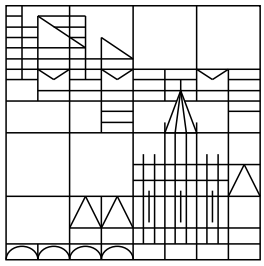

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Rate of Stability in Hyperbolic Thermoelasticity

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Abstract: In this paper we consider the system of hyperbolic thermoelasticity in one dimension with Dirichlet-Neumann boundary conditions. First, the roots of the characteristic polynomial are investigated analytically applying appropriate scalings. Then we prove the exponential decay of the associated energy and describe the optimal rate of stability. Finally, we turn to the system of classical thermoelasticity. There we use the same energy as for the previous system to derive an analogous result.

Keywords: hyperbolic thermoelasticity, second sound, classical thermoelasticity, energy decay, rate of stability, exponential stability.

AMS subject classification: 74 F 05, 74 H 40

1 Introduction

The equations of thermoelasticity are used to model the behaviour of elastic and heat conductive media. Let $u = u(t,x)$, $\vartheta = \vartheta(t,x)$, and $q = q(t,x)$ ($t \geqslant 0$, $x \in (0,L)$ for some $L > 0$) be the displacement, the temperature difference to a fixed reference temperature, and the heat flux, respectively, then the differential equations for (u, ϑ, q) are first

$$u_{tt} - \alpha u_{xx} + \beta \vartheta_x = 0 \quad \text{in } [0,\infty) \times (0,L), \tag{1.1a}$$

$$\vartheta_t + \gamma q_x + \delta u_{tx} = 0 \quad \text{in } [0,\infty) \times (0,L), \tag{1.1b}$$

where (1.1a) is an equation of motion and (1.1b) describes the conservation of energy. The positive coefficients α, β, γ, δ depend on the material. For a physical derivation of (1.1) we refer to [1].

These two equations have to be completed by a heat equation. Here we want to use Cattaneo's law of heat propagation

$$\tau q_t + q + \kappa \vartheta_x = 0 \quad \text{in } [0,\infty) \times (0,L) \tag{1.2}$$

with positive constants κ, τ. The system (1.1) - (1.2) is purely hyperbolic, but lightly damped, and it models thermal disturbances as wave-like pulses propagating with finite speed, the so-called second sound. This is important in some physical applications such as in laser cleaning (a physical review is given in [8]) or in pulsed laser heating (cf. [12]).

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If we use FOURIER's law

$$q + \kappa\vartheta_x = 0 \quad \text{in } [0,\infty) \times (0,L), \tag{1.3}$$

instead of (1.2) we get the hyperbolic-parabolic system of classical thermoelasticity

$$\begin{aligned} u_{tt} - \alpha u_{xx} + \beta\vartheta_x &= 0 \quad \text{in } [0,\infty) \times (0,L), \\ \vartheta_t - \gamma\kappa\vartheta_{xx} + \delta u_{tx} &= 0 \quad \text{in } [0,\infty) \times (0,L), \end{aligned}$$

including the paradox of infinite propagation speed of heat pulses.

There are many, but mostly qualitative results on the exponential stability of these two systems (also in higher dimensions and in the non-linear case) proving that the associated energy decays exponentially, but the rate of stability is not described more precisely (cf. [6], [5], [11] for the classical system and [3], [9], [10] for the hyperbolic one as well as the references cited there). For a survey on hyperbolic thermoelasticity see e. g. [2]. In [4] it is shown that the rate of stability in case of classical thermoelasticity is given by the minimal real part of all the characteristic roots using their asymptotic behaviour.

We will present an analogue result for the hyperbolic system: in the first part we will investigate the high and low frequency behaviour of the roots of the fourth-order characteristic polynomial. As solution formulas are rather complex – one would need CARDAN's solution of the cubic – appropriate scalings and the Implicit Functions Theorem will be used to derive an expansion of the roots. Exemplarly we will display the characteristic roots for silicon.

The second part will give both a proof of the exponential stability of the hyperbolic system and the optimal rate of stability.

Proving our main result we will observe that in the case of hyperbolic thermoelasticity we do not have to make further assumptions on the decay rate as done in [4] where it was supposed that

$$d_{ct} < \frac{\pi^2}{L^2}\gamma\kappa \tag{1.4}$$

holds for the decay rate d_{ct} of the classical system. So in the last part we will show that assumption (1.4) can be dropped if we consider higher order energy terms. Actually we will not use the natural energy for the classical system as usually done in the literature, but we will apply the same energy as in hyperbolic thermoelasticity.

2 The characteristic roots

We consider the system of hyperbolic thermoelasticity

$$u_{tt} - \alpha u_{xx} + \beta\vartheta_x = 0 \quad \text{in } [0,\infty) \times (0,L), \tag{2.1a}$$

$$\vartheta_t + \gamma q_x + \delta u_{tx} = 0 \quad \text{in } [0,\infty) \times (0,L), \tag{2.1b}$$

$$\tau q_t + q + \kappa\vartheta_x = 0 \quad \text{in } [0,\infty) \times (0,L) \tag{2.1c}$$

with the initial data

$$u(0,x) = u^0(x), \quad u_t(0,x) = u^1(x), \quad \vartheta(0,x) = \vartheta^0(x), \quad q(0,x) = q^0(x) \tag{2.2}$$

for $x \in (0,L)$ and the DIRICHLET-NEUMANN boundary conditions

$$u(t,0) = u(t,L) = 0, \quad q(t,0) = q(t,L) = 0 \tag{2.3}$$

for all $t \geqslant 0$. Note that due to (1.3) the second condition in (2.3) turns to the NEUMANN condition $\vartheta_x(t,0) = \vartheta_x(t,L) = 0$ in the classical case. Further we assume

$$\int_0^L \vartheta^0(x)\,\mathrm{d}x = 0.$$

From (2.1b) and (2.3) we get the conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \vartheta(t,x)\,\mathrm{d}x = 0,$$

thus it holds for all $t \geqslant 0$

$$\int_0^L \vartheta(t,x)\,\mathrm{d}x = 0. \tag{2.4}$$

Let

$$s_j(x) := \sqrt{\frac{2}{L}}\sin\left(\sqrt{\lambda_j}x\right) \quad \text{and} \quad c_j(x) := \sqrt{\frac{2}{L}}\cos\left(\sqrt{\lambda_j}x\right)$$

with

$$\lambda_j := \frac{j^2\pi^2}{L^2}.$$

Then we can expand the solution (u,ϑ,q) in the series

$$u(t,x) = \sum_{j=1}^{\infty} u_j(t)s_j(x), \quad \vartheta(t,x) = \sum_{j=1}^{\infty} \vartheta_j(t)c_j(x), \quad q(t,x) = \sum_{j=1}^{\infty} q_j(t)s_j(x), \tag{2.5}$$

because both u and q satisfy the homogenious DIRICHLET boundary condition. Hence, the series for ϑ follows from (2.1) and starts at $j = 1$, since we have (2.4).
The coefficients $u_j(t)$, $\vartheta_j(t)$, and $q_j(t)$ can be represented as linear combinations

$$u_j(t) = \sum_{k=1}^{4} u_j^k e^{-\varrho_j^k t}, \quad \vartheta_j(t) = \sum_{k=1}^{4} \vartheta_j^k e^{-\varrho_j^k t}, \quad q_j(t) = \sum_{k=1}^{4} q_j^k e^{-\varrho_j^k t}, \tag{2.6}$$

where ϱ_j^k, $k = 1,2,3,4$, are the roots of the underlying characteristic polynomial χ_j,

$$\chi_j(\varrho) := \varrho^4 - A\varrho^3 + B\lambda_j\varrho^2 - C\lambda_j\varrho + D\lambda_j^2$$

with

$$A := \frac{1}{\tau}, \qquad B := \alpha + \beta\delta + \frac{\gamma\kappa}{\tau},$$
$$C := \frac{\alpha + \beta\delta}{\tau}, \qquad D := \frac{\alpha\gamma\kappa}{\tau}.$$

For simplicity we have assumed that the roots ϱ_j^k, $k = 1, 2, 3, 4$, are pairwise different. We will discuss this point later.

Now we will investigate the roots ϱ^k, $k = 1, 2, 3, 4$, of the polynomial $\chi(\xi, \cdot)$ with

$$\chi(\xi, \varrho) := \varrho^4 - A\varrho^3 + B\xi^2\varrho^2 - C\xi^2\varrho + D\xi^4$$

for all $\xi \geqslant 0$:

Lemma 2.1. (i) *Define*

$$\Lambda_1 := \frac{1}{2}\left(B - \sqrt{B^2 - 4D}\right), \qquad d_1 := \frac{1}{2}\frac{C - \Lambda_1 A}{\sqrt{B^2 - 4D}},$$
$$\Lambda_2 := \frac{1}{2}\left(B + \sqrt{B^2 - 4D}\right), \qquad d_2 := \frac{1}{2}\frac{\Lambda_2 A - C}{\sqrt{B^2 - 4D}}.$$

Then the four roots $\varrho^k = \varrho^k(\xi)$, $k = 1, 2, 3, 4$, of the characteristic polynomial $\chi(\xi, \cdot)$ satisfy the following expansion for $\xi \to \infty$

$$\varrho^{1/2}(\xi) = d_1 - \frac{4(\Lambda_1 d_2 - \Lambda_2 d_1)d_1 d_2}{(\Lambda_2 - \Lambda_1)^2}\,\frac{1}{\xi^2} + \mathcal{O}\big(\tfrac{1}{\xi^4}\big)$$
$$\pm i\left(\sqrt{\Lambda_1}\,\xi + \left(\frac{2d_1 d_2\sqrt{\Lambda_1}}{\Lambda_2 - \Lambda_1} - \frac{d_1^2}{2\sqrt{\Lambda_1}}\right)\frac{1}{\xi} + \mathcal{O}\big(\tfrac{1}{\xi^3}\big)\right),$$
$$\varrho^{3/4}(\xi) = d_2 + \frac{4(\Lambda_1 d_2 - \Lambda_2 d_1)d_1 d_2}{(\Lambda_2 - \Lambda_1)^2}\,\frac{1}{\xi^2} + \mathcal{O}\big(\tfrac{1}{\xi^4}\big)$$
$$\pm i\left(\sqrt{\Lambda_2}\,\xi - \left(\frac{2d_1 d_2\sqrt{\Lambda_2}}{\Lambda_2 - \Lambda_1} + \frac{d_2^2}{2\sqrt{\Lambda_2}}\right)\frac{1}{\xi} + \mathcal{O}\big(\tfrac{1}{\xi^3}\big)\right).$$

(ii) *For $\xi \to 0$ we get:*

$$\varrho^{1/2}(\xi) = \frac{\beta\gamma\delta\kappa}{2(\alpha + \beta\delta)}\,\xi^2 + \mathcal{O}(\xi^4) \pm i\left(\sqrt{\alpha + \beta\delta}\,\xi + \mathcal{O}(\xi^3)\right),$$
$$\varrho^3(\xi) = \frac{\alpha\gamma\kappa}{\alpha + \beta\delta}\,\xi^2 + \mathcal{O}(\xi^4),$$
$$\varrho^4(\xi) = \frac{1}{\tau} - \gamma\kappa\,\xi^2 + \mathcal{O}(\xi^4).$$

Proof. (i) Since the polynomial $\chi(\xi, \cdot)$ has real valued coefficients, we can assume without loss of generality that

$$d^1 := \frac{1}{2}\left(\varrho^1 + \varrho^2\right), \qquad b^1 := \varrho^1\varrho^2, \tag{2.7a}$$
$$d^2 := \frac{1}{2}\left(\varrho^3 + \varrho^4\right), \qquad b^2 := \varrho^3\varrho^4 \tag{2.7b}$$

are real valued functions depending on ξ, and by definition we get:

$$A = 2\left(d^1 + d^2\right), \tag{2.8a}$$
$$\xi^2 B = b^1 + b^2 + 4d^1 d^2, \tag{2.8b}$$
$$\xi^2 C = 2\left(b^1 d^2 + b^2 d^1\right), \tag{2.8c}$$
$$\xi^4 D = b^1 b^2. \tag{2.8d}$$

We define

$$h := \frac{1}{\xi^2}$$

and consider therefore the limiting process $h \to 0$. To this purpose we use the scaling

$$\tilde{b}^1 := hb^1 \quad \text{and} \quad \tilde{b}^2 := hb^2.$$

Then we get from (2.8)

$$\begin{aligned} A &= 2\left(d^1 + d^2\right), \\ B &= \tilde{b}^1 + \tilde{b}^2 + 4hd^1 d^2, \\ C &= 2\left(\tilde{b}^1 d^2 + \tilde{b}^2 d^1\right), \\ D &= \tilde{b}^1 \tilde{b}^2. \end{aligned}$$

Now let $F \in C^1(\mathbb{R}^5, \mathbb{R}^4)$ be defined by

$$F(h, d^1, d^2, \tilde{b}^1, \tilde{b}^2) := \begin{pmatrix} d^1 + d^2 - \frac{A}{2} \\ \tilde{b}^1 + \tilde{b}^2 + 4hd^1 d^2 - B \\ \tilde{b}^1 d^2 + \tilde{b}^2 d^1 - \frac{C}{2} \\ \tilde{b}^1 \tilde{b}^2 - D \end{pmatrix},$$

where we also want to allow $h < 0$. We calculate

$$F(0, d_1, d_2, \Lambda_1, \Lambda_2) = 0$$

and

$$\begin{aligned} \det \frac{\partial F}{\partial(d^1, d^2, \tilde{b}^1, \tilde{b}^2)}(0, d_1, d_2, \Lambda_1, \Lambda_2) &= -B^2 + 4D \\ &= -\frac{\pi^4}{\tau^2 L^4}\big((\alpha\tau - \gamma\kappa)^2 + 2\alpha\beta\delta\tau^2 + \beta^2\delta^2\tau^2 + 2\beta\delta\gamma\kappa\tau\big) \\ &\neq 0. \end{aligned}$$

By the Implicit Functions Theorem there exists a neighbourhood $U \subset \mathbb{R}$ of $h = 0$ and a function $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T \in C^1(U, \mathbb{R}^4)$ such that

$$\varphi(0) = (d_1, d_2, \Lambda_1, \Lambda_2)^T, \quad F\big(h, \varphi(h)\big) = 0$$

for all $h \in U$, and

$$\varphi'(0) = -\left(\frac{\partial F}{\partial(d^1, d^2, \tilde{b}^1, \tilde{b}^2)}\right)^{-1} \frac{\partial F}{\partial h}(0, d_1, d_2, \Lambda_1, \Lambda_2)$$

$$= \begin{pmatrix} -\frac{4(\Lambda_1 d_2 - \Lambda_2 d_1) d_1 d_2}{(\Lambda_2 - \Lambda_1)^2} \\ \frac{4(\Lambda_1 d_2 - \Lambda_2 d_1) d_1 d_2}{(\Lambda_2 - \Lambda_1)^2} \\ \frac{4\Lambda_1 d_1 d_2}{\Lambda_2 - \Lambda_1} \\ -\frac{4\Lambda_2 d_1 d_2}{\Lambda_2 - \Lambda_1} \end{pmatrix}.$$

From (2.7a) we get

$$\varrho^{1/2} = d^1 \pm i\sqrt{\frac{\tilde{b}^1}{h} - (d^1)^2},$$

and an analogous formula holds for ϱ^3 and ϱ^4 using d^2 and $\tilde{b}^2$. Especially, we see $\varrho^2 = \overline{\varrho^1} \in \mathbb{C} \setminus \mathbb{R}$ and $\varrho^4 = \overline{\varrho^3} \in \mathbb{C} \setminus \mathbb{R}$ for sufficiently small h, i. e. for sufficiently large ξ.
Since we have $d^k = \varphi_k$, $k = 1, 2$, by construction, the asserted expansion of the real parts follows. On the other hand

$$\operatorname{Im} \varrho^{2k-1} = \frac{\sqrt{\tilde{b}^k - h(d^k)^2}}{\sqrt{h}} =: \frac{I_k}{\sqrt{h}}$$

for $k = 1, 2$ and h small enough holds. Differentiating the the function I_k we get by definition

$$I_k'(0) = \frac{\varphi_{k+2}'(0) - \varphi_k^2(0)}{2\sqrt{\varphi_{k+2}(0)}},$$

hence it follows for the imaginary parts

$$\operatorname{Im} \varrho^{2k-1} = \frac{1}{\sqrt{h}} \left(\sqrt{\varphi_{k+2}(0)} + \frac{\varphi_{k+2}'(0) - \varphi_k^2(0)}{2\sqrt{\varphi_{k+2}(0)}}\, h + \mathcal{O}(h^2) \right).$$

This is the asserted expansion of the imaginary parts $\operatorname{Im} \varrho^1$ and $\operatorname{Im} \varrho^3$, respectively, and so of $-\operatorname{Im} \varrho^2$ and $-\operatorname{Im} \varrho^4$, respectively.

(ii) Now let

$$h := \xi^2,$$

and we use as appropriate scaling

$$\tilde{d}^1 := \frac{d^1}{h}, \quad \tilde{b}^1 := \frac{b^1}{h}, \quad \text{and} \quad \tilde{\varrho}^3 := \frac{\varrho^3}{h}.$$

From (2.8) we can conclude

$$\begin{aligned}
A &= 2h\tilde{d}^1 + h\tilde{\varrho}^3 + \varrho^4,\\
B &= \tilde{b}^1 + \tilde{\varrho}^3\varrho^4 + 2\tilde{d}^1(h\tilde{\varrho}^3 + \varrho^4),\\
C &= \tilde{b}^1\big(h\tilde{\varrho}^3 + \varrho^4\big) + 2h\tilde{d}^1\tilde{\varrho}^3\varrho^4,\\
D &= \tilde{b}^1\tilde{\varrho}^3\varrho^4.
\end{aligned}$$

The function $G \in C^1(\mathbb{R}, \mathbb{R}^4)$ with

$$G\big(h, \tilde{d}^1, \tilde{b}^1, \tilde{\varrho}^3, \varrho^4\big) := \begin{pmatrix} 2h\tilde{d}^1 + h\tilde{\varrho}^3 + \varrho^4 - A \\ \tilde{b}^1 + \tilde{\varrho}^3\varrho^4 + 2\tilde{d}^1(h\tilde{\varrho}^3 + \varrho^4) - B \\ \tilde{b}^1(h\tilde{\varrho}^3 + \varrho^4) + 2h\tilde{d}^1\tilde{\varrho}^3\varrho^4 - C \\ \tilde{b}^1\tilde{\varrho}^3\varrho^4 - D \end{pmatrix}$$

fulfils the conditions

$$G\big(0, \tfrac{1}{2A}(B - \tfrac{C}{A} - \tfrac{AD}{C}), \tfrac{D}{A}, \tfrac{D}{C}, A\big) = 0$$

and

$$\det \frac{\partial G}{\partial\big(\tilde{d}^1, \tilde{b}^1, \tilde{\varrho}^3, \varrho^4\big)}\big(0, \tfrac{1}{2A}(B - \tfrac{C}{A} - \tfrac{AD}{C}), \tfrac{D}{A}, \tfrac{D}{C}, A\big) = -2A^2C \neq 0,$$

and so the Implicit Functions Theorem can be applied again. Observing the definition of A, B, C, and D we get the assertion for ϱ^1, ϱ^2, and ϱ^3. If we compute analogously to (i) the derivative

$$\frac{\mathrm{d}}{\mathrm{d}h}\varrho^4(0) = -\frac{AB - C}{A^2} = -\gamma\kappa,$$

we finally get the expansion of ϱ^4. □

Remark 2.2. Note that for $\xi \to 0$ only ϱ^4 depends on τ in zeroth and first order approximation, and ϱ^1, ϱ^2, and ϱ^3 correspond in this approximation with the roots of the characteristic polynomial in the classical case accordingly (cf. [5] and the expansion given therein). Since the roots with small values of ξ (i. e. $\xi^2 = \lambda_1 = \frac{\pi^2}{L^2}$) determine the rate of stability, we expect no significant difference between the hyperbolic and the classical system in this respect. This is conspicuously affirmed by the numerical results in [4]. We also want to mention that we get in the case of physically realisticly dimensioned constants (cf. [4] or the subsequently given constants for silicon)

$$\min\left\{\frac{\pi^2\beta\gamma\delta\kappa}{2L^2(\alpha+\beta\delta)}, \frac{\pi^2\alpha\gamma\kappa}{L^2(\alpha+\beta\delta)}, \frac{1}{\tau} - \frac{\pi^2}{L^2}\gamma\kappa\right\} = \frac{\pi^2\beta\gamma\delta\kappa}{2L^2(\alpha+\beta\delta)} < \frac{\pi^2}{L^2}\gamma\kappa.$$

This corresponds with condition (1.4).

Figure 1 shows the characteristic roots ϱ_j^k for $1 \leqslant j \leqslant 500\,000$ and $k = 1, 2, 3, 4$ for a thin wafer of isotropic silicon at a reference temperature of $300\,K$. There we used the following material data (cf. [9] or [4]) – physical units are neglected:

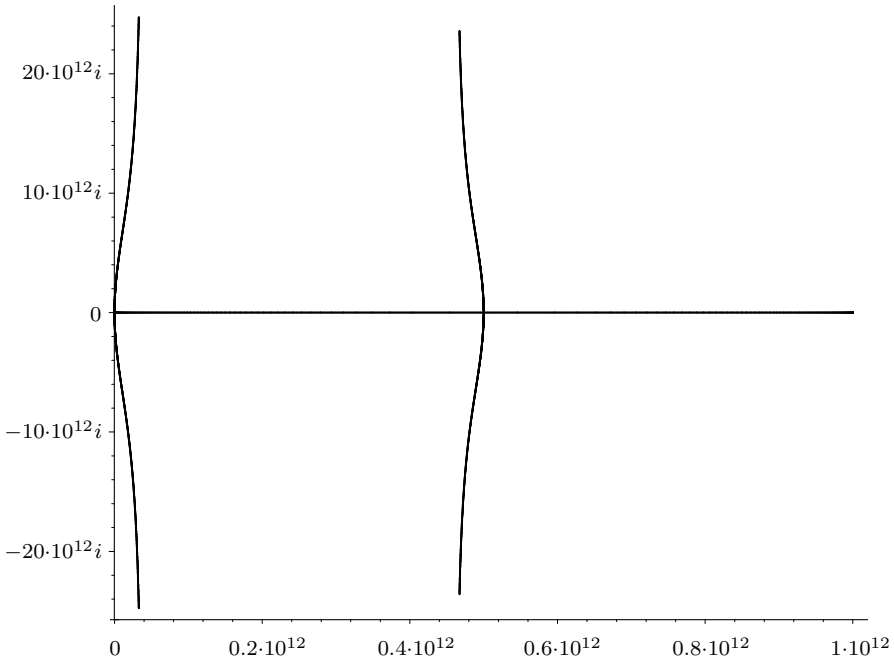

Figure 1: Characteristic roots for silicon.

α	β	γ	δ	κ
$9.62\cdot10^{7}$	392	$5.99\cdot10^{-7}$	164	148

The relaxation time τ as well as the thickness L of the wafer is given in [9]:

τ	L
$1\cdot10^{-12}$	$6.25\cdot10^{-4}$

Thereby the dimensions have to be regarded: in [4] it is proved that all characteristic roots have positive real part bounded from below by 0.7462. But this cannot be seen in figure 1 as too large scales are used.

3 Exponential decay and rate of stability

In this section we will give the optimal decay rate for the system of hyperbolic thermoelasticty. Actually, this will be a byproduct of the proof of the exponential stability of the system. First we make the following

Definition 3.1. *The energy $\mathcal{E}$ is defined by*

$$\mathcal{E}(t) := \int_0^L \left(u_{tx}^2 + u_{xx}^2 + \vartheta_t^2 + \vartheta_x^2 + q_t^2\right)(t,x)\,\mathrm{d}x.$$

A constant $d \in \mathbb{R}$ is called decay rate if there is a positive constant C so that for all $t \geqslant 0$ and for all initial data the inequality

$$\mathcal{E}(t) \leqslant Ce^{-2dt}\mathcal{E}(0)$$

holds. Finally we call

$$d_{ss} := \sup\{d\colon d \text{ is a decay rate}\}$$

the rate of stability.

Remark 3.2. For the ordinary differential equation

$$f''(t) + 2f'(t) + f(t) = 0$$

with the energy $f(t)^2 + f'(t)^2$ the set of all decay rates coincides with the open interval $(-\infty, 1)$. Hence, the rate of stability is not a decay rate. Actually, this is due to the fact that 1 is a double root of the underlying characteristic polynomial, and a fundamental system for the differential equation is given by e^{-t} and te^{-t}.

The ansatz (2.5), (2.6) suggests that

$$\varrho_{ss} := \inf\left\{\operatorname{Re}\varrho\colon \text{there is } j \in \mathbb{N} \text{ such that } \chi_j(\varrho) = 0\right\}$$

corresponds with d_{ss}, and indeed we get

Theorem 3.3 (Hyperbolic system).
The energy $\mathcal{E}$ of the hyperbolic system decays exponentially with the rate of stability

$$d_{ss} = \varrho_{ss}.$$

Furthermore, if all ϱ which are multiple roots of some polynomial χ_j, $j \in \mathbb{N}$, satisfy the condition

$$\operatorname{Re}\varrho > \varrho_{ss}$$

then d_{ss} is the maximal decay rate.

Proof. For simplicity we choose $L = \pi$. Hence, we get $\lambda_j = j^2$. First we derive from (2.1):

$$\begin{aligned}
u_j(0) &= -j^{-2}\left\langle u^0_{xx}, s_j\right\rangle,\\
u'_j(0) &= j^{-1}\left\langle u^1_x, c_j\right\rangle,\\
u''_j(0) &= \alpha\left\langle u^0_{xx}, s_j\right\rangle + \beta\left\langle \vartheta^0_x, s_j\right\rangle,\\
u'''_j(0) &= -\alpha j\left\langle u^1_x, c_j\right\rangle + \beta j\left\langle \vartheta^0_t, c_j\right\rangle
\end{aligned}$$

using for example ϑ^0_t instead of $\vartheta_t(0,\cdot\,)$ and $\langle f, g\rangle := \int_0^\pi f(x)g(x)\,\mathrm{d}x$ for functions $f, g \in L^2\big((0,\pi),\mathbb{R}\big)$. In the same way we get

$$\begin{aligned}
\vartheta_j(0) &= -j^{-1}\left\langle \vartheta^0_x, s_j\right\rangle,\\
\vartheta'_j(0) &= \left\langle \vartheta^0_t, c_j\right\rangle,\\
\vartheta''_j(0) &= -\gamma j\left\langle q^0_t, s_j\right\rangle - \alpha\delta j\left\langle u^0_{xx}, s_j\right\rangle + \beta\delta j\left\langle \vartheta^0_x, s_j\right\rangle,\\
\vartheta'''_j(0) &= \frac{\gamma}{\tau} j\left\langle q^0_t, s_j\right\rangle - \frac{\gamma\kappa}{\tau} j^2\left\langle \vartheta^0_t, c_j\right\rangle + \alpha\delta j^2\left\langle u^1_x, c_j\right\rangle - \beta\delta j^2\left\langle \vartheta^0_t, c_j\right\rangle
\end{aligned}$$

and

$$
\begin{aligned}
q_j(0) &= -\frac{1}{\gamma} j^{-1} \left\langle \vartheta_t^0, c_j \right\rangle - \frac{\delta}{\gamma} j^{-1} \left\langle u_x^1, c_j \right\rangle, \\
q_j'(0) &= \left\langle q_t^0, s_j \right\rangle, \\
q_j''(0) &= -\frac{1}{\tau} \left\langle q_t^0, s_j \right\rangle + \frac{\kappa}{\tau} j \left\langle \vartheta_t^0, c_j \right\rangle, \\
q_j'''(0) &= \frac{1}{\tau^2} \left\langle q_t^0, s_j \right\rangle - \frac{\kappa}{\tau^2} j \left\langle \vartheta_t^0, c_j \right\rangle - \frac{\gamma\kappa}{\tau} j^2 \left\langle q_t^0, s_j \right\rangle - \frac{\alpha\delta\kappa}{\tau} j^2 \left\langle u_{xx}^0, s_j \right\rangle + \frac{\beta\delta\kappa}{\tau} j^2 \left\langle \vartheta_x^0, s_j \right\rangle.
\end{aligned}
$$

From (2.6) we obtain

$$
\begin{pmatrix} u_j(0) \\ u_j'(0) \\ u_j''(0) \\ u_j'''(0) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ -\varrho_j^1 & -\varrho_j^2 & -\varrho_j^3 & -\varrho_j^4 \\ (\varrho_j^1)^2 & (\varrho_j^2)^2 & (\varrho_j^3)^2 & (\varrho_j^4)^2 \\ -(\varrho_j^1)^3 & -(\varrho_j^2)^3 & -(\varrho_j^3)^3 & -(\varrho_j^4)^3 \end{pmatrix}}_{=: B_j} \begin{pmatrix} u_j^1 \\ u_j^2 \\ u_j^3 \\ u_j^4 \end{pmatrix}
$$

with the VANDERMONDE-determinant

$$
\det B_j = \prod_{1 \leqslant n < m \leqslant 4} (\varrho_j^m - \varrho_j^n).
$$

In Lemma 2.1 we have proved that the characteristic roots satisfy

$$
\operatorname{Re} \varrho_j^1 = \operatorname{Re} \varrho_j^2 \sim 1, \tag{3.1a}
$$

$$
\operatorname{Re} \varrho_j^3 = \operatorname{Re} \varrho_j^4 \sim 1, \tag{3.1b}
$$

$$
\operatorname{Im} \varrho_j^1 = -\operatorname{Im} \varrho_j^2 \sim j, \tag{3.1c}
$$

$$
\operatorname{Im} \varrho_j^3 = -\operatorname{Im} \varrho_j^4 \sim j, \tag{3.1d}
$$

where we use the notation $f(j) \sim g(j)$ if and only if $\lim_{j\to\infty} \frac{f(j)}{g(j)} \neq 0$ exists. Observing Lemma 2.1 (i) and the definition of Λ_1 and Λ_2 it is easy to see that the four roots of χ_j are pairwise different for sufficiently large j. Hence

$$
\det B_j \sim j^6.
$$

Applying CRAMER's rule we get

$$
u_j^1 = \frac{1}{\det B_j} \det \begin{pmatrix} u_j(0) & 1 & 1 & 1 \\ u_j'(0) & -\varrho_j^2 & -\varrho_j^3 & -\varrho_j^4 \\ u_j''(0) & (\varrho_j^2)^2 & (\varrho_j^3)^2 & (\varrho_j^4)^2 \\ u_j'''(0) & -(\varrho_j^2)^3 & -(\varrho_j^3)^3 & -(\varrho_j^4)^3 \end{pmatrix}
$$

and after some easy calculation

$$
u_j^1 \sim u_j(0) + j^{-1} u_j'(0) + j^{-2} u_j''(0) + j^{-3} u_j'''(0), \tag{3.2}
$$

and (3.2) holds analogously for $u_j^2, \dots, q_j^4$.

Now we want to consider the finite set of all those $j \in \mathbb{N}$ for which the polynomial χ_j has at least one multiple root. Exemplary we assume that all polynomials χ_j have single roots except χ_J which has exactly three different roots ϱ_J^k, $k = 1, 2, 3$, with the double root $\varrho_J^3 = \varrho_J^4$. By a well-known result in the theory of ordinary differential equations we can represent $u_J(t)$ by

$$u_J(t) = \sum_{k=1}^{3} u_J^k e^{-\varrho_J^k t} + u_J^4\, t e^{-\varrho_J^3 t}.$$

Analogous representations hold for $\vartheta_J(t)$ and $q_J(t)$. The occuring factors $u_J^1, \dots, q_J^4$ depend linearly on the initial data $\langle u^0, s_J\rangle$, $\langle u^1, s_J\rangle$, $\langle \vartheta^0, c_J\rangle$, and $\langle q^0, s_J\rangle$.

Hence, treating each summand of $\mathcal{E}$ separately we find that all $d < \varrho_{ss}$ are decay rates. This we want to prove with the following two examples. Note that $te^{-\varrho_J^3 t}$ can be estimated by Ce^{-dt} with a constant C which does not depend on t. Let $d < \varrho_{ss}$, than

$$\begin{aligned}
\int_0^\pi u_{xx}^2(t,x)\,\mathrm{d}x &= \sum_{j\neq J}\left|\sum_{k=1}^{4} j^2 u_j^k e^{-\varrho_j^k t}\right|^2 + \left|\sum_{k=1}^{3} J^2 u_J^k e^{-\varrho_J^k t} + J^2 u_J^4\, t e^{-\varrho_J^3 t}\right|^2 \\
&\leqslant C_1 e^{-2dt} \sum_{j=1}^{\infty}\sum_{k=1}^{4}\left|j^2 u_j^k\right|^2 \\
&\leqslant C_2 e^{-2dt} \sum_{j=1}^{\infty} \left|\left\langle u_x^1, c_j\right\rangle\right|^2 + \left|\left\langle u_{xx}^0, s_j\right\rangle\right|^2 + \left|\left\langle \vartheta_t^0, c_j\right\rangle\right|^2 + \left|\left\langle \vartheta_x^0, s_j\right\rangle\right|^2,
\end{aligned}$$

and similarly

$$\begin{aligned}
\int_0^\pi q_t^2(t,x)\,\mathrm{d}x &= \sum_{j\neq J}\left|\sum_{k=1}^{4} \varrho_j^k q_j^k e^{-\varrho_j^k t}\right|^2 + \left|\sum_{k=1}^{3} \varrho_J^k u_J^k e^{-\varrho_J^k t} + \varrho_J^3 u_J^4\, t e^{-\varrho_J^3 t}\right|^2 \\
&\leqslant C_3 e^{-2dt} \sum_{j=1}^{\infty}\left|\varrho_j^k q_j^k\right|^2 \\
&\leqslant C_4 e^{-2dt} \sum_{j=0}^{\infty} \left|\left\langle u_x^1, c_j\right\rangle\right|^2 + \left|\left\langle u_{xx}^0, s_j\right\rangle\right|^2 + \left|\left\langle \vartheta_t^0, c_j\right\rangle\right|^2 + \left|\left\langle \vartheta_x^0, s_j\right\rangle\right|^2 + \left|\left\langle q_t^0, c_j\right\rangle\right|^2
\end{aligned}$$

with positive constants C_i, $i = 1, 2, 3, 4$ which are independent of the initial data.

Since it is clear that $d > \varrho_{ss}$ cannot be a decay rate, the first statement of the Theorem follows. The second one directly results from the last estimates where e^{-dt} can be replaced by $e^{-\varrho_{ss} t}$ because under the more stringend condition on the multiple roots the appearing term $te^{-\varrho_J^3 t}$ is now dominated by $Ce^{-\varrho_{ss} t}$. □

Remark 3.4. Using physically realistically dimensioned coefficients in (2.1), the additional condition is satisfied. For this we refer to the numerical results in [4]. Especially for silicon this also can be seen in figure 1 where the only double root – if it exists – lies in a neighbourhood of $0.5{\cdot}10^{12}$.

Remark 3.5. The LJAPUNOV-functional $\mathcal{F}$ with

$$\mathcal{F}(t) := \int_0^L \left(u_x^2 + u_t^2 + u_{xt}^2 + u_{tt}^2 + \vartheta^2 + \vartheta_t^2 + q^2 + q_t^2\right)(t,x)\,\mathrm{d}x$$

is equivalent to $\mathcal{E}$, i. e. there are constants $C_1 > 0$ and $C_2 > 0$ with

$$C_1\mathcal{E}(t) \leqslant \mathcal{F}(t) \leqslant C_2\mathcal{E}(t)$$

for all $t \geqslant 0$. To see this, one has to use (2.1) and POINCARÉ's inequality for ϑ and u_t which holds because of (2.4) and (2.3) respectively. Note that u_x is bounded by u_{xx} because of (2.3) and YOUNG's inequality.

Usually $\mathcal{F}$ is applied as the associated energy to the system of hyperbolic thermoelasticity as in [9], [10], and [3].

4 Classical thermoelasticity

Typically, the energy E with

$$E(t) := \int_0^L \left(u_t^2 + u_x^2 + \vartheta^2\right)(t,x)\,\mathrm{d}x,$$

is typically used for the system of classical thermoelasticity

$$u_{tt} - \alpha u_{xx} + \beta\vartheta_x = 0 \quad \text{in } [0,\infty)\times(0,L), \tag{4.1a}$$

$$\vartheta_t - \gamma\kappa\vartheta_{xx} + \delta u_{tx} = 0 \quad \text{in } [0,\infty)\times(0,L) \tag{4.1b}$$

with DIRICHLET-NEUMANN boundary conditions and given initial data (cf. [7], [6], [11], [4] for example). In the cited paper [4] the problem appears that the summand $\int \vartheta^2(t,x)\,\mathrm{d}x$ cannot be estimated in the same manner as the other terms and one has to apply GRONWALL's Lemma to succeed. Furthermore, the additional assumption (1.4) on the rate of stability d_{ct} is made.

We will now use the energy $\mathcal{E}$ also for the classical system. Note that in this cases $\mathcal{E}$ is still equivalent to $\mathcal{F}$ and equals

$$\mathcal{E}(t) = \int_0^L \left(u_{tx}^2 + u_{xx}^2 + \vartheta_t^2 + \vartheta_x^2 + \kappa^2\vartheta_{tx}^2\right)(t,x)\,\mathrm{d}x$$

because of FOURIER's law (1.3). So we will consider higher order energy terms than in E contained, but we will get by without the assumption (1.4) or GRONWALL's Lemma.

We mention that in the classical case the characteristic polynomial is given by ψ_j with

$$\psi_j(\varrho) := -\varrho^3 + \gamma\kappa\lambda_j\varrho^2 - (\alpha+\beta\delta)\lambda_j\varrho + \alpha\gamma\kappa\lambda_j^4.$$

The ansatz (2.5) for u and ϑ is possible again where the coefficients $u_j(t)$ and $\vartheta_j(t)$ can be represented by

$$u_j(t) = \sum_{k=1}^{3} u_j^k e^{-\varrho_j^k t}, \quad \vartheta_j(t) = \sum_{k=1}^{3} \vartheta_j^k e^{-\varrho_j^k t},$$

if ϱ_j^k, $k = 1, 2, 3$, are pairwise different roots of ψ_j.

For the classical system define

$$\varrho_{ct} := \inf \left\{ \operatorname{Re} \varrho \colon \text{there is } j \in \mathbb{N} \text{ such that } \psi_j(\varrho) = 0 \right\},$$

and let d_{ct} denote the rate of stability. Than we have

Theorem 4.1 (Classical system).
The energy $\mathcal{E}$ of the classical system decays exponentially with the rate of stability

$$d_{ct} = \varrho_{ct}.$$

Furthermore, if all ϱ which are multiple roots of some polynomial ψ_j, $j \in \mathbb{N}$, satisfy the condition

$$\operatorname{Re} \varrho > \varrho_{ct}$$

then d_{ct} is the maximal decay rate.

Proof. We choose $L = \pi$ again. Moreover, we assume for simplicity that none of the polynomials ψ_j has multiple roots. Otherwise we can argue as in the proof of Theorem 3.3.
From (4.1) we conclude

$$\begin{aligned}
u_j(0) &= -j^{-2} \left\langle u_{xx}^0, s_j \right\rangle, \\
u_j'(0) &= j^{-1} \left\langle u_x^1, c_j \right\rangle, \\
u_j''(0) &= \alpha \left\langle u_{xx}^0, s_j \right\rangle - \beta \left\langle \vartheta_x^0, s_j \right\rangle,
\end{aligned}$$

as well as

$$\begin{aligned}
\vartheta_j(0) &= -\frac{1}{\gamma\kappa} j^{-2} \left\langle \vartheta_t^0, c_j \right\rangle - \frac{\delta}{\gamma\kappa} j^{-2} \left\langle u_x^1, c_j \right\rangle, \\
\vartheta_j'(0) &= -j^{-1} \left\langle \vartheta_{tx}^0, s_j \right\rangle, \\
\vartheta_j''(0) &= \gamma\kappa j \left\langle \vartheta_{tx}^0, s_j \right\rangle - \alpha\delta j \left\langle u_{xx}^0, s_j \right\rangle + \beta\delta j \left\langle \vartheta_x^0, s_j \right\rangle.
\end{aligned}$$

For classical thermoelasticity the behaviour of the roots ϱ_j^k is well known (cf. [5] or [7]):

$$\begin{aligned}
\varrho_j^1 &\sim j^2, \\
\operatorname{Re} \varrho_j^2 = \operatorname{Re} \varrho_j^3 &\sim 1, \\
\operatorname{Im} \varrho_j^2 = -\operatorname{Im} \varrho_j^3 &\sim j.
\end{aligned}$$

Hence, it follows (cf. [4], p. 8):

$$u_j^1 \sim j^{-2} u_j(0) + j^{-3} u_j'(0) + j^{-4} u_j''(0), \tag{4.2a}$$

$$u_j^{2,3} \sim u_j(0) + j^{-1} u_j'(0) + j^{-3} u_j''(0), \tag{4.2b}$$

and (4.2a) holds analogously for ϑ_j^1 and (4.2b) for $\vartheta_j^{2/3}$, respectively.

Finally, we only have to show that ϱ_{ct} is a decay rate. For this purpose we estimate each summand of the energy $\mathcal{E}$ separately. For example we get

$$\begin{aligned}\int_0^\pi u_{tx}^2(t,x)\,\mathrm{d}x &= \sum_{j=1}^{\infty}\left|\sum_{k=1}^{3}\varrho_j^k j u_j^k e^{-\varrho_j^k t}\right|^2\\ &\leqslant C_5 e^{-2\varrho_{ct}t}\sum_{j=1}^{\infty}\sum_{k=1}^{3}\left|\varrho_j^k j u_j^k\right|^2\\ &\leqslant C_6 e^{-2\varrho_{ct}t}\sum_{j=1}^{\infty}\left|\left\langle u_x^1, c_j\right\rangle\right|^2+\left|\left\langle u_{xx}^0, s_j\right\rangle\right|^2+j^{-2}\left|\left\langle \vartheta_x^0, s_j\right\rangle\right|^2\end{aligned}$$

and

$$\begin{aligned}\int_0^\pi \vartheta_{tx}^2(t,x)\,\mathrm{d}x &= \sum_{j=1}^{\infty}\left|\sum_{k=1}^{3}\varrho_j^k j \vartheta_j^k e^{-\varrho_j^k t}\right|^2\\ &\leqslant C_7 e^{-\varrho_{ct}t}\sum_{j=1}^{\infty}\sum_{k=1}^{3}\left|\varrho_j^k j \vartheta_j^k\right|^2\\ &\leqslant C_8 e^{-2\varrho_{ct}t}\sum_{j=1}^{\infty}\left|\left\langle u_x^1, c_j\right\rangle\right|^2+\left|\left\langle u_{xx}^0, s_j\right\rangle\right|^2+\left|\left\langle \vartheta_t^0, c_j\right\rangle\right|^2+\left|\left\langle \vartheta_x^0, s_j\right\rangle\right|^2+\left|\left\langle \vartheta_{tx}^0, s_j\right\rangle\right|^2\end{aligned}$$

with $C_i > 0$ for $i = 5, 6, 7, 8$. □

Remark 4.2. For physically realistic coefficients there is no multiple root, cf. [7], p. 241.

Thus it is possible to weaken the assumptions for proving the optimality of d_{ct} if one takes energy terms of higher derivative order into account, i. e. additionally the time derivatives of u_x, u_t, and ϑ as well as $q = -\kappa\vartheta_x$ together with its time derivative.

At the same time we managed to estimate the same energy for both hyperbolic and classical thermoelastisity. This might be important for a comparison between the two systems.

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