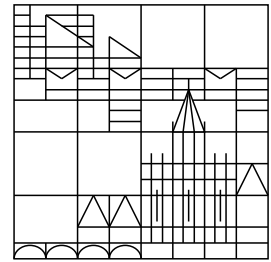


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ABSTRACT. We investigate the equivariant intersection cohomology of a toric variety. Considering the defining fan of the variety as a finite topological space with the subfans being the open sets (that corresponds to the “toric” topology given by the invariant open subsets), equivariant intersection cohomology provides a sheaf (of graded modules over a sheaf of graded rings) on that “fan space”. We prove that this sheaf is a “*minimal extension sheaf*”, i.e., that it satisfies three relatively simple axioms which are known to characterize such a sheaf up to isomorphism. In the verification of the second of these axioms, a key role is played by “equivariantly formal” toric varieties, where equivariant and “usual” (non-equivariant) intersection cohomology determine each other by Künneth type formulæ. Minimal extension sheaves can be constructed in a purely formal way and thus also exist for non-rational fans. As a consequence, we can extend the notion of an *equivariantly formal fan* even to this general setup. In this way, it will be possible to introduce “virtual” intersection cohomology for equivariantly formal non-rational fans.

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Introduction

The relation between algebraic geometry and topology is especially close in the case of smooth compact toric varieties: The canonical homomorphism from the Chow-theoretic intersection ring to the integral homology intersection ring is an isomorphism. Even when we allow quotient singularities, the result remains valid if we replace integer with rational coefficients.

As is well known¹, toric varieties admit a description in terms of combinatorial-geometric objects: cones, fans, and polytopes. In the case considered above, these combinatorial-geometric objects allow us to compute the intersection ring explicitly. In the opposite direction, properties of the intersection ring known from algebraic geometry yield consequences for the describing geometric object. A most striking application is Stanley's beautiful proof – using the hard Lefschetz theorem for projective toric varieties – for the necessity of the conditions that characterize the face numbers of simplicial polytopes, conjectured by McMullen (see, e.g., [Fu, §5.6]). That example should suffice to explain the great interest of toric varieties in studying the combinatorial geometry of fans – as a variation of the title of Hirzebruch's landmark book, one might call this an *Application of Topological Methods from Algebraic Geometry to Combinatorial Convexity*.

Stated in terms of the defining fan, a toric variety $X = X_{\Delta}$ is compact and \mathbb{Q} -smooth (i.e., a rational homology manifold) if the fan Δ is *complete* and *simplicial*. When studying the (rational) cohomology algebra of such a toric variety X , a crucial role is played by the (rational) *Stanley-Reisner ring*

$$S_{\Delta} := \mathbb{Q}[\Delta^1]/I,$$

where Δ^1 is the set of *rays* (i.e., one-dimensional cones) of Δ , and $\mathbb{Q}[\Delta^1]$ is the polynomial algebra $\mathbb{Q}[(t_{\rho})_{\rho \in \Delta^1}]$ on free generators t_{ρ} that

¹For the basic theory of toric varieties, we refer the reader to the pertinent monographs [Ew], [Fu], or [Oda₁].

are in one-to-one correspondence to the rays ρ , and where I is the homogeneous ideal

$$I := \left\langle \prod_{j=1}^k t_{\rho_j} ; \sum_{j=1}^k \rho_j \notin \Delta \right\rangle$$

generated by those square-free monomials where the rays corresponding to the factors do not span a cone of the fan. The theorem of Jurkiewicz and Danilov [Oda₁, 3.3, p. 134] describes the (rational) cohomology algebra $H^\bullet(X)$ as the quotient of S_Δ modulo the ideal \tilde{J} , the image of the homogeneous ideal

$$J := \left\langle \sum_{\rho \in \Delta^1} \chi(v_\rho) \cdot t_\rho ; \chi \in \text{Hom}(\mathbb{T}, \mathbb{C}^*) \right\rangle \subset \mathbb{Q}[\Delta^1],$$

where v_ρ denotes the unique primitive lattice vector spanning the ray ρ , and \mathbb{T} , the acting torus. We thus have isomorphisms

$$(\dagger) \quad H^\bullet(X_\Delta, \mathbb{Q}) \cong S_\Delta / \tilde{J} \cong \mathbb{Q}[\Delta^1] / (I + J).$$

The ideals I and J are most naturally interpreted using the one-to-one correspondence between rays and \mathbb{T} -invariant irreducible divisors on X : The monomials generating I correspond to sets of divisors with empty intersection, and the generators of J , to \mathbb{T} -invariant principal divisors. In fact, identifying a product of generators with the intersection of the corresponding divisors, this correspondence extends to isomorphisms between the rings $\mathbb{Q}[\Delta^1]/(I + J) \cong H^\bullet(X)$ and the rational Chow ring of X . – We note that the ideal I , and hence the Stanley-Reisner ring, depends only on the combinatorial structure of the fan.

The isomorphism $\mathbb{Q}[\Delta^1]/(I + J) \cong H^\bullet(X)$ is a morphism of graded rings that multiplies degrees by two. As a consequence, the odd-dimensional cohomology of X vanishes. Furthermore, it turns out that the Betti numbers are explicitly computable in terms of combinatorial data of the fan.

Besides of being a graded algebra that encodes the combinatorics of the fan Δ , the Stanley-Reisner ring S_Δ has a second, more geometric, interpretation. To give it, we consider, for a completely arbitrary fan Δ , the graded algebra $\mathcal{A}^\bullet(\Delta)$ of (*rational*) Δ -*piecewise polynomial functions* on the support $|\Delta|$ of the fan², i.e., functions $f: |\Delta| \rightarrow \mathbb{Q}$ whose restriction to each cone $\sigma \in \Delta$ extends to a – unique – polynomial function f_σ on the linear span $V_\sigma := \sigma + (-\sigma)$ of σ . Since the algebra

²Unless otherwise stated, cones and fans are always considered as subsets of the *rational* vector space $N_\mathbb{Q}$ generated by the one parameter subgroups of the torus.

of polynomial functions on V_σ is $S^\bullet(V_\sigma^*)$, the symmetric \mathbb{Q} -algebra of the dual vector space V_σ^* , we arrive at the formal definition

$$\mathcal{A}^\bullet(\Delta) := \{f: |\Delta| \rightarrow \mathbb{Q}; \forall \sigma \in \Delta \exists f_\sigma \in S^\bullet(V_\sigma^*) : f|_\sigma = f_\sigma|_\sigma\} .$$

In addition to this graded \mathbb{Q} -algebra of piecewise polynomial functions, we also have to consider the graded \mathbb{Q} -algebra

$$A^\bullet = S^\bullet(V^*) \cong \mathbb{Q}[u_1, \dots, u_n]$$

of (globally) polynomial functions on $V := M_{\mathbb{Q}}$, where u_1, \dots, u_n denotes a basis of $V^* := M_{\mathbb{Q}}$ (see section 0 for notations). The obvious restriction homomorphism $A^\bullet \rightarrow \mathcal{A}^\bullet(\Delta)$, $f \mapsto f|_{|\Delta|}$ of graded algebras is injective if the fan Δ contains at least one n -dimensional cone, thus making A^\bullet a graded subalgebra of $\mathcal{A}^\bullet(\Delta)$.

In the sequel, the algebra $\mathcal{A}^\bullet(\Delta)$ will be used without any restriction on the fan. But assuming again that Δ is complete and simplicial for the moment, we obtain a homomorphism $\mathbb{Q}[\Delta^1] \rightarrow \mathcal{A}^\bullet(\Delta)$ of graded \mathbb{Q} -algebras by associating to each generator t_ρ of $\mathbb{Q}[\Delta^1]$ the unique piecewise linear function on $|\Delta|$ that takes the value 1 at the vector v_ρ and vanishes on all other rays. This homomorphism is surjective, and its kernel coincides with the ideal I , so we get the identification

$$(*) \quad S_\Delta \xrightarrow{\cong} \mathcal{A}^\bullet(\Delta)$$

that provides the geometric interpretation of the Stanley-Reisner ring.

Under that identification, the ideal \tilde{J} in S_Δ corresponds to the homogeneous ideal $\mathfrak{m} \cdot \mathcal{A}^\bullet(\Delta)$ generated by the (globally) linear functions. (In the (sub-)algebra A^\bullet , the ideal generated by linear functions is the unique homogeneous maximal ideal $\mathfrak{m} = A^{>0}$ of polynomial functions vanishing at the origin.) Using the above isomorphism (\dagger), we may look at $H^\bullet(X)$ as quotient modulo that ideal. We can thus describe the cohomology algebra of a compact, \mathbb{Q} -smooth toric variety X_Δ by an isomorphism

$$H^\bullet(X_\Delta) \cong (A^\bullet/\mathfrak{m}) \otimes_{A^\bullet} \mathcal{A}^\bullet(\Delta) .$$

On the other hand, the ring $\mathcal{A}^\bullet(\Delta)$ itself admits a direct topological interpretation: There is a natural isomorphism

$$(**) \quad H_{\mathbb{T}}^\bullet(X_\Delta) \xrightarrow{\cong} \mathcal{A}^\bullet(\Delta)$$

with the *equivariant cohomology algebra* of X_Δ (see, e.g., [BriVe], [GoKoMPH], and the first section of this article). Hence, the two isomorphisms (*) and (**) together yield a third, topological, interpretation of the Stanley-Reisner ring.

The case of *non-simplicial* fans is much more elusive, as the above results do not remain valid. In fact, the cohomology becomes quite

a delicate object that is difficult to compute. In particular, the Betti numbers are not always determined by the combinatorial type of the fan.

Fortunately, for compact toric varieties with arbitrary singularities, intersection cohomology (with respect to “middle perversity”) is known to behave much better than “usual” cohomology in many respects, e.g., Poincaré duality still holds, there is a Hard Lefschetz Theorem in the projective case, and intersection cohomology Betti numbers are combinatorial invariants. With the crucial role of the Stanley-Reisner ring for cohomology and with its various interpretations in mind, when looking for a substitute on the fan-theoretic side that could play a similar role for intersection cohomology, we were lead to investigate the properties of the *equivariant intersection cohomology* $IH_{\mathbb{T}}^{\bullet}(X_{\Delta})$. This graded vector space is endowed with a natural structure of a graded module over the graded ring $H_{\mathbb{T}}^{\bullet}(X_{\Delta})$.

The aim of this article is to find some analogue of the combinatorial description provided by the above isomorphism (**) in the case of a completely arbitrary fan Δ , replacing equivariant cohomology with equivariant intersection cohomology and keeping possible generalizations to the case of non-rational fans in mind. It is convenient to adopt a sheaf-theoretic point of view: The finite family consisting of all \mathbb{T} -invariant open subsets of the toric variety X is a topology on X , and associating to such a “toric” open subset $U \subset X$ the graded vector space $IH_{\mathbb{T}}^{\bullet}(U)$ yields a presheaf on this topological space. Since open toric subvarieties of $X = X_{\Delta}$ are in one-to-one correspondence to subfans of the defining fan Δ , the “toric” topology on X_{Δ} corresponds to the “fan topology” on the finite set Δ , namely, the topology given by the collection of the subfans $\Lambda \preceq \Delta$, together with the empty set, as family of open sets. For such a subfan Λ , we then understand the graded A^{\bullet} -module $IH_{\mathbb{T}}^{\bullet}(X_{\Lambda})$ as the module of sections over Λ of the *equivariant intersection cohomology (pre-)sheaf* $\mathcal{I}H_{\mathbb{T}}^{\bullet}$ on the “fan space” Δ .

To be more precise, we show that the presheaf

$$\mathcal{I}H_{\mathbb{T}}^{\bullet}: \Lambda \mapsto \mathcal{I}H_{\mathbb{T}}^{\bullet}(\Lambda) := IH_{\mathbb{T}}^{\bullet}(X_{\Lambda}, \mathbb{Q}) , \quad \text{for } \Lambda \preceq \Delta ,$$

is in fact a sheaf of modules over the *sheaf of rational piecewise polynomial functions on the support of the fan*

$$\mathcal{A}^{\bullet}: \Lambda \mapsto \mathcal{A}^{\bullet}(\Lambda) := \{f : |\Lambda| \longrightarrow \mathbb{Q} ; \Lambda\text{-piecewise polynomial}\} ,$$

and we prove that the sheaf $\mathcal{I}H_{\mathbb{T}}^{\bullet}$ has the three properties stated in the following definition.

We call a sheaf \mathcal{E}^\bullet of \mathcal{A}^\bullet -modules a *minimal extension sheaf* (of the constant sheaf \mathbb{Q}) if it satisfies the following conditions³:

- (N) Normalization: For the zero cone $o := \{0\}$, there is an isomorphism $\mathcal{E}^\bullet(o) \cong \mathcal{A}^\bullet(o) (= \mathbb{Q}^\bullet)$.
- (PF) Pointwise Freeness: For each cone $\sigma \in \Delta$, the module $\mathcal{E}^\bullet(\sigma) := \mathcal{E}^\bullet(\langle \sigma \rangle)$ is free over $\mathcal{A}^\bullet(\sigma)$.
- (LME) Local Minimal Extension mod \mathfrak{m} : For each cone $\sigma \in \Delta$, the restriction mapping

$$\varphi_\sigma: \mathcal{E}^\bullet(\sigma) \rightarrow \mathcal{E}^\bullet(\partial\sigma)$$

induces an isomorphism

$$\overline{\varphi}_\sigma: \frac{\mathcal{E}^\bullet(\sigma)}{\mathfrak{m} \cdot \mathcal{E}^\bullet(\sigma)} \xrightarrow{\cong} \frac{\mathcal{E}^\bullet(\partial\sigma)}{\mathfrak{m} \cdot \mathcal{E}^\bullet(\partial\sigma)}$$

of graded vector spaces.

Restated in other words, a minimal extension sheaf \mathcal{E}^\bullet on a fan Δ is characterized as follows: It is a sheaf of graded \mathcal{A}^\bullet -modules satisfying the equality $\mathcal{E}^\bullet(o) = \mathbb{Q}^\bullet$ and having the property that for each cone $\sigma \in \Delta$, the module $\mathcal{E}^\bullet(\sigma)$ is free and of minimal rank over $\mathcal{A}^\bullet(\sigma)$ such that the restriction $\varphi_\sigma: \mathcal{E}^\bullet(\sigma) \rightarrow \mathcal{E}^\bullet(\partial\sigma)$ is surjective.

It is not difficult to see that such a sheaf \mathcal{E}^\bullet can be constructed recursively on the k -skeletons $\Delta^{\leq k}$ of the fan Δ in a purely formal way, starting from $\mathcal{E}^0 = \mathbb{Q}$, and that the resulting sheaf is *uniquely determined up to isomorphism*.

Using this notion of minimal extension sheaves, the central result of the article can be stated as follows:

MAIN THEOREM. *The equivariant intersection cohomology sheaf $\mathcal{I}H_{\mathbb{T}}^\bullet$ on a fan space Δ is a minimal extension sheaf of \mathbb{Q}^\bullet .*

The fact that $\mathcal{I}H_{\mathbb{T}}^\bullet$ is a minimal extension sheaf has interesting consequences that are discussed in [BBFK₂]: In particular, it leads to a simple proof of the inductive ‘‘Local-Global-Formula’’ for the computation of intersection cohomology Betti numbers of compact toric varieties, but also, *mutatis mutandis*, it opens a way to introduce the analogous sheaf for arbitrary (not necessarily rational) fans in real vector spaces. On the other hand, the ‘‘minimal complexes’’ of [BeLu] occur now naturally as the ‘‘cellular Čech complexes’’ of the sheaf $\mathcal{I}H_{\mathbb{T}}^\bullet$.

The article is organized as follows: After fixing our (more or less standard) notation, we first recall in section 1 the definition and some basic properties of the equivariant cohomology of a toric variety; in

³For a sheaf \mathcal{F} on Δ and a cone σ , we simply write $\mathcal{F}(\sigma)$ to denote the space of sections $\mathcal{F}(\langle \sigma \rangle)$ over the affine subfan $\langle \sigma \rangle$ generated by σ .

particular, we look at the *presheaf* $\mathcal{H}_{\mathbb{T}}^{\bullet}$ that it defines on the fan space. For later use, we also touch upon the equivariant Chern class of toric line bundles. In section 2, we introduce the equivariant intersection cohomology and prove that it vanishes in odd degrees, which implies that it actually defines a *sheaf* $\mathcal{I}H_{\mathbb{T}}^{\bullet}$ on the fan space.

The characteristic properties (N), (PF), and (LME) of $\mathcal{I}H_{\mathbb{T}}^{\bullet}$, formalized in the notion of a minimal extension sheaf, are discussed in section 3. Whereas property (N) is easily seen to hold, the other two require considerably more work. Property (PF) is established in section 4 by proving that any *contractible affine toric variety is equivariantly formal* (with respect to intersection cohomology): The “usual” intersection cohomology $IH^{\bullet}(X)$ is the quotient of the equivariant theory $IH_{\mathbb{T}}^{\bullet}(X)$ modulo the homogeneous A^{\bullet} -submodule $\mathfrak{m} \cdot IH_{\mathbb{T}}^{\bullet}(X)$. The constructions used in the proof are also crucial for proving the remaining property (LME) in section 5. The article concludes with a discussion of “*equivariantly formal fans*”, i.e., fans Δ such that the corresponding toric variety X_{Δ} is equivariantly formal. This property holds both for complete fans and for “*affine fans*” consisting of a full dimensional cone and its faces; more generally, it holds if and only if the non-equivariant intersection cohomology vanishes in odd degrees.

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0. Preliminaries

We use the following *notation*: We denote with $\mathbb{T} \cong (\mathbb{C}^*)^n$ the complex *algebraic torus* of dimension n , with $N := \text{Hom}(\mathbb{C}^*, \mathbb{T}) \cong \mathbb{Z}^n$ the lattice of its one parameter subgroups, and with $M := \text{Hom}(\mathbb{T}, \mathbb{C}^*) \cong \mathbb{Z}^n$ the dual lattice of characters. We recall that there are natural isomorphism $\mathbb{T} \cong N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$. Let Δ be a fan in the *rational* vector space $V := N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. For two cones σ and τ in Δ , we write $\tau \prec \sigma$ if τ is a *proper* face of σ , and $\tau \prec_1 \sigma$ if τ is a *facet* (i.e., a face of codimension 1) of σ . – The symbol $\Lambda \prec \Delta$ indicates that a proper subset Λ of Δ actually is a *subfan*.

A cone $\sigma \in \Delta$ *generates* the “*affine*” subfan $\langle \sigma \rangle := \{\sigma\} \cup \partial\sigma$ consisting of σ and the *boundary subfan* $\partial\sigma := \{\tau \in \Delta; \tau \prec \sigma\}$. Occasionally, if there is no risk of confusion, we simply write σ instead of $\langle \sigma \rangle$ for notational convenience.

For a cone $\sigma \in \Delta$, we denote with $V_\sigma := \sigma + (-\sigma)$ the linear span of σ in V , and with \mathbb{T}_σ , the subtorus of \mathbb{T} corresponding to the sublattice $N_\sigma := N \cap V_\sigma$ as lattice of its one parameter subgroups (i.e., we have $\mathbb{T}_\sigma \cong N_\sigma \otimes_{\mathbb{Z}} \mathbb{C}^*$). The associated lattice of characters is $M_\sigma := M/N_\sigma^\perp$. Let us denote with V_σ^* the dual of V_σ in $V^* := M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$. The symmetric \mathbb{Q} -algebra $A_\sigma^\bullet := S^\bullet(V_\sigma^*)$ is naturally isomorphic to the algebra of polynomial functions $f: V_\sigma \rightarrow \mathbb{Q}$. The restriction $f \mapsto f|_\sigma$ thus provides an isomorphism

$$(0.1) \quad A_\sigma^\bullet \xrightarrow{\cong} \mathcal{A}^\bullet(\sigma) := \mathcal{A}^\bullet(\langle \sigma \rangle)$$

with the algebra of sections over $\langle \sigma \rangle$ of the sheaf \mathcal{A}^\bullet of piecewise polynomial functions described in the introduction. This is compatible with the induced homomorphisms $A_\sigma^\bullet \rightarrow A_\tau^\bullet$ and $\mathcal{A}^\bullet(\sigma) \rightarrow \mathcal{A}^\bullet(\tau)$ for each face $\tau \prec \sigma$. – In view of the cohomological interpretation discussed in section 1, we endow the algebra A_σ^\bullet – just as we did with $A^\bullet := S^\bullet(V^*)$ – with the grading that doubles the usual degrees, so linear polynomials (i.e., elements of V_σ^*) get the degree 2, etc.

If F^\bullet is a graded A^\bullet -module, we denote with \overline{F}^\bullet its residue class module

$$(0.2) \quad \overline{F}^\bullet := F^\bullet / (\mathfrak{m} \cdot F^\bullet) \cong \mathbb{Q}^\bullet \otimes_{A^\bullet} F^\bullet \quad (\text{with } \mathfrak{m} := A^{>0} \text{ and } \mathbb{Q}^\bullet := A^\bullet / \mathfrak{m}),$$

where \mathfrak{m} is the unique homogeneous maximal ideal of A^\bullet , and \mathbb{Q}^\bullet is the graded algebra concentrated in degree zero with $\mathbb{Q}^0 := \mathbb{Q}$. – By means of the natural epimorphism $A^\bullet \rightarrow A_\sigma^\bullet$ extending the projection $V^* \rightarrow V_\sigma^*$ (and corresponding to the restriction of polynomial functions from $V := N_{\mathbb{Q}}$ to V_σ), every A_σ^\bullet -module F_σ^\bullet also is an A^\bullet -module, and there is a canonical isomorphism $\overline{F}_\sigma^\bullet \cong F_\sigma^\bullet / (\mathfrak{m}_\sigma \cdot F_\sigma^\bullet)$.

In the sequel, we shall freely use the following basic facts on finitely generated graded A^\bullet -modules F^\bullet : Given a family (f_1, \dots, f_r) of homogeneous elements in F^\bullet , it generates F^\bullet over A^\bullet if and only if the system of residue classes $(\overline{f}_1, \dots, \overline{f}_r)$ modulo $\mathfrak{m} \cdot F^\bullet$ generates the vector space \overline{F}^\bullet . If F^\bullet is free, then (f_1, \dots, f_r) is part of a basis of F^\bullet over A^\bullet if and only if $(\overline{f}_1, \dots, \overline{f}_r)$ is linearly independent over \mathbb{Q} . Furthermore, every homomorphism $\varphi: F^\bullet \rightarrow G^\bullet$ of graded A^\bullet -modules induces a homomorphism $\overline{\varphi}: \overline{F}^\bullet \rightarrow \overline{G}^\bullet$ of graded vector spaces which is surjective if and only if φ is so. If F^\bullet is free, then every homomorphism $\psi: \overline{F}^\bullet \rightarrow \overline{G}^\bullet$ can be lifted to a homomorphism $\varphi: F^\bullet \rightarrow G^\bullet$; if both F^\bullet and G^\bullet are free, then φ is an isomorphism if and only if that holds for $\overline{\varphi}$.

For the affine toric variety X_σ associated to σ , the torus \mathbb{T}_σ is the *isotropy subtorus* at any point in the unique closed orbit B_σ , and this orbit B_σ is \mathbb{T} -isomorphic to the quotient torus $\mathbb{T}/\mathbb{T}_\sigma$, looked at as a

homogeneous space. Any splitting $\mathbb{T} \cong \mathbb{T}' \times \mathbb{T}_\sigma$ of the torus \mathbb{T} into the isotropy subtorus \mathbb{T}_σ and a complementary subtorus $\mathbb{T}' \cong \mathbb{T}/\mathbb{T}_\sigma$ extends to an equivariant splitting of affine toric varieties: The choice of an arbitrary base point x_o in the open dense orbit B_o determines an embedding of \mathbb{T} into X_σ . Denoting with Z_σ the closure of \mathbb{T}_σ in X_σ with respect to this embedding, we obtain an isomorphism

$$(0.3) \quad \mathbb{T}' \times Z_\sigma \xrightarrow{\cong} X_\sigma, (t, z) \mapsto tz.$$

The unique point $x_\sigma \in B_\sigma \cap Z_\sigma$ will sometimes be referred to as the *distinguished point* in the orbit B_σ . We remark that Z_σ , being the \mathbb{T}_σ -toric variety corresponding to σ considered as N_σ -cone, is equivariantly contractible to its fixed point x_σ , so X_σ has the closed orbit B_σ as equivariant deformation retract. – We refer to the above splitting (0.3) as the *affine orbit splitting*.

1. Equivariant Cohomology of Toric Varieties

Before proceeding to study the equivariant *intersection* cohomology, we first look at the “usual” \mathbb{T} -equivariant cohomology $H_{\mathbb{T}}^\bullet(X)$ of a toric variety $X = X_\Delta$. We recall the definition: With respect to a fixed identification $\mathbb{T} \cong (\mathbb{C}^*)^n$, a universal \mathbb{T} -bundle is given by the principal $(\mathbb{C}^*)^n$ -bundle

$$E\mathbb{T} := (\mathbb{C}^\infty \setminus \{0\})^n \longrightarrow B\mathbb{T} := (\mathbb{P}_\infty)^n,$$

the limit of the finite-dimensional approximations

$$E_m\mathbb{T} := (\mathbb{C}^{m+1} \setminus \{0\})^n \longrightarrow B_m\mathbb{T} := (\mathbb{P}_m)^n$$

for $m \rightarrow \infty$. One considers the associated bundles

$$X_{\mathbb{T}} := E\mathbb{T} \times_{\mathbb{T}} X \longrightarrow B\mathbb{T} \quad \text{and} \quad X_{\mathbb{T},m} := E_m\mathbb{T} \times_{\mathbb{T}} X \longrightarrow B_m\mathbb{T}$$

with fibre X and defines the (*rational*) *equivariant cohomology algebra* of X as follows:

$$H_{\mathbb{T}}^\bullet(X) := H_{\mathbb{T}}^\bullet(X, \mathbb{Q}) := H^\bullet(X_{\mathbb{T}}, \mathbb{Q}).$$

For the homogeneous part of some fixed degree $q \geq 0$, there is the description

$$H_{\mathbb{T}}^q(X, \mathbb{Q}) \cong \lim_{m \rightarrow \infty} H^q(X_{\mathbb{T},m}, \mathbb{Q}).$$

The bundle projection $X_{\mathbb{T}} \rightarrow B\mathbb{T}$ makes $H_{\mathbb{T}}^\bullet(X)$ an algebra over the cohomology algebra $H^\bullet(B\mathbb{T}) \cong \mathbb{Q}[u_1, \dots, u_n]$ of the classifying space $B\mathbb{T} = (\mathbb{P}_\infty)^n$ of \mathbb{T} , a polynomial algebra on n free generators of degree 2.

To relate $H_{\mathbb{T}}^\bullet(X)$ with the combinatorial data encoded in the defining fan Δ for X , we first recall the following result.

LEMMA 1.1. *For each cone σ , there are natural isomorphisms*

$$\mathcal{A}^\bullet(\sigma) \cong A_\sigma^\bullet \cong H^\bullet(B\mathbb{T}_\sigma) \cong H_{\mathbb{T}}^\bullet(X_\sigma),$$

i.e., they are compatible with maps defined by face relations $\tau \prec \sigma$.

PROOF. The isomorphism on the left-hand side has been discussed in (0.1). The one in the middle is the special case $V = V_\sigma$, $\mathbb{T} = \mathbb{T}_\sigma$ of the isomorphism of graded \mathbb{Q} -algebras

$$A^\bullet = S^\bullet(V^*) \xrightarrow{\cong} H^\bullet(B\mathbb{T})$$

induced from the *Chern class homomorphism*

$$c: M \longrightarrow H^2(B\mathbb{T}), \quad \chi \mapsto c_1(L_\chi)$$

that associates to a character $\chi \in M \subset V^* = M_{\mathbb{Q}}$ the first Chern class of the line bundle $L_\chi := E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}_\chi \rightarrow E\mathbb{T} \times_{\mathbb{T}} \{\text{pt}\} = B\mathbb{T}$. Here \mathbb{C}_χ denotes the one-dimensional \mathbb{T} -module with weight χ .

For a cone $\sigma \in \Delta$ and the corresponding subtorus \mathbb{T}_σ of \mathbb{T} , the restriction mapping $f \mapsto f|_{V_\sigma}$ and the natural mapping $B\mathbb{T}_\sigma \hookrightarrow B\mathbb{T}$ induce a commutative diagram

$$\begin{array}{ccc} A^\bullet & \longrightarrow & A_\sigma^\bullet \\ \downarrow \cong & & \downarrow \cong \\ H^\bullet(B\mathbb{T}) & \longrightarrow & H^\bullet(B\mathbb{T}_\sigma) \end{array}$$

since we have $L_\chi|_{B\mathbb{T}_\sigma} = L_{\chi|_{\mathbb{T}_\sigma}}$ (the restriction of L_χ to $B\mathbb{T}_\sigma$ is just the line bundle associated to the character $\chi|_{\mathbb{T}_\sigma}$) and Chern classes are functorial, thus proving the naturality of the isomorphism in the middle.

We now discuss the isomorphism $H^\bullet(B\mathbb{T}_\sigma) \cong H_{\mathbb{T}}^\bullet(X_\sigma)$ on the right hand side: The affine orbit splitting $(X_\sigma, \mathbb{T}) \cong \mathbb{T}' \times (Z_\sigma, \mathbb{T}_\sigma)$ of (0.3) and the \mathbb{T}_σ -equivariant contraction $Z_\sigma \simeq \{x_\sigma\}$ onto the distinguished point induce isomorphisms

$$\begin{aligned} H^\bullet(B\mathbb{T}_\sigma) &\cong H^\bullet(E\mathbb{T}_\sigma \times_{\mathbb{T}_\sigma} \{x_\sigma\}) \cong H^\bullet(E\mathbb{T}_\sigma \times_{\mathbb{T}_\sigma} Z_\sigma) \\ &= H_{\mathbb{T}_\sigma}^\bullet(Z_\sigma) \cong H_{\mathbb{T}}^\bullet(\mathbb{T}' \times Z_\sigma) = H_{\mathbb{T}}^\bullet(X_\sigma). \end{aligned}$$

The whole construction is natural with respect to some face relation $\tau \prec \sigma$. This is easily seen from any splitting of the torus in the form $\mathbb{T} = \mathbb{T}' \times \mathbb{T}_\sigma = \mathbb{T}' \times (\mathbb{T}'' \times \mathbb{T}_\tau) = (\mathbb{T}' \times \mathbb{T}'') \times \mathbb{T}_\tau$, since the choices of \mathbb{T}' and \mathbb{T}'' do not play any role. \square

1.A Equivariant Cohomology as a Presheaf on the Fan

On the fan space Δ defined in the introduction, we now consider the presheaf $\mathcal{H}_{\mathbb{T}}^{\bullet}$ of graded algebras that is given by

$$\mathcal{H}_{\mathbb{T}}^{\bullet}: \Lambda \mapsto H_{\mathbb{T}}^{\bullet}(X_{\Lambda}) \quad \text{for } \Lambda \preceq \Delta.$$

Inverting the isomorphisms of Lemma 1.1 and using the fact that \mathcal{A}^{\bullet} clearly is a sheaf on Δ , we obtain the following result:

COROLLARY 1.2. *There is a homomorphism of presheaves $\mathcal{H}_{\mathbb{T}}^{\bullet} \rightarrow \mathcal{A}^{\bullet}$ that is an isomorphism on the stalks, so \mathcal{A}^{\bullet} is the associated sheaf to the presheaf $\mathcal{H}_{\mathbb{T}}^{\bullet}$. \square*

Note here that the stalk \mathcal{F}_{σ} of a presheaf \mathcal{F} on Δ coincides with $\mathcal{F}(\sigma)$, since for each point σ of the fan space, the basic open set $\langle \sigma \rangle$ is its smallest open neighbourhood.

In the simplicial case, that homomorphism turns out to be an isomorphism:

THEOREM 1.3. *For a simplicial fan Δ , the homomorphism*

$$\mathcal{H}_{\mathbb{T}}^{\bullet} \longrightarrow \mathcal{A}^{\bullet}$$

of graded presheaves is an isomorphism, so $\mathcal{H}_{\mathbb{T}}^{\bullet}$ is a sheaf (and actually flabby).

PROOF. We proceed by induction on the number of cones in the fan Δ . For $\Delta = \{\emptyset\}$, the assertion is obvious. For the induction step, we choose a maximal cone $\sigma \in \Delta$ and consider the Mayer-Vietoris sequences associated to $\Lambda := \Delta \setminus \{\sigma\}$ and $\langle \sigma \rangle$, both for $\mathcal{H}_{\mathbb{T}}^{\bullet}$ and for \mathcal{A}^{\bullet} . It suffices to prove that in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H}_{\mathbb{T}}^{2q}(\Delta) & \longrightarrow & \mathcal{H}_{\mathbb{T}}^{2q}(\Lambda) \oplus \mathcal{H}_{\mathbb{T}}^{2q}(\sigma) & \xrightarrow{\alpha} & \mathcal{H}_{\mathbb{T}}^{2q}(\Lambda \cap \sigma) \longrightarrow \mathcal{H}_{\mathbb{T}}^{2q+1}(\Delta) \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \mathcal{A}^{2q}(\Delta) & \longrightarrow & \mathcal{A}^{2q}(\Lambda) \oplus \mathcal{A}^{2q}(\sigma) & \xrightarrow{\beta} & \mathcal{A}^{2q}(\Lambda \cap \sigma) \longrightarrow 0 \end{array}$$

obtained from Corollary 1.2, the rows are exact, the vertical arrows are isomorphisms, and $\mathcal{H}_{\mathbb{T}}^{2q+1}(\Delta)$ vanishes. Applying the induction hypothesis to the fans Λ and $\Lambda \cap \langle \sigma \rangle$, and Lemma 1.1 to $\langle \sigma \rangle$, we see that the assertion holds for the second and the third arrow, and we obtain the leading 0 in the upper row. Furthermore, since the fan Δ is simplicial, it is not difficult to see that the sheaf \mathcal{A}^{\bullet} is flabby; hence, the map β , and thus also α , is surjective. This proves $\mathcal{H}_{\mathbb{T}}^{2q+1}(\Delta) = 0$, as, by induction hypothesis, we know that $\mathcal{H}_{\mathbb{T}}^{2q+1}(\Lambda) \oplus \mathcal{H}_{\mathbb{T}}^{2q+1}(\sigma)$ vanishes. By the Five Lemma, the first vertical arrow is an isomorphism as well, thus proving our claim. \square

In the non-simplicial case, we still have some partial results which are proved by similar arguments:

REMARK 1.4. (i) If a fan Δ can be successively built up from cones $\sigma_1, \dots, \sigma_r$ such that σ_{i+1} intersects $\sigma_1 \cup \dots \cup \sigma_i$ in a single proper face, then the homomorphism $\mathcal{H}_{\mathbb{T}}^{\bullet} \rightarrow \mathcal{A}^{\bullet}$ is an isomorphism.

(ii) For an arbitrary toric variety, we have isomorphisms of sheaves

$$\mathcal{H}_{\mathbb{T}}^q \cong \mathcal{A}^q \quad \text{in all degrees } q \leq 2;$$

in particular, $\mathcal{H}_{\mathbb{T}}^2$ is a sheaf, and there is an isomorphism

$$H_{\mathbb{T}}^2(X_{\Delta}) \cong \mathcal{A}^2(\Delta).$$

On the other hand, this does not carry over to degree $q = 3$, as follows from the next example:

EXAMPLE 1.5. There is a three-dimensional toric variety X with $H_{\mathbb{T}}^3(X) \cong \mathbb{Q}$.

Since we know that $\mathcal{H}_{\mathbb{T}}^3(\sigma) \cong \mathcal{A}^3(\sigma)$ vanishes on each basic open set, the associated sheaf is the zero sheaf. Hence the example implies the following statement:

COROLLARY 1.6. *In the non-simplicial case, the presheaf $\mathcal{H}_{\mathbb{T}}^{\bullet}$ is in general not a sheaf.*

PROOF OF EXAMPLE 1.5. We consider the fan Δ generated by the four “vertical” facets of a cube centred at the origin, and write it in the form $\Delta_1 \cup \Delta_2$, where Δ_1 is generated by two adjacent facets, and Δ_2 is generated by the other two. According to Remark 1.4, i), we have isomorphisms $\mathcal{H}_{\mathbb{T}}^{\bullet}(\Delta_j) \cong \mathcal{A}^{\bullet}(\Delta_j)$ and $\mathcal{H}_{\mathbb{T}}^{\bullet}(\Delta_1 \cap \Delta_2) \cong \mathcal{A}^{\bullet}(\Delta_1 \cap \Delta_2)$. We now show that the homomorphism

$$\mathcal{A}^2(\Delta_1) \oplus \mathcal{A}^2(\Delta_2) \longrightarrow \mathcal{A}^2(\Delta_1 \cap \Delta_2)$$

in the appropriate Mayer-Vietoris sequence is not surjective. In fact, its cokernel is one-dimensional: The fan $\Delta_1 \cap \Delta_2$ is the union $\langle \tau_1 \rangle \cup \langle \tau_2 \rangle$ of the subfans generated by the two opposite two-dimensional “vertical” cones τ_j that are spanned by the “outer” vertical edges of the two adjacent facets. The vector space $\mathcal{A}^2(\Delta_1 \cap \Delta_2) = \mathcal{A}_{\tau_1}^2 \oplus \mathcal{A}_{\tau_2}^2$ is four-dimensional. The restriction homomorphisms from $\mathcal{A}^2(\Delta_j)$ to $\mathcal{A}^2(\Delta_1 \cap \Delta_2)$ both map onto the three-dimensional subspace consisting of all piecewise linear functions for which the differences between the values at the top vertex and at the bottom vertex on both edges agree. Applying that $\mathcal{H}_{\mathbb{T}}^{\bullet}(\Lambda) = \mathcal{A}^{\bullet}(\Lambda)$ holds for $\Lambda = \Delta_j$ and for $\Lambda = \Delta_1 \cap \Delta_2$, we get $H_{\mathbb{T}}^3(X_{\Delta}) \cong \mathbb{Q}$. \square

1.B. Toric Line Bundles and their Equivariant Chern Class

A line bundle $L \rightarrow X$ on a toric variety X is called a *toric line bundle* if there is a \mathbb{T} -action by bundle automorphisms on the total space such that the bundle projection is equivariant. We obtain an induced line bundle $L_{\mathbb{T}} := E\mathbb{T} \times_{\mathbb{T}} L$ on $X_{\mathbb{T}}$, whose Chern class $c_1(L_{\mathbb{T}}) \in H^2(X_{\mathbb{T}}) = H_{\mathbb{T}}^2(X)$ is called the *equivariant Chern class* of L on X and is denoted with $c_1^{\mathbb{T}}(L)$. This class can be considered as a lift of the “usual” Chern class $c_1(L) \in H^2(X)$ to $H_{\mathbb{T}}^2(X) = H^2(X_{\mathbb{T}})$: Each fibre map $X \hookrightarrow X_{\mathbb{T}}$ of the bundle $X_{\mathbb{T}} \rightarrow B\mathbb{T}$ induces the same “edge homomorphism” $H_{\mathbb{T}}^2(X) \rightarrow H^2(X)$, and that edge homomorphism maps $c_1^{\mathbb{T}}(L)$ to $c_1(L)$.

Now let Δ be a *purely n -dimensional fan* (i.e., a fan generated by its n -dimensional cones). For use in section 6, we have to determine the Δ -piecewise linear function $\psi := \psi_L \in \mathcal{A}^2(\Delta)$ corresponding to the equivariant Chern class $c_1^{\mathbb{T}}(L) \in H_{\mathbb{T}}^2(X)$ of a toric line bundle L on X with respect to the isomorphism $H_{\mathbb{T}}^2(X_{\Delta}) \rightarrow \mathcal{A}^2(\Delta)$. It is convenient to introduce the notation $\check{\mathbb{T}} := \mathbb{T} \times \mathbb{C}^*$, $\check{N} := N \oplus \mathbb{Z}$, and $\check{V} := V \oplus \mathbb{Q}$.

The total space of L is in a natural way a toric variety with acting torus $\check{\mathbb{T}}$, where the second factor acts on the fibres by scalar multiplication. The associated principal \mathbb{C}^* -bundle L_0 , obtained from L by removing the zero section, is an invariant open subset. Let us first describe the fan $\check{\Delta}_0$ in \check{V} corresponding to L_0 . The fan is determined by the following properties: The projection $p: \check{V} \rightarrow V$ maps $|\check{\Delta}_0|$ homeomorphically onto $|\Delta|$, inducing a bijection between the cones in $\check{\Delta}_0$ and Δ . In order to assure local triviality of the projection, we have to require the equality

$$p(\check{N}_{\check{\sigma}}) = N_{p(\check{\sigma})}$$

for each cone $\check{\sigma}$ in $\check{\Delta}_0$. Then L is the $\check{\mathbb{T}}$ -toric variety given by the fan

$$\check{\Delta} := \check{\Delta}_0 \cup \{\check{\sigma} + \rho; \check{\sigma} \in \check{\Delta}_0\}, \quad \rho := \mathbb{Q}_{\geq 0} \cdot (0_V, 1_{\mathbb{Q}})$$

spanned by $\check{\Delta}_0$ and the “vertical” ray ρ in \check{V} . The support $|\check{\Delta}_0|$ is the graph of a piecewise linear function $\psi := \psi_L: |\Delta| \rightarrow \mathbb{Q}$ taking integral values at lattice points. Vice versa, such a function clearly determines a fan $\check{\Delta}_0$ of the above type. We note that ψ_L is the composition of the Δ -piecewise linear homeomorphism $(p|_{\check{\Delta}_0})^{-1}: |\Delta| \rightarrow |\check{\Delta}_0|$ and the linear projection $\check{V} \rightarrow \mathbb{Q}$ onto the second factor.

REMARK 1.7. If, as above, the projection $p: \check{V} \rightarrow V$ maps $|\check{\Delta}_0|$ homeomorphically onto $|\Delta|$, but if the condition $p(\check{N}_{\check{\sigma}}) = N_{p(\check{\sigma})}$ is not satisfied, then L_0 is called a *toric Seifert bundle*. Replacing the lattice $\check{N} = N \times \mathbb{Z}$ by $N \times \frac{1}{m}\mathbb{Z}$ for a suitable $m \in \mathbb{N}_{>0}$, we see that the above condition holds for that new lattice. On the level of tori and toric varieties, that means passing from $\mathbb{T} \times \mathbb{C}^*$ to $\mathbb{T} \times \mathbb{C}^*/C_m \cong \mathbb{T} \times \mathbb{C}^*$

and from L_0 to L_0/C_m , respectively, where the group $C_m \subset \mathbb{C}^*$ of m -th roots of unity acts on L_0 as subgroup of the second factor in $\mathbb{T} \times \mathbb{C}^*$.

LEMMA 1.8. *For a purely n -dimensional fan Δ , the function $\psi_L \in \mathcal{A}^2(\Delta)$ is the image of the equivariant Chern class $c_1^{\mathbb{T}}(L)$ of L under the isomorphism $H_{\mathbb{T}}^2(X_{\Delta}) \rightarrow \mathcal{A}^2(\Delta)$ from Remark 1.4, (ii).*

PROOF. For a cone $\sigma \in \Delta^n$, let $\psi_{\sigma} \in M$ be the character which coincides with ψ_L on $\sigma \cap N$. Since the map $\mathcal{H}_{\mathbb{T}}^2 \rightarrow \mathcal{A}^2$ is an isomorphism of sheaves, it suffices to show that the “local” equivariant Chern class $c_1^{\mathbb{T}}(L|_{X_{\sigma}}) \in H_{\mathbb{T}}^2(X_{\sigma})$ is mapped onto $\psi_{\sigma} \in M \cong \mathcal{A}^2(\sigma)$. Observing that the inclusion $x_{\sigma} \hookrightarrow X_{\sigma}$ of the fixed point $x_{\sigma} \in X_{\sigma}$ induces an isomorphism $H_{\mathbb{T}}^2(X_{\sigma}) \xrightarrow{\cong} H_{\mathbb{T}}^2(x_{\sigma})$, we may further restrict our attention to the fibre $L_{x_{\sigma}}$ of L over x_{σ} . As a \mathbb{T} -module, this fibre is nothing but $\mathbb{C}_{\psi_{\sigma}}$, and the character $\psi_{\sigma} \in M \cong H^2(B\mathbb{T}) \cong H_{\mathbb{T}}^2(x_{\sigma})$ is the Chern class of that bundle. This completes the proof. \square

2. The Equivariant Intersection Cohomology Sheaf

For a non-simplicial fan Δ , the equivariant cohomology presheaf $\mathcal{H}_{\mathbb{T}}^{\bullet}$ is no longer a sheaf (see Example 1.5), so Theorem 1.3 fails to be true in the general case. The situation is much better behaved for *intersection* cohomology, though we do not have such a nice combinatorial interpretation as is given by the Stanley-Reisner ring in the simplicial case.

First let us recall how to describe equivariant intersection cohomology: Following the approach of F. Kirwan⁴ and using the notation of section 1, one defines the q -th (*rational*) *equivariant intersection cohomology group* of X as the limit

$$IH_{\mathbb{T}}^q(X) := IH_{\mathbb{T}}^q(X; \mathbb{Q}) := \lim_{m \rightarrow \infty} IH^q(X_{\mathbb{T}, m}; \mathbb{Q})$$

and sets

$$IH_{\mathbb{T}}^{\bullet}(X) := \bigoplus_{q=0}^{\infty} IH_{\mathbb{T}}^q(X).$$

This construction provides a presheaf

$$\mathcal{I}H_{\mathbb{T}}^{\bullet}: \Lambda \mapsto \mathcal{I}H_{\mathbb{T}}^{\bullet}(\Lambda) := IH_{\mathbb{T}}^{\bullet}(X_{\Lambda}) \quad \text{for } \Lambda \preceq \Delta$$

on the fan space Δ . In order to prove that it is in fact a sheaf, we verify the following basic result:

VANISHING LEMMA 2.1. *The equivariant intersection cohomology $IH_{\mathbb{T}}^q(X)$ of a toric variety X vanishes in odd degrees q .*

⁴See [Ki], formula (2.12) and the surrounding text; for a more “sophisticated” approach, see [Bry] and [Jo].

PROOF. Let $\hat{\Delta}$ be a simplicial refinement of the defining fan Δ for X , and denote with $\hat{X} \rightarrow X$ the corresponding equivariant \mathbb{Q} -resolution of singularities. By theorem 1.3, the assertion holds for \hat{X} , since then $IH_{\mathbb{T}}^{\bullet}(\hat{X}) \cong H_{\mathbb{T}}^{\bullet}(\hat{X}) \cong \mathcal{A}_{\hat{\Delta}}^{\bullet}$. By the equivariant version of the Decomposition Theorem of Beilinson, Bernstein, Deligne, and Gabber as stated in [Ki, p. 394] (see also [BeLu, 5.3] or [BreLu]), we may interpret $IH_{\mathbb{T}}^q(X)$ as a subspace of $IH_{\mathbb{T}}^q(\hat{X})$. That proves the assertion. \square

THEOREM 2.2. *The presheaf $\mathcal{I}H_{\mathbb{T}}^{\bullet}$ on Δ is a sheaf of \mathcal{A}^{\bullet} -modules.*

PROOF. Since there are only finitely many open subsets in Δ , it suffices to verify the sheaf axioms for two open subsets $\Lambda_1, \Lambda_2 \preceq \Delta$. We thus have to prove the exactness of the sequence

$$0 \longrightarrow \mathcal{I}H_{\mathbb{T}}^q(\Lambda_1 \cup \Lambda_2) \longrightarrow \mathcal{I}H_{\mathbb{T}}^q(\Lambda_1) \oplus \mathcal{I}H_{\mathbb{T}}^q(\Lambda_2) \longrightarrow \mathcal{I}H_{\mathbb{T}}^q(\Lambda_1 \cap \Lambda_2) .$$

That follows from the Lemma 2.1: The exactness is obvious if q is odd; for even q , the vector space $\mathcal{I}H_{\mathbb{T}}^{q-1}(\Lambda_1 \cap \Lambda_2)$ vanishes and thus, the sequence is part of the exact Mayer-Vietoris sequence for $IH_{\mathbb{T}}^{\bullet}$.

As a consequence, $\mathcal{I}H_{\mathbb{T}}^{\bullet}$ is a sheaf of \mathcal{A}^{\bullet} -modules, since \mathcal{A}^{\bullet} is the associated sheaf to the presheaf $\mathcal{H}_{\mathbb{T}}^{\bullet}$, and each $IH_{\mathbb{T}}^{\bullet}(X_{\Lambda})$ is an $H_{\mathbb{T}}^{\bullet}(X_{\Lambda})$ -module. \square

3. Minimal Extension Sheaves

We now proceed towards (re-)stating and verifying the three properties that actually characterize the sheaf $\mathcal{I}H_{\mathbb{T}}^{\bullet}$ up to isomorphism.

For the ease of notation, we write E_{Λ}^{\bullet} for the module $\mathcal{E}^{\bullet}(\Lambda)$ of sections of a sheaf \mathcal{E}^{\bullet} over a subfan $\Lambda \preceq \Delta$; in particular, we do so for the residue class module $\overline{E}_{\Lambda}^{\bullet} := E_{\Lambda}^{\bullet}/(\mathfrak{m} \cdot E_{\Lambda}^{\bullet}) \cong \mathbb{Q}^{\bullet} \otimes_{A^{\bullet}} E_{\Lambda}^{\bullet}$, see (0.2). Using this notation, we restate the properties (N) (*Normalization*), (PF) (*Pointwise Freeness*), and (LME) (*Local Minimal Extension mod \mathfrak{m}*) from the introduction.

DEFINITION 3.1. A sheaf \mathcal{E}^{\bullet} of graded \mathcal{A}^{\bullet} -modules on the fan Δ is called a *minimal extension sheaf* (of \mathbb{Q}^{\bullet}) if it satisfies the following conditions:

(N): One has $E_o^{\bullet} \cong A_o^{\bullet} = \mathbb{Q}^{\bullet}$ for the zero cone o .

(PF): For each cone $\sigma \in \Delta$, the module E_{σ}^{\bullet} is *free* over A_{σ}^{\bullet} .

(LME): For each cone $\sigma \in \Delta$, the restriction mapping $\varphi_{\sigma}: E_{\sigma}^{\bullet} \rightarrow E_{\partial\sigma}^{\bullet}$ induces an isomorphism

$$\overline{\varphi}_{\sigma}: \overline{E}_{\sigma}^{\bullet} \xrightarrow{\cong} \overline{E}_{\partial\sigma}^{\bullet}$$

of graded vector spaces.

Condition (LME) implies that \mathcal{E}^\bullet is minimal in the set of all flabby sheaves of graded \mathcal{A}^\bullet -modules that satisfy the conditions (N) and (PF), cf. Remark 3.3, whence the name “minimal extension sheaf”. – Furthermore, let us note that on a simplicial subfan Λ , the restriction $\mathcal{E}^\bullet|_\Lambda$ of such a sheaf is isomorphic to $\mathcal{A}^\bullet|_\Lambda$, so it is locally free of rank one. In the non-simplicial case, however, the rank of the stalks of \mathcal{E}^\bullet is not constant, so \mathcal{E}^\bullet can not be a locally free \mathcal{A}^\bullet -module.

We now show how to construct such a sheaf on an arbitrary fan space.

PROPOSITION 3.2. *On every fan Δ , there exists a minimal extension sheaf \mathcal{E}^\bullet , and it is unique up to isomorphism.*

PROOF. We define the sheaf \mathcal{E}^\bullet inductively on the k -skeleton subfans

$$\Delta^{\leq k} := \bigcup_{j \leq k} \Delta^j, \quad \Delta^j := \{\sigma \in \Delta; \dim \sigma = j\},$$

starting with $E_o^\bullet := \mathbb{Q}^\bullet$ for $k = 0$. Suppose that for some $k > 0$, the sheaf \mathcal{E}^\bullet has already been constructed on $\Delta^{\leq k-1}$, so in particular, for each k -dimensional cone σ , the module $E_{\partial\sigma}^\bullet$ is given. It thus suffices to define E_σ^\bullet together with a restriction homomorphism $E_\sigma^\bullet \rightarrow E_{\partial\sigma}^\bullet$ inducing an isomorphism on the quotients modulo \mathfrak{m} . This is achieved by setting

$$E_\sigma^\bullet := A_\sigma^\bullet \otimes_{\mathbb{Q}} \overline{E}_{\partial\sigma}^\bullet$$

with the restriction map being induced by some \mathbb{Q}^\bullet -linear section $s : \overline{E}_{\partial\sigma}^\bullet \rightarrow E_{\partial\sigma}^\bullet$ of the residue class map $E_{\partial\sigma}^\bullet \rightarrow \overline{E}_{\partial\sigma}^\bullet$.

The unicity is proved similarly in an inductive manner; we refer to our companion article [BBFK₂] for details. \square

REMARK 3.3. A minimal extension sheaf \mathcal{E}^\bullet is flabby and vanishes in odd degrees.

PROOF. Since the fan space Δ is covered by finitely many affine fans, it suffices to prove that for each cone $\sigma \in \Delta$, the restriction map φ_σ is surjective. Using the results on graded modules recalled in the “Preliminaries”, that is a consequence of condition (LME). – The vanishing of \mathcal{E}^{2q+1} follows immediately from the same condition, since A_σ^\bullet and thus $E_\sigma^\bullet \cong A_\sigma^\bullet = \mathbb{Q}^\bullet$ “live” only in even degrees. \square

The properties of minimal extension sheaves are investigated in [BBFK₂]; let us quote just one result:

The sheaf \mathcal{A}^\bullet is a minimal extension sheaf if and only if the fan Δ is simplicial.

Our aim here is to show that $\mathcal{I}H_{\mathbb{T}}^{\bullet}$ represents this unique isomorphism class of minimal extension sheaves.

MAIN THEOREM 3.4. *The equivariant intersection cohomology sheaf $\mathcal{I}H_{\mathbb{T}}^{\bullet}$ on Δ is a minimal extension sheaf.*

PROOF. The Normalization property is obviously satisfied, since we have

$$\mathcal{I}H_{\mathbb{T}}^{\bullet}(o) = IH_{\mathbb{T}}^{\bullet}(\mathbb{T}) \cong \mathbb{Q}^{\bullet}.$$

The Pointwise Freeness condition will be verified in Corollary 4.5, and the Local Minimal Extension requirement, in Proposition 5.1. \square

4. Equivariantly Formal Toric Varieties and Pointwise Freeness of $\mathcal{I}H_{\mathbb{T}}^{\bullet}$.

We now proceed to verify the condition

(PF) *For each cone σ , the A_{σ}^{\bullet} -module $IH_{\mathbb{T}}^{\bullet}(X_{\sigma})$ is free.*

This follows immediately from the isomorphism

$$IH_{\mathbb{T}}^{\bullet}(X_{\sigma}) \cong A_{\sigma}^{\bullet} \otimes IH^{\bullet}(Z_{\sigma})$$

obtained in the proof of Corollary 4.5 below, where Z_{σ} is the contractible affine \mathbb{T}_{σ} -toric variety occurring in the orbit splitting $X_{\sigma} \cong \mathbb{T}' \times Z_{\sigma}$ as in (0.3). The crucial point is that Z_{σ} satisfies the conditions – of course, with respect to the acting torus \mathbb{T}_{σ} – stated in the following result, a more general version of which can be found in [**GoKoMP**h].

LEMMA 4.1. *For a toric variety X , the following statements are equivalent:*

- i) *The Künneth formula $IH_{\mathbb{T}}^{\bullet}(X) \cong H^{\bullet}(B\mathbb{T}) \otimes IH^{\bullet}(X)$ holds (which, as above, implies that $IH_{\mathbb{T}}^{\bullet}(X)$ is a free A^{\bullet} -module).*
- ii) *With $\overline{IH_{\mathbb{T}}^{\bullet}(X)} := IH_{\mathbb{T}}^{\bullet}(X) / (\mathfrak{m} IH_{\mathbb{T}}^{\bullet}(X))$, each inclusion $X \hookrightarrow X_{\mathbb{T}}$ as a fibre induces an isomorphism*

$$(4.1) \quad \overline{IH_{\mathbb{T}}^{\bullet}(X)} \cong IH^{\bullet}(X)$$

of graded vector spaces.

- iii) *The non-equivariant intersection cohomology $IH^q(X)$ vanishes in odd degrees q .*

PROOF. Since condition i) says that for intersection cohomology, $X_{\mathbb{T}}$ behaves like the product $X \times B\mathbb{T}$, the implication “i) \Rightarrow ii)” is obvious, and “ii) \Rightarrow iii)” follows immediately from the Vanishing Lemma 2.1. For the implication “iii) \Rightarrow i)”, we observe that the

assumption implies the degeneration of the intersection cohomology Leray-Serre spectral sequence

$$E_2^{p,q} = H^p(B\mathbb{T}) \otimes IH^q(X) \Rightarrow IH_{\mathbb{T}}^{p+q}(X)$$

associated to the fibering $X_{\mathbb{T}} \rightarrow B\mathbb{T}$ at the E_2 -level: Since $H^p(B\mathbb{T})$ vanishes for odd p , the spectral terms $E_2^{p,q}$ vanish for odd total degrees $k = p + q$, and consequently, the differentials $d_2^{p,q}: E_2^{p,q} \rightarrow E_2^{p+2,q-1}$ are trivial. By induction on r , that holds also for every $r \geq 2$. \square

Thus, for toric varieties satisfying these properties, the equivariant and the non-equivariant theory determine each other in a simple way. Following [GoKoMPH], we use the following terminology:

DEFINITION 4.2. A toric variety X – and its defining fan – are called *equivariantly formal* (for intersection cohomology), or $IH_{\mathbb{T}}$ -formal for short, if and only if X satisfies one – and hence all – of the above three conditions.

We point out that there is an analogous notion for “ordinary” cohomology, and that a toric variety may be $IH_{\mathbb{T}}$ -formal, but not $H_{\mathbb{T}}$ -formal (e.g., a compact toric threefold with $b_3 \neq 0$). As we are mainly dealing with intersection cohomology, however, we use “equivariantly formal” only in the sense of “ $IH_{\mathbb{T}}$ -formal”.

The most important cases when toric varieties are equivariantly formal are the compact case and the contractible affine case. From the theorem of Jurkiewicz and Danilov cited in the introduction, it follows that a rationally smooth compact toric variety is equivariantly formal. The proof in the compact case is now an easy consequence of the famous Decomposition Theorem:

PROPOSITION 4.3. *A compact toric variety X is equivariantly formal.*

PROOF. Let $\pi: \hat{X} \rightarrow X$ denote a toric \mathbb{Q} -resolution as in the proof of the Vanishing Lemma 2.1. By the “classical” (i.e, non-equivariant) Decomposition Theorem of Beilinson, Bernstein, Deligne and Gabber, we know that $IH^{\bullet}(X)$ is isomorphic to a direct summand of $H^{\bullet}(\hat{X})$, and, according to the theorem of Jurkiewicz and Danilov, that module vanishes in odd degrees. \square

We now state the analogous result for the contractible affine case where it is considerably more difficult to handle:

THEOREM 4.4. *A contractible affine toric variety X is equivariantly formal.*

This result immediately yields the “Pointwise Freeness” property (PF).

COROLLARY 4.5. *For each cone $\sigma \in \Delta$, the A_σ^\bullet -module $\mathcal{I}H_{\mathbb{T}}^\bullet(X_\sigma)$ is free.*

PROOF OF COROLLARY 4.5. The affine orbit splitting $X_\sigma \cong \mathbb{T}' \times Z_\sigma$ as in (0.3) yields the isomorphism

$$(4.2) \quad IH_{\mathbb{T}}^\bullet(X_\sigma) \cong IH_{\mathbb{T}' \times \mathbb{T}_\sigma}^\bullet(\mathbb{T}' \times Z_\sigma) \cong IH_{\mathbb{T}_\sigma}^\bullet(Z_\sigma).$$

By the theorem, the contractible affine \mathbb{T}_σ -toric variety Z_σ is $IH_{\mathbb{T}_\sigma}^\bullet$ -formal, i.e., we have isomorphisms

$$IH_{\mathbb{T}_\sigma}^\bullet(Z_\sigma) \cong H^\bullet(B\mathbb{T}_\sigma) \otimes IH^\bullet(Z_\sigma) = A_\sigma^\bullet \otimes IH^\bullet(Z_\sigma),$$

showing that $IH_{\mathbb{T}_\sigma}^\bullet(Z_\sigma)$ is a free A_σ^\bullet -module and thus proving the corollary. \square

PROOF OF THEOREM 4.4. Writing $X := X_\sigma$ for ease of notation, we first notice that X is a distinguished neighbourhood of its (unique) fixed point x . According to the attachment condition (see [Bo, V.4.2]), we thus have an isomorphism

$$(4.3) \quad IH^\bullet(X) \cong \tau_{<n} IH^\bullet(X^*),$$

where $X^* := X \setminus \{x\}$ is the ‘‘punctured’’ toric variety obtained by removing the fixed point. Hence, it suffices to prove that $IH^q(X^*)$ vanishes in odd degrees $q < n$. The basic idea to reach that aim is to pass from X^* to a projective toric variety Y having X as its ‘‘affine cone’’ and then to compare $IH^\bullet(X^*)$ with $IH^\bullet(Y)$, keeping in mind that Y is equivariantly formal.

Such a projective toric variety Y is obtained as quotient of X^* modulo the action of any one parameter subgroup of \mathbb{T} having the following property: The orbits of the induced \mathbb{C}^* -action on X have the fixed point x as common ‘‘source’’, so an equivariant contraction is provided by $(t, x) \mapsto t \cdot x$ for $t \rightarrow 0$, where $t \in]0, 1] \subset \mathbb{C}^*$ is the parameter. A one parameter subgroup satisfies these conditions if and only if the representing lattice vector $\alpha \in N = \text{Hom}(\mathbb{C}^*, \mathbb{T})$ lies in the relative interior of σ . If, in addition, the lattice vector α is primitive, then the induced \mathbb{C}^* -action on X^* is free, possibly up to some finite isotropy along lower-dimensional \mathbb{T} -orbits. Let $F \subset \alpha(\mathbb{C}^*)$ be the finite subgroup generated by these isotropy groups. Then the equivariant mapping $X^* \rightarrow Y$ factors through the quotient X^*/F , making $X^*/F \rightarrow Y$ a principal \mathbb{C}^* -bundle (cf. Remark 1.7). Replacing X^* with X^*/F does not change the intersection cohomology: In fact, there is an identification

$$(4.4) \quad IH^\bullet(X^*/F) \cong IH^\bullet(X^*)^F = IH^\bullet(X^*)$$

(see [Ki₂, Lemma 2.12]), the equality holding since the action of F on $IH^\bullet(X^*)$ is trivial: It is induced from a – necessarily trivial – action of the connected group \mathbb{T} on the rational vector space $IH^*(X^*)$.

The quotient Y is a toric variety for the torus $\overline{\mathbb{T}} := \mathbb{T}/\alpha(\mathbb{C}^*)$, whose lattice of one parameter subgroups is $\overline{N} := N/(\mathbb{Z}\cdot\alpha)$. The defining fan for Y is the image

$$\Phi := p(\partial\sigma) := \{p(\tau) ; \tau \prec \sigma\}$$

of the boundary fan $\partial\sigma$ under the quotient mapping $p: V \rightarrow W$ from $V = N_{\mathbb{Q}}$ onto the quotient vector space $W := \overline{N}_{\mathbb{Q}} := N_{\mathbb{Q}}/(\mathbb{Q}\cdot\alpha)$.

We now proceed to proving that $\tau_{<n}IH^\bullet(X^*)$ vanishes in odd degrees, thus showing that X is equivariantly formal. By the identification (4.4), we may assume without loss of generality that F is the trivial subgroup and thus, that $X^* \rightarrow Y$ is a principal \mathbb{C}^* -bundle over the projective toric variety Y . We next consider the toric line bundle $L \rightarrow Y$ obtained from $X^* \rightarrow Y$ by adding a zero section opposite to the fixed point of X (“section at infinity”) to that \mathbb{C}^* -bundle: Its total space L is the toric variety associated to the fan

$$\Sigma := \partial\sigma \cup \{\tau + \mathbb{Q}_{\geq 0}\cdot(-\alpha) ; \tau \prec \sigma\}$$

in $V = N_{\mathbb{Q}}$, with the projection $N \rightarrow \overline{N}$ inducing a mapping of fans $\Sigma \rightarrow \Phi$ and thus, a toric morphism $L \rightarrow Y$. The line bundle $L \rightarrow Y$ is ample (see, e.g., [Oda₁, 2.12, p. 82]), so a suitable tensor power is very ample. If L is already very ample itself, then the one point compactification of L is the projective cone over Y with respect to the projective embedding determined by the sections of L , the corresponding complete fan being $\Sigma \cup \{\sigma\}$.

Let us now look at the following commutative diagram whose top row is part of the long exact IH^\bullet -sequence associated to the pair (L, X^*) :

$$\begin{array}{ccccccc} \dots & \longrightarrow & IH^{q-1}(X^*) & \longrightarrow & IH^q(L, X^*) & \longrightarrow & IH^q(L) & \longrightarrow & IH^q(X^*) & \longrightarrow & \dots \\ & & & & \uparrow \cong & & \uparrow \cong & & & & \\ & & & & IH^{q-2}(Y) & \xrightarrow{\lambda} & IH^q(Y) & & & & \end{array}$$

Here the first vertical isomorphism is the Thom isomorphism for the line bundle, the second one is induced by the bundle projection, and

$$\lambda: IH^\bullet(Y)[-2] \rightarrow IH^\bullet(Y)$$

is the homomorphism given by cup product multiplication with $c_1(L) \in H^2(Y)$. The resulting long exact sequence

$$\dots \longrightarrow IH^{q-1}(X^*) \longrightarrow IH^{q-2}(Y) \xrightarrow{\lambda} IH^q(Y) \longrightarrow IH^q(X^*) \longrightarrow \dots$$

is the Gysin sequence associated to the bundle $X^* \rightarrow Y$.

Using the fact that Y is equivariantly formal and thus has vanishing intersection cohomology in odd degrees, the Gysin sequence decomposes into shorter exact sequences

$$0 \longrightarrow IH^{q-1}(X^*) \longrightarrow IH^{q-2}(Y) \xrightarrow{\lambda} IH^q(Y) \longrightarrow IH^q(X^*) \longrightarrow 0$$

if q is even. Hence, it suffices to show that λ is injective for $q \leq n$.

Since L is ample, we may apply the hard Lefschetz theorem for intersection cohomology: a suitable tensor power $L^{\otimes m}$ of L is very ample, and since $c_1(L^{\otimes m}) = mc_1(L)$, the assertion of the theorem holds for L as well. Hence, we know that

$$\lambda^k: IH^{n-1-k}(Y) \longrightarrow IH^{n-1+k}(Y)$$

is an isomorphism for every $k \geq 0$; as a consequence, we see that

$$\lambda: IH^{q-2}(Y) \longrightarrow IH^q(Y)$$

is injective for $q \leq n$ and surjective for $q \geq n$. \square

For later use, we note that the properties of λ yield an isomorphism

$$(4.5) \quad \text{coker } \lambda \cong \tau_{<n} IH^\bullet(X^*).$$

5. Local Minimal Extension Property of $\mathcal{I}H_{\mathbb{T}}^\bullet$.

To complete the proof of our Main Theorem stating that the sheaf $\mathcal{I}H_{\mathbb{T}}^\bullet$ is a minimal extension sheaf, we have to verify the condition (LME). We restate it as follows:

PROPOSITION 5.1. *For each cone σ , the restriction mapping*

$$IH_{\mathbb{T}}^\bullet(X_\sigma) \rightarrow IH_{\mathbb{T}}^\bullet(X_{\partial\sigma})$$

induces an isomorphism between the quotients modulo the maximal ideal \mathfrak{m} of A^\bullet .

PROOF. By the ‘‘relative affine orbit splitting’’ $(X_\sigma, X_{\partial\sigma}) \cong \mathbb{T} \times (Z_\sigma, Z_{\partial\sigma})$ and the formula (4.2) as well as its analogue $IH_{\mathbb{T}}^\bullet(X_{\partial\sigma}) \cong IH_{\mathbb{T}_\sigma}^\bullet(Z_{\partial\sigma})$, we see that it is sufficient to consider the case of an n -dimensional cone σ . We use the same notations as in the previous section; in particular, we write $X = X_\sigma$ and $X^* = X_\sigma^*$. Furthermore, by the arguments of the preceding section, we may replace \mathbb{T} , X_σ , and $X_{\partial\sigma}$ by \mathbb{T}/F , X_σ/F , and $X_{\partial\sigma}/F$, respectively, where $F \subset \mathbb{T}$ is a suitable finite subgroup, without changing the base ring $H^\bullet(B\mathbb{T}) \cong H^\bullet(B(\mathbb{T}/F))$ and the above homomorphism. Hence we may assume that $X^* \rightarrow Y$ is a principal \mathbb{C}^* -bundle.

First we collect in a big commutative diagram all the objects we have to consider:

$$\begin{array}{ccccc}
IH_{\mathbb{T}}^{\bullet}(X) & \longrightarrow & IH_{\mathbb{T}}^{\bullet}(X^*) & \xleftarrow[\pi_{\mathbb{T}}]{\cong} & IH_{\mathbb{T}}^{\bullet}(Y) \\
\downarrow & & \downarrow & & \downarrow \\
\overline{IH_{\mathbb{T}}^{\bullet}(X)} & \longrightarrow & \overline{IH_{\mathbb{T}}^{\bullet}(X^*)} & \xleftarrow[\pi_{\mathbb{T}}]{} & \overline{IH_{\mathbb{T}}^{\bullet}(Y)} \\
\downarrow \cong & & \cap & & \downarrow \cong \\
IH^{\bullet}(X) & \longrightarrow & IH^{\bullet}(X^*) & \xleftarrow[\pi]{} & IH^{\bullet}(Y) \\
& \cong \searrow & \cup & & \downarrow \\
& & \tau_{<n} IH^{\bullet}(X^*) & \xleftarrow[\cong]{} & \text{coker } \lambda.
\end{array}$$

Here, λ again denotes the homomorphism given by the cup product with the Chern class $c_1(L)$ of the line bundle $L \rightarrow Y$ as in the previous section.

Once having established the diagram, the proof is achieved as soon as we have identified the quotient

$$\overline{IH_{\mathbb{T}}^{\bullet}(X^*)} := (A^{\bullet}/\mathfrak{m}) \otimes_{A^{\bullet}} IH_{\mathbb{T}}^{\bullet}(X^*) \cong IH_{\mathbb{T}}^{\bullet}(X^*) / \mathfrak{m} \cdot IH_{\mathbb{T}}^{\bullet}(X^*)$$

– this is just the image of the “edge homomorphism” $IH_{\mathbb{T}}^{\bullet}(X^*) \rightarrow IH^{\bullet}(X^*)$ relating the equivariant and the non-equivariant theory – with $\tau_{<n} IH^{\bullet}(X^*)$.

To that end, we consider the right hand side of the diagram, carefully keeping track of the different module structures in the top row: Whereas $IH_{\mathbb{T}}^{\bullet}(X)$ and $IH_{\mathbb{T}}^{\bullet}(X^*)$ are both modules over $A^{\bullet} = S^{\bullet}(M_{\mathbb{Q}}) \cong H^{\bullet}(B\mathbb{T})$, we have to look at $IH_{\mathbb{T}}^{\bullet}(Y)$ as a module over the ring $B^{\bullet} := S^{\bullet}(\overline{M}_{\mathbb{Q}}) \cong H^{\bullet}(B\overline{\mathbb{T}})$, where $\overline{M} := \overline{N}^*$. In particular, the “overlined” modules in the second row are quotients modulo the maximal ideals of the respective base rings: For X and X^* , this is the ideal $\mathfrak{m} = \mathfrak{m}_A := A^{>0}$, whereas for Y , we have to consider

$$\overline{IH_{\mathbb{T}}^{\bullet}(Y)} := (B^{\bullet}/\mathfrak{m}_B) \otimes_{B^{\bullet}} IH_{\mathbb{T}}^{\bullet}(Y) \cong IH_{\mathbb{T}}^{\bullet}(Y) / \mathfrak{m}_B \cdot IH_{\mathbb{T}}^{\bullet}(Y)$$

(with $\mathfrak{m}_B := B^{>0}$). As the projective $\overline{\mathbb{T}}$ -toric variety Y is equivariantly formal, we may identify this graded \mathbb{Q}^{\bullet} -module with $IH^{\bullet}(Y)$. We may now consider B^{\bullet} as a subring of A^{\bullet} since $\overline{M} = \overline{N}^*$ is canonically isomorphic to a submodule of $N^* = M$. Hence, writing the horizontal arrow $\overline{\pi}_{\mathbb{T}}: \overline{IH_{\mathbb{T}}^{\bullet}(Y)} \rightarrow \overline{IH_{\mathbb{T}}^{\bullet}(X^*)}$ as

$$IH_{\mathbb{T}}^{\bullet}(Y) / \mathfrak{m}_B \cdot IH_{\mathbb{T}}^{\bullet}(Y) \longrightarrow IH_{\mathbb{T}}^{\bullet}(X^*) / \mathfrak{m}_A \cdot IH_{\mathbb{T}}^{\bullet}(X^*),$$

we see that it is an epimorphism, being induced by the horizontal isomorphism $\pi_{\mathbb{T}}$ in the top row. Thus, using the isomorphism in the

bottom row, we are done if we can show that the kernel of π equals the image of the “hard Lefschetz homomorphism” λ .

Before we do that let us give some further remarks on the big diagram: The isomorphism

$$(5.1) \quad IH_{\mathbb{T}}^{\bullet}(Y) \xrightarrow[\pi_{\mathbb{T}}]{\cong} IH_{\mathbb{T}}^{\bullet}(X^*)$$

in the top row is obtained as follows: The bundle projection $p: X^* \rightarrow Y$ induces a compatible family of projections

$$X_{\mathbb{T},m}^* \rightarrow Y_{\mathbb{T},m} \cong Y_{\mathbb{T},m} \times B_m \mathbb{C}^* \rightarrow Y_{\mathbb{T},m}$$

between the finite-dimensional approximations of $X_{\mathbb{T}}$ and $Y_{\mathbb{T}}$. As these projections are *placid* maps, there is a (unique) induced homomorphism

$$\pi_{\mathbb{T}}: IH_{\mathbb{T}}^{\bullet}(Y) \rightarrow IH_{\mathbb{T}}^{\bullet}(X^*)$$

(for a discussion, see [GoMPH, §4] or [BBFGK, 3.3]). We now note that the bundle is locally trivial and that there is a finite open affine covering of Y by toric subvarieties V_i such that the restricted bundle $U_i := p^{-1}(V_i) \rightarrow V_i$ actually is trivial. By (4.2), we thus have isomorphisms $IH_{\mathbb{T}}^{\bullet}(U_i) \cong IH_{\mathbb{T}}^{\bullet}(V_i)$; gluing these by a Mayer-Vietoris argument yields the result. – We note that, by construction, the isomorphism (5.1) is a morphism of B^{\bullet} -modules; it thus induces an isomorphism of the quotients modulo the homogeneous maximal ideal $\mathfrak{m}_B = B^{>0}$, while $\overline{IH_{\mathbb{T}}^{\bullet}(X^*)}$ refers to the bigger base ring A^{\bullet} and thus is a quotient of $\overline{IH_{\mathbb{T}}^{\bullet}(Y)}$.

We recall why we obtain the other isomorphisms occurring in the diagram: By the attachment condition, the restriction mapping $IH^{\bullet}(X) \rightarrow IH^{\bullet}(X^*)$ factors through the inclusion of $\tau_{<n} IH^{\bullet}(X^*)$, inducing the oblique isomorphism. The two vertical isomorphisms follow from the fact that both X and Y are equivariantly formal. The lower horizontal isomorphism has been obtained in (4.5) in the previous section.

We now continue with the proof of Proposition 5.1. First of all, we lift the “hard Lefschetz map” $\lambda: IH^{\bullet}(Y)[-2] \rightarrow IH^{\bullet}(Y)$ to a map in equivariant intersection cohomology: We make the line bundle $L \rightarrow Y$ a $\overline{\mathbb{T}}$ -toric line bundle by choosing some sublattice N_0 complementary to $\mathbb{Z} \cdot \alpha$, thus providing a direct sum decomposition $N = \mathbb{Z}\alpha \oplus N_0$. From section 1.B, we recall that the $\overline{\mathbb{T}}$ -equivariant Chern class $c_1^{\overline{\mathbb{T}}}(L) \in H_{\overline{\mathbb{T}}}^2(Y)$ of L is a lifting of the “usual” Chern class $c_1(L) \in H^2(Y)$. It follows that the cup product with $c_1^{\overline{\mathbb{T}}}(L)$ yields a homomorphism $\lambda_{\overline{\mathbb{T}}}: IH_{\overline{\mathbb{T}}}^{\bullet}(Y)[-2] \rightarrow IH_{\overline{\mathbb{T}}}^{\bullet}(Y)$ that is a lifting of the mapping $\lambda: IH^{\bullet}(Y)[-2] \rightarrow IH^{\bullet}(Y)$ given by the cup product with $c_1(L)$.

We further recall that $\mathcal{I}H_{\mathbb{T}}^\bullet$ is a sheaf on the defining fan Φ for Y , and it is a module over the “structure sheaf” ${}_{\Phi}\mathcal{A}^\bullet$ corresponding to the fan Φ . Using the canonical isomorphism $H_{\mathbb{T}}^2(Y) \rightarrow \mathcal{A}^2(\Phi)$, we may identify the equivariant Chern class $c_{\mathbb{T}}^{\overline{1}}(L) \in H^2(Y_{\mathbb{T}})$ with the Φ -piecewise linear function $\psi := \psi_L \in A^2(\Phi)$ (see Remark 1.4, (ii), and Lemma 1.8).

Now the quotient projection $N \rightarrow \overline{N}$ induces an isomorphism

$${}_{\Phi}\mathcal{A}^\bullet(\Phi) \xrightarrow{\cong} {}_{\Delta}\mathcal{A}^\bullet(\partial\sigma),$$

and the image of $\psi \in \mathcal{A}^2(\Phi)$ coincides with the restriction to $\partial\sigma$ of the “global” linear form $f \in A^2$, the projection $f: N_{\mathbb{Q}} = \mathbb{Q}\cdot\alpha \oplus (N_0)_{\mathbb{Q}} \rightarrow \mathbb{Q}$ mapping α to -1 and having $(N_0)_{\mathbb{Q}}$ as kernel, cf. section 1.B.

On the other hand, we have an isomorphism of polynomial rings $A^\bullet = B^\bullet[f]$ and thus, the equality $\mathfrak{m} = \mathfrak{m}_A = (\mathfrak{m}_B, f) := \mathfrak{m}_B + A^\bullet \cdot f$ for the homogeneous maximal ideals. The proof of the assertion that the restriction homomorphism $IH_{\mathbb{T}}^\bullet(X) \rightarrow IH_{\mathbb{T}}^\bullet(X^*)$ induces an isomorphism on the quotients with respect to the submodules generated by \mathfrak{m}_A is now obtained as follows: Under the inverse of the isomorphism (5.1), the submodule $\mathfrak{m}_A \cdot IH_{\mathbb{T}}^\bullet(X^*) = (\mathfrak{m}_B, f) \cdot IH_{\mathbb{T}}^\bullet(X^*)$ is mapped onto $(\mathfrak{m}_B, \psi) \cdot IH_{\mathbb{T}}^\bullet(Y)$. We thus have an isomorphism of quotients

$$(5.2) \quad \frac{IH_{\mathbb{T}}^\bullet(X^*)}{\mathfrak{m}_A \cdot IH_{\mathbb{T}}^\bullet(X^*)} \cong \frac{IH_{\mathbb{T}}^\bullet(Y)}{(\mathfrak{m}_B, \psi) \cdot IH_{\mathbb{T}}^\bullet(Y)}.$$

As explained above, the mapping $IH_{\mathbb{T}}^\bullet(Y)[-2] \rightarrow IH_{\mathbb{T}}^\bullet(Y)$ given as multiplication by $\psi \in \mathcal{A}^2(\Phi)$ lifts the “Hard Lefschetz homomorphism” $\lambda: IH^\bullet(Y)[-2] \rightarrow IH^\bullet(Y)$ to the equivariant theory. Since Y , as a projective \mathbb{T} -toric variety, is equivariantly formal, we may eventually rewrite the right hand side of the above isomorphism (5.2) as follows:

$$\frac{IH_{\mathbb{T}}^\bullet(Y)}{(\mathfrak{m}_B, \psi) \cdot IH_{\mathbb{T}}^\bullet(Y)} \cong \frac{IH^\bullet(Y)}{\lambda(IH^\bullet(Y))} = \text{coker } \lambda.$$

The proof is now achieved using the isomorphisms (4.5), (4.3), and the fact that the contractible affine toric variety X is $IH_{\mathbb{T}}^\bullet$ -formal. \square

6. Some Results on Equivariantly Formal Fans

We recall from Lemma 4.1 that in the case of an equivariantly formal toric variety, the equivariant intersection cohomology determines the “usual” intersection cohomology in a straightforward way, namely, as the quotient by the A^\bullet -submodule generated by the homogeneous maximal ideal \mathfrak{m} of the ring A^\bullet . Since the equivariant intersection cohomology sheaf $\mathcal{I}H_{\mathbb{T}}^\bullet$ is a minimal extension sheaf, we thus have an

isomorphism $IH^\bullet(X_\Delta) \cong \overline{\mathcal{E}^\bullet(\Delta)} := \mathcal{E}^\bullet(\Delta)/\mathfrak{m} \cdot \mathcal{E}^\bullet(\Delta)$, where as usual \mathcal{E}^\bullet denotes a minimal extension sheaf on Δ .

It is convenient to call a (rational) fan Δ *equivariantly formal* if the toric variety X_Δ has that property. In Proposition 6.1, ii) below, we shall see that equivariant formality can be characterized by the freeness of the A^\bullet -module $IH_\mathbb{T}^\bullet(X)$. Using the isomorphism $IH_\mathbb{T}^\bullet(X_\Delta) \cong \mathcal{E}^\bullet(\Delta)$ together with the fact that minimal extension sheaves exist on non-rational fans as well, this allows to introduce the notion of a (“virtually”) equivariantly formal fan in the non-rational case, thus eventually leading to a notion of “virtual” intersection cohomology for arbitrary equivariantly formal fans.

So far, we know that complete fans and n -dimensional affine fans are of this type. In order to study further examples, we first collect some properties pertinent to equivariantly formal fans.

- PROPOSITION 6.1. i) *The A^\bullet -module $\mathcal{I}H_\mathbb{T}^\bullet(\Delta)$ is torsion-free if and only if we have Δ is purely n -dimensional.*
 ii) *The A^\bullet -module $\mathcal{I}H_\mathbb{T}^\bullet(\Delta)$ is free if and only if Δ is equivariantly formal.*
 iii) *An equivariantly formal fan Δ is purely n -dimensional.*
 iv) *If Δ has an equivariantly formal subdivision $\hat{\Delta}$, then Δ itself is equivariantly formal.*

We note explicitly that, as a consequence of ii), a notion of “virtual” equivariant formality may be defined even for not necessarily rational fans in \mathbb{R}^n via minimal extension sheaves – no toric varieties are needed.

PROOF. i) “ \Leftarrow ”: Since $\mathcal{I}H_\mathbb{T}^\bullet$ is a sheaf, we see that we have a natural inclusion

$$\mathcal{I}H_\mathbb{T}^\bullet(\Delta) \subset \bigoplus_{\sigma \in \Delta^{\max}} \mathcal{I}H_\mathbb{T}^\bullet(\sigma)$$

of A^\bullet -modules. By Corollary 4.5, each $\mathcal{I}H_\mathbb{T}^\bullet(\sigma)$ is a free A_σ^\bullet -module. For $\dim \sigma = n$, we have $A^\bullet \cong A_\sigma^\bullet$, so the right hand side is a free A^\bullet -module. Moreover, every submodule of a torsion-free module is again torsion-free.

“ \Rightarrow ”: If $\sigma \in \Delta$ is a maximal cone of dimension $d < n$, let $\Delta' := \Delta \setminus \{\sigma\}$. The product of some non-zero polynomial function $h \in A_\sigma^\bullet$ vanishing on $\partial\sigma$ (such a function can be obtained as a product $h = \prod_{\tau \prec_1 \sigma} \ell_\tau$ of non-zero linear functions $\ell_\tau \in A_\sigma^2$ with $\ell_\tau|_\tau = 0$) and of a non-zero section in $\mathcal{I}H_\mathbb{T}^\bullet(\sigma)$ yields a non-zero section $f \in \mathcal{I}H_\mathbb{T}^\bullet(\sigma)$ (recall that $\mathcal{I}H_\mathbb{T}^\bullet(\sigma)$ is a free A_σ^\bullet -module!) that vanishes on $\partial\sigma$. We extend it trivially outside of σ and thus get a non-trivial torsion element, since it is “killed” by every non-zero global linear function in A^2 vanishing on σ .

ii): “ \Leftarrow ”: This is clear by Lemma 4.1, i).

“ \Rightarrow ”: We only sketch the argument, leaving details for future exposition: Consider the intersection cohomology version of the *Eilenberg-Moore spectral sequence* (see, e.g., [MCI, § 7.2.1]) that computes the intersection cohomology of the pull back of a bundle. Here we look at the bundle $X_{\mathbb{T}} \rightarrow B\mathbb{T}$ and take as map the inclusion of a one point set $\{\text{pt}\}$ into $B\mathbb{T}$. Since $B\mathbb{T}$ is simply connected, the spectral sequence converges: We have $E_2^{p,q} \Rightarrow IH^{p+q}(X)$ with

$$E_2^{p,q} \cong \text{Tor}_{H^\bullet(B\mathbb{T})}^{p,q}(H^\bullet(\text{pt}), IH_{\mathbb{T}}^\bullet(X)) \cong \text{Tor}_{A^\bullet}^{p,q}(\mathbb{Q}^\bullet, IH_{\mathbb{T}}^\bullet(X)).$$

Here A^\bullet , \mathbb{Q}^\bullet , and $IH_{\mathbb{T}}^\bullet(X)$, respectively, are considered as differential graded algebras resp. modules with trivial differential, and we can compare with the classical Tor functors of commutative algebra: If M^\bullet, N^\bullet are graded A^\bullet -modules, the corresponding Tor-modules are again in a natural way graded A^\bullet -modules:

$$\text{Tor}_p^{A^\bullet}(M^\bullet, N^\bullet) = \bigoplus_{q=0}^{\infty} \text{Tor}_p^q(M^\bullet, N^\bullet)$$

such that

$$\text{Tor}_{A^\bullet}^{p,q}(M^\bullet, N^\bullet) \cong \text{Tor}_{-p}^q(M^\bullet, N^\bullet).$$

Since $IH_{\mathbb{T}}^\bullet(X)$ is a free A^\bullet -module, we obtain that

$$E_2^{0,\bullet} \cong \mathbb{Q}^\bullet \otimes_{A^\bullet} IH_{\mathbb{T}}^\bullet(X) \cong \overline{IH_{\mathbb{T}}^\bullet(X)}$$

and $E_2^{p,q} = \{0\}$ for $p \neq 0$. So in particular, we have $E_2^{p,q} = \{0\}$ for $p+q$ odd, and hence also $IH^{p+q}(X) = \{0\}$ in that case.

iii) This follows immediately from i) and ii), since free modules are torsion-free.

iv) According to the Decomposition Theorem of Beilinson, Bernstein, Deligne, and Gabber, we know that $IH^\bullet(X_\Delta)$ is a direct summand of $IH^\bullet(X_{\hat{\Delta}})$, thus inheriting the property that the “usual” intersection cohomology vanishes in odd degrees. \square

The following example shows that the condition $\Delta^{\max} = \Delta^n$ of i) is not sufficient for equivariant formality:

EXAMPLE 6.2. Let Δ be a fan consisting of two equivariantly formal subfans Δ_1 and Δ_2 intersecting in a single cone τ . If the codimension of τ is at least 2, then Δ is not equivariantly formal.

PROOF. We intend to prove that the intersection cohomology Betti number Ib_3 is non-zero. With $X := X_\Delta$ and $X_i := X_{\Delta_i}$, we consider the following part of the exact Mayer-Vietoris sequence:

$$0 \rightarrow IH^1(X_\tau) \rightarrow IH^2(X) \rightarrow IH^2(X_1) \oplus IH^2(X_2) \rightarrow IH^2(X_\tau) \rightarrow IH^3(X) \rightarrow 0.$$

The zeroes at both ends are due to the fact that the toric varieties X_1 and X_2 are equivariantly formal.

The “affine orbit splitting” (0.3) provides an isomorphism $X_\tau \cong (\mathbb{C}^*)^k \times Z_\tau$ with $x := \text{codim } \tau$, where Z_τ , as a contractible affine \mathbb{T}_τ -toric variety, is known to be equivariantly formal. By the Künneth formula, we have $IH^\bullet(X_\tau) \cong H^\bullet((\mathbb{C}^*)^k) \otimes_{\mathbb{Q}} IH^\bullet(Z_\tau)$; in particular, we get $Ib^1(X_\tau) = k$ and $Ib^2(X_\tau) = \binom{k}{2} + Ib^2(Z_\tau)$. By the results of [BBFK, §4], the Betti number $Ib^2(X_\Phi)$ of an arbitrary d -dimensional toric variety given by a non-degenerate fan Λ (i.e., such that Λ spans V) is determined by the number $a := \#\Lambda^{(1)}$ of rays, namely, we have $Ib^2(X_\Lambda) = a - d$. Denoting with $a_i := \#\Delta_i^{(1)}$ and $a_3 := \#\tau^{(1)}$ the respective number of rays, we clearly have $\#\Delta^{(1)} = a_1 + a_2 - a_3$. We thus obtain $Ib^2(X_i) = a_i - n$, $Ib^2(X) = a_1 + a_2 - a_3 - n$, and $Ib^2(Z_\tau) = a_3 - (n - k)$. Since the Euler characteristic of an exact sequence vanishes, we obtain $Ib^3(X) = k(k - 1)/2 > 0$. \square

We now state a necessary “topological” condition for a fan to be equivariantly formal, thus obviously providing many examples of toric varieties not having that property.

PROPOSITION 6.3. *If Δ is an equivariantly formal fan, then the complement $N_{\mathbb{R}} \setminus |\Delta|$ of the support $|\Delta|$ is connected.*

This is a consequence of the following inequality:

LEMMA 6.4. *The intersection cohomology Betti number $Ib^{2n-1}(X_\Delta)$ satisfies the inequality*

$$Ib^{2n-1}(X_\Delta) \geq b_0(N_{\mathbb{R}} \setminus |\Delta|) - 1.$$

PROOF. It is known (see, e.g., [KaFi, 3.5]) that the natural homomorphism

$$IH^{2n-1}(X_\Delta) \longrightarrow H_1^{\text{cld}}(X_\Delta)$$

is surjective (and even an isomorphism, though we do not need this stronger result). To investigate the target, let $\tilde{\Delta}$ be a completion of the fan. The set of cones $\tilde{\Delta} \setminus \Delta$ defines a closed invariant subvariety \tilde{A} of the compact toric variety $\tilde{X} := X_{\tilde{\Delta}}$ that has the same number $b_0(\tilde{A})$ of connected components as $N_{\mathbb{R}} \setminus |\Delta|$. Combining that with the isomorphism $H_1^{\text{cld}}(X_\Delta) \cong H_1(\tilde{X}, \tilde{A})$ and with the exact sequence

$$\dots \longrightarrow H_1(\tilde{X}, \tilde{A}) \longrightarrow H_0(\tilde{A}) \longrightarrow H_0(\tilde{X}) \longrightarrow H_0(\tilde{X}, \tilde{A}) = 0$$

eventually yields the following chain of inequalities

$$Ib^{2n-1}(X_\Delta) \geq b^{2n-1}(X_\Delta) \geq b_1(X_\Delta) = b_1(\tilde{X}, \tilde{A}) \geq b_0(\tilde{A}) - 1 = b_0(N_{\mathbb{R}} \setminus |\Delta|)$$

that proves the assertion. \square

Suppose a fan Δ is obtained from an equivariantly formal fan Δ_0 by adding some n -dimensional cone together with its faces. Then is natural to ask if Δ is again equivariantly formal. The above example 6.2 shows that $|\Delta_0| \cap \sigma$ should not be of too small dimension.

Some partial positive results are given by the following propositions.

PROPOSITION 6.5. *If an equivariantly formal fan Δ_0 and an n -dimensional cone σ intersect in a single simplicial cone $\tau \in \Delta$ that is a facet of σ , then the enlarged fan*

$$\Delta := \Delta_0 \cup \langle \sigma \rangle$$

is equivariantly formal.

PROOF. We have to prove that the toric variety $X := X_\Delta$ has vanishing intersection cohomology in each odd degree q if $X_0 := X_{\Delta_0}$ has. To that end, we look at the exact sequence

$$\dots \longrightarrow IH^{q-1}(X_0) \longrightarrow IH^q(X, X_0) \longrightarrow IH^q(X) \longrightarrow IH^q(X_0) = 0$$

where the final term vanishes since q is odd. It clearly suffices to prove that $IH^q(X, X_0)$ vanishes. We may identify this relative group with $IH^q(X_\sigma, X_\tau)$ by excision. We thus consider the analogous exact sequence

$$\dots \longrightarrow IH^{q-1}(X_\tau) \longrightarrow IH^q(X_\sigma, X_\tau) \longrightarrow IH^q(X_\sigma) = 0.$$

As τ is simplicial, there is an isomorphism $X_\tau \cong (\mathbb{C}^{n-1}/F) \times \mathbb{C}^*$, where F is a finite subgroup of $(\mathbb{C}^*)^{n-1}$ acting diagonally on \mathbb{C}^{n-1} . By the Künneth formula, we thus have isomorphisms $IH^\bullet(X_\tau) \cong \mathbb{Q}^\bullet \otimes_{\mathbb{Q}} H^\bullet(\mathbb{C}^*) \cong H^\bullet(S^1)$, so $IH^q(X_\tau)$ vanishes for each $q > 1$. For $q \geq 3$, these facts immediately yield $IH^q(X_\sigma, X_\tau) = 0$. The remaining case $q = 1$ follows from the fact that the restriction mapping $IH^0(X_\sigma) \rightarrow IH^0(X_\tau)$ is an isomorphism. \square

If the cone σ is simplicial, we may even allow that it meets Δ in several facets:

PROPOSITION 6.6. *If an equivariantly formal fan Δ_0 and an n -dimensional simplicial cone σ intersect in a subfan Λ generated by facets $\tau_i \prec_1 \sigma$ for $i = 1, \dots, \ell$, (i.e., each τ_i is a cone of Δ_0), then the enlarged fan*

$$\Delta := \Delta_0 \cup \langle \sigma \rangle$$

is equivariantly formal.

PROOF. As in the proof of Proposition 6.5, we have to show that $IH^\bullet(X, X_0) \cong IH^q(X_\sigma, X_\Lambda)$ vanishes in odd degrees. There is an isomorphism

$$(X_\sigma, X_\Lambda) \cong (\mathbb{C}^{n-\ell} \times (\mathbb{C}^\ell, \mathbb{C}^\ell \setminus \{0\})) / F,$$

where F is a finite subgroup of $(\mathbb{C}^*)^n$ acting diagonally on \mathbb{C}^n , such that $\mathbb{T} \cong (\mathbb{C}^*)^n/F$. As passing to the quotient by F does not influence the rational (intersection) cohomology (see [Bre, Thm. II.19.2]), we obtain

$$IH^q(X_\sigma, X_\Lambda) \cong H^q(X_\sigma, X_\Lambda) \cong H^q(\mathbb{C}^\ell, \mathbb{C}^\ell \setminus \{0\}) = 0 \quad \text{for } q \neq 0, 2\ell,$$

thus proving the assertion. \square

Certain fans obtained from complete ones by removing a few n -dimensional cones are equivariantly formal. This is implied by the following result, stated in [Oda₂, Th. 4.2] and rephrased here in terms of equivariant formality:

THEOREM 6.7. [Ishida] *The subfan supported by the complement of the star of a single ray in a complete fan is equivariantly formal.*

COROLLARY 6.8. i) *A purely n -dimensional fan with convex support is equivariantly formal.*

ii) *A purely n -dimensional fan with open convex complement is equivariantly formal. In particular, if X is a complete toric variety and x_σ , a fixed point, then $X \setminus \{x_\sigma\}$ is equivariantly formal*

PROOF. We choose a vector v such that either $-v$ lies in the interior of the support (case i), or that v lies in the interior of the complement (case ii), and complete the given fan Δ by adding the new ray $\rho := \mathbb{Q}_{>0}v$ together with all cones of the form $\rho + \tau$, where τ is a cone in the boundary of the support. The “old” fan then is the complement of the star of the “new” ray. \square

Proposition 6.3 above shows that the completeness hypothesis in the second part of ii) cannot be omitted: If Δ is a non-complete fan and $\sigma \in \Delta$, an n -dimensional cone, then the support of the subfan $\Delta \setminus \{\sigma\}$ that is defining $X_\Delta \setminus \{x_\sigma\}$ has a non-connected complement, since $|\Delta \setminus \{\sigma\}|$ is obtained from $|\Delta|$ by removing the interior of σ , which is separated by the facets from the exterior and thus, from the complement of $|\Delta|$.

Using Proposition 6.2 and some homological algebra, we shall prove in the companion article [BBFK₂] the “topological” characterization of equivariantly formal fans given in the theorem below. To formulate it, we use the following notation: For a purely n -dimensional fan Δ , we denote with $\partial\Delta$ the subfan supported by the topological boundary of $|\Delta|$ (this is the subfan generated by those $(n-1)$ -dimensional cones that are contained in only one n -dimensional cone). Fix a euclidean norm in the real vector space $N_{\mathbb{R}}$, and denote with S

resp. with ∂S the intersection of the unit sphere with the support of the fan Δ resp. of $\partial\Delta$. Furthermore let $Z := S \cup c(\partial S)$ be the compact topological space obtained by patching together S and the real cone $c(\partial S)$ over ∂S along their respective boundaries.

THEOREM 6.9. *A purely n -dimensional fan Δ is equivariantly formal if and only if the following conditions hold: The pair $(S, \partial S)$ has the real homology of an $(n - 1)$ -ball modulo its boundary, and Z is a real homology manifold outside the vertex of the cone $c(\partial S)$. The last condition is satisfied e.g. if S is a manifold with boundary.*

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