Universität Konstanz

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Konstanzer Schriften in Mathematik

Nr. 278, Februar 2011

ISSN 1430-3558

NECESSITY OF PARAMETER-ELLIPTICITY FOR MULTI-ORDER SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract. In this paper we investigate parameter-ellipticity conditions for multi-order systems of differential equations on a bounded domain. Under suitable assumptions on smoothness and on the order structure of the system, it is shown that parameter-dependent a priori-estimates imply the conditions of parameter-ellipticity, i.e., interior ellipticity, conditions of Shapiro-Lopatinskii type, and conditions of Vishik-Lyusternik type. The mixed-order systems considered here are of general form; in particular, it is not assumed that the diagonal operators are of the same order. This paper is a continuation of an article by the same authors where the sufficiency was shown, i.e., a priori-estimates for the solutions of parameter-elliptic multi-order systems were established.

1. Introduction and main results

In this paper, we will study multi-order boundary value problems defined over a bounded domain in $\mathbb{R}^n$. Under rather general assumptions on the structure of the system, it was shown in the paper [DF] that parameter-ellipticity implies uniform a priori-estimates for the solutions. Now we will show that the conditions of parameter-ellipticity are also necessary.

Parameter-elliptic boundary value problems and a priori estimates for them were treated, e.g., in [ADF] (scalar problems), [DFM] (systems of homogeneous type), and [F] (multi-order systems). The notion of parameter-ellipticity for general multi-order systems was introduced by Kozhevnikov ([K1], [K2]) and by Denk, Mennicken, and Volevich ([DMV]). The mentioned papers had restrictions on the orders of the operators which excluded, for instance, boundary conditions of Dirichlet type (see [ADN, Section 2], [G, p. 448]). In the paper [DF], these restrictions were removed.

Let us consider in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with boundary Γ the boundary value problem

$$\begin{aligned} A(x,D)u(x) - \lambda u(x) &= f(x) \quad \text{in } \Omega, \\ B(x,D)u(x) &= g(x) \quad \text{on } \Gamma. \end{aligned} \tag{1.1}$$

Here $A(x,D) = \big(A_{jk}(x,D)\big)_{j,k=1,\dots,N}$ is an $N \times N$-matrix of linear differential operators, $N \in \mathbb{N}$, $N \geq 2$, $u(x) = (u_1(x),\dots,u_N(x))^T$ and $f(x) = (f_1(x),\dots,f_N(x))^T$ are defined on Ω (^{T} denoting the transpose), whereas $B(x,D) = \big(B_{jk}(x,D)\big)_{\substack{j=1,\dots,\widetilde{N}\\ k=1,\dots,N}}$ is an $\widetilde{N} \times N$-matrix of boundary operators, and $g(x) = (g_1(x),\dots,g_{\widetilde{N}}(x))^T$ is defined on Γ.

Date: February 24, 2011.

1991 *Mathematics Subject Classification.* Primary 35J55; Secondary 35S15.

Key words and phrases. Parameter-ellipticity, multi-order systems, a priori estimates.

To describe the order structure of the boundary value problem (A, B), let $\{s_j\}_{j=1}^N$ and $\{t_j\}_{j=1}^N$ denote sequences of integers satisfying $s_1 \geq \cdots \geq s_N$, $t_1 \geq \cdots \geq t_N \geq 0$, and put $m_j := s_j + t_j$ $(j = 1, \ldots, N)$. We assume

$$m_1 = \cdots = m_{k_1} > m_{k_1+1} = \cdots = m_{k_{d-1}} > m_{k_{d-1}+1} = \cdots = m_{k_d} > 0,$$

where $k_d = N$. We set $\widetilde{m}_j := m_{k_j}$ $(j = 1, \ldots, d)$, and assume that $2N_r := \sum_{j=1}^{k_r} m_j$ is even for $r = 1, \ldots, d$. We also set $k_0 := 0$ and $N_0 := 0$. Further, let $\{\sigma_j\}_{j=1}^{\widetilde{N}}$, $\widetilde{N} := N_d$, be a sequence of integers satisfying $\max_j \sigma_j < s_N$. It was shown in [DF, Section 2] that we may also assume $s_j \geq 0$ $(j = 1, \ldots, N)$ and $\sigma_j < 0$ $(j = 1, \ldots, \widetilde{N})$. Define $\kappa_0 := \max\{t_1, -\sigma_1, \ldots, -\sigma_{\widetilde{N}}\}$. Concerning (A, B), we will assume that

$$\begin{aligned} &\operatorname{ord} A_{jk} \leq s_j + t_k \quad (j, k = 1, \ldots, N), \\ &\operatorname{ord} B_{jk} \leq \sigma_j + t_k \quad (j = 1, \ldots, \widetilde{N},\ k = 1, \ldots, N). \end{aligned}$$

Using the standard multi-index notation $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = -i\frac{\partial}{\partial x_j}$, we write $A_{jk}(x, D) = \sum_{|\alpha| \leq s_j + t_k} a_\alpha^{jk}(x) D^\alpha$ for $x \in \Omega$ and $j, k = 1, \ldots, N$ and $B_{jk}(x, D) = \sum_{|\alpha| \leq \sigma_j + t_k} b_\alpha^{jk}(x) D^\alpha$ for $x \in \Gamma$, $j = 1, \ldots, \widetilde{N}$, and $k = 1, \ldots, N$. With respect to the smoothness, we will suppose

(S) (1) Γ is of class $C^{\kappa_0 - 1, 1} \cap C^{s_1}$,
(2) $a_\alpha^{jk} \in C^{s_j}(\overline{\Omega})$ $(|\alpha| \leq s_j + t_k)$ if $s_j > 0$ and
$a_\alpha^{jk} \in C^0(\overline{\Omega})$ $(|\alpha| = s_j + t_k)$, $a_\alpha^{jk} \in L_\infty(\Omega)$ $(|\alpha| < s_j + t_k)$ if $s_j = 0$,
(3) $b_\alpha^{jk} \in C^{-\sigma_j - 1, 1}(\Gamma)$ $(|\alpha| \leq \sigma_j + t_k)$.

Let $\mathring{A}_{jk}(x, \xi)$ consist of all terms in $A_{jk}(x, \xi)$ which are exactly of order $s_j + t_k$, and set

$$\mathring{A}(x, \xi) := \big(\mathring{A}_{jk}(x, \xi)\big)_{j,k=1,\ldots,N} \quad (x \in \overline{\Omega},\ \xi \in \mathbb{R}^n).$$

Analogously, define $\mathring{B}(x, \xi) = \big(\mathring{B}_{jk}(x, \xi)\big)_{\substack{j=1,\ldots,\widetilde{N} \\ k=1,\ldots,N}}$ for $x \in \Gamma$, $\xi \in \mathbb{R}^n$. The operator $B(x, D)$ is said to be essentially upper triangular if $\mathring{B}_{jk}(x, D) = 0$ for $j = N_{\ell-1} + 1, \ldots, N_\ell$, $k = 1, \ldots, k_{\ell-1}$, $\ell = 2, \ldots, d$.

To formulate the ellipticity conditions, let

$$\mathcal{A}_{11}^{(r)}(x, \xi) := \big(\mathring{A}_{jk}(x, \xi)\big)_{j,k=1,\ldots,k_r} \quad (r = 1, \ldots, d).$$

Let I_ℓ denote the $\ell \times \ell$ unit matrix, $\widetilde{I}_\ell := I_{k_\ell - k_{\ell-1}}$, and $\widetilde{I}_{\ell,0} := \operatorname{diag}\big(0 \cdot \widetilde{I}_1, \ldots, 0 \cdot \widetilde{I}_{\ell-1}, \widetilde{I}_\ell\big)$. In the following, let $\mathcal{L} \subset \mathbb{C}$ be a closed sector in the complex plane with vertex at the origin. The following condition is taken from [DF, Section 2] (cf. also [DMV, Section 3]).

(E) For each $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n \setminus \{0\}$, $\lambda \in \mathcal{L}$, and $r = 1, \ldots, d$ we have

$$\det\big(\mathcal{A}_{11}^{(r)}(x, \xi) - \lambda \widetilde{I}_{r,0}\big) \neq 0.$$

If condition (E) holds, the operator $A(x, D) - \lambda I_N$ is said to be parameter-elliptic in $\mathcal{L}$. In order to formulate conditions of Shapiro-Lopatinskii type, for $x^0 \in \Gamma$ we rewrite the boundary value problem (1.1) in terms of local coordinates associated to x^0. In these coordinates $x^0 = 0$, and the positive x_n-axis coincides with the direction of the inner normal to Γ. We will keep the notation for A and B in the new coordinates. In local coordinates associated to $x^0 \in \Gamma$, let

$$\mathcal{B}_{r,1}^{(r,r)}(0,\xi',D_n) := \big(\mathring{B}_{jk}(0,\xi',D_n)\big)_{\substack{j=1,\dots,N_r\\ k=1,\dots,k_r}} \quad (r=1,\dots,d),$$
$$\mathcal{B}_{r,1}^{(1,r)}(0,\xi',D_n) := \big(\mathring{B}_{jk}(0,\xi',D_n)\big)_{\substack{j=N_{r-1}+1,\dots,N_r\\ k=1,\dots,k_r}} \quad (r=2,\dots,d),$$

The following conditions (see [DF, Section 2]) are of Shapiro-Lopatinskii type and of Vishik-Lyusternik type, respectively (cf. also [DV, Section 2.3]).

(SL) For each $x^0 \in \Gamma$ rewrite (1.1) in local coordinates associated to x^0. Then for $r = 1,\dots,d$, the boundary value problem on the half-line,

$$\begin{aligned} \mathcal{A}_{11}^{(r)}(0,\xi',D_n)v(x_n) - \lambda\widetilde{I}_{r,0}v(x_n) &= 0 \quad (x_n>0),\\ \mathcal{B}_{r,1}^{(r,r)}(0,\xi',D_n)v(x_n) &= 0 \quad (x_n=0),\\ |v(x_n)| &\to 0 \quad (x_n\to\infty) \end{aligned} \tag{1.2}$$

has only the trivial solution for $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$, $\lambda \in \mathcal{L}$.

(VL) For each $x^0 \in \Gamma$ rewrite (1.1) in local coordinates associated to x^0. Then for $r = 2,\dots,d$, the boundary value problem on the half-line,

$$\begin{aligned} \mathcal{A}_{11}^{(r)}(0,0,D_n)v(x_n) - \lambda\widetilde{I}_{r,0}v(x_n) &= 0 \quad (x_n>0),\\ \mathcal{B}_{r,1}^{(1,r)}(0,0,D_n)v(x_n) &= 0 \quad (x_n=0),\\ |v(x_n)| &\to 0 \quad (x_n\to\infty) \end{aligned} \tag{1.3}$$

has only the trivial solution for $\lambda \in \mathcal{L}\setminus\{0\}$.

We will show that (E), (SL), and (VL) are necessary for a priori estimates to hold. In order to formulate these estimates, we will introduce parameter-dependent norms. For $G \subset \mathbb{R}^\ell$ open, $\ell \in \mathbb{N}$, $s \in \mathbb{N}$ and $1 < p < \infty$, let $\|u\|_{s,p,G}$ denote the norm in the standard Sobolev space $W_p^s(G)$. For $\lambda \in \mathbb{C}\setminus\{0\}$ and $j = 1,\dots,d$ set

$$|\!|\!|u|\!|\!|_{s,p,G}^{(j)} := \|u\|_{s,p,G} + |\lambda|^{s/\widetilde{m}_j}\|u\|_{0,p,G} \quad (u \in W_p^s(G)).$$

For $s < 0$, $s \in \mathbb{Z}$, and $j = 1,\dots,d$, let $H_p^s(\mathbb{R}^n)$ be the Bessel-potential space equipped with the parameter-dependent norm $|\!|\!|u|\!|\!|_{s,p,\mathbb{R}^n}^{(j)} := \|F^{-1}\langle\xi,\lambda\rangle_j^s Fu\|_{0,p,\mathbb{R}^n}$ where F denotes the Fourier transform in $\mathbb{R}^n$ ($x \to \xi$) and where $\langle\xi,\lambda\rangle_j^s := \big(|\xi|^2 + |\lambda|^{2/\widetilde{m}_j}\big)^{1/2}$. For $G \subset \mathbb{R}^n$ open, set $|\!|\!|u|\!|\!|_{s,p,G}^{(j)} := \inf\{|\!|\!|v|\!|\!|_{s,p,\mathbb{R}^n}^s : v \in H_p^s(\mathbb{R}^n),\ v|_G = u\}$. Finally, for $s \in \mathbb{N}$ we define the parameter-dependent norm on the boundary by

$$|\!|\!|v|\!|\!|_{s-1/p,p,\partial G}^{(j)} := \|v\|_{s-1/p,p,\partial G} + |\lambda|^{(s-1/p)/\widetilde{m}_j}\|v\|_{0,p,\partial G} \quad (v \in W_p^{s-1/p}(\partial G)).$$

For $j = 1,\dots,N$, let $\pi_1(j) := r$ if $k_{r-1} < j \le k_r$. Similarly, for $j = 1,\dots,N_d$ let $\pi_2(j) := r$ if $N_{r-1} < j \le N_r$. Note that, by definition, $\widetilde{m}_{\pi_1(j)} = m_j$ for $j = 1,\dots,N$.

The aim of the paper is to show the following result.

Theorem 1.1. *Let* (S) *hold, let $1 < p < \infty$, and assume that there exist constants $C_0, C_1 > 0$ such that for all $\lambda \in \mathcal{L}$, $|\lambda| \ge C_0$ and all $u \in \prod_{j=1}^N W_p^{t_j}(\Omega)$ the a priori* *estimate*

$$\sum_{j=1}^{N} \|u_j\|^{(\pi_1(j))}_{t_j,p,\Omega} \le C_1\Big(\sum_{j=1}^{N} \|f_j\|^{(\pi_1(j))}_{-s_j,p,\Omega} + \sum_{j=1}^{\widetilde N} \|g_j\|^{(\pi_2(j))}_{-\sigma_j-1/p,p,\Gamma}\Big) \tag{1.4}$$

holds for $f := A(x,D)u - \lambda u$ *and* $g := B(x,D)u$. *Assume further that* $B(x,D)$ *is essentially upper triangular. Then the parameter-ellipticity conditions* (E), (SL), *and* (VL) *are satisfied.*

Remark 1.2. In [DF], the following result was shown, where we refer to [DF] for the definitions of properly parameter-elliptic and compatible: Let (S), (E), (SL), and (VL) hold. Assume further that (A,B) is properly parameter-elliptic, that $B(x,D)$ is essentially upper triangular, and that $A(x^0,D)$ and $B(x^0,D)$ are compatible at every $x_0 \in \Gamma$. Then there exist $C_0, C_1 > 0$ such that for all $\lambda \in \mathcal{L}$, $|\lambda| \ge C_0$, the boundary value problem (1.1) has a unique solution $u \in \prod_{j=1}^N W_p^{t_j}(\Omega)$ for every $f \in \prod_{j=1}^N H_p^{-s_j}(\Omega)$ and every $g \in \prod_{j=1}^{\widetilde N} W_p^{-\sigma_j-1/p}(\Gamma)$, and the a priori estimate (1.4) holds.

In this sense, the sufficiency of parameter-ellipticity for the validity of the a priori estimate was shown in [DF] while Theorem 1.1 states the necessity of the conditions (E), (SL), and (VL).

2. Proof of the necessity

Throughout this section, we assume condition (S) to hold, and fix a closed sector $\mathcal{L} \subset \mathbb{C}$. In the following, C stands for a generic constant which may vary from inequality to inequality but which is independent of the functions appearing in the inequality and independent of λ. Let $B_\delta(x^0) := \{x \in \mathbb{R}^n : |x - x^0| < \delta\}$, and let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$, $\mathbb{R}_+ := (0,\infty)$. We start with some useful remarks on negative-order Sobolev spaces where $C_0^\infty(\overline{\mathbb{R}^n_+})$ stands for the set of all restrictions of functions in $C_0^\infty(\mathbb{R}^n)$ to $\mathbb{R}^n_+$.

Lemma 2.1. *Let* $s \in \mathbb{N}$, $1 < p < \infty$, $j \in \{1,\dots,N\}$. *Then for all* $v \in L_p(\mathbb{R}^n)$ *and all* $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$, *we have:*

a) $\|v\|^{(j)}_{-s,p,\mathbb{R}^n} \le C\|v\|_{-s,p,\mathbb{R}^n}$,

b) $\|D^\alpha v\|^{(j)}_{-s,p,\mathbb{R}^n} \le |\lambda|^{(|\alpha|-s)/\widetilde m_j}\|v\|_{0,p,\mathbb{R}^n}$ *for all* $|\alpha| \le s$,

c) for each $\phi \in C_0^\infty(\mathbb{R}^n)$ *there exists a constant* $C_\phi > 0$ *independent of* v *such that* $\|v\phi\|_{-s,p,\mathbb{R}^n} \le C_\phi\|v\|_{-s,p,\mathbb{R}^n}$ *and* $\|v\phi\|^{(j)}_{-s,p,\mathbb{R}^n} \le C_\phi\|v\|^{(j)}_{-s,p,\mathbb{R}^n}$.

The same assertions hold if we replace $\mathbb{R}^n$ *by* $\mathbb{R}^n_+$ *and* $C_0^\infty(\mathbb{R}^n)$ *in c) by* $C_0^\infty(\overline{\mathbb{R}^n_+})$.

Proof. a) We have

$$\|v\|^{(j)}_{-s,p,\mathbb{R}^n} = \|F^{-1}\langle\xi,\lambda\rangle_j^{-s}Fv\|_{0,p,\mathbb{R}^n} = \Big\|F^{-1}\,\frac{\langle\xi\rangle^s}{\langle\xi,\lambda\rangle_j^s}\,\langle\xi\rangle^{-s}Fv\Big\|_{0,p,\mathbb{R}^n}.$$

Now the assertion follows immediately from the Mikhlin-Lizorkin multiplier theorem.

b) Similarly,

$$|\lambda|^{(s-|\alpha|)/\widetilde m_j}\|D^\alpha v\|^{(j)}_{-s,p,\mathbb{R}^n} = \|F^{-1}m(\xi,\lambda)Fv\|_{0,p,\mathbb{R}^n}$$

with $m(\xi,\lambda) := |\lambda|^{(s-|\alpha|)/\widetilde m_j}\xi^\alpha\langle\xi,\lambda\rangle_j^{-s}$. Noting that m is infinitely smooth in ξ and quasi-homogeneous in (ξ,λ) of degree 0 in the sense that $m(\rho\xi,\rho^{\widetilde m_j}\lambda) = m(\xi,\lambda)$

for $\rho > 0$, we see that we may apply the Mikhlin-Lizorkin theorem to obtain the statement in b).

c) We make use of the dual pairing of $H_p^{-s}(\mathbb{R}^n)$ and $W_q^s(\mathbb{R}^n)$, $\frac{1}{p}+\frac{1}{q}=1$, and get

$$\|v\phi\|_{-s,p,\mathbb{R}^n} = \sup_\zeta |\langle v\phi,\zeta\rangle| = \sup_\zeta \Big| \int v(x)\phi(x)\zeta(x)dx \Big| = \sup_\zeta |\langle v, \phi\zeta\rangle|,$$

where the supremum is taken over all $\zeta \in C_0^\infty(\mathbb{R}^n)$ with $\|\zeta\|_{s,q,\mathbb{R}^n} \le 1$. Now we make use of $\|\phi\zeta\|_{s,q,\mathbb{R}^n} \le C_\phi \|\zeta\|_{s,q,\mathbb{R}^n}$ with $C_\phi := C_{s,q} \sup\{|D^\alpha \phi(x)| : |\alpha| \le s,\, x \in \mathbb{R}^n\}$ where $C_{s,q}$ is a constant depending on s and q only. We obtain $\sup_\zeta |\langle v,\phi\zeta\rangle| \le C_\phi \sup_\zeta |\langle v,\zeta\rangle| = C_\phi \|v\|_{-s,p,\mathbb{R}^n}$.

For the parameter-dependent norms $|\!|\!|\cdot|\!|\!|^{(j)}_{-s,p,\mathbb{R}^n}$ we again consider the dual pairing between $H_p^{-s}(\mathbb{R}^n)$ and $W_q^s(\mathbb{R}^n)$, but now with respect to the parameter-dependent norm $|\!|\!|\cdot|\!|\!|^{(j)}_{s,q,\mathbb{R}^n}$ on $W_q^s(\mathbb{R}^n)$. Then the result follows in exactly the same way, noting that

$$|\!|\!|\phi\zeta|\!|\!|^{(j)}_{s,q,\mathbb{R}^n} = \|\phi\zeta\|_{s,q,\mathbb{R}^n} + |\lambda|^{s/\widetilde{m}_j}\|\phi\zeta\|_{0,p,\mathbb{R}^n} \le C_\phi\Big(\|\zeta\|_{s,q,\mathbb{R}^n} + |\lambda|^{s/\widetilde{m}_j}\|\zeta\|_{0,p,\mathbb{R}^n}\Big).$$

Finally, in the case of $\mathbb{R}^n_+$ instead of $\mathbb{R}^n$ the assertions of the lemma follow easily from the results in $\mathbb{R}^n$ and the fact that there exists an extension operator $E\colon u \mapsto Eu$ which is continuous as an operator from $H_p^r(\mathbb{R}^n_+)$ to $H_p^r(\mathbb{R}^n)$ for all $|r| \le s$ (see [T, p. 218]). □

The following lemma will allow us to consider the model problem in $\mathbb{R}^n$ for the proof of the necessity.

Lemma 2.2. *Assume that there exist constants $C_0, C_1 > 0$ such that for all $u \in \prod_{j=1}^N W_p^{t_j}(\mathbb{R}^n)$ and all $\lambda \in \mathcal{L}$, $|\lambda| \ge C_0$, the a priori estimate* (1.4) *holds. Let $x^0 \in \overline{\Omega}$. Then there exist an $x^1 \in \Omega$, a $\delta > 0$ with $\overline{B_\delta(x^1)} \subset \Omega$, and a $\widetilde{\lambda} > 0$ such that for all $\lambda \in \mathcal{L}$ with $|\lambda| \ge \widetilde{\lambda}$ and all $u \in \prod_{j=1}^N W_p^{t_j}(\mathbb{R}^n)$ with $\operatorname{supp} u \subset B_\delta(x^1)$, we have*

$$\sum_{j=1}^N |\!|\!|u|\!|\!|^{(\pi_1(j))}_{t_j,p,\mathbb{R}^n} \le C \sum_{j=1}^N |\!|\!|f_j^0|\!|\!|^{(\pi_1(j))}_{-s_j,p,\mathbb{R}^n}, \tag{2.1}$$

where we have set $f^0 := (\mathring{A}(x^0,D)-\lambda)u$.

Proof. In [DF, Prop. 4.1] it was shown that for any $\varepsilon > 0$ there exist a $\delta_0 > 0$ and a $\lambda_0 > 0$ such that for $\lambda \in \mathcal{L}$, $|\lambda| \le \lambda_0$, and all $u \in \prod_{j=1}^N W_p^{t_j}(\mathbb{R}^n)$ with $\operatorname{supp} u \subset B_\delta(x^0) \cap \overline{\Omega}$ we have

$$\sum_{j=1}^N |\!|\!|f_j - f_j^0|\!|\!|^{(\pi_1(j))}_{-s_j,p,\Omega} \le \varepsilon \sum_{j=1}^N \|u_j\|_{t_j,p,\Omega}$$

where $f := (A(x,D)-\lambda)u$. Let ε be sufficiently small. If $x^0 \in \Omega$ we choose $x^1 := x^0$ and $\delta := \frac{1}{2}\min\{\delta_0, \operatorname{dist}(x^0,\Gamma)\}$. If $x^0 \in \Gamma$ we choose $x^1 \in B_\delta(x^0) \cap \Omega$ and $\delta > 0$ sufficiently small such that $\overline{B_\delta(x^1)} \subset B_\delta(x^0) \cap \Omega$. In both cases, the statement of the lemma follows easily by arguments similar to those used in the proof of [AV, Lemma 4.2]. □

Proposition 2.3. *Under the assumptions of Lemma* 2.2*, condition* (E) *is satisfied, i.e., for* $r = 1, \dots, d$*,* $x^0 \in \overline{\Omega}$*,* $\xi^0 \in \mathbb{R}^n \setminus \{0\}$*, and* $\lambda^0 \in \mathcal{L}$ *we have*

$$\det \big(\mathcal{A}_{11}^{(r)}(x^0,\xi^0) - \lambda^0 \widetilde{I}_{r,0}\big) \neq 0.$$

Proof. Assume that (E) does not hold. Then there exist $r \in \{1,\dots,d\}$, $x^0 \in \overline{\Omega}$, $\xi^0 \in \mathbb{R}^n \setminus \{0\}$, $\lambda^0 \in \mathcal{L}$, and a vector $h \in \mathbb{C}^{k_r} \setminus \{0\}$ such that $(\mathcal{A}_{11}^{(r)}(x^0,\xi^0) - \lambda^0 \widetilde{I}_{r,0})h = 0$.

Let us first consider the case $\lambda^0 = 0$. We choose $x^1 \in \Omega$, $\delta > 0$ and $\widetilde{\lambda} > 0$ according to Lemma 2.1. Let $\phi \in C_0^\infty(B_\delta(x^1))$ with $\phi \not\equiv 0$, and for $\rho > 1$ set

$$u_j(x) := \begin{cases} \phi(x) e^{i\rho\xi^0\cdot x} \rho^{-t_j} h_j, & j = 1,\dots,k_r, \\ 0, & j = k_r+1,\dots,N, \end{cases}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^n$. We are now going to use (2.1) to arrive at a contradiction. Indeed, we easily see that for $j = 1,\dots,k_r$,

$$\|u_j\|_{t_j,p,\mathbb{R}^n} \geq |h_j|\, |\xi_\ell^0|^{t_j} \|\phi\|_{0,p,\mathbb{R}^n} - C\rho^{-1}$$

where $\xi_\ell^0 \neq 0$. We further choose μ with

$$\widetilde{m}_{r+1} < \mu < \widetilde{m}_r \;\text{ if }\; r < d \;\text{ and }\; \widetilde{m}_d/2 < \mu < \widetilde{m}_d \;\text{ if }\; r = d, \tag{2.2}$$

and choose $\lambda \in \mathcal{L}$ with $|\lambda| = \rho^\mu$. Then it is clear that

$$|\lambda|^{t_j/\widetilde{m}_j} \|u_j\|_{0,p,\mathbb{R}^n} = \rho^{-t_j(1-\mu/\widetilde{m}_j)} |h_j|\, \|\phi\|_{0,p,\mathbb{R}^n}.$$

Thus we have shown that

$$\sum_{j=1}^N |\!|\!| u_j |\!|\!|_{t_j,p,\mathbb{R}^n}^{(\pi_1(j))} \geq \frac{1}{2}\Big(\sum_{j=1}^{k_r} |h_j|\, |\xi_\ell^0|^{t_j}\Big) \|\phi\|_{0,p,\mathbb{R}^n} \tag{2.3}$$

for sufficiently large ρ.

Turning next to the right-hand side of (2.1), let $j \in \{1,\dots,N\}$. Then

$$\begin{aligned} |\!|\!| f_j^0 |\!|\!|_{-s_j,p,\mathbb{R}^n}^{(\pi_1(j))} &= \Big|\!\Big|\!\Big| \sum_{k=1}^{k_r} \mathring{A}_{jk}(x^0,D)u_k - \delta_{jk}\lambda u_k \Big|\!\Big|\!\Big|_{-s_j,p,\mathbb{R}^n}^{(\pi_1(j))} \\ &\leq \Big|\!\Big|\!\Big| \sum_{k=1}^{k_r} \sum_{|\alpha| = s_j+t_k} a_\alpha^{jk}(x^0) \sum_\beta \binom{\alpha}{\beta} \rho^{-t_k} h_k D^\beta(e^{i\rho\xi^0\cdot x}) D^{\alpha-\beta}\phi \Big|\!\Big|\!\Big|_{-s_j,p,\mathbb{R}^n}^{(\pi_1(j))} \\ &\quad + \sum_{k=1}^{k_r} \delta_{jk}\rho^{-r_k+\mu} |h_k|\, |\!|\!| e^{i\rho\xi^0\cdot x}\phi |\!|\!|_{-s_j,p,\mathbb{R}^n}^{(\pi_1(j))} =: I_1 + I_2, \end{aligned}$$

where δ_{jk} denotes the Kronecker delta and where $\sum_\beta = \sum_{\beta<\alpha}$ if $j \leq k_r$ and $\sum_\beta = \sum_{\beta\leq\alpha}$ if $r < d$ and $j > k_r$. (Here we used the fact that $\mathcal{A}_{11}^{(r)}(x^0,\xi^0)h = 0$.)

It is clear that $I_2 \to 0$ as $\rho \to \infty$. Hence fixing our attention next upon I_1, we see that $I_1 \leq \sum_{k=1}^{k_r} \sum_{|\alpha|=s_j+t_k} \sum_\beta I_{1,k}^{\alpha,\beta}$ with

$$I_{1,k}^{\alpha,\beta} := \Big|\!\Big|\!\Big| \binom{\alpha}{\beta} a_\alpha^{jk}(x^0) \rho^{-t_k} D^\beta(e^{i\rho\xi^0\cdot x}) D^{\alpha-\beta}\phi \Big|\!\Big|\!\Big|_{-s_j,p,\mathbb{R}^n}^{(\pi_1(j))}.$$

To establish $I_{1,k}^{\alpha,\beta} \to 0$ $(\rho \to \infty)$ and, in consequence, a contradiction, it remains to show that for all appearing indices we have

$$\rho^{-t_k} |\!|\!| D^\beta(e^{i\rho\xi^0\cdot x}) D^{\alpha-\beta}\phi |\!|\!|_{-s_j,p,\mathbb{R}^n}^{(\pi_1(j))} \to 0 \quad (\rho \to \infty).$$

Let $j \in \{1, \dots, k_r\}$, $|\alpha| = s_j + t_k$, and $\beta < \alpha$. If $|\beta| \le t_k$, we apply Lemma 2.1 b) to obtain

$$\rho^{-t_k} \|D^\beta(e^{i\rho\xi^0\cdot x})D^{\alpha-\beta}\phi\|^{(\pi_1(j))}_{-s_j,p,\mathbb{R}^n} \le C\rho^{-t_k-s_j\mu/m_j}\|D^\beta(e^{i\rho\xi^0\cdot x})D^{\alpha-\beta}\phi\|_{0,p,\mathbb{R}^n}$$
$$\le C\rho^{-t_k+|\beta|-s_j\mu/m_j}|(\xi^0)^\beta|\,\|D^{\alpha-\beta}\phi\|_{0,p,\mathbb{R}^n} \to 0 \quad (\rho\to\infty).$$

Note here that $s_j = 0$ implies $|\beta| < t_k$.

If $|\beta| \ge t_k$, we write $\beta = \beta_1 + \beta_2$ with $|\beta_1| = t_k$, $|\beta_2| < s_j$ and fix $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi = 1$ on $\operatorname{supp}\phi$. Then, using Lemma 2.1 b) and c),

$$\begin{aligned}\rho^{-t_k} \|D^\beta(e^{i\rho\xi^0\cdot x})D^{\alpha-\beta}\phi\|^{(\pi_1(j))}_{-s_j,p,\mathbb{R}^n} &= \rho^{-t_k}\|D^\beta(e^{i\rho\xi^0\cdot x}\psi)D^{\alpha-\beta}\phi\|^{(\pi_1(j))}_{-s_j,p,\mathbb{R}^n}\\ &\le C\rho^{-t_k}\|D^{\beta_2}(D^{\beta_1}e^{i\rho\xi^0\cdot x}\psi)\|^{(\pi_1(j))}_{-s_j,p,\mathbb{R}^n}\\ &\le C\rho^{-t_k-(s_j-|\beta_2|)\mu/m_j}\|D^{\beta_1}e^{i\rho\xi^0\cdot x}\psi\|_{0,p,\mathbb{R}^n}\\ &\le C\rho^{-(s_j-|\beta_2|)\mu/m_j}\big(\|\psi\|_{0,p,\mathbb{R}^n}+C\rho^{-1}\big)\to 0 \quad (\rho\to\infty).\end{aligned}$$

Now let $j \in \{k_r+1,\dots,N\}$. Again by Lemma 2.1 b), we have for $|\alpha| = s_j + t_k$ and $|\beta| \le |\alpha|$

$$\rho^{-t_k}\|D^\beta(e^{i\rho\xi^0\cdot x})D^{\alpha-\beta}\phi\|^{(\pi_1(j))}_{-s_j,p,\mathbb{R}^n} \le C\rho^{-s_j\mu/m_j+s_j}\|D^{\alpha-\beta}\phi\|_{0,p,\mathbb{R}^n}\to 0 \quad (\rho\to\infty)$$

as $\mu/m_j > 1$.

Finally, the case $\lambda^0 \ne 0$ can be dealt with by arguing in a manner similar to that above, except now we take $\lambda = \lambda^0\rho^{\widetilde{m}_r}$. □

To prove the necessity of (SL) and (VL), we transform the problem to the half-space. For this let $x^0 \in \Gamma$ and assume that (A, B) is given in local coordinates associated to x^0. Let $\{U, \Phi\}$ be a chart on Γ such that $x^0 = 0 \in U$, $\Phi(0) = 0$, and Φ is a diffeomorphism of class $C^{\kappa_0-1,1}\cap C^{s_1}$ mapping U onto an open set in $\mathbb{R}^n$ with $\Phi(U\cap\Omega)\subset\mathbb{R}^n_+$, $\Phi(U\cap\Gamma)\subset\mathbb{R}^{n-1}$. We denote the push-forward of the operators $A(x,D)$ and $B(x,D)$ by $\widetilde{A}(y,D)$ and $\widetilde{B}(y,D)$, respectively, where $y = \Phi(x)$.

Replacing $\Phi(x)$ by $D\Phi(0)^{-1}\Phi(x)$, it is easily seen that we may assume the Jacobian $D\Phi(0)$ to be equal to I_n. Then we have $\mathring{\widetilde{A}}_{jk}(0,\xi) = \mathring{A}_{jk}(0,\xi)$ and $\mathring{\widetilde{B}}_{jk}(0,\xi) = \mathring{B}_{jk}(0,\xi)$. In particular, (SL) and (VL) are satisfied for $(\widetilde{A},\widetilde{B})$ at 0 if only if this holds for (A, B) at $x^0 = 0$ (see also [DHP, p. 205]).

Lemma 2.4. *Under the assumptions of Lemma 2.2, let $x^0 \in \Gamma$ and assume (A,B) to be written in coordinates associated to x^0. Then there exist a $\delta > 0$ and a $\widetilde{\lambda} > 0$ such that for all $u \in \prod_{j=1}^N W_p^{t_j}(\mathbb{R}^n_+)$ with $\operatorname{supp} u \subset B_\delta(0)\cap\overline{\mathbb{R}^n_+}$ and all $\lambda\in\mathcal{L}$ with $|\lambda| \ge \widetilde{\lambda}$, we have*

$$\sum_{j=1}^N \|u_j\|^{(\pi_1(j))}_{t_j,p,\mathbb{R}^n_+} \le C\Big(\sum_{j=1}^N \|f_j^0\|^{(\pi_1(j))}_{-s_j,p,\mathbb{R}^n_+} + \sum_{j=1}^{\widetilde{N}} \|g_j^0\|^{(\pi_2(j))}_{-\sigma_j-1/p,p,\mathbb{R}^{n-1}}\Big), \tag{2.4}$$

where $f^0 := (\mathring{A}(0,D)-\lambda)u$, $g^0 := \mathring{B}(0,D)u$.

Proof. Let Φ be as above, and let $\widetilde{A}(y,D)$ and $\widetilde{B}(y,D)$ be the push-forward of $A(x,D)$ and $B(x,D)$, respectively. Then

$$\Phi_*\big[(a^{jk}_\alpha(x)-a^{jk}_\alpha(0))D^\alpha u_k\big] = (\widetilde{a}^{jk}_\alpha(y)-\widetilde{a}^{jk}_\alpha(0))D^\alpha_y\widetilde{u}_k + \sum_{|\beta|<|\alpha|}\widetilde{a}^{jk}_{\alpha,\beta}(y)D^\beta_y\widetilde{u}_k.$$

It was shown in the proof of [DF, Prop. 4.1], that for each $\varepsilon > 0$ there exist a $\delta_0 > 0$ and a $\lambda_0 > 0$ such that for all $u \in \prod_{j=1}^N W_p^{t_j}(\Omega)$ with $\operatorname{supp} u \subset B_{\delta_0}(0) \cap \overline{\Omega}$ and all $\lambda \in \mathcal{L}$, $|\lambda| \geq \lambda_0$, we have

$$\| \Phi_* \big[(a_\alpha^{jk}(x) - a_\alpha^{jk}(0)) D^\alpha u_k \big] \|_{-s_j,p,\mathbb{R}^n_+}^{(\pi_1(j))} \leq \varepsilon \| \widetilde{u}_k \|_{t_k,p,\mathbb{R}^n_+}$$

for $|\alpha| = s_j + t_k$, and

$$\| \Phi_* \big[a_\alpha^{jk}(x) D^\alpha u_k \big] \|_{-s_j,p,\mathbb{R}^n_+}^{(\pi_1(j))} \leq \varepsilon \| \widetilde{u}_k \|_{t_k,p,\mathbb{R}^n_+}$$

for $|\alpha| < s_j + t_k$. From this we easily obtain that for all $\varepsilon > 0$ there exist $\delta_0, \lambda_0 > 0$ such that for all $u \in \prod_{j=1}^N W_p^{t_j}(\Omega)$ with $\operatorname{supp} u \subset B_{\delta_0}(0)$,

$$\sum_{j=1}^N \| \widetilde{f}_j \|_{-s_j,p,\mathbb{R}^n_+} \leq C \sum_{j=1}^N \| \widetilde{f_j^0} \|_{-s_j,p,\mathbb{R}^n_+} + \varepsilon \sum_{j=1}^N \| u_j \|_{t_j,p,\mathbb{R}^n_+}$$

where we have set $f := (A(x,D) - \lambda)u$, $\widetilde{f} := \Phi_* f$, $\widetilde{f^0} := \Phi_* f^0$.

To estimate g^0, we first remark that we may assume b_α^{jk} to be defined on $\overline{\Omega}$ with $b_\alpha^{jk} \in C^{-\sigma_j-1,1}(\overline{\Omega})$. We define the function h on Ω by $h_j := \sum_{j=1}^{\widetilde{N}} b_\alpha^{jk}(x) D^\alpha u_k$, $h_j^0 := \sum_{j=1}^{\widetilde{N}} b_\alpha^{jk}(0) D^\alpha u_k$ and set $\widetilde{h} := \Phi_* h$, $\widetilde{h^0} := \Phi_* h^0$. In the same way as above, we obtain

$$\begin{aligned} \sum_{j=1}^{\widetilde{N}} \| \widetilde{g}_j \|_{-\sigma_j-1/p,p,\mathbb{R}^{n-1}}^{(\pi_2(j))} &\leq \sum_{j=1}^{\widetilde{N}} \| \widetilde{h}_j \|_{-\sigma_j-1/p,p,\mathbb{R}^{n-1}}^{(\pi_2(j))} \\ &\leq C \sum_{j=1}^{\widetilde{N}} \| \widetilde{h_j^0} \|_{-\sigma_j-1/p,p,\mathbb{R}^{n-1}}^{(\pi_2(j))} + \varepsilon \sum_{j=1}^N \| \widetilde{u} \|_{t_j,p,\mathbb{R}^n_+}. \end{aligned}$$

Finally, it was shown in [DF, p. 362-363] that there exist constants $c_1, c_2 > 0$ such that for all $u \in \prod_{j=1}^N W_p^{t_j}(\Omega)$ with $\operatorname{supp} u \subset B_\delta(x^0)$, $B_{2\delta}(x^0) \subset U$, we have

$$\begin{aligned} c_1 \| u_j \|_{t_j,p,\Omega}^{(\pi_1(j))} &\leq \| \widetilde{u}_j \|_{t_j,p,\mathbb{R}^n_+}^{(\pi_1(j))} &&\leq c_2 \| u_j \|_{t_j,p,\Omega}^{(\pi_1(j))}, \\ c_1 \| f_j \|_{-s_j,p,\Omega}^{(\pi_1(j))} &\leq \| \widetilde{f}_j \|_{-s_j,p,\mathbb{R}^n_+}^{(\pi_1(j))} &&\leq c_2 \| f_j \|_{-s_j,p,\Omega}^{(\pi_1(j))}, \\ c_1 \| g_j \|_{-\sigma_j-1/p,p,\Gamma}^{(\pi_1(j))} &\leq \| \widetilde{g}_j \|_{-\sigma_j-1/p,p,\mathbb{R}^{n-1}}^{(\pi_1(j))} &&\leq c_2 \| g_j \|_{-\sigma_j-1/p,p,\Gamma}^{(\pi_1(j))}. \end{aligned} \tag{2.5}$$

Therefore, from the a priori-estimate (1.4) we obtain that for each $\varepsilon > 0$ there exist $\delta, \widetilde{\lambda} > 0$ such that for $u \in \prod_{j=1}^N W_p^{t_j}(\Omega)$ with $\operatorname{supp} u \subset B_\delta(0)$ and $\lambda \in \mathcal{L}$, $|\lambda| \geq \widetilde{\lambda}$, we have

$$\begin{aligned} \sum_{j=1}^N \| \widetilde{u}_j \|_{t_j,p,\mathbb{R}^n_+}^{(\pi_1(j))} &\leq C \sum_{j=1}^N \| u_j \|_{t_j,p,\Omega}^{(\pi_1(j))} \leq C \Big(\sum_{j=1}^N \| f_j \|_{-s_j,p,\Omega}^{(\pi_1(j))} + \sum_{j=1}^{\widetilde{N}} \| g_j \|_{-\sigma_j-1/p,p,\Gamma}^{(\pi_2(j))} \Big) \\ &\leq C \Big(\sum_{j=1}^N \| \widetilde{f}_j \|_{-s_j,p,\mathbb{R}^n_+}^{(\pi_1(j))} + \sum_{j=1}^{\widetilde{N}} \| \widetilde{g}_j \|_{-\sigma_j-1/p,p,\mathbb{R}^{n-1}}^{(\pi_2(j))} \Big) + \varepsilon \sum_{j=1}^N \| \widetilde{u}_j \|_{t_j,p,\mathbb{R}^n_+}^{(\pi_1(j))}. \end{aligned}$$

Taking ε small enough and λ large enough and noting (2.5) and $\mathring{\widetilde{A}}(0,D) = \mathring{A}(0,D)$ and $\mathring{\widetilde{B}}(0,D) = \mathring{B}(0,D)$, we obtain the assertion of the Lemma. □

Proposition 2.5. *Assume that there exist constants $C_0, C_1 > 0$ such that for all $u \in \prod_{j=1}^N W_p^{t_j}(\mathbb{R}^n)$ and all $\lambda \in \mathcal{L}$, $|\lambda| \geq C_0$, the a priori estimate* (1.4) *holds. Further, let $x^0 \in \Gamma$, and assume that $B(x^0, D)$ is essentially upper triangular. Then condition* (SL) *holds at x^0.*

Proof. Let (A, B) be written in coordinates associated to x^0 and assume that (SL) does not hold. Then there exist $r \in \{1, \dots, d\}$, $\lambda^0 \in \mathcal{L}$, $\xi_0' = (\xi_1^0, \dots, \xi_{n-1}^0) \in \mathbb{R}^{n-1} \setminus \{0\}$, and $v \not\equiv 0$ satisfying (1.2). By Proposition 2.3, we know that the polynomial $\det(\mathcal{A}_{11}^{(r)}(0, \xi_0', \tau) - \lambda^0 \widetilde{I}_{r,0})$ as a function of τ has no real roots. Therefore, $v = v(x_n)$ is infinitely smooth and decays exponentially for $x_n \to \infty$, in particular, $v \in L_p(\mathbb{R}_+)$.

Again, let us first consider the case $\lambda^0 = 0$. We choose $\phi' \in C_0^\infty(\mathbb{R}^{n-1})$ such that $\phi' \not\equiv 0$ and $\operatorname{supp} \phi' \subset B_\delta(0)$ with δ from Lemma 2.4, $\psi \in C_0^\infty([0, \delta))$ with $0 \leq \psi \leq 1$ and $\psi(x_n) = 1$ for $0 \leq x_n \leq \delta/2$, and $\lambda \in \mathcal{L}$ with $|\lambda| = \rho^\mu$ where μ satisfies (2.2). For $x \in \mathbb{R}_+^n$, we set $w(x) := e^{i\xi_0' \cdot x'} v(x_n)$, $\phi(x) := \phi'(x')\psi(x_n)$, and

$$u_j(x) := \begin{cases} \rho^{-t_j + 1/p} w_j(\rho x) \phi(x), & j = 1, \dots, k_r, \\ 0, & j = k_r + 1, \dots, N. \end{cases} \tag{2.6}$$

We will show that (2.4) leads to a contradiction for large ρ. For this we first remark that for $j = 1, \dots, k_r$

$$\begin{aligned} \rho \|v_j(\rho x_n)\psi(x_n)\|_{0,p,\mathbb{R}_+}^p &= \rho \int_0^\infty |v_j(\rho x_n)\psi(x_n)|^p dx_n \\ &= \int_0^\infty |v_j(y_n)\psi(y_n/\rho)|^p dy_n \nearrow \|v_j\|_{0,p,\mathbb{R}_+}^p \quad (\rho \to \infty). \end{aligned}$$

Therefore, for $\rho \geq \rho_0$, ρ_0 being sufficiently large, we have

$$\frac{1}{2} \rho^{-1/p} \|v_j\|_{0,p,\mathbb{R}_+} \leq \|v_j(\rho x_n)\psi(x_n)\|_{0,p,\mathbb{R}_+} \leq \rho^{-1/p} \|v_j\|_{0,p,\mathbb{R}_+}.$$

In the same way, we see that for any $\zeta \in C_0^\infty(\overline{\mathbb{R}_+^n})$ and $\alpha \in \mathbb{N}_0^n$ we have

$$\|D^\alpha w_j(\rho x)\zeta(x)\|_{0,p,\mathbb{R}_+^n} \leq C_\zeta \rho^{|\alpha| - 1/p} \|v_j\|_{|\alpha|,p,\mathbb{R}_+}$$

with a constant C_ζ depending on ζ but not on v or ρ.

Turning now to the left-hand side of (2.4), the above considerations show that for ρ sufficiently large,

$$|\!|\!| u_j |\!|\!|_{t_j,p,\mathbb{R}_+^n}^{(\pi_1(j))} \geq \|u_j\|_{t_j,p,\mathbb{R}_+^n} \geq \frac{1}{2} |\xi_\ell^0|^{t_j} \|\phi'\|_{0,p,\mathbb{R}^{n-1}} \|v_j\|_{0,p,\mathbb{R}_+}. \tag{2.7}$$

On the right-hand side of (2.4), the terms $|\!|\!| f_j^0 |\!|\!|_{-s_j,p,\mathbb{R}_+^n}^{(\pi_1(j))}$ can be estimated in the same way as in the proof of Proposition 2.3. Indeed, we have

$$f_j^0(x) = \sum_{k=1}^{k_r} \Big(\sum_{|\alpha| = s_j + t_k} \sum_\beta a_\alpha^{jk}(0) \binom{\alpha}{\beta} \rho^{-t_k + 1/p} (D^\beta w)(\rho x)(D^{\alpha - \beta}\phi)(x) + \delta_{jk} \lambda u_k \Big)$$

where $\sum_\beta = \sum_{\beta < \alpha}$ if $j \leq k_r$ and $\sum_\beta = \sum_{\beta \leq \alpha}$ if $j > k_r$. Here we used the fact $\mathring{A}(0, D) w(x) = e^{i\xi_0' \cdot x'} \mathring{A}(0, \xi_0', D_n) v(x_n) = 0$. From this we obtain in the same way as in the proof of Proposition 2.3

$$\sum_{j=1}^N |\!|\!| f_j^0 |\!|\!|_{-s_j,p,\mathbb{R}_+^n}^{(\pi_1(j))} \to 0 \quad (\rho \to \infty). \tag{2.8}$$

To estimate g_j^0, we first remark that

$$\mathring{B}_{jk}(0,\rho\xi_0',D_n)v_k(\rho x_n)\psi(x_n)\big|_{x_n=0} = \rho^{\sigma_j+t_k}\mathring{B}_{jk}(0,\xi_0',D_n)v_k(x_n)\big|_{x_n=0}$$

by homogeneity and as $\psi(x_n)=1$ near $x_n=0$. Therefore, for $j=1,\dots,N_r$ we have

$$\begin{aligned} g_j &= \sum_{k=1}^{k_r} \rho^{-t_k+1/p}\mathring{B}_{jk}(0,D)w_k(\rho x)\phi(x)\big|_{x_n=0} \\ &= \sum_{k=1}^{k_r}\sum_{|\alpha|=\sigma_j+t_k}\sum_{\beta<\alpha} b_\alpha^{jk}(0)\binom{\alpha}{\beta}\rho^{-t_k+1/p}D^\beta w_k(\rho x)D^{\alpha-\beta}\phi(x)\big|_{x_n=0} \end{aligned}$$

due to $\sum_{k=1}^{k_r}\mathring{B}_{jk}(0,\xi_0',D_n)v_k(x_n)|_{x_n=0}=0$. For $j=1,\dots,N_r$, $|\alpha|=\sigma_j+t_k$, and $\beta<\alpha$ we can estimate

$$\begin{aligned} \rho^{-t_k+1/p}&\big\|D^\beta w(\rho x)D^{\alpha-\beta}\phi(x)\big|_{x_n=0}\big\|_{-\sigma_j-1/p,p,\mathbb{R}^{n-1}} \\ &\le \rho^{-t_k+1/p}\big\|D^\beta w(\rho x)D^{\alpha-\beta}\phi(x)\big\|_{-\sigma_j,p,\mathbb{R}^n_+} \\ &\le C\rho^{-t_k+1/p}\sum_{|\gamma|\le-\sigma_j}\big\|D^\gamma\big[D^\beta w(\rho x)D^{\alpha-\beta}\phi(x)\big]\big\|_{0,p,\mathbb{R}^n_+} \\ &\le C_\phi\rho^{-t_k-\sigma_j+|\beta|}\|v\|_{0,p,\mathbb{R}_+}\to 0 \quad (\rho\to\infty). \end{aligned} \tag{2.9}$$

Further, for $|\lambda|=\rho^\mu$ we obtain

$$\begin{aligned} |\lambda|^{(-\sigma_j-1/p)/\widetilde{m}_{\pi_2(j)}}&\|g_j\|_{0,p,\mathbb{R}^{n-1}} \\ &= |\lambda|^{(-\sigma_j-1/p)/\widetilde{m}_{\pi_2(j)}}\Big\|\sum_{k=1}^{k_r}\sum_{|\alpha|=\sigma_j+t_k} b_\alpha^{jk}(0)D^\alpha u_k(x)\big|_{x_n=0}\Big\|_{0,p,\mathbb{R}^{n-1}} \\ &\le C\rho^{(\sigma_j+1/p)(1-\mu/\widetilde{m}_{\pi_2(j)})}\to 0 \quad (\rho\to\infty) \end{aligned}$$

as $\mu/\widetilde{m}_{\pi_2(j)}<1$ and $\sigma_j\le -1$. From this and (2.9) we see that for $j=1,\dots,N_r$

$$|\!|\!|g_j^0|\!|\!|^{(\pi_2(j))}_{-\sigma_j-1/p,p,\mathbb{R}^{n-1}}\to 0 \quad (\rho\to\infty). \tag{2.10}$$

Finally, for $j>N_r$ we have $g_j^0=0$ as $B(0,D)$ is assumed to be essentially upper triangular. From (2.7), (2.8), and (2.10) we obtain a contradiction to the a priori-estimate (2.4).

In the case $\lambda^0\neq 0$, the result follows from similar considerations where we now set $\lambda=\lambda^0\rho^{\widetilde{m}_r}$ again. □

Proposition 2.6. *Under the assumptions of Proposition* 2.5*, condition* (VL) *holds at* x^0*.*

Proof. The proof is similar to the proof of Proposition 2.5, and we only indicate some changes and additional remarks. Assuming v to be a nontrivial solution of (1.3), define $\lambda:=\rho^{\widetilde{m}_r}\lambda^0$ and u as in (2.6), but now setting $\xi_0'=0$, i.e., we set

$$u(x):=\begin{cases}\rho^{-t_j+1/p}\phi(x)v_j(\rho x_n), & j=1,\dots,k_r,\\ 0, & j=k_r+1,\dots,N.\end{cases}$$

Now the left-hand side of (2.4) can be estimated from below by

$$|\!|\!|u_j|\!|\!|^{(\pi_2(j))}_{t_j,p,\mathbb{R}^n_+}\ge\|u_j\|_{t_j,p,\mathbb{R}^n_+}\ge\|D_n^{t_j}u_j\|_{0,p,\mathbb{R}^n_+}\ge\frac12\|v_j\|_{0,p,\mathbb{R}_+}\|\phi'\|_{0,p,\mathbb{R}^{n-1}}$$

for $\rho \geq \rho_0$. To estimate f_j^0, note that

$$f_j^0 = \rho^{s_j+1/p}\phi(x)\Big[\sum_{k=1}^{k_r} \sum_{\substack{|\alpha|=s_j+t_k\\ \alpha'=0}} a_\alpha^{jk}(0)(D_n^{\alpha_n}v_k)(\rho x_n) - \delta_{jk}\rho^{\widetilde{m}_r-m_j}\lambda^0 v_k(\rho x_n)\Big]$$

$$+ \phi(x)\sum_{k=1}^{k_r} \sum_{\substack{|\alpha|=s_j+t_k\\ \alpha'=0}} \sum_{\beta_n<\alpha_n} \rho^{-t_k+1/p} a_\alpha^{jk}(0)\binom{\alpha_n}{\beta_n} D_n^{\beta_n} v(\rho x_n) D_n^{\alpha_n-\beta_n}\psi(x_n)$$

$$+ \sum_{k=1}^{k_r} \sum_{\substack{|\alpha|=s_j+t_k\\ \alpha'\neq 0}} \rho^{-t_k+1/p} a_\alpha^{jk}(0) D_{x'}^{\alpha'}\phi(x') D_n^{\alpha_n}(v_k(\rho x_n)\psi(x_n)).$$

Here $\alpha' = (\alpha_1, \dots, \alpha_{n-1})$. For $\pi_1(j) < r$ we have $\widetilde{m}_r - m_j < 0$, and the term in brackets tends to the j-th row of $A(0,0,D_n)v(\rho x_n)$. For $\pi_1(j) = r$, the term equals the j-th row of $A(0,0,D_n)v(\rho x_n) - \lambda v(\rho x_n)$. All other terms are of lower order with respect to ρ and can be estimated in the same way as in the proof of Proposition 2.5. As $(\mathcal{A}_{11}^{(r)}(0,0,D_n) - \lambda^0 \widetilde{I}_{r,0})v = 0$ by assumption, we obtain (2.8) again.

By considerations similar to those above, the estimate of g_j^0 is reduced to the estimate of

$$\rho^{-t_k+1/p} \|\!| D_{x'}^{\alpha'}\phi(x') D_n^{\alpha_n} v_k(\rho x_n)\big|_{x_n=0} \|\!|^{(\pi_2(j))}_{-\sigma_j-1/p,p,\mathbb{R}^{n-1}}. \tag{2.11}$$

Here $j = 1, \dots, N_r$, $|\alpha| = \sigma_j + t_k$, and $\alpha_n < \sigma_j + t_k$ if $\pi_2(j) = r$. Taking into account $\sigma_j < 0$ and therefore $\alpha_n \leq t_k - 1$, we may estimate

$$\begin{aligned} &\rho^{-t_k+1/p} \| D_{x'}^{\alpha'}\phi(x') D_n^{\alpha_n} v_k(\rho x_n)\big|_{x_n=0} \|_{-\sigma_j-1/p,p,\mathbb{R}^{n-1}} \\ &\qquad \leq \rho^{-t_k+\alpha_n+1/p} \|\!| D_{x'}^{\alpha'}\phi(x')(D_n^{\alpha_n}v_k)(0)\|\!|_{-\sigma_j-1/p,p,\mathbb{R}^{n-1}} \\ &\qquad \leq C_{\phi'}\rho^{-t_k+\alpha_n+1/p} |(D_n^{\alpha_n}v_k)(0)| \to 0 \quad (\rho\to\infty) \end{aligned}$$

for $j = 1, \dots, N_r$. Similarly, for $j = 1, \dots, N_{r-1}$, i.e. for $\pi_2(j) < r$, we have

$$\begin{aligned} &|\lambda|^{(-\sigma_j-1/p)\widetilde{m}_r/\widetilde{m}_{\pi_2(j)}} \| D_{x'}^{\alpha'}\phi(x') D_n^{\alpha_n} v_k(\rho x_n)\big|_{x_n=0} \|_{0,p,\mathbb{R}^{n-1}} \\ &\qquad \leq C\rho^{(\sigma_j+1/p)(1-\widetilde{m}_r/\widetilde{m}_{\pi_2(j)})} |(D_n^{\alpha_n}v_k)(0)| \to 0 \quad (\rho\to\infty), \end{aligned}$$

as $\sigma_j \leq -1$ and $\widetilde{m}_r/\widetilde{m}_{\pi_2(j)} < 1$. Therefore, we see that in all cases the expression in (2.11) tends to zero for $\rho \to \infty$ which finally leads to a contradiction. □

Now the proof of Theorem 1.1, i.e. of the necessity of the parameter-ellipticity conditions (E), (SL) and (VL), follows from Propositions 2.3, 2.5, and 2.6, respectively.

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