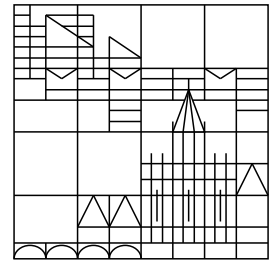


Universität Konstanz



Unique Tensor Factorization of Algebras

Michael Nüsken

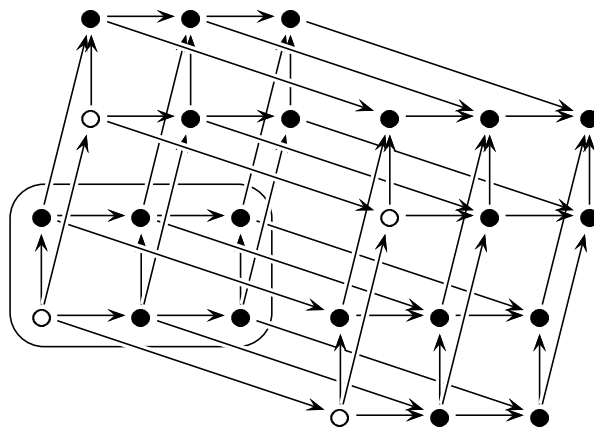
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Unique Tensor Factorization of Algebras

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Zusammenfassung

Tensorproduktzerlegung von Algebren ist bekanntermaßen in vielen Fällen nicht eindeutig. Wie hier gezeigt wird, hat aber eine additiv unzerlegbare, endlich-dimensionale \mathbb{C} -Algebra A eine im wesentlichen eindeutige Tensorfaktorisierung

$$A = A_1 \otimes \cdots \otimes A_r$$

in nicht-triviale, \otimes -unzerlegbare Faktoren A_i . Damit ist der Halbring der Isomorphieklassen endlich-dimensionaler \mathbb{C} -Algebren ein polynomialer Halbring $\mathbb{N}[\mathcal{X}]$. Der Körper \mathbb{C} der komplexen Zahlen kann sogar durch einen beliebigen Körper der Charakteristik Null ersetzt werden, wenn man sich auf SCHURsche Algebren beschränkt.

Abstract

Tensor product decomposition of algebras is known to be non-unique in many cases. But, as will be shown here, an additively indecomposable, finite-dimensional \mathbb{C} -algebra A has an essentially unique tensor factorization

$$A = A_1 \otimes \cdots \otimes A_r$$

into non-trivial, \otimes -indecomposable factors A_i . Thus the semiring of isomorphism classes of finite-dimensional \mathbb{C} -algebras is a polynomial semiring $\mathbb{N}[\mathcal{X}]$. Moreover, the field \mathbb{C} of complex numbers can be replaced by an arbitrary field of characteristic zero if one restricts oneself to SCHURian algebras.

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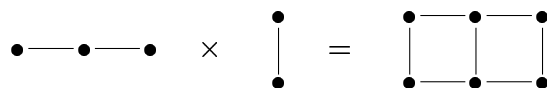
This dissertation grew out of joint work with V. STRASSEN. He suggested the main result and reduced its proof to the case of basic algebras (see Appendix C), while I obtained a proof for the basic case. Later I saw that my proof could be adapted to yield the general result directly, and in fact not only for \mathbb{C} -algebras, but also for SCHURIAN algebras over an arbitrary field of characteristic zero.

I wish to thank all the people who supported this work directly or indirectly. First of all, I would like to thank my Doktorvater Prof. Dr. V. STRASSEN. His assistance and criticism were always helpful and his demand for clarity and precision still challenges me. I have to apologize to and thank all the people who were drawn into discussions by me, sometimes without understanding much of the story. The most important of these listeners were W. GLAS, and Prof. Dr. H.-J. STOSS. In addition, Prof. Dr. G. BARTHEL, Dr. J. HAUSEN, and A. WINTER were involved in various conversations. Furthermore, I am indebted to S. SCHÖPF and A. HÖHMANN for linguistic advice and proof-reading.

0 Einleitung

Eine natürliche Zahl hat eine eindeutige Primfaktorzerlegung, wie wir alle wissen. Auch gewisse Algebren haben eine eindeutige Zerlegung als Tensorprodukt von nicht weiter zerlegbaren Algebren; darüber berichten wir hier.

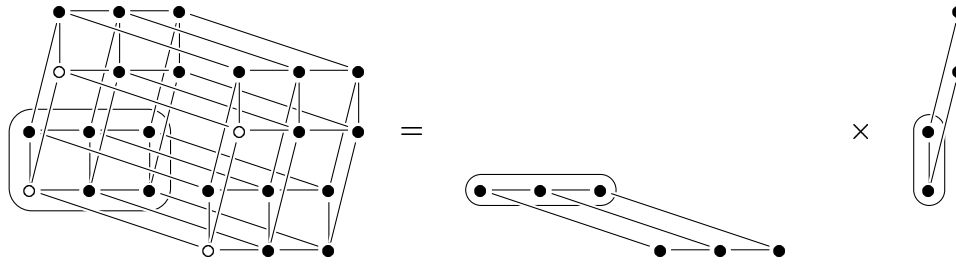
Bevor wir über Algebren sprechen, möchte ich eine ähnliche Geschichte über Graphen erzählen. Die Graphen, die wir betrachten, sind endliche, ungerichtete Graphen ohne Schleifen und Mehrfachkanten. Tatsächlich können Graphen miteinander multipliziert werden: Dazu verwenden wir das *kartesische Produkt von Graphen* $\Gamma_1 \times \Gamma_2$. Die Punktmenge des Produktgraphen ist das kartesische Produkt der Punktmenge der Faktoren. Zwei Punkte darin, etwa (α_1, α_2) und (β_1, β_2) , sind durch eine Kante verbunden genau, wenn entweder die zweiten Koordinaten gleich und die ersten Koordinaten in Γ_1 verbunden sind (also $\alpha_2 = \beta_2$ und $\{\alpha_1, \beta_1\} \in \Gamma_1$) oder umgekehrt die ersten Koordinaten gleich und die zweiten in Γ_2 verbunden sind. Um das zu veranschaulichen, betrachten wir folgendes Beispiel:



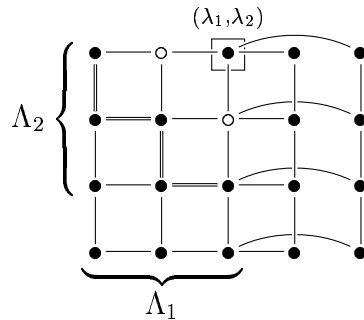
Zuerst zeichnen wir die neue Punktmenge und dann kopieren wir den ersten Faktor in jede waagrechte Schicht und den zweiten Faktor in jede senkrechte Schicht. Dieses Produkt ist nun (bis auf Isomorphie) assoziativ, kommutativ und es hat ein neutrales Element: den trivialen Graphen, einen Punkt. Bezüglich dieses Produktes nennen wir einen Graphen \times -*unzerlegbar* genau, wenn er nicht trivial ist und (bis auf Isomorphie) keine echte Zerlegung besitzt. Mit anderen Worten: In jeder Zerlegung eines \times -unzerlegbaren Graphen in zwei Faktoren ist genau einer trivial. SABIDUSSI (1960) hat gezeigt, daß tatsächlich jeder zusammenhängende Graph eine eindeutige Faktorisierung besitzt. (Mit Faktorisierung meinen wir hier immer eine Produktzerlegung in unzerlegbare Faktoren. Eindeutige Faktorisierung bedeutet, daß die Faktoren einer solchen Zerlegung bis auf Reihenfolge und Isomorphie eindeutig bestimmt sind.) IMRICH (1967) hat einen kurzen Beweis geliefert, den wir hier skizzieren. Wir beginnen mit einem Isomorphismus zweier Produkte $\Gamma_1 \times \Gamma_2$ und $\Delta_1 \times \Delta_2$ zusammenhängender Graphen. Nun zeigt sich, daß ein Teilgraph der Gestalt $\Gamma_1 \times \{\alpha_2\}$ auf einen Teilgraphen der Gestalt $\Lambda_1 \times \Lambda_2$ abgebildet wird.

$$\begin{array}{ccc} \Gamma_1 \times \Gamma_2 & \xrightarrow{\cong} & \Delta_1 \times \Delta_2 \\ \cup & & \cup \quad \cup \\ \Gamma_1 \times \{\alpha_2\} & \xrightarrow{\cong} & \Lambda_1 \times \Lambda_2 \end{array}$$

Mit anderen Worten: Eine Schicht auf der linken Seite entspricht einem Rechteck auf der rechten. Dazu beobachten wir zunächst eine bemerkenswerte Eigenschaft einer solchen Schicht: Ein Punkt außerhalb der Schicht ist immer mit höchstens



einem Punkt in der Schicht verbunden. Anders ausgedrückt: Wenn ein Punkt mit zwei Punkten einer Schicht verbunden ist, so gehört er selbst dazu. Natürlich ist das genauso richtig für das Bild der Schicht auf der rechten Seite (also in $\Delta_1 \times \Delta_2$). Wir sammeln nun alle Spalten, die von diesem Bild getroffen werden, um Λ_1 zu definieren, und alle Zeilen für Λ_2 . Um zu zeigen, daß das Bild tatsächlich ein Rechteck ist, müssen wir zeigen, daß jeder Punkt (λ_1, λ_2) des Rechtecks $\Lambda_1 \times \Lambda_2$ im Bild der Schicht liegt. Aber wir wissen zunächst nur, daß ein Punkt in $\Lambda_1 \times \{\lambda_2\}$ und ein Punkt in $\{\lambda_1\} \times \Lambda_2$ im Bild der Schicht liegen. Da Γ_1 zusammenhängend ist, ist auch das Bild der Schicht zusammenhängend. Wir können also diese beiden Punkte innerhalb des Bildes verbinden. Betrachten wir folgendes Beispiel:



Darin bezeichnet das kleine Quadrat den Punkt (λ_1, λ_2) , mit dem wir anfangen, die Punkte in der zugehörigen Spalte und Zeile sind durch einen doppelt gezeichneten Weg verbunden. Wir sehen zwei Punkte (als Kreise), die nicht selbst auf dem Weg liegen, aber jeweils mit zwei seiner Punkte verbunden sind. Da der Weg ganz zum Bild gehört, schließen wir mit Hilfe der obigen Schichteigenschaft, daß diese zwei Punkte auch zum Bild der Schicht gehören. Aber jetzt ist der ausgesuchte Punkt selbst mit diesen beiden Punkten verbunden. Darum liegt auch er in der Schicht. Durch Induktion bekommt man mit dieser Idee die Rechteckaussage.

Nehmen wir nun zusätzlich an, daß alle Faktoren Γ_i und Δ_j \times -unzerlegbare Graphen sind. Wir wollen zeigen, daß die beiden Faktorisierungen tatsächlich gleich sind (bis auf Reihenfolge und Isomorphie der Faktoren). Als Graph ist das Bild der Schicht isomorph zu Γ_1 . Andererseits wissen wir, daß dieses ein Rechteck

ist, ein Produkt. Daher muß einer der beiden Faktoren trivial sein. Sagen wir, Λ_2 sei gerade der Einpunktgraph $\{\beta_2\}$. Dann ist das Schichtbild enthalten in der Schicht $\Delta_1 \times \{\beta_2\}$. Aber diese bildet — mutatis mutandis — auf der linken Seite ein Rechteck. Weil auch $\Delta_1 \times$ -unzerlegbar ist, ist dieses Rechteck in einer Schicht enthalten. Aber es enthält ja bereits die Schicht $\Gamma_1 \times \{\alpha_2\}$. Das kann aber nur sein, wenn Gleichheit vorliegt. Also ist Λ_1 gleich Δ_1 . Die Schicht $\Gamma_1 \times \{\alpha_2\}$ links ist damit isomorph zur Schicht $\Delta_1 \times \{\beta_2\}$ rechts. Also gilt $\Gamma_1 \simeq \Delta_1$. Entsprechend erhalten wir $\Gamma_2 \simeq \Delta_2$. (Beachte, daß eine Γ_2 -Schicht jetzt nicht auf eine Δ_1 -Schicht abgebildet werden kann, da eine waagrechte und eine senkrechte Schicht sich in genau einem Punkt treffen.) Damit ist der betrachtete Fall erledigt. Der allgemeine Fall von SABIDUSSIS Satz kann mit den gleichen Ideen gezeigt werden. Soviel zu Graphen, wenden wir uns nun Algebren zu.

Alle hier betrachteten Algebren sind endlich-dimensionale, assoziative k -Algebren mit Eins. Man denke zum Beispiel an volle Matrixalgebren, oder an endlich-dimensionale Quotienten von Polynomalgebren. Nicht nur Graphen kann man multiplizieren: Für Algebren verwenden wir das *Tensorprodukt von Algebren* $A_1 \otimes A_2$. Das ist die folgendermaßen erklärte Algebra: Wir versehen das Tensorprodukt der unterliegenden Vektorräume mit dem Produkt, das zerfallende Tensoren koordinatenweise multipliziert, $a'_1 \otimes a'_2 \cdot a_1 \otimes a_2 = a'_1 a_1 \otimes a'_2 a_2$. Die Faktoren A_i können wir stets als Unteralgebren von $A_1 \otimes A_2$ auffassen, etwa für $i = 1$ vermöge $a \mapsto a \otimes 1$. Auch dieses Produkt von Algebren verhält sich gut: es ist (bis auf Isomorphie) assoziativ, kommutativ und es hat ein neutrales Element: die triviale Algebra, welche einfach der Grundkörper ist. Wir nennen eine Algebra \otimes -unzerlegbar genau, wenn sie nicht trivial ist und (bis auf Isomorphie) keine echte Zerlegung besitzt. Mit anderen Worten: In jeder Zerlegung einer \otimes -unzerlegbaren Algebra in zwei Faktoren ist genau einer der beiden trivial. Für volle Matrixalgebren drückt das Tensorprodukt die altbekannte Tatsache aus, daß man pq -reihige Matrizen blockweise multiplizieren kann, indem man sie als p -reihige Matrizen auffaßt, deren Einträge q -reihige Matrizen sind.

$$k^{p \times p} \otimes k^{q \times q} \simeq k^{pq \times pq}.$$

Ein anderes einfaches Beispiel ist folgendes: Wir tensorieren einen endlich-dimensionalen Quotienten einer reellen Polynomalgebra mit dem Körper der komplexen Zahlen. Dann bekommen wir den entsprechenden Quotienten der komplexen Polynomalgebra:

$$\mathbb{R}[T]/\langle f \rangle \otimes \mathbb{C} \simeq \mathbb{C}[T]/\langle f \rangle.$$

Daraus ergibt sich unmittelbar, daß wir keine eindeutige Faktorisierung für beliebige Algebren erwarten können: Setze $f = T^2 + 1$. Dann ist die linke Seite $\mathbb{C} \otimes \mathbb{C}$, während die rechte nach dem Chinesischen Restsatz $\mathbb{C} \oplus \mathbb{C}$ ist. Nochmal umgeschrieben erhalten wir einen Isomorphismus zweier Tensorzerlegungen:

$$\mathbb{C} \otimes \mathbb{C} \simeq (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{C}.$$

Alle Faktoren sind \otimes -unzerlegbar, weil ihre Vektorraumdimensionen alle 2 sind (also prim). Nun ist \mathbb{C} ein Körper, während $\mathbb{R} \oplus \mathbb{R}$ noch nicht einmal eine Divisionsalgebra ist. Also sind die beiden Zerlegungen wesentlich verschieden. Um Schwierigkeiten dieser Art aus dem Wege zu gehen, setzen wir von nun an voraus, daß der Grundkörper k der Körper \mathbb{C} der komplexen Zahlen ist.

Natürlich ist das Tensorprodukt nicht nur wegen seiner Allgegenwart in der Algebra wichtig, sondern auch wegen seiner engen Beziehung zum direkten Produkt geometrischer und topologischer Strukturen. Das ist besonders deutlich in der algebraischen Geometrie beim Übergang von einer affinen algebraischen Varietät zu ihrem Koordinatenring. Natürlich sind diese Algebren in der Regel nicht endlich-dimensional. Ein anderes Beispiel kommt aus der Topologie, wenn man von topologischen Räumen mit endlich-dimensionalem, kommutativem Kohomologie-Ring mit Körperkoeffizienten zu eben diesem Kohomologie-Ring übergeht. Aber andererseits haben diese Algebren typischerweise zusätzliche Struktur wie Kommutativität oder eine Graduierung.

Auch durch eine topologische Frage angeregt, haben BODY & DOUGLAS (1979) ein erstes Ergebnis über eindeutige Faktorisierung erhalten. Mit einer sehr eleganten Methode haben sie eindeutige Faktorisierung für graduierte, lokale Algebren gezeigt. Wir deuten die wesentlichen Schritte ihres Beweises an. Zunächst bemerken wir, daß die Automorphismengruppe einer endlich-dimensionalen Algebra eine lineare algebraische Gruppe ist. In der Automorphismengruppe einer graduierten Algebra $B = \bigoplus_{\ell} B^{(\ell)}$ finden wir wie folgt immer einen eindimensionalen Untertorus. Zu $\lambda \in \mathbb{C}^*$ definiert die Zuordnung $\sum b^{(\ell)} \mapsto \sum \lambda^{\ell} b^{(\ell)}$ einen Automorphismus der Algebra B . Betrachten wir ein Tensorprodukt A von s graduierten Algebren B_i , so finden wir s unabhängige eindimensionale Tori, die zusammen einen s -dimensionalen Untertorus der Automorphismengruppe von A aufspannen. Nehmen wir an, wir haben zwei Tensorzerlegungen einer Algebra A , deren Faktoren wir vermöge der natürlichen Einbettungen als Unteralgebren von A verstehen,

$$B_1 \otimes \cdots \otimes B_s = C_1 \otimes \cdots \otimes C_t,$$

$$S \qquad \qquad T$$

mit Tori S und T in der Automorphismengruppe von A . (Also, $S \simeq (\mathbb{C}^*)^s$, $T \simeq (\mathbb{C}^*)^t$.) Es zeigt sich, daß die beiden Zerlegungen genau dann übereinstimmen, wenn die Tori gleich sind. Sind alle Faktoren \otimes -unzerlegbar, dann genügt es sogar, daß die Tori S und T vertauschen! Nun sagt ein klassisches Ergebnis von BOREL, daß maximale Tori in einer linearen Gruppe konjugiert sind. Erweitern wir also S und T zu maximalen Tori. Durch eine Konjugation können wir nun erreichen, daß diese gleich sind. (Das bedeutet, daß wir eine der Zerlegungen durch eine isomorphe ersetzen.) Aber jetzt sind die beiden Tori S und T in diesem maximalen Torus enthalten. Da dieser eine kommutative Gruppe ist, vertauschen die Tori, und die Zerlegungen sind gleich. Das beweist schon den Satz.

Während diese Methode nicht auf endlich-dimensionale Algebren beschränkt ist, ist es sehr wichtig, daß wir Graduierungen haben. Die Automorphismengruppe einer beliebigen lokalen Algebren enthält typischerweise überhaupt keinen nicht-trivialen Untertorus.

Mit einer sehr viel elementareren Methode hat HORST (1987) eindeutige Faktorisierung für beliebige lokale Algebren gezeigt. Das Herzstück ihres Beweises ist eine Verfeinerungseigenschaft. Beginnen wir mit einem Isomorphismus zweier Produkte $B_1 \otimes B_2$ und $C_1 \otimes C_2$. Die Algebren C_j sind natürlich in $C_1 \otimes C_2$ eingebettet. Aber wir haben auch Projektionen. Weil B_2 eine lokale Algebra ist und der Grundkörper \mathbb{C} algebraisch abgeschlossen, ist der Quotient der Algebra B_2 nach ihrem maximalen Ideal gerade die triviale Algebra \mathbb{C} . Das liefert uns eine Projektion $B_2 \rightarrow \mathbb{C}$, $b_2 \mapsto \overline{b_2}$. Tensorieren wir diese mit B_1 , so bekommen wir eine Projektion $B_1 \otimes B_2 \rightarrow B_1$, die einen zerfallenden Tensor $b_1 \otimes b_2$ auf $\overline{b_2}b_1$ abbildet. Wir haben also:

$$\begin{array}{ccccc}
 & & & & C_2 \\
 & & & \swarrow & \\
 & & & C_1 \otimes C_2 & \\
 & \xleftarrow{\simeq} & & \swarrow & \\
 B_1 \otimes B_2 & & & C_1 & \\
 \swarrow & & & & \\
 B_1 & & & &
 \end{array}$$

In B_1 können wir nun die Bilder B_{1j} von C_j betrachten. Es ist leicht zu sehen, daß diese Algebren B_{11} und B_{12} zusammen B_1 als Algebra erzeugen. (Natürlich erzeugen C_1 und C_2 ihr Tensorprodukt $C_1 \otimes C_2$. Und die Projektion $B_1 \otimes B_2 \rightarrow B_1$ ist surjektiv. Also kann jedes Element von B_1 durch Elemente ausgedrückt werden, die aus C_1 oder C_2 stammen.) Das bedeutet, daß der Morphismus

$$\mu : \begin{array}{ccc} B_{11} \otimes B_{12} & \longrightarrow & B_1, \\ b_{11} \otimes b_{12} & \longmapsto & b_{11} \cdot b_{12}, \end{array}$$

surjektiv ist. Nicht so leicht zu sehen ist, daß er sogar ein Isomorphismus ist. Mit anderen Worten: Die Zerlegung $C_1 \otimes C_2$ induziert eine Zerlegung von B_1 . Entsprechend bekommen wir Zerlegungen der anderen Faktoren B_2 , C_1 und C_2 . Nun sind beide Seiten in je vier Faktoren zerlegt: B_{ij} links und C_{ji} rechts. Zusätzlich kann man $B_{ij} \simeq C_{ji}$ zeigen. Also haben wir tatsächlich eine gemeinsame Verfeinerung der gegebenen Zerlegungen gefunden. Und daraus folgt natürlich eindeutige Faktorisierung.

Diese Methode läßt mehr Spielraum für Verallgemeinerungen. Um den Multiplikationsmorphismus μ zu formulieren, brauchen wir nur, daß B_2 lokal ist. Wir dürfen also hoffen, daß μ sogar dann noch ein Isomorphismus ist, wenn die anderen Algebren nicht mehr lokal sind. Wenn wir nur \oplus -unzerlegbare Algebren betrachten, ist das tatsächlich wahr.

Proposition *Es sei $B_1 \otimes B_2 \leftarrow C_1 \otimes C_2$ ein Isomorphismus \oplus -unzerlegbarer Algebren und B_2 sei lokal. Dann ist der natürliche Morphismus*

$$\mu : B_{11} \otimes B_{12} \longrightarrow B_1$$

ein Isomorphismus.

Diese Erweiterung von HORSTs Satz ist ein Kernstück des Beweises des folgenden Ergebnisses.

Satz *Eine \oplus -unzerlegbare Algebra hat eine eindeutige Faktorisierung.*

Die folgende strukturelle Version zeigt, warum nicht jede Algebra eine eindeutige Faktorisierung hat, obwohl der Grundkörper \mathbb{C} ist. Betrachten wir den Halbring \mathcal{U} aller Isomorphieklassen von Algebren. Die Operationen sind induziert durch direkte Summe und Tensorprodukt.

Korollar *Der Halbring \mathcal{U} ist ein Polynomhalbring $\mathbb{N}[\mathcal{X}]$ über der Menge \mathcal{X} der Isomorphieklassen $\{\oplus, \otimes\}$ -unzerlegbarer Algebren.*

Nach dem Satz von GAUSS ist $\mathbb{Z}[\mathcal{X}]$ ein faktorieller Ring. Aber schon der Polynomhalbring $\mathbb{N}[T]$ in einer Unbestimmten ist nicht mehr faktoriell. (Siehe Example A.1.)

Zum Beweis des Satzes brauchen wir ein Werkzeug, um zu messen, wie nicht-lokal eine Algebra ist. Der Graph $\Delta(A)$ einer Algebra A tut das. Er hat folgende Eigenschaften:

- Zuerst charakterisiert er (fast) lokale Algebren: Die Algebra A ist (fast) lokal genau dann, wenn ihr Graph $\Delta(A)$ trivial ist.
- Und die Algebra A ist \oplus -unzerlegbar genau, wenn der Graph zusammenhängend ist.
- Der Graph verhält sich gut im Zusammenhang mit dem Tensorprodukt:

$$\Delta(A_1 \otimes A_2) \simeq \Delta(A_1) \times \Delta(A_2).$$

Das Produkt auf der rechten Seite ist das kartesische Produkt von Graphen.

- Zu jedem Teilgraphen Λ des Graphen $\Delta(A)$ kann man eine eingeschränkte Algebra $A \downarrow \Lambda$ finden so, daß ihr Graph $\Delta(A \downarrow \Lambda)$ bijektiv auf den Teilgraphen Λ bezogen ist. Unter Mißachtung der Eins kann man die Algebra $A \downarrow \Lambda$ als Unteralgebra von A betrachten.
- Auch die Einschränkung verträgt sich gut mit den Produkten:

$$(A_1 \otimes A_2) \downarrow (\Lambda_1 \times \Lambda_2) \simeq (A_1 \downarrow \Lambda_1) \otimes (A_2 \downarrow \Lambda_2).$$

- Außerdem gilt $A \downarrow \Delta(A) \simeq A$.

Mit Hilfe dieser Eigenschaften skizzieren wir nun den ersten Schritt im Beweis des Satzes. Wir beginnen wieder mit einem Isomorphismus zweier Produkte von, diesmal, \oplus -unzerlegbaren Algebren B_i, C_j mit nicht-trivialen Graphen:

$$B_1 \otimes B_2 \xrightarrow{\simeq} C_1 \otimes C_2$$

Der Einfachheit halber seien die Faktoren zusätzlich \otimes -unzerlegbar. An diesem Punkt können wir nicht ohne weiteres HORSTs Idee verwenden, die Zerlegung $C_1 \otimes C_2$ nach B_1 zu projizieren, da B_2 hier ganz bestimmt nicht lokal ist. Wenn wir aber den Graphen berechnen, bekommen wir einen Isomorphismus von Graphen und wie in IMRICHs Beweis können wir eine Schicht auf der linken Seite betrachten. Die Graphen sind zusammenhängend, also bekommen wir rechts ein Rechteck $\Lambda_1 \times \Lambda_2$.

$$\begin{array}{ccc} \Delta(B_1) \times \Delta(B_2) & \xrightarrow{\simeq} & \Delta(C_1) \times \Delta(C_2) \\ \cup & & \cup \quad \cup \\ \Delta(B_1) \times \{\beta_2\} & \xrightarrow{\simeq} & \Lambda_1 \times \Lambda_2 \end{array}$$

Im Unterschied zu vorher können wir nicht annehmen, daß die Schicht ein \times -unzerlegbarer Graph ist. (Jeder Graph ist nämlich (bis auf Isomorphie) Graph einer \otimes -unzerlegbaren Algebra.) Wir müssen damit auskommen, daß B_1 \otimes -unzerlegbar ist. Aber mit der Einschränkung bekommen wir erneut einen Isomorphismus von Algebren:

$$B_1 \otimes (B_2 \downarrow \beta_2) \xrightarrow{\simeq} (C_1 \downarrow \Lambda_1) \otimes (C_2 \downarrow \Lambda_2).$$

Beachte, daß links die Algebra B_1 selbst erscheint! Für diese Gleichung können wir nun viel eher die Proposition anwenden. Die Algebra $B_2 \downarrow \beta_2$ hat nämlich den trivialen Graphen $\{\beta_2\}$ und ist daher (fast) lokal.

Wenn nun der Satz schon bewiesen wäre, würden wir bekommen, daß einer der Faktoren auf der rechten Seite lokal ist: Zerlege alle auftretenden Algebren in unzerlegbare Faktoren. Links finden wir genau einen nicht-lokalen Faktor, nämlich B_1 . Also muß er auch rechts genau einmal auftauchen. Mit anderen Worten: B_1 ist Faktor einer Algebra $C_j \downarrow \Lambda_j$. Die andere ist dann ein Produkt lokaler Algebren, also lokal. Und der zugehörige Graph wäre ein Punkt.

Nun betrachten wir zuerst den Fall, daß die Algebra $B_2 \downarrow \beta_2$ sogar trivial ist, also der Grundkörper. Dann steht auf der linken Seite die \otimes -unzerlegbare Algebra B_1 und auf der rechten ein Produkt. Also ist einer der Faktoren $C_j \downarrow \Lambda_j$ trivial und der zugehörige Graph Λ_j ein Punkt. (Der Graph des Grundkörpers \mathbb{C} ist ein Punkt, weil \mathbb{C} lokal ist.) Im Fall, daß $B_2 \downarrow \beta_2$ lokal ist, wenden wir die Proposition an: Die Projektion des Produktes rechts liefert eine Zerlegung von B_1 mit, ferner, zugehöriger Graphzerlegung $\Lambda_1 \times \Lambda_2$ (jedenfalls was die Punktmenge betrifft).

Aber B_1 hat keine echte Zerlegung. Daher ist einer der Faktoren trivial und der entsprechende Graph Λ_j ein Punkt. Der allgemeine Fall schließlich kann mit Hilfe der eindeutigen Primfaktorzerlegung natürlicher Zahlen auf den vorigen zurückgeführt werden. In jedem Fall ist ein Λ_j ein Punkt, sagen wir $\Lambda_2 = \{\gamma_2\}$. Wie in IMRICHs Beweis wiederholen wir die Argumentation von der anderen Seite, um $\Lambda_1 = \Delta(C_1)$ zu bekommen. Als Ergebnis sieht der eingeschränkte Algebrenisomorphismus nun so aus:

$$B_1 \otimes (B_2 \downarrow \beta_2) \xrightarrow{\simeq} C_1 \otimes (C_2 \downarrow \gamma_2).$$

Wir haben jetzt auf beiden Seiten ein Produkt aus einer \otimes -unzerlegbaren, nicht-lokalen Algebra und einer lokalen Algebra. Mit Hilfe der Proposition kann man daraus $B_1 \simeq C_1$ schließen.

Die Festlegung auf den Grundkörper \mathbb{C} ist tatsächlich etwas zu drastisch. Alle Beweise funktionieren auch für einen beliebigen Körper der Charakteristik Null, vorausgesetzt wir beschränken uns auf sogenannte SCHURsche Algebren. Nach dem Satz von WEDDERBURN ist der Quotient einer Algebra nach ihrem Radikal eine Summe von Matrixalgebren über Divisionsalgebren. Eine Algebra ist SCHURsch, genau wenn jede dieser Divisionsalgebren trivial ist:

$$A/\text{rad } A \simeq \bigoplus k^{n_i \times n_i}.$$

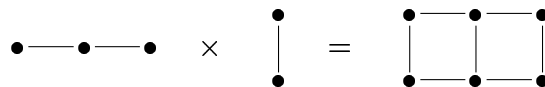
Sogar für Körper endlicher Charakteristik erlaubt noch eine große Klasse von Algebren eindeutige Faktorisierung. Dafür ist es wichtig, HORSTs Methode und die Proposition zu vermeiden, denn diese versagen definitiv in endlicher Charakteristik. (Siehe Example A.5.)

Zum Abschluß soll noch gesagt werden, daß der Satz einige seiner Vorgänger einschließt: Da ein Faktor einer lokalen Algebra wieder lokal ist, erhalten wir HORSTs Satz über eindeutige Faktorisierung lokaler Algebren. Auch SABIDUSSIS Satz über eindeutige Faktorisierung von Graphen kann man wieder zurückbekommen. Wenn man noch bemerkt, daß eine volle Matrixalgebra $\mathbb{C}^{n \times n}$ durch ihren Rang n würdig vertreten wird, auch was Produkte angeht, dann bekommen wir sogar die eindeutige Primfaktorzerlegung natürlicher Zahlen wieder zurück.

0 Introduction

A natural number has a unique prime factor decomposition, as we all know. Also certain algebras can be written *uniquely* as a tensor product of indecomposable factors; this is what we report here.

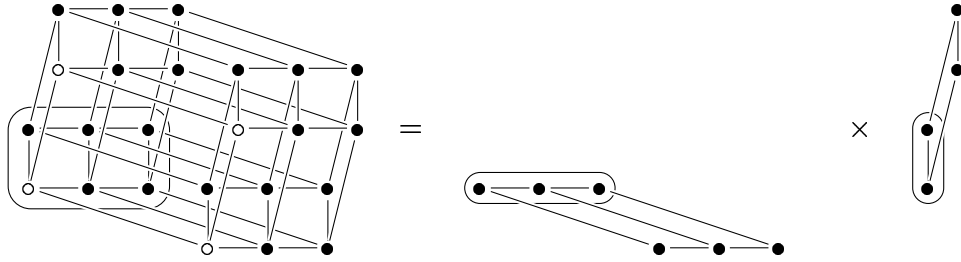
Before we start talking about algebras, I will tell you a similar story about graphs. The graphs considered here are just finite, undirected graphs without loops or multiple edges. In fact, graphs can be multiplied: We use the *cartesian product of graphs* $\Gamma_1 \times \Gamma_2$. The vertex set of the product graph is the cartesian product of the vertex sets of the factors. Two vertices, say (α_1, α_2) and (β_1, β_2) , then are joined by an edge iff either the second coordinates are equal and the first coordinates are joined by an edge in Γ_1 (ie. $\alpha_2 = \beta_2$ and $\{\alpha_1, \beta_1\} \in \Gamma_1$) or, vice versa, the first coordinates are equal and the second coordinates are joined by an edge in Γ_2 . To illustrate this look at the following example:



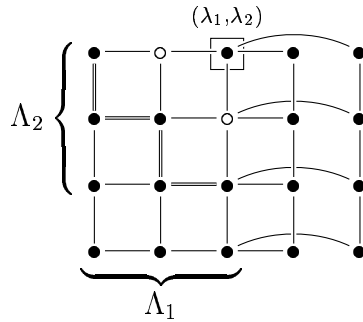
First, we draw the new vertex set. Then, we copy the first factor in each horizontal slice and the second factor in each vertical slice. This is a nice product: It is (up to isomorphism) associative, commutative and it has a neutral element: the trivial graph, a point. With respect to this product, we call a graph \times -*indecomposable* iff it is not trivial and it has (up to isomorphism) no proper decomposition. In other words: In each decomposition of a \times -indecomposable graph into two factors, exactly one of them is trivial. SABIDUSSI (1960) showed that a connected graph indeed has a unique factorization. (By a factorization we always mean a product decomposition into indecomposable factors. Unique factorization means that the factors of such a decomposition are uniquely determined up to order and isomorphism.) IMRICH (1967) provides us with a short proof that can be sketched here. We start with an isomorphism of two products $\Gamma_1 \times \Gamma_2$ and $\Delta_1 \times \Delta_2$ of connected graphs. Now it turns out that a subgraph of the form $\Gamma_1 \times \{\alpha_2\}$ is mapped to a subgraph of the form $\Lambda_1 \times \Lambda_2$.

$$\begin{array}{ccc} \Gamma_1 \times \Gamma_2 & \xrightarrow{\cong} & \Delta_1 \times \Delta_2 \\ \cup & & \cup \quad \cup \\ \Gamma_1 \times \{\alpha_2\} & \xrightarrow{\cong} & \Lambda_1 \times \Lambda_2 \end{array}$$

In other words, a horizontal slice on the left corresponds to a rectangle on the right. Therefore, we first observe a remarkable property of such a slice: A vertex outside the slice always is joined to at most one vertex in the slice. In other words, if a vertex is joined to two vertices of a slice, then it belongs itself to the slice. Of course, this now is true as well for the image of the slice on the right (ie. in $\Delta_1 \times \Delta_2$). We collect all columns that are hit by the image to define Λ_1 and all hit rows to define Λ_2 . In order to prove that the image is indeed a rectangle,



we have to show that any vertex (λ_1, λ_2) in the rectangle $\Lambda_1 \times \Lambda_2$ is in the image of the slice. But we only know that some vertex in $\Lambda_1 \times \{\lambda_2\}$ and some vertex in $\{\lambda_1\} \times \Lambda_2$ belong to the image. Now, since Γ_1 is connected, so is the image of the slice. Hence these two vertices can be connected by a path *in* the image. Consider the following example:



Herein, the small square indicates the vertex (λ_1, λ_2) we start with, the vertices in the row and the column are connected by a double line path. We see two vertices (marked as circles) which are not on the path but which are each joined to two of its vertices. Since the path belongs entirely to the image, we infer, using the above slice property, that these two vertices are as well in the image of the slice. But now the chosen vertex is joined to these two vertices. Hence it is also in the slice. Using induction this idea proves the rectangle statement.

Now, suppose additionally that all the factors Γ_i and Δ_j are \times -indecomposable graphs. We want to show that the two factorizations are in fact the same (up to order and isomorphism of the factors). As a graph the image of the slice is isomorphic to Γ_1 . Thus it is a \times -indecomposable graph. On the other hand, we know that it is a rectangle, a product. Thus one of the factors Λ_j must be trivial. Say, Λ_2 is just the point $\{\beta_2\}$. Thus the slice image is contained in the slice $\Delta_1 \times \{\beta_2\}$. But this forms — mutatis mutandis — a rectangle on the left. Since Δ_1 is also \times -indecomposable, the rectangle is contained in a slice. But it already contains the slice $\Gamma_1 \times \{\alpha_2\}$. This is only possible if equality holds. Thus Λ_1 is equal to Δ_1 . The slice $\Gamma_1 \times \{\alpha_2\}$ on the left is thus isomorphic to the

slice $\Delta_1 \times \{\beta_2\}$ on the right. Hence we have $\Gamma_1 \simeq \Delta_1$. Analogously, we obtain $\Gamma_2 \simeq \Delta_2$. (Note that a Γ_2 -slice cannot be mapped to a Δ_1 -slice anymore, since a horizontal and a vertical slice intersect in exactly one point.) This completes the considered case. The general case of SABIDUSSI's theorem can be proved with the same idea. Well, so much about graphs. Let us turn to algebras.

All algebras considered here are finite-dimensional, associative k -algebras with a unit. For example, think of full matrix algebras $k^{n \times n}$, or of finite-dimensional quotients of polynomial algebras. Not only graphs can be multiplied: For algebras we use the *tensor product of algebras* $A_1 \otimes A_2$. This is the algebra defined as follows: Endow the tensor product of the underlying vector spaces with the product that multiplies split tensors component-wise, $a'_1 \otimes a'_2 \cdot a_1 \otimes a_2 = a'_1 a_1 \otimes a'_2 a_2$. The factors A_i can always be viewed as subalgebras of $A_1 \otimes A_2$, for example in case $i = 1$ via $a \mapsto a \otimes 1$. This product of algebras also behaves well: It is (up to isomorphism) associative, commutative, and it has a neutral element: the trivial algebra, which is just the ground field k . We call an algebra \otimes -*indecomposable* iff it is not trivial and has (up to isomorphism) no proper decomposition. In other words: In each decomposition of a \otimes -indecomposable algebra into two factors exactly one of the factors is trivial. For matrix algebras the tensor product expresses the well known fact that we can multiply $pq \times pq$ -matrices block-wise by interpreting them as $p \times p$ -matrices whose entries are $q \times q$ -matrices:

$$k^{p \times p} \otimes k^{q \times q} \simeq k^{pq \times pq}.$$

Another simple example is as follows: We tensor a finite-dimensional quotient of a real polynomial algebra with the field of complex numbers, both regarded as real algebras. Then we obtain the corresponding quotient of the complex polynomial algebra:

$$\mathbb{R}[T]/\langle f \rangle \otimes \mathbb{C} \simeq \mathbb{C}[T]/\langle f \rangle.$$

This immediately shows that we cannot expect tensor factorization to be unique for arbitrary algebras: Take $f = T^2 + 1$. Then the left hand side is $\mathbb{C} \otimes \mathbb{C}$ while the right-hand side is $\mathbb{C} \oplus \mathbb{C}$ by the Chinese Remainder Theorem. Rewriting again, we obtain isomorphism of two tensor decomposition:

$$\mathbb{C} \otimes \mathbb{C} \simeq (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{C}.$$

All the factors are \otimes -indecomposable, since their vector space dimensions are all 2 (hence prime). While \mathbb{C} is a field, $\mathbb{R} \oplus \mathbb{R}$ is not even a division algebra. Thus the two factorization are essentially different. To circumvent difficulties of this kind, from now on, we suppose that the ground field k is the field \mathbb{C} of complex numbers.

Of course, the tensor product is not only important due its omni presence in algebra, but also because of its strong connection to the direct product of geometric or topological structures. This is very clear in algebraic geometry, when

we pass from an affine algebraic variety to its coordinate ring. Of course, these algebras are usually not finite-dimensional. Another example comes from topology, when passing from topological spaces with finite-dimensional, commutative cohomology algebra with field coefficients to this cohomology algebra. But on the other hand, these algebras typically have additional structure like commutativity or a grading.

Also inspired by a topological question, a first result on unique factorization of algebras was obtained by BODY & DOUGLAS (1979). By a very elegant method they showed unique factorization for graded, local algebras. We indicate the major steps of their proof. First, note that the automorphism group of a finite-dimensional algebra is a linear algebraic group. Now, given a graded algebra $B = \bigoplus_{\ell} B^{(\ell)}$, we always find a one-dimensional subtorus in its automorphism group as follows. For $\lambda \in \mathbb{C}^*$, the assignment $\sum b^{(\ell)} \mapsto \sum \lambda^{\ell} b^{(\ell)}$ defines an automorphism of the algebra B . Considering a tensor product A of s graded algebras B_i , we find s independent one-dimensional subtori spanning an s -dimensional subtorus S of the automorphism group of A . Suppose we have two tensor decompositions of an algebra A , whose factors we consider as subalgebras of A by the natural embeddings,

$$B_1 \otimes \cdots \otimes B_s = C_1 \otimes \cdots \otimes C_t,$$

$S \qquad \qquad \qquad T$

with tori S and T in the automorphism group of A . (Thus, $S \simeq (\mathbb{C}^*)^s$, $T \simeq (\mathbb{C}^*)^t$.) It turns out that the two decompositions coincide iff the tori are equal. If all factors are \otimes -indecomposable, then it is even sufficient that the tori S and T commute! Now, a classical result of BOREL states that *maximal* tori in a linear group are conjugate. So embed S and T in maximal tori. By applying a conjugation we can then assume that these maximal tori are equal. (This means that we replace one of the decompositions by an isomorphic one.) But now both tori, S and T , are contained in this maximal torus. Since this is a commutative group, the tori commute, and the decompositions coincide. This already proves the theorem. While this method is not restricted to finite-dimensional algebras, it is very important that we have a grading. Arbitrary local algebras typically admit no non-trivial torus at all in their automorphism group.

Using a much more elementary method HORST (1987) proved unique factorization for arbitrary local algebras. The heart of her proof is a refinement property. Suppose we have an isomorphism of two products $B_1 \otimes B_2$ and $C_1 \otimes C_2$. The algebras C_j are naturally embedded in $C_1 \otimes C_2$. But we also have projections. Since B_2 is a local algebra and the ground field \mathbb{C} is algebraically closed, the quotient of B_2 by its maximal ideal is the trivial algebra \mathbb{C} . This gives us a projection $B_2 \rightarrow \mathbb{C}$, $b_2 \mapsto \overline{b_2}$. Tensoring this with B_1 yields a projection $B_1 \otimes B_2 \rightarrow B_1$

mapping a split tensor $b_1 \otimes b_2$ to $\overline{b_2}b_1$. Thus we have:

$$\begin{array}{ccc}
 & & C_2 \\
 & & \swarrow \\
 & B_1 \otimes B_2 & \xleftarrow{\simeq} C_1 \otimes C_2 \\
 \swarrow & & \swarrow \\
 B_1 & & C_1
 \end{array}$$

In B_1 we can now consider the images B_{1j} of C_j . It is easy to see that these algebras B_{11} and B_{12} together generate B_1 as an algebra. (Clearly, C_1 and C_2 generate their tensor product $C_1 \otimes C_2$. And the projection $B_1 \otimes B_2 \rightarrow B_1$ is surjective. Thus any element of B_1 can be expressed by elements stemming from C_1 and C_2 .) This means that the morphism

$$\mu : \begin{array}{ccc} B_{11} \otimes B_{12} & \longrightarrow & B_1, \\ b_{11} \otimes b_{12} & \longmapsto & b_{11} \cdot b_{12}, \end{array}$$

is surjective. It is not so easy to see, that it is even an isomorphism. In other words: The decomposition $C_1 \otimes C_2$ induces a decomposition of B_1 . Likewise, we obtain decompositions of the other factors B_2 , C_1 and C_2 . Now, both sides are decomposed into four factors: B_{ij} on the left and C_{ji} on the right. Additionally, it can be shown that $B_{ij} \simeq C_{ji}$. Thus we have indeed found a common refinement of the given decompositions. And this implies of course unique factorization.

This method leaves more space for generalizations. In order to formulate the multiplication morphism μ , we only need that B_2 is local. Thus we can hope that μ is still an isomorphism even if the other algebras are not local. If we only consider \oplus -indecomposable algebras, this is indeed true.

Proposition *Suppose that $B_1 \otimes B_2 \leftarrow C_1 \otimes C_2$ is an isomorphism of \oplus -indecomposable algebras and B_2 is local. Then the natural morphism*

$$\mu : B_{11} \otimes B_{12} \longrightarrow B_1$$

is an isomorphism.

This extension of HORST's theorem is a main item in the proof of the following result.

Theorem *A \oplus -indecomposable algebra has a unique factorization.*

The following structural version shows why even with ground field \mathbb{C} we do not have unique factorization for arbitrary algebras. Consider the semiring \mathcal{U} of all isomorphism classes of algebras. Its operations are induced by direct sum and tensor product of algebras.

Corollary *The semiring \mathcal{U} is a polynomial semiring $\mathbb{N}[\mathcal{X}]$ over the set \mathcal{X} of isomorphism classes of $\{\oplus, \otimes\}$ -indecomposable algebras.*

By GAUSS's Theorem $\mathbb{Z}[\mathcal{X}]$ is a factorial ring. But already the polynomial semiring $\mathbb{N}[T]$ in one variable is not. (See Example A.1.)

For the proof of the theorem, we need a tool to measure the 'non-localness' of an algebra. The graph $\Delta(A)$ of an algebra A does this. It has the following properties:

- First, it (almost) characterizes local algebras: The algebra A is (almost) local iff its graph $\Delta(A)$ is trivial.
- And the algebra A is \oplus -indecomposable iff its graph $\Delta(A)$ is connected.
- The graph behaves well with respect to the tensor product:

$$\Delta(A_1 \otimes A_2) \simeq \Delta(A_1) \times \Delta(A_2).$$

The product on the right is the cartesian product of graphs.

- To each subgraph Λ of the graph $\Delta(A)$, one can associate a restricted algebra $A \downarrow \Lambda$ such that its graph $\Delta(A \downarrow \Lambda)$ is bijectively related to the subgraph Λ . Up to the unit, the algebra $A \downarrow \Lambda$ can be viewed as a subalgebra of A .
- The restriction also behaves well with respect to the products:

$$(A_1 \otimes A_2) \downarrow (\Lambda_1 \times \Lambda_2) \simeq (A_1 \downarrow \Lambda_1) \otimes (A_2 \downarrow \Lambda_2).$$

- Furthermore, $A \downarrow \Delta(A) \simeq A$.

Using these properties, we are going to sketch the first step of the proof of the theorem. Again, we start with an isomorphism of two products of, this time, \oplus -indecomposable algebras B_i, C_j with non-trivial graphs:

$$B_1 \otimes B_2 \xrightarrow{\simeq} C_1 \otimes C_2$$

For simplicity, we additionally suppose that the factors are also \otimes -indecomposable. At this point, we cannot simply use HORST's idea to project the decomposition $C_1 \otimes C_2$ to B_1 , since B_2 is definitely not local here. But, calculating the graph, we get an isomorphism of graphs and, as in IMRICH's proof, we consider a slice on the left. The graphs are connected, hence on the right this slice becomes a rectangle $\Lambda_1 \times \Lambda_2$.

$$\begin{array}{ccc} \Delta(B_1) \times \Delta(B_2) & \xrightarrow{\simeq} & \Delta(C_1) \times \Delta(C_2) \\ \cup & & \cup \quad \cup \\ \Delta(B_1) \times \{\beta_2\} & \xrightarrow{\simeq} & \Lambda_1 \times \Lambda_2 \end{array}$$

Unlike above, we cannot assume here that the slice is a \times -indecomposable graph. (In fact, every graph occurs (up to isomorphism) as the graph of some \otimes -indecomposable algebra.) So we must content ourselves with the fact that B_1 is \otimes -indecomposable. Using the restriction we again obtain an isomorphism of algebras:

$$B_1 \otimes (B_2 \downarrow \beta_2) \xrightarrow{\simeq} (C_1 \downarrow \Lambda_1) \otimes (C_2 \downarrow \Lambda_2).$$

Note that on the left the algebra B_1 itself occurs! This equation now is much more likely to be plugged into the Proposition. Observe that the algebra $B_2 \downarrow \beta_2$ has the trivial graph $\{\beta_2\}$ and is thus (almost) local.

If the Theorem was already proved, we would find that one of the factors on the right is local: Decompose all algebras appearing in the isomorphism into \otimes -indecomposable factors. On the left, there is exactly one non-local (prime) factor, namely B_1 . Thus it must appear exactly once on the right. In other words, B_1 is a factor of one $C_j \downarrow \Lambda_j$. The other then is a product of local algebras and thus local. And the corresponding graph would be a point.

Now, let us first consider the case that the algebra $B_2 \downarrow \beta_2$ is even trivial, ie. it coincides with the ground field. Then on the left we have the \otimes -indecomposable algebra B_1 and on the right a product. Hence, one of the factors $C_j \downarrow \Lambda_j$ is trivial and the corresponding graph Λ_j is indeed a point. (The graph of the ground field \mathbb{C} is a point, since \mathbb{C} is local.) In case $B_2 \downarrow \beta_2$ is local, we apply the Proposition: The projection of the product on the right yields a decomposition of B_1 with, moreover, corresponding graph decomposition $\Lambda_1 \times \Lambda_2$ (concerning the vertices only). But B_1 does not have a proper decomposition. Thus one of the factors is trivial and, again, the corresponding graph Λ_j is a point. Finally, the general case is reduced to the latter with the help of the unique prime factorization of natural numbers. In any case, one Λ_j is a point, say, $\Lambda_2 = \{\gamma_2\}$. As in IMRICH's proof, we repeat the argument from the other side to see that $\Lambda_1 = \Delta(C_1)$. As a result, the restricted algebra isomorphism now looks as follows:

$$B_1 \otimes (B_2 \downarrow \beta_2) \xrightarrow{\simeq} C_1 \otimes (C_2 \downarrow \gamma_2).$$

On both sides, we have a product of a \otimes -indecomposable, non-local algebra and a local algebra. The Proposition then allows to conclude $B_1 \simeq C_1$.

The restriction to the ground field \mathbb{C} is indeed a bit too drastic. All proofs still work for an arbitrary field of characteristic zero, provided we restrict ourselves to so-called SCHURian algebras. Due to WEDDERBURN's Theorem the quotient of an algebra by its radical is a sum of matrix algebras over division algebras. An algebra is SCHURian iff each of these division algebras is trivial:

$$A/\text{rad } A \simeq \bigoplus k^{n_i \times n_i}.$$

For a field of finite characteristic, there is still a large class of algebras allowing unique factorization. It is then important to avoid the use of HORST's method and the Proposition, since these definitely fail in finite characteristic. (See Example A.5.)

Finally, we remark that the theorem contains several of its predecessors: Noticing that a factor of a local algebra is again local, we can deduce HORST's unique factorization of local algebras. Also, SABIDUSSI's unique factorization of connected graphs can be obtained. Moreover, if we observe that the rank n of a full matrix algebra $\mathbb{C}^{n \times n}$ is a worthy representative, also with respect to products, then we even get back the unique prime factorization of natural numbers.

Contents of the sections

Preliminaries The appropriate class of algebras is described. We recall the definitions of sum and product of algebras and introduce the universal semiring. We summarize major properties of the JACOBSON radical, define the associated graded algebra of an algebra and prove Surjectivity Criterion 1.2. A translation into a functorial language will later enable us to argue locally.

Graphs The definition of the cartesian product of graphs and its most important properties are given, as well as the precise description of the graph of an algebra. The connectedness criterion Corollary 2.3(i) implies that the set \mathcal{M} of \oplus -indecomposable algebras is a monoid with respect to the operation induced by the (tensor) product. Furthermore, the multiplicity of an algebra and a transformation for algebras are discussed. A special case of this transformation yields MORITA equivalent algebras, but we also need the inner restriction. Graphs and multiplicities allow us to characterize local algebras as those whose graph is a point and whose multiplicity is 1.

Uniqueness for Local Algebras We reprove some results of HORST (1987) concerning unique factorization of finite-dimensional, local algebras, thereby completing her proof of what we call Separation Lemma 3.1.

Loops As we have tried to illustrate above, local factors cause most harm in our proof. The key result in this section, Proposition 4.1, allows us to prove Corollary 4.2, Corollary 4.3, and Corollary 4.4 which are mainly concerned with the separation of local and non-local factors. Only these three results alone are needed in the sequel.

Uniqueness Here everything is merged in to prove the Unique Factorization Theorem 5.4.

Corollaries Some known results are derived from the Unique Factorization Theorem 5.4. The main idea is to exhibit other monoids as ‘good’ submonoids of the monoid of \oplus -indecomposable algebras.

Frontiers Some limiting examples are listed here.

A Generalized Separation Lemma This section is devoted to not necessarily finite-dimensional algebras. A slightly different formulation of Separation Lemma 3.1 is presented.

A Reduction This is V. STRASSEN’s reduction of the Unique Factorization Theorem to the basic case.

1 Preliminaries

All algebras considered in the main text are finite-dimensional, associative algebras with unit element 1 over a field k of characteristic zero; morphisms of algebras map 1 to 1. All modules encountered are finite-dimensional.

As usual, the JACOBSON *radical* $\text{rad} A$ of an algebra A is defined as the intersection of all maximal left ideals. It is the largest nilpotent (left) ideal and the smallest ideal with semi-simple residue algebra. Hence it is the unique two-sided ideal that is nilpotent with semi-simple residue algebra.

Schurian Algebras and a Semiring

If k is algebraically closed, no further assumptions on algebras are necessary. For a general field k we restrict ourselves to SCHURian algebras. An algebra A is called SCHURian iff, for some (n_i) ,

$$A/\text{rad} A \simeq \bigoplus_i k^{n_i \times n_i}.$$

By WEDDERBURN's Theorem the *residue algebra* $A^{(0)} := A/\text{rad} A$ always is sum of matrix algebras over finite-dimensional division algebras. We require that these division algebras coincide with the ground field. (The major *raison d'être* of this assumption is Graph Lemma 2.2(iii). See Frontier A.4.) Over an algebraically closed field k , there is no finite-dimensional division algebra apart from k itself, hence any algebra over an algebraically closed field is SCHURian.

SCHURian algebras can also be characterized by their modules: In general by FITTING's Lemma, the endomorphism algebra of an indecomposable projective P is local, hence its residue algebra $\text{End}_A^{(0)}(P)$ is a division algebra. Now, an algebra is SCHURian iff for all its indecomposable projectives we have $\text{End}_A^{(0)}(P) = k$. (This follows from Theorem 3.5.2 with $M = A$ and Proposition 3.3.11 in DROZD & KIRICHENKO (1994).) Moreover, we have the following lemma.

1.1 Lemma *Let A be a SCHURian algebra and P a projective A -module. Then P is indecomposable iff $\text{End}_A^{(0)}(P) = k$. \square*

For an algebraically closed field k this generalizes to all finite-dimensional modules. For other fields, however, it needs not.

Furthermore, the lemma implies that a local, SCHURian algebra A has the residue algebra $A^{(0)} = k$, since A is the opposite endomorphism algebra of its only indecomposable projective.

By the *sum* $A_1 \oplus A_2$ of two algebras we mean their direct product; by the *product* $A_1 \otimes A_2$ their tensor product, ie. the tensor product of the underlying vector spaces (over k) with the multiplication induced by $(a'_1 \otimes a'_2) \cdot (a_1 \otimes a_2) =$

$a'_1 a_1 \otimes a'_2 a_2$.¹ With these two operations, the isomorphism classes of SCHURian algebras over a fixed field k form a commutative semiring \mathcal{U} . It is easy to extend \mathcal{U} to a ring. Our main objective is to prove that this ring is a polynomial ring $\mathbb{Z}[\mathcal{X}]$ with integer coefficients and that the semiring \mathcal{U} is the positive cone $\mathbb{N}[\mathcal{X}]$ therein.

An algebra A is \oplus -*indecomposable* iff in each direct decomposition $A = A_1 \oplus A_2$ exactly one of the summands is trivial, ie. $A_i = 0$. (Especially the trivial algebra 0 is not considered to be \oplus -indecomposable.) Similarly an algebra is \otimes -*indecomposable* iff in each tensor product decomposition $A = A_1 \otimes A_2$ exactly one of the factors is trivial, ie. $A_i = k$.

Radicals

The radical of a tensor product of SCHURian algebras is easily determined; we have

$$\text{rad}(A_1 \otimes A_2) = (\text{rad } A_1) \otimes A_2 + A_1 \otimes (\text{rad } A_2).$$

The right-hand side of this equation is a nilpotent ideal in $A_1 \otimes A_2$. The corresponding quotient algebra $A_1/\text{rad } A_1 \otimes A_2/\text{rad } A_2$ is semi-simple due to the definition of SCHURian. Together we infer that it is the radical of $A_1 \otimes A_2$.

The *associated graded algebra* $A^{(*)} = \bigoplus A^{(\ell)}$ is built from the spaces $A^{(\ell)} = \text{rad}^\ell A / \text{rad}^{\ell+1} A$. This is functorial for morphisms that map the radical into the radical and commutes with taking tensor products,

$$(A_1 \otimes A_2)^{(*)} = A_1^{(*)} \otimes A_2^{(*)}$$

as graded algebras via $(a_1 \otimes a_2)^{(\ell_1 + \ell_2)} \leftarrow a_1^{(\ell_1)} \otimes a_2^{(\ell_2)}$. In particular, $(A_1 \otimes A_2)^{(0)} = A_1^{(0)} \otimes A_2^{(0)}$ and $(A_1 \otimes A_2)^{(1)} = (A_1^{(1)} \otimes A_2^{(0)}) \oplus (A_1^{(0)} \otimes A_2^{(1)})$.

A surjective morphism $\varphi : A \rightarrow A'$ maps the radical $\text{rad } A$ onto the radical $\text{rad } A'$. This extends to the following surjectivity criterion.

1.2 Surjectivity Criterion *Let $\varphi : A \rightarrow A'$ be a morphism. Then the following are equivalent:*

- (1) φ respects radicals, ie. $\varphi(\text{rad } A) \subset \text{rad } A'$, and $\varphi^{(0)}$ and $\varphi^{(1)}$ are surjective.
- (2) φ is surjective.

¹In categorial language this is a ‘commutative’ coproduct. In fact, the following universal property characterizes the product: Take a pair of *commuting* algebra morphisms $\varphi_i : A_i \rightarrow C$, ie. $\varphi_1(a_1)\varphi_2(a_2) = \varphi_2(a_2)\varphi_1(a_1)$ for all $a_i \in A_i$. Then there is a unique morphism $\varphi : A_1 \otimes A_2 \rightarrow C$ satisfying $\varphi \iota_i = \varphi_i$. Here $\iota_i : A_i \rightarrow A_1 \otimes A_2$ denotes the canonical injections, for instance, $\iota_1(a_1) = a_1 \otimes 1$.

Proof: Since the radical $\text{rad } A'$ is nilpotent, it suffices to show that for $\ell \in \mathbb{N}$ the morphism

$$A \xrightarrow{\varphi} A' \rightarrow A'/\text{rad}^\ell A'$$

is surjective. For $\ell = 1$ this is true, since $\varphi^{(0)}$ is surjective. Suppose $\ell > 1$ and show that for $a' \in A'$ there is an $a \in A$ such that $a' - \varphi(a) \in \text{rad}^\ell A'$. By induction hypothesis we can assume $a' \in \text{rad}^{\ell-1} A'$. Without loss of generality $a' = r'b'$ with $r' \in \text{rad } A'$, $b' \in \text{rad}^{\ell-2} A'$. Using surjectivity of $\varphi^{(1)}$ and induction hypothesis we find $r, b \in A$ with $r' - \varphi(r) \in \text{rad}^2 A'$ and $b' - \varphi(b) \in \text{rad}^{\ell-1} A'$. Let $a := rb$. But now $a' - \varphi(a) \in \text{rad}^\ell A'$ as desired. \square

In this context it should be mentioned that not only surjective maps respect radicals but also the injections of a factor into a tensor product.

The *radical* $\text{Rad}_A(V, V')$ is, for V and V' indecomposable, the set of all non-isomorphisms $V \rightarrow V'$. This definition can be linearly extended to arbitrary modules; then $\text{Rad}_A(V, V')$ consists of all those module morphisms φ such that for every inclusion ι of an indecomposable direct summand to V and every projection π to an indecomposable direct factor of V' the composition $\pi\varphi\iota$ is no isomorphism. For an indecomposable module V we have $\text{Rad}_A(V, V) = \text{rad } \text{End}_A(V)$, since the elements of the left-hand side are the non-isomorphisms and the elements of the right-hand side are the non-units ($\text{End}_A(V)$ is local by FITTING'S Lemma).

The *radical powers*, we need, are only defined for projectives. They are given by the recursion: $\text{Rad}_A^0(P, P') = \text{Hom}_A(P, P')$, $\text{Rad}_A^1(P, P') = \text{Rad}_A(P, P')$, and for $\ell \geq 2$

$$\begin{aligned} \text{Rad}_A^\ell(P, P') &= \sum_R \text{Rad}_A^{\ell-1}(R, P') \circ \text{Rad}_A(P, R) \\ &= \sum_R \{g \circ h \mid g \in \text{Rad}_A^{\ell-1}(R, P'), h \in \text{Rad}_A(P, R)\}, \end{aligned}$$

where sums range over the indecomposable projectives. Then for $\ell > 0$ and P, P' indecomposable, $\text{Rad}_A^\ell(P, P')$ is the set of module morphisms $P \rightarrow P'$ generated linearly by compositions of at least ℓ non-isomorphisms between indecomposable projectives.

We define the *associated graded homomorphism functor* $\text{Hom}_A^{(*)}$ only for projectives P', P . Let $\text{Hom}_A^{(*)}(P, P') := \bigoplus_\ell \text{Hom}_A^{(\ell)}(P, P')$, where $\text{Hom}_A^{(\ell)}(P, P') = \text{Rad}_A^\ell(P, P')/\text{Rad}_A^{\ell+1}(P, P')$. In fact, the associated graded homomorphism functor is a refinement of the notion of the associated graded algebra: For idempotents $e', e \in A$, we have $\text{Hom}_A^{(*)}(Ae, Ae') \simeq e'(A^{\text{op}})^{(*)}e$.

1.3 Lemma *Let P'_i, P_i be projective A_i -modules. Then:*

- (i) $\text{Hom}_{A_1 \otimes A_2}(P_1 \otimes P_2, P'_1 \otimes P'_2) = \text{Hom}_{A_1}(P_1, P'_1) \otimes \text{Hom}_{A_2}(P_2, P'_2)$.
- (ii) If P_1, P_2 are indecomposable, so is $P_1 \otimes P_2$.
- (iii) $\text{Hom}_{A_1 \otimes A_2}^{(*)}(P_1 \otimes P_2, P'_1 \otimes P'_2) = \text{Hom}_{A_1}^{(*)}(P_1, P'_1) \otimes \text{Hom}_{A_2}^{(*)}(P_2, P'_2)$.

(iv) If P'_i, P_i are indecomposable, then $P_1 \otimes P_2 \simeq P'_1 \otimes P'_2$ as $A_1 \otimes A_2$ -modules implies $P_i \simeq P'_i$ as A_i -modules.

$$(v) \operatorname{Hom}_{A_1 \otimes A_2}^{(1)}(P_1 \otimes P_2, P'_1 \otimes P'_2) = \operatorname{Hom}_{A_1}^{(1)}(P_1, P'_1) \otimes \operatorname{Hom}_{A_2}^{(0)}(P_2, P'_2) \\ \oplus \operatorname{Hom}_{A_1}^{(0)}(P_1, P'_1) \otimes \operatorname{Hom}_{A_2}^{(1)}(P_2, P'_2).$$

Proof:

(i) (V. STRASSEN) We show “ \subset ”. Suppose $\varphi = \sum \varphi_{1\nu} \otimes \varphi_{2\nu} \in \operatorname{Hom}_{A_1 \otimes A_2}(P_1 \otimes P_2, P'_1 \otimes P'_2)$ with linear maps $\varphi_{i\nu}$ and the sum as short as possible. We are going to show that the $\varphi_{i\nu}$ are actually morphisms of A_i -modules. Take $a_2 \in A_2$. Since φ commutes with $1 \otimes a_2$, we have $\sum \varphi_{1\nu} \otimes (a_2 \varphi_{2\nu}(v_2) - \varphi_{2\nu}(a_2 v_2)) = 0$ for all $v_2 \in P_2$. Since the $\varphi_{1\nu}$ are linearly independent by the minimality of the sum, $a_2 \varphi_{2\nu}(v_2) - \varphi_{2\nu}(a_2 v_2) = 0$ for all a_2 and v_2 . Hence, the $\varphi_{2\nu}$ are A_2 -morphisms. The assertion now follows by symmetry.

(ii) By (i) $\operatorname{End}_{A_1 \otimes A_2}(P_1 \otimes P_2) = \operatorname{End}_{A_1}(P_1) \otimes \operatorname{End}_{A_2}(P_2)$. Combine this equality, Lemma 1.1 and $k \otimes k = k$.

(iii) For $\varphi_i \in \operatorname{Rad}_{A_i}^{\ell_i}(P_i, P'_i)$ we wish to map $(\varphi_1 \otimes \varphi_2)^{(\ell_1 + \ell_2)} \leftarrow \varphi_1^{(\ell_1)} \otimes \varphi_2^{(\ell_2)}$. This induces a well defined isomorphism, since for $\ell \in \mathbb{N}$ we have

$$\operatorname{Rad}_{A_1 \otimes A_2}^{\ell}(P_1 \otimes P_2, P'_1 \otimes P'_2) = \sum_{\ell_1 + \ell_2 = \ell} \operatorname{Rad}_{A_1}^{\ell_1}(P_1, P'_1) \otimes \operatorname{Rad}_{A_2}^{\ell_2}(P_2, P'_2).$$

Indeed, that follows from the cases $\ell = 0$ and $\ell = 1$. The case $\ell = 0$ is (i). So suppose $\ell = 1$. By linearity, we only need to deal with indecomposable P_i, P'_i . Then by (ii) $P'_1 \otimes P'_2, P_1 \otimes P_2$ are indecomposable, too. We can further assume $P_i = P'_i$ for both i : if $P_i \not\simeq P'_i$ for an i , there are no isomorphisms. This means $\operatorname{Rad}(P_i, P'_i) = \operatorname{Hom}(P_i, P'_i)$, and so we are back in the case $\ell = 0$. The case $P_i \simeq P'_i$ can be reduced to $P_i = P'_i$. Thus we are in the following situation: P_1, P_2 are indecomposable projectives and we have to show

$$\operatorname{Rad}_{A_1 \otimes A_2}(P_1 \otimes P_2, P_1 \otimes P_2) = \operatorname{Hom}_{A_1}(P_1, P_1) \otimes \operatorname{Rad}_{A_2}(P_2, P_2) \\ + \operatorname{Rad}_{A_1}(P_1, P_1) \otimes \operatorname{Hom}_{A_2}(P_2, P_2)$$

For the inclusion “ \supset ” simply note that $\varphi_1 \otimes \varphi_2$ is no isomorphism if one of the φ_i is no isomorphism. To prove “ \subset ”, take $\varphi = \sum_{\nu} \psi_{1\nu} \otimes \psi_{2\nu} \in \operatorname{Rad}_{A_1 \otimes A_2}(P_1 \otimes P_2, P_1 \otimes P_2)$ with $\psi_{i\nu} \in \operatorname{Hom}_{A_i}(P_i, P_i)$. Following Lemma 1.1, we can write $\psi_{i\nu} = z_{i\nu} \mathbb{1}_{P_i} + \varphi_{i\nu}$ with $z_{i\nu} \in k$ so that $\varphi_{i\nu}$ is no isomorphism. Expansion yields $\varphi = (\sum z_{1\nu} z_{2\nu}) \mathbb{1}_{P_1 \otimes P_2} + \psi$ with ψ in the right-hand side of the asserted equation. Especially, ψ is no isomorphism due to the proven inclusion. Since $P_1 \otimes P_2$ is indecomposable and φ is no isomorphism, $\sum z_{1\nu} z_{2\nu}$ vanishes.

(iv) For indecomposable P'_i, P_i we have $P_i \simeq P'_i$ iff $\operatorname{Hom}_{A_i}^{(0)}(P_i, P'_i) \neq 0$. For completion of the proof apply (iii) in degree $\ell = 0$.

(v) This is (iii) in degree $\ell = 1$. □

2 Graphs

In this paper, a *graph* Γ is always a finite, undirected graph without multiple edges or loops. It is given by a finite vertex set $V(\Gamma)$ and a set of edges $E(\Gamma)$ consisting of unordered pairs of vertices. In abuse of notation, we usually write $\gamma \in \Gamma$ or $\Lambda \subset \Gamma$ instead of $\gamma \in V(\Gamma)$ or $\Lambda \subset V(\Gamma)$, respectively. An *induced subgraph* Λ of Γ consists of a subset of the vertex sets and all edges whose vertices are in Λ . Putting two graphs Γ_ν ‘side by side’ yields the *disjoint union* $\Gamma_1 \uplus \Gamma_2$.

Morphisms of graphs are simply maps between the vertex sets that respect edges. In other words: a morphism $\delta : \Gamma \rightarrow \Gamma'$ of graphs is given by a map $\delta : V(\Gamma) \rightarrow V(\Gamma')$ such that for every edge $e \in E(\Gamma)$ its image $\delta(e)$ is an edge in Γ' . Since all graph morphisms we encounter are injective, we do not need to worry about the case that the vertices of an edge are identified by δ . An isomorphism of graphs is then a bijective map between the vertex sets which respects edges and non-edges. Given one-point graphs Γ', Γ , there is a unique (iso)morphism $\Gamma \rightarrow \Gamma'$; in the sequel we denote it by ε (possibly decorated with an index).

A Graph Product

The vertex set of the *cartesian product* $\Gamma_1 \times \Gamma_2$ of graphs Γ_i is the cartesian product of the vertex sets of the Γ_i . Two vertices (γ_1, γ_2) and (γ'_1, γ'_2) are joined by an edge iff $\{\gamma_1, \gamma'_1\} \in E(\Gamma_1)$ and $\gamma_2 = \gamma'_2$ or $\gamma_1 = \gamma'_1$ and $\{\gamma_2, \gamma'_2\} \in E(\Gamma_2)$. In other words, to obtain the edge set, use a copy of the edge set of Γ_1 for each vertex of Γ_2 and vice versa. For example, the product of the *one edge graph* $\Theta = (\{1, 2\}, \{\{1, 2\}\})$ with itself is a *square*: $\Theta \times \Theta$ has four vertices and four edges.

$$\bullet \text{---} \bullet \quad \times \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad = \quad \begin{array}{cc} \bullet & \text{---} & \bullet \\ | & & | \\ \bullet & \text{---} & \bullet \end{array}$$

This product is associative and commutative (up to isomorphisms). A Γ_1 -*slice* in a product $\Gamma_1 \times \Gamma_2$ is an induced subgraph $\Gamma_1 \times \gamma_2$ with vertex set $V(\Gamma_1) \times \{\gamma_2\}$ for some $\gamma_2 \in \Gamma_2$. By definition of the product, such a Γ_1 -slice is isomorphic to Γ_1 . Similarly, we define Γ_2 -slices. In a product of several graphs, a slice is an induced subgraph on all points with some prescribed coordinates. A $\Gamma_1 \times \Gamma_2$ -*rectangle* is an induced subgraph $\Lambda_1 \times \Lambda_2$ of $\Gamma_1 \times \Gamma_2$, ie. an induced subgraph whose vertex set is a product of subsets of the vertex sets of the Γ_i .

A graph Γ is *connected* iff every pair of vertices $\gamma', \gamma \in \Gamma$ can be connected by a path, ie. there is a sequence $(\gamma_1, \dots, \gamma_n)$ of vertices starting with γ , ending with γ' , and such that γ_ν and $\gamma_{\nu+1}$ are joined by an edge for all ν . Note that $\Gamma_1 \times \Gamma_2$ is connected iff both Γ_1 and Γ_2 are connected.

2.1 Cartesian Product Lemma (IMRICH (1967)) *Let $\delta : \Gamma_1 \times \Gamma_2 \rightarrow \Delta_1 \times \Delta_2$ be an isomorphism of connected graphs. Then:*

- (i) *Every Γ_1 -slice is mapped onto a $\Delta_1 \times \Delta_2$ -rectangle.*

- (ii) If every Γ_1 -slice is mapped into a Δ_1 -slice or into a Δ_2 -slice then either all Γ_1 -slices are mapped into Δ_1 -slices or all Γ_1 -slices are mapped into Δ_2 -slices.
- (iii) If every Γ_1 -slice is mapped onto a Δ_1 -slice then $\delta = \delta_1 \times \delta_2$, where $\delta_j : \Gamma_j \rightarrow \Delta_j$ are graph isomorphisms.

Proof:

(i) This is based on the fact that a vertex in one Γ_1 -slice is joined to at most one vertex in another Γ_1 -slice. In other words, if a vertex is joined to two vertices in a Γ_1 -slice then it belongs to that slice, too. For example consider an edge in each Δ_i , regarded as one edge subgraphs. Their product is a square in $\Delta_1 \times \Delta_2$. If three corners of this square belong to the δ -image of the Γ_1 -slice $\Gamma_1 \times \gamma_2$, then the fourth corner does so, too. Induction on the length of paths joining vertices in $\delta(\Gamma_1 \times \gamma_2)$ yields the assertion. Indeed, take a path in the ‘slice’ $\delta(\Gamma_1 \times \gamma_2)$ of minimal length connecting two given vertices. We show that this slice contains the entire rectangle spanned by the vertices of the path, ie. it contains all vertices whose first coordinate is a first coordinate of a path vertex and the second analogously, but possibly for another path vertex. By induction hypothesis, the rectangle spanned by all but the final vertex and the rectangle spanned by all but the first vertex are in the slice. If the first and the last edge are parallel (ie. they are both in Γ_1 -slices or both in Γ_2 -slices), we are done. Otherwise the remaining corner is joined to two vertices in the slice, and is thus contained therein.

(ii) Suppose this is wrong. Then there are adjacent vertices $\gamma_{2j} \in \Gamma_2$ such that $\Gamma_1 \times \gamma_{2j}$ is mapped into a Δ_j -slice. Take two adjacent vertices $\gamma_{1j} \in \Gamma_1$. Then for some j the edge between $\delta(\gamma_{11}, \gamma_{21})$ and $\delta(\gamma_{11}, \gamma_{22})$ is in the Δ_j -slice that contains $\delta(\Gamma_1 \times \gamma_{2j})$. Thus three of the four vertices $(\gamma_{1j}, \gamma_{2j'})$ of the square are in this slice and so should be the fourth. It is not, however.

(iii) Two adjacent Γ_1 -slices are bijectively related by the joining edges. The set of Γ_1 -slices is bijectively related to the set of Δ_1 -slices and each Γ_1 -slice is mapped bijectively to the corresponding Δ_1 -slice. Putting this together with the fact that Γ_2 and Δ_2 are connected finishes the proof. \square

This lemma is strong enough to prove a unique factorization theorem for connected graphs. IMRICH (1967) does all this for set systems whose edges are non-empty, non-singleton subsets of the vertex set.

Algebras and Graphs

The vertices of the *graph* $\Delta(A)$ of an algebra A are the isomorphism classes of its indecomposable projective modules. A complete list of these vertices can be obtained from any maximal direct A -module decomposition $A = \bigoplus P_\nu$. Two different vertices $[P']$ and $[P]$ are joined by an edge iff $\text{Hom}_A^{(1)}(P', P) \neq 0$ or

$\text{Hom}_A^{(1)}(P, P') \neq 0$.² In fact, this graph is the GABRIEL quiver $Q(A)$ of the algebra without directions, multiplicities, and loops. The only algebra with an empty graph is the trivial algebra 0 . The graph $\Delta(A)$ of a local algebra A is a point. An isomorphism $\varphi : A \rightarrow A'$ induces an isomorphism $\Delta(\varphi) : \Delta(A) \rightarrow \Delta(A')$ by $\Delta(\varphi)([P]) = [\varphi P] = [P^{(\varphi^{-1})}]$ for indecomposable projectives $P \subset A$. Here $P^{(\varphi^{-1})}$ denotes the scalar restriction of P through φ^{-1} , it is isomorphic to φP via φ . This construction is functorial for isomorphisms.

2.2 Graph Lemma

- (i) $\Delta(A_1 \oplus A_2) = \Delta(A_1) \uplus \Delta(A_2)$.
- (ii) $\Delta(\varphi_1 \oplus \varphi_2) = \Delta(\varphi_1) \uplus \Delta(\varphi_2)$.
- (iii) $\Delta(A_1 \otimes A_2) = \Delta(A_1) \times \Delta(A_2)$.
- (iv) $\Delta(\varphi_1 \otimes \varphi_2) = \Delta(\varphi_1) \times \Delta(\varphi_2)$.

Proof:

(i) For an indecomposable projective A_i -module P denote by P^{π_i} the scalar restriction through the canonical projection $\pi_i : A_1 \oplus A_2 \rightarrow A_i$. Consider the map

$$\begin{array}{ccc} \Delta(A_1) \uplus \Delta(A_2) & \longrightarrow & \Delta(A_1 \oplus A_2), \\ [P] & \longmapsto & [P^{\pi_i}] \end{array}$$

for $[P] \in \Delta(A_i)$. Decomposing $A_i = \bigoplus P_{i\nu}$ shows $A_1 \oplus A_2 = \bigoplus P_{1\nu}^{\pi_1} \oplus \bigoplus P_{2\lambda}^{\pi_2}$ and the map's surjectivity. Furthermore, $\text{Hom}_A(P_{i\nu}^{\pi_i}, P_{i\nu'}^{\pi_i}) = \text{Hom}_{A_i}(P_{i\nu}, P_{i\nu'})$ and $\text{Hom}_A(P_{i\nu}^{\pi_i}, P_{j\lambda}^{\pi_j}) = 0$ for $i \neq j$. Thus the map is injective and a graph isomorphism.

(ii) Just check $(\varphi_1 \oplus \varphi_2)(P_1 \oplus P_2) = \varphi_1 P_1 \oplus \varphi_2 P_2$.

(iii) Consider the map

$$\begin{array}{ccc} \Delta(A_1) \times \Delta(A_2) & \longrightarrow & \Delta(A_1 \otimes A_2), \\ ([P_1], [P_2]) & \longmapsto & [P_1 \otimes P_2]. \end{array}$$

By Lemma 1.3(ii) it is well defined. Choose decompositions $A_i = \bigoplus n_{i\nu} P_{i\nu}$ with pairwise non-isomorphic, indecomposable (projective) A_i -modules $P_{i\nu}$. Then $A_1 \otimes A_2 = \bigoplus n_{1\nu} n_{2\lambda} P_{1\nu} \otimes P_{2\lambda}$ is a decomposition into indecomposable (projective) $A_1 \otimes A_2$ -modules. By Lemma 1.3(iv) the modules $P_{1\nu} \otimes P_{2\lambda}$ are pairwise non-isomorphic. Hence the map is bijective. And it is a graph isomorphism due to Lemma 1.3(v).

(iv) Check $(\varphi_1 \otimes \varphi_2)(P_1 \otimes P_2) = \varphi_1 P_1 \otimes \varphi_2 P_2$. □

We now obtain the following Connectedness Criterion and the fact that the class of \oplus -indecomposable algebras is closed with respect to the tensor product.

²Given a maximal 1-decomposition $1 = \sum e \in A$ each vertex α of $\Delta(A)$ can be represented by one of the idempotents e as $\alpha = [Ae]$. Two different vertices $[Ae]$ and $[Ae']$ are joined by an edge iff $e'A^{(1)}e \neq 0$ or $eA^{(1)}e' \neq 0$.

2.3 Corollary

- (i) An algebra A is \oplus -indecomposable iff its graph $\Delta(A)$ is connected.
- (ii) The tensor product of two \oplus -indecomposable algebras is again \oplus -indecomposable.

Proof:

(i) By Graph Lemma 2.2(i), we only need to show that A is not \oplus -indecomposable if $\Delta(A)$ is disconnected. To this end, we assume $\Delta(A) = \Delta_1 \uplus \Delta_2$ with non-empty Δ_i . Write $A = A_1 \oplus A_2$ as A -modules with $A_i \subset \sum \{P \subset A \mid [P] \in \Delta_i\}$. Then $A \simeq \text{End}_A^{\text{op}} A = \bigoplus \text{Hom}_A(A_i, A_j)$, thus it suffices to show $\text{Hom}_A(A_i, A_j) = 0$ for $i \neq j$. There are no isomorphisms between direct summands of A_1 and A_2 , hence $\text{Hom}_A(A_i, A_j) = \text{Rad}_A(A_i, A_j)$. The fact that there are no edges between Δ_i and Δ_j implies $\text{Rad}_A(A_i, A_j) = \text{Rad}_A^2(A_i, A_j)$. Now $\text{Rad}_A(A_i, A_j)$ is a module over $A' := \text{End}_A A_i \otimes \text{End}_A^{\text{op}} A_j$ with radical $\text{Rad}_{A'}^2(A_i, A_j)$. Combining this with NAKAYAMA's Lemma yields $\text{Rad}_A(A_i, A_j) = 0$ as desired.

(ii) Combine (i) and Graph Lemma 2.2(iii) with the fact that the cartesian product of two connected graphs is again connected. \square

Transforming Algebras

Suppose $\Delta(A) = \{[P_\nu] \mid \nu\}$ with pairwise non-isomorphic indecomposable projectives P_ν . For a projective A -module $V = \bigoplus m_\nu P_\nu$, its *multiplicity* $\text{mult}(V) : \Delta(A) \rightarrow \mathbb{N}$, $[P_\nu] \mapsto m_\nu$, is well defined by the KRULL-SCHMIDT Theorem. Given an isomorphism $\varphi : A \rightarrow A'$, we obtain $\text{mult}(P) = \text{mult}(\varphi P) \circ \Delta(\varphi)$ for projectives $P \subset nA$. In addition, we have a product for multiplicities: for multiplicities $m_i : \Delta_i \rightarrow \mathbb{N}$ their product $m_1 \cdot m_2 : \Delta_1 \times \Delta_2 \rightarrow \mathbb{N}$ is defined by $(m_1 \cdot m_2)(\gamma_1, \gamma_2) = m_1(\gamma_1) \cdot m_2(\gamma_2)$. If $A = A_1 \otimes A_2$, we have $\text{mult}(A) = \text{mult}(A_1) \cdot \text{mult}(A_2)$. (We could have observed this as early as in the proof of Graph Lemma 2.2(iii).)

An algebra A is *basic* iff its multiplicity $\text{mult}(A)$ is constantly 1. Now we can completely characterize local algebras (in terms of graphs and multiplicities): An algebra A is local iff it is basic and $\Delta(A)$ is a point.

To any multiplicity $m : \Delta(A) \rightarrow \mathbb{N}$, we associate the *transformed algebra* $M(A, m)$. It is the opposite endomorphism algebra $\text{End}_A^{\text{op}} V$ of a projective V with multiplicity m . This transformed algebra is, of course, only determined up to isomorphism. The *support* $\text{supp } m$ of a multiplicity m is the induced subgraph of $\Delta(A)$ whose vertices are those, where m is non-zero. The transformed algebra $M(A, m)$ is a MORITA *transform* of A iff m is strictly positive, ie. $\text{supp } m = \Delta(A)$.

Another special case is the *inner restriction* $A \downarrow \Lambda$: it is obtained from an induced subgraph Λ of $\Delta(A)$ as the transform $M(A, \text{mult}(A)|_\Lambda)$ of A with the multiplicity of A restricted to Λ .

Before we present the main properties of this transformation, we need another tool. Given an A -module V , we have the *evaluation functor*

$$e_V : \begin{array}{ccc} \text{mod}_A & \longrightarrow & \text{mod}_B, \\ W & \longmapsto & \text{Hom}_A(V, W), \end{array}$$

where $B := \text{End}_A^{\text{op}} V$ acts naturally on $\text{Hom}_A(V, W)$. Note that, as B -modules, $e_V V = B$. The category $\text{add}_A V$ generated by V consists of direct summands of multiples of V . It is the smallest full subcategory of mod_A containing V that is closed with respect to direct sums, direct summands, and isomorphism. The category $\text{add}_B B$ generated by V 's image B in mod_B is exactly the category proj_B of projective B -modules. It turns out that $\text{add}_A V$ and $\text{add}_B B$ do not actually differ.

2.4 Lemma *The restricted functor $e_V : \text{add}_A V \rightarrow \text{add}_B B$ is an equivalence.*

Proof: (This is YONEDA's Lemma combined with the fact that V is a generator of $\text{add}_A V$.) We only need to deal with indecomposable modules because e_V is a linear functor. To prove that $e_V V$ is full and faithful, suppose that V_1 is a direct summand of V , $\iota : V_1 \rightarrow V$ an embedding, and $\pi : V \rightarrow V_1$ a (split) projection with $\pi\iota = \mathbb{1}_{V_1}$. Then the inverse of the linear map $e_V : \text{Hom}_A(V_1, V_2) \rightarrow \text{Hom}_A(e_V V_1, e_V V_2)$, $\varphi \mapsto (\psi \mapsto \varphi\psi)$, is given by $\Phi \mapsto \Phi(\pi)\iota$. Since $e_V V = B$, e_V is almost surjective. \square

2.5 Transform Lemma *Let V be a projective A -module with multiplicity $\text{mult}(V) = m$.*

- (i) *The map $\vartheta : \text{supp } m \rightarrow \Delta(\text{M}(A, m))$, $[P] \mapsto [e_V P]$, is a bijective graph morphism. If $\text{supp } m = \Delta(A)$ it is an isomorphism. Furthermore, the multiplicity of a projective W in $\text{add}_A V$ does not really change: $\text{mult}(e_V W) \circ \vartheta = \text{mult}(W)$.*
- (ii) *Suppose that $\varphi : A \rightarrow A'$ is an isomorphism. Let $m' := m \circ \Delta(\varphi)^{-1}$. Then there is an isomorphism $\text{M}(\varphi, m) : \text{M}(A, m) \rightarrow \text{M}(A', m')$ with $\Delta(\text{M}(\varphi, m)) \circ \vartheta = \vartheta' \circ \Delta(\varphi)$.*
- (iii) $\text{M}(A_1 \otimes A_2, m_1 \cdot m_2) = \text{M}(A_1, m_1) \otimes \text{M}(A_2, m_2)$.
- (iv) $\text{M}(A, \text{mult}(A)) = A$.

Proof:

(i) Abbreviate $B := \text{M}(A, m) = \text{End}_A^{\text{op}} V$. By Lemma 2.4, the functor $e_V = \text{Hom}_A(V, \cdot)$ is an equivalence between $\text{add}_A V$ and proj_B . Hence a maximal direct decomposition $V = \bigoplus m([P_\nu])P_\nu$ (with pairwise non-isomorphic projectives P_ν) gives us a maximal direct decomposition $B = \bigoplus m([P_\nu])e_V P_\nu$, which proves that ϑ is well defined and bijective.

In order to verify that ϑ is a graph morphism, take P_1, P_2 with $[P_\nu] \in S$, let $R_\nu := e_V P_\nu = \text{Hom}_A(V, P_\nu)$, and show that $\dim \text{Hom}_B^{(1)}(R_1, R_2) \geq \dim \text{Hom}_A^{(1)}(P_1, P_2)$. Clearly, $e_V : \text{Rad}_A(P_1, P_2) \rightarrow \text{Rad}_B(R_1, R_2)$ is an isomorphism. Then $\dim \text{Rad}_B^2(R_1, R_2) \leq \dim \text{Rad}_A^2(P_1, P_2)$ because A has more projectives than the transform B . [$\text{Rad}_B^2(R_1, R_2) = \sum_{[R'] \in \Delta(B)} \text{Rad}_B(R', R_2) \circ \text{Rad}_B(R_1, R') \simeq \sum_{[P'] \in S} \text{Rad}_A(P', P_2) \circ \text{Rad}_A(P_1, P') \subset \text{Rad}_A^2(P_1, P_2)$.]

If m is strictly positive, ϑ is an isomorphism of graphs, since A and B then have an equal number of projectives (up to isomorphism).

(ii) This is evident if we use $V' = V^{(\varphi^{-1})}$ (and $P' = P^{(\varphi^{-1})}$).

(iii) Let $V_i = \bigoplus m_i([P_{i\nu}])P_{i\nu}$, where $\Delta(A_i) = \{[P_{i\nu}] \mid \nu\}$. By Lemma 1.3(i), we only need that $V_1 \otimes V_2 = \bigoplus m_1([P_{1\nu}])m_2([P_{2\lambda}])P_{1\nu} \otimes P_{2\lambda}$.

(iv) We have $A \simeq \text{End}_A^{\text{op}} A = M(A, \text{mult}(A))$ by mapping a to the right multiplication ϱ_a with a . \square

Using (iii) we conclude $(A_1 \otimes A_2) \downarrow (\Lambda_1 \times \Lambda_2) = (A_1 \downarrow \Lambda_1) \otimes (A_2 \downarrow \Lambda_2)$.

2.6 Corollary *Let A be an arbitrary algebra.*

(i) *If A is \otimes -indecomposable, then $\text{mult}(A)$ is primitive (ie. its values do not have a non-trivial common divisor), or A is a matrix algebra with prime multiplicity.*

(ii) $M(A, 1)$ is basic.

Proof:

(i) Suppose $\text{mult}(A) = p \cdot m$ with $p \in \mathbb{N}_{>1}$ prime. By Transform Lemma 2.5(iii) and (iv), $A = M(k \otimes A, p \cdot m) = M(k, p) \otimes M(A, m)$. However, $M(k, p) = k^{p \times p}$ and $M(A, m)$ are non-trivial, or $A = k^{p \times p}$.

(ii) $\text{mult}(M(A, 1)) \circ \vartheta = 1$. \square

And now for something completely different:

2.7 Lemma *Let $m : \Lambda_1 \times \Lambda_2 \rightarrow \mathbb{N}$ be the product of $m_i : \Lambda_i \rightarrow \mathbb{N}$. If all values of $m = m_1 \cdot m_2$ are divisible by a fixed $n \in \mathbb{N}$, then there is a decomposition $n = n_1 \cdot n_2$ such that all values of m_i are divisible by n_i .*

Proof (Following an idea of W. GLAS.): In \mathbb{Z} , the ideal I generated by the values of m is the product of the ideals I_i generated by the values of m_i . Write $I = \langle g \rangle$, ie. g is the greatest common divisor of the values of m , and $I_i = \langle g_i \rangle$. Then we have $g = g_1 \cdot g_2$. Since n divides g , we can factor it as required. \square

3 Uniqueness for Local Algebras

For the sake of completeness, we reprove the results of HORST (1987), HORST (1990) concerning finite-dimensional, local algebras; in particular her uniqueness theorem for tensor decompositions of local algebras. Her main tool is (a variant of) the following lemma, which allows to “separate” the tensor factors in the proof of the refinement property of Theorem 3.2. HORST’s proof of the lemma seems to be incomplete.

A morphism $\varphi : A \rightarrow B$ of local algebras is *horizontal* iff for every ℓ with $\text{rad}^\ell A \neq 0$ we have $\varphi(\text{rad}^\ell A) \not\subset \text{rad}^{\ell+1} B$ or, equivalently, $\varphi^{(\ell)} \neq 0$ for every such ℓ . For example, sections are always horizontal. We call a morphism of local algebras *trivial* iff it factors through k or, equivalently, iff it maps the radical to zero. Recall $\text{char } k = 0$.

3.1 Separation Lemma *Let $\varphi : A \rightarrow B_1 \otimes B_2$ be a morphism of local algebras with $\pi_1\varphi$ horizontal. Then $\pi_2\varphi$ is trivial.*

Proof (with V. STRASSEN): Otherwise choose a counter-example of minimal dimension.

Let $\ell > 0$ be maximal with $0 \neq \pi_2\varphi(\text{rad } A) \subset \text{rad}^\ell B_2$. Then $\text{rad}^{\ell+1} B_2 = 0$. Else we could take $B'_2 := B_2/\text{rad}^{\ell+1} B_2$ instead of B_2 . We would still have $\ker \pi_2\varphi \neq \text{rad } A$ but the dimension of B'_2 is smaller.³

Choose n minimal with $\text{rad}^{n+1} A = 0$. Then $\text{rad}^{n+1} B_1 = 0$. Otherwise we could replace B_1 with $B'_1 := B_1/\text{rad}^{n+1} B_1$ yielding a smaller counter-example: $\pi_2\varphi$ does not change, and $\pi_1\varphi$ remains horizontal, since $\text{rad}^{n+1} A = 0$.

Now we show $\pi_1\varphi(\text{rad}^n A) = 0$. For $a \in \text{rad } A$ we can write

$$\varphi(a) = b \otimes 1 + 1 \otimes c + x$$

with $b := \pi_1\varphi(a) \in \text{rad } B_1$, $c := \pi_2\varphi(a) \in \text{rad}^\ell B_2$, and $x \in \text{rad } B_1 \otimes \text{rad } B_2$. For elements $a_0, \dots, a_n \in \text{rad } A$ expansion of $0 = \varphi(a_0 \cdots a_n)$ yields

$$(*) \quad 0 = \sum_{i=0}^n b_0 \cdots b_{i-1} b_{i+1} \cdots b_n \otimes c_i.$$

The other terms are zero: products with two or more ‘ c ’s are in $B_1 \otimes (\text{rad}^{2\ell} B_2) = 0$, products with one ‘ c ’ and an ‘ x ’ are in $B_1 \otimes (\text{rad}^{\ell+1} B_2) = 0$, and products without a ‘ c ’ are in $(\text{rad}^{n+1} B_1) \otimes B_2 = 0$.

Fix $a_0 \in \text{rad } A$ with $c_0 \neq 0$. This is possible, since we are dealing with a counter-example. Now we use backward induction on r to show

$$\forall a_1, \dots, a_n \in \text{rad } A : \quad b_0^r b_{r+1} \cdots b_n = 0.$$

³We even have $\dim \text{rad}^\ell B_2 = 1$. Otherwise take $c = \pi_2\varphi(a) \neq 0$. Choose a 1-codimensional subspace $I \subset \text{rad}^\ell B_2$ not containing c . Then I is an ideal in B_2 and thus we can replace B_2 by B_2/I and obtain a smaller counter-example.

Take $n \geq r \geq 0$. From (*) letting $a_1 := \dots a_r := a_0$ we get

$$0 = b_0^r b_{r+1} \cdots b_n \otimes \left(\sum_{i=0}^r c_0 \right) + \sum_{i=r+1}^n b_0^{r+1} b_{r+1} \cdots b_{i-1} b_{i+1} \cdots b_n \otimes c_i.$$

Due to characteristic zero $(r+1)c_0 \neq 0$. By induction hypothesis we are done.

For $r = 0$ we see that any product of n elements in $\pi_1 \varphi(\text{rad } A)$ is zero, or, in other words, $\pi_1 \varphi(\text{rad}^n A) = 0 = \text{rad}^{n+1} B_1$. However, this contradicts the assumption that $\pi_1 \varphi$ is horizontal. \square

Given an isomorphism $\varphi : B_1 \otimes B_2 \rightarrow C_1 \otimes C_2$, we use some associated objects: we have the injections $\iota_j : B_j \rightarrow B_1 \otimes B_2$ and the projections $\pi_i : C_1 \otimes C_2 \rightarrow C_i$ provided the corresponding other factor of $C_1 \otimes C_2$ is local. Patching these together we obtain morphisms $\varphi_{ij} = \pi_i \varphi \iota_j : B_j \rightarrow C_i$, which are usually no isomorphisms. Thus the subalgebras $C_{ij} := \text{im } \varphi_{ij}$ of C_i are proper in general. Corresponding objects come with φ^{-1} ; the image B_{ji} of $\varphi_{ji}^{-1} := (\varphi^{-1})_{ji}$ is a subalgebra of B_j .

3.2 Theorem (HORST (1987), HORST (1990)) *Let $\varphi : B_1 \otimes B_2 \rightarrow C_1 \otimes C_2$ be an isomorphism of local algebras. Then:*

- (i) *We have $B_{ji} \simeq C_{ij}$. More precisely, the corresponding restrictions of φ_{ij} and φ_{ji}^{-1} are inverse isomorphisms between B_{ji} and C_{ij} .*
- (ii) *The natural morphism $\mu : C_{11} \otimes C_{12} \rightarrow C_1$, $c_{11} \otimes c_{12} \mapsto c_{11} \cdot c_{12}$, is an isomorphism.*

Proof: We claim that any concatenation $\varphi_{ijk} := \varphi_{ij}^{-1} \varphi_{jk}$ or $\varphi_{ijk}^{-1} := \varphi_{ij} \varphi_{jk}^{-1}$ with $i \neq k$ is a trivial morphism. We will prove this in several steps. By symmetry we only deal with φ_{211} .

The morphism $\varphi_{11} \varphi_{121}^n$ is trivial for large n . Indeed, choose n such that $\text{im } \varphi_{121}^{2n+1} = \text{im } \varphi_{121}^n$. Take the morphism $\psi : \text{im } \varphi_{121}^n \subset B_1 \rightarrow C_1 \otimes C_2$. Now $\varphi_{121}^n \varphi_{12}^{-1} \pi_2 \psi : \text{im } \varphi_{121}^n \rightarrow \text{im } \varphi_{121}^n$ is an isomorphism hence $\pi_2 \psi$ is a section and horizontal. By Separation Lemma 3.1, $\pi_1 \psi$ is trivial. Since $\pi_1 \psi = \varphi_{11}|_{\text{im } \varphi_{121}^n}$, $\varphi_{11} \varphi_{121}^n$ is trivial, too.

Similarly, the morphism $\varphi_{211} \varphi_{111}^n$ is trivial for large n .

The morphism $\tilde{\mu} := \mu(\varphi_{111}^n \otimes \varphi_{121}^n) : B_1 \otimes B_1 \rightarrow B_{11} \otimes B_{12} \rightarrow B_1$ is surjective for every $n \in \mathbb{N}$. Indeed, by Surjectivity Criterion 1.2 we only need to show that $\tilde{\mu}^{(1)}$ is surjective. Write $\varphi^{(1)}(b^{(1)} \otimes 1) = \alpha^+(b^{(1)}) \otimes 1 + 1 \otimes \gamma^+(b^{(1)})$, $\varphi^{(1)}(1 \otimes b^{(1)}) = \beta^+(b^{(1)}) \otimes 1 + 1 \otimes \delta^+(b^{(1)})$, briefly: $\varphi^{(1)} \sim \begin{bmatrix} \alpha^+ & \beta^+ \\ \gamma^+ & \delta^+ \end{bmatrix}$. Similarly write $(\varphi^{-1})^{(1)} \sim \begin{bmatrix} \alpha^- & \beta^- \\ \gamma^- & \delta^- \end{bmatrix}$. Then $\tilde{\mu}^{(1)} = [(\alpha^- \alpha^+)^n (\beta^- \gamma^+)^n]$. $[\tilde{\mu}^{(1)}(b^{(1)} \otimes 1) = (\mu(\varphi_{111}^n \otimes \varphi_{121}^n)(b \otimes 1))^{(1)} = (\varphi_{111}^n(b))^{(1)} = (\alpha^- \alpha^+)^n (b^{(1)}) \otimes 1.]$ On the other hand, $\mathbb{1}_{B_1^{(1)} \otimes B_1^{(0)}}^{2n} = (\alpha^- \alpha^+ + \beta^- \gamma^+)^{2n} = (\alpha^- \alpha^+)^n \eta + (\beta^- \gamma^+)^n \vartheta$ for some maps η, ϑ , since

$\alpha^- \alpha^+$ and $\beta^- \gamma^+ = \mathbb{1} - \alpha^- \alpha^+$ commute. Now $\mathbb{1}_{B_1^{(1)} \otimes B_1^{(0)}} = \tilde{\mu}^{(1)} \begin{bmatrix} \eta \\ \vartheta \end{bmatrix}$ shows that $\tilde{\mu}^{(1)}$ is surjective.

Together we obtain the above claim that φ_{211} is trivial. Indeed, we already know that $\varphi_{211} \varphi_{111}^n$ and $\varphi_{211} \varphi_{121}^n$ are trivial provided n is large enough. Since $\tilde{\mu}$ is surjective, we have $\text{rad } B_1 = \tilde{\mu}(\text{rad}(B_1 \otimes B_1))$. Now calculate $\varphi_{211}(\text{rad } B_1) = \varphi_{211} \mu(\varphi_{111}^n \otimes \varphi_{121}^n)(\text{rad } B_1 \otimes B_1 + B_1 \otimes \text{rad } B_1) \subset \varphi_{211} \varphi_{111}^n(\text{rad } B_1) \cdot B_2 + B_2 \cdot \varphi_{211} \varphi_{121}^n(\text{rad } B_1) = 0$.

(i) By symmetry we only show that $\varphi_{212} : B_{21} \rightarrow C_{12} \rightarrow B_{21}$ is the identity. Take any $b = \varphi_{21}^{-1}(c) \in B_{21}$, $c \in C_1$. Then we have $\varphi_{212}(b) - b = \varphi_{21}^{-1} \pi_1 \varphi(\iota_2 \pi_2 \varphi^{-1} \iota_1(c) - \varphi^{-1} \iota_1(c))$. The term in parentheses is in the kernel of π_2 , which is $(\text{rad } B_1) \otimes B_2 = (B_1 \otimes B_2) \cdot \iota_1(\text{rad } B_1)$. And this kernel is mapped by $\varphi_{21}^{-1} \pi_1 \varphi$ to $B_2 \cdot \varphi_{211}(\text{rad } B_1)$, which is zero, since φ_{211} is trivial.

(ii) Since π_1 is surjective, so is μ . For the injectivity we show that

$$\chi : \begin{array}{ccc} C_{11} & & C_1 \\ \otimes & \xrightarrow{\mu} & C_1 \\ C_{12} & & C_2 \end{array} \begin{array}{c} \nearrow \iota_1 \\ \searrow \varphi^{-1} \end{array} \begin{array}{ccc} C_1 & & B_1 \\ \otimes & \xrightarrow{\varphi^{-1}} & \otimes \\ C_2 & & B_2 \end{array} \begin{array}{ccc} \xrightarrow{\varphi_{11}} & C_{11} \\ & \otimes \\ \xrightarrow{\varphi_{12}} & C_{12} \end{array}$$

is surjective. Using Surjectivity Criterion 1.2, we only need that $\chi^{(1)}$ is surjective. By definition we have $\chi_{ij} = \varphi_{1i1}^{-1} : C_{1j} \rightarrow C_{1i}$. By (i) χ_{11} and χ_{22} are isomorphic while χ_{12} and χ_{21} are trivial by the above. Thus $\chi^{(1)}$ is given by the matrix $\begin{bmatrix} \chi_{11}^{(1)} & 0 \\ 0 & \chi_{22}^{(1)} \end{bmatrix}$, which describes an isomorphism, since both non-trivial entries do. This proves the theorem. \square

The refinement property in this theorem enables us to prove a corresponding unique factorization property. First we provide the induction step.

3.3 Lemma *Let $\varphi : B_1 \otimes B' \rightarrow C_1 \otimes \cdots \otimes C_s$ be an isomorphism of local algebras and suppose that B_1, C_1, \dots, C_s are \otimes -indecomposable. Then there are an index j and isomorphisms*

$$\begin{aligned} \varphi_1 : B_1 &\longrightarrow C_j, \\ \varphi' : B' &\longrightarrow C' := C_1 \otimes \cdots \otimes C_{j-1} \otimes C_{j+1} \otimes \cdots \otimes C_s. \end{aligned}$$

Proof: We write B_{-1}, C_{-j} instead of B', C' , respectively. From an appropriately permuted version of φ mapping $B_1 \otimes B_{-1} \rightarrow C_j \otimes C_{-j}$ we derive the morphisms $\varphi_{y,x}, \varphi_{x,y}^{-1}$, and their images, the subalgebras $C_{y,x} \subset C_y, B_{x,y} \subset B_x$.

First choose j with $B_{1,j} = B_1$. This is possible, since we otherwise obtain a contradiction: By Theorem 3.2(ii) $B_{1,j} \otimes B_{1,-j} \rightarrow B_1$ is an isomorphism. Since we assume $B_{1,j} \neq B_1$ and B_1 is \otimes -indecomposable, the factor $B_{1,j}$ is trivial. On the other hand, $B_{1,1} \otimes \cdots \otimes B_{1,s} \rightarrow B_1$ is surjective (since $C_1 \otimes \cdots \otimes C_s \rightarrow B_1$ is). Consequently, B_1 is trivial. This is a contradiction.

By Theorem 3.2(i), $C_{j,1} \simeq B_{1,j}$, hence non-trivial. Furthermore, $C_{j,1} \otimes C_{j,-1} \rightarrow C_j$ is an isomorphism and, since C_j is \otimes -indecomposable, $C_{j,1} = C_j$. This already gives us an isomorphism $B_1 = B_{1,j} \rightarrow C_{j,1} = C_j$, namely $\varphi_{j,1}$. Moreover, $B_{-1,j} \simeq C_{j,-1}$ is trivial, $B_{-1,j} \otimes B_{-1,-j} \simeq B_{-1}$, and hence $B_{-1,-j} = B_{-1}$. Analogously $C_{-j,-1} = C_{-j}$. Thus $\varphi_{-j,-1} : B_{-1} = B_{-1,-j} \rightarrow C_{-j,-1} = C_{-j}$ is an isomorphism. \square

Now we are able to prove a first unique factorization theorem.

3.4 Unique Factorization for Local Algebras (HORST) *The set $\mathcal{M}_{\text{local}}$ of isomorphism classes of local algebras is a free commutative monoid over the set $\mathcal{X}_{\text{local}}$ of isomorphism classes of \otimes -indecomposable local algebras as a basis.*

Proof: Clearly $\mathcal{M}_{\text{local}}$ is a commutative monoid with the multiplication that is induced by the tensor product.

We prove by induction on r : If $B_1 \otimes \dots \otimes B_r \simeq C_1 \otimes \dots \otimes C_s$ with local \otimes -indecomposable algebras, then there is a bijection $\sigma : \{1, \dots, r\} \rightarrow \{1, \dots, s\}$ such that $B_i \simeq C_{\sigma i}$.

There is nothing to do for $r = 0$, so suppose $r > 0$. By Lemma 3.3, there are an index j and isomorphisms $B_1 \simeq C_j$ and $B_2 \otimes \dots \otimes B_r \simeq C_1 \otimes \dots \otimes C_{j-1} \otimes C_{j+1} \otimes C_s$. By induction hypothesis, we get a bijection $\sigma : \{2, \dots, r\} \rightarrow \{1, \dots, j-1, j+1, \dots, s\}$ with $B_i \simeq C_{\sigma i}$ for all $i > 1$. Extending σ by $\sigma(1) = j$ completes the proof. \square

Unlike some other types of algebras, the class of local algebras is not closed with respect to direct sums.

4 Loops

In this section, we always start with \oplus -indecomposable algebras.

In the previous section we have seen that unique factorization holds for local algebras. In general, even algebras without local factors can lead to local factors in some steps of the proofs in the following section. Our next aim is to obtain some results that enable us to separate non-local and local factors.

The key result is a generalization of Theorem 3.2(ii).

4.1 Proposition *Let $\varphi : B_1 \otimes B_2 \rightarrow C_1 \otimes C_2$ be an isomorphism and let C_2 be local. Recall the definition of the subalgebras C_{11} and C_{12} of C_1 . Then the natural morphism*

$$\mu : \begin{array}{ccc} C_{11} \otimes C_{12} & \longrightarrow & C_1, \\ c_{11} \otimes c_{12} & \longmapsto & c_{11} \cdot c_{12}, \end{array}$$

is an isomorphism.

Before we prove this proposition, we discuss its further use.

The final question we are about to answer in this paper is: Is a tensor product decomposition of a \oplus -indecomposable algebra unique? Some partial answers are given in the following corollaries. For example, we ask whether a \otimes -indecomposable algebra tensored with a local algebra can decompose into two non-local factors. It cannot, of course, since the global question is answered positively.

4.2 Corollary *Let $\varphi : B_1 \otimes B_2 \rightarrow C_1 \otimes C_2$ be an isomorphism and suppose that B_1 is \otimes -indecomposable and B_2 is local. Then C_1 or C_2 is local.*

Proof: By Proposition 4.1 we have $B_{11} \otimes B_{12} \simeq B_1$, thus one of the factors is trivial, say $B_{11} = k$. Note that $(\varphi_{11}^{-1})^{(0)}$ is injective, since B_2 is local and therefore $\pi_1^{(0)}$ injective. Together we obtain that $C_1^{(0)}$ is embedded in $B_{11}^{(0)} = k$, hence it is k . Thus C_1 is local. \square

Next we ask whether a decomposition into a local factor and a factor not admitting any further non-trivial local (tensor) factor is unique. Again the answer should be “yes”.

4.3 Corollary *Let $\varphi : B_1 \otimes B_2 \rightarrow C_1 \otimes C_2$ be an isomorphism and suppose that B_1, C_1 do not allow any non-trivial, local factor while B_2, C_2 are local. Then $\varphi_{11} : B_1 \rightarrow C_1$ is an isomorphism and $\Delta(\varphi) = \Delta(\varphi_{11}) \times \varepsilon$.⁴ Furthermore, B_2 and C_2 are isomorphic.*

⁴Recall that ε denotes the unique isomorphism between one-point graphs.

Proof: By Proposition 4.1 we have an isomorphism $B_{11} \otimes B_{12} \rightarrow B_1$. Since $B_{12} = \varphi_{12}C_2$ is local, this factor must be trivial: $B_{12} = k$. Similarly $C_{12} = k$. In other words, $\varphi_{11} : B_1 \rightarrow C_1$ and $\varphi_{11}^{-1} : C_1 \rightarrow B_1$ are both surjective and thus isomorphic.

Moreover, for $[P] \in \Delta(B_1)$ we have by definition $\Delta(\varphi_{11})[P] = [\varphi_{11}P]$ and $\Delta(\varphi)[P \otimes B_2] = [\varphi(P \otimes B_2)]$. Thus to show $\Delta(\varphi) = \Delta(\varphi_{11}) \times \varepsilon$, we only need $\varphi(P \otimes B_2) \simeq \varphi_{11}P \otimes C_2$. The projection $C_1 \otimes C_2 \rightarrow C_1$ maps $\varphi(P \otimes B_2) = \varphi((B_1 \otimes B_2) \cdot (P \otimes 1)) = (C_1 \otimes C_2) \cdot \varphi(P \otimes 1)$ to $\varphi_{11}P$. Now for some projective R we have $\varphi(P \otimes B_2) \simeq R \otimes C_2$. Dividing out the radicals of the C_2 -modules yields two isomorphic C_1 -modules, namely $\varphi_{11}P$ and R .

Finally, we show that B_2 and C_2 are isomorphic: Take any $\beta \in \Delta(B_1)$ and set $\gamma := \Delta(\varphi_{11})\beta$. Then by Transform Lemma 2.5(iii) we get $M(B_1 \downarrow \beta, 1) \otimes B_2 \simeq M(C_1 \downarrow \gamma, 1) \otimes C_2$ and $M(B_1 \downarrow \beta, 1) \simeq M(C_1 \downarrow \gamma, 1)$. Using Unique Factorization for Local Algebras 3.4 we obtain $B_2 \simeq C_2$. \square

Thirdly, we ask whether it is possible to cancel a common local factor.

4.4 Corollary *Let $\varphi : B_1 \otimes B_2 \rightarrow C_1 \otimes C_2$ be an isomorphism and suppose that B_2, C_2 are local and isomorphic. Then there is an isomorphism $\varphi_1 : B_1 \rightarrow C_1$ such that $\Delta(\varphi) = \Delta(\varphi_1) \times \varepsilon$.*

Warning! It is possible that φ_{11} is not an isomorphism: an automorphism $\varphi : B_1 \otimes B_2 \rightarrow B_1 \otimes B_2$ can swap a common local tensor factor of B_1 and B_2 .

Proof: Write $B_1 = B_n \otimes B_\ell$ and $C_1 = C_n \otimes C_\ell$ such that B_n, C_n do not allow local factors and B_ℓ, C_ℓ are local. Applying Corollary 4.3 to $\varphi : B_n \otimes (B_\ell \otimes B_2) \rightarrow C_n \otimes (C_\ell \otimes C_2)$, we obtain an isomorphism $\varphi_{nn} : B_n \rightarrow C_n$ with $\Delta(\varphi) = \Delta(\varphi_{nn}) \times \varepsilon_{\ell 2}$. Then $B_\ell \otimes B_2$ and $C_\ell \otimes C_2$ are also isomorphic. By Unique Factorization for Local Algebras 3.4, there is an isomorphism $\psi : B_\ell \rightarrow C_\ell$. Hence $\varphi_{nn} \otimes \psi : B_1 = B_n \otimes B_\ell \rightarrow C_n \otimes C_\ell = C_1$ is an isomorphism. And $\Delta(\varphi) = \Delta(\varphi_{nn}) \times \varepsilon_{\ell 2} = \Delta(\varphi_{nn}) \times \varepsilon_\ell \times \varepsilon_2 = \Delta(\varphi_{nn} \otimes \psi) \times \varepsilon_2$, using Graph Lemma 2.2(iv) for the last equality. \square

Idempotents

The proof of the proposition is formulated with primitive idempotents and not with indecomposable projectives.

In general, a direct decomposition $M = \bigoplus M_\nu \in \text{mod}_A$ corresponds bijectively to a 1-decomposition $1 = \sum e_\nu \in \text{End}_A^{\text{op}} M$. The idempotents are the projections onto summands, the summands are the images of the idempotents. Of course, maximal decompositions correspond to each other. A direct decomposition is maximal iff all its summands are indecomposable, whereas a 1-decomposition is maximal iff all its idempotents are primitive. The right multiplication $\varrho : A \rightarrow \text{End}_A^{\text{op}} A$, $a \mapsto (x \mapsto xa)$, is an isomorphism. (This can be

regarded as a base or consequence of Lemma 2.4.) Thus, to an A -module decomposition of the algebra A , we can associate a decomposition of its unit element via ϱ . In the following, an idempotent will always be an element of the algebra itself rather than of its endomorphism algebra.

In accordance with the correspondence between idempotents and projectives, we call two idempotents e' , e *isomorphic* iff $Ae' \simeq Ae$. This holds iff there are elements $a', a \in A$ such that $e = a'a$ and $e' = aa'$.

Given two only slightly different 1-decompositions, we can identify them:

4.5 Point Correction *Let $1 = \sum e_\nu$, $1 = \sum f_\nu$ be 1-decompositions in an algebra A with $e_\nu - f_\nu \in \text{rad } A$ for all ν . Then there is a unit $u \in A$ such that $\kappa_u(e_\nu) = f_\nu$ for all ν , where κ_u denotes the conjugation $a \mapsto uau^{-1}$.*

Proof: Since the (so-called) tops $Ae_\nu/\text{rad } Ae_\nu$ and $Af_\nu/\text{rad } Af_\nu$ are equal, the projectives Ae_ν and Af_ν are isomorphic. (NAKAYAMA's Lemma implies that any lift of the identity to $Ae_\nu \rightarrow Af_\nu$ is an isomorphism.)

Thus we have an A -module automorphism φ of A satisfying $\varphi \varrho_{e_\nu} = \varrho_{f_\nu} \varphi$, where ϱ_b denotes the right multiplication with b . Hence, the conjugation κ_φ with φ is an automorphism of the (oppositional) endomorphism algebra $\text{End}_A^{\text{op}} A$ of A mapping ϱ_{e_ν} to ϱ_{f_ν} . Since $\varrho : A \rightarrow \text{End}_A^{\text{op}} A$ is an isomorphism, we find a unit $u \in A$ such that $\varphi = \varrho_{u^{-1}}$. Now $\varrho^{-1} \kappa_\varphi \varrho = \kappa_u$ is an automorphism of the algebra A mapping e_ν to f_ν . \square

Proof of the Proposition

We have seen that Proposition 4.1 is a powerful tool that allows us to handle a considerable amount of the interaction between local and non-local factors in a tensor product decomposition. Hence it is not a surprise that its proof is tricky. Nevertheless, the basic idea is quite easy: We construct the inverse of μ in the same way as C. HORST did in the local case. After tidying up a bit, we use a PIERCE-decomposition $A = \bigoplus f' Af$ to break up the question into several local ones; one for each pair f', f of primitive idempotents. The case that Af' and Af are isomorphic turns out to be the most troublesome. At first, it seems that Theorem 3.2(ii) does the job. At a closer look, we additionally need Rigidity Lemma 4.6.

Proof (Proposition 4.1): Before we begin with the proof, we tidy up the situation: Choose maximal 1-decompositions $1 = \sum_\nu e_{1\nu} \in B_1$ and $1 = \sum_\lambda e_{2\lambda} \in B_2$ and set $f_{1\nu\lambda} := \pi_1 \varphi(e_{1\nu} \otimes e_{2\lambda})$ for each pair $(e_{1\nu}, e_{2\lambda})$. In the following we suppress the indices ν, λ ; it will always be clear which pair (e_1, e_2) is related to f_1 . We may assume that $\varphi(e_1 \otimes e_2) = f_1 \otimes 1$ for all e_1, e_2 . If this is not already true, use Point Correction 4.5 to choose a unit $u \in C_1 \otimes C_2$ with $\kappa_u \varphi(e_1 \otimes e_2) = f_1 \otimes 1$, where κ_u denotes the conjugation with u . Let $\varphi' := \kappa_{\pi_1(u) \otimes 1}^{-1} \kappa_u \varphi$. Then $\pi_1 \varphi' = \pi_1 \varphi$ and $\varphi'(e_1 \otimes e_2) = f_1 \otimes 1$. (The first holds, since $\pi_1 \kappa_v = \kappa_{\pi_1(v)} \pi_1$ for any unit v . And

with $\kappa_{\pi_1(u)}(f_1) = \kappa_{\pi_1(u)}\pi_1\varphi(e_1 \otimes e_2) = \pi_1\kappa_u\varphi(e_1 \otimes e_2) = \pi_1(f_1 \otimes 1) = f_1$ we get $\varphi'(e_1 \otimes e_2) = \kappa_{\pi_1(u)\otimes 1}^{-1}(f_1 \otimes 1) = f_1 \otimes 1$.) Thus we can replace φ by φ' without changing C_{11} , C_{12} , or μ .

Clearly μ is a surjective morphism. To prove its injectivity, we will show that

$$\chi : \begin{array}{ccccccc} C_{11} & & & & & & \\ \otimes & \xrightarrow{\mu} & C_1 & \xrightarrow{\iota_1} & C_1 & \xrightarrow{\varphi^{-1}} & B_1 & \longrightarrow & C_{11} \\ C_{12} & & & & \otimes & & \otimes & \psi & \otimes \\ & & & & C_2 & & B_2 & \longrightarrow & C_{12} \end{array}$$

is surjective, where $\psi = \varphi_{11} \otimes \varphi_{12}$. For later reference, we list the idempotents obtained by moving the $e_1 \otimes e_2$ with the morphisms that build χ :

$$\begin{array}{ccccccc} f_{11} & & & & f_1 & e_1 & \longmapsto & f_{11} \\ \otimes & \longmapsto & f_1 & \xrightarrow{\iota_1} & \otimes & \otimes & \psi & \otimes \\ f_{12} & & & & 1 & e_2 & \longmapsto & f_{12} \end{array} .$$

It is clear that the e_1 , e_2 , and f_1 are primitive idempotents. In fact, the $f_{11} = \varphi_{11}(e_1)$ and $f_{12} = \varphi_{12}(e_2)$ are primitive in C_{11} and C_{12} , respectively, but we will not use this. Note that $\mu\psi = \pi_1\varphi$: $\mu\psi(b_1 \otimes b_2) = \pi_1\varphi(b_1 \otimes 1) \cdot \pi_1\varphi(1 \otimes b_2) = \pi_1\varphi(b_1 \otimes b_2)$. Using this verify $\mu(f_{11} \otimes f_{12}) = f_1$. Thus χ maps the chosen idempotents to themselves. For later use, observe $\mu\chi = \mu$. ($\mu\chi = \mu\psi\varphi^{-1}\iota_1\mu = \pi_1\iota_1\mu = \mu$.) So in the end we will know that χ is the identity.

In order to show that χ is surjective we apply the Surjectivity Criterion 1.2. To this end, it suffices to show that χ respects radicals and that $\chi^{(0)}$ and $\chi^{(1)}$ are surjective. Since any surjective morphism respects radicals and so does ι_1 , χ respects radicals. Surjectivity of $\iota_1^{(0)}$ implies that $\chi^{(0)} = \psi^{(0)}(\varphi^{-1})^{(0)}\iota_1^{(0)}\mu^{(0)}$ is surjective. Therefore and since ι_1 is even a section, it suffices to check that $C_1^{(0)}$ and $(C_1 \otimes C_2)^{(0)}$ have the same dimension. This is clear, however, because C_2 is local.

It remains to verify that $\chi^{(1)}$ is surjective. Write $(C_{11} \otimes C_{12})^{(1)} = \bigoplus f'(C_{11} \otimes C_{12})^{(1)}f$, where f' and f run over all $f_{11} \otimes f_{12}$. (We treat $A^{(*)}$ and its homogeneous parts as A -bimodules.) So we can deal with appropriate restrictions of $\chi^{(1)}$. We show: For any such pair f', f , the restriction⁵

$$\chi_{f'f}^{(1)} : f'(C_{11} \otimes C_{12})^{(1)}f \longrightarrow f'(C_{11} \otimes C_{12})^{(1)}f$$

of the map $\chi^{(1)}$ is surjective. We distinguish two cases: either f' and f are isomorphic, or they are not.

If f' and f are not isomorphic, then $\iota_{1f'f}^{(1)} : f'_1 C_1^{(1)} f_1 \rightarrow (f'_1 \otimes 1)(C_1 \otimes C_2)^{(1)}(f_1 \otimes 1)$ is surjective. Indeed, its domain and target spaces are isomorphic because $C_2^{(0)} = k$ and $f'_1 C_1^{(0)} f_1 = 0$. (The latter are classes of C_1 -module isomorphisms

⁵Keep in mind that $\chi(f) = f$. And be careful on the order: we never talk about $(\chi_{ff})^{(1)}$. In general, even $eA^{(1)}e \neq (eAe)^{(1)}$.

from $C_1 f_1$ to $C_1 f'_1$ modulo non-isomorphisms.) Since ι_1 is a section, so is $\iota_1^{(1)}_{f'f}$. Hence $\chi_{f'f}^{(1)}$ is a composition of surjective maps.

If f' and f are isomorphic, we show that $\mu_{f'f}$ is isomorphic. Then, since $\mu\chi = \mu$ and $\chi(f) = f$, we have $\mu_{f'f}\chi_{f'f} = \mu_{f'f}$, and so $\chi_{f'f}$ is the identity. Consequently, $\chi_{f'f}^{(1)}$ is the identity, too.⁶ In fact, it is sufficient to deal with the case $f' = f$.⁷

We still have to show that the morphism

$$\mu_{ff} : f_{11}C_{11}f_{11} \otimes f_{12}C_{12}f_{12} \longrightarrow f_1C_1f_1, \quad c_{11} \otimes c_{12} \longmapsto c_{11}c_{12},$$

is isomorphic. Having seen the local case, we may hope to get this isomorphism from an appropriately restricted version of φ . This is only partially true, as we will see immediately.

To the restriction $\tilde{\varphi} : e_1B_1e_1 \otimes e_2B_2e_2 \longrightarrow f_1C_1f_1 \otimes C_2$ of φ , we can apply Theorem 3.2(ii), since $\tilde{\varphi}$ is an isomorphism and all these algebras are local because the idempotents e_1 , e_2 , and f_1 are primitive. We obtain an isomorphism

$$f_1C_{11}f_1 \otimes f_1C_{12}f_1 \longrightarrow f_1C_1f_1, \quad c_{11} \otimes c_{12} \longmapsto c_{11} \cdot c_{12},$$

since $f_1C_{1\nu}f_1 = \text{im } \tilde{\varphi}_{1\nu}$. To close the gap between this isomorphism and μ_{ff} , we apply the following Rigidity Lemma 4.6 to the restriction $B_1 \otimes e_2B_2e_2 \rightarrow f_{12}C_1f_{12} \otimes C_2$ of φ . Indeed, $e_2B_2e_2$ is local and the maximal 1-decomposition $1 = \sum e_i \in B_1$ is mapped properly: $\varphi(e_1 \otimes e_2) = f_1 \otimes 1$ for all e_i . Now $\pi_1\varphi(1 \otimes e_2B_2e_2) = f_{12}C_{12}f_{12}$, while $\pi_1\varphi(e_1 \otimes e_2B_2e_2) = f_1C_{12}f_1$. Thus we get an isomorphism

$$f_{12}C_{12}f_{12} \longrightarrow f_1C_{12}f_1, \quad c_{12} \longmapsto f_1c_{12}f_1.$$

By symmetry, we obtain another such isomorphism identifying $f_{11}C_{11}f_{11}$ with $f_1C_{11}f_1$. Use $f_1 = f_{11} \cdot f_{12}$ and the fact that elements of C_{11} and C_{12} commute in order to verify $f_1c_{11}f_1 \cdot f_1c_{12}f_1 = c_{11} \cdot c_{12}$. These three isomorphisms thus fit together to μ_{ff} . \square

4.6 Rigidity Lemma *Let $\varphi : B_1 \otimes B_2 \rightarrow C_1 \otimes C_2$ be an isomorphism. Recall that B_1 is \oplus -indecomposable and suppose that B_2 and C_2 are local. Assume*

⁶Take any $y \in f'(C_{11} \otimes C_{12})^{(1)}f$, say $y = (f'df)^{(1)}$. Then $\chi^{(1)}(y) = (\chi(f'df))^{(1)} = (f'df)^{(1)} = y$.

⁷Otherwise, since f' and f are isomorphic, so are $f'_{1\nu}$ and $f_{1\nu}$ by Lemma 1.3(iv). Say $f'_{1\nu} = a_{1\nu}a'_{1\nu}$, $f_{1\nu} = a'_{1\nu}a_{1\nu}$. Then $f_{1\nu}C_{1\nu}f_{1\nu} \rightarrow f'_{1\nu}C_{1\nu}f'_{1\nu}$, $c_{1\nu} \mapsto a_{1\nu}c_{1\nu}$, and $f_1C_1f_1 \rightarrow f'_1C_1f'_1$, $c_1 \mapsto a_{11}a_{12}c_1$, are linear isomorphisms. Moreover, $\mu_{f'f}(a_{11}c_{11} \otimes a_{12}c_{12}) = a_{11}a_{12}\mu_{ff}(c_{11} \otimes c_{12})$ because C_{11} and C_{12} commute in C_1 . Hence $\mu_{f'f}$ is isomorphic if μ_{ff} is isomorphic.

further that $1 = \sum e_i \in B_1$ and $1 = \sum f_i \in C_1$ are maximal 1-decompositions satisfying $\varphi(e_1 \otimes 1) = f_1 \otimes 1$. Then the map

$$\pi_1\varphi(1 \otimes B_2) \longrightarrow \pi_1\varphi(e_1 \otimes B_2), \quad c_{12} \longmapsto f_1 c_{12} f_1,$$

is an isomorphism.

Proof: Again we can define a restriction $\tilde{\varphi} : e_1 B_1 e_1 \otimes B_2 \rightarrow f_1 C_1 f_1 \otimes C_2$ of φ . Now $\pi_1\varphi(1 \otimes B_2) = \text{im } \varphi_{12}$, while $\pi_1\varphi(e_1 \otimes B_2) = \text{im } \tilde{\varphi}_{12}$. Since $\tilde{\varphi}_{12}(b_2) = \pi_1\varphi(e_1 \otimes b_2) = f_1 \cdot \pi_1\varphi(1 \otimes b_2) \cdot f_1 = f_1 \cdot \varphi_{12}(b_2) \cdot f_1$, we have $\ker \varphi_{12} \subset \ker \tilde{\varphi}_{12}$ and obtain a surjective morphism $\text{im } \varphi_{12} \rightarrow \text{im } \tilde{\varphi}_{12}$ mapping $\varphi_{12}(b_2)$ to $\tilde{\varphi}_{12}(b_2) = f_1 \varphi_{12}(b_2) f_1$ as desired. The injectivity follows from the other inclusion of the kernels which, in turn, is a consequence of the following

Claim Take $b_2 \in B_2$, two non-isomorphic primitive idempotents $e', e \in B_1$ with $\varphi(e \otimes 1) = f \otimes 1$, $\varphi(e' \otimes 1) = f' \otimes 1$, and $\ell \in e' B_1 e \setminus \text{rad}^2 B_1$ or $\ell \in e B_1 e' \setminus \text{rad}^2 B_1$. If now $\pi_1\varphi(e' \otimes b_2) \neq 0$, then also $\pi_1\varphi(\ell \otimes b_2) \neq 0$.

Indeed, assume $b_2 \in \ker \tilde{\varphi}_{12} \setminus \ker \varphi_{12}$, or, explicitly, $\pi_1\varphi(e_1 \otimes b_2) = 0$ but $\pi_1\varphi(1 \otimes b_2) \neq 0$. Since the graph of B_1 is connected by Corollary 2.3(i) (B_1 is \oplus -indecomposable), we find idempotents e', e in the given 1-decomposition with $\pi_1\varphi(e' \otimes b_2) \neq 0$ and $\pi_1\varphi(e \otimes b_2) = 0$ such that the corresponding vertices in $\Delta(B_1)$ are adjacent. Note that e' and e cannot be isomorphic.⁸ Then $e' B_1^{(1)} e$ or $e B_1^{(1)} e'$ is non-trivial. (Note that $e' B_1^{(1)} e \simeq \text{Hom}_{B_1}^{(1)}(B_1 e', B_1 e)$.) In the first case, choose any $\ell \in e' B_1 e \setminus \text{rad}^2 B_1$. By the claim, $\pi_1\varphi(\ell \otimes b_2) \neq 0$. However, since $\ell = \ell e$, we have $\pi_1\varphi(\ell \otimes b_2) = \pi_1\varphi(\ell \otimes 1) \cdot \pi_1\varphi(e \otimes b_2) = 0$. This is a contradiction. The second case is analogous. Thus up to the claim we are done.

Now we prove the claim. Let $\varphi' : e' B_1 e' \otimes B_2 \rightarrow f' C_1 f' \otimes C_2$ be a restriction of φ , and define $B'_{ji} := \text{im } \varphi'^{-1}_{ji}$.

First we reduce our task to the case $b_2 \in B'_{21}$: Since the natural map $B'_{21} \otimes B'_{22} \rightarrow B_2$ is surjective, we can write $b_2 = b_{21} \cdot 1 + \sum_{\lambda} b_{21,\lambda} b_{22,\lambda}$ with $b_{21}, b_{21,\lambda} \in B'_{21}$ and $b_{22,\lambda} \in \text{rad } B'_{22}$ for all λ . (Observe that B'_{22} is local and $B'_{22}{}^{(0)} = k$ by Lemma 1.1.) By Theorem 3.2, the map $B'_{22} \rightarrow f' C_1 f' \otimes C_2 \rightarrow C_2$ is a section and therefore the map $B'_{22} \rightarrow f' C_1 f' \otimes C_2 \rightarrow f' C_1 f'$ is trivial by Separation Lemma 3.1. Hence $\pi_1\varphi(e' \otimes b_{22,\lambda}) = 0$. So $\pi_1\varphi(e' \otimes b_2) = \pi_1\varphi(e' \otimes b_{21})$ and $\pi_1\varphi(\ell \otimes b_2) = \pi_1\varphi(\ell \otimes b_{21})$. Now the restricted case yields the assertion.

So assume $b_{21} := b_2 \in B'_{21}$ in the following. Let $\nu(b_{21})$ be the natural number such that $b_{21} \in \text{rad}^{\nu(b_{21})} B_2 \setminus \text{rad}^{\nu(b_{21})+1} B_2$, and observe $\ell \in \text{rad}^1 B_1 \setminus \text{rad}^2 B_1$. We show

$$\varphi(\ell \otimes b_{21}) - \iota_1 \pi_1\varphi(\ell \otimes b_{21}) \in \text{rad}^{\nu(b_{21})+2}(C_1 \otimes C_2).$$

⁸Otherwise, $e' = aa'$, $e = a'a$ for some $a', a \in B_1$ and $\pi_1\varphi(e' \otimes b_2) = \pi_1\varphi(a \otimes 1)\pi_1\varphi(e \otimes b_2)\pi_1\varphi(a' \otimes 1) = 0$.

This finishes the proof: since $\ell \otimes b_{21} \notin \text{rad}^{\nu(b_{21})+2}(B_1 \otimes B_2)$, φ is an isomorphism, and any radical power contains zero, we obtain $\pi_1 \varphi(\ell \otimes b_{21}) \neq 0$.

We treat this difference in two steps:

$$\begin{aligned} d_1 &:= \varphi(e' \otimes b_{21} - \varphi^{-1} \iota_1 \pi_1 \varphi(e' \otimes b_{21})) \cdot \varphi(\ell \otimes 1) \in \text{rad}^{\nu(b_{21})+2}(C_1 \otimes C_2), \\ d_2 &:= \iota_1 \pi_1 \varphi(e' \otimes b_{21}) \cdot (\varphi(\ell \otimes 1) - \iota_1 \pi_1 \varphi(\ell \otimes 1)) \in \text{rad}^{\nu(b_{21})+2}(C_1 \otimes C_2). \end{aligned}$$

Then the above difference, which is equal to $d_1 + d_2$, is in the desired radical power.

For the first step, apply the following Distortion Lemma 4.7 to the isomorphism $\varphi' : e' B_1 e' \otimes B_2 \rightarrow f' C_1 f' \otimes C_2$. Since $b_{21} \in \text{rad}^{\nu(b_{21})} B_{21}$, the difference $\beta(b_{21}) - \gamma(b_{21}) = e' \otimes b_{21} - \varphi^{-1} \iota_1 \pi_1 \varphi(e' \otimes b_{21})$ is in $\text{rad}^{\nu(b_{21})+1}(e' B_1 e' \otimes B_2) \subset \text{rad}^{\nu(b_{21})+1}(B_1 \otimes B_2)$, where β and γ are defined as in Distortion Lemma 4.7. And φ of this difference is the first factor of d_1 .

For the second step, we only need $\varphi(\ell \otimes 1) - \iota_1 \pi_1 \varphi(\ell \otimes 1) \in \text{rad}^2(C_1 \otimes C_2)$. But this element is in the kernel of π_1 and in $f' C_1 f \otimes C_2$. Thus it is in $f' C_1 f \otimes \text{rad} C_2$, which is a subset of $\text{rad}^2(C_1 \otimes C_2)$, for $f' C_1 f \subset \text{rad} C_1$, since f' and f are non-isomorphic. \square

4.7 Distortion Lemma *Let $\varphi : B_1 \otimes B_2 \rightarrow C_1 \otimes C_2$ be an isomorphism of local algebras. Then B_{21} and $\varphi^{-1} C_{12}$ are almost equally embedded in $B := B_1 \otimes B_2$. Precisely this means: Define the two embeddings*

$$\begin{aligned} \beta : B_{21} &\longrightarrow B, & b &\longmapsto 1 \otimes b, \\ \gamma : B_{21} \simeq C_{12} &\longrightarrow B, & b &\longmapsto \varphi^{-1} \iota_1 \varphi_{12}(b). \end{aligned}$$

For $\nu \in \mathbb{N}$ and an element $b \in \text{rad}^\nu B_{21}$ we then have $\beta(b) - \gamma(b) \in \text{rad}^{\nu+1} B$.

Proof: By Theorem 3.2(ii), we can write $B = B_1 \otimes B_{22} \otimes B_{21}$ and obtain the projections $\varrho_1 : B \rightarrow B_1 \otimes B_{22}$ and $\varrho_2 : B \rightarrow B_{21}$. Clearly $\varrho_2 \beta = \mathbb{1}_{B_{21}}$. Even $\varrho_2 \gamma = \mathbb{1}_{B_{21}}$. Indeed, $\varrho_2 = \pi_{21} \pi_2$ using the projection $\pi_{21} : B_2 = B_{22} \otimes B_{21} \rightarrow B_{21}$. With Theorem 3.2(i), we calculate $\varrho_2 \gamma(b) = \pi_{21} \varphi_{21}^{-1} \varphi_{12}(b) = \pi_{21}(b) = b$ for $b \in B_{21}$.

We proceed by induction on ν . First, for any $b \in B_{21}$ we have $\beta(b) - \gamma(b) \in \ker \varrho_2$, completing the case $\nu = 0$. If $b \in \text{rad} B_{21}$, we have $\varrho_1 \beta(b) = 0$ and $\varrho_1 \gamma(b) = 0$. For the latter use Separation Lemma 3.1 to see that $\varrho_1 \gamma$ is trivial. Thus the difference $\beta(b) - \gamma(b)$ is in the intersection of the kernels of ϱ_1 and ϱ_2 , which in turn is in the square of the radical. Thus the case $\nu = 1$ is proved as well.

Now suppose $\nu > 1$ and take $b \in \text{rad}^\nu B_{21}$. We may assume $b = b' b''$ with $b' \in \text{rad} B_{21}$ and $b'' \in \text{rad}^{\nu-1} B_{21}$. We obtain $\beta(b) - \gamma(b) = (\beta(b') - \gamma(b')) \beta(b'') + \gamma(b') (\beta(b'') - \gamma(b'')) \in \text{rad}^{\nu+1} B$, since by induction hypothesis the terms in parentheses are in a higher radical power than the corresponding element. \square

5 Uniqueness

As in the previous section, we always start with \oplus -indecomposable algebras.

5.1 Lemma *Let $\varphi : B_1 \otimes B_2 \rightarrow C_1 \otimes C_2$ be an isomorphism.*

- (i) *Suppose that B_1 is \otimes -indecomposable and $\Delta(B_1)$ is not a point. Then either every $\Delta(B_1)$ -slice is mapped into a $\Delta(C_1)$ -slice or every $\Delta(B_1)$ -slice is mapped into a $\Delta(C_2)$ -slice.*
- (ii) *Assume that the first case of (i) applies and that C_1 is also \otimes -indecomposable. Then $\Delta(\varphi) = \delta_1 \times \delta_2$, where $\delta_i : \Delta(B_i) \rightarrow \Delta(C_i)$. Moreover, $\text{mult}(B_i) = \text{mult}(C_i) \circ \delta_i$.*
- (iii) *If $\Delta(\varphi) = \delta_1 \times \delta_2$, where $\delta_i : \Delta(B_i) \rightarrow \Delta(C_i)$, and $\text{mult}(B_i) = \text{mult}(C_i) \circ \delta_i$, then there are isomorphisms $\varphi_i : B_i \rightarrow C_i$ such that $\Delta(\varphi_i) = \delta_i$ for $i \in \{1, 2\}$.*

Proof:

(i) We obtain an isomorphism $\Delta(\varphi) : \Delta(B_1) \times \Delta(B_2) \rightarrow \Delta(C_1) \times \Delta(C_2)$ using Graph Lemma 2.2(iii). By Cartesian Product Lemma 2.1(i), the slice $\Delta(B_1) \times \beta_2$ is mapped onto a rectangle $\Lambda_1 \times \Lambda_2$. We show that either Λ_1 or Λ_2 is just a point. Applying Transform Lemma 2.5(iii), we get $B_1 \otimes (B_2 \downarrow \beta_2) \simeq (C_1 \downarrow \Lambda_1) \otimes (C_2 \downarrow \Lambda_2)$. Since $\Delta(B_2 \downarrow \beta_2)$ is a point, the value of $\text{mult}(B_2 \downarrow \beta_2)$ divides the values of $\text{mult}(C_1 \downarrow \Lambda_1) \cdot \text{mult}(C_2 \downarrow \Lambda_2)$. By Lemma 2.7, we can write $\text{mult}(B_2 \downarrow \beta_2) = n_1 \cdot n_2$ such that n_j divides all values of $\text{mult}(C_j \downarrow \Lambda_j)$. Let $m_j := \frac{1}{n_j} \text{mult}(C_j \downarrow \Lambda_j)$. Using Transform Lemma 2.5(iii) and (iv), then $B_1 \otimes M(B_2 \downarrow \beta_2, 1) \simeq M(C_1 \downarrow \Lambda_1, m_1) \otimes M(C_2 \downarrow \Lambda_2, m_2)$. Now Corollary 4.2 tells us that one of the factors on the right is local, say $M(C_2 \downarrow \Lambda_2, m_2)$. So its graph is a point. Using Transform Lemma 2.5(i) twice, this graph is Λ_2 . And so the rectangle is contained in a slice. Consequently, each $\Delta(B_1)$ -slice is mapped into a $\Delta(C_1)$ -slice or into a $\Delta(C_2)$ -slice. Cartesian Product Lemma 2.1(ii) completes the first part.

(ii) Since every $\Delta(B_1)$ -slice is mapped into a $\Delta(C_1)$ -slice, $\Delta(C_1)$ cannot be a point and thus, by the first part, all $\Delta(C_1)$ -slices are mapped into $\Delta(B_1)$ - or $\Delta(B_2)$ -slices. The latter case cannot occur, since $\Delta(\varphi)$ is bijective and $\Delta(B_1)$ is not a point. By Cartesian Product Lemma 2.1(iii), $\Delta(\varphi)$ is a product as stated.

By Corollary 2.6(i), $\text{mult}(B_1)$ and $\text{mult}(C_1)$ are primitive. Since $\Delta(\varphi) = \delta_1 \times \delta_2$, we obtain $\text{mult}(B_1) \cdot \text{mult}(B_2) = (\text{mult}(C_1)\delta_1) \cdot (\text{mult}(C_2)\delta_2)$. Since the left factors are primitive, the right ones coincide, and so do the left ones.

(iii) Choose any point $(\beta_1, \beta_2) \in \Delta(B_1) \times \Delta(B_2)$ and set $(\gamma_1, \gamma_2) := (\delta_1\beta_1, \delta_2\beta_2)$. Restricting $\Delta(\varphi)$ then yields graph isomorphisms

$$\begin{aligned} \delta_1 \times \varepsilon : \Delta(B_1) \times \beta_2 &\longrightarrow \Delta(C_1) \times \gamma_2, \\ \varepsilon \times \delta_2 : \beta_1 \times \Delta(B_2) &\longrightarrow \gamma_1 \times \Delta(C_2). \end{aligned}$$

These are induced by algebra isomorphisms

$$\begin{aligned}\psi_1 &: B_1 \otimes M(B_2 \downarrow \beta_2, 1) \longrightarrow C_1 \otimes M(C_2 \downarrow \gamma_2, 1), \\ \psi_2 &: M(B_1 \downarrow \beta_1, 1) \otimes B_2 \longrightarrow M(C_1 \downarrow \gamma_1, 1) \otimes C_2.\end{aligned}$$

Indeed, by Transform Lemma 2.5, the graph isomorphism $\delta_1 \times \varepsilon$ is induced by an isomorphism $B_1 \otimes (B_2 \downarrow \beta_2) \rightarrow C_1 \otimes (C_2 \downarrow \gamma_2)$. By assumption, $\text{mult}(B_2 \downarrow \beta_2)$ and $\text{mult}(C_2 \downarrow \gamma_2)$ have the same value. Using Transform Lemma 2.5(iii) and (iv) yields the desired isomorphism ψ_1 with $\Delta(\psi_1) = \delta_1 \times \varepsilon$ by Transform Lemma 2.5(i). Similarly we obtain ψ_2 .

By Corollary 4.3, ψ_1 gives us an isomorphism $\varphi_1 : B_1 \rightarrow C_1$ with $\Delta(\varphi_1) = \delta_1$. Since $\gamma_1 = \Delta(\varphi_1)\beta_1$, the first factors in ψ_2 are isomorphic: $M(B_1 \downarrow \beta_1, 1) \simeq M(C_1 \downarrow \gamma_1, 1)$. Hence we can apply Corollary 4.4 to obtain an isomorphism $\varphi_2 : B_2 \rightarrow C_2$ with $\Delta(\varphi_2) = \delta_2$. \square

5.2 Lemma *Let $\varphi : B_1 \otimes B' \rightarrow C_1 \otimes \cdots \otimes C_s$ be an isomorphism and suppose that B_1, C_1, \dots, C_s are \otimes -indecomposable and $\Delta(B_1)$ is not a point. Then there are an index j and isomorphisms*

$$\begin{aligned}\varphi_1 &: B_1 \longrightarrow C_j, \\ \varphi' &: B' \longrightarrow C' := C_1 \otimes \cdots \otimes C_{j-1} \otimes C_{j+1} \otimes \cdots \otimes C_s,\end{aligned}$$

such that

$$\Delta(\varphi) = \Delta(\zeta \circ (\varphi_1 \otimes \varphi')),$$

where $\zeta : C_j \otimes C' \rightarrow C_1 \otimes \cdots \otimes C_s$ is defined by $\zeta(c_j \otimes c_1 \otimes \cdots \otimes c_{j-1} \otimes c_{j+1} \otimes \cdots \otimes c_s) = c_1 \otimes \cdots \otimes c_s$.

Proof: First we show that there exists an index j such that $\Delta(\varphi)$ maps every $\Delta(B_1)$ -slice into a $\Delta(C_j)$ -slice of $\Delta(C_1) \times \cdots \times \Delta(C_s)$. Otherwise, by Lemma 5.1(i) the slice $\Delta(B_1) \times \beta'$ is mapped for any j into a slice $\Delta(C_1) \times \cdots \times \Delta(C_{j-1}) \times \gamma_j \times \Delta(C_{j+1}) \times \cdots \times \Delta(C_s)$. Therefore $\Delta(B_1) \times \beta'$ is mapped to the point $(\gamma_1, \dots, \gamma_s)$. But $\Delta(\varphi)$ is bijective, hence $\Delta(B_1)$ is a point, which contradicts the assumption.

We may assume that $j = 1$. (If necessary, replace φ by $\zeta^{-1} \circ \varphi$.) Then we have the following situation: $\varphi : B_1 \otimes B' \rightarrow C_1 \otimes C'$ is an isomorphism, B_1, C_1 are \otimes -indecomposable, and $\Delta(B_1)$ is not a point. Moreover, $\Delta(\varphi)$ maps $\Delta(B_1)$ -slices into $\Delta(C_1)$ -slices, hence by Lemma 5.1(ii) we have $\Delta(\varphi) = \delta_1 \times \delta'$ with $\delta_1 : \Delta(B_1) \rightarrow \Delta(C_1)$ and $\delta' : \Delta(B') \rightarrow \Delta(C')$ and appropriate multiplicities. Now Lemma 5.1(iii) gives isomorphisms $\varphi_1 : B_1 \rightarrow C_1$ and $\varphi' : B' \rightarrow C'$ such that $\Delta(\varphi) = \delta_1 \times \delta' = \Delta(\varphi_1) \times \Delta(\varphi') = \Delta(\varphi_1 \otimes \varphi')$, using Graph Lemma 2.2(iv) for the last equality. This is the assertion of the present lemma in case $j = 1$. \square

Using this and Lemma 3.3 inductively we obtain the following Theorem.

5.3 Theorem *Let B_1, \dots, B_r and C_1, \dots, C_s be \otimes -indecomposable algebras and let $\varphi : B_1 \otimes \dots \otimes B_r \rightarrow C_1 \otimes \dots \otimes C_s$ be an isomorphism. Then there are a bijection $\sigma : \{1, \dots, r\} \rightarrow \{1, \dots, s\}$ and isomorphisms $\psi_i : B_i \rightarrow C_{\sigma i}$. Moreover, it can be assured that $\Delta(\varphi) = \Delta(\zeta \circ (\psi_1 \otimes \dots \otimes \psi_r))$, where the algebra isomorphism $\zeta : C_{\sigma 1} \otimes \dots \otimes C_{\sigma r} \rightarrow C_1 \otimes \dots \otimes C_s$ is defined by $\zeta(c_{\sigma 1} \otimes \dots \otimes c_{\sigma r}) := c_1 \otimes \dots \otimes c_s$.*

Proof: We prove this by induction on r . The case $r = 0$ being evident, we suppose $r > 0$. For convenience, sort the B_i by descending graph size and by descending multiplicities, where the graphs are just points. We claim that for some j there are isomorphisms $\psi_1 : B_1 \rightarrow C_j$ and $\varphi' : B_2 \otimes \dots \otimes B_r \rightarrow C_1 \otimes \dots \otimes C_{j-1} \otimes C_{j+1} \otimes \dots \otimes C_s$ such that $\Delta(\varphi) = \Delta(\zeta_1 \circ (\psi_1 \otimes \varphi'))$ with ζ_1 as in Lemma 5.2. If $\Delta(B_1)$ is not a point, this is the statement of Lemma 5.2. Otherwise all graphs $\Delta(B_i)$ are points; in particular, the statement about the graph maps is trivial. If $\text{mult}(B_1)$ is not 1, then, by Corollary 2.6(i), $B_1 \simeq k^{p \times p}$ for some prime p . By Lemma 2.7, there is an index j such that $\text{mult}(C_j)$ is divisible by p . Using Corollary 2.6(i) again, $C_j \simeq k^{p \times p}$ too, and we therefore have an isomorphism $\psi_1 : B_1 \rightarrow C_j$. But then we have $\varphi' : B_2 \otimes \dots \otimes B_r \simeq M(B_1, 1) \otimes B_2 \otimes \dots \otimes B_r \simeq C_1 \otimes \dots \otimes C_{j-1} \otimes M(C_j, 1) \otimes C_{j+1} \otimes \dots \otimes C_s \simeq C_1 \otimes \dots \otimes C_{j-1} \otimes C_{j+1} \otimes \dots \otimes C_s$. In the remaining case all graphs and multiplicities are trivial; in other words, the algebras are local. So Lemma 3.3 completes the claim.

By induction hypothesis, we get a bijection $\sigma' : \{2, \dots, r\} \rightarrow \{1, \dots, j-1, j+1, \dots, s\}$, isomorphisms $\psi_i : B_i \rightarrow C_{\sigma' i}$, and a permuting morphism ζ' such that $\Delta(\varphi') = \Delta(\zeta' \circ (\psi_2 \otimes \dots \otimes \psi_r))$. Extend σ' to a bijection $\sigma : \{1, \dots, r\} \rightarrow \{1, \dots, s\}$ by setting $\sigma(1) = j$, and define $\zeta := \zeta_1 \circ (\mathbb{1}_{C_1} \otimes \zeta')$. This completes the proof. \square

In particular, we obtain the following structural version of the unique factorization property.

5.4 Unique Factorization Theorem *The set \mathcal{M} of isomorphism classes of \oplus -indecomposable algebras is a free commutative monoid over the set \mathcal{X} of isomorphism classes of $\{\oplus, \otimes\}$ -indecomposable algebras as a basis.*

Proof: By Corollary 2.3(ii), the set \mathcal{M} together with the multiplication induced by the tensor product is a commutative monoid. By Theorem 5.3, we can define an isomorphism from \mathcal{M} to the free commutative monoid with basis \mathcal{X} . \square

5.5 Corollary *The semiring \mathcal{U} of isomorphism classes of algebras is the positive cone $\mathbb{N}[\mathcal{X}]$ of the polynomial ring $\mathbb{Z}[\mathcal{X}]$.*

Proof: Since the monoid \mathcal{M} is free over \mathcal{X} , it suffices to show that the additive monoid of \mathcal{U} is free over \mathcal{M} . That is the additive decomposition of algebras should be unique. But additive decomposition of an algebra A is equivalent to decomposing it as an $A^{\text{op}} \otimes A$ -module. Such a decomposition is unique by the KRULL-SCHMIDT Theorem. \square

Of course, $\mathbb{Z}[\mathcal{X}]$ is a factorial ring by GAUSS' theorem. This does not mean, however, that its positive cone $\mathbb{N}[\mathcal{X}]$ is factorial, see Example A.1.

It seems to be a difficult task to describe all $\{\oplus, \otimes\}$ -indecomposable algebras in a constructive way. Yet we can list some of them.

To do so we use the GABRIEL quiver $Q(A)$ of an algebra A and the algebra kQ of a quiver Q . (See p.46 for definitions and references.) In fact, the graph $\Delta(A)$ is obtained from the quiver $Q(A)$ by omitting directions, multiplicities, and loops. In contrast to the graph, the quiver allows to identify a trivial algebra: If the quiver $Q(A)$ of an algebra A is a point and the multiplicity $\text{mult}(A)$ is 1, then $A = k$. This is due to the fact that for any SCHURian algebra A there is a surjective morphism $M(kQ(A), \text{mult}(A)) \rightarrow A$.

Thus if A is a non-trivial algebra A whose quiver $Q(A)$ is connected and \times -indecomposable and whose multiplicity $\text{mult}(A)$ is primitive, then $[A] \in \mathcal{X}$. Similarly, $[k^{p \times p}] \in \mathcal{X}$ for p prime. Moreover, $[kQ] \in \mathcal{X}$ for any non-trivial, connected, cycle-free quiver Q . (Note that $k(Q_1 \times Q_2) \rightarrow kQ_1 \otimes kQ_2$ is usually not injective, while the quivers of these two algebras are equal.)

6 Corollaries

Knowing that the monoid \mathcal{M} is free, we will now examine several submonoids for freeness. Some remarks on commutative monoids will help us.

For the next few paragraphs let \mathcal{F} denote a free commutative monoid. First, note that ‘the’ free commutative monoid over a set B can be constructed as the set $F(B)$ of all maps $B \rightarrow \mathbb{N}$ with finite support endowed with point-wise addition. Of course, not every submonoid of a free monoid is free, for example, $\mathbb{N} \setminus \{1\}$ is a non-free submonoid of the free monoid $(\mathbb{N}, +)$. There is an easy way to derive freeness.

6.1 Lemma *Let \mathcal{F} be a free commutative monoid and $\iota : \mathcal{E} \rightarrow \mathcal{F}$ a split monoid embedding, ie. there is a retraction $\varrho : \mathcal{F} \rightarrow \mathcal{E}$ with $\varrho\iota = \mathbb{1}_{\mathcal{E}}$. Then \mathcal{E} is free.*

Proof: An element e in \mathcal{E} has a maximal decomposition in \mathcal{E} , since the number of factors is delimited by the number of factors of the maximal decomposition of e in \mathcal{F} .

Suppose $e = \prod_{\nu} e_{i\nu}$ is a maximal decomposition in \mathcal{E} for $i \in \{1, 2\}$. Decompose $\iota(e_{i\nu}) = \prod_{\mu} f_{i\nu\mu}$ maximally in \mathcal{F} . Since $e_{i\nu} = \prod_{\mu} \varrho(f_{i\nu\mu})$ is indecomposable, $\varrho(f_{i\nu\mu}) \neq 1$ for exactly one μ . But in $\prod_{\nu, \mu} f_{1\nu\mu} = \prod_{\nu, \mu} f_{2\nu\mu}$, the factors on the left-hand side are just a permutation of the factors on the right-hand side. Hence the decompositions of e coincide up to order. \square

Unfortunately, the intersection of two free submonoids of a free monoid needs not be a free monoid. Call a submonoid $\mathcal{S} \subset \mathcal{F}$ *saturated* iff it is closed with respect to taking factors: For all $f_i \in \mathcal{F}$ with $f_1 f_2 \in \mathcal{S}$, the factors f_i are in \mathcal{S} . Equivalently, all indecomposable elements in \mathcal{S} are still indecomposable in \mathcal{F} . In other words, \mathcal{S} is a submonoid of \mathcal{F} generated by part of a basis of \mathcal{F} . Clearly, a saturated submonoid has a retraction to the inclusion, and thus by Lemma 6.1 it is free. Even more is true: Suppose $\mathcal{E} \subset \mathcal{F}$ is a free submonoid, and $\mathcal{S} \subset \mathcal{F}$ is saturated. Then saturatedness is inherited when intersecting with \mathcal{E} : $\mathcal{S} \cap \mathcal{E} \subset \mathcal{E}$ is saturated. Loosely speaking: if a property (represented by \mathcal{S}) is stable with respect to taking factors then we can add it without interfering with freeness. Referring to the monoid \mathcal{M} , commutativity is such a property, see below.

... On Numbers

6.2 Fundamental Theorem of Arithmetics *The monoid of positive naturals with multiplication is free.*

Proof: The map $\iota : \mathbb{N}' \rightarrow \mathcal{M}$, $n \mapsto [k^{n \times n}]$, is an injective monoid morphism. The map $\varrho : \mathcal{M} \rightarrow \mathbb{N}'$, $[A] \mapsto \sum \text{mult}(A)$, is a retraction to ι . Hence \mathbb{N}' is free. \square

In fact, the naturals are even saturated in \mathcal{M} . (Use graphs and multiplicities to see that.)

... On Graphs

On the set of isomorphism classes of quivers we define a *cartesian product* analogous to the one for graphs. (A *quiver* is a directed graph possibly with loops and multiplicities. It is *connected* if its underlying graph (omit directions) is connected. And it is *cycle-free* if there is no directed path returning to its origin.)

Given a quiver Q , we define the quiver algebra kQ as follows: Consider the set $Q_{\geq 0}$ of paths in Q (of length at least 0), and note that, for each vertex of Q , there is a ‘lazy’ path using no edge at all. We endow the vector space kQ over the basis $Q_{\geq 0}$ with the multiplication induced by the composition of paths. Specifically, for $w', w \in Q_{\geq 0}$, let $w' \cdot w$ be the concatenation of w' and w if w' starts where w ends, and otherwise let $w' \cdot w$ be 0. (Compare the definition of $K(\mathcal{D})$ on page 63 of DROZD & KIRICHENKO (1994).)

Conversely, we can define the GABRIEL *quiver* $Q(A)$ of an algebra A . (This quiver is also known as the *ordinary quiver* or as the *diagram* of an algebra.) To do so, take the set of vertices of $Q(A)$ to be the set of isomorphism classes of indecomposable projectives, ie. the vertex set of $\Delta(A)$, and put $\dim \text{Hom}_A^{(1)}(P, P')$ arrows from $[P]$ to $[P']$. Like the graph, the quiver behaves well with respect to sum and product of algebras: sum corresponds to disjoint union, product to the cartesian product. Furthermore, for any SCHURian algebra there is a surjection $kQ(A) \rightarrow M(A, 1)$ whose kernel is an admissible ideal in $kQ(A)$, ie. it is generated by combinations of paths of length at least 2, and it contains all paths of some length N . Finally, $Q(kQ/I) = Q$ provided I is an admissible ideal in kQ . (See DROZD & KIRICHENKO (1994), Theorem 3.6.6.)

6.3 Corollary *The monoid of isomorphism classes of finite, connected, cycle-free quivers is free.*

Proof: By a *square* in a quiver Q we mean a tuple of four arrows $\ell_i \in Q$ such that $\ell_1\ell_2$ and $\ell_3\ell_4$ are paths with the same head and tail vertices. Given a cycle-free quiver Q , define

$$\begin{aligned} I(Q) &:= \langle \{\ell_1\ell_2 - \ell_3\ell_4 \mid (\ell_i) \text{ is a square in } Q\} \rangle \subset kQ, \\ A(Q) &:= kQ/I(Q). \end{aligned}$$

Then the map $Q \mapsto A(Q)$ induces a monoid embedding. Clearly $Q' \simeq Q$ implies $A(Q') \simeq A(Q)$. If Q is a point without arrows, then $A(Q) = k$.

Now suppose $Q = Q_1 \times Q_2$. In order to show $A(Q) \simeq A(Q_1) \otimes A(Q_2)$, observe that the left-hand side is the quotient of kQ by $I(Q)$ while the right-hand side is isomorphic to the quotient of kQ by the preimage J of $I(Q_1) \otimes kQ_2 + kQ_1 \otimes I(Q_2)$ under the obvious map $kQ \rightarrow kQ_1 \otimes kQ_2$. (Map a vertex (α_1, α_2) to $\alpha_1 \otimes \alpha_2$, an arrow (α_1, ℓ_2) to $\alpha_1 \otimes \ell_2$ and an arrow (ℓ_1, α_2) to $\ell_1 \otimes \alpha_2$. Since the vertices and the arrows generate kQ as an algebra this defines a unique morphism.) The kernel of this map is generated by the squares that are products of an arrow of

each Q_i . Thus J is generated by all those squares that are either in a Q_i -slice or a product of an arrow of each factor. $I(Q)$, on the other hand, is generated by all squares in Q . Thus it remains to see that any square in Q is either in a Q_1 - or in a Q_2 -slice, or it is a product of two arrows, one from each factor. This is easy to verify.

For every quiver Q the quiver of $A(Q)$ is Q . (Observe that $I(Q) \subset kQ_{\geq 2}$ and $kQ_{\geq n} = 0$ for $n = \#Q$.) By Lemma 6.1 this proves the assertion. \square

This proof can be adapted to arbitrary connected quivers by enlarging the ideal $I(Q)$. Two arbitrary paths are *equal modulo squares* if it is possible to produce the second from the first by exchanging subpaths l_1l_2 with l_3l_4 from a square (l_i) . We call a cycle c in Q *irreducible modulo squares* if no path that is equal to c modulo squares has a proper cyclic subpath. For each irreducible cycle c in Q add c to $I(Q)$ if c is not a loop, otherwise add c^2 . This works because an irreducible cycle in $Q = Q_1 \times Q_2$ always is in a slice. Since we want $Q(A(Q)) = Q$, in case of a loop we need to take c^2 instead of c . Moreover, any path whose length exceeds the number of arrows is in the ideal. So we have:

6.4 Corollary (MARYLAND (1978), FEIGENBAUM (1986)) *The monoid of isomorphism classes of finite, connected quivers is free.* \square

Let's turn back to one of our starting points. We have considered quivers, but what about graphs? Not surprisingly we have:

6.5 Corollary (SABIDUSSI (1960)) *The monoid of isomorphism classes of finite, connected, loop-free graphs is free.*

To proof this map an undirected graph to the quiver that consists of a forth and a back arrow for each edge in the graph. Then by Corollary 6.4 we get the result.

... On Algebras

The next result is due to POWER (1990). To a partial pre-order⁹ R on the set $\{1, \dots, n\}$, we associate the *incidence algebra* $A(R)$: the elements of $A(R)$ are all $n \times n$ -matrices a over k with $a_{\beta\alpha} \neq 0$ only if $\beta R\alpha$. If R is connected and anti-symmetric, ie. a connected partial order, $A(R)$ is called a (connected) *triangular matrix algebra*. (The relation should be reflexive and transitive to assure that $A(R)$ is an algebra. Connectedness implies that $A(R)$ is \oplus -indecomposable. Due to antisymmetry, $A(R)$ is basic.)

6.6 Corollary (POWER (1990), Theorem 3.1) *The monoid of isomorphism classes of triangular matrix algebras (over k) is free.*

⁹A partial pre-order is a reflexive and transitive binary relation

Proof: Consider an algebra $A := A(R)$ and for $\beta R \alpha$ let $e^{\beta\alpha} \in A$ denote the matrix with 1 at (β, α) and 0 elsewhere. The quiver $Q := Q(A)$ perfectly reflects the relation R : We identify the vertex $[Ae^{\alpha\alpha}]$ of the quiver with $\alpha \in \{1, \dots, n\}$. Two vertices in Q are connected by a directed path iff they are in relation R . Thus, from $A(R)$ we can recover the relation. For an arbitrary algebra A , we define $R(A)$ by: $\beta R(A)\alpha$ iff there is a path from α to β in $Q(A)$. Then $A \mapsto R(A) \mapsto A(R(A))$ induces a retraction to the natural inclusion. Indeed, we can define a product of relations: $(\beta_1, \beta_2)(R_1 \times R_2)(\alpha_1, \alpha_2)$ iff for both i we have $\beta_i R_i \alpha_i$. Now the maps $A \mapsto R(A)$ and $R \mapsto A(R)$ induce monoid morphisms. \square

As we have just seen, the map $R \mapsto A(R)$ induces a split monoid embedding. In fact, we do not need antisymmetry of R .

6.7 Corollary (CHANG, JONSSON & TARSKI (1964)) *The monoid of isomorphism classes of partial pre-orders is free.* \square

For a certain class of graded algebras, we also have a split embedding. Call an algebra *naturally graded* iff $A \simeq A^{(*)}$.

6.8 Corollary *The monoid of isomorphism classes of naturally graded, \oplus -indecomposable, SCHURian algebras is free.* \square

Finally, we turn our attention to some ‘saturated’ properties of algebras: each of the submonoids of isomorphism classes of SCHURian algebras with one or more of the following properties is saturated in \mathcal{M} . Thus each combination of them leads to a new free monoid.

- Commutative,
- one-point ($\Delta(A)$ is a point),
- basic ($\text{mult}(A) \equiv 1$),
- cycle-free ($Q(A)$ has no cycles),
- loop-free ($Q(A)$ has no loops),
- cycle-resistant ($Q(A)$ has a point which is in no cycle),
- loop-resistant ($Q(A)$ has a point without a loop),
- local-free (A has no local tensor factor),
- local (one-point and basic) [HORST (1987)],
- ‘relational’ (basic and cycle-free).

We can also start with the monoids of Corollary 6.6 or Corollary 6.8, and add one or several of these properties. You may notice that this list can be easily extended by imposing other suitable conditions on the quiver $Q(A)$ and the multiplicity $\text{mult}(A)$ or on the algebra itself.

A Frontiers (Examples)

This section collects some limiting results and examples. We examine the necessity of various conditions used before.

An \oplus -indecomposable, SCHURian algebra has a unique factorization. However, an arbitrary SCHURian algebra may still have different decompositions.

A.1 Example $\mathbb{N}[\mathcal{X}]$ is not factorial.

Proof: Simply examine $(X + 1)(X^2 - X + 2)(X + 2)$ for $X \in \mathcal{X}$. This yields $(X^3 + X + 2)(X + 2) = X^4 + 2X^3 + X^2 + 4X + 4 = (X + 1)(X^3 + X^2 + 4)$. The smallest algebra obtained this way has dimension 48 using $X = [k[T]/T^2]$. NAKAYAMA & HASHIMOTO (1950) use $X^5 + X^4 + X^3 + X^2 + X + 1 = (X + 1)(X^2 + X + 1)(X^2 - X + 1)$ for a similar purpose. \square

The Unique Factorization Theorem 5.4 should not be misunderstood: it does not mean that an \oplus -indecomposable, SCHURian algebra has a unique factorization in the class of all k -algebras.

A.2 Example Suppose that A is a finite-dimensional, central, simple k -algebra and $n = \prod p_i$ is the prime factor decomposition of its dimension. Then we have

$$A^{\text{op}} \otimes A \simeq k^{n \times n} \simeq \bigotimes k^{p_i \times p_i}.$$

To be specific, take the real quaternion algebra $A = \mathbb{H}$ with $k = \mathbb{R}$ and $n = 2 \cdot 2$.

In our reasoning, the condition ‘SCHURian’ guarantees that the tensor product of two \oplus -indecomposable algebras is again \oplus -indecomposable (Corollary 2.3(ii)). Yet this cannot be expected in general.

A.3 Example We have $\mathbb{C} \otimes \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C} \simeq \mathbb{C} \otimes (\mathbb{R} \oplus \mathbb{R})$ as \mathbb{R} -algebras.

Proof: Write $\mathbb{C} = \mathbb{R}[T]/\langle T^2 + 1 \rangle$ for the second factor on the left to obtain $\mathbb{C} \otimes \mathbb{C} = \mathbb{C}[T]/\langle T^2 + 1 \rangle = \mathbb{C} \oplus \mathbb{C}$ using the Chinese Remainder Theorem. \square

The last example shows that we cannot completely omit ‘SCHURian’. (As an \mathbb{R} -algebra \mathbb{C} is not SCHURian.) In fact, if we want our algebras to behave well in connection with graphs and multiplicities, this condition is even necessary.

A.4 Frontier Suppose D is a finite-dimensional division algebra over a field k such that $D \otimes D$ is a division algebra, too. Then $D = k$.

Proof: Otherwise there is an $x \in D \setminus k$. Now $L := k[x]$ is a non-trivial subfield of D . Since $L \otimes L$ is contained in the division algebra $D \otimes D$, it is a field. Thus the algebra morphism $L \otimes L \rightarrow L$ is injective. (It cannot be trivial, since $1 \otimes 1 \mapsto 1$.) Thus $\dim_k L = 1$, which contradicts $L \neq k$. \square

Take an algebra A which behaves well in connection with graphs and multiplicities, ie. we have at least $\Delta(A \otimes A) = \Delta(A) \times \Delta(A)$ and $\text{mult}(A \otimes A) = \text{mult}(A) \cdot \text{mult}(A)$. For an indecomposable projective P of A , $P \otimes P$ now has to be indecomposable. Thus its endomorphism algebra B is local, and the residue algebra D is a division algebra. Furthermore, the residue algebra of $P \otimes P$'s endomorphism algebra $B \otimes B$ is $D \otimes D$ by Lemma 1.3(iii). Since $P \otimes P$ is indecomposable, $B \otimes B$ is local, hence $D \otimes D$ is a division algebra. But then $D = k$. Thus A is SCHURIAN.

The presented proofs of Separation Lemma 3.1, Theorem 3.2, or Proposition 4.1 all use characteristic zero. This is inevitable:

A.5 Example *Let k be any field of non-zero characteristic p . Take an element $\gamma \in k \setminus \{0, 1\}$, let $B_i = C_i = k[T]/T^p$ for $i \in \{1, 2\}$, and denote by t the class of T . Then there is an isomorphism $\varphi : B_1 \otimes B_2 \rightarrow C_1 \otimes C_2$ mapping $t \otimes 1 \mapsto t \otimes 1 + 1 \otimes t$ and $1 \otimes t \mapsto t \otimes 1 + \gamma \otimes t$. Now all φ_{ij} are isomorphisms, and so $C_{ij} = C_i$.*

This shows that Theorem 3.2 and Proposition 4.1 do not hold in finite characteristic. And Separation Lemma 3.1 fails, too: the morphism $\psi := \varphi \iota_1$ is a morphism of local algebras with $\pi_1 \psi = \varphi_{11}$ being horizontal, even isomorphic, but $\pi_2 \psi = \varphi_{21}$ is isomorphic, too, and is therefore not trivial.

Proof (Example A.5): Due to the characteristic φ is well defined. Indeed, since the algebras are commutative, we only need to verify $(t \otimes 1 + \gamma \otimes t)^p = 0$.

To prove that φ is isomorphic, we show that it is surjective. By Surjectivity Criterion 1.2, this follows from the surjectivity of $\varphi^{(1)}$. Since $\gamma \neq 1$, $\varphi^{(1)}((t \otimes 1)^{(1)}) = (t \otimes 1)^{(1)} + (1 \otimes t)^{(1)}$, and $\varphi^{(1)}((t \otimes 1)^{(1)}) = (t \otimes 1)^{(1)} + \gamma(1 \otimes t)^{(1)}$ are linearly independent. And $(C_1 \otimes C_2)^{(1)}$ is two-dimensional.

Since $\varphi_{ij}(t) = t$ for $(i, j) \neq (2, 2)$ and $\varphi_{22}(t) = \gamma t$ with $\gamma \neq 0$, all φ_{ij} are isomorphisms. \square

Dealing with \mathbb{C} -algebras one might hope that it is possible to generalize the unique factorization property to algebras of arbitrary dimension, maybe subject to a delimiting condition like ‘NOETHERIAN’. Any trial to generalize Unique Factorization Theorem 5.4 or Unique Factorization for Local Algebras 3.4 to algebras of arbitrary dimension has to bypass the following frontier:

A.6 Example *There are commutative, NOETHERIAN, finitely generated, reduced \mathbb{C} -algebras A_0, A_1, A_2 such that*

$$\begin{aligned} A_0 \otimes A_1 &\simeq A_0 \otimes A_2, \\ A_1 &\not\simeq A_2. \end{aligned}$$

This statement is strongly connected with the ZARISKI cancellation problem. It is known by examples of DANIELEWSKI (not published) and FIESELER (1994) that for affine varieties $X \times Z \simeq Y \times Z$ does not imply $X \simeq Y$. Generalizing the examples of DANIELEWSKI, TOM DIECK (1992) constructs families of \mathbb{Q} -homology planes without the cancellation property.

Proof: Let $a \in \mathbb{N}_{\geq 2}$, $k \in \mathbb{N}_{\geq 1}$, and $j \in \mathbb{N}_{< a}$. The affine variety $W(a, k) := V_{\mathbb{C}^3}(x^k z + y^a - 1)$ is stable under the action $H(j)$ of the cyclic group of order a given by $\lambda \cdot (x, y, z) := (\lambda^j x, \lambda y, \lambda^{-jk} z)$. The corresponding quotient $W(a, k)_j$ is again an affine variety due to the Finiteness Theorem of Invariant Theory (see for example Section 3.2 in KRAFT (1984)). Using TOM DIECK (1992)¹⁰ we obtain:

- While $jk \equiv 1 \pmod{a}$, the isomorphism type of $W(a, k)_j \times \mathbb{C}$ does not depend on k or j .
- While $2/a + 1/k \leq 1$, even the homeomorphism type of $W(a, k)_j$ is characterized by (a, k, j) .

Now choose two varieties $W(a, k)_j$ with same a falling in both cases and pass to coordinate rings. □

The following easy example shows that Separation Lemma 3.1 (or Corollary B.2) does not hold for local algebras of arbitrary dimension.

A.7 Example *Let L be any non-trivial local k -algebra with residue algebra k . Denote by $L^{\mathbb{N}}$ the tensor product of countably many copies of L . There is an obvious isomorphism*

$$\varphi : L^{\mathbb{N}} \longrightarrow L^{\mathbb{N}} \otimes L.$$

Then $\pi_1 \varphi$ is horizontal while $\pi_2 \varphi$ is surjective and hence not trivial.

That is cheated, you might say, since $L^{\mathbb{N}}$ is so ‘big’. Clearly, the algebra $L^{\mathbb{N}}$ is not NOETHERIAN, not even for the smallest possible L , namely $L = k[T]/T^2$. Yet we could still hope to have a suitable generalization of Separation Lemma 3.1 to NOETHERIAN algebras, for example as in Corollary B.2 without the assumption that A is ARTINIAN. But even that is wrong:

A.8 Example *Let B_1, B_2 be local k -algebras with residue algebra k such that $\text{rad}^\ell B_1 \neq 0$ for all $\ell \in \mathbb{N}$ and $B_2 \not\cong k$. Let $A := B_1 \otimes B_2$ and consider the identical morphism*

$$\varphi : A \longrightarrow B_1 \otimes B_2.$$

Then $\pi_1 \varphi$ is horizontal while $\pi_2 \varphi$ is surjective and hence not trivial.

¹⁰There, Remark (3.6) has to be modified: the action (of $H(j)$) is trivial on the factor \mathbb{C} iff $jk \equiv 1 \pmod{a}$.

To be specific, take for B_1 the ring $k[[T]]$ of formal power series or the local ring $k[T]_{\langle T \rangle}$ and let $B_2 = k[T]/T^2$. Then all algebras are commutative NOETHERIAN. If $k = \mathbb{C}$, we can also take the ring $k\{T\}$ of convergent power series for B_1 . Then the algebras are even local analytic.

B A Generalized Separation Lemma

In this section, an algebra is an associative k -algebra with unit element. The ground field k can be an arbitrary field of characteristic zero.

B.1 Lemma *Let $\varphi : A \rightarrow B_1 \otimes B_2$ be a morphism of local algebras with residue algebra k . Recall $\text{char } k = 0$. Let $N \subset \text{rad } A$ be a nilpotent set such that $\pi_2\varphi(N) \not\subset \bigcap \text{rad}^\ell B_2$. If $N^{n+1} = 0$, then $\pi_1\varphi(N^n) \subset \text{rad}^{n+1} B_1$.*

For commutative, NOETHERIAN algebras KRULL's Intersection Theorem states that the intersection of the radical powers is trivial. The radical of a finite-dimensional algebra is nilpotent. The following Corollary B.2 can completely replace Lemma 1.1.2 in HORST (1990) and Lemma 4 in HORST (1987). A morphism $\varphi : A \rightarrow B$ of local algebras is *trivial* iff it maps the radical to zero. It is *horizontal* iff, for every ℓ with $\text{rad}^\ell A \neq 0$, we have $\varphi(\text{rad}^\ell A) \not\subset \text{rad}^{\ell+1} B$. For example, sections are always horizontal.

B.2 Corollary (HORST (1987)) *Let $\varphi : A \rightarrow B_1 \otimes B_2$ be a morphism of local, commutative, NOETHERIAN algebras with residue algebra k . Recall $\text{char } k = 0$. Moreover, assume A ARTINIAN. Then, if $\pi_1\varphi$ is horizontal, $\pi_2\varphi$ is trivial.*

Proof: Suppose φ is a counter-example. Since A is ARTINIAN, the set $N := \text{rad } A$ is nilpotent. According to KRULL's Intersection Theorem, $\bigcap \text{rad}^\ell B_2 = 0$. Especially $\pi_2\varphi(N) = \pi_2\varphi(\text{rad } A) \not\subset 0 = \bigcap \text{rad}^\ell B_2$, since $\pi_2\varphi$ is not trivial. Take n such that $\text{rad}^n A \neq 0$, $\text{rad}^{n+1} A = 0$. Now Lemma B.1 yields $\pi_1\varphi(\text{rad}^n A) \subset \text{rad}^{n+1} B_1$, ie. $\pi_1\varphi$ is not horizontal, which is a contradiction. \square

Similarly, we obtain Separation Lemma 3.1. Indeed, $\text{rad } B_2$ is even nilpotent there, so we do not need KRULL's Intersection Theorem and commutativity.

Proof (Lemma B.1, with V. STRASSEN): We have $0 \neq \pi_2\varphi(\text{rad } A) \subset \text{rad } B_2$, since $B_2^{(0)} = k$.

Choose $\ell > 0$ maximal with $\pi_2\varphi(N) \subset \text{rad}^\ell B_2$. There is such an $\ell < \infty$, since $\pi_2\varphi(N) \not\subset \bigcap \text{rad}^\ell B_2$ and $\pi_2\varphi(N) \subset \pi_2\varphi(\text{rad } A) \subset \text{rad } B_2$.

For $a \in N$ write $\varphi(a) = b \otimes 1 + 1 \otimes c + x$ with $b := \pi_1\varphi(a) \in \text{rad } B_1$, $c := \pi_2\varphi(a) \in \text{rad}^\ell B_2$, and $x \in \text{rad } B_1 \otimes \text{rad } B_2$. Indeed, we only need to verify the statement on x : $x \in \ker \pi_1 \cap \ker \pi_2 = \text{rad } B_1 \otimes \text{rad } B_2$.

Let $I := (\text{rad}^{n+1} B_1) \otimes B_2 + B_1 \otimes (\text{rad}^{\ell+1} B_2)$. For $a_0, \dots, a_n \in N$ expansion of $0 = \varphi(a_0 \cdots a_n)$ yields

$$(*) \quad 0 \equiv \sum_{i=0}^n b_0 \cdots b_{i-1} b_{i+1} \cdots b_n \otimes c_i \quad \text{mod } I.$$

All other terms vanish modulo I : terms with two 'c's are in $B_1 \otimes \text{rad}^{2\ell} B_2 = 0$, terms with a 'c' and an 'x' are in $B_1 \otimes \text{rad}^{\ell+1} B_2$, and terms without a 'c' are in $(\text{rad}^{n+1} B_1) \otimes B_2$.

Fix $a_0 \in N$ with $c_0 = \pi_2\varphi(a_2) \notin \text{rad}^{\ell+1} B_2$. Then, for $r \in \{n, n-1, \dots, 0\}$, we have

$$\forall a_1, \dots, a_n \in N : b_0^r b_{r+1} \cdots b_n \in \text{rad}^{n+1} B_1.$$

This we prove by backward induction on r : Take $n \geq r \geq 0$. Applying (*) with $a_1 := \dots a_r := a_0$ yields

$$b_0^r b_{r+1} \cdots b_n \otimes \sum_{i=0}^r c_0 \in I,$$

since, by induction hypothesis, the other summands are in $(\text{rad}^{n+1} B_1) \otimes B_2$. Since $(r+1)c_0 \notin \text{rad}^{\ell+1} B_2$ due to the characteristic of k , we obtain the assertion.

For $r = 0$ we see that $\pi_1\varphi(N^n) \subset \text{rad}^{n+1} B_1$. \square

Using the following Lemma B.3, more general corollaries can be derived. The (*lower*) *nilradical* (or *prime radical*) $\text{nil } A$ of an algebra A is the intersection of all its prime ideals. Here an ideal $P \triangleleft A$ is *prime* iff, for all ideals $I_1, I_2 \triangleleft A$ with $I_1 \cdot I_2 \subset P$, we have $I_1 \subset P$ or $I_2 \subset P$.

B.3 Lemma *Let A be an algebra. Then:*

- (i) $\text{nil } A = \bigcap \{P \triangleleft A \text{ prime}\} \subset \text{rad } A$.
- (ii) $\text{nil } A$ is nil, ie. all its elements are nilpotent, and contains all nilpotent ideals.
- (iii) If A satisfies the ascending chain condition for two-sided ideals, then $\text{nil } A$ is nilpotent.
- (iv) If A is commutative, then $\text{nil } A$ is exactly the set of nilpotent elements.

For proofs of these statements see ROWEN (1988, §2.6). [For (i), use Remark 2.1.14 and Definition 2.5.1. For (ii), combine Proposition 2.6.15 and Definition 2.6.3. Lemma 2.6.22 is (iii). And (iv) follows from (ii): the set N of nilpotent elements is a nil ideal containing $\text{nil } A$, for an element $a \in N$ the ideal Aa is nilpotent (since A is commutative) and hence contained in $\text{nil } A$.]

C A Reduction

This is V. STRASSEN's reduction of the Unique Factorization Theorem to the basic case.

Endomorphisms

Let k be an algebraically closed field of characteristic 0. By an algebra we always mean a finite dimensional k -algebra. Let A be an algebra. We denote the set of isomorphism classes of indecomposable projective A -modules by $\pi(A)$. Thus if $A = P_1^{\oplus n_1} \oplus \dots \oplus P_r^{\oplus n_r}$, where A is considered as left A -module and the P_i are indecomposable and pairwise nonisomorphic, then $\pi(A) = \{[P_1], \dots, [P_r]\}$. When $\varphi : A \rightarrow A'$ is an isomorphism of algebras, the map $\pi(\varphi) : \pi(A') \rightarrow \pi(A)$ is defined by $\pi(\varphi)([Q]) := [Q^\varphi]$, where Q^φ denotes the scalar restriction of the A' -module Q via φ . It is easy to see that $\pi(\varphi)$ is well defined and bijective. In fact, if $A' = \bigoplus Q_i$ is a decomposition into indecomposable modules, then also $A = \bigoplus \varphi^{-1}Q_i$ is a decomposition into indecomposable modules and we have the isomorphism of A -modules

$$(C.1) \quad \varphi : \varphi^{-1}Q_i \rightarrow Q_i^\varphi.$$

Given a projective A -module V , we define its *multiplicity* $\mu(V) : \pi(A) \rightarrow \mathbb{N}$ by $\mu(V)([P_i]) := m_i$, where $\pi(A) = \{[P_1], \dots, [P_r]\}$ with pairwise nonisomorphic P_i and $V \simeq P_1^{\oplus m_1} \oplus \dots \oplus P_r^{\oplus m_r}$. (By Krull-Schmidt this makes sense.) Clearly the map μ (i.e. $V \mapsto \mu(V)$) is constant on isomorphism classes and induces an isomorphism of the additive monoid of classes of projective A -modules (addition being induced by \oplus) with the additive monoid $\mathbb{N}^{\pi(A)}$. Let $\varphi : A \rightarrow A'$ be an isomorphism and $W \in \text{mod } A'$. Since scalar restriction commutes with direct sum, we have

$$(C.2) \quad \mu(W) = \mu(W^\varphi) \circ \pi(\varphi).$$

Note that V is a projective generator of $\text{mod } A$ if and only if $\mu(V)$ is strictly positive.

Given a strictly positive function $m : \pi(A) \rightarrow \mathbb{N}$ (i.e. $m : \pi(A) \rightarrow \mathbb{N}'$), we define $M(A, m) := \text{End}_A V$, where V is a projective A -module with multiplicity m . (The reader will notice that $M(A, m)^{op}$ is a typical *Morita transform* of A .) We write $M(A)$ for $M(A, 1)$. Thus $M(A) = \text{End}_A(P_1 \oplus \dots \oplus P_r)$, when $A = P_1^{\oplus n_1} \oplus \dots \oplus P_r^{\oplus n_r}$ with pairwise non-isomorphic indecomposable P_i . Clearly, $M(A, m)$ depends on the choice of V , but the isomorphism type of $M(A, m)$ does not. We call (A, m) and (A', m') isomorphic, $(A, m) \simeq (A', m')$, when there is an isomorphism $\varphi : A \rightarrow A'$ such that $m' = m \circ \pi(\varphi)$. Recall that an algebra B is *basic* when $B/\text{rad } B$ is a direct sum (= direct product) of copies of the ground field k . Equivalently, B is basic when $\mu(B) = 1$.

C.3 Proposition *Take algebras A, A' and basic algebras B, B' . Then:*

- (i) $M(A)$ is basic.
- (ii) $M(M(B, m)) \simeq B$ for any m .
- (iii) $M(M(A), m) \simeq A$ for some m .
- (iv) $M(B, m) \simeq M(B', m')$ if and only if $(B, m) \simeq (B', m')$.

Proof: (i): This is true by definition of a basic algebra, see DROZD & KIRICHENKO (1994), Theorem 3.5.4, statement 3).

(ii): This follows from DROZD & KIRICHENKO (1994), Theorem 3.5.6, by taking $B_1 = B$ and $B_2 = M(B, m)$. Then statement 2) of Theorem 3.5.6 is satisfied, since m is strictly positive, and statement 1) is our assertion, since $M(B) \simeq B$ for basic B .

(iii): This is the statement of DROZD & KIRICHENKO (1994), Corollary 3.5.7. (The strict positivity of m is not explicitly mentioned, but follows from the proof of Corollary 3.5.7.)

(iv): This is the content of DROZD & KIRICHENKO (1994), Exercise 3.16, statement c). \square

Tensor products

Let A_1, A_2 be algebras.

C.4 Lemma *Let $V_i, W_i \in \text{mod } A_i$. Then*

$$\text{Hom}_{A_1 \otimes A_2}(V_1 \otimes V_2, W_1 \otimes W_2) = \text{Hom}_{A_1}(V_1, W_1) \otimes \text{Hom}_{A_2}(V_2, W_2).$$

Proof: We show “ \subset ”. Suppose $\varphi = \sum \varphi_1^j \otimes \varphi_2^j \in \text{Hom}_{A_1 \otimes A_2}(V_1 \otimes V_2, W_1 \otimes W_2)$ with linear maps φ_i^j and the sum as short as possible. We are going to show that the φ_i^j are actually morphisms of A_i -modules. Take $a_2 \in A_2$. Since φ commutes with $1 \otimes a_2$, we have for all $v_2 \in V_2$

$$\sum \varphi_1^j \otimes (a_2 \varphi_2^j(v_2) - \varphi_2^j(a_2 v_2)) = 0.$$

Since the φ_1^j are linearly independent (by the minimality of the sum),

$$a_2 \varphi_2^j(v_2) - \varphi_2^j(a_2 v_2) = 0$$

for all a_2 and v_2 . Hence the φ_2^j are A_2 -morphisms. The lemma now follows by symmetry. \square

C.5 Corollary *Let V_i be A_i -modules. Then*

$$\text{End}_{A_1 \otimes A_2}(V_1 \otimes V_2) = \text{End}_{A_1}(V_1) \otimes \text{End}_{A_2}(V_2)$$

as algebras.

C.6 Corollary *Let V_i be indecomposable A_i -modules. Then $V_1 \otimes V_2$ is an indecomposable $A_1 \otimes A_2$ -module.*

Proof: Use Fitting's Lemma and Corollary C.5. \square

C.7 Corollary *Let V_i, W_i be indecomposable A_i -modules, $V_1 \otimes V_2 \simeq W_1 \otimes W_2$ as $A_1 \otimes A_2$ -modules. Then $V_1 \simeq W_1$ and $V_2 \simeq W_2$.*

Proof: Take an isomorphism $\varphi = \sum \varphi_1^j \otimes \varphi_2^j : V_1 \otimes V_2 \rightarrow W_1 \otimes W_2$, where φ_i^j are A_i -morphisms. Then $\sum \varphi^{-1}(\varphi_1^j \otimes \varphi_2^j)$ is the unit element of $\text{End}_{A_1 \otimes A_2}(V_1 \otimes V_2)$. By Corollary C.6 and Fittings Lemma $\varphi^{-1}(\varphi_1^j \otimes \varphi_2^j)$ is an automorphism for some j . Then $\varphi_i^j : V_i \rightarrow W_i$ are isomorphisms of A_i -modules. \square

C.8 Corollary *Let A_1, A_2 be algebras. Then:*

- (i) $\pi(A_1 \oplus A_2) = \pi(A_1) \cup \pi(A_2)$ (disjoint) via $[P_i] \mapsto [P_i^{\varphi_i}]$, where $[P_i] \in \pi(A_i)$ and φ_i is the projection $A_1 \oplus A_2 \rightarrow A_i$.
- (ii) $\pi(A_1 \otimes A_2) = \pi(A_1) \times \pi(A_2)$ via $([P_1], [P_2]) \mapsto [P_1 \otimes P_2]$.

Proof: (i) being clear, we prove (ii). For each $p_i \in \pi(A_i)$ we choose a module $P_i \in p_i$. Suppose $A_i \simeq \bigoplus_{p_i \in \pi(A_i)} P_i^{\oplus n_{ip_i}}$. Then $A_1 \otimes A_2 \simeq \bigoplus_{p_1, p_2} P_1^{\oplus n_{1p_1}} \otimes P_2^{\oplus n_{2p_2}} = \bigoplus_{p_1, p_2} (P_1 \otimes P_2)^{\oplus n_{1p_1} n_{2p_2}}$, and by Corollary C.6 and Corollary C.7 the $P_1 \otimes P_2$ are indecomposable and pairwise non-isomorphic. Hence the $[P_1 \otimes P_2]$ bijectively represent $\pi(A_1 \otimes A_2)$. \square

C.9 Corollary *Let A_1, A_2 be algebras, $m_i : \pi(A_i) \rightarrow \mathbb{N}^r$. Then:*

- (i) $M(A_1, m_1) \oplus M(A_2, m_2) \simeq M(A_1 \oplus A_2, m)$, where $m(p_i) := m_i(p_i)$ for $p_i \in \pi(A_i) \subset \pi(A_1 \oplus A_2)$.
- (ii) $M(A_1, m_1) \otimes M(A_2, m_2) \simeq M(A_1 \otimes A_2, m)$, where $m(p_1, p_2) := m_1(p_1) \cdot m_2(p_2)$.

Proof: (ii): For each $p_i \in \pi(A_i)$ we choose a module $P_i \in p_i$. Then Corollary C.5 and Corollary C.8 (ii) give $M(A_1, m_1) \otimes M(A_2, m_2) = \text{End}_{A_1}(\bigoplus_{p_1} P_1^{m_1(p_1)}) \otimes \text{End}_{A_2}(\bigoplus_{p_2} P_2^{m_2(p_2)}) = \text{End}_{A_1 \otimes A_2} \bigoplus_{p_1, p_2} (P_1 \otimes P_2)^{m_1(p_1) m_2(p_2)} = \text{End}_{A_1 \otimes A_2} \bigoplus_{p_1, p_2} (P_1 \otimes P_2)^{m(p_1, p_2)} = M(A_1 \otimes A_2, m)$. \square

C.10 Corollary *Let A_1, A_2 be algebras. Then $A_1 \otimes A_2$ is indecomposable with respect to \oplus if and only if both A_1 and A_2 are.*

Proof: Given an algebra A , we turn $\pi(A)$ into a directed graph by drawing an arrow from $[P]$ to $[Q]$ when $\text{Hom}_A(P, Q) \neq 0$. By Corollary C.8 (ii) and Lemma C.4 the graph $\pi(A_1 \otimes A_2)$ is the direct product of the graphs $\pi(A_1)$ and $\pi(A_2)$. Thus it suffices to prove that A is indecomposable with respect to \oplus if and only

if the graph $\pi(A)$ is weakly connected (i.e. $\pi(A)$ is connected as an undirected graph). Now given a nontrivial decomposition $A = B \oplus C$, there is no nonzero morphism of A -modules from B to C (since $\varphi(b) = \varphi(1_B b) = 1_B \varphi(b) = 0$) or from C to B . Choose decompositions $B = \bigoplus P_\iota$ and $C = \bigoplus Q_\kappa$ into indecomposable A -modules. From the matrix description of morphisms of direct sums of modules it is clear that $\text{Hom}_A(P_\iota, Q_\kappa) = 0$ and $\text{Hom}_A(Q_\kappa, P_\iota) = 0$ for all ι, κ . Hence the sets $\{[P_\iota]\}_\iota$ and $\{[Q_\kappa]\}_\kappa$ form a nontrivial disconnected partition of $\pi(A)$. The argument is easily inverted. \square

A Basic Reference

The following result is proved elsewhere by M. Nüsken:

C.11 Proposition *Let B_1, \dots, B_d and $B'_1, \dots, B'_{d'}$ be basic algebras, indecomposable with respect to direct sum and tensor product of algebras, and for each i let $B_i = \bigoplus_\iota P_{i\iota}$ be a decomposition into indecomposable submodules. Let $\varphi : B_1 \otimes \dots \otimes B_d \rightarrow B'_1 \otimes \dots \otimes B'_{d'}$ be an isomorphism. Then $d = d'$ and there are a permutation $\sigma \in S_d$ and isomorphisms $\psi_i : B_i \rightarrow B'_{\sigma_i}$ such that $\varphi(P_{1\nu_1} \otimes \dots \otimes P_{d\nu_d}) = \psi(P_{1\nu_1} \otimes \dots \otimes P_{d\nu_d})$ for all (ν_i) , where the algebra isomorphism $\psi : B_1 \otimes \dots \otimes B_d \rightarrow B'_1 \otimes \dots \otimes B'_d$ is defined by $\psi(b_1 \otimes \dots \otimes b_d) := \psi_{\sigma^{-1}1} b_{\sigma^{-1}1} \otimes \dots \otimes \psi_{\sigma^{-1}d} b_{\sigma^{-1}d}$.*

C.12 Corollary *Let B_1, \dots, B_d and $B'_1, \dots, B'_{d'}$ be basic algebras, indecomposable with respect to direct sum and tensor product of algebras and let $\varphi : B_1 \otimes \dots \otimes B_d \rightarrow B'_1 \otimes \dots \otimes B'_{d'}$ be an isomorphism. Then $d = d'$ and there are a permutation $\sigma \in S_d$ and isomorphisms $\psi_i : B_i \rightarrow B'_{\sigma_i}$ such that $\pi(\varphi) = \pi(\psi)$, where the algebra isomorphism $\psi : B_1 \otimes \dots \otimes B_d \rightarrow B'_1 \otimes \dots \otimes B'_d$ is defined by $\psi(b_1 \otimes \dots \otimes b_d) := \psi_{\sigma^{-1}1} b_{\sigma^{-1}1} \otimes \dots \otimes \psi_{\sigma^{-1}d} b_{\sigma^{-1}d}$.*

Multilinear Forms

Take a nonempty set Ω and, for each $\omega \in \Omega$, a nonempty finite set $\pi(\omega)$. We consider the polynomial ring $\mathbb{Z}[(X_{\omega p})_{\omega \in \Omega, p \in \pi(\omega)}]$. A polynomial f is called *positive multilinear* of degree n , when there are pairwise different $\omega_1, \dots, \omega_d \in \Omega$ and a function $m : \pi(\omega_1) \times \dots \times \pi(\omega_d) \rightarrow \mathbb{N}'$ such that

$$(C.13) \quad f = \sum_{p \in \pi(\omega_1) \times \dots \times \pi(\omega_d)} m(p) X_{\omega_1 p_1} \cdot \dots \cdot X_{\omega_d p_d}.$$

We allow $d = 0$, in which case f is a positive integer. In place of the cumbersome equation (C.13) we will prefer to write

$$(C.14) \quad f \sim (\omega_1, \dots, \omega_d, m)$$

and we will call $(\omega_1, \dots, \omega_d, m)$ a *parameter sequence* of f . It is not uniquely determined by f , but may be replaced by $(\omega_{\sigma^{-1}1}, \dots, \omega_{\sigma^{-1}d}, \sigma m)$, where σ is any permutation of $\{1, \dots, d\}$ and $\sigma m : \pi(\omega_{\sigma^{-1}1}) \times \dots \times \pi(\omega_{\sigma^{-1}d}) \rightarrow \mathbb{N}^d$ is given by $(\sigma m)q := m(q_{\sigma 1}, \dots, q_{\sigma d})$. Moreover, in this way we get all possible parameter sequences of f .

The set $\{\omega_1, \dots, \omega_d\}$ is called the support of f . Positive multilinear forms are called disjoint, when their supports are disjoint. We denote the set of positive multilinear forms of $\mathbb{Z}[(X_{\omega p})]$ by F . The set F is closed neither under addition nor under multiplication. However, the product of two disjoint elements of F again belongs to F . Indeed, when $f \sim (\omega_1, \dots, \omega_d, m)$ and $f' \sim (\omega'_1, \dots, \omega'_{d'}, m')$ and f, f' have disjoint supports, then

$$(C.15) \quad ff' \sim (\omega_1, \dots, \omega_d, \omega'_1, \dots, \omega'_{d'}, m''),$$

where $m''(p, p') = m(p)m(p')$.

C.16 Lemma *Let $f \in F$ and $f = f_1 f_2$ with $f_1, f_2 \in \mathbb{Z}[(X_{\omega p})]$. Then either $f_1, f_2 \in F$ or $-f_1, -f_2 \in F$. In the first case, f_1 and f_2 are disjoint.*

Proof: Easy. □

Suppose that G is a group of automorphisms of $\mathbb{Z}[(X_{\omega p})]$ with the following properties:

- (1) G stabilizes F and preserves degree and disjointness.
- (2) Given $f_1, \dots, f_m, f \in F$, there is a $g \in G$ such that $g(f)$ is disjoint from f_1, \dots, f_m .
- (3) Given $f_1, \dots, f_m, f, f' \in F$ such that f_1, \dots, f_m are disjoint from f, f' and such that $f' = g(f)$ for some $g \in G$, there is an $h \in G$ fixing all f_j such that $f' = h(f)$.

We define a multiplication in the set \mathcal{S} of G -orbits of F in the following way: The product of two orbits s_1 and s_2 is the orbit $Gf_1 f_2$, where $f_1 \in s_1$ and $f_2 \in s_2$ are chosen disjoint. By property (2) such a choice is possible. Moreover the resulting orbit is independent of this choice. (Indeed, in order to show $Gf_1 f_2 = Gf'_1 f'_2$, say, we first use properties (2) and (3) to find an $h \in G$ such that $h(f_2)$ is disjoint from f_1, f'_1, f'_2 and such that $hf_1 = f_1$. Then $Gf_1 f_2 = Gh(f_1 f_2) = Gf_1 h(f_2)$, so that we may assume that f_2 is disjoint from f_1, f'_1, f'_2 to begin with. By property (3) there is an $h \in G$ fixing f_2 and mapping f_1 to f'_1 . Hence $Gf_1 f_2 = Gh(f_1 f_2) = Gf'_1 f_2$, and by a similar argument $Gf'_1 f_2 = Gf'_1 f'_2$.) Clearly, the multiplication is associative and commutative with unit element $1 := G1 = \{1\}$.

C.17 Proposition *The commutative monoid \mathcal{S} is factorial and 1 is its only unit (invertible element). In other words: \mathcal{S} is a free commutative monoid over its set of irreducible elements.*

Proof: By property (1) the degree function is defined on \mathcal{S} . This implies that 1 is the only unit of \mathcal{S} , and that every element of \mathcal{S} different from 1 is a product of irreducible elements. The uniqueness of such a factorization (up to order) is a consequence of the following claim: When s_1 is irreducible and $s_1|s_2s_3$, then $s_1|s_2$ or $s_1|s_3$ (i.e. every irreducible element is prime). In order to prove this claim, suppose $s_1s' = s_2s_3$. We may choose f_1, f', f_2, f_3 from s_1, s', s_2, s_3 respectively, such that f_1, f' as well as f_2, f_3 are disjoint and that $f_1f' = f_2f_3$. (Note that all elements of s_2s_3 are products of disjoint elements of s_2 and s_3 , since being such a product is preserved by G .) By Lemma C.16 the polynomial f_1 is irreducible in $\mathbb{Z}[(X_{\omega p})]$. Since $\mathbb{Z}[(X_{\omega p})]$ is a factorial ring, we have $f_1f'' = f_2$ (say) for some $f'' \in \mathbb{Z}[(X_{\omega p})]$. Lemma C.16 tells us that $f'' \in F$ and f_1 is disjoint from f'' . Hence $s_1|s_2$. This proves the claim. \square

Proof of the Reduction

Let \mathcal{B} be a set of representatives (a transversal) of the ‘set’ of all isomorphism classes of basic algebras that are $\{\oplus, \otimes\}$ -indecomposable. Let $\Omega := \{(B, \nu) : B \in \mathcal{B}, \nu \in \mathbb{N}\}$ and for $\omega = (B, \nu) \in \Omega$ let $\pi(\omega) := \pi(B)$ be the set of isomorphism classes of indecomposable projective B -modules.

If for every $\omega = (B, \nu) \in \Omega$ an automorphism φ_ω of B is given, we may define an automorphism g of $\mathbb{Z}[(X_{\omega p})]$ by

$$(C.18) \quad g(X_{\omega p}) := X_{\omega, \pi(\varphi_\omega^{-1})p}.$$

In particular, when $f \in F$ and $f \sim (\omega_1, \dots, \omega_d, m)$, the polynomial $g(f)$ is again in F and

$$(C.19) \quad g(f) \sim (\omega_1, \dots, \omega_d, m \circ (\pi(\varphi_{\omega_1}) \times \dots \times \pi(\varphi_{\omega_d}))).$$

Varying (φ_ω) , we get a group G_0 of automorphisms of $\mathbb{Z}[(X_{\omega p})]$.

If a permutation τ of Ω is given with the property that for any $(B, \nu) \in \Omega$ we have $\tau(B, \nu) = (B, \nu')$ for some $\nu' \in \mathbb{N}$ (*admissible permutation*), we may define an automorphism g of $\mathbb{Z}[(X_{\omega p})]$ by

$$(C.20) \quad g(X_{\omega p}) := X_{\tau(\omega)p}.$$

In particular, when $f \in F$ and $f \sim (\omega_1, \dots, \omega_d, m)$, the polynomial $g(f)$ is again in F and

$$(C.21) \quad g(f) \sim (\tau\omega_1, \dots, \tau\omega_d, m).$$

Varying τ , we get a group G_1 of automorphisms of $\mathbb{Z}[(X_{\omega p})]$. We let G be the group of automorphisms of $\mathbb{Z}[(X_{\omega p})]$ generated by $G_0 \cup G_1$. Then $G = G_0 \cdot G_1$, since G_1 normalizes G_0 . (Actually G is a wreath product, but this need not concern us.)

G has the properties (1)–(3) above. Indeed, (1) is clear from (C.19) and (C.21). Property (2) follows from (C.21) alone. In order to prove (3), we may assume that g is either in G_0 or in G_1 . (Write $g = g_0 g_1$ with $g_j \in G_j$ and note that g_0 does not change supports, so that $g_1(f)$ is disjoint from f_1, \dots, f_m .) Let $f \sim (\omega_1, \dots, \omega_d, m)$ and $f' \sim (\omega'_1, \dots, \omega'_d, m')$. When g is in G_0 , we define h by (φ'_ω) with $\varphi'_{\omega_i} = \varphi_{\omega_i}$ for $i = 1, \dots, d$ and $(\varphi'_\omega) = id_B$ for $\omega = (B, \nu)$ not in the support of f . When g is in G_1 , we define h by any admissible permutation τ' of Ω satisfying

$$\tau'(\omega) = \begin{cases} \tau(\omega) & \text{when } \omega \in \{\omega_1, \dots, \omega_d\} \\ \omega & \text{when } \omega \notin \{\omega_1, \dots, \omega_d, \omega'_1, \dots, \omega'_d\}. \end{cases}$$

Such a τ' exists, since (C.21) implies $\tau(\omega_i) = \omega'_i$.

Thus we have at our disposal the factorial monoid \mathcal{S} of the last section. We also have the commutative monoid \mathcal{M} consisting of the isomorphism classes of algebras that are \oplus -indecomposable, under the multiplication induced by \otimes . (See Corollary C.10.)

C.22 Proposition $\mathcal{S} \simeq \mathcal{M}$

Proof: Define a map

$$j : \mathcal{S} \rightarrow \mathcal{M}$$

in the following way: Given $s \in \mathcal{S}$, take $f \in s$ and let $f \sim (\omega_1, \dots, \omega_d, m)$ with $\omega_i = (B_i, \nu_i)$. Since $\pi(\omega_1) \times \dots \times \pi(\omega_d) = \pi(B_1) \times \dots \times \pi(B_d) = \pi(B_1 \otimes \dots \otimes B_d)$ by Corollary C.8 (ii), we may use the function $m : \pi(B_1 \otimes \dots \otimes B_d) \rightarrow \mathbb{N}'$ to define

$$j(s) := [M(B_1 \otimes \dots \otimes B_d, m)],$$

where $[B]$ denotes the isomorphism class of the algebra B . (For $d = 0$ one has to interpret $B_1 \otimes \dots \otimes B_d$ as k .)

First we show $j(s) \in \mathcal{M}$. Indeed, Corollary C.10 implies that $B_1 \otimes \dots \otimes B_d$ is indecomposable with respect to \oplus . Therefore, by Proposition C.3 (ii) and by Corollary C.9 (i), the algebra $M(B_1 \otimes \dots \otimes B_d, m)$ is \oplus -indecomposable, hence $j(s) \in \mathcal{M}$.

Next we observe that $j(s)$ is independent of the choice of parameter sequence of f . According to Proposition C.3 (iv) we have to show $(B_1 \otimes \dots \otimes B_d, m) \simeq (B_{\sigma^{-1}1} \otimes \dots \otimes B_{\sigma^{-1}d}, \sigma m)$. Now $\varphi : B_1 \otimes \dots \otimes B_d \rightarrow B_{\sigma^{-1}1} \otimes \dots \otimes B_{\sigma^{-1}d}$ with $\varphi(b_1 \otimes \dots \otimes b_d) := b_{\sigma^{-1}1} \otimes \dots \otimes b_{\sigma^{-1}d}$ defines an isomorphism of algebras and by our identifications and (C.1) we have $(\sigma m)[Q_1 \otimes \dots \otimes Q_d] = m[Q_{\sigma 1} \otimes \dots \otimes Q_{\sigma d}] = m[\varphi^{-1}(Q_1 \otimes \dots \otimes Q_d)] = m[(Q_1 \otimes \dots \otimes Q_d)^\varphi] = (m \circ \pi(\varphi))[Q_1 \otimes \dots \otimes Q_d]$, which gives the required isomorphism of pairs.

In order to show that $j(s)$ is independent of the choice of $f \in s$, replace f by $g(f)$ for some $g \in G$. We may assume that $g(f)$ is of the form (C.19) or (C.21). In the last case $M(B_1 \otimes \dots \otimes B_d, m)$ and hence $j(s)$ do not change at all. In the first case, by Proposition C.3 (iv) and (C.19), we have to show $(B_1 \otimes \dots \otimes B_d, m) \simeq (B_1 \otimes \dots \otimes B_d, m')$, where $m' := m \circ (\pi(\varphi_{\omega_1}) \times \dots \times \pi(\varphi_{\omega_d}))$. But $\varphi_{\omega_1} \otimes \dots \otimes \varphi_{\omega_d}$ is an automorphism of $B_1 \otimes \dots \otimes B_d$ and $\pi(\varphi_{\omega_1}) \times \dots \times \pi(\varphi_{\omega_d}) = \pi(\varphi_{\omega_1} \otimes \dots \otimes \varphi_{\omega_d})$, since our identifications and (C.1) give $(\pi(\varphi_{\omega_1}) \times \dots \times \pi(\varphi_{\omega_d}))[Q_1 \otimes \dots \otimes Q_d] = [\varphi_{\omega_1}^{-1} Q_1 \otimes \dots \otimes \varphi_{\omega_d}^{-1} Q_d] = [(\varphi_{\omega_1} \otimes \dots \otimes \varphi_{\omega_d})^{-1} Q_1 \otimes \dots \otimes Q_d] = \pi(\varphi_{\omega_1} \otimes \dots \otimes \varphi_{\omega_d})[Q_1 \otimes \dots \otimes Q_d]$. Thus $\varphi_{\omega_1} \otimes \dots \otimes \varphi_{\omega_d}$ is an isomorphism of pairs and we have proved that j is well defined.

Next we show that j is injective. Suppose $j(s) = j(s')$ with $f' \in s'$, where $f' \sim (\omega'_1, \dots, \omega'_d, m')$ with $\omega'_i = (B'_i, \nu'_i)$. Then by Proposition C.3 (iv) we have $(B_1 \otimes \dots \otimes B_d, m) \simeq (B'_1 \otimes \dots \otimes B'_d, m')$, hence there is an isomorphism of algebras $\varphi : B_1 \otimes \dots \otimes B_d \rightarrow B'_1 \otimes \dots \otimes B'_d$ such that $m' = m \circ \pi(\varphi)$. By Corollary C.12 we have $d = d'$ and we may assume that there are a permutation $\sigma \in S_d$ and isomorphisms $\varphi_i : B_i \rightarrow B'_{\sigma_i}$ such that $\varphi(b_1 \otimes \dots \otimes b_d) := \varphi_{\sigma^{-1}1} b_{\sigma^{-1}1} \otimes \dots \otimes \varphi_{\sigma^{-1}d} b_{\sigma^{-1}d}$. Note that since B_i and B'_{σ_i} are isomorphic they are actually equal. Hence by an argument that we have used above (when showing the independence of $j(s)$ of the parameter sequence) we see that $(B'_1 \otimes \dots \otimes B'_d, m') \simeq (B'_{\sigma_1} \otimes \dots \otimes B'_{\sigma_d}, \sigma^{-1} m') = (B_1 \otimes \dots \otimes B_d, m' \circ \pi(\chi))$ via $\chi : B'_1 \otimes \dots \otimes B'_d \rightarrow B'_{\sigma_1} \otimes \dots \otimes B'_{\sigma_d} = B_1 \otimes \dots \otimes B_d$, where $\chi(b'_1 \otimes \dots \otimes b'_d) := b'_{\sigma_1} \otimes \dots \otimes b'_{\sigma_d}$. Thus, replacing the given parameter sequence $(\omega'_1, \dots, \omega'_d, m')$ of f' by the parameter sequence $(\omega'_{\sigma_1}, \dots, \omega'_{\sigma_d}, \sigma^{-1} m')$ and the isomorphism φ by $\chi \circ \varphi$, we may assume that $\omega'_i = (B_i, \nu'_i)$ and that there are automorphisms φ_i of B_i such that $\varphi = \varphi_1 \otimes \dots \otimes \varphi_d$ and therefore $m' = m \circ \pi(\varphi_1 \otimes \dots \otimes \varphi_d) = m \circ (\pi(\varphi_1) \times \dots \times \pi(\varphi_d))$. Replacing f by $g(f)$, where g is defined by (C.18) through

$$\varphi_\omega := \begin{cases} \varphi_i & \text{when } \omega = \omega_i \\ id_B & \text{when } \omega = (B, \nu) \text{ is not in the support of } f, \end{cases}$$

and keeping in mind (C.19), we may assume $m = m'$. Finally $f' = g(f)$ (and therefore $s' = s$), when g is defined by (C.20) through any admissible τ satisfying $\tau(\omega_i) = \omega'_i$. This proves that j is injective.

In order to show that j is surjective, suppose $[A] \in \mathcal{M}$ and set $B := M(A)$. Since A is \oplus -indecomposable and $A = M(B, m')$ for some m' by Proposition C.3 (iii), we deduce from Corollary C.9 (i) that B is \oplus -indecomposable. Thus $B \simeq B_1 \otimes \dots \otimes B_d$ for certain (not necessarily distinct) $B_i \in \mathcal{A}$, and therefore $(B, m') \simeq (B_1 \otimes \dots \otimes B_d, m)$ for a suitable $m : \pi(B_1) \times \dots \times \pi(B_d) = \pi(B_1 \otimes \dots \otimes B_d) \rightarrow \mathbb{N}$. Set $\omega_i := (B_i, i)$ and define $f \in F$ by $f \sim (\omega_1, \dots, \omega_d, m)$. If $s \in \mathcal{S}$ denotes the orbit Gf , we have $j(s) = [M(B_1 \otimes \dots \otimes B_d, m)] = [M(B, m')] = [A]$. Thus j is surjective.

It remains to prove that j is a monoid morphism. Take $s, s' \in \mathcal{S}$ and $f \in s$, $f' \in s'$ with disjoint supports. Let $f \sim (\omega_1, \dots, \omega_d, m)$ and $f' \sim (\omega'_1, \dots, \omega'_d, m')$

with $\omega_i = (B_i, \nu_i)$ and $\omega'_i = (B'_i, \nu'_i)$. Then by (C.15) and Corollary C.9 (ii) we have $j(ss') = [M(B_1 \otimes \dots \otimes B_d \otimes B'_1 \otimes \dots \otimes B'_{d'}, m'')] = [M(B_1 \otimes \dots \otimes B_d, m) \otimes M(B'_1 \otimes \dots \otimes B'_{d'}, m')] = j(s)j(s')$. Hence j is a monoid morphism and the Proposition is proved. \square

C.23 Corollary *The set \mathcal{M} of isomorphism classes of \oplus -indecomposable algebras is a free commutative monoid over the set \mathcal{Y} of isomorphism classes of $\{\oplus, \otimes\}$ -indecomposable algebras.*

Let $\mathbb{Z}[\mathcal{Y}]$ be the polynomial ring over \mathcal{Y} as set of indeterminates and let $\mathbb{N}[\mathcal{Y}]$ be the positive cone of polynomials with nonnegative coefficients in $\mathbb{Z}[\mathcal{Y}]$. Let \mathcal{A} be the commutative semiring of isomorphism classes of algebras, where $+$ and \cdot are induced by \oplus and \otimes . The inclusion $\mathcal{Y} \hookrightarrow \mathcal{A}$ induces a morphism $\Phi : \mathbb{N}[\mathcal{Y}] \rightarrow \mathcal{A}$ of semirings.

C.24 Theorem Φ is an isomorphism.

Proof: Since the set of monomials of $\mathbb{Z}[\mathcal{Y}]$ may be identified with \mathcal{M} , it suffices to show that the additive monoid of \mathcal{A} is a free commutative monoid in \mathcal{M} . This is well known. \square

Thus we may identify \mathcal{A} with the the positive cone of nonnegative polynomials of $\mathbb{Z}[\mathcal{Y}]$ via the canonical isomorphism Φ . In this way $\mathbb{Z}[\mathcal{Y}]$ becomes the universal ring (Grothendieck ring) of \mathcal{A} . In other words: The universal ring of \mathcal{A} is a polynomial ring.

Names and Symbols

$A, B', B_i, B_{ij},$	algebras
C', C_j, C_{ji}	
$a, a', a_i, b, b', b'',$	algebra elements
$b_2, b_{21}, c, d_1, d_2,$	
ℓ, x	
$A_1 \oplus A_2$	sum of the algebras A_1 and A_2
$A_1 \otimes A_2$	(tensor) product of the algebras A_1 and A_2
$A \downarrow S$	inner restriction of the algebra A to the subgraph S of $\Delta(A)$, see p.26
$A^{(*)}, A^{(\ell)}$	associated graded algebra of A and its ℓ -th homogeneous part of A , $= \text{rad}^\ell A / \text{rad}^{\ell+1} A$
\mathbb{C}	the field of complex numbers
$E(\Gamma)$	edge set of the graph Γ
$e, e', e_i, f, f', f_j,$	idempotents in an algebra
f_{1i}	
$\text{Hom}_A^{(*)}(\cdot, \cdot)$	associated graded homomorphism functor, see p.21
$\text{Hom}_A^{(\ell)}(\cdot, \cdot)$	$= \text{Rad}_A^\ell / \text{Rad}_A^{\ell+1}$, see p.21
i, j	index (identifying a factor)
k	ground field
kQ	quiver algebra of the quiver Q over k
ℓ, ℓ_1, ℓ_2	index (denoting a degree)
\mathcal{M}	the monoid of isomorphism classes of \oplus -indecomposable, SCHURIAN algebras
$M(A, m)$	transform of A with multiplicity $m : \Delta(A) \rightarrow \mathbb{N}$, see p.26
m, m', m_1, m_2	multiplicity functions, see $\text{mult}(A)$
$m_1 \cdot m_2$	product of the multiplicities m_1 and m_2
$\text{mult}(A)$	multiplicity of A , see p.26
\mathbb{N}	the semiring of natural numbers (including 0)
\mathbb{N}'	the set of natural numbers without 0
n, n_1, n_2	natural numbers
P, P', P_1, P_2, R	indecomposable projectives
$P^{(\varphi^{-1})}$	scalar restriction of P via the algebra morphism φ^{-1}
\mathbb{Q}	the field of rational numbers
Q	quiver
$Q_{\geq n}$	set of all paths in Q of length at least n
$Q(A)$	GABRIEL quiver of an algebra A
\mathbb{R}	the field of real numbers
$\text{Rad}_A(\cdot, \cdot)$	(functor) radical of A , see p.21
$\text{rad } A$	JACOBSON radical of the algebra A
r, s	counts

$\text{supp } m$	support of the multiplicity m
\mathcal{U}	the semiring of isomorphisms classes of SCHURian algebras
u	unit in an algebra
$V(\Gamma)$	vertex set of the graph Γ
V, V_i, W	(projective) modules
v_i	elements of a projective
\mathbb{Z}	the ring of integers
$z_{i\nu}$	field elements
$\alpha, \beta, \gamma, \gamma', \gamma_i$	vertices in a graph
β, γ	morphisms in Distortion Lemma 4.7
$\Gamma, \Gamma_i, \Delta_j$	graphs
$\Gamma_1 \uplus \Gamma_2$	disjoint union of the graphs Γ_1 and Γ_2
$\Gamma_1 \times \Gamma_2$	cartesian product of the graphs Γ_1 and Γ_2
$\Delta(A)$	graph of the algebra A , see p.24
$\delta, \delta_1, \delta_2$	graph morphism
$\varepsilon, \varepsilon.$	the only map (and iso) between two one point graphs
ζ, ζ_1	permutation morphisms of tensor products of algebras
Θ	the one edge graph
ϑ, ϑ'	special graph maps in Transform Lemma 2.5
ι_i	canonicle inclusion $A_i \rightarrow A_1 \otimes A_2$
κ_u	conjugation with u : $\kappa_u(a) = uau^{-1}$
Λ, Λ_j	subgraphs
μ	multiplication morphism $C_{11} \otimes C_{12} \rightarrow C_1$
ν, λ	indices (for a list)
π_i	canonicle projection $A_1 \otimes A_2 \rightarrow A_i$ if A_{3-i} is local
ϱ	the canonicle isomorphism $A \rightarrow \text{End}_A^{\text{op}} A$
σ, σ'	permutation
$\varphi, \varphi', \varphi_1, \varphi_2, \psi,$	algebra morphisms
$\psi_1, \psi_2, \tilde{\varphi}, \tilde{\psi}, \chi$	

References

- BODY, Richard & DOUGLAS, Roy (1979), “Tensor products of graded algebras and unique factorization”, *Am. J. Math.*, **101**, 909–914. Zbl. 422.16001.
- CHANG, C.C., JONSSON, B., & TARSKI, A. (1964), “Refinement properties for relational structures”, *Fundam. Math.*, **55**, 249–281. Zbl. 171.25805.
- DROZD, Yuriy A. & KIRICHENKO, Vladimir V. (1994), *Finite dimensional algebras*. Springer-Verlag, Berlin. Zbl. 816.16001, mat 9:dr67/f46.
- FEIGENBAUM, Joan (1986), “Directed Cartesian-product graphs have unique factorizations that can be computed in polynomial time”, *Discrete Appl. Math.*, **15**, 105–110. Zbl. 637.05018.
- FIESELER, Karl-Heinz (1994), “On complex affine surfaces with \mathbb{C}^+ -action.”, *Comment. Math. Helv.*, **69**(1), 5–27. Zbl. 806.14033.
- HORST, Camilla (1987), “A cancellation theorem for artinian local algebras”, *Math. Ann.*, **276**, 657–662. Zbl. 595.13007.
- HORST, Camilla (1990), “On product decompositions of complex spaces”, *Note Mat.*, **10**(1), 157–213. Zbl. 790.32010.
- IMRICH, W. (1967), “Kartesisches Produkt von Mengensystemen und Graphen”, *Stud. Sci. Math. Hungar.*, **2**, 285–290. Zbl. 153.32301.
- KRAFT, Hanspeter (1984), *Geometrische Methoden in der Invariantentheorie*, Aspects of Mathematics, **D1**. Friedr. Vieweg & Sohn, Braunschweig-Wiesbaden. Zbl. 569.14003, mat 9:kr13:ma/g26a.
- MARYLAND, Wallace jun. (1978), “Decomposition of finite connected partially ordered sets.”, *J. Comb. Inf. Syst. Sci.*, **3**, 238–244. Zbl. 404.06002.
- NAKAYAMA, Tadasi & HASHIMOTO, Junji (1950), “On a problem of G. Birkoff”, *Proc. Amer. Math. Soc.*, **1**, 141–142. Zbl. 036.29604.
- POWER, S. C. (1990), “Classifications of tensor products of triangular operator algebras”, *Proc. Lond. Math. Soc.*, **61**(3), 571–614. Zbl. 783.47060.
- ROWEN, Louis H. (1988), *Ring theory, Volume I*, Pure and Applied Mathematics, **127**. Academic Press, Inc., Boston, MA etc. Zbl. 651.16001, mat 9:ro96:da/r46-1.
- SABIDUSSI, Gert (1960), “Graph multiplication”, *Math. Z.*, **72**, 446–457. Zbl. 093.37603.

TOM DIECK, Tammo (1992), “Homology planes without cancellation property.”,
Arch. Math., **59**(2), 105–114. Zbl. 780.57016.