

Notation A.1.40. We denote dimension of a projective variety $\mathfrak{V}$ by $\dim(\mathfrak{V})$.

Proposition A.1.41. *Let $\mathfrak{V}, \mathfrak{W} \subseteq \mathbb{P}^l_{\mathbb{K}}$ be two projective varieties such that $\mathfrak{W}$ is irreducible and $\mathfrak{V} \subseteq \mathfrak{W}$. If $\dim(\mathfrak{V}) = \dim(\mathfrak{W})$, then $\mathfrak{V} = \mathfrak{W}$.*

Proof. Contraposition of [CLO15, Chapter 9, §4, Proposition 10 (ii)]. ∎

Theorem A.1.42. *Let $\mathfrak{V}, \mathfrak{W} \subseteq \mathbb{P}^l_{\mathbb{K}}$ be two projective varieties over an algebraically closed field $\mathbb{K}$ such that $\mathfrak{V} \subseteq \mathfrak{W}$ and $\mathfrak{V}$ is irreducible. If $\dim(\mathfrak{V}) = \dim(\mathfrak{W})$, then $\mathfrak{V}$ is an irreducible component of $\mathfrak{W}$.*

Proof. We observe that $\mathfrak{V}$ is a subvariety of some irreducible component $\mathfrak{W}'$ of $\mathfrak{W}$ since $\mathfrak{V}$ is assumed to be irreducible. Hence, the dimension of $\mathfrak{W}'$ is at least as great as the dimension of $\mathfrak{V}$ but not greater than the dimension of $\mathfrak{W}$. Since $\dim(\mathfrak{V}) = \dim(\mathfrak{W})$, it thus follows $\dim(\mathfrak{V}) = \dim(\mathfrak{W}')$ and we conclude $\mathfrak{V} = \mathfrak{W}'$ by Proposition A.1.41. ∎

Proposition A.1.43. *Let $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ be an irreducible projective variety over an algebraically closed field $\mathbb{K}$ and $f \in \mathbb{K}[Y]$ a form. If $\mathfrak{V} \not\subseteq \mathcal{V}(f)$ and $\dim(\mathfrak{V}) > 0$, then* $\dim(\mathfrak{V} \cap \mathcal{V}(f)) = \dim(\mathfrak{V}) - 1$.

Proof. See [CLO15, Chapter 9, §4, Proposition 10 (i)]. ∎

Definition A.1.44. For a projective variety $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ over an algebraically closed field $\mathbb{K}$ with graded homogeneous coordinate ring $\mathbb{K}[\mathfrak{V}] = \bigoplus_{t \geq 0} \mathbb{K}[W]_t$, the *Hilbert function* of $\mathfrak{V}$ is given by

$$\begin{aligned} h_{\mathfrak{V}} \colon \mathbb{N}_0 &\to \mathbb{N}_0 \\ t &\mapsto \dim(\mathbb{K}[\mathfrak{V}]_t). \end{aligned}$$

Proposition A.1.45. *Let $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ be a projective variety over an algebraically closed field $\mathbb{K}$ with Hilbert function $h_{\mathfrak{V}}$, then there exists a unique polynomial $p_{\mathfrak{V}}$ such that $h_{\mathfrak{V}}(t) = p_{\mathfrak{V}}(t)$ for all sufficiently large $t \in \mathbb{N}_0$.*

Proof. See [Har92, Proposition 13.2.]. ■

Definition A.1.46. Let $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ be a projective variety over an algebraically closed field $\mathbb{K}$ with Hilbert function $h_{\mathfrak{V}}$. The unique polynomial $p_{\mathfrak{V}}$ such that $h_{\mathfrak{V}}(t) = p_{\mathfrak{V}}(t)$ for all sufficiently large $t \in \mathbb{N}_0$ is called the *Hilbert polynomial* of $\mathfrak{V}$.

Definition A.1.47. Let $\mathbb{K}$ be an algebraically closed field.

(i) Let $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ be an irreducible projective variety with Hilbert polynomial $p_{\mathfrak{V}}$. The *degree* of $\mathfrak{V}$ is $\dim(\mathfrak{V})!$ times the leading coefficient of $p_{\mathfrak{V}}$.

(ii) The *degree* of a (not necessarily irreducible) projective variety $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ is the sum of the degrees of the irreducible components of $\mathfrak{V}$ with dimension $\dim(\mathfrak{V})$.

Notation A.1.48. We denote the degree of a projective variety $\mathfrak{V} \subseteq \mathbb{P}^l_{\mathbb{K}}$ by $\deg(\mathfrak{V})$.

Theorem A.1.49. Bézout's Theorem
Let $\mathfrak{V}, \mathfrak{W} \subseteq \mathbb{P}^l_{\mathbb{K}}$ be two irreducible projective varieties over an algebraically closed field $\mathbb{K}$ such that $\dim(\mathfrak{V} \cap \mathfrak{W}) = \dim(\mathfrak{V}) + \dim(\mathfrak{W}) - l$. Moreover, let $\mathfrak{O}_1, \ldots, \mathfrak{O}_s$ $(s \in \mathbb{N})$ be the irreducible components of $\mathfrak{V} \cap \mathfrak{W}$, then there exist $c_1, \ldots, c_s \in \mathbb{N}$ such that

$$\deg(\mathfrak{V}) \cdot \deg(\mathfrak{W}) = \sum_{i=1}^{s} c_i \deg(\mathfrak{O}_i).$$

Proof. See [Har92, Theorem 18.4.] for a modern take on Bézout's results [Bé06]. ■

Proposition A.1.50. *For $l \geq 2$, let $\mathfrak{V} \subseteq \mathbb{P}^{l-1}_{\mathbb{K}}$ be a projective variety over an algebraically closed field $\mathbb{K}$ and ψ an embedding of $\mathbb{P}^{l-1}_{\mathbb{K}}$ in $\mathbb{P}^l_{\mathbb{K}}$, then the degree of $\mathfrak{V}$ coincides with the degree of the cone over $\mathfrak{V}$ with vertex $[y] \in \mathbb{P}^l_{\mathbb{K}} \backslash \psi(\mathfrak{V})$.*

Proof. See [Har92, Example 18.16.]. ■

A.1.3 A Relation between Affine and Projective Varieties

Proposition A.1.51. *The map*

$$\begin{aligned} \phi\colon \mathbb{K}^l &\to \mathbb{P}^l_{\mathbb{K}} \\ \boldsymbol{y} &\mapsto [1:\boldsymbol{y}] \end{aligned}$$

embeds $\mathbb{K}^l$ *in the* l*-dimensional projective space* $\mathbb{P}^l_{\mathbb{K}}$ *over* $\mathbb{K}$ *and*

$$\phi(\mathbb{K}^l) = \left\{ [y] \in \mathbb{P}^l_{\mathbb{K}} \mid y_0 \neq 0 \right\}.$$

Proof. See [CLO15, Chapter, §2, Proposition 2]. ■

Definition A.1.52. The *projective closure* $\overline{\mathfrak{V}}$ of an affine variety $\mathfrak{V} \subseteq \mathbb{K}^l$ is the smallest (w.r.t. $\subseteq$) projective variety in $\mathbb{P}^l_{\mathbb{K}}$ that contains $\phi(\mathfrak{V})$.

Proposition A.1.53. *Let* $\mathfrak{V} \subseteq \mathbb{K}^l$ *be an affine variety with projective closure* $\overline{\mathfrak{V}} \subseteq \mathbb{P}^l_{\mathbb{K}}$*, then* $\overline{\mathfrak{V}} \cap \phi(\mathbb{K}^l) = \phi(\mathfrak{V})$*.*

Proof. See [CLO15, Chapter 8, §4, Proposition 7 (i)]. ■

Theorem A.1.54. *Let* $\mathfrak{V} \subseteq \mathbb{K}^l$ *be an affine variety. If* $\mathfrak{V}$ *is irreducible, then also the projective closure* $\overline{\mathfrak{V}} \subseteq \mathbb{P}^l_{\mathbb{K}}$ *of* $\mathfrak{V}$ *is irreducible.*

Proof. See [CLO15, Chapter 8, §4, Proposition 7 (iii)]. ■

Theorem A.1.55. *Let* $\mathfrak{V} \subseteq \mathbb{K}^l$ *be a non-empty affine variety with projective closure* $\overline{\mathfrak{V}} \subseteq \mathbb{P}^l_{\mathbb{K}}$*, then* $\dim(\mathfrak{V}) = \dim(\overline{\mathfrak{V}})$*.*

Proof. See [CLO15, Chapter 9, §3, Theorem 12 (ii)]. ■

We recall that any polynomial $g \in \mathbb{K}[\boldsymbol{Y}]$ can be *homogenized* into a form $g^h \in \mathbb{K}[Y]$ of the same degree by multiplying each term of g with an appropriate power of Y_0. The *homogenization* of an ideal $I \subseteq \mathbb{K}[\boldsymbol{Y}]$ is thus given by $I^h := \langle g^h \mid g \in I \rangle \subseteq \mathbb{K}[Y]$.

Lemma A.1.56. *The homogenization of an ideal* $I \subseteq \mathbb{K}[\boldsymbol{Y}]$ *is a homogenous ideal.*

Proof. See [CLO15, Chapter 8, §4, Proposition 2]. ■

Proposition A.1.57. *The projective closure of an affine variety* $\mathfrak{V} \subseteq \mathbb{K}^l$ *coincides with* $\mathcal{V}(\boldsymbol{I}(\mathfrak{V})^h)$*.*

Proof. See [CLO15, Chapter 8, §4, Proposition 7 (ii)]. ■

Theorem A.1.58. *The projective closure of an affine variety* $\mathfrak{V} = \boldsymbol{\mathcal{V}}(I) \subseteq \mathbb{K}^l$ *with ideal* $I \subseteq \mathbb{K}[\boldsymbol{Y}]$ *over an algebraically closed field* $\mathbb{K}$ *coincides with* $\mathcal{V}(I^h)$*.*

Proof. See [CLO15, Chapter 8, §4, Theorem 8]. ∎

The rest of this section is dedicated to a method that allows us to compute the homogenization of an a priori fixed ideal.

Definition A.1.59. A *monomial order* $\leq$ on $\mathbb{K}[\boldsymbol{Y}]$ is a relation on $\mathbb{N}_0^l$ such that

(i) the relation $\leq$ is a well-order on $\mathbb{N}_0^l$ and[1]

(ii) if $\alpha < \beta$ for $\alpha, \beta \in \mathbb{N}_0^l$, then $\alpha + \gamma < \beta + \gamma$ for $\gamma \in \mathbb{N}_0^l$.

Notation A.1.60. For a monomial order $\leq$ on $\mathbb{K}[\boldsymbol{Y}]$ and $\alpha, \beta \in \mathbb{N}_0^l$, we set $\boldsymbol{Y}^\alpha \leq \boldsymbol{Y}^\beta$ if $\alpha \leq \beta$. Moreover, we denote the leading term of $g \in \mathbb{K}[\boldsymbol{Y}]$ by $\mathrm{LT}(g)$.

Definition A.1.61. The *lexicographic order* $\leq_{\text{lex}}$ on $\mathbb{K}[\boldsymbol{Y}]$ is defined for $\alpha, \beta \in \mathbb{N}_0^l$ by setting $\alpha <_{\text{lex}} \beta$ if the first non-zero entry of $\beta - \alpha \in \mathbb{N}_0^l$ is positive.

Proposition A.1.62. *The lexicographic order is a monomial order.*

Proof. See [CLO15, Chapter 2, §2, Proposition 4]. ∎

We recall that an ideal I in a commutative ring R with identity element 0 for addition is called *trivial* if $I = \{0\}$. In all other cases, I is called *non-trivial*.

Definition A.1.63. Let $\leq$ be a monomial order on $\mathbb{K}[\boldsymbol{Y}]$ and $G \subseteq \mathbb{K}[\boldsymbol{Y}]$ a finite subset of a non-trivial ideal $I \subseteq \mathbb{K}[\boldsymbol{Y}]$. If $\langle \mathrm{LT}(g) \mid g \in G\rangle = \langle \mathrm{LT}(g) \mid \text{non-zero } g \in I\rangle$, then G is called a *Gröbner basis* of I w.r.t. $\leq$.

Theorem A.1.64. *Let* $\leq$ *be a monomial order on* $\mathbb{K}[\boldsymbol{Y}]$ *and* $I \subseteq \mathbb{K}[\boldsymbol{Y}]$ *a non-trivial ideal, then there exists a Gröbner basis of* I *w.r.t.* $\leq$ *and any such generates* I*.*

Proof. See [CLO15, Chapter 2, §5, Corollary 6]. ∎

Theorem A.1.65. A Division Algorithm
Let $\leq$ *be a monomial order on* $\mathbb{K}[\boldsymbol{Y}]$ *and* $(g_1, \ldots, g_s)$ *(*$s \in \mathbb{N}$*) an* s*-tuple of polynomials in* $\mathbb{K}[\boldsymbol{Y}]$*, then, for any* $g \in \mathbb{K}[\boldsymbol{Y}]$*, there exist* $q_1, \ldots, q_s, r \in \mathbb{K}[\boldsymbol{Y}]$ *such that*

$$g = q_1 g_1 + \ldots + q_s g_s + r$$

and r *is the zero polynomial or no leading term of any* g_i *divides any monomial of* r*.*

Proof. See the proof of [CLO15, Chapter 2, §3, Theorem 3] for an algorithm which determines appropriate $q_1, \ldots, q_s$ and r. ∎

[1] That is, $\leq$ is a total order such that every non-empty set has a least element.

Proposition A.1.66. *Let $\leq$ be a monomial order on $\mathbb{K}[\boldsymbol{Y}]$ and $\{g_1, \ldots, g_s\} \subseteq \mathbb{K}[\boldsymbol{Y}]$ ($s \in \mathbb{N}$) a Gröbner basis of a non-trivial ideal $I \subseteq \mathbb{K}[\boldsymbol{Y}]$ w.r.t. $\leq$, then, for any $g \in \mathbb{K}[\boldsymbol{Y}]$, there exists a unique $r \in \mathbb{K}[\boldsymbol{Y}]$ such that r is the zero polynomial or no leading term of $g_1, \ldots, g_s$ divides any monomial of r and $g = q_1 g_1 + \ldots + q_s g_s + r$ for some $q_1, \ldots, q_s \in \mathbb{K}[\boldsymbol{Y}]$.*

Proof. See [CLO15, Chapter 2, §6, Proposition 1]. ∎

Definition A.1.67. Let $\leq$ be a monomial order on $\mathbb{K}[\boldsymbol{Y}]$, $G := \{g_1, \ldots, g_s\} \subseteq \mathbb{K}[\boldsymbol{Y}]$ ($s \in \mathbb{N}$) a Gröbner basis of a non-trivial ideal $I \subseteq \mathbb{K}[\boldsymbol{Y}]$ w.r.t. $\leq$ and $g \in \mathbb{K}[\boldsymbol{Y}]$. The unique $r \in \mathbb{K}[\boldsymbol{Y}]$ such that r is the zero polynomial or no leading term of $g_1, \ldots, g_s$ divides any monomial of r and $g = q_1 g_1 + \ldots + q_s g_s + r$ for some $q_1, \ldots, q_s \in \mathbb{K}[\boldsymbol{Y}]$ is called the *remainder* on division of g by G w.r.t. $\leq$.

Notation A.1.68. We denote the remainder on division of g by G w.r.t. $\leq$ by $\overline{g}^{G,\leq}$.

Definition A.1.69. Let $\leq$ be a monomial order on $\mathbb{K}[\boldsymbol{Y}]$ and $\mathrm{LCM}(g, h)$ the least common multiple of the leading terms $\mathrm{LT}(g)$, $\mathrm{LT}(h)$ of non-zero $g, h \in \mathbb{K}[\boldsymbol{Y}]$. The *$S$-polynomial* of g and h is given by

$$S(g, h) := \frac{\mathrm{LCM}(g, h)}{\mathrm{LT}(g)} g - \frac{\mathrm{LCM}(g, h)}{\mathrm{LT}(h)} h \in \mathbb{K}[\boldsymbol{Y}].$$

Theorem A.1.70. Buchberger's Criterion
Let $\leq$ be a monomial order on $\mathbb{K}[\boldsymbol{Y}]$ and $I \subseteq \mathbb{K}[\boldsymbol{Y}]$ a non-trivial ideal that is generated by a finite set $G \subseteq \mathbb{K}[\boldsymbol{Y}]$, then G is a Gröbner basis of I w.r.t. $\leq$ if and only if $\overline{S(g, h)}^{G,\leq}$ is zero for all distinct $g, h \in G$.

Proof. See [CLO15, Chapter 2, §6, Theorem 6] for an interpretation of Buchberger's result [Buc95, Theorem 6.2]. ∎

Algorithm A.1.71. Buchberger's Algorithm
Input: A non-trivial ideal $I = \langle g_1, \ldots, g_s \rangle \subseteq \mathbb{K}[\boldsymbol{Y}]$ and a monomial order $\leq$ on $\mathbb{K}[\boldsymbol{Y}]$.

(1) Set $G := (g_1, \ldots, g_s)$.

(2) Set $G' := \emptyset$.

(3) While $G \neq G'$,

  (a) set $G' := G$ and

  (b) for each pair of distinct $g, h \in G'$ compute $r := \overline{S(g, h)}^{G,\leq}$.

  (c) If r is not the zero polynomial, then set $G := G \cup \{r\}$.

Output: A Gröbner basis G of I w.r.t. $\leq$.

Proof. See [CLO15, Chapter 2 §7 Theorem 2] for a variation of Buchberger's original consideration [Buc95, Algorithm 6.2]. ■

Remark A.1.72. *Buchberger's algorithm terminates in finite time.*

Definition A.1.73. Let $\leq$ be a monomial order on $\mathbb{K}[\boldsymbol{Y}]$ and $G \subseteq \mathbb{K}[\boldsymbol{Y}]$ a Gröbner basis of a non-trivial ideal $I \subseteq \mathbb{K}[\boldsymbol{Y}]$ w.r.t. $\leq$. If for every $h \in G$, the coefficient of the leading monomial of h is one and if no monomial of h lies in $\langle \mathrm{LT}(g) \mid g \in G, g \neq h\rangle$, then G is called a *reduced* Gröbner basis of I.

Theorem A.1.74. Buchberger's Algorithm for Reduced Gröbner Bases *Let $\leq$ be a monomial order on $\mathbb{K}[\boldsymbol{Y}]$ and $I \subseteq \mathbb{K}[\boldsymbol{Y}]$ a non-trivial ideal, then there exists a unique reduced Gröbner basis of I w.r.t. $\leq$. In particular, the reduced Gröbner basis of I w.r.t. $\leq$ can be algorithmically determined.*

Proof. See [CLO15, Chapter 2, §7, Theorem 5] and [Buc95, Algorithm 6.3] for an algorithm which determines the reduced Gröbner basis of I w.r.t. $\leq$. ■

Remark A.1.75. *Buchberger's algorithm for reduced Gröbner bases terminates in finite time.*

Definition A.1.76. A *graded* monomial order $\leq_{\mathrm{gr}}$ on $\mathbb{K}[\boldsymbol{Y}]$ is a monomial order on $\mathbb{K}[\boldsymbol{Y}]$ such that $\alpha < \beta$ for all $\alpha, \beta \in \mathbb{N}_0^l$ with $|\alpha| < |\beta|$.

Definition A.1.77. The *graded lexicographic order* $\leq_{\mathrm{grlex}}$ on $\mathbb{K}[\boldsymbol{Y}]$ is defined for $\alpha, \beta \in \mathbb{N}_0^l$ by setting $\alpha <_{\mathrm{grlex}} \beta$ if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and $\alpha <_{\mathrm{lex}} \beta$.

Corollary A.1.78. *The graded lexicographic order is a graded monomial order.*

Proof. The assertion follows from Proposition A.1.62. See also [CLO15, pp. 416]. ■

We recall that a non-empty finite set $G \subseteq \mathbb{K}[\boldsymbol{Y}]$ can be *homogenized* into a set of forms by setting $G^h := \{g^h \mid g \in G\} \subseteq \mathbb{K}[Y]$. Moreover, we observe that a graded monomial order $\leq_{\mathrm{gr}}$ on $\mathbb{K}[\boldsymbol{Y}]$ can be extended to a monomial order $\leq_{\mathrm{grh}}$ on $\mathbb{K}[Y]$ by setting $(c, \alpha) <_{\mathrm{grh}} (d, \beta)$ if $\alpha < \beta$ or $\alpha = \beta$ and $c < d$ for $\alpha, \beta \in \mathbb{N}_0^l$ and $c, d \in \mathbb{N}_0$.

Theorem A.1.79. *Let $I \subseteq \mathbb{K}[\boldsymbol{Y}]$ be a non-trivial ideal and G a Gröbner basis of I w.r.t. a graded monomial order $\leq_{gr}$ on $\mathbb{K}[\boldsymbol{Y}]$, then G^h is a Gröbner basis of the non-trivial ideal $I^h \subseteq \mathbb{K}[Y]$ w.r.t. the extended monomial order $\leq_{grh}$ on $\mathbb{K}[Y]$.*

Proof. See [CLO15, Chapter 8, §4, Theorem 4]. ■

Theorem A.1.80. *Let $\mathfrak{V} = \boldsymbol{\mathcal{V}}(I) \subseteq \mathbb{K}^l$ be an affine variety with non-trivial ideal $I \subseteq \mathbb{K}[\boldsymbol{Y}]$ over an algebraically closed field $\mathbb{K}$. Moreover, let $G \subseteq \mathbb{K}[\boldsymbol{Y}]$ be a Gröbner basis of I w.r.t. a graded monomial order on $\mathbb{K}[\boldsymbol{Y}]$, then the projective closure of $\mathfrak{V}$ in $\mathbb{P}_{\mathbb{K}}^l$ coincides with $\mathcal{V}(G^h)$.*

Proof. Theorem A.1.58 yields $\overline{\mathfrak{V}} = \mathcal{V}(I^h)$ and $I^h = \langle G^h \rangle$ by Theorem A.1.79 since G^h is a Gröbner basis of the non-trivial $I^h \subseteq \mathbb{K}[Y_0, \dots, Y_l]$ w.r.t. the extended monomial order on $\mathbb{K}[Y]$. Hence, $\overline{\mathfrak{V}} = \mathcal{V}(I^h) = \mathcal{V}(\langle G^h \rangle) = \mathcal{V}(G^h)$. ■

A.2 Topology

An interested reader may consult [Ovc18], [Par22] and [Voi20] for a comprehensive overview on topology. We here only give a brief introduction. Throughout this section, we let **$\mathbb{K}$ denote either the field of the real or the complex numbers**.

Definition A.2.1. Let $\mathcal{X}$ be a non-empty set and τ a family of subsets of $\mathcal{X}$. If

(i) $\emptyset, \mathcal{X} \in \tau$,

(ii) τ is closed under arbitrary unions and

(iii) τ is closed under finite intersections,

then τ is a *topology* on $\mathcal{X}$ and $(\mathcal{X}, \tau)$ is a *topological space*. Moreover, any $U \in \tau$ is called *open* and any $W \subseteq \mathcal{X}$ with open complement $W^c := \mathcal{X} \backslash W \in \tau$ is called *closed*.

Definition A.2.2. Let $(\mathcal{X}, \tau)$ be a topological space and $A \subseteq \mathcal{X}$. The *interior* of A is the open set that is given by the union of all $U \in \tau$ that are contained in A. Moreover, the *closure* of A is the closed set that is given by the intersection of all closed sets $W \subseteq \mathcal{X}$ that contain A. The *boundary* of A is the intersection of the closure of A with the closure of the complement $A^c := \mathcal{X} \backslash A$.

Notation A.2.3. Let $(\mathcal{X}, \tau)$ be a topological space and $A \subseteq \mathcal{X}$. We denote the interior of A by $\mathring{A}$, the closure of A by $\overline{A}$ and the boundary of A by ∂A.

Lemma A.2.4. *Let $(\mathcal{X}, \tau)$ be a topological space and $A \subseteq \mathcal{X}$, then the interior $\mathring{A}$ is the greatest (w.r.t. $\subseteq$) open subset of $\mathcal{X}$ that is contained in A. Moreover, the closure $\overline{A}$ is the smallest (w.r.t. $\subseteq$) closed subset of $\mathcal{X}$ that contains A.*

Proof. See [Par22, Proposition 3.10 1) and Proposition 3.25 1)]. ■

Lemma A.2.5. *Let $(\mathcal{X}, \tau)$ be a topological space, $A \subseteq \mathcal{X}$ and $a \in \mathcal{X}$. It holds $a \in \mathring{A}$ if and only if there exists some $U \in \tau$ such that $a \in U \subseteq A$. Moreover, it holds $a \in \overline{A}$ if and only if $U \cap A \neq \emptyset$ for any $U \in \tau$ with $a \in U$.*

Proof. See [Par22, Proposition 3.10 1) and Proposition 3.25 1)]. ■

Lemma A.2.6. *Let $(\mathcal{X}, \tau)$ be a topological space and $A \subseteq \mathcal{X}$, then $\mathring{A} \cap \partial A = \emptyset$ and*

$$\overline{A} = \mathring{A} \cup \partial A.$$

Proof. See [Par22, Proposition 4.15]. ■

Example A.2.7. Zariski Topology
For $l \in \mathbb{N}$, let K^l be either $\mathbb{C}^l$ or $\mathbb{P}^l_{\mathbb{C}}$. The *Zariski topology* on K^l is the topology on K^l that is obtained by setting the closed sets to be given by the affine respectively projective varieties (cf. Lemma A.1.2 and Lemma A.1.21). We observe $\overline{U} = \mathcal{V}(\mathcal{I}(U))$ for any $U \subseteq K^l$ (see [CLO15, Chapter 4, §4, Proposition 1]) and note that the projective closure of an affine variety $\mathfrak{V} \subseteq \mathbb{C}^l$ in $\mathbb{P}^l_{\mathbb{C}}$ (cf. Definition A.1.52) is the Zariski closure of the set $\phi(\mathfrak{V})$ in $\mathbb{P}^l_{\mathbb{C}}$.

Example A.2.8. Subspace Topology
Let $(\mathcal{X}, \tau)$ be a topological space and $\mathcal{S} \subseteq \mathcal{X}$, then τ induces a topology on $\mathcal{S}$ by setting $U \subseteq \mathcal{S}$ to be open if and only if there exists some $W \in \tau$ such that $U = W \cap \mathcal{S}$. Equivalently, we may also set $U \subseteq \mathcal{S}$ to be closed if and only if there exists some closed $W \subseteq \mathcal{X}$ such that $U = W \cap \mathcal{S}$. Either way, this gives us the same topology, the *subspace topology* $\tau_{\mathcal{S}}$ on $\mathcal{S}$, and we observe that the closure of a subset $U \subseteq \mathcal{S}$ w.r.t. $\tau_{\mathcal{S}}$ coincides with the intersection of $\mathcal{S}$ with the closure of U w.r.t. τ.

Definition A.2.9. Let $(\mathcal{X}, \tau)$ be a topological space. A subfamily $\mathcal{B}$ of τ is a *basis* of τ if every open set in $\mathcal{X}$ can be written as a (possibly empty) union of sets from $\mathcal{B}$.

Proposition A.2.10. *We let $\mathcal{X}$ be a set and $\mathcal{B}$ a family of subsets of $\mathcal{X}$, then $\mathcal{B}$ is a basis for some unique topology τ on $\mathcal{X}$ if and only if*

(i) for any $a \in \mathcal{X}$, there exists some $B \in \mathcal{B}$ with $a \in B$ and

(ii) for any $B_1, B_2 \in \mathcal{B}$ such that $B_1 \cap B_2 \neq \emptyset$, there exists some $B_b \in \mathcal{B}$ with $b \in B_b \subseteq B_1 \cap B_2$ for each $b \in B_1 \cap B_2$.

Proof. See [Par22, Proposition 3.14 and Proposition 3.15]. ∎

Example A.2.11. Product Topology
Let $(\mathcal{X}_i, \tau_i)_{i=1,\ldots,s}$ be a collection of $s \in \mathbb{N}$ topological spaces, then

$$\mathcal{B} := \{U_1 \times \ldots \times U_s \mid U_i \in \tau_i \text{ for } i = 1, \ldots, s\}$$

is a basis for a unique topology on $\mathcal{X}_1 \times \ldots \times \mathcal{X}_s$ by Proposition A.2.10, which we call the *product topology* τ_{Prod} on $\mathcal{X}_1 \times \ldots \times \mathcal{X}_s$.

Definition A.2.12. Let $(\mathcal{X}, \tau)$ be a topological space. If there exists a metric δ on $\mathcal{X}$ such that the collection $\{B_r(a)\}_{a \in \mathcal{X}, r>0}$ of open balls $B_r(a) := \{b \in \mathcal{X} \mid \delta(a, b) < r\}$ with radius $r > 0$ and center $a \in \mathcal{X}$ is a basis of τ, then $(\mathcal{X}, \tau)$ is called *metrizable.*

Example A.2.13. Metric Spaces
Let $(\mathcal{X}, \delta)$ be a metric space and τ_δ the topology that is induced by the metric δ via the basis $\{B_r(a)\}_{a \in \mathcal{X}, r>0}$, then the topological space $(\mathcal{X}, \tau_\delta)$ is metrizable by construction and we see that $U \subseteq \mathcal{X}$ is open if and only if for any $a \in U$, there exists some $r > 0$ such that $B_r(a) \subseteq U$. Moreover, $W \subseteq \mathcal{X}$ is closed if and only if for any $a \in W$ and

any $r > 0$, the intersection $B_r(a) \cap W$ is non-empty. Hence, for $A \subseteq \mathcal{X}$, we have

$$\begin{aligned}\mathring{A} &= \{a \in \mathcal{X} \mid \exists r > 0\colon B_r(a) \subseteq A\},\\ \overline{A} &= \{a \in \mathcal{X} \mid \forall r > 0\colon B_r(a) \cap A \neq \emptyset\}.\end{aligned}$$

Example A.2.14. Euclidean Topology
Let $\mathcal{X}$ be a finite-dimensional $\mathbb{K}$-vector space and δ the Euclidean metric on $\mathcal{X}$, then $(\mathcal{X}, \tau_\delta)$ is a metrizable topological space and τ_δ is called the *Euclidean topology* on X, which is often denoted by τ_{Eucl}.

We recall that a sequence $(a_m)_{m\in\mathbb{N}}$ in a metric space $(\mathcal{X}, \delta)$ *converges* to $a \in \mathcal{X}$ if for any $\varepsilon > 0$, there exists some $M \in \mathbb{N}$ such that $\delta(a_m, a) < \varepsilon$ for all $m \geq M$.

Proposition A.2.15. *Let $(\mathcal{X}, \delta)$ be a metric space and $A \subseteq \mathcal{X}$, then A is closed if and only if for any sequence $(a_m)_{m\in\mathbb{N}} \subseteq A$ that converges to $a \in \mathcal{X}$, it follows $a \in A$.*

Proof. ($\Rightarrow$) Let $(a_m)_{m\in\mathbb{N}} \subseteq A$ be a sequence that converges to some $a \in \mathcal{X}$. For any $U \in \tau$ with $a \in U$, we fix $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq U$ and let $M \in \mathbb{N}$ be such that $\delta(a_m, a) < \varepsilon$ for all $m \geq M$. We conclude $U \cap A \supseteq B_\varepsilon(a) \cap A \neq \emptyset$ since $a_m \in B_\varepsilon(a) \cap A$ for all $m \geq M$. Therefore, it follows $a \in \overline{A} = A$ since A is closed.
($\Leftarrow$) It suffices to show $\overline{A} \subseteq A$. To this end, we let $a \in \overline{A}$ be arbitrary but fixed and observe $B_r(a) \cap A \neq \emptyset$ for all $r > 0$. Thus, we set $r_m := \frac{1}{m}$ and fix $a_m \in B_{r_m}(a) \cap A$ for $m \in \mathbb{N}$. Moreover, for any a priori fixed $\varepsilon > 0$, we choose $M \in \mathbb{N}$ sufficiently large such that $r_M \leq \varepsilon$ and conclude $\delta(a_m, a) < r_M \leq \varepsilon$ for all $m \geq M$. This shows that the sequence $(a_m)_{m\in\mathbb{N}} \subseteq A$ converges to a. Hence, $a \in A$. ∎

Definition A.2.16. A map $f\colon \mathcal{X} \to \mathcal{Y}$ on two topological spaces $(\mathcal{X}, \tau)$ and $(\mathcal{Y}, \sigma)$ is *continuous* (w.r.t. τ and σ) if $f^{-1}(U) \in \tau$ for all $U \in \sigma$.

Remark A.2.17. *Recalling that $f^{-1}(\mathcal{Y}\backslash U) = \mathcal{X}\backslash f^{-1}(U)$ for all $U \subseteq \mathcal{Y}$, we see that continuity can be equivalently defined by demanding that the preimage $f^{-1}(W) \subseteq \mathcal{X}$ of any closed $W \subseteq \mathcal{Y}$ is closed.*

Example A.2.18. Let $(\mathcal{X}_i, \tau_i)_{i=1,\dots,s}$ be a collection of $s \in \mathbb{N}$ topological spaces and consider the projection

$$\begin{aligned}\pi\colon \mathcal{X}_1 \times \ldots \times \mathcal{X}_s &\to \mathcal{X}_1 \times \ldots \times \mathcal{X}_{s-1},\\ (a_1, \ldots, a_s) &\mapsto (a_1, \ldots, a_{s-1})\end{aligned}$$

onto the first $s-1$ components. Thus, we see that π is continuous w.r.t. the product topologies on $\mathcal{X}_1 \times \ldots \mathcal{X}_s$ and $\mathcal{X}_1 \times \ldots \times \mathcal{X}_{s-1}$ by construction.

Example A.2.19. For $l \in \mathbb{N}$, the polynomial $g \in \mathbb{R}[Y_1, \ldots, Y_l]$ can be interpreted as the polynomial map $g\colon \mathbb{R}^l \to \mathbb{R}, y \mapsto g(y)$ which is continuous w.r.t. the Euclidean topologies on $\mathbb{R}^l$ and $\mathbb{R}$.

Lemma A.2.20. *If $f\colon \mathcal{X} \to \mathcal{Y}$ is a continuous map on two topological spaces $(\mathcal{X}, \tau)$ and $(\mathcal{Y}, \sigma)$, then $\overline{f(\overline{A})} = \overline{f(A)}$ for all $A \subseteq \mathcal{X}$.*

Proof. ($\subseteq$) The minimality of the closure w.r.t. $\subseteq$ (cf. Lemma A.2.4) yields that it suffices to show $f(\overline{A}) \subseteq \overline{f(A)}$. To this end, we argue by a proof by contradiction and, therefore, let $a \in \overline{A}$ be such that $b := f(a) \notin \overline{f(A)}$. This allows us to fix some $U \in \sigma$ with $b \in U$ such that $U \cap f(A) = \emptyset$. We set $W := f^{-1}(U)$ and observe that W is open since f is continuous. It thus follows $W \cap A \neq \emptyset$ since we have $a \in \overline{A} \cap W$ by construction. Hence, we fix some $a' \in W \cap A$ and conclude $b' := f(a') \in f(W \cap A) \subseteq f(W) \cap f(A) \subseteq U \cap f(A)$ which contradicts $U \cap f(A) = \emptyset$.
($\supseteq$) We deduce $f(A) \subseteq f(\overline{A})$ from $A \subseteq \overline{A}$ and thus $\overline{f(A)} \subseteq \overline{f(\overline{A})}$ follows. ∎

Proposition A.2.21. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map on two metric spaces $(\mathcal{X}, \delta_{\mathcal{X}})$ and $(\mathcal{Y}, \delta_{\mathcal{Y}})$. If $(a_m)_{m\in\mathbb{N}} \subseteq \mathcal{X}$ is a sequence that converges to some $a \in \mathcal{X}$, then the sequence $(f(a_m))_{m\in\mathbb{N}} \subseteq \mathcal{Y}$ converges to $f(a) \in \mathcal{Y}$.*

Proof. See [Par22, Proposition 2.13]. ∎

Corollary A.2.22. *For $l \in \mathbb{N}$, let $\mathbb{R}^l$ be endowed with the Euclidean topology. For $W \subseteq \mathbb{R}^l$ and $f \in \mathbb{R}[Y_1, \ldots, Y_l]$, it then holds $f(\boldsymbol{y}) \geq 0$ for all $\boldsymbol{y} \in W$ if and only if $f(\boldsymbol{y}) \geq 0$ for all $\boldsymbol{y} \in \overline{W}$.*

Proof. ($\Rightarrow$) For $\boldsymbol{y} \in \overline{W}$, we fix $\left(\boldsymbol{y}^{(m)}\right)_{m\in\mathbb{N}} \subseteq W$ such that $\boldsymbol{y}^{(m)} \to \boldsymbol{y}$ as $m \to \infty$. Hence, Proposition A.2.21 yields that the non-negative sequence $\left(f\left(\boldsymbol{y}^{(m)}\right)\right)_{m\in\mathbb{N}} \subseteq \mathbb{R}$ converges to $f(\boldsymbol{y})$ since the interpretation of f as a polynomial map is continuous w.r.t. the Euclidean topologies on $\mathbb{R}^l$ and $\mathbb{R}$ by Example A.2.19. We conclude $f(\boldsymbol{y}) \geq 0$.
($\Leftarrow$) Immediate consequence of $W \subseteq \overline{W}$. ∎

Example A.2.23. Quotient Topology
Let $(\mathcal{X}, \tau)$ be a topological space and $\sim$ an equivalence relation on $\mathcal{X}$. The *quotient set* $\mathcal{X} / \sim$ is the set of all equivalence classes $[a]$ of elements $a \in \mathcal{X}$ w.r.t. $\sim$. The map $\psi\colon \mathcal{X} \to \mathcal{X} / \sim, a \mapsto [a]$ is called the *quotient map* and is surjective. We set $U \subseteq \mathcal{X} / \sim$ to be open if and only if $\psi^{-1}(U)$ is open in $\mathcal{X}$. Equivalently, we may also set $W \subseteq \mathcal{X} / \sim$ to be closed if and only if $\psi^{-1}(W) \subseteq \mathcal{X}$ is closed. Either way, this gives us the same topology, the *quotient topology* τ_{Quot} on $\mathcal{X} / \sim$, with respect to which ψ is continuous.

In the special case that $\mathcal{X}$ is a vector space with a linear subspace $\mathcal{M}$, we may induce an equivalence relation $\sim_{\mathcal{M}}$ on $\mathcal{X}$ by setting $a \sim_{\mathcal{M}} b$ for $a, b \in \mathcal{X}$ if $a - b \in \mathcal{M}$ and write $\mathcal{X} / \mathcal{M}$ instead of $\mathcal{X} / \sim_{\mathcal{M}}$.

Example A.2.24. Euclidean Topology on $\mathbb{P}^l_{\mathbb{K}}$
For $l \in \mathbb{N}$, we let 0 denote the origin of $\mathbb{K}^{l+1}$ and observe that $\mathbb{P}^l_{\mathbb{K}}$ is the quotient set $\mathbb{K}^{l+1} \backslash \{0\} / \sim$ w.r.t. the equivalence relation $\sim$ that is defined by setting $y \sim y'$ for

$y, y' \in \mathbb{K}^{l+1} \backslash \{0\}$ if there exists some $\lambda \in \mathbb{K}^\times$ such that $y = \lambda y'$ (cf. Notation A.1.17). Example A.2.23 thus implies that the Euclidean topology on $\mathbb{K}^{l+1} \backslash \{0\}$ induces a topology on $\mathbb{P}^l_\mathbb{K}$ with respect to which the quotient map $\psi\colon \mathbb{K}^{l+1} \backslash \{0\} \to \mathbb{P}^l_\mathbb{K}, y \mapsto [y]$ is continuous. Moreover, for $A \subseteq \mathbb{P}^l_\mathbb{K}$ and $[y] \in \mathbb{P}^l_\mathbb{K}$, we observe $[y] \in \overline{A}$ if and only if there exist $\left(\left[y^{(m)}\right]\right)_{m\in\mathbb{N}} \subseteq A$ and $(\lambda_m)_{m\in\mathbb{N}} \subseteq \mathbb{K}^\times$ such that $\lambda_m y^{(m)} \to y$ as $m \to \infty$.

Indeed, if $[y] \in \overline{A}$, then we set $\mathcal{B}_{\frac{1}{m}} := \left\{[u] \in \mathbb{P}^l_\mathbb{K} \mid \exists \lambda \in \mathbb{K}^\times \colon \lambda u \in B_{\frac{1}{m}}(y)\right\}$ for $m \in \mathbb{N}$, where $B_{\frac{1}{m}}(y)$ denotes the intersection of $\mathbb{K}^{l+1} \backslash \{0\}$ and the open ball of radius $\frac{1}{m}$ with center y. We observe that $\mathcal{B}_{\frac{1}{m}}$ is a well-defined set and deduce that

$$\psi^{-1}\left(\mathcal{B}_{\frac{1}{m}}\right) = \bigcup_{\lambda \in \mathbb{K}^\times} B_{\frac{1}{|\lambda| m}}\left(\frac{1}{\lambda} y\right)$$

is open as the union of open sets. Hence, $\mathcal{B}_{\frac{1}{m}}$ is an open set in $\mathbb{P}^l_\mathbb{K}$ that contains $[y]$. This thus allows us to fix some $\left[y^{(m)}\right] \in A \cap \mathcal{B}_{\frac{1}{m}}$ since $[y] \in \overline{A}$. We hence obtain a sequence $\left(\left[y^{(m)}\right]\right)_{m\in\mathbb{N}} \subseteq A$ with a corresponding sequence $(\lambda_m)_{m\in\mathbb{N}} \subseteq \mathbb{K}^\times$ such that $\lambda_m y^{(m)} \in B_{\frac{1}{m}}(y)$ for all $m \in \mathbb{N}$. In particular, $\lambda_m y^{(m)} \to y$ as $m \to \infty$ follows.

Vice versa, if we assume that there exist some sequences $\left(\left[y^{(m)}\right]\right)_{m\in\mathbb{N}} \subseteq A$ and $(\lambda_m)_{m\in\mathbb{N}} \subseteq \mathbb{K}^\times$ such that $\lambda_m y^{(m)} \to y$ as $m \to \infty$, then $\left(\lambda_m y^{(m)}\right)_{m\in\mathbb{N}}$ is a sequence in $B := \psi^{-1}(A)$ that converges to y. We therefore conclude $y \in \overline{B}$ which implies $[y] \in \psi(\overline{B}) \subseteq \overline{\psi(\overline{B})} = \overline{\psi(B)} = \overline{A}$ by Lemma A.2.20 and the surjectivity of ψ.

In the special case that $\mathbb{K} = \mathbb{C}$, we see that the set of real points of a projective variety in $\mathbb{P}^l_\mathbb{C}$ is closed w.r.t. the Euclidean topology (cf. [Man20, Proposition 2.2.11]).

Definition A.2.25. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map on two topological spaces $(\mathcal{X}, \tau)$, $(\mathcal{Y}, \sigma)$.

(i) If $f(U) \in \sigma$ for all $U \in \tau$, then f is *open* (w.r.t. τ and σ).

(ii) If $f(W) \subseteq \mathcal{Y}$ is closed for all closed $W \subseteq \mathcal{X}$, then f is *closed* (w.r.t. τ and σ).

Lemma A.2.26. *If $f\colon \mathcal{X} \to \mathcal{Y}$ is an open bijective map on two topological spaces $(\mathcal{X}, \tau)$ and $(\mathcal{Y}, \sigma)$, then f is closed.*

Proof. For closed $W \subseteq \mathcal{X}$, we have $\mathcal{X} \backslash W \in \tau$ and the bijectivity of f implies $f(\mathcal{X} \backslash W) = f(\mathcal{X}) \backslash f(W) = \mathcal{Y} \backslash f(W)$. It therefore follows $\mathcal{Y} \backslash f(W) \in \sigma$ since f is open and thus $f(W) \subseteq \mathcal{Y}$ is closed. ∎

Definition A.2.27. A continuous map $f\colon \mathcal{X} \to \mathcal{Y}$ on two topological spaces $(\mathcal{X}, \tau)$ and $(\mathcal{Y}, \sigma)$ is a *homeomorphism* if f is bijective and open.

Lemma A.2.28. *If $f\colon \mathcal{X} \to \mathcal{Y}$ is a homeomorphism on two topological spaces $(\mathcal{X}, \tau)$ and $(\mathcal{Y}, \sigma)$, then $f(\mathring{A}) = \overset{\circ}{f(A)}$ for all $A \subseteq \mathcal{X}$.*

Proof. ($\subseteq$) Since the map f is open, $f(\mathring{A})$ is an open set that is contained in $f(A)$. Hence, $f(\mathring{A}) \subseteq \overset{\circ}{f(A)}$.

($\supseteq$) We know that $f^{-1}(f(\mathring{A}))$ is an open subset of A since f is bijective and open. Hence, we have $f^{-1}(f(\mathring{A})) \subseteq \mathring{A}$ which implies $f(\mathring{A}) = f(f^{-1}(f(\mathring{A}))) \subseteq f(\mathring{A})$ by the bijectivity of f. ∎

Definition A.2.29. Let $\mathbb{K}$ be endowed with the Euclidean topology and $(\mathcal{X}, +, *)$ a $\mathbb{K}$-vector space. A topology τ on $\mathcal{X}$ is a *linear topology* and (X, τ), more precisely $(X, +, *, \tau)$, is a *topological* $\mathbb{K}$*-vector space* if $+\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $*\colon \mathbb{K} \times \mathcal{X} \to \mathcal{X}$ are continuous w.r.t. the product topology on $\mathcal{X} \times \mathcal{X}$, respectively, $\mathbb{K} \times \mathcal{X}$ and τ.

Proposition A.2.30. *Let $\mathcal{X}$ be a $\mathbb{K}$-vector space and δ a metric on $\mathcal{X}$, then $(\mathcal{X}, \tau_\delta)$ is a topological $\mathbb{K}$-vector space.*

Proof. See [Ovc18, Theorem 2.1]. ∎

Example A.2.31. Normed Spaces
Let $(\mathcal{X}, ||\cdot||)$ be a normed $\mathbb{K}$-vector space, then $||\cdot||$ induces a metric δ on $\mathcal{X}$ by setting $\delta(a, b) := ||a - b||$ for $a, b \in \mathcal{X}$. This turns the normed $\mathbb{K}$-vector space $(\mathcal{X}, ||\cdot||)$ into the topological $\mathbb{K}$-vector space $(\mathcal{X}, \tau_\delta)$.

In the special case that $\mathcal{X}$ is finite-dimensional, we know that for any two norms $||\cdot||_1$ and $||\cdot||_2$ on $\mathcal{X}$, there exist some $c_1, c_2 > 0$ such that $c_1||a||_1 \leq ||a||_2 \leq c_2||a||_1$ for all $a \in \mathcal{X}$ (cf. [Ovc18, Theorem 4.9]). Therefore, all norms on $\mathcal{X}$ induce the same topology (cf. [Ovc18, Corollary 4.2]). In particular, we may set $l := \dim(\mathcal{X})$, identify $\mathcal{X}$ with $\mathbb{K}^l$ and endow $\mathbb{K}^l$ with the Euclidean norm, or any other norm we like.

Proposition A.2.32. *Let $(\mathcal{X}, \tau)$ be a topological $\mathbb{K}$-vector space and $\mathcal{M}$ a linear subspace, then $(\mathcal{X} / \mathcal{M}, \tau_{Quot})$ is a topological $\mathbb{K}$-vector space.*

Proof. See [Voi20, Theorem 7.9]. ∎

Definition A.2.33. A topological space $(\mathcal{X}, \tau)$ is *Hausdorff* if for all $a, b \in \mathcal{X}$, $a \neq b$, there exist disjoint open sets U and W in $\mathcal{X}$ such that $a \in U$ and $b \in W$.

Example A.2.34. Metric spaces are Hausdorff and therefore any finite-dimensional $\mathbb{K}$-vector space can be interpreted as a Hausdorff topological space by Example A.2.31.

Definition A.2.35. A topological $\mathbb{K}$-vector space $(\mathcal{X}, \tau)$ with identity element 0 is *locally convex* if for any $A \subseteq \mathcal{X}$ for which there exists some $U \in \tau$ with $0 \in U \subseteq A$, there exists some convex set $A' \subseteq \mathcal{X}$ and some $U' \in \tau$ such that $0 \in U' \subseteq A' \subseteq A$.

Example A.2.36. Normed spaces are locally convex and thus any finite-dimensional $\mathbb{K}$-vector space can be interpreted as a locally convex topological $\mathbb{K}$-vector space by Example A.2.31.

Definition A.2.37. A set A in a topological $\mathbb{K}$-vector space $(\mathcal{X}, \tau)$ with identity element 0 is *bounded* if for all $U \in \tau$ with $0 \in U$, there exists a $\lambda \in \mathbb{K}$ with $A \subseteq \lambda U$.

Definition A.2.38. A map $f\colon \mathcal{X} \to \mathcal{Y}$ on two topological $\mathbb{K}$-vector spaces $(\mathcal{X}, \tau)$ and $(\mathcal{Y}, \sigma)$ is *bounded* (w.r.t. τ and σ) if $f(A) \subseteq \mathcal{Y}$ is bounded for each bounded $A \subseteq \mathcal{X}$.

Proposition A.2.39. *If $f\colon \mathcal{X} \to \mathcal{Y}$ is a continuous linear map on two topological $\mathbb{K}$-vector spaces $(\mathcal{X}, \tau)$ and $(\mathcal{Y}, \sigma)$, then f is bounded.*

Proof. See [Voi20, Lemma 3.4 (a)]. ∎

Proposition A.2.40. *If $f\colon \mathcal{X} \to \mathcal{Y}$ is a linear map on two finite-dimensional normed spaces $(\mathcal{X}, ||\cdot||_{\mathcal{X}})$ and $(\mathcal{Y}, ||\cdot||_{\mathcal{Y}})$ over $\mathbb{K}$, then f is continuous and bounded.*

Proof. We deduce from [Ovc18, Theorem 4.10, Theorem 4.5 and Theorem 2.11] that f is continuous. Thus, f is also bounded by Proposition A.2.39. ∎

Corollary A.2.41. *If $f\colon \mathcal{X} \to \mathcal{Y}$ is a linear map on two finite-dimensional normed spaces $(\mathcal{X}, ||\cdot||_{\mathcal{X}})$ and $(\mathcal{Y}, ||\cdot||_{\mathcal{Y}})$ over $\mathbb{K}$, then the kernel of f is closed.*

Proof. The linear map f is continuous by Proposition A.2.40 and the kernel of f is the preimage of the closed set $\{0_{\mathcal{Y}}\} \subseteq \mathcal{Y}$, where $0_{\mathcal{Y}}$ is the identity element of $\mathcal{Y}$. ∎

We recall that a *Banach space* is a normed space $(\mathcal{X}, ||\cdot||)$ such that the induced metric space $(\mathcal{X}, \delta)$ is complete.[2]

Proposition A.2.42. *Let $(\mathcal{X}, ||\cdot||)$ be a normed space. If $\mathcal{X}$ is finite-dimensional, then $(\mathcal{X}, ||\cdot||)$ is a Banach space.*

Proof. See [Ovc18, Theorem 4.13]). ∎

We recall that $\mathcal{F}_{n+1,2d}$ denotes the $\mathbb{R}$-vector space of all forms in $\mathbb{R}[X_0, \ldots, X_n]$ of degree $2d$ for $n, d \geq 1$. Moreover, $\mathrm{Sym}_{k+1}(\mathbb{R})$ denotes the $\mathbb{R}$-vector space of symmetric $(k+1) \times (k+1)$ matrices with entries in $\mathbb{R}$ for $k := k(n,d) := \binom{n+d}{n} - 1$.

Corollary A.2.43. *For $n, d \geq 1$ and $k := k(n,d) := \binom{n+d}{n} - 1$, the finite-dimensional $\mathbb{R}$-vector spaces $\mathcal{F}_{n+1,2d}$ and $\mathrm{Sym}_{k+1}(\mathbb{R})$ are Banach spaces.*

Proof. Example A.2.31 illustrates how the finite-dimensional $\mathbb{R}$-vector spaces $\mathcal{F}_{n+1,2d}$ and $\mathrm{Sym}_{k+1}(\mathbb{R})$ can be interpreted as normed spaces. The assertion therefore follows from Proposition A.2.42. ∎

Theorem A.2.44. Open Mapping Theorem
If $f\colon \mathcal{X} \to \mathcal{Y}$ is a bounded linear map on two Banach spaces $(\mathcal{X}, ||\cdot||_{\mathcal{X}})$ and $(\mathcal{Y}, ||\cdot||_{\mathcal{Y}})$ over $\mathbb{K}$, then f is open.

Proof. See [Ovc18, Theorem 6.3]. ∎

[2] That is, any Cauchy sequence in $\mathcal{X}$ converges w.r.t. the metric δ in $\mathcal{X}$.

Corollary A.2.45. *Let* $(\mathcal{X}, ||\cdot||_{\mathcal{X}})$, $(\mathcal{Y}, ||\cdot||_{\mathcal{Y}})$ *be two finite-dimensional normed spaces over* $\mathbb{K}$ *and* $f\colon \mathcal{X} \to \mathcal{Y}$ *a linear map, then* f *is open.*

Proof. Proposition A.2.40 and Proposition A.2.42 together yield that f is a bounded linear map between the two Banach spaces $(\mathcal{X}, ||\cdot||_{\mathcal{X}})$ and $(\mathcal{Y}, ||\cdot||_{\mathcal{Y}})$ over $\mathbb{K}$. Thus, f is open by Theorem A.2.44. ■

We recall that a subset A of a topological space $(\mathcal{X}, \tau)$ is *compact* if every open cover of A contains a finite subcover of A.

Lemma A.2.46. *Let* $(\mathcal{X}, \tau)$ *be a topological space. If* $A \subseteq \mathcal{X}$ *is compact and* $W \subseteq \mathcal{X}$ *closed, then* $A \cap W$ *is compact.*

Proof. See [Par22, Corollary 5.2]. ■

Proposition A.2.47. *Let* $f\colon \mathcal{X} \to \mathbb{R}$ *be a continuous map on a topological space* $(\mathcal{X}, \tau)$*. If* $\mathcal{X}$ *is compact, then* f *is bounded and attains a minimum and a maximum.*

Proof. See [Par22, Corollary 5.24]. ■

The result below is a special case of the Bolzano–Weierstraß theorem for finite-dimensional normed spaces.

Theorem A.2.48. Heine–Borel Theorem
Let $(\mathcal{X}, ||\cdot||)$ *be a finite-dimensional normed space and* $A \subseteq \mathcal{X}$*, then* A *is compact if and only if* A *is closed and bounded.*

Proof. See [Ovc18, Theorem 4.14]. ■

Example A.2.49. The Closed Unit Ball in $\mathbb{R}^l$
For $l \in \mathbb{N}$, we endow $\mathbb{R}^l$ with the Euclidean topology and denote the Euclidean norm on $\mathbb{R}^l$ by $||\cdot||$. Moreover, we set 0 to be the origin of $\mathbb{R}^l$ and observe that the *closed unit ball* $B_1(0) := \{y \in \mathbb{R}^l \mid ||y|| \leq 1\} \subseteq \mathbb{R}^l$ is closed and bounded. Thus, $B_1(0)$ is compact by Theorem A.2.48.

Definition A.2.50. A topological space $(\mathcal{X}, \tau)$ is *locally compact* if for any $a \in \mathcal{X}$, there exists some compact $A \subseteq \mathcal{X}$ and some $U \in \tau$ such that $0 \in U \subseteq A$.

Example A.2.51. Example A.2.31, combined with the Heine–Borel theorem given in Theorem A.2.48, implies that any finite-dimensional $\mathbb{K}$-vector space can be interpreted as a locally compact topological space.

Theorem A.2.52. *Let* $(\mathcal{X}, \tau)$ *be a locally compact Hausdorff topological space. If* $W \subseteq \mathcal{X}$ *is closed, then* W *is locally compact w.r.t. the subspace topology* τ_W*.*

Proof. See [Par22, Proposition 11.7]. ■

A.3 Convex Geometry

An interested reader may consult [HW20], [Bar02] and [Sch24, Section 8.1] for a comprehensive overview on convex geometry. We here only give a brief introduction to this topic for finite-dimensional $\mathbb{R}$-vector spaces. To this end, we recall that any finite-dimensional $\mathbb{R}$-vector space $\mathcal{X}$ is isomorphic to $\mathbb{R}^l$ for $l := \dim(\mathcal{X})$. Therefore, it suffices to examine $\mathbb{R}^l$ for a **natural number l** in this section.

Notation A.3.1. For $a, b \in \mathbb{R}^l$, we set

$$\begin{aligned} [a,b] &:= \left\{ \lambda a + (1-\lambda)b \in \mathbb{R}^l \mid 0 \leq \lambda \leq 1 \right\}, \\ ]a,b[&:= \left\{ \lambda a + (1-\lambda)b \in \mathbb{R}^l \mid 0 < \lambda < 1 \right\}. \end{aligned}$$

Definition A.3.2. A set $A \subseteq \mathbb{R}^l$ is *convex* if $[a,b] \subseteq A$ for all $a, b \in A$.

Definition A.3.3. A linear combination $\sum_{i=1}^{s} \lambda_i a_i$ of $a_1, \ldots, a_s \in \mathbb{R}^l$ $(s \in \mathbb{N})$ with scalars $0 \leq \lambda_1, \ldots, \lambda_s \leq 1$ such $\sum_{i=1}^{s} \lambda_i = 1$ is called a *convex combination*.

Theorem A.3.4. *A set $A \subseteq \mathbb{R}^l$ is convex if and only if every convex combination of elements in A lies in A.*

Proof. See [HW20, Theorem 1.2]. ■

Definition A.3.5. The *convex hull* of $A \subseteq \mathbb{R}^l$ is the intersection of all convex sets in $\mathbb{R}^l$ that contain A.

Notation A.3.6. We denote the convex hull of $A \subseteq \mathbb{R}^l$ by $\operatorname{conv}(A)$.

Theorem A.3.7. *The convex hull of $A \subseteq \mathbb{R}^l$ is a convex set that is given by*

$$\operatorname{conv}(A) = \left\{ \sum_{i=1}^{s} \lambda_i a_i \mid s \in \mathbb{N}, a_1, \ldots, a_s \in A, 0 \leq \lambda_1, \ldots, \lambda_s \leq 1, \sum_{i=1}^{s} \lambda_i = 1 \right\}.$$

Proof. See [HW20, Theorem 1.2] and Theorem A.3.4. ■

Definition A.3.8. The *dimension* of a convex set $A \subseteq \mathbb{R}^l$ is the dimension of the smallest affine subspace of $\mathbb{R}^l$ that contains A. If the dimension of A is l, then A is *full-dimensional*.

Definition A.3.9. A non-empty convex subset F of a closed convex set $A \subseteq \mathbb{R}^l$ is a *face* of A if $a, b \in A$ and $]a,b[\cap F \neq \emptyset$ implies $a, b \in F$. The 0-dimensional faces of A are called *extreme points*. Any face other than A is a *proper* face of A.

We recall that an affine hyperplane $H \subseteq \mathbb{R}^l$ is a *supporting hyperplane* of a set $A \subseteq \mathbb{R}^l$ if $A \cap H \neq \emptyset$ and A is contained in one of the two closed halfspaces induced by H.

Definition A.3.10. A face F of a closed convex set $A \subseteq \mathbb{R}^l$ is *exposed* if $F = A \cap H$ for some supporting hyperplane H of A.

Definition A.3.11. A non-empty set $C \subseteq \mathbb{R}^l$ is a *(convex) cone* if $C + C \subseteq C$ and $\lambda C \subseteq C$ for all $\lambda \geq 0$.

Lemma A.3.12. *A cone $C \subseteq \mathbb{R}^l$ is a convex set that contains the origin.*

Proof. For $a, b \in C$ and $0 \leq \lambda \leq 1$, we have $\lambda a + (1-\lambda)b \in C$ by the cone property. This shows that C is convex. Moreover, we know that there exists some $y \in C$ since C is assumed to be non-empty. For any such $y \in C$, we deduce $\lambda a \in C$ for all $\lambda \geq 0$ by the cone property. Setting $\lambda := 0$, we thus conclude that C contains the origin. ∎

Definition A.3.13. A cone $C \subseteq \mathbb{R}^l$ is *pointed* if $C \cap -C = \{(0, \ldots, 0)\}$.

Definition A.3.14. The *ray* spanned by a non-zero $y \in \mathbb{R}^l$ is given by

$$R_y := \mathbb{R}_{\geq 0} y := \{\lambda y \mid \lambda \geq 0\}.$$

Definition A.3.15. Let $A \subseteq \mathbb{R}^l$ be a convex set and $a \in A$ a non-zero element. The ray R_a spanned by a is an *extreme ray* of A if $A \backslash R_a$ is convex.

Theorem A.3.16. Klee's Theorem
If $A \subseteq \mathbb{R}^l$ is a closed convex set which contains no straight lines, then A is the convex hull of the union of its extreme points and extreme rays.

Proof. See [Kle57, 2.5. Theorem]. ∎

Corollary A.3.17. Krein–Milman's Theorem for Cones
If $C \subseteq \mathbb{R}^l$ is a closed cone which contains no straight lines, then C is the convex hull of its extreme rays.

Proof. The origin is the only extreme point of C. Thus, this modification of the Krein–Milman's theorem [KM40] follows from Klee's theorem (cf. Theorem A.3.16). ∎

Definition A.3.18. The *recession cone* of a non-empty convex set $A \subseteq \mathbb{R}^l$ is given by

$$\mathrm{rc}(A) := \left\{ y \in \mathbb{R}^l \mid \forall a \in A \; \forall \lambda \geq 0 \colon a + \lambda y \in A \right\}.$$

Theorem A.3.19. *If $C \subseteq \mathbb{R}^l$ is a cone, then* $\mathrm{rc}(C) = C$.

Proof. ($\subseteq$) For $y \in \mathrm{rc}(C)$, we have $a + \lambda y \in C$ for all $a \in C$ and all $\lambda \geq 0$. In the special case that a is the origin and $\lambda = 1$, we therefore conclude $y = a + \lambda y \in C$.
($\supseteq$) For $y \in C$, we have $a + \lambda y \in C$ for all $a \in C$ and all $\lambda \geq 0$ by the cone property. Hence, $y \in \mathrm{rc}(C)$. ∎

Theorem A.3.20. Dieudonné's Theorem in $\mathbb{R}^l$
Let $A, B \subseteq \mathbb{R}^l$ be two non-empty closed convex sets. If A or B is locally compact and $\operatorname{rc}(A) \cap \operatorname{rc}(B)$ is a linear subspace of $\mathbb{R}^l$, then $A - B$ is closed.

Proof. We recall from Example A.2.36 that $\mathbb{R}^l$ can be interpreted as a locally convex topological $\mathbb{R}$-vector space and refer to [Z02̆, Theorem 1.1.8] for an interpretation of Dieudonné's results [Die66]. ∎

We denote the $\mathbb{R}$-vector space of all homomorphisms $L\colon \mathbb{R}^l \to \mathbb{R}$ by $\operatorname{Hom}(\mathbb{R}^l, \mathbb{R})$ and recall that $\operatorname{Hom}(\mathbb{R}^l, \mathbb{R})$ is isomorphic to $\mathbb{R}^l$.

Definition A.3.21. The *dual cone* of $A \subseteq \mathbb{R}^l$ is given by

$$A^\vee := \left\{ L \in \operatorname{Hom}(\mathbb{R}^l, \mathbb{R}) \mid \forall a \in A\colon L(a) \geq 0 \right\}.$$

Lemma A.3.22. *The dual cone of $A \subseteq \mathbb{R}^l$ is a closed cone.*

Proof. Convexity: The homomorphism $L\colon \mathbb{R}^l \to \mathbb{R}, y \mapsto 0$ is contained in $A^\vee$, which is therefore non-empty. Moreover, we observe for all $a \in A$ and any $L_1, L_2 \in A^\vee$, $\lambda \geq 0$ that $(L_1 + L_2)(a) = L_1(a) + L_2(a) \geq 0$ and $\lambda L_1(a) \geq 0$.
Closure: For $(L_m)_{m\in\mathbb{N}} \subseteq A^\vee$ and $L \in \operatorname{Hom}(\mathbb{R}^l, \mathbb{R})$ such that $L_m \to L$ as $m \to \infty$, we know that $(L_m(a))_{m\in\mathbb{N}}$ is a non-negative sequence in $\mathbb{R}$ that converges to $L(a)$ for all $a \in A$. Thus, $L(a) \geq 0$ for all $a \in A$ which shows $L \in A^\vee$. Therefore, $A^\vee$ is closed by Proposition A.2.15. ∎

Proposition A.3.23. *If $A \subseteq \mathbb{R}^l$ is a closed convex set and $y \in \mathbb{R}^l \backslash A$, then there exists some $L \in \operatorname{Hom}(\mathbb{R}^l, \mathbb{R})$ and some $c \in \mathbb{R}$ such that $L(y) < c \leq L(a)$ for all $a \in A$.*

Proof. See [Bar02, (1.3) Theorem]. ∎

Theorem A.3.24. *If $C \subseteq \mathbb{R}^l$ is a closed cone, then $(C^\vee)^\vee = C$.*

Proof. ($\subseteq$) For a proof by contradiction, we let $y \in (C^\vee)^\vee$ be such that $y \notin C$. Proposition A.3.23 thus allows us to fix some $L \in \operatorname{Hom}(\mathbb{R}^l, \mathbb{R})$ and some $c \in \mathbb{R}$ such that $L(y) < c \leq L(a)$ for all $a \in C$. In particular, $c = 0$ may be chosen since C is a cone. Hence, we have $0 \leq L(a)$ for all $a \in C$ which shows $L \in C^\vee$. Therefore, it follows $0 \leq L(y)$ since we chose $y \in (C^\vee)^\vee$. This contradicts $L(y) < 0$.
($\supseteq$) For $y \in C$ and $L \in C^\vee$, we have $L(y) \geq 0$ by definition. ∎

A.4 Real Algebraic Geometry

An interested reader may consult [BCR98] and [BPR06] for a comprehensive overview on real algebraic geometry. We here only give a brief introduction to this topic following [Sch18b, Paragraph 4.13].

Definition A.4.1. Let R be a field.

(i) An *order* on R is a total order relation $\leq$ such that $0 \leq a$ and $0 \leq b$ implies $0 \leq ab$ for all $a, b \in R$ and also $a \leq b$ implies $a + c \leq b + c$ for all $a, b, c \in R$.

(ii) The field R is a *real field* if there exists an order on R.

(iii) The field R is *real closed* if R is a real field and there does not exist a real algebraic extension $\mathfrak{R} \supsetneq R$.

Definition A.4.2. The *canonical valuation ring* B of a real closed field R containing $\mathbb{R}$ is given by $B := \{a \in R \mid \exists m \in \mathbb{N}\colon -m < a < m\}$ and induces the maximal ideal $\mathfrak{m}_B := \{a \in R \mid \forall m \in \mathbb{N}\colon |ma| < 1\}$. Any non-zero element in $\mathfrak{m}_B$ is called an *infinitesimal* element of R.

Example A.4.3. The Field of Puiseux series in T with coefficients in $\mathbb{R}$
A *Puiseux series* in the variable T with coefficients in $\mathbb{R}$ is of the form

$$\sum_{i=l}^{\infty} a_i T^{\frac{i}{m}}$$

for $l \in \mathbb{Z}$, $m \in \mathbb{N}$ and $a_i \in \mathbb{R}$ for $i \geq l$. The *valuation* or *order* of a non-zero Puiseux series $\overline{a} = \sum_{i=l}^{\infty} a_i T^{\frac{i}{m}}$ in T with coefficients in $\mathbb{R}$ and $a_l \neq 0$ is given by $o(\overline{a}) := \frac{l}{m}$ and the order of zero is set to be infinity. The *field of Puiseux series* in T with coefficients in $\mathbb{R}$ is a real closed field (cf. [BPR06, Theorem 2.113.]) containing $\mathbb{R}$, which we denote by $\mathbb{R}\langle\langle T\rangle\rangle$. The canonical valuation ring of $\mathbb{R}\langle\langle T\rangle\rangle$ is moreover given by

$$B = \{\overline{a} \in \mathbb{R}\langle\langle T\rangle\rangle \mid o(\overline{a}) \geq 0\}$$

and induces the maximal ideal $\mathfrak{m}_B = \{\overline{a} \in \mathbb{R}\langle\langle T\rangle\rangle \mid o(\overline{a}) > 0\}$.

Theorem A.4.4. A Tarski-Transfer Principle
Let R be a real closed field with a real closed extension $\mathfrak{R}$. If $B(Y_1, \ldots, Y_l)$ ($l \in \mathbb{N}$) is a (finite) Boolean combination of polynomial equalities and inequalities with variables $Y_1, \ldots, Y_l$ and coefficients in R, then $B(y)$ holds true for some $y \in R^l$ if and $B(y)$ holds true for some $y \in \mathfrak{R}^l$.

Proof. This result is a consequence of the Tarski–Seidenberg principle presented in [Tar98; Sei54]. For a proof, we refer to [BCR98, Proposition 4.1.1.]. ■

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