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Sabine Cornelsen
Yefim Dinitz
Dorothea Wagner

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Planarity of the 2-level Cactus Model

Sabine Cornelsen[1], Yefim Dinitz[2], and Dorothea Wagner[1]

[1]University of Konstanz, Dept. of Computer and Information Science
{Sabine.Cornelsen,Dorothea.Wagner}@uni-konstanz.de
[2]Ben-Gurion University, Dept. of Computer Science
dinitz@cs.bgu.ac.il

Abstract

The 2-level cactus introduced by Dinitz and Nutov in [5] is a data structure that represents the minimum and minimum+1 edge-cuts of an undirected connected multi-graph G in a compact way. In this paper, we study planarity of the 2-level cactus, which can be used e.g. in graph drawing. We give a new sufficient planarity criterion in terms of projection paths over a spanning subtree of a graph. Using this criterion, we show that the 2-level cactus of G is planar if the cardinality of a minimum edge-cut of G is not equal to 2, 3 or 5. On the other hand, we give examples for non-planar 2-level cacti of graphs with these connectivities.

1 Introduction

Edge connectivity is a fundamental structural property of a graph. In the last decade, not only the properties of minimum cuts but also the number [13, 11] and structure [1] of near minimum cuts were examined. Galil and Italiano [8] and Dinitz and Westbrook [3, 7] developed models for all 1 and 2 cuts and all 2 and 3 cuts, respectively. Based on these two models, Dinitz and Nutov introduced the so called 2-level cactus model – a data structure that represents the minimum and minimum+1 edge cuts of an undirected multi-graph with connectivity $\lambda \geq 3$ in a compact way [5]. There is no other so compact model, and no other compact graph model for these cuts known, for the best of our knowledge. The above models imply, in particular, fast incremental maintenance algorithms [8, 7, 5].

The 2-level cactus model (or "2–level cactus", for simplicity) generalizes the cactus model of all minimum cuts [4]. In case of odd connectivity $\lambda > 3$, the 2-level cactus is really a cactus, that is a connected graph in which every edge is contained in at most one simple cycle. Some cuts, however, are represented only implicitly in the graph of the model. In order to reduce this implicitness, we add some auxiliary edges. We call the resulting graph the extended 2-level cactus. The main question considered in this paper is whether the modeling graph is planar, for both odd and even cases.

The proof of planarity is based on properties of the set of projection paths of auxiliary edges, that is the set of (shortest) paths in the 2-level cactus between the end nodes of auxiliary edges. To obtain planarity, we give a new sufficient planarity criterion, generalizing a corollary to the criterion of MacLane [10].

The question of planarity is not only of graph theoretical interest, it is also useful for algorithmic purposes. But the main reason, why we study planarity of the 2-level cactus, is an application in graph drawing. We are interested in drawing all small cuts ("bottlenecks") of a graph. Clearly, for a presentation on a screen, planarity is a key property. A first approach in visualizing the minimum cuts is done in [2], utilizing the cactus model for all minimum cuts [4]. The 2-level cactus might be a next step in this direction.

This paper is organized as follows. Sect. 2 first introduces MacLane's planarity criterion. Then, we generalize a corollary of MacLane's criterion. Sect. 3 introduces the 2-level cactus. In Sect. 4, we show, first, that it is planar for connectivity $\lambda = 1$ and for any even connectivity $\lambda \geq 4$. In case of odd connectivity, we show that the 2-level cactus together with the auxiliary edges is planar if $\lambda > 5$. We also give examples of non-planar (extended) 2-level cacti of graphs with connectivity $\lambda = 2$, $\lambda = 3$, and $\lambda = 5$. We conclude the paper by Sect. 5 with some remarks on how we would like to choose the faces of a planar embedding and what can be guaranteed about that.

2 Planarity of trees with additional edges

Let $E_1 \Delta E_2 = E_1 \setminus E_2 \cup E_2 \setminus E_1$ be the *ring sum* of two sets E_1 and E_2. Let $\mathfrak{E}_G$ be the vector space on the subsets of an edge-set E of a graph G over $\mathbb{F}_2$ under the ring sum operation Δ. The set $\mathfrak{Z}_G$ of all cycles and unions of edge-disjoint cycles is a subspace of the vector space $\mathfrak{E}_G$ and is called the *cycle space* of G. A *2-basis* of G is a basis of the cycle space of G, such that every edge occurs in at most two elements of this basis.

Theorem 1 (Planarity Criterion of MacLane [10]) *A graph is planar if and only if it has a 2-basis. Moreover, any 2-basis of a 2-connected graph consists of all but one facial cycle of some of its planar representations.*

A short proof of MacLane's planarity criterion can be found in [12].

A basis of the cycle space can be constructed from a spanning tree: Let T be a spanning tree of a connected graph G. For an edge $e = \{v, w\}$ in $G - T$, let p_e denote the set of edges on the path in T between v and w; called the *projection path* of e. Then, $\{\{e\} \cup p_e | e \text{ edge in } G - T\}$ is a basis of the cycle space $\mathfrak{Z}_G$. Thus, there is the following immediate corollary of MacLane's Planarity Criterion.

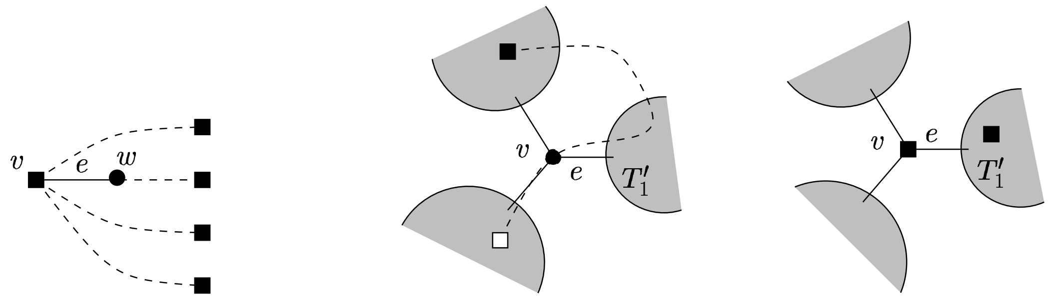
Corollary 1 *A graph G is planar if there is a spanning tree T of G such that every edge in T is contained in at most two projection paths.*

The following lemma gives a sufficient planarity criterion under somewhat weaker conditions.

Lemma 1 *A graph G is planar if there is a spanning tree T of G such that for any edge $e = \{v, w\}$ in T, the number of projection paths that contain e and one more edge in T incident to v is at most two.*

Proof: Let $G = (V, B \dot{\cup} S)$[1] be a connected graph such that $T = (V, B)$ is a spanning tree of G that fulfills the condition of the lemma. We use Kuratowski's Theorem [9].

[1]With $M_1 \dot{\cup} M_2$ we denote the disjoint union of two sets M_1 and M_2.



No K_5 | Case: v is not a node | No $K_{3,3}$ Case: v is a node

Figure 1: Illustration of the proof of Lemma 1. Rectangularly shaped vertices are nodes.

G **does not contain a** K_5**:** Suppose G contains a subdivision of a K_5 as a subgraph. We call the vertices of this K_5 nodes. Let T' be the smallest subtree of T that contains all nodes of the K_5. Let v be a leaf of T'. Thus, v is a node. Let $e = \{v, w\}$ be the edge incident to v in T'. At most one of the four subdivision paths of the K_5 that connect v to the other nodes can contain vertex w. This situation is illustrated in Fig. 1 (left). Thus, there are at least three subdivision paths that (i) connect the two connected components T_1 and T_2 of $T - \{e\}$ and that (ii) do not contain w. Such a subdivision path contains at least one edge $s \in S$ with one end vertex in T_1 and one end vertex in T_2. The projection path of s contains e and one more edge incident to w. Hence, there are at least three projection paths containing e and another edge incident to w. This contradicts the precondition.

G **does not contain a** $K_{3,3}$**:** Suppose G contains a subdivision of a $K_{3,3}$. We call the vertices of this $K_{3,3}$ nodes. We distinguish the two parts of the $K_{3,3}$ as white nodes and black nodes. Let T' be the smallest subtree of T that contains all nodes of the $K_{3,3}$. Let v be a vertex that has maximum degree in T'.

Let us first consider the case that $\deg_{T'}(v) \geq 3$ and v is not a node as illustrated in Fig. 1 (middle). At most one subdivision path can contain v. So, there is at least one connected component T_1' of $T' - \{v\}$ that contains none of the end nodes of such a subdivision path. Let e be the edge that connects T_1' to v in T'. Each subdivision path incident to a node in T_1' and a node that is not in T_1' must contain an edge from S whose projection path contains e and another edge incident to v. If T_1' contains b black nodes and w white nodes, there are $b(3-w) + w(3-b)$ such paths. As we have $1 \leq b + w \leq 4$, there are at least 3 such paths. This contradicts the precondition.

Now consider the case that $\deg_{T'}(v) \geq 3$ and v is a node – say a black node as illustrated in Fig. 1 (right). Let T_1' be a connected component of $T' - \{v\}$ that contains at least one black node. Let e be the edge that connects T_1' to v in T'. If T_1' contains b black nodes and w white nodes, there are $b(3-w) + w(3-b-1)$ subdivision paths between nodes in T_1' and nodes not in $T_1' + \{v\}$. As we have $1 \leq b + w \leq 3$, there are at least 3 such paths and thus at least 3 projection paths containing e and another edge incident to v – a contradiction.

If $\deg_{T'}(v) = 2$, tree T' is a path. Without loss of generality, we can assume that v is the third node in the path. Then, we can use the same argumentation as in the previous case. □

3 The 2-level cactus model

Throughout the rest of this paper, let $G = (V, E)$ be an undirected connected multi-graph. Even though there might be several edges of G that are incident to the same two vertices v and w, we denote each of them by $\{v, w\}$. For two subsets $S, T \subset V$ let $E(S,T) := \{\{s,t\} \in E|\ s \in S,\ t \in T\}$ denote the set of edges in E that are incident to a vertex in S and to a vertex in T and let $c(S,T) := |E(S,T)|$ be the cardinality of this set. A non-empty proper subset S of V *induces* the *cut* $E(S, \overline{S})$ of G. A *2-cut* is a cut of cardinality 2. Let $\lambda = \min_{\emptyset \subsetneq S \subsetneq V} c(S, \overline{S})$ denote the minimum cardinality of a cut of G.

A cut that is induced by S *divides* a subset T of V if none of the two sets $S \cap T$ and $\overline{S} \cap T$ is empty. Two cuts $E(S, \overline{S})$ and $E(T, \overline{T})$ are *crossing*, if none of the four *corner sets* $S \cap T$, $S \cap \overline{T}$, $\overline{S} \cap T$, and $\overline{S} \cap \overline{T}$ is empty. If not, they are *parallel*. A cut that is induced by a corner set is called a *corner cut*. $G' = (V', E')$ results from G by *shrinking* a subset S of V, if $V' = (V \setminus S)\dot{\cup}\{v_S\}$ and $E' = (E \setminus E(V,S)) \cup \{\{v, v_S\};\ \{v,s\} \in E(\overline{S}, S)\}$, that is every incidence of an edge in E to a vertex in S is replaced by an incidence to v_S, omitting loops. Two cuts C and C' *induce the quotient graph* that results from G by shrinking the four corner sets.

For $\lambda \geq 3$, Dinitz and Nutov developed in [5] a compact model for the λ and $(\lambda+1)$-cuts of a graph G, called the 2-level cactus model. In the following, we briefly sketch this model and summarize those properties that we use to prove the planarity of the 2-level cactus.

Generally, a *model* for a family F of cuts of G is a triple $(\mathcal{G}, \varphi, \mathcal{F})$ such that the model graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is an undirected multi-graph, $\mathcal{F}$ is a set of cuts of $\mathcal{G}$, and $\varphi : V \longrightarrow \mathcal{N}$ is a mapping with $\varphi^{-1}(\mathcal{F}) = F$, where $\varphi^{-1}(E(S, \overline{S}))$ is defined to be $E(\varphi^{-1}(S), \overline{\varphi^{-1}(S)})$. We say that a cut $\mathcal{C}$ of $\mathcal{G}$ *models* the cut $\varphi^{-1}(\mathcal{C})$. The elements of $\mathcal{N}$ are called nodes and a *node* $\nu \in \mathcal{N}$ with $\varphi^{-1}(\nu) = \emptyset$ is called an *empty node.*

In [5], models for the set F of λ and $(\lambda+1)$-cuts are built in the following way. Set F is divided into the set of those λ-cuts not crossing any other λ-cut in F, called the set of all *basic cuts* F^{bas}, the set of all remaining cuts in F that do not cross any cut in F^{bas}, called the set of all *local cuts* F^{loc}, and the set of all cuts that cross at least one of the cuts in F^{bas}, called the set of all *global cuts* F^{glb}. F^{bas} can be modeled by the tree $\mathcal{T}^{\text{bas}}$. See Fig. 2a,b for an illustration to the construction of such a tree.

For a node $\nu \in \mathcal{N}$, let $V_\nu^1, \ldots, V_\nu^k$ be the subsets of V that are mapped on the connected components of $\mathcal{T}^{\text{bas}} - \nu$, and let G_ν be the *quotient graph of* ν that is the graph resulting from G by shrinking the sets $V_\nu^1, \ldots, V_\nu^k$ into a single vertex each. Given any subset $\widetilde{F}^{\text{loc}}$ of the local cuts, it can be partitioned into F_ν^{loc}, $\nu \in \mathcal{N}$, where F_ν^{loc} is defined as the set of those cuts in $\widetilde{F}^{\text{loc}}$, that do not divide any of the sets V_ν^i. Assume, for all $\nu \in \mathcal{N}$, there is a model for the cut set F_ν^{loc} (they can be considered as cuts of the graph G_ν). Then, a model for $F^{\text{bas}} \cup \widetilde{F}^{\text{loc}}$ is built by "implanting" for every node $\nu \in \mathcal{N}$ with $F_\nu^{\text{loc}} \neq \emptyset$ a model $\mathcal{G}_\nu$ – called *local model* – for F_ν^{loc} into $\mathcal{T}^{\text{bas}}$. See Fig. 2c,d,e for illustration. The nodes μ of $\mathcal{G}_\nu$ with $\varphi^{-1}(\mu) = V_\nu^i$, for some i, are called *halo nodes.* We also call the vertices of G_ν that correspond to the shrunken sets V_ν^i *halo vertices.* Finally, depending on whether λ is even or odd, the global cuts and the local cuts in $F^{\text{loc}} \setminus \widetilde{F}^{\text{loc}}$ are added in a suitable way.

Odd connectivity. Every global cut is modeled by a 2-cut of $\mathcal{T}^{\text{bas}}$. Moreover, for a 2-cut $\{e_1, e_2\}$ in $\mathcal{T}^{\text{bas}}$, let $p_{e_1e_2}$ be the set of edges on the path between e_1 and e_2 in $\mathcal{T}^{\text{bas}}$, with e_1 and e_2 included. If cut $\{e_1, e_2\}$ models a global cut, then, any 2-cut $\{e_1', e_2'\} \subset p_{e_1e_2}$ does also model a cut in $F^{\text{loc}} \cup F^{\text{glb}}$ ([5], Lemma 5.1). The cuts that are modeled by a 2-cut of $\mathcal{T}^{\text{bas}}$ are

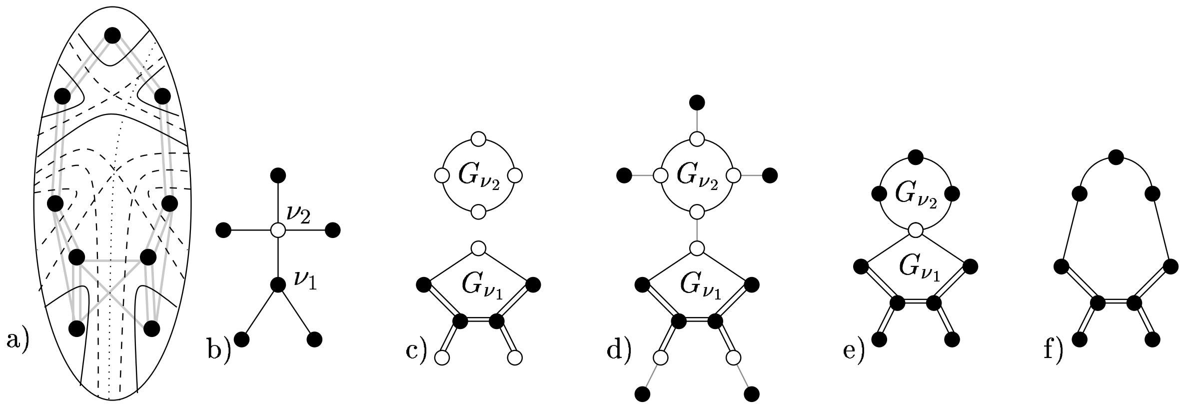


Figure 2: a) A set of cuts of a graph (the graph edges are shown grey). Continuous curves indicate basic cuts and dashed curves indicate local cuts. The dotted curve represents a global cut. b) Tree $\mathcal{T}^{\text{bas}}$ and c) two local models $\mathcal{G}_{\nu_1}$ and $\mathcal{G}_{\nu_2}$ that d) are implanted instead of ν_1 and ν_2 into $\mathcal{T}^{\text{bas}}$. White nodes are empty nodes. The remainders of the edges in $\mathcal{T}^{\text{bas}}$ are grey. e) They can be contracted at the end of the implantation process. f) The opening of the white halo node in e).

called *degenerate* cuts. Let P be the set of inclusion-maximal sets $p_{e_1e_2}$, such that $\{e_1, e_2\}$ models a $(\lambda + 1)$-cut of G. The elements of P are called *generating paths.*

Lemma 2 ([6] Lemma 5.4 and proof of Lemma 5.5)

1. *Any two generating paths have at most one edge in common.*
2. *Let $\lambda > 3$. Let $p \in P$ and $\{v, w\} \in p$. Except $\{v, w\}$, there are at most two edges incident to v in $\mathcal{T}^{\text{bas}}$ such that there exists a generating path containing $\{v, w\}$ and such an edge.*

For the local models, we consider only $\lambda > 3$. Let $\widetilde{F}^{\text{loc}}$ be the set of non-degenerate local cuts plus the set of corner cuts of non-degenerate local cuts. Let ν be a node of $\mathcal{T}^{\text{bas}}$.

Lemma 3 ([5] Lemma 5.4) *Let C, C' be two crossing non-degenerate cuts. Then, the quotient graph induced by C and C' is a simple cycle with $\frac{\lambda+1}{2}$ edges between adjacent vertices.*

From this lemma and the fact that for $\lambda > 3$ no degenerate cut in $\widetilde{F}^{\text{loc}}$ crosses another cut in $\widetilde{F}^{\text{loc}}$, Dinitz and Nutov conclude in [5] that there exists a tree of cycles which is a suitable local model for each node ν with $F_\nu^{\text{loc}} \neq \emptyset$. Implanting these local models into $\mathcal{T}^{bas}$ results in a cactus tree type graph $\mathcal{G}$, which will be called *2-level cactus.*

To make the generating paths, and thus the 2-cuts of $\mathcal{G}$ modeling the global cuts, visible in a drawing of the 2-level cactus $\mathcal{G}$, let us extend $\mathcal{G}$ as follows. For each generating path p, consider the corresponding sequence of edges in $\mathcal{G}$ and add an auxiliary edge e_p connecting the first and last end node of this sequence to $\mathcal{E}$. We call the result *extended 2-level cactus* $\mathcal{G}^+$. The set of edges on the shortest path in $\mathcal{G}$ between the two end nodes of e_p is called the *projection path* of e_p. Note, that it follows from Lemma 6 Item 1 in Sect. 4 that the projection paths are unique up to multiple edges.

Even connectivity. For a node ν of $\mathcal{T}^{\text{bas}}$, the local model $\mathcal{G}_\nu$ is either a simple cycle or it can be described as a tree $\mathcal{T}_\nu$ plus the halo nodes, where each halo node is connected by two additional edges to $\mathcal{T}_\nu$. In the latter case, the following property holds. For a halo node μ, let p_μ be the set of edges on the path in $\mathcal{T}_\nu$ between the two end nodes of the two edges incident to μ, and let P be the set of these paths. Lemma 5.6 in [5] gives the following properties of the paths in P.

Lemma 4

1. *Two elements of P have at most one edge in common.*

2. *An edge of $\mathcal{T}_\nu$ is contained in at most two elements of P.*

Let C be a global cut. Then, there is exactly one non-empty node ν of $\mathcal{T}^{\text{bas}}$ such that C divides $\varphi^{-1}(\nu)$. Moreover, C contains one or two sets of $\frac{\lambda}{2}$ edges corresponding to an edge of a cycle that was implanted instead a node μ in the neighborhood of ν in $\mathcal{T}^{\text{bas}}$. To model these cuts, the halo node of $\mathcal{G}_\nu$ that was implanted into edge $\{\nu, \mu\}$ of $\mathcal{T}^{\text{bas}}$ is "opened", which means the halo node is deleted and corresponding pairs of edges are merged. See Fig. 2f for illustration.

Suppose now that every edge in the tree $\mathcal{T}_\nu$ of a local model that is contained in k elements of P is replaced by $3-k$ parallel edges and that every tree edge of $\mathcal{T}^{\text{bas}}$ that was not contracted after implanting is replaced by a pair of parallel edges. Then, the 2- and 3-cuts of the resulting 2-level cactus $\mathcal{G}$ model the λ- and $(\lambda+1)$-cuts of G, respectively. This is the 2-level cactus tree model for the even case.

4 Planarity of the 2-level cactus

Odd connectivity. In Sect. 2, we have shown that a tree with additional edges, such that the projection paths fulfill the properties of Lemma 2 is planar. We cannot apply Item 2 of Lemma 2 for $\lambda = 3$ and, it turns out that in this case the 2-level cactus is not planar in general. For example (see Fig. 3a,b), if we take $G = K_4$, then the extended 2-level cactus is K_5.

In case of $\lambda \geq 5$, the tree $\mathcal{T}^{\text{bas}}$ extended by the above auxiliary edges is planar, by Lemma 2. However, when $\lambda = 5$, implanting the local model might destroy planarity. An example is shown in Fig. 3c,d. The 2-level cactus in Fig. 3d contains a subdivision of a K_5 with the 5 white nodes as nodes of the K_5.

To show that for $\lambda \geq 7$, implanting the local models preserves planarity, we will use the following trick. First, we will consider all cycles of $\mathcal{G}$ and modify them in $\mathcal{G}^+$. Then, we show that Lemma 1 can be applied to the thus modified extended 2-level cactus $\mathcal{G}^+$. Second, we will restore the original extended 2-level cactus $\mathcal{G}^+$, and show that planarity is preserved. We start with the following observation (see also [6] Lemma 5.2).

Lemma 5 *Let e_1 and e_2 be two edges of a generating path. Let V_1, V_2 and V' be the set of vertices of G that are mapped on the connected components of $\mathcal{G}-\{e_1, e_2\}$ such that V' induces the $(\lambda+1)$-cut that is modeled by $\{e_1, e_2\}$. Then, there are exactly $\frac{\lambda-1}{2}$ edges connecting V_1 and V_2.*

Proof: Let ϵ, ϵ_1 and ϵ_2 be the number of edges between V_1 and V_2, V_1 and V', and V_2 and V', respectively. Then we have $\epsilon + \epsilon_1 = \lambda$, $\epsilon + \epsilon_2 = \lambda$ and $\epsilon_1 + \epsilon_2 = \lambda + 1$. Thus $\epsilon = \frac{\lambda-1}{2}$. $\square$

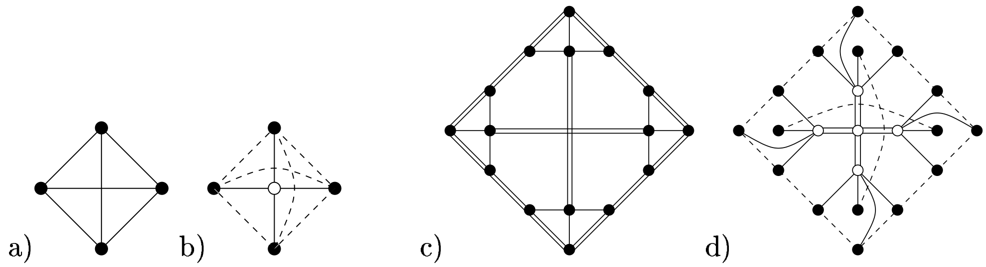


Figure 3: Example of planar graphs of a) connectivity 3 and c) connectivity 5 and their non-planar extended 2-level cacti b) and d). φ is represented by the location of the vertices and nodes. In the extended 2-level cacti, white nodes are empty nodes, black edges are tree-edges, dashed edges represent the generating paths, and double edges are those of the implanted local model.

An edge of an implanted local model is called a tree-edge if it is contained in a 2-cycle, and it is called a cycle-edge if it is contained in a cycle of length greater than 2. Applying Lemma 3 and Lemma 5, we can show

Lemma 6

1. *A projection path contains at most one edge of each simple cycle.*
2. *Let $\lambda \geq 5$. Each cycle-edge is contained in at most one projection path.*
3. *Let $\lambda \geq 7$. Each tree-edge of an implanted local model is contained in at most two projection paths.*

Proof:

1. Let p be a projection path that contains the edges $\{v_1, v_2\}, \ldots, \{v_{l-1}, v_l\}$ of a simple cycle $c = v_1, \ldots, v_k$. Suppose p contains more than one edge of c. As p takes the shortest path on c, it follows that $k > l$. Let V_1, V_2, V_l, and V_k be the subsets of V that are mapped on the connected components of $\mathcal{G} - \{\{v_1, v_2\}, \{v_{l-1}, v_l\}, \{v_l, v_{l+1}\}, \{v_k, v_1\}\}$ such that a vertex of V_i is mapped on v_i. Then, the two cuts that are induced by $V_1 \cup V_2$ and by $V_2 \cup V_l$, respectively, are crossing $(\lambda + 1)$-cuts. By Lemma 3, there are no edges between V_1 and V_l. On the other hand, let us consider the generating path $e_1, e_2, \ldots, e_r$, corresponding to p. By Lemma 5 applied to the pair $\{e_1, e_r\}$, there are at least $\frac{\lambda-1}{2}$ edges between V_1 and V_l, a contradiction.

2. Let c be the set of edges of a simple cycle in a local model $\mathcal{G}_\nu$ and let $\{v_1, v_2\}$ be an edge in c. For $i = 1, 2$, let $\mathcal{N}_i$ be the set of nodes in the connected component of $\mathcal{G} - c$ that contains v_i, and let V_i be the subset of V that is mapped on $\mathcal{N}_i$. Suppose edge $\{v_1, v_2\}$ is contained in at least two projection paths p_1 and p_2. Let $e_i^j, i, j = 1, 2$, be the edge on p_i with end nodes in $\mathcal{N}_j$ that was incident to ν in $\mathcal{T}^{\text{bas}}$ before implanting $\mathcal{G}_\nu$. By Lemma 2 Item 1, we know that $e_1^1 \neq e_2^1$ or $e_1^2 \neq e_2^2$, so we may assume $e_1^2 \neq e_2^2$. Let v_{p_1} and v_{p_2} be the end-nodes of p_1 and p_2 in $\mathcal{N}_2$. For $i = 1, 2$, let $e_i \in p_i$ be the edge incident to v_{p_i}, and let V_{p_1} and V_{p_2} be the subsets of V_2 that are mapped on the connected components of $\mathcal{G} - \{e_1, e_2\}$ that contain v_{p_i}. See Fig. 4a for illustration.

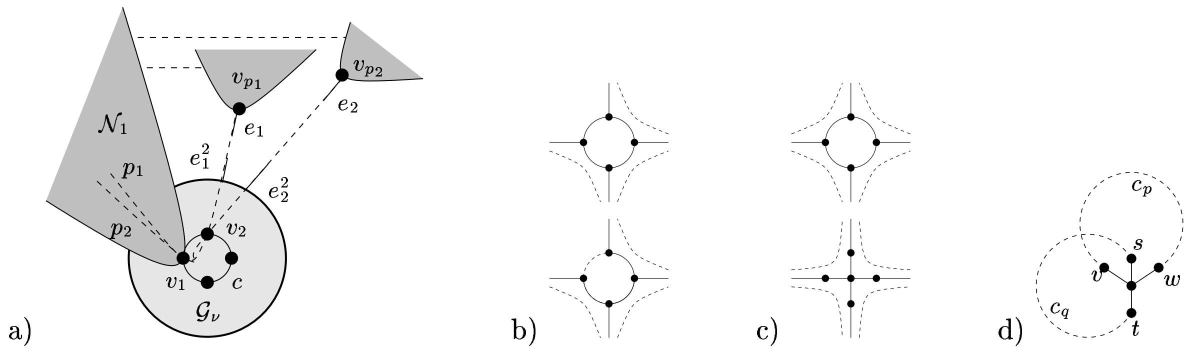


Figure 4: a) Illustration of the proof of Lemma 6 Item 2. b), c) Modification of the cycles in the local model b) in case not all and c) in case all cycle edges are contained in a projection path, that is indicated by a dashed curve. d) Illustration of the final proof of planarity in the odd case.

Then, by Lemma 5 there are $\frac{\lambda-1}{2}$ edges between V_1 and V_{p_1} and between V_1 and V_{p_2}, respectively, and thus, there are at least $\lambda - 1$ edges connecting V_1 and V_2. On the other hand, by Lemma 3, $c(V_1, V_2) = \frac{\lambda+1}{2}$, contradicting the precondition that $\lambda \geq 5$.

3. Let e, e' be a 2-cycle of a local model, and let ϵ be the number of projection paths containing either e or e'. Let V_1 and V_2 be the sets that are mapped on the connected components of $\mathcal{G} - \{e, e'\}$. With the same argumentation as in Item 2, per projection path that contains e or e', there are at least $\frac{\lambda-1}{2}$ distinct edges between V_1 and V_2. As $\{V_1, V_2\}$ induces a $(\lambda+1)$-cut, ϵ must fulfill the inequality $\epsilon \cdot \frac{\lambda-1}{2} \leq \lambda + 1$. For $\lambda \geq 7$, it follows $\epsilon < 3$. □

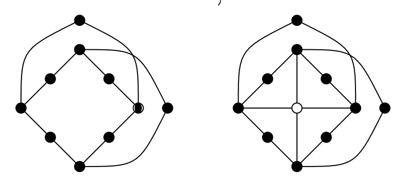
Let us consider the following modifications on the cycles. Let c be a simple cycle. If c contains edges that are not contained in a projection path, we choose one of these edges and declare it to be an auxiliary edge. If each edge of c is contained in a projection path, we replace c by a star. That means, we delete all edges of c, add an additional vertex v_c to $\mathcal{G}^+$ and connect all vertices of c to v_c. See Fig. 4b,c for illustration. We call the so modified graph $\mathcal{G}'$ and we denote by $S' \supseteq S$ the set of all auxiliary edges of $\mathcal{G}'$ and by $\mathcal{T}'$ the spanning tree of $\mathcal{G}'$ that is induced by the edges that are not in S'.

The only case that e is contained in the projection path of an edge in $S' \setminus S$ is if e is a cycle-edge of a local model. Thus, by the above lemma and by Lemma 2, $\mathcal{G}'$ with its spanning tree $\mathcal{T}'$ fulfills the condition of Lemma 1. Hence, $\mathcal{G}'$ is planar.

It remains to show that we can restore the deleted cycle-edges into $\mathcal{G}'$ without producing edge crossings. Let c be the set of edges of a cycle of $\mathcal{G}$ that we have replaced by a star. Let $\{v, w\}$ be an edge of c. We will show, that in any embedding of $\mathcal{G}'$, edges $\{v, v_c\}$ and $\{w, v_c\}$ are neighboring in the cyclic order around v_c. Suppose not: then v and w divide the adjacency list of v_c into two non-empty parts S and T. As the graph induced by $c - \{v, w\}$ is connected, there is an edge $\{s, t\}$ in c with $s \in S$ and $t \in T$. Let p be the projection path that contains edge $\{v, w\}$. Let $e_p \in S$ be the auxiliary edge that represents p and let p' be the projection path of e_p in $\mathcal{G}'$. Similarly, let q be the projection path that contains edge $\{s, t\}$, let e_q be the corresponding edge in S and q' the corresponding path in $\mathcal{G}'$. Then, on one hand, the two cycles induced by the edge sets $c_p = p' \cup e_p$ and $c_q = q' \cup e_q$ are vertex disjoint except v_c: If not, the graph that is induced by $c_p \cup c_q$ has at least four faces and thus, the graph induced by $p' \cup q'$, which is a sub graph of tree $\mathcal{T}'$, would contain a cycle. But on the other hand, there is an edge of c_q in the inner part of c_p and an edge of c_q in the outer part of c_p, contradicting the planarity of $\mathcal{G}'$. See Fig. 4d for illustration.

Even connectivity. The corollary to MacLane's planarity criterion shows that a tree with additional edges, such that the projection paths fulfill the properties of Lemma 4 is planar. Thus, the local models are planar. Because the local models are stuck together in a tree-structure, proceeding from a leaf, we can open the halo nodes without losing planarity.

Connectivities One and Two. Finally, for completeness, we consider the prototypes of the 2-level cactus. The cut model of the inclusion minimal 1- and 2-cuts was introduced in [8]; it is a tree of edges and cycles and thus, is planar. In the case $\lambda = 2$, a cut model for all minimum and minimum+1 cuts is described in [3]; it is constructed in a similar way as the 2-level cactus in the even case. In general, it is not planar. An example is a K_4 with every edge broken into two by a new vertex. This graph and its non-planar 2-level cactus is shown on the right. We summarize this section in the following theorem.



Theorem 2 *For a multi-graph with edge-connectivity $\lambda = 1$ or with even edge-connectivity $\lambda \geq 4$ the 2-level cactus is planar, for odd edge-connectivity $\lambda \geq 7$ the extended 2-level cactus is planar, and this list is exact.*

5 Embedding with specified faces

For an extended planar 2-level cactus (the odd case), we want to examine whether it can be embedded such that

(a) the auxiliary edges are all on the same (outer) face, and

(b) each auxiliary edge together with its projection path is the boundary of a face.

In this case, $\mathcal{G}$ is completely contained inside a planar drawing of $\mathcal{G}^+$, and the projection path that corresponds to an auxiliary edge can be discovered more easily. Moreover, the interior of each simple cycle of a local model will then be empty. In the even case we wish that:

(a) the halo nodes of any local model $\mathcal{G}_\nu$ are all on the same (outer) face of $\mathcal{G}_\nu$.

In this case, the tree structure of $\mathcal{T}^{\text{bas}}$ can be better recognized in a planar drawing of $\mathcal{G}$.

In the rest of this section, we assume w.l.o.g. that those edges of $\mathcal{G}^+$ or $\mathcal{G}_\nu$ that are not contained in a cycle, i.e., the bridges of $\mathcal{G}^+$ or $\mathcal{G}_\nu$ are contracted. Let us consider the general problem: Given a planar connected graph $G = (V, S \dot\cup B)$ such that every edge of G is contained in a cycle and such that $T = (V, B)$ is a spanning tree of G, is there an embedding of G such that

(a) all edges in S are on the same face, and

(b) for each edge $e \in S$, the cycle on the set $e \cup p_e$ is the boundary of a face?

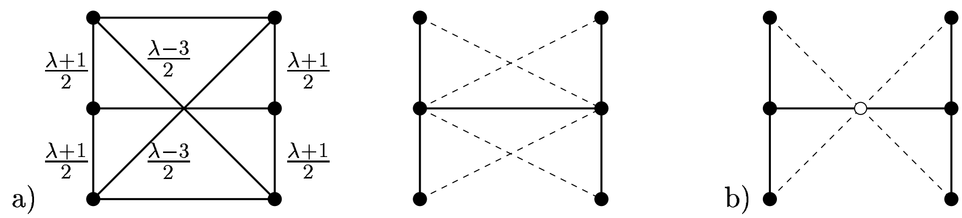


Figure 5: a) A graph of odd connectivity $\lambda > 3$ and its 2-level cactus $\mathcal{G}$. Vertical edges refer to $\frac{\lambda+1}{2}$ edges, diagonal edges refer to $\frac{\lambda-3}{2}$ edges, and horizontal edges refer to one edge each. There is no embedding of $\mathcal{G}$ with the property that, for every simple cycle c that is generated by a dashed edge and its projection path, there is a face f such that c is part of the boundary of f. b) Splitting the edge that is contained in more than 2 generating paths achieves this property.

We will first prove that

Lemma 7 *Any embedding of G that fulfills property (a) also fulfills property (b) and vice versa.*

Then, we will give examples for which the above mentioned properties of the (extended) planar 2-level cactus are not true in general, but we will show that in the odd case, the extended 2-level cactus $\mathcal{G}^+$ can be modified in such a way, that it has an embedding in which

(c) for each auxiliary edge e, the cycle on the edge set $e \cup p_e$ is a part of the boundary of some face f.

Finally, in the even case we can conclude that Property (a) is true for each 2-connected component of any local model.

Proof: of Lemma 7.

$(a) \Rightarrow (b)$**:** Let $e \in S$ be an edge and let p_e be the set of edges on its projection path. Suppose the cycle on the edge set $c = \{e\} \cup p_e$ is not the boundary of a face of G. Then, there is an edge inside and outside of c. Both of them lie on a cycle. Without loss of generality, these cycles do not contain e and are completely contained inside c or outside c (including c). These two cycles both contain an edge of S. Thus, there is an edge of S inside c and another one outside c, contradicting (a).

$(b) \Rightarrow (a)$**:** From Euler's formula we can conclude that there are $|S| + |B| - |V| + 2 = |S| + |B| - (|V| - 1) + 1 = |S| + 1$ faces and $|S|$ faces are bounded by the cycle on the edge set $\{e\} \cup p_e$, $e \in S$. Totally, these faces are incident to each edge in S exactly once. Every edge in S is incident to exactly two faces. Thus, the remaining face is incident to all edges in S. □

We now consider an extended 2-level cactus, the odd case. As there might be edges that are contained in four projection paths, not all 2-level cacti can fulfill property (b). For example, see Fig. 5a.

Now, let us modify the 2-level cactus in the following way: Each edge e in $\mathcal{T}^\lambda$ that is contained in more than two projection paths is subdivided by an empty node. Let e_p be

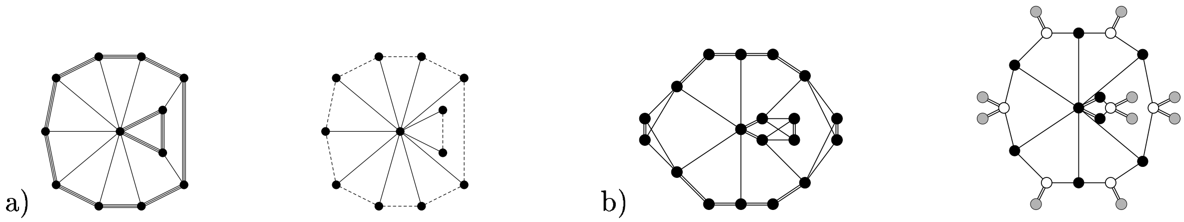


Figure 6: a) A graph of odd connectivity $\lambda = 7$ and its 2-level cactus $\mathcal{G}$. There is no embedding of $\mathcal{G}$ such that all dashed auxiliary edges are on the same face. b) A graph of even connectivity $\lambda = 4$ and its 2-level cactus $\mathcal{G}$. All non-grey nodes belong to a single local model $\mathcal{G}_\nu$. There is no embedding of $\mathcal{G}_\nu$ such that its halo nodes (the white nodes) are on the same face. With an increasing number of vertices, both examples can be extended to arbitrary odd or even connectivity, respectively.

the auxiliary edge that represents a projection path that contains e. If e_p is adjacent to e, replace the end node of e_p that is incident to e by the newly inserted empty node. See Fig. 5b for illustration. Note, that the resulting 2-level cactus $\dot{\mathcal{G}}^+$ is still a suitable model for the minimum and minimum+1 cuts of G, and its size is still linear in $|V|$. By Lemma 2 Item 2, after these modifications, the 2-level cactus has the property that each edge is contained in at most two projection paths. By MacLane's criterion, Property (b) (and thus, Property (a)) is true for the biconnected components of $\dot{\mathcal{G}}^+$ and thus, Property (c) is true for $\dot{\mathcal{G}}^+$. However, Properties (a) and (b) are not true for arbitrary extended planar 2-level cacti. For example, see Fig. 6a.

Again by MacLane's criterion, we obtain that in the even case the biconnected components of a local model fulfill Property (a), but also in this case, Property (a) need not to be true for arbitrary local models (for example, see Fig. 6b).

Notice that, with an increasing number of vertices, the above examples can be extended to have any greater odd or even connectivity.

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