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This note grew out of discussions about the preprint [Si].

Chang-Skjelbred (s.[CS]) and G. Bredon (s.[Br]) have proved independently that the fixed point components of a G-action, $G = S^1$ or $\mathbb{Z}_p$, p odd, on a Poincaré duality space again fulfil Poincaré duality (with coefficients in $\mathbb{Q}$ or $\mathbb{Z}_p$ respectively). By now there are several further versions and variants of proofs for this result (s.[Ra], [AP], [Ha]). We will use the approach given in [AP] to obtain certain consequences of Poincaré duality, which improve the classical relations between the Betti numbers of the total space and its fixed point set. Results in this direction are already stated in a recent preprint by A. Sikora ([Si], (1.2.1), (1.3.2)) but for $\mathbb{Z}_p$-actions Sikora's proof is incomplete. We can only prove the desired statement for $\mathbb{Z}_p$-action under certain additional assumptions. The methods of proof, even for the S^1-case which is due to Sikora, are somewhat different from Sikora's.

1. S^1-actions

Let X be a finite-dimenional G-CW-complex, $G = S^1$, such that $H^*(X;\mathbb{Q})$ fulfils Poincaré duality with formal dimension $fd_{\mathbb{Q}}(X) = n$.

(1.1) Theorem. *If*

(i) n *is even, or*

(ii) $n = 2m+1, X^G \neq 0, \;\; b_i(X) = 0 \;\;$ *for* $\;\; 0 < i \leq m, \;\; i$ *even*

then

$$\sum_i b_i(X^G) \equiv \sum_i b_i(X) \mod 4 \quad , \quad where$$

$b_i(\ )$ *denotes the* $i-th$ *Betti number with coefficients in* $\mathbb{Q}$*.*

(1.2) Remark. Since for a T-space X, $T = S^1 \times \cdots \times S^1$, with finitely many orbit types one can always choose a subcircle $S^1 \subset T$, such that $X^{S^1} = X^T$, the above theorem generalizes immediately to finite-dimensional T-CW-complexes with finitely many orbit types. Alternatively one could use an induction argument on the dimension of T.

Proof of (1.1). Here we always take $\mathbb{Q}$ as coefficients. $H^*_G(X)$ can be described as the cohomology of a differential graded $\mathbb{Q}[t]$-'algebra' $H^*(X)\tilde{\otimes}\mathbb{Q}[t]$ (s.[AP], (3.5.6), (3.5.9)) (note that multiplication is $\mathbb{Q}[t]$-bilinear, but may be commutative and associative only up to homotopy over $\mathbb{Q}[t]$). The evaluation at $t = 0$ gives a map of differential graded algebras $H^*(X)\tilde{\otimes}\mathbb{Q}[t] \longrightarrow H^*(X)$, where the target has trivial differential and the usual cup-product. Taking the evaluation of $H^*(X)\tilde{\otimes}\mathbb{Q}[t]$ at $t = 1$ and homology gives $H^{(*)}(X^G)$, where $H^{(*)}(-) := \oplus_i H^i(\ \ )$. Since $H^*(X)$ fulfils Poincaré duality one gets the associated isomorphism

$$H^*(X) \ \xrightarrow{\mathcal{D}}\ Hom_{\mathbb{Q}}(H^*(X), \mathbb{Q}) \ , \quad a \longmapsto \mathcal{D}_a$$

where $\mathcal{D}_a(X) := \sigma_X(a \cup X)$, with $\sigma_X : H^*(X) \longrightarrow \mathbb{Q}$ the orientation (cf. [AP], Chap.5 for notation and otherwise).
Using the extended orientation $\tilde{\sigma} : H^*(X)\tilde{\otimes}\mathbb{Q}[t] \longrightarrow \mathbb{Q}[t]$, $x \otimes t^i \longmapsto \sigma(x)t^i$ we similarly get a $\mathbb{Q}[t]$-morphism

$$H^*(X)\tilde{\otimes}\mathbb{Q}[t] \xrightarrow{\tilde{\mathcal{D}}} Hom_{\mathbb{Q}[t]}(H^*(X)\tilde{\otimes}\mathbb{Q}[t],\ \mathbb{Q}[t]),$$

which, evaluated at $t=0$, gives the above $\mathcal{D}$.

Since $\mathcal{D}$ is an isomorphism, so is $\tilde{\mathcal{D}}$ (s., e.g., [AP] (A.7.3)).

Hence the evaluation of $\tilde{\mathcal{D}}$ at $t=1$ is also an isomorphism. One therefore gets a differential $\mathbb{Z}_2$-graded (note that $|t|=2$) $\mathbb{Q}$-algebra $H_1^{(*)} := (H^*(X)\tilde{\otimes}\mathbb{Q}[t])_{t=1}$, the cohomology of which is just $H^{(*)}(X^G)$. $H_1^{(*)}$ inherits an orientation $\tilde{\sigma}_1 : H_1^{(*)} \longrightarrow \mathbb{Q}$ from the extended orientation $\tilde{\sigma} : H^*(X)\tilde{\otimes}\mathbb{Q}[t] \longrightarrow \mathbb{Q}[t]$, and fulfils Poincaré duality.

Now the statement of Theorem (1.1) in case (i) follows immediately from the following lemma. □

(1.3) Lemma. *Let $(A^{(*)},\delta)$ be a diffential $\mathbb{Z}_2$-graded Poincaré algebra of even formal dimension over a field k (i.e., $A^{(*)}$ fulfils Poincaré duality with respect to an orientation $\sigma_A : A^{(*)} \longrightarrow k$, which is compatible with δ and vanishes on A^{odd}, and δ is a derivation (of degree $1 \bmod 2$) with respect to the multiplication in $A^{(*)}$) then the same holds for*

$$H^{(*)} = H(A^{(*)},\delta\ )\ ,\ \textit{and}\ \ \dim H^{(*)} \equiv \dim A^{(*)} \mod 4.$$

Proof of (1.3). Let $Z^{(*)}$ and $B^{(*)}$ denote the cycles and boundaries of $A^{(*)}$. Then clearly $\dim H^{(*)} = \dim Z^{(*)} - \dim B^{(*)}$. On the other hand $\dim A^{(*)} = \dim Z^{(*)} + \dim B^{(*)}$, since $B^{(*)}$ is the orthogonal complement of $Z^{(*)}$ in $A^{(*)}$ with respect to the pairing given by the product and the orientation. Since $A^{(*)}$ has even formal dimension, not only the first but also the second equality holds for the even and odd degree part separately. Hence $\dim A^{(*)} - \dim H^{(*)} = 2\dim B^{(*)}$, and this again holds for the even and odd degree part separately. Since the Euler characteristics of A^* and H^* coincide , one gets $A^{even} - H^{even} = A^{odd} - H^{odd}$. So it follows that

$\dim A^{(*)} \equiv \dim H^{(*)} \bmod 4$. By standard arguments one sees that $H^{(*)}$ is again a $\mathbb{Z}_2$-graded Poincaré algebra of even formal dimension. □

The proof for case (ii) of Theorem (1.1) is a little more involved, due to the fact that the congruence in Lemma (1.3) does not follow in general in case of odd formal dimension.

One has to make use of the additional structure of $H_1^{(*)}$, which comes from the filtration $\mathcal{F}_j(H^*(X)\tilde{\otimes}\mathbb{Q}[t]) := \oplus_{i=0}^{j} H^i(X)\tilde{\otimes}\mathbb{Q}[t]$. This filtration on $H^*(X)\tilde{\otimes}\mathbb{Q}[t]$ induces a filtration on $H_1^{(*)}$, and the induced boundary on $H_1^{(*)}$ lowers the filtration degree (s.[AP] for details). Because of the assumption about the vanishing of certain Betti numbers in (ii) one can decompose $H^{(*)}$ into a direct sum $H_1^{(*)} = \mathbb{Q} \oplus \bar{H}_1^{odd} \oplus \bar{H}_1^{even} \oplus \mathbb{Q}$, where the boundary $\tilde{\delta}_1$ vanishes on the two summands $\mathbb{Q}$ (which come from the zero degree part and top degree part of $H^*(X)$, respectively) and maps $\bar{H}_1^{even}$ to $\bar{H}_1^{odd}$. Note that if $\tilde{\delta}_1 : \bar{H}_1^{odd} \longrightarrow \mathbb{Q}$ were non-zero, the cohomology of $(H_1^*, \tilde{\delta}_1)$ would vanish (since $1 \in H_1^{(*)}$ would be a boundary). So the condition '$X^G \neq \phi$' implies that $\tilde{\delta}_1$ vanishes on $\bar{H}_1^{odd}$ and by duality $\tilde{\delta}_1$ vanishes on the top degree part. Let $\bar{Z}_1^{even}$ be the cycles in $\bar{H}_1^{even}$. Since $\bar{H}_1^{even}$ does not contain any non-zero boundaries, $\mathbb{Q} \oplus \bar{Z}_1^{even}$ is already the even degree part of the homology of $(H_1^{(*)}, \tilde{\delta}_1)$. We want to show that $\dim(\bar{H}_1^{even}/\bar{Z}_1^{even}) \equiv 0 \bmod 2$. Using an Euler characteristic argument as above this then implies the statement of Theorem (1.1). To prove the above congruence we define a non-degenerate skew-symmetric bilinear form on $\bar{H}_1^{even}/\bar{Z}_1^{even}$. Let $h, h' \in \bar{H}_1^{even}$, define $s(h,h') := \tilde{\sigma}_1(\tilde{\delta}_1 h \cup h')$, where '$\cup$' denotes the product in $H_1^{(*)}$, and $\tilde{\sigma}_1 : H_1^{(*)} \longrightarrow \mathbb{Q}$ the orientation. The form s is skew-symmetric since $\tilde{\sigma}_1$ is a chain map and $\tilde{\delta}_1$ is a derivation, it is clearly well defined, and it is non-degenerate, since $\bar{H}_1^{odd}$ is Poincaré dual to $\bar{H}_1^{even}$. This finishes the proof of Theorem (1.1). □

2. $\mathbb{Z}_p$-actions

The $\mathbb{Z}_p$-case is more complicated than the S^1-case since the exterior part of the cohomology $H^*(BG;\mathbb{Z}_p)$ 'mixes up' odd and even degrees (for p odd). In order to avoid this we first consider cohomology with integral coefficients and prove a suitable variant of the usual Localization Theorem and Evaluation Theorem (cf.[AP], [tD](4.45) Exercise 6), which allows under certain extra assumptions to imitate the arguments for the S^1-case. Alternatively we use results from [Ha] later on to 'separate' odd and even degree terms in the case of $\mathbb{Z}_p$-coefficients if the Bockstein operator vanishes on $H^*(X;\mathbb{Z}_p)$. Let $H^*_G(X;\mathbb{Z}) = H^*(X \times_G EG, \mathbb{Z})$ denote the equivariant cohomology of X.

(2.1) Theorem

Let X be a finite-dimenional $G-CW$-complex, $G = \mathbb{Z}_p$, and let $j: X^G \hookrightarrow X$ be the inclusion of the fixed point set; $k = \mathbb{F}_p$. The map j induces an isomorphism

$$j^*: H^*_G(X;\mathbb{Z}) \otimes_{H^*_G} k[t,t^{-1}] \longrightarrow H^*(X^G;k) \otimes_k k[t,t^{-1}],$$

where $k[t,t^{-1}]$ is considered as a H^*_G–module via the canonical morphismen

$$H^*_G := H^*(BG;\mathbb{Z}) = \mathbb{Z}[t]/(pt) \longrightarrow k[t,t^{-1}], \quad t \mapsto t.$$

In particular the evaluation of j^* at $t=1$ gives an isomorphism

$$j^*_1: H^*_G(X;\mathbb{Z}) \otimes_{H^*_G} k \xrightarrow{\cong} H^{(*)}(X^G;k),$$

where k is considered as a H^*_G–module via

$$\mathbb{Z}[t]/(pt) \longrightarrow k;\ t \longmapsto 1,$$

and $H^{(*)}(X^G;k) = H^{even}(X^G;k) \oplus H^{odd}(X^G;k)$ viewed as a $\mathbb{Z}_2$–graded algebra.

Proof. The functor $- \otimes_{H_G^*} k[t,t^{-1}]$ is just localization with respect to the multiplicative subset $S := \{t^r, r \in \mathbf{N}\} \subset H_G^*$. Since, for a free G–space Y, $H_G^*(Y;\mathbb{Z}) \cong H^*(Y/G;\mathbb{Z})$, one obtains that $j\colon X^G \to X$ induces an isomorphism for the localized equivariant cohomology. It remains to identify $H_G^*(X^G;\mathbb{Z}) \otimes_{H_G^*} k[t,t^{-1}]$. Clearly $H_G^*(X^G;\mathbb{Z}) = H^*(X^G \times BG;\mathbb{Z})$. We claim that $H^*(X^G \times BG;\mathbb{Z}) \otimes_{H_G^*} k[t,t^{-1}] \cong H^*(X^G;\mathbb{Z}_p) \otimes_k k[t,t^{-1}]$, which immediately gives the desired result. If $H^*(X^G;\mathbb{Z})$ has no p–torsion then the above claim follows directly from the Künneth formula, but in the presence of p–torsion there is a slight problem with the multiplicative structure.

The following commutative diagram

$$\begin{array}{ccccc} H^*(Y \times BG;\mathbb{Z}) & \longleftarrow & H^*(BG;\mathbb{Z}) & \longrightarrow & k[t,t^{-1}] \\ \downarrow & & \downarrow & & \downarrow id \\ H^*(Y \times BG;k) & \longleftarrow & H^*(BG;k) & \longrightarrow & k[t,t^{-1}] \end{array}$$

(where the first two vertical maps are induced by reducing the coefficients mod p, the two left horizontal maps are induced by the projection $Y \times BG \to BG$, and $H^*(BG;k) \cong \Lambda(s) \otimes k[t] \longrightarrow k[t,t^{-1}]$ is given by $s \longmapsto 0$, $t \longmapsto t$) induces a natural transformation of functors (in the variable Y)

$$H^*(Y \times BG;\mathbb{Z}) \otimes_{H_G^*} k[t,t^{-1}] \to H^*(Y \times BG;k) \otimes_{H^*(BG;k)} k[t,t^{-1}].$$

Both functors are cohomology theories in Y , the first since it is the localization of the cohomology theory $H^*(- \times BG;\mathbb{Z})$ with respect to $S = \{t^r, r \in \mathbf{N}\} \subset H^*(BG;\mathbb{Z})$, the second because of the isomorphism

$$H^*(Y \times BG; k) \otimes_{H^*(BG;k)} k[t,t^{-1}] \cong H^*(Y;k) \otimes_k \ k[t,t^{-1}],$$

and the above natural transformation is an isomorphism for Y a point. Hence we get a natural isomorphism

$$H^*(Y \times BG; \mathbb{Z}) \otimes_{H^*_G} k[t,t^{-1}] \xrightarrow{\cong} H^*(Y;k) \otimes_k \ k[t,t^{-1}].$$

This proves the theorem. □

We remark in passing that the following Corollary is an immediate consequence of Theorem (2.1) and contains, as a special case, the standard 'strong inequalities' for $\mathbb{Z}_p$–action (see, e.g., [AP], 3.10).

(2.2) Corollary. *Let X be a finite-dimensional $G - CW$-complex, $G = \mathbb{Z}_p$; $k = \mathbb{F}_p$. Let $A_2^{*,*} := (S^{-1}E_2^{*,*})_1$ be the localized and at $t = 1$ evaluated E_2-term of the Leray-Serre spectral sequence of the Borel construction which inherits a $\mathbb{Z}_2 \times \mathbb{Z}$-grading. Then*

$$\sum_i \dim_k H^{m+2i}(X^G;k) \leq \sum_i (\dim_k A_2^{0,m+2i} + \dim_k A_2^{1,m+2i+1}).$$

If G acts trivially on $H^*(X;\mathbb{Z})$, then (additively) $H^d(X;k) \cong A_2^{0,d} \oplus A_2^{1,d+1}$.

Proof. Using Theorem, (2.1) the proof is more or less standard (cf., e.g., [AP],(3.10.7)).
Put $A_r^{*,*} := (S^{-1}E_r^{*,*})_1$; then $A_r^{*,*}$ is $\mathbb{Z}_2 \times \mathbb{Z}$-graded. Clearly $\dim_k A_\infty^{i,j} \leq \dim A_2^{i,j}$. On the other hand, using Theorem (2.1), one gets that
$\sum_i \dim_k \ H^{m+2i}(X^G;k) \leq \sum_i (\dim_k \ A_\infty^{0,m+2i} + \dim_k \ A_\infty^{1,m+2i+1})$.
Hence the desired inequality follows. The last statement in the corollary is

a consequence of the universal coefficient formula. □

Let X be a finite-dimensional $G-CW$-complex, $G = \mathbb{Z}_p$, such that $H^*(X;\mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module without p-torsion. Assume also that the induced G-action on $H^*(X;\mathbb{Z})$ is trivial. Then the E_2-term of the Leray-Serre spectral sequence of the Borel construction $X \longrightarrow X_G := X \times_G EG \longrightarrow BG$ is given by $E_2^{*,*} = H^*(BG;\mathbb{Z}) \otimes H^*(X;\mathbb{Z})$. Since localization is exact we can localize the spectral sequence with respect to $S := \{t^i, i \in \mathbb{N}\} \subset \mathbb{Z}[t]/(pt) = H^*(BG;\mathbb{Z})$ and obtain $S^{-1}E_2^{*,*} = k[t,t^{-1}] \otimes H^*(X;\mathbb{Z})$. (Note that this is just the Tate cohomology of G with coefficients in $H^*(X;\mathbb{Z})$.) Since for $\mathbb{Z}$-graded $k[t]$-modules the evaluation at $t = 1$ is also exact (see, e.g.,[AP],(A.7.2)), one can evaluate the localized spectral sequence at $t = 1$ and obtain $k \otimes H^{(*)}(X;\mathbb{Z}) = H^{(*)}(X;k)$ as $\mathbb{Z}_2$-graded E_2-term. This is a $\mathbb{Z}_2$-graded Poincaré algebra over k, if $H^*(X;k)$ fulfils Poincaré duality. If $H^*(X;k)$, and hence $H^{(*)}(X;k)$, has even formal dimension then an iterated application of Lemma (1.2) (which holds for any ground field) gives: $\dim_k \ H^{(*)}(X;k) \equiv \dim_k(S^{-1}\ E_\infty^{*,*})_1 \mod 4$. But, by Theorem (2.1), $(S^{-1}\ E_\infty^{*,*})_1$ is the graded algebra associated to a filtration of $H^{(*)}(X^G;k)$. Hence $\dim_k \ H^{(*)}(X^G;k) \equiv \dim_k \ H^{(*)}(X;k) \mod 4$.

This gives an analogue of case (i) of Theorem (1.2) for $\mathbb{Z}_p$-actions. To get the analogue of case (ii), one observes that for each $\mathbb{Z}_2$- graded Poincaré algebra $(S^{-1}E_r^{*,*})_1$ one has a decomposition $(S^{-1}E_r^{*,*})_1 = k \oplus A_r^{odd} \oplus A_r^{even} \oplus k$, such that d_r maps A_r^{even} to A_r^{odd} and is zero otherwise. So again an iteration of the argument used for the S^1-case above gives the desired result. Note that the assumption 'p odd' is essential for the argument using the non-degenerate skew symmetric bilinear form. Altogether one has the following result:

(2.3) Theorem. *Let X be a finite-dimensional $G - CW$-complex, $G = \mathbb{Z}_p$, such that $H^*(X;\mathbb{Z})$ has no p-torsion, G acts trivially on $H^*(X;\mathbb{Z})$ and $H^*(X;k)$, $k = \mathbb{F}_p$, fulfils Poincaré duality. If*

(i) the formal dimension, $fdH^*(X;k)$, of $H^*(X;k)$ is even, or

(ii) p odd, $fdH^*(X;k) = 2m+1$, $X^G \neq \emptyset$ and $H^i(X;k) = 0$ for $0 < i \leq m$, i even,

then

$$\dim_k \ H^{(*)}(X^G;k) \equiv \dim_k \ H^{(*)}(X;k) \mod 4. \qquad \square$$

(2.4) Remark. Theorem (2.3) can be generalized to the situation where one replaces the assumption 'G acts trivially on $H^*(X;\mathbb{Z})$' by 'G acts nicely on $H^*(X;\mathbb{Z})$' in the sense of A. Sikora (s.[Si]), i.e., as G-module $H^*(X;\mathbb{Z})$ splits into a direct sum $H^*(X;\mathbb{Z}) = F^* \oplus T^*$, where F is a free G-module and T is a trivial G-module. The result then is that $\dim_k \ H^{(*)}(X^G;k) \equiv \dim_k \ T^{(*)} \mod 4$, and the proof is analogous, using the fact that $(S^{-1}E_2^{*,*})_1 \cong T^{(*)} \quad (H^*(BG;H^*(X;\mathbb{Z})) \cong H^*(BG) \otimes T^*)$ in this case. It suffices here to assume that $T^i = 0$ instead of $H^i(X;k) = 0$ for $0 < i < m$, i *even*.

(2.5) Remark. It is not obvious how to generalize Theorem (2.3) to p-tori (analogous to Corollary (1.2)). One difficulty comes from the fact that the conclusion of Theorem (2.3) is not strong enough in order to apply an induction argument, e.g., $H^*(X^{\mathbb{Z}_p};\mathbb{Z})$ may have p-torsion. Note also that, contrary to the (real) torus case, $X^{\mathbb{Z}_p}$ might be different from X^G for all $\mathbb{Z}_p \subset G = (\mathbb{Z}_p)^r$.

(2.6) Remark. If one tries to prove a generalization of Theorem (2.3), for p odd, by using k-coefficient, one runs into the following problem: The E_2-term of the Leray-Serre spectral sequence of the Borel construction, localized with respect to $S = \{ t^i \ , \ i \in N\} \subset H^*(B\ \mathbb{Z}_p; k) \cong k[t] \otimes \Lambda(s)$, is the free module $S^{-1}E_2^{*,*} = H^*(B\ \mathbb{Z}_p; k) \otimes H^*(X; k)$ over $S^{-1}H^*(B\ \mathbb{Z}_p; k) = k[t, t^{-1}] \otimes \Lambda(s)$, if the action is cohomologically trivial. But, although the terms $S^{-1}E_r^{*,*}$ for $r > 2$ are free over $k[t, t^{-1}]$ it is not clear that they are free over $H^*(B\ \mathbb{Z}_p; k)$ in general. It follows from [Ha], Chap.9 that this is indeed the case, if the Bockstein operator, associated to the coefficient sequence $0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Z}_p^2 \longrightarrow \mathbb{Z}_p \longrightarrow 0$, is trivial. (Note that this holds, if and only if $H^*(X; \mathbb{Z})$ has no direct summand of the form $\mathbb{Z}_p$.) But if all $S^{-1}E_r^{*,*}$ are free $k[t, t^{-1}] \otimes \Lambda(s)$, then evaluating at $s = 0$ and $t = 1$ commutes with taking homology with respect to the differentials in the (localized) spectral sequence. Hence one ends up with a situation which is completely analogous to the case discussed above. One therefore gets the following result (and the corresponding generalization of Corollary (2.2), too).

(2.7) Theorem. *Let X be a finite-dimensional $G-CW$-complex, $G = \mathbb{Z}_p, p$ odd, such that $H^*(X; k)$ fulfils Poincaré duality, $H^*(X; k)$ - as a G-module - splits into a direct sum $F^* \oplus T^*$, where F^* is a free G-module and T^* is a trivial G-module. Assume further that $\beta(H^*(X; k)) = 0$, where β is the Bockstein operator associated to $0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Z}_p^2 \longrightarrow \mathbb{Z}_p \longrightarrow 0$.*
If

(i) $fdH^*(X; k)$ is even, or

(ii) $fdH^*(X; k) = 2m + 1$, $T^i = 0$ for $0 < i \leq m, i$ even,

then

$$\dim_k\ H^{(*)}(X^G; k) \equiv \dim_k T^{(*)} \mod 4. \qquad \square$$

(2.8) Remark. It is shown in [Ha], Chap.6 and 9, that in case of a trivial Bockstein operator $H^*(X;k)$ decomposes into a direct sum $H^*(X;k) \cong F^* \oplus T^* \oplus S^*$, where F^* is a free G-module, T^* is a trivial G-module, and S^* is a direct sum of G-modules of the form $ker\ \epsilon$ where $\epsilon : k[G] \to k$ is the standard augmentation. Since $H^*(G; ker\ \epsilon)$ evaluated at $t = 1$ and $s = 0$ is isomorphic to k, one gets for the E_2-term of the spectral sequence of X_G :

$$\dim_k (E_2)^{0,i}_{t=1,s=0} = \dim_k T^i; \quad \dim_k (E_2)^{1,i}_{t=1,s=0} = \frac{1}{p-1} \dim_k S^i.$$

Using the same argument as for Theorem (2.7), one gets in case (i), that

$$\dim_k H^{(*)}(X^G;k) \equiv \dim_k T^{(*)} + \frac{1}{p-1} \dim_k S^{(*)} \mod 4.$$

If one wants to generalize the case (ii) in a similar way one has to assume that $T^i = 0$ for $0 < i \leq m$, i even and $S^j = 0$ for $0 < j \leq m$, j odd. This is due to the fact $(H^*(G; ker\ \epsilon))_{t=1,s=0}$ is concentrated in odd degree. The examples in [Si], p.7 show that without any assumption on the G-module structure of $H^*(X;k)$ the desired conclusion in case (ii) does not hold.

The following example is of some independent interest in 3-manifolds (cf.[Si], (1.6.2)). It is not covered by Theorem (2.7), but a direct inspection of the localized and at $t = 1$ evaluated spectral sequence (with $\mathbb{Z}_p$ coefficients) gives the desired congruence (cf.[Su]).

(2.9) Example. Let X be a finite-dimensional $G-CW$-complex, $G = \mathbb{Z}_p, p$ odd, such that $H^i(X;k) \cong \mathbb{Z}_p$ for $i = 0, 1, 2, 3$ and zero otherwise and $H^*(X;k)$ fulfils Poincaré duality. Then

$$\dim_k H^{(*)}(X^G;k) \equiv \dim_k H^{(*)}(X;k) \mod 4.$$

Proof. Let $1, a, b, c$ be the respective generators of $H^i(X;k), i = 0,1,2,3,$ such that $a \cup b = c$.

The localized and at $t = 1$ evaluated E_2-term of the Leray-Serre spectral sequence of the Borel construction has then following form. (Note that the G-action on $H^*(X;k)$ must be trivial, since $\dim H^i(X;k) \leq 1$ for all i.)

$$(S^{-1}E_2^{*,*})_1 \cong H^{(*)}(X;k) \oplus s \cup H^{(*)}(X;k),$$
$$\text{where } \ s \in (S^{-1}H^*(BG;k))_1 \cong (k[t,t^{-1}] \otimes \Lambda(s))_1 \cong \Lambda(s)$$

This is a differential graded algebra over $\Lambda(s)$, which fulfils Poincaré duality. The boundary δ, induced by d_2, is a derivation and is $\Lambda(s)$-linear.

If $X^G = \emptyset$, then the above congruence holds. Assume that $X^G \neq \emptyset$. Then $1 \in H^{(*)}(X;k) \oplus s \cup H^{(*)}(X;k)$ can not be a boundary, so $\delta(a) = 0$. Also, clearly $\delta(1) = 0$. Since δ is $\Lambda(s)$-linear, one gets $\delta(s \cup 1) = 0$, $\delta(s \cup a) = 0$. By Poincaré duality $\delta(s \cup c) = 0$ (otherwise $\delta(s \cup c)$ would have a complementary factor of the form αa , $\alpha \in k$, i.e., $(\alpha a) \cup \delta(s \cup c) = c$; but this would implies $\delta(\alpha a \cup \delta(s \cup c)) = 0$, since $\delta a = 0$). So $0 = \delta(s \cup c) = \delta(s \cup a \cup b) = -s \cup \delta(a \cup b) = s \cup a \cup \delta b$. Furthermore $0 = \delta(b \cup b) = 2\delta(b) \cup b$, but $\delta(b)$ must be of the form $\delta(b) = \nu a;\ \mu, \nu \in\ k$. So $2\delta(b) \cup b = \nu c = 0$, hence $\nu = 0$ and $\delta(b) = 0$. It therefore follows that δ (being a derivation) vanishes altogether. Similarly one sees that there are no higher non-zero differentials in the spectral sequence. So in this case the spectral sequence collapses at E_2 and $\dim_k\ H^{(*)}(X^G;k) = \dim_k H^{(*)}(X;k)$.

Returning to the general situation one might expect that $(S^{-1}E_r^{*,*})_1$ is always a free module over $\Lambda(s)$, as this is the case for $r = 2$, assuming that $H^*(X;k)$ decomposes into free, trivial and G-modules of type $\ker \epsilon$, as in (2.8). But the following algebraic example shows, that this seems unlikely.

(2.10) Example. Let $A^* \cong H^*(S^1 \times S^3; \mathbb{Z}_p)$, and let $1, a, b, c$ denote generators in A^i, for $i = 0, 1, 3, 4$ respectively, such that $a \cup b = c$. Define a derivation on $\tilde{A}^* := A^* \oplus s \cup A^*$ by $\tilde{\delta}(b) = a$, and zero otherwise. Then $(\tilde{A}^*, \tilde{\delta})$ is a differential graded algebra, which is free over $\Lambda(s)$, and which fulfils Poincaré duality. But $H(\tilde{A}^*, \tilde{\delta})$ has the equivalence classes $[1], [s \cup 1], [a], [c], [s \cup b], [s \cup c]$ as a k-basis, and is not a free $\Lambda(s)$ - module.

The differential described in Example (2.10) can not actually come from a $\mathbb{Z}_p$-action on $S^1 \times S^3$, since this would contradict Theorem (2.7). But it shows that the property "strong duality" (s.[Si]) is not inherited by E_{r+1} from E_r for purely algebraic reasons. So even in case of cohomologically trivial actions it is not clear whether Theorem (2.7) holds if one drops the assumption on the Bockstein operator completely.

(2.11) Example. The standard free linear involution on S^{2m} does not fulfil the hypothesis of Theorem (2.3), since it is not cohomologically trivial (with $\mathbb{Z}$ coefficients), but it would fulfil the hypothesis of Theorem (2.7) for $p = 2$, so the assumption 'p odd' is essential for Theorem (2.7), case (i). Of course, the standard free linear involution on S^{2m+1} shows that the assumption 'p odd' is essential already for Theorem (2.3), case (ii).

(2.12) Remark. To simplify the presentation we have made the assumption that X is a $G - CW$-complex. But using Čech cohomology and the usual somewhat more technical machinery (see, e.g., [AP]) one can extend all the result to general G-spaces, which fulfil the hypothesis (LT) for the Localization Theorem (s.[AP], p.208).

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