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On symmetric div-quasiconvex hulls and divsym-free L^∞-truncations

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Abstract. We establish that for any non-empty, compact set $K \subset \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ the 1- and ∞-symmetric div-quasiconvex hulls $K^{(1)}$ and $K^{(\infty)}$ coincide. This settles a conjecture in a recent work of Conti, Müller & Ortiz [Arch. Ration. Mech. Anal. 235 (2020)] in the affirmative. As a key novelty, we construct an L^∞-truncation that preserves both symmetry and solenoidality of matrix-valued maps in L^1. For comparison, we moreover give a construction of $\mathcal{A}$-free truncations in the regime $1 < p < \infty$ which, however, does not apply to the case $p = 1$.

1. Introduction

1.1. Aim and scope

One of the key problems in continuum mechanics is the mathematical description of the plasticity behaviour of solids. Such solids are usually modelled by reference configurations $\Omega \subset \mathbb{R}^3$ subject to loads or forces and corresponding *velocity fields* $v\colon \Omega \to \mathbb{R}^3$. The (elasto)plastic behaviour of the material is mathematically described in terms of the stress tensor $\sigma\colon \Omega \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ and is dictated by the precise target $K \subset \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ where it takes values; K is usually referred to as the *elastic domain*. When ideal plasticity is assumed and potential hardening effects are excluded, K is a compact set in $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$ with non-empty interior. As prototypical examples, in the von Mises or Tresca models used for the description of metals or alloys, we have $K = \{\sigma \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} : \mathbf{f}(\sigma^D) \le \theta\}$ with a threshold $\theta > 0$, the deviatoric stress $\sigma^D := \sigma - \frac{1}{3}\operatorname{tr}(\sigma) E_{3\times 3}$ and *convex* $\mathbf{f}\colon \mathbb{R}^{3\times 3}_{\mathrm{sym}} \to \mathbb{R}$. Generalising this to $K = \{\sigma \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} : \mathbf{f}(\sigma^D) + \vartheta \operatorname{tr}(\sigma) \le \theta\}$ for $\vartheta > 0$ as in the Drucker–Prager or Mohr–Coulomb models for concrete or sand (cf. [13, 27]), such models take into account persisting volumetric changes induced by the hydrostatic pressure as plasticity effects. In all of these models, K is a *convex* set. This opens the gateway to the techniques from convex analysis, and we refer to [21, 27] for more detail.

As the main motivation for the present paper, the convexity assumption on the elastic domain K is *not satisfied* by all materials. A prominent example where the non-convexity

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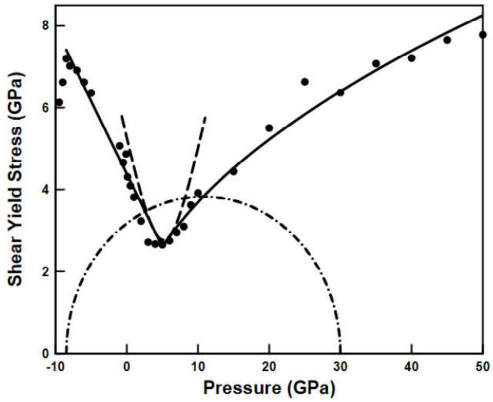

Figure 1. Molecular dynamics computations for fused silica glass linking pressure and shear yield stress, taken from Schill et al. [38, Fig. 17(b)]. Within the framework of limit analysis [27], the non-convexity of the critical state line (thick line) is linked to the instability for microstructure formation (cf. [38, Sec. 4]) and so a suitable relaxation is required.

of K can be observed explicitly is fused silica glass (cf. Meade & Jeanloz [30]). Slightly more generally, for amorphous solids being deformed subject to shear, experiments on the molecular dynamics (cf. Maloney & Robbins [28]) exhibit the formation of characteristic patterns in the underlying deformation fields. As a possible explanation of this phenomenon, the emergence of such patterns on the *microscopic* level displays the effort of the material to cope with the enduring *macroscopic* deformations. Within the framework of limit analysis [27], Schill et al. [38] offer a link between the non-convexity of K and the appearance of such fine microstructure; cf. Figure 1. Working from plastic dissipation principles, the corresponding static problem is identified in [38] as

$$\sup_{\sigma}\inf_{v}\left\{\int_{\Omega}\sigma\cdot\nabla v\,\mathrm{d}x : \sigma\in \mathrm{L}^{\infty}_{\mathrm{div}}(\Omega;K),\ v\in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^{n}),\ v=g \text{ on } \partial\Omega\right\} \tag{1.1}$$

for given boundary data $g\colon\partial\Omega\to\mathbb{R}^3$. Here, $\mathrm{L}^{\infty}_{\mathrm{div}}(\Omega;K)$ is the space of all $\mathrm{L}^{\infty}(\Omega;K)$-maps which are rowwise divergence-free (or solenoidal) in the sense of distributions; note that, even if we admitted general $\sigma\in\mathrm{L}^{\infty}(\Omega;K)$ in (1.1), the variational principle would be non-trivial only for $\sigma\in\mathrm{L}^{\infty}_{\mathrm{div}}(\Omega;K)$. Stability under microstructure formation, in turn, is linked to the existence of solutions of (1.1); cf. Müller [32] for a discussion of the underlying principles. Regarding the existence of solutions, the direct method of the calculus of variations requires semicontinuity, and it is here where the set K must be relaxed. By the constraints on σ, this motivates the passage to the *symmetric div-quasiconvex hull* of K as studied by Conti, Müller & Ortiz [10]. In the present paper, we complete the characterisation of such hulls (cf. Theorem 1.1 below) and thereby answer a conjecture posed in [10] in the affirmative. To state our result, we pause and introduce the requisite terminology first.

1.2. Divsym-quasiconvexity and the main result

Following [10], we call a Borel measurable, locally bounded function $F\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ *symmetric div-quasiconvex* if

$$F(\xi) \le \int_{\mathbb{T}_n} F(\xi + \varphi(x))\,\mathrm{d}x \tag{1.2}$$

holds for all $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and all admissible test maps

$$\varphi \in \mathcal{T} := \big\{\varphi \in \mathrm{C}^\infty(\mathbb{T}_n;\mathbb{R}^{n\times n}_{\mathrm{sym}}) : \mathrm{div}(\varphi) = 0,\ \textstyle\int_{\mathbb{T}_n} \varphi\,\mathrm{d}x = 0\big\}, \tag{1.3}$$

where $\mathbb{T}_n$ denotes the n-dimensional torus. Here, the divergence is understood in the rowwise (or equivalently, columnwise) manner. Accordingly, the *symmetric div-quasiconvex (or divsym-quasiconvex) envelope* of a Borel measurable, locally bounded function $F\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ is defined as the largest symmetric div-quasiconvex function below F; more explicitly,

$$\mathcal{Q}_{\mathrm{sdqc}}F(\xi) := \inf\big\{\textstyle\int_{\mathbb{T}_n} F(\xi + \varphi(x))\,\mathrm{d}x : \varphi \in \mathcal{T}\big\}. \tag{1.4}$$

Divsym-quasiconvexity is a strictly weaker notion than convexity, which can be seen by Tartar's [43] example $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \ni \xi \mapsto (n-1)|\xi|^2 - \mathrm{tr}(\xi)^2$. The discussion in Section 1.1 necessitates a notion of divsym-quasiconvexity *for sets*. Inspired by the separation theory from convex analysis, we call a compact set $K \subset \mathbb{R}^{n\times n}_{\mathrm{sym}}$ *symmetric div-quasiconvex* provided that, for each $\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}} \setminus K$, there exists a symmetric div-quasiconvex $g \in \mathrm{C}(\mathbb{R}^{n\times n}_{\mathrm{sym}};[0,\infty))$ such that $g(\xi) > \max_K g$. The relaxation of the elastic domains $K \subset \mathbb{R}^{n\times n}_{\mathrm{sym}}$ in turn is defined in terms of the symmetric div-quasiconvex envelopes of distance functions. For a compact subset $K \subset \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and $1 \le p < \infty$, put $f_p(\xi) := \mathrm{dist}^p(\xi, K)$. The *$p$-symmetric div-quasiconvex hull* of K then is defined by

$$K^{(p)} := \big\{\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}} : \mathcal{Q}_{\mathrm{sdqc}} f_p(\xi) = 0\big\}, \tag{1.5}$$

whereas we set for $p = \infty$,

$$K^{(\infty)} := \big\{\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}} : g(\xi) \le \max\nolimits_K g \text{ for all symmetric div-quasiconvex } g \in \mathrm{C}(\mathbb{R}^{n\times n}_{\mathrm{sym}};[0,\infty))\big\}. \tag{1.6}$$

Both (1.5) and (1.6) are the natural generalisations of the usual convex hulls to the symmetric div-quasiconvex context, and one easily sees that $K^{(\infty)}$ is the smallest symmetric div-quasiconvex, compact set containing K. If the distance function to K is nicely coercive, which is in particular satisfied for compact sets, then the definition of $K^{(\infty)}$ can be viewed as the limiting object of $K^{(p)}$, since in this case

$$K^{(p)} = \{\xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}} : g(\xi) \le \max\nolimits_K g \text{ for all symmetric div-quasiconvex } g \in \mathrm{C}(\mathbb{R}^{n\times n}_{\mathrm{sym}};[0,\infty)) \text{ with } g(z) \le C(1+|z|^p) \text{ for all } z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}\}.$$

By our discussion in Section 1.1, it is particularly important to understand the properties of the symmetric div-quasiconvex hulls. In [10], Conti, Müller & Ortiz established that $K^{(p)}$ is independent of $1 < p < \infty$. Specifically, they conjectured in [10, Rem. 3.9] that $K^{(1)} = K^{(\infty)}$ in analogy with the usual quasiconvex envelopes (see Zhang [49] or Müller [32, Thm. 4.10]). The present paper answers this question in the affirmative, leading us to our main result:

Theorem 1.1 (Main result). *Let $K \subset \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ be compact. Then $K^{(1)} = K^{(\infty)}$ and so*

$$K^{(p)} = K^{(1)} = K^{(\infty)} \quad \textit{for all } 1 \leq p \leq \infty. \tag{1.7}$$

Let us note that the p-symmetric div-quasiconvex hulls satisfy the antimonotonicity property with respect to inclusions, i.e., if $1 \leq p \leq q \leq \infty$, then $K^{(q)} \subset K^{(p)}$. For Theorem 1.1, it thus suffices to establish $K^{(1)} \subset K^{(\infty)}$, and this is exactly what will be achieved in Section 5. We wish to point out that, for the present paper, our focus is on compact sets K and not on potentially unbounded ones, for which even in the usual quasiconvex case only a few contributions are available; see, e.g., [16, 33, 45, 46, 50].

From a proof perspective, any underlying argument must use an L^∞-truncation of suitable recovery sequences, simultaneously keeping track of the differential constraint. Contrary to routine mollification, truncations leave the input functions unchanged on a large set and display an important tool in the study of non-linear problems [1, 3, 19, 20, 31, 47]. It is here where Theorem 1.1 cannot be established by analogous means to [10, Sec. 3], where a higher-order truncation argument in the spirit of Acerbi & Fusco [2] and Zhang [48] is employed. More precisely, for $1 < p < q < \infty$, the critical inclusion $K^{(p)} \subset K^{(q)}$ is established in [10] by passing to the corresponding potentials of divsym-free fields, and as these potentials are of second order, performing a $\mathrm{W}^{2,\infty}$-truncation on the potentials; this will be referred to as *potential truncation*. The underlying potential operators are obtained as suitable Fourier multiplier operators, which is why they only satisfy strong L^p–L^p-bounds for $1 < p < \infty$ (cf. Lemma 2.2 below). It is well known that such Fourier multiplier operators do not map $\mathrm{L}^1 \to \mathrm{L}^1$ boundedly (cf. Ornstein [35] and, more recently, [9, 17, 26]), and so this approach is bound to fail in view of Theorem 1.1. In the regime $1 < p < \infty$, this strategy can readily be employed in the general context of $\mathcal{A}$-quasiconvex hulls in the sense of Fonseca & Müller [18] (cf. Proposition 6.1 and Section 6) but is not even required for the inclusion $K^{(p)} \subset K^{(q)}$, $p < q$, and can be established by more elementary means; cf. Lemma 5.2 and its proof for the simplifying argument.

1.3. A truncation theorem and its context

The key tool in establishing Theorem 1.1 therefore consists in the following truncation result, allowing us to truncate a div-free L^1-map $u\colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ while still preserving the constraint $\mathrm{div}(u) = 0$:

Theorem 1.2 (Main truncation theorem). *There exists a constant* $C > 0$ *solely depending on the underlying space dimension* $n = 3$ *with the following property: for all* $u \in L^1(\mathbb{R}^3; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *with* $\operatorname{div}(u) = 0$ *in* $\mathcal{D}'(\mathbb{R}^3; \mathbb{R}^3)$ *and all* $\lambda > 0$ *there exists* $u_\lambda \in L^1(\mathbb{R}^3; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *satisfying the*

(a) L^∞*-bound:*

$$\|u_\lambda\|_{L^\infty(\mathbb{R}^3)} \leq C\lambda;$$

(b) *strong stability:*

$$\|u - u_\lambda\|_{L^1(\mathbb{R}^3)} \leq C \int_{\{|u|>\lambda\}} |u|\,\mathrm{d}x;$$

(c) *small change:*

$$\mathcal{L}^3(\{u \neq u_\lambda\}) \leq C\lambda^{-1} \int_{\{|u|>\lambda\}} |u|\,\mathrm{d}x;$$

(d) *differential constraint:* $\operatorname{div}(u_\lambda) = 0$ *in* $\mathcal{D}'(\mathbb{R}^3; \mathbb{R}^3)$.

The same remains valid when replacing the underlying domain $\mathbb{R}^3$ *by the torus* $\mathbb{T}_3$.

The way in which Theorem 1.2 implies Theorem 1.1 can be accomplished by analogous means to [10] (also see the discussion by the third author [37]), and is sketched for the reader's convenience in Section 5. Here we heavily rely on the *strong stability property* from item (b), without which the proof of Theorem 1.1 is not clear to us. The detailed construction that underlies the proof of Theorem 1.2, reminiscent of a geometric version of the Whitney smoothing or extension procedure [44], is explained in Section 3 and carried out in detail in Section 4. Here we understand by *geometric* that the construction is directly tailored to the problem at our disposal, meaning that the solenoidality constraint $\operatorname{div}(u) = 0$ is visible in our construction in terms of the Gauß–Green theorem on certain simplices. The line of argument employed in the proof can also be applied to higher dimensions, but to focus on the essentials for the physically relevant case we here stick to $n = 3$ dimensions for expository reasons.

Working on a higher a priori regularity level, *Lipschitz truncations* that preserve solenoidality constraints are not new and have been studied most notably by Breit et al. [6, 8], originally developed for problems from mathematical fluid mechanics and since then have been fruitfully used in a variety of related problems; see, e.g., Süli et al. [12,42]. Let us note that the two key approaches in [6, 8] either hinge on locally correcting divergence contributions on certain bad sets [6] or performing the potential truncation [8]. Whereas the ansatz in [6] in principle may be expected to work in the present setting apart from technical intricacies (cf. Remark 6.3), the key drawback of the potential truncation is the non-availability of the strong stability estimate. This is essentially a consequence of singular integrals only mapping $L^\infty \to \mathrm{BMO}$ in general but *not* $L^\infty \to L^\infty$; see Section 6 and Proposition 6.1, where the corresponding potential truncations are revisited and discussed in the general framework of constant rank operators $\mathcal{A}$ a lá Schulenberger & Wilcox [39] or Murat [34]. A related result in this context can be found in the work of Gallenmüller [22] which, however, works subject to different hypotheses from ours. Note that although the potential truncation does not yield the strong stability, the techniques developed in the framework of Theorem 1.2 suggest that a refinement of the method could make strong stability estimates work for a large class of operators; see Conjecture 6.4 for the precise statement.

To conclude, let us note that by Müller's improvement [31, Thm. 2] of the aforementioned Zhang truncation lemma [48, Lem. 3.1] for convex sets, one might wonder whether an analogous result can be achieved in the framework discussed in the present paper. Even though the underlying mollification strategy in [31] should be compatible with our approach, the precise technical implementation needs some refinement and will be deferred to future work. Still, such a result will only concern convex (and not symmetric div-quasiconvex) sets, as even Müller's original result for convex sets seems to be open for quasiconvex sets.

1.4. Organisation of the paper

Apart from this introductory section, the paper is organised as follows. In Section 2 we fix notation and gather auxiliary material on maximal operators and basic facts from harmonic analysis. Section 3 then explains the idea underlying the construction employed in the proof of Theorem 1.2, and is then carried out in detail in Section 4. Section 5 is devoted to the proof of Theorem 1.1, and the paper is concluded in Section 6 by revisiting potential truncations. The appendix gathers various instrumental computations that underlie some of the results presented in Section 4.

2. Preliminaries

2.1. Notation

The linear operators between two finite-dimensional real vector spaces V, W are denoted $\mathcal{L}(V;W)$. We denote by $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$ the n-dimensional Lebesgue or $(n-1)$-dimensional Hausdorff measures, respectively. For notational brevity, we will also write $\mathrm{d}^{n-1} = \mathrm{d}\mathcal{H}^{n-1}$. Given n- or $(n-1)$-dimensional measurable subsets Ω and Σ of $\mathbb{R}^n$ with $\mathcal{L}^n(\Omega)$, $\mathcal{H}^{n-1}(\Sigma) \in (0,\infty)$, respectively, we use the shorthand

$$\fint_\Omega u \,\mathrm{d}x := \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega u \,\mathrm{d}x \quad \text{and} \quad \fint_\Sigma v \,\mathrm{d}^{n-1}x := \frac{1}{\mathcal{H}^{n-1}(\Sigma)} \int_\Sigma v \,\mathrm{d}^{n-1}x$$

for $\mathcal{L}^n$- or $\mathcal{H}^{n-1}$-measurable maps $u\colon \Omega \to \mathbb{R}^m$ and $v\colon \Sigma \to \mathbb{R}^m$. As we will mostly assume $n = 3$, we denote by $B_r(z)$ the open ball of radius r centred at $z \in \mathbb{R}^3$, whereas we reserve the notation $\mathbb{B}_r(z)$ to denote the corresponding open balls in the symmetric (3×3)-matrices $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$; moreover, we put $\omega_3 := \mathcal{L}^3(B_1(0))$. By *cubes* Q we understand non-degenerate cubes throughout, and use $\ell(Q)$ to denote their side length. Lastly, for $x_1,\dots,x_j \in \mathbb{R}^3$, we denote by $\langle x_1,\dots,x_j\rangle$ the convex hull of the vectors $x_1,\dots,x_j$, and if x_1, x_2, x_3 do not lie on a joint line, by $\mathrm{aff}(x_1,x_2,x_3)$ the affine hyperplane containing x_1, x_2, x_3.

2.2. Maximal operator, bad sets and Whitney covers

For a finite-dimensional real vector space V, $w \in L^1(\mathbb{R}^n; V)$ and $R > 0$, we recall the (restricted) *centred Hardy–Littlewood maximal operators* to be defined by

$$\begin{aligned} \mathcal{M}_R w(x) &:= \sup_{0<r<R} \fint_{B_r(x)} |w| \, dy, \qquad x \in \mathbb{R}^n, \\ \mathcal{M} w(x) &:= \sup_{r>0} \fint_{B_r(x)} |w| \, dy, \qquad x \in \mathbb{R}^n. \end{aligned} \tag{2.1}$$

Note that, by lower semicontinuity of $\mathcal{M}_R w$, the superlevel sets $\{\mathcal{M}_R w > \lambda\}$ are open for all $\lambda > 0$. Moreover, we record that $\mathcal{M}$ is of weak-(1, 1) type, meaning that there exists $c = c(n) > 0$ such that

$$\mathcal{L}^n(\{\mathcal{M}w > \lambda\}) \leq \frac{c}{\lambda} \|w\|_{L^1(\mathbb{R}^n)} \quad \text{for all } w \in L^1(\mathbb{R}^n; V). \tag{2.2}$$

See [25,41] for more background information. Now let $\Omega \subset \mathbb{R}^n$ be open. Then there exists a *Whitney cover* $\mathcal{W} = (Q_j)$ for Ω. By this we understand a sequence of open cubes Q_j with the following properties:

- (W1) $\Omega = \bigcup_{j\in\mathbb{N}} Q_j$.
- (W2) $\frac{1}{5}\ell(Q_j) \leq \operatorname{dist}(Q_j, \Omega^\complement) \leq 5\ell(Q_j)$ for all $j \in \mathbb{N}$.
- (W3) *Finite overlap:* there exists a number $\mathtt{N} = \mathtt{N}(n) > 0$ such that at most $\mathtt{N}$ elements of $\mathcal{W}$ overlap; i.e., for each $i \in \mathbb{N}$,
$$\left|\{j \in \mathbb{N} : Q_j \in \mathcal{W} \text{ and } Q_i \cap Q_j \neq \emptyset\}\right| \leq \mathtt{N}.$$
- (W4) *Comparability for touching cubes:* there exists a constant $c(n) > 0$ such that if $Q_i, Q_j \in \mathcal{W}$ satisfy $Q_i \cap Q_j \neq \emptyset$, then
$$\frac{1}{c(n)}\ell(Q_i) \leq \ell(Q_j) \leq c(n)\ell(Q_i).$$

Whenever such a Whitney cover is considered, we tacitly understand x_j to be the *centre* of the corresponding cube Q_j. Based on the Whitney cover $\mathcal{W}$ from above, we choose a partition of unity (φ_j) subject to $\mathcal{W}$ with the following properties:

- (P1) For any $j \in \mathbb{N}$, $\varphi_j \in C_c^\infty(Q_j; [0, 1])$.
- (P2) $\sum_{j\in\mathbb{N}} \varphi_j = 1$ in Ω.
- (P3) For each $l \in \mathbb{N}$, there exists a constant $c = c(n, l) > 0$ such that
$$|\nabla^l \varphi_j| \leq \frac{c}{\ell(Q_j)^l} \quad \text{for all } j \in \mathbb{N}.$$

2.3. Differential operators and projection maps

For the following sections, we require some terminology for differential operators and a suitable projection property to be gathered in the sequel. Let $\mathcal{A}$ be a constant coefficient, linear and homogeneous differential operator of order $k \in \mathbb{N}$ on $\mathbb{R}^n$ (or $\mathbb{T}_n$) between $\mathbb{R}^d$ and $\mathbb{R}^N$, so $\mathcal{A}$ has a representation

$$\mathcal{A}u = \sum_{|\alpha|=k} \mathcal{A}_\alpha \partial^\alpha u, \quad u: \mathbb{R}^n \to \mathbb{R}^d, \tag{2.3}$$

with fixed $\mathcal{A}_\alpha \in \mathcal{L}(\mathbb{R}^d; \mathbb{R}^N)$ for $|\alpha| = k$. Following [34, 39] we say that $\mathcal{A}$ has *constant rank* (in $\mathbb{R}$) provided the rank of the Fourier symbol

$$\mathcal{A}[\xi] = \sum_{|\alpha|=k} \mathcal{A}_\alpha \xi^\alpha : \mathbb{R}^d \to \mathbb{R}^N$$

is independent of $\xi \in \mathbb{R}^n \setminus \{0\}$. A constant coefficient differential operator $\mathbb{A}$ of order $j \in \mathbb{N}$ on $\mathbb{R}^n$ (or $\mathbb{T}_n$) between $\mathbb{R}^\ell$ and $\mathbb{R}^d$ consequently is called a *potential* of $\mathcal{A}$ provided, for each $\xi \in \mathbb{R}^n \setminus \{0\}$, the Fourier symbol sequence

$$\mathbb{R}^\ell \xrightarrow{\mathbb{A}[\xi]} \mathbb{R}^d \xrightarrow{\mathcal{A}[\xi]} \mathbb{R}^N$$

is exact at every $\xi \in \mathbb{R}^n \setminus \{0\}$, i.e., $\mathbb{A}[\xi](\mathbb{R}^\ell) = \ker(\mathcal{A}[\xi])$ for each such ξ. We moreover say that $\mathcal{A}$ has *constant rank* (in $\mathbb{C}$) provided $\mathcal{A}[\xi]: \mathbb{C}^d \to \mathbb{C}^N$ has rank independent of $\xi \in \mathbb{C}^n \setminus \{0\}$. If we only speak of *constant rank*, then we tacitly understand constant rank in $\mathbb{R}$. In Section 6 we require the following two auxiliary results, ensuring both the existence of potentials and suitable projection operators.

Lemma 2.1 (Existence of potentials, [36, Thm. 1, Lem. 5]). *Let $\mathcal{A}$ be a differential operator with constant rank over $\mathbb{R}$. Then $\mathcal{A}$ possesses a potential $\mathbb{A}$. Moreover, if $u \in \mathrm{C}^\infty(\mathbb{T}_n; \mathbb{R}^d)$ satisfies $\int_{\mathbb{T}_n} u \, \mathrm{d}x = 0$ and $\mathcal{A}u = 0$, there exists $v \in \mathrm{C}^\infty(\mathbb{T}_n; \mathbb{R}^\ell)$ with $\mathbb{A}v = u$. Equally, for each $u \in \mathcal{S}(\mathbb{R}^n; \mathbb{R}^d)$ with $\mathcal{A}u = 0$ there exists $v \in \mathcal{S}(\mathbb{R}^n; \mathbb{R}^\ell)$ with $\mathbb{A}v = u$.*

Lemma 2.2 (Projection maps on the torus, [18, Lem. 2.14]). *Let $1 < p < \infty$ and let $\mathcal{A}$ be a differential operator of order k with constant rank in $\mathbb{R}$. Then there is a bounded, linear projection map $P_\mathcal{A}: \mathrm{L}^p(\mathbb{T}_n; \mathbb{R}^d) \to \mathrm{L}^p(\mathbb{T}_n; \mathbb{R}^d)$ with the following properties:*

(a) *$P_\mathcal{A} u \in \ker \mathcal{A}$ and $P_\mathcal{A} \circ P_\mathcal{A} = P_\mathcal{A}$.*

(b) *$\|u - P_\mathcal{A} u\|_{\mathrm{L}^p(\mathbb{T}_n)} \leq C_{\mathcal{A},p} \|\mathcal{A}u\|_{\mathrm{W}^{-k,p}(\mathbb{T}_n)}$ whenever $\fint_{\mathbb{T}_n} u \, \mathrm{d}x = 0$.*

(c) *If $(u_j) \subset \mathrm{L}^p(\mathbb{T}_n; \mathbb{R}^d)$ is bounded and p-equiintegrable, i.e.,*

$$\lim_{\varepsilon \searrow 0} \left(\sup_{j \in \mathbb{N}} \sup_{E: \mathcal{L}^n(E) < \varepsilon} \int_E |u_j|^p \, \mathrm{d}x \right) = 0,$$

then also $(P_\mathcal{A} u_j)$ is p-equiintegrable.

As alluded to in the introduction, Lemma 2.2 does not extend to $p = 1$ in general, the reason being Ornstein's non-inequality [35]; also see [9, 17, 26] for more recent approaches to the matter and Grafakos [25, Thm. 4.3.4] for a full characterisation of L^1-multipliers.

3. On the construction of divsym-free truncations

Before embarking on the proof of Theorem 1.2 in Section 4, we comment on the underlying idea and how it is implemented in conceptually easier settings (see Sections 3.2 and 3.3 below). To elaborate on the connections to divsym-truncations, we premise a discussion of the general framework first.

3.1. Potential truncations versus $\mathcal{A}$-free truncations

We start by streamlining terminology as follows: Let Ω either be $\mathbb{T}_n$ or $\mathbb{R}^n$. Given a constant rank differential operator $\mathbb{A}$ on Ω between $\mathbb{R}^\ell$ and $\mathbb{R}^d$ and $1 \leq p \leq \infty$, we define Sobolev-type spaces

$$\mathrm{W}^{\mathbb{A},p}(\Omega) := \{u \in \mathrm{L}^p(\Omega;\mathbb{R}^\ell) : \mathbb{A}u \in \mathrm{L}^p(\Omega;\mathbb{R}^d)\}.$$

A family of operators $(S_\lambda)_{\lambda>0}$ with $S_\lambda\colon \mathrm{W}^{\mathbb{A},p}(\Omega) \to \mathrm{W}^{\mathbb{A},\infty}(\Omega)$ is called a $\mathrm{W}^{\mathbb{A},p}$–$\mathrm{W}^{\mathbb{A},\infty}$-*truncation* provided there exists a constant $c = c(\mathbb{A}, p) > 0$ such that, for all $u \in \mathrm{W}^{\mathbb{A},p}(\Omega)$ and $\lambda > 0$,

(a) $\|S_\lambda u\|_{\mathrm{L}^\infty(\Omega)} + \|\mathbb{A}S_\lambda u\|_{\mathrm{L}^\infty(\Omega)} \leq c\lambda$;

(b) $\|u - S_\lambda u\|_{\mathrm{L}^p(\Omega)} + \|\mathbb{A}u - \mathbb{A}S_\lambda u\|_{\mathrm{L}^p(\Omega)} \leq c \int_{\{|u|+|\mathbb{A}u|>\lambda\}} |u|^p + |\mathbb{A}u|^p \,\mathrm{d}x$;

(c) $\mathcal{L}^n(\{u \neq S_\lambda u\}) \leq \frac{c}{\lambda^p} \int_{\{|u|+|\mathbb{A}u|>\lambda\}} |u|^p + |\mathbb{A}u|^p \,\mathrm{d}x$.

If $\mathbb{A} = \nabla^k$, then we simply speak of a $\mathrm{W}^{k,p}$–$\mathrm{W}^{k,\infty}$-*truncation*. Conversely, if $\mathbb{A}$ is a potential of the differential operator $\mathcal{A}$ having form (2.3) and $1 \leq p \leq \infty$, we define $\mathrm{L}^p_{\mathcal{A}}(\Omega) := \{u \in \mathrm{L}^p(\Omega;\mathbb{R}^d) : \mathcal{A}u = 0\}$. A family of operators $(T_\lambda)_{\lambda>0}$ with $T_\lambda\colon \mathrm{L}^p_{\mathcal{A}}(\Omega) \to \mathrm{L}^\infty_{\mathcal{A}}(\Omega)$ is called an $\mathcal{A}$-*free* L^p–L^∞-*truncation* (or simply $\mathcal{A}$-free L^∞-truncation) provided there exists $c = c(\mathcal{A}, p) > 0$ such that the following hold for all $u \in \mathrm{L}^\infty_{\mathcal{A}}(\Omega)$ and $\lambda > 0$:

(a) $\|T_\lambda u\|_{\mathrm{L}^\infty(\Omega)} \leq c\lambda$.

(b) $\|u - T_\lambda u\|_{\mathrm{L}^p(\Omega)} \leq c \int_{\{|u|>\lambda\}} |u|^p \,\mathrm{d}x$.

(c) $\mathcal{L}^n(\{u \neq T_\lambda u\}) \leq \frac{c}{\lambda^p} \int_{\{|u|>\lambda\}} |u|^p \,\mathrm{d}x$.

The original $\mathrm{W}^{1,p}$–$\mathrm{W}^{1,\infty}$-truncations as in Acerbi & Fusco [2] leave $u \in \mathrm{W}^{1,p}(\Omega)$ unchanged on $\{\mathcal{M}u \leq \lambda\} \cap \{\mathcal{M}(\nabla u) \leq \lambda\}$. Here, the functions satisfy the Lipschitz estimate

$$\begin{aligned}|u(x) - u(y)| &\lesssim |x - y|(\mathcal{M}(\nabla u)(x) + \mathcal{M}(\nabla u)(y)) \\ &\lesssim \lambda|x - y|\end{aligned}$$

for $\mathcal{L}^n$-a.e. $x, y \in \{\mathcal{M}(\nabla u) \leq \lambda\}$ and thus can be extended to a $c\lambda$-Lipschitz function $S_\lambda u$ by virtue of McShane's extension theorem [15, Chap. 3.1.1., Thm. 1]. Note that, if u is divergence-free, then $S_\lambda u$ is not in general. In view of preserving differential constraints, this necessitates a more flexible approach that allows the action of differential operators to be handled geometrically. Instead of appealing to the McShane extension, one may directly perform a Whitney-type extension [44] and truncate $u \in \mathrm{W}^{1,1}(\Omega)$ on the bad set $\mathcal{O}_\lambda = \{\mathcal{M}u > \lambda\} \cup \{\mathcal{M}(\nabla u) > \lambda\}$ via

$$\begin{aligned} \widetilde{\mathbf{S}}_\lambda u(x) &= \begin{cases} \sum_{j\in\mathbb{N}} \varphi_j (u)_{Q_j}, & x \in \mathcal{O}_\lambda, \\ u(x), & x \in \mathcal{O}_\lambda^\complement, \end{cases} \\ \text{or} \quad \mathbf{S}_\lambda u(x) &= \begin{cases} \sum_{j\in\mathbb{N}} \varphi_j u(y_j), & x \in \mathcal{O}_\lambda, \\ u(x), & x \in \mathcal{O}_\lambda^\complement, \end{cases} \end{aligned} \tag{3.1}$$

where $y_j \in \mathcal{O}_\lambda^\complement$ are chosen suitably and (φ_j) is a partition of unity subordinate to the Whitney covering of $\mathcal{O}_\lambda$ (cf. Section 2.2). Then $\widetilde{\mathbf{S}}_\lambda$ and $\mathbf{S}_\lambda$ define $\mathrm{W}^{1,1}$–$\mathrm{W}^{1,\infty}$-truncations; cf. [11,41]. Setting $v = \nabla u$, this formula gives a curl-free L^1–L^∞-truncation, as $\operatorname{curl}(v) = 0 \Leftrightarrow v = \nabla u$ for some function u. Using (P1)–(P3), we can, however, rewrite $\tilde{v} := \nabla \mathbf{S}_\lambda u$ purely in terms of v, i.e.,

$$\tilde{v}(x) = \begin{cases} \displaystyle\sum_{i,j\in\mathbb{N}} \varphi_i \nabla\varphi_j \int_0^1 v(ty_j + (1-t)y_i)\cdot(y_j - y_i)\,\mathrm{d}t, & x \in \mathcal{O}_\lambda, \\ v(x), & x \in \mathcal{O}_\lambda^\complement. \end{cases} \tag{3.2}$$

To see the validity of (3.2), we first note that (φ_i) is a partition of unity on $\mathcal{O}_\lambda$, i.e., $\sum_{i\in\mathbb{N}} \varphi_i(y) = 1$ for $y \in \mathcal{O}_\lambda$ and also that, due to the same fact, $\sum_{j\in\mathbb{N}} \nabla\varphi_j(y) = 0$ for any $y \in \mathcal{O}_\lambda$. Using this fact at $(*)$, we conclude

$$\begin{aligned} \tilde{v}(x) &= \nabla \mathbf{S}_\lambda u(x) = \sum_{j\in\mathbb{N}} \nabla\varphi_j u(y_j) \\ &\overset{(*)}{=} \sum_{i,j\in\mathbb{N}} \varphi_i \nabla\varphi_j (u(y_j) - u(y_i)) \\ &= \sum_{i,j\in\mathbb{N}} \varphi_i \nabla\varphi_j \int_0^1 \nabla u(ty_j + (1-t)y_i)\cdot(y_j - y_i)\,\mathrm{d}t \\ &\overset{\nabla u = v}{=} \sum_{i,j\in\mathbb{N}} \varphi_i \nabla\varphi_j \int_0^1 v(ty_j + (1-t)y_i)\cdot(y_j - y_i)\,\mathrm{d}t, \end{aligned} \tag{3.3}$$

which is (3.2). The previous calculation yields that we may skip the step of going to the potential u of v, as the truncation $\tilde{v}$ *does not* depend on the choice of u.

3.2. The construction of divergence-free truncations

In an intermediate step, we explain how (3.2) gives rise to divergence-free L^1–L^∞-truncations. Here, given a divergence-free map $w \in (\mathrm{L}^1 \cap \mathrm{C}^\infty)(\Omega; \mathbb{R}^3)$, we may write $w = \operatorname{curl}(v)$ for some $v \in \mathrm{W}^{\operatorname{curl},1}(\Omega)$.

The key observation is that the truncation formula (3.2) does not only give a curl-free L^1–L^∞-truncation, but is stronger and gives a $\mathrm{W}^{\operatorname{curl},1}$–$\mathrm{W}^{\operatorname{curl},\infty}$-truncation, if we redefine the bad set to be $\widetilde{\mathcal{O}}_\lambda := \{\mathcal{M}v > \lambda\} \cup \{\mathcal{M}\,\mathrm{curl}(v) > \lambda\}$. Temporarily accepting this fact and hereafter that

$$S_\lambda^{\mathrm{curl}} v = \begin{cases} \displaystyle\sum_{i,j\in\mathbb{N}} \varphi_i \nabla\varphi_j \int_0^1 v(ty_j + (1-t)y_i)\cdot(y_j - y_i)\,\mathrm{d}t, & x \in \widetilde{\mathcal{O}}_\lambda, \\ v(x), & x \in \widetilde{\mathcal{O}}_\lambda^{\complement}, \end{cases} \tag{3.4}$$

defines a $\mathrm{W}^{\mathrm{curl},1}$–$\mathrm{W}^{\mathrm{curl},\infty}$-truncation of $v \in \mathrm{W}^{\mathrm{curl},1}(\Omega;\mathbb{R}^3)$, we may then apply $S_\lambda^{\mathrm{curl}}$ to v. Most importantly, we here *directly truncate the curl instead of the full gradients*, and so are in a position to use that $w = \mathrm{curl}(v) \in \mathrm{L}^1$, which is a crucial difference from the potential truncation displayed in Section 6 below. Returning to $\widetilde{w} := \mathrm{curl}(S_\lambda^{\mathrm{curl}} v)$, we then arrive at the requisite truncation. For $n = 3$, this can be written explicitly for $y \in \mathcal{O}_\lambda$ via

$$\begin{aligned} \widetilde{w}(y) &= (\widetilde{w}_1(y), \widetilde{w}_2(y), \widetilde{w}_3(y)) \\ &= \mathrm{curl}(S_\lambda^{\mathrm{curl}} v)(y) \\ &= \sum_{i,j\in\mathbb{N}} \mathrm{curl}(\varphi\nabla\varphi_j) \int_0^1 v(ty_j + (1-t)y_i)\cdot(y_j - y_i)\,\mathrm{d}t, \end{aligned} \tag{3.5}$$

and for future comparison with divsym-free truncations, we carry out the computation for $\widetilde{w}_1$. For brevity, we put $A(i,j) := \int_0^1 v(ty_j + (1-t)y_i)\cdot(y_j - y_i)\,\mathrm{d}t$. Then, artificially introducing a third variable k, we obtain

$$\begin{aligned} &\widetilde{w}_1(y) \\ &\quad = \sum_{i,j\in\mathbb{N}} \big(\partial_2(\varphi_i\partial_3\varphi_j) - \partial_3(\varphi_i\partial_2\varphi_j)\big) A(i,j) \\ &\quad = 2\sum_{i,j\in\mathbb{N}} \partial_2\varphi_i\partial_3\varphi_j A(i,j) \quad (\text{permuting } i \leftrightarrow j \text{ and using } A(i,j) = -A(j,i)) \\ &\quad = 2\sum_{i,j,k\in\mathbb{N}} \varphi_k\partial_2\varphi_i\partial_3\varphi_j (A(i,j) + A(j,k) + A(k,i)) \quad (\text{by } \textstyle\sum_l \nabla\varphi_l = 0,\ l \in \{i,j,k\}). \end{aligned}$$

Instead of using the fundamental theorem of calculus, we use Stokes' theorem to write

$$(A(i,j) + A(j,k) + A(k,i)) = \fint_{\langle x_i,x_j,x_k\rangle} \mathrm{curl}\, v\cdot((y_i - y_j)\times(y_j - y_k))\,\mathrm{d}\mathcal{H}^2$$

for the triangle $\langle x_i, x_j, x_k\rangle$ with vertices x_i, x_j and x_k. Since $\mathrm{curl}\, v = w$, we then arrive at

$$\widetilde{w}_1(y) = \sum_{i,j,k\in\mathbb{N}} \varphi_k\partial_2\varphi_i\partial_3\varphi_j \fint_{\langle x_i,x_j,x_k\rangle} w\cdot((y_i - y_j)\times(y_j - y_k))\,\mathrm{d}\mathcal{H}^2. \tag{3.6}$$

Using formula (3.6), instead of going to the potential of div, we may directly construct truncations of div-free functions.

Pursuing the strategy explained above, the reader might notice that the effective difficulty for div-free fields is to verify that (3.4) defines a $\mathrm{W}^{\mathrm{curl},1}$–$\mathrm{W}^{\mathrm{curl},\infty}$-truncation. For divsym-free L^1-fields, the main argument (to be explained in Section 3.3 and carried out in detail in Section 4) will be centred around constructing the more involved $\mathrm{W}^{\mathrm{curl}\,\mathrm{curl}^\top,1}$–$\mathrm{W}^{\mathrm{curl}\,\mathrm{curl}^\top,\infty}$-truncations rather than $\mathrm{W}^{\mathrm{curl},1}$–$\mathrm{W}^{\mathrm{curl},\infty}$-truncations; see Section 3.3 below for the definition of $\mathrm{curl}\,\mathrm{curl}^\top$. To motivate the need for such truncations, a quick homological discussion in the div-free case is in order. By the construction in (3.4)ff., we are able to formulate an $\mathcal{A}$-free L^1–L^∞-truncation of the annihilator $\mathcal{A}$ of curl, which is div in three dimensions. As discussed by the third author [37], this approach works for all potential–annihilator pairs along the exact sequence of exterior derivatives. This is the exact sequence of differential operators starting with ∇, that is,

$$\begin{aligned}0 \to \mathrm{C}^{\infty,0}(\mathbb{T}_n;\mathbb{R}) \xrightarrow{\nabla} \mathrm{C}^{\infty,0}(\mathbb{T}_n;\mathbb{R}^n) \xrightarrow{\mathrm{curl}} \mathrm{C}^{\infty,0}(\mathbb{T}_n;\mathbb{R}^{n\times n}_{\mathrm{skew}}) \to \cdots\\ \to \mathrm{C}^{\infty,0}(\mathbb{T}_n;\mathbb{R}^n) \xrightarrow{\mathrm{div}} \mathrm{C}^{\infty,0}(\mathbb{T}_n;\mathbb{R}) \to 0,\end{aligned}$$

where $\mathrm{C}^{\infty,0}(\mathbb{T}_n;\mathbb{R}^m)$ denotes the space of smooth functions on the torus with average 0. To summarise the above procedure for div-free fields, one

(D1) *first* picks a suitable $\mathrm{W}^{\nabla,1}$–$\mathrm{W}^{\nabla,\infty}$-truncation as in (3.1);

(D2) *second* rewrites it by considering gradients only as in (3.2);

(D3) *third* shows that the resulting operator as in (3.4) defines a $\mathrm{W}^{\mathrm{curl},1}$–$\mathrm{W}^{\mathrm{curl},\infty}$-truncation.

This consequently gives rise to a div-free L^1–L^∞-truncation.

3.3. Truncations involving the symmetric gradient

Let $n = 3$. Regarding divsym-free L^1–L^∞-truncations, we now aim to modify the procedure (D1)–(D3) above. Here we work from the exact sequence

$$\begin{aligned}0 \to \mathrm{C}^{\infty,0}(\mathbb{T}_3;\mathbb{R}^3) \xrightarrow{\varepsilon} \mathrm{C}^{\infty,0}(\mathbb{T}_3;\mathbb{R}^{3\times 3}_{\mathrm{sym}}) \xrightarrow{\mathrm{curl}\,\mathrm{curl}^\top} \mathrm{C}^{\infty,0}(\mathbb{T}_3;\mathbb{R}^{3\times 3}_{\mathrm{sym}})\\ \xrightarrow{\mathrm{div}} \mathrm{C}^{\infty,0}(\mathbb{T}_3;\mathbb{R}^3) \to 0,\end{aligned} \tag{3.7}$$

where $\mathrm{curl}\,\mathrm{curl}^\top$ is the potential of the divergence of symmetric matrices, defined in $n = 3$ dimensions by

$$\mathrm{curl}\,\mathrm{curl}^\top v = \begin{pmatrix} w_{2323} & w_{2331} & w_{2312}\\ w_{3123} & w_{3131} & w_{3112}\\ w_{1223} & w_{1231} & w_{1212}\end{pmatrix} \quad \text{for } v \in \mathrm{C}^2(\mathbb{R}^3;\mathbb{R}^{3\times 3}_{\mathrm{sym}}),$$

where

$$w_{abcd} := \partial_a\partial_c v_{bd} + \partial_b\partial_d v_{ac} - \partial_a\partial_d v_{bc} - \partial_b\partial_c v_{ad}.$$

Note that in dimension $n = 3$, the exact sequence starting with symmetric gradients has three non-zero elements (ε, $\operatorname{curl}\operatorname{curl}^\top$ and the symmetric divergence); in higher dimensions it is longer, and for simplicity we therefore restrict ourselves to $n = 3$. We then proceed by analogy with (D1)–(D3), namely

(DS1) *first* pick a suitable $W^{\varepsilon,1}$–$W^{\varepsilon,\infty}$-truncation;

(DS2) *second* rewrite it by considering symmetric gradients only;

(DS3) *third* show that the resulting operator defines a $W^{\operatorname{curl}\operatorname{curl}^\top,1}$–$W^{\operatorname{curl}\operatorname{curl}^\top,\infty}$-truncation.

Regarding (DS1), we note that $W^{\mathbb{A},1}$–$W^{\mathbb{A},\infty}$-truncations are also known in settings where $\mathbb{A} \neq \nabla$. In this work, we use that such a truncation exists for the symmetric gradient, i.e., $\mathbb{A} = \varepsilon = \frac{1}{2}(\nabla + \nabla^\top)$ (cf. [4, 14]). As an analogue of formula (3.1), we now use

$$\mathbf{S}_\lambda^\varepsilon u(x) = \begin{cases} \sum_{j\in\mathbb{N}} \varphi_j(x) P_j u(x), & x \in \mathcal{O}_\lambda, \\ u(x), & x \in \mathcal{O}_\lambda^\complement, \end{cases} \tag{3.8}$$

with suitable projections P_j onto the rigid deformations, so the nullspace of the symmetric gradient ε. Such projections can be obtained via

$$P_j u(x) = \fint_{Q_j} u(\xi) + \tfrac{1}{2}(\nabla - \nabla^\top) u(\xi)(x - \xi)\,\mathrm{d}\mu_j(\xi)$$

for suitable measures μ_j, so that $(\nabla - \nabla^\top)$ becomes invisible after integrating by parts. As an adaptation of (3.3) and hereafter (3.4), one may then follow (DS2) to obtain

$$S_\lambda^{\operatorname{curl}\operatorname{curl}^\top} v(x)_{ab} = \begin{cases} \frac{1}{2}\sum_{i,j\in\mathbb{N}} \varphi_i \partial_a \varphi_j (G_b(i,j) + H_b(i,j)) & \\ \qquad + \varphi_i \partial_b \varphi_j (G_a(i,j) + H_a(i,j)), & x \in \mathcal{O}_\lambda, \\ v_{ab}(x), & x \in \mathcal{O}_\lambda^\complement, \end{cases}$$

for $a, b \in \{1, 2, 3\}$ as a substitute for (3.4), where G_a, G_b and H_a, H_b are defined in terms of v and the previously mentioned measures μ_j. In view of (DS3), we then need to establish that the resulting operator in fact yields a $W^{\operatorname{curl}\operatorname{curl}^\top,1}$–$W^{\operatorname{curl}\operatorname{curl}^\top,\infty}$-truncation, and this is *in essence* what we establish in Section 4. More precisely, we directly prove that when applying $\operatorname{curl}\operatorname{curl}^\top$ to $S_\lambda^{\operatorname{curl}\operatorname{curl}^\top} v$ and rewriting the result purely in terms of $w = \operatorname{curl}\operatorname{curl}^\top(v)$ (just as (3.6) rewrites $\operatorname{curl}(S_\lambda^{\operatorname{curl}} v)$ purely in terms of w), we obtain the requisite truncation operator. Omitting the details of the derivation, the truncation operator is written down explicitly in (4.5), and the entire Section 4 is centred around establishing that it features the desired properties.

4. Construction of the truncation and the proof of Theorem 1.2

In this section we establish Theorem 1.2. As a main ingredient, we will prove the following variant for smooth maps that will be shown to imply Theorem 1.2 in Section 4.7:

Proposition 4.1. *Let* $w \in (\mathrm{C}^\infty \cap \mathrm{L}^1)(\mathbb{R}^3;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *satisfy* $\operatorname{div}(w) = 0$. *Then there exists a constant* $c > 0$ *such that for all* $\lambda > 0$ *there exists an open set* $\mathcal{U}_\lambda \subset \mathbb{R}^3$ *and a function* $w_\lambda \in (\mathrm{L}^1 \cap \mathrm{L}^\infty)(\mathbb{R}^3;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *with the following properties:*

(a) $w = w_\lambda$ *on* $\mathcal{U}_\lambda^\complement$ *and* $\mathcal{L}^3(\{w \neq w_\lambda\}) < \frac{c}{\lambda}\int_{\{|w|>\frac{\lambda}{2}\}} |w|\,\mathrm{d}x$.

(b) $\operatorname{div}(w_\lambda) = 0$ *in* $\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$.

(c) $\|w_\lambda\|_{\mathrm{L}^\infty(\mathbb{R}^3)} \leq c\lambda$.

4.1. A short outline of the proof of Proposition 4.1

As the proof of Proposition 4.1 involves several rather technical steps, let us briefly outline its strategy:

(a) In Section 4.2 we define the truncation pointwise (which is derived by following the steps explained in Sections 3.2 and 3.3) and collect auxiliary properties of the terms involved in Lemma 4.2.

(b) Lemma 4.3 is designed to bound single terms appearing as a summand when proving in Lemma 4.4 that our truncation actually maps into L^∞.

(c) We then show that the truncation is actually a smooth function on the bad set $\mathcal{O}_\lambda$. Therefore, we can check the constraint $\operatorname{div}(T_\lambda w) = 0$ pointwise in $\mathcal{O}_\lambda$ (cf. Lemma 4.5), which involves a technical computation given in the appendix.

(d) Consequently, the truncation is div-free both in the interior of $\mathcal{O}_\lambda$ and its complement. To show global solenoidality, we verify that the distributional divergence is actually an L^1-function; cf. Lemma 4.6. We then conclude that $\operatorname{div}(T_\lambda w) \in \mathrm{L}^1$ and $\operatorname{div}(T_\lambda w) = 0$ almost everywhere, hence $\operatorname{div}(T_\lambda w) = 0$.

(e) Finally, we conclude by estimating the measure of the bad set to get a bound on the measure of the set $\{w \neq T_\lambda w\}$; cf. Lemma 4.8.

4.2. Definition of T_λ

Let $w = (w_1, w_2, w_3) \in (\mathrm{C}^\infty \cap \mathrm{L}^1)(\mathbb{R}^3;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ satisfy $\operatorname{div}(w) = 0$. In view of locally redefining our given map w on $\mathcal{O}_\lambda = \{\mathcal{M}w > \lambda\}$, we put

$$\begin{aligned} \mathfrak{A}_{\alpha,\beta}(i,j,k)(y) &:= \fint_{\langle x_i,x_j,x_k\rangle} ((y-\xi)_\beta w_\alpha(\xi) - (y-\xi)_\alpha w_\beta(\xi))\nu_{ijk}\,\mathrm{d}^2\xi, \\ \mathfrak{B}_\alpha(i,j,k) &:= \fint_{\langle x_i,x_j,x_k\rangle} w_\alpha(\xi)\cdot \nu_{ijk}\,\mathrm{d}^2\xi \end{aligned} \tag{4.1}$$

provided the simplex $\langle x_i, x_j, x_k\rangle$ (i.e., the convex hull of x_i, x_j, x_k) is non-degenerate; if it is degenerate, we then define $\mathfrak{A}_{\alpha,\beta}(i,j,k) := 0$ and $\mathfrak{B}_\alpha(i,j,k) := 0$. Here and in what follows, we use

$$\nu_{x_i,x_j,x_k} := \nu_{ijk} := \tfrac{1}{2}(x_i - x_j)\times(x_k - x_j), \tag{4.2}$$

provided the simplex $\langle x_i, x_j, x_k\rangle$ is non-degenerate. Consider a three-tuple

$$(\alpha,\beta,\gamma)\in\{(1,2,3),(2,3,1),(3,1,2)\}.$$

For $(i,j,k)\in\mathbb{N}^3$ and centre points $x_l\in Q_l$ for $l\in\{i,j,k\}$, we then define

$$\begin{aligned}\widetilde{w}_{\alpha\beta}^{(k)} &= 3\sum_{i,j\in\mathbb{N}}(\partial_\gamma\varphi_j\partial_\alpha\varphi_i\mathfrak{B}_\alpha(i,j,k)+\partial_\beta\varphi_j\partial_\gamma\varphi_i\mathfrak{B}_\beta(i,j,k))\\ &\quad+\sum_{i,j\in\mathbb{N}}(\partial_{\beta\gamma}\varphi_j\partial_\gamma\varphi_i-\partial_{\gamma\gamma}\varphi_j\partial_\beta\varphi_i)\mathfrak{A}_{\beta,\gamma}(i,j,k)\\ &\quad+\sum_{i,j\in\mathbb{N}}(\partial_{\alpha\gamma}\varphi_j\partial_\gamma\varphi_i-\partial_{\gamma\gamma}\varphi_j\partial_\alpha\varphi_i)\mathfrak{A}_{\gamma,\alpha}(i,j,k)\\ &\quad+\sum_{i,j\in\mathbb{N}}(\partial_{\alpha\gamma}\varphi_j\partial_\beta\varphi_i+\partial_{\beta\gamma}\varphi_j\partial_\alpha\varphi_i-2\partial_{\alpha\beta}\varphi_j\partial_\gamma\varphi_i)\mathfrak{A}_{\alpha,\beta}(i,j,k).\end{aligned}\tag{4.3}$$

We define $\widetilde{w}_{\beta\alpha}^{(k)}=\widetilde{w}_{\alpha\beta}^{(k)}$ by symmetry. For the diagonal terms, we put

$$\begin{aligned}\widetilde{w}_{\alpha\alpha}^{(k)} &= 6\sum_{i,j\in\mathbb{N}}\partial_\beta\varphi_j\partial_\gamma\varphi_i\mathfrak{B}_\alpha(i,j,k)\\ &\quad+2\sum_{i,j\in\mathbb{N}}(\partial_{\gamma\gamma}\varphi_j\partial_\beta\varphi_i-\partial_{\beta\gamma}\varphi_j\partial_\gamma\varphi_i)\mathfrak{A}_{\gamma,\alpha}(i,j,k)\\ &\quad+2\sum_{i,j\in\mathbb{N}}(\partial_{\beta\beta}\varphi_j\partial_\gamma\varphi_i-\partial_{\beta\gamma}\varphi_j\partial_\beta\varphi_i)\mathfrak{A}_{\alpha,\beta}(i,j,k).\end{aligned}\tag{4.4}$$

Note that, since by (W3) at most $\mathtt{N}$ cubes Q_j overlap, each of the sums in (4.3) and (4.4) is, in a neighbourhood of each point $x\in\mathcal{O}_\lambda$, actually a *finite* sum and hence $\widetilde{w}^{(k)}:=(w_{\alpha\beta}^{(k)})_{\alpha\beta}$ is well defined. Based on (4.3), we define the truncation operator T_λ by

$$T_\lambda w := w-\sum_k\varphi_k(w-\widetilde{w}^{(k)}) = \begin{cases} w & \text{in } \mathcal{O}_\lambda^{\complement},\\ \sum_k \varphi_k\widetilde{w}^{(k)} & \text{in } \mathcal{O}_\lambda.\end{cases}\tag{4.5}$$

Note that on $\mathcal{O}_\lambda$, $T_\lambda w$ is a locally finite sum of C^∞-maps and thus equally is of class $\mathrm{C}^\infty(\mathcal{O}_\lambda;\mathbb{R}^{3\times3}_{\mathrm{sym}})$.

4.3. Auxiliary properties of $\mathfrak{A}_{\alpha,\beta}$ and $\mathfrak{B}_\alpha$

In this section we record some useful properties and auxiliary bounds on the maps $\mathfrak{A}_{\alpha,\beta}(i,j,k)$ and the (constant) maps $\mathfrak{B}_\alpha(i,j,k)$ that will play an instrumental role in the proof of Proposition 4.1. We begin by gathering elementary properties of $\mathfrak{A}_{\alpha,\beta}$ and $\mathfrak{B}_\alpha$ to be utilised crucially when performing index permutations for the sums appearing in (4.5):

Lemma 4.2. *Let $w \in \mathrm{C}^1(\mathbb{R}^3; \mathbb{R}^{3\times3}_{\mathrm{sym}})$ satisfy $\operatorname{div}(w) = 0$, $i, j, k, l \in \mathbb{N}$ and define $\mathfrak{A}_{\alpha\beta}$, $\mathfrak{B}_\alpha$ for $\alpha, \beta \in \{1, 2, 3\}$ by (4.1). Then the following hold:*

(a) $\partial_\alpha \mathfrak{A}_{\alpha,\beta}(i, j, k) = -\mathfrak{B}_\beta(i, j, k)$.

(b) $\partial_\beta \mathfrak{A}_{\alpha,\beta}(i, j, k) = \mathfrak{B}_\alpha(i, j, k)$.

(c) Antisymmetry of $\mathfrak{A}_{\alpha,\beta}$: $\mathfrak{A}_{\alpha,\beta}(i, j, k) = -\mathfrak{A}_{\alpha,\beta}(j, i, k) = \mathfrak{A}_{\alpha,\beta}(j, k, i)$.

(d) Antisymmetry of $\mathfrak{B}_\alpha$: $\mathfrak{B}_\alpha(i, j, k) = -\mathfrak{B}_\alpha(j, i, k) = \mathfrak{B}_\alpha(j, k, i)$.

(e) $\operatorname{div}_\xi((y - \xi)_\beta w_\alpha(\xi) - (y - \xi)_\alpha w_\beta(\xi)) = 0$.

(f) $\mathfrak{B}_\alpha(i, j, k) - \mathfrak{B}_\alpha(l, j, k) - \mathfrak{B}_\alpha(i, l, k) - \mathfrak{B}_\alpha(i, j, l) = 0$.

(g) $\mathfrak{A}_{\alpha,\beta}(i, j, k) - \mathfrak{A}_{\alpha,\beta}(l, j, k) - \mathfrak{A}_{\alpha,\beta}(i, l, k) - \mathfrak{A}_{\alpha,\beta}(i, j, l) = 0$.

Proof. Properties (a)–(d) are immediate consequences of the definitions. Property (e) holds, since

$$\begin{aligned}
&\operatorname{div}_\xi((y - \xi)_\beta w_\alpha(\xi) - (y - \xi)_\alpha w_\beta(\xi)) \\
&\quad = -w_{\alpha\beta}(\xi) - \xi_\beta \operatorname{div}(w_\alpha) + w_{\beta\alpha}(\xi) + \xi_\alpha \operatorname{div}(w_\beta) \\
&\quad = 0.
\end{aligned}$$

To prove (f) we use that by the definition of $\mathfrak{B}_\alpha$ and the Gauß–Green theorem we have

$$\begin{aligned}
&\mathfrak{B}_\alpha(i, j, k) - \mathfrak{B}_\alpha(l, j, k) - \mathfrak{B}_\alpha(i, l, k) - \mathfrak{B}_\alpha(i, j, l) \\
&\quad = \int_{\langle x_i, x_j, x_k, x_m\rangle} \operatorname{div}(w_\alpha)\,\mathrm{d}x \\
&\quad = 0.
\end{aligned}$$

Note that this calculation also holds in the case that one or multiple simplices are degenerate. Analogously, we can prove (g) by applying the Gauß–Green theorem as well as (e) to get

$$\begin{aligned}
&\mathfrak{A}_{\alpha,\beta}(i, j, k) - \mathfrak{A}_{\alpha,\beta}(l, j, k) - \mathfrak{A}_{\alpha,\beta}(i, l, k) - \mathfrak{A}_{\alpha,\beta}(i, j, l) \\
&\quad = \int_{\langle x_i, x_j, x_k, x_m\rangle} \operatorname{div}_\xi((y - \xi)_\beta w_\alpha(\xi) - (y - \xi)_\alpha w_\beta(\xi))\,\mathrm{d}x \\
&\quad = 0.
\end{aligned}$$

The proof is complete. ∎

Lemma 4.3. *Let $u \in (\mathrm{L}^1 \cap \mathrm{C}^1)(\mathbb{R}^3; \mathbb{R}^3)$ satisfy $\operatorname{div}(u) = 0$ and $z_0 \in \{\mathcal{M}_{2R} u \le \lambda\}$, where $R > 0$. In addition, let $x_1, x_2, x_3 \in B_R(z_0)$. Then*

$$\left| \fint_{\langle x_1, x_2, x_3\rangle} u(\xi) \cdot \nu_{123}\,\mathrm{d}^2\xi \right| \le C\lambda R^2. \tag{4.6}$$

Moreover, if $w \in (\mathrm{L}^1 \cap \mathrm{C}^1)(\mathbb{R}^3; \mathbb{R}^{3\times3}_{\mathrm{sym}})$ satisfies $\operatorname{div}(w) = 0$ and the cubes Q_i, Q_j, Q_k have non-empty intersection, $y \in Q_i \cap Q_j \cap Q_k$, we have for $\mathfrak{A}_{\alpha,\beta}$ and $\mathfrak{B}_\alpha$ as defined in (4.1),

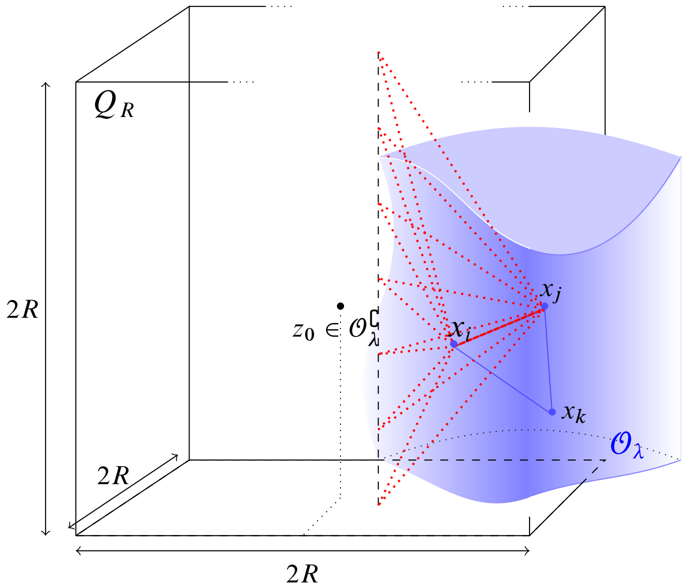

Figure 2. The construction in the proof of Lemma 4.3. The point $z_0 \in \mathcal{O}_\lambda^\complement$ is chosen such that it is close to x_i, x_j and x_k respectively. Instead of estimating the integral on the triangle with vertices x_i, x_j and x_k directly, we estimate integrals along triangles with vertices x_i, x_j and $z \in Q_R(z_0)$ (the triangles with red dashed lines) and use Gauß's theorem.

(a) $|\mathfrak{A}_{\alpha,\beta}(i,j,k)(y)| \le C\lambda\ell(Q_i)^3$;

(b) $|\mathfrak{B}_\alpha(i,j,k)| \le C\lambda\ell(Q_i)^2$.

The constant $C = C(3)$ is a dimensional constant that does not depend on u, i, j, k and the shape of $\mathcal{O}_\lambda$.

Proof. Let $x_1, x_2, x_3, z_0 \in \mathbb{R}^3$ be according to the assumption, $z_0 = (z_0^1, z_0^2, z_0^3)$. Then, using that $\operatorname{div} u = 0$, we find by Gauß's theorem for an arbitrary $\eta \in \mathbb{R}^3$,

$$\left| \fint_{\langle x_1,x_2,x_3\rangle} u \cdot \nu_{123}\, \mathrm{d}^2\xi \right| \le \left(\int_{\langle \eta,x_2,x_3\rangle} + \int_{\langle x_1,\eta,x_3\rangle} + \int_{\langle x_1,x_2,\eta\rangle} \right) |u|\, \mathrm{d}^2\xi. \tag{4.7}$$

Recalling from Section 2.1 that $\operatorname{aff}(x_i, x_j, x_k)$ denotes the affine hyperplane containing x_i, x_j, x_k, we now establish the existence of some $\eta \in \mathbb{R}^3 \setminus \operatorname{aff}(x_i, x_j, x_k)$ such that the right-hand side of (4.7) is bounded by $CR^2\lambda$ for some $C > 0$ solely depending on the underlying space dimension $n = 3$. Denote by $Q_R(z_0)$ the cube centred at z_0 with faces parallel to the coordinate planes and side length $2R$ so that $B_R(z_0) \subset Q_R(z_0) \subset B_{\sqrt{3}R}(z_0)$; cf. Figure 2.

Then, with the maximal operator $\mathcal{M}_{2R}$ from (2.1),

$$\begin{aligned}
&\int_{B_R(z_0)} \int_{\langle x_1,x_2,z\rangle} |u(\xi)|\,\mathrm{d}^2\xi\,\mathrm{d}z \\
&\quad\le \int_{Q_R(z_0)} \int_{\langle x_1,x_2,z\rangle} |u(\xi)|\,\mathrm{d}^2\xi\,\mathrm{d}z \\
&\quad= \int_{z_0^1-R}^{z_0^1+R} \int_{z_0^2-R}^{z_0^2+R} \int_{z_0^3-R}^{z_0^3+R} \int_{\langle x_1,x_2,(z^1,z^2,z^3)\rangle} |u(\xi)|\,\mathrm{d}^2\xi\,\mathrm{d}z^3\,\mathrm{d}z^2\,\mathrm{d}z^1 \\
&\quad\le \int_{z_0^1-R}^{z_0^1+R} \int_{z_0^2-R}^{z_0^2+R} \int_{Q_R(z_0)} |u|\,\mathrm{d}x\,\mathrm{d}z^2\,\mathrm{d}z^1 \\
&\quad\le \omega_3(\sqrt{3}R)^3 \int_{z_0^1-R}^{z_0^1+R} \int_{z_0^2-R}^{z_0^2+R} \fint_{B_{\sqrt{3}R}(z_0)} |u|\,\mathrm{d}x\,\mathrm{d}z^2\,\mathrm{d}z^1 \\
&\quad\le \omega_3(2R)^3(2R)^2\mathcal{M}_{2R}u(z_0) \\
&\quad\le c\lambda R^5.
\end{aligned} \tag{4.8}$$

Here $c > 0$ is a constant solely depending on the space dimension $n = 3$. In consequence, by Markov's inequality,

$$\begin{aligned}
\mathcal{L}^3(\mathcal{U}_{x_1,x_2,\cdot}[u,\lambda';B_R(z_0)]) &:= \mathcal{L}^3\big(\big\{z\in B_R(z_0) : \textstyle\int_{\langle x_1,x_2,z\rangle}|u(\xi)|\,\mathrm{d}^2\xi > \lambda'\big\}\big) \\
&\overset{(4.8)}{\le} c\frac{\lambda}{\lambda'}R^5 \quad \text{for any } \lambda' > 0,
\end{aligned}$$

where $\mathcal{U}_{x_1,x_2,\cdot}[u,\lambda';B_R(z_0)]$ is defined in the obvious manner. The same argument works equally well for the remaining simplices that appear in (4.7), and therefore, setting

$$\mathcal{U} := \mathcal{U}_{x_1,x_2,\cdot}[u,\lambda';B_R(z_0)] \cup \mathcal{U}_{\cdot,x_2,x_3}[u,\lambda';B_R(z_0)] \cup \mathcal{U}_{x_1,\cdot,x_3}[u,\lambda';B_R(z_0)],$$

with an obvious definition of the sets appearing on the right-hand side, we obtain

$$\mathcal{L}^3(\mathcal{U}) \le \frac{4c\lambda}{\lambda'}R^5.$$

We still have the freedom to choose $\lambda' > 0$ and consequently put $\lambda' := \frac{16}{\omega_3}c\lambda R^2$ so that $\mathcal{L}^3(\mathcal{U}^\complement) \ge \frac{3}{4}\mathcal{L}^3(B_R(z_0))$. We may thus pick $\eta \in B_R(z_0)\setminus \mathrm{aff}(x_i,x_j,x_k)$ such that $\eta\in\mathcal{U}^\complement$, and by definition of $\mathcal{U}$, this choice of η gives

$$\left|\fint_{\langle x_1,x_2,x_3\rangle} u\cdot\nu_{123}\,\mathrm{d}^2\xi\right| \le c\lambda R^2$$

with some purely dimension-dependent constant $c > 0$. This completes the proof of (4.6). The estimates in (a) and (b) are consequences of (4.6). For (a) note that there is $z_0 \in \mathcal{O}_\lambda^\complement$ with $\mathrm{dist}(z_0,Q_i)\le C\ell(Q_i)$ and $Q_i\cap Q_j\cap Q_k \subset B_{C\ell(Q_i)}(z_0)$ by (W2) and (W4). Moreover, $\mathcal{M}w(z_0)\le\lambda$ by definition of $\mathcal{O}_\lambda$ and therefore, for fixed $y\in Q_i$,

$$\mathcal{M}_{2R}((y-\cdot)_\beta w_\alpha(\cdot) - (y-\cdot)_\alpha w_\beta)(z_0) \le 2\sup_{z\in B_{2R}(z_0)}|y-z|\cdot\mathcal{M}w(z_0).$$

Setting $R = C\ell(Q_i)$ and using Lemma 4.2 (e) yields estimate (b). The estimate for $\mathfrak{B}_\alpha$ directly uses the existence of a point $z_0 \in \mathcal{O}_\lambda^{\complement}$, such that $Q_i, Q_j, Q_k \subset B_{C\ell(Q_i)}(z_0)$ and that w_α is divergence-free. Applying (4.6) in this setting yields (b). ∎

4.4. Elementary properties of T_λ

We now record various properties of T_λ that play an instrumental role in the proof of Theorem 1.2. Throughout this section, we tacitly suppose that $w \in (\mathrm{C}^\infty \cap \mathrm{L}^1)(\mathbb{R}^3; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$, and begin by providing the corresponding L$^\infty$-bounds.

Lemma 4.4. *There exists a purely dimensional constant $c > 0$ such that*

$$\|T_\lambda w\|_{\mathrm{L}^\infty(\mathbb{R}^3)} \le c\lambda \quad \textit{holds for all } \lambda > 0. \tag{4.9}$$

Proof. Since $|w| \le \lambda$ on $\mathcal{O}_\lambda^{\complement}$, it suffices to prove $\|T_\lambda w\|_{\mathrm{L}^\infty(\mathcal{O}_\lambda)} \le c\lambda$ for some suitable $c > 0$. Hence let $x \in \mathcal{O}_\lambda$. Then, by (W1) and (W3), $x \in Q_k$ for some $k \in \mathbb{N}$, and there are only finitely many cubes Q_i, Q_j such that $Q_i \cap Q_j \cap Q_k \neq \emptyset$; note that the number of such cubes solely depends on the underlying space dimension $n = 3$. For any choice of $\alpha', \beta', \gamma' \in \{1, 2, 3\}$ and $\ell_1 + \ell_2 = 2$ we have

$$|\varphi_k \partial_{\beta'}^{\ell_1} \varphi_i \partial_{\gamma'}^{\ell_2} \varphi_j| \le c \frac{\mathbb{1}_{Q_i \cap Q_j \cap Q_k}}{\ell(Q_k)^2} \tag{4.10}$$

and similarly, if $\ell_1 + \ell_2 = 3$,

$$|\varphi_k \partial_{\beta'}^{\ell_1} \varphi_i \partial_{\gamma'}^{\ell_2} \varphi_j| \le c \frac{\mathbb{1}_{Q_i \cap Q_j \cap Q_k}}{\ell(Q_k)^3}, \tag{4.11}$$

which is seen by combining (W4) and (P3). Again, $c > 0$ is a purely dimensional constant. By definition of $\widetilde{w}^{(k)}$ (cf. (4.3) and (4.4)), on $\mathcal{O}_\lambda$ every summand in (4.5) containing some $\mathfrak{B}_\delta(i, j, k)$, $\delta \in \{\alpha, \beta, \gamma\}$ is of the form $\varphi_k \partial_{\beta'}^{\ell_1} \varphi_i \partial_{\gamma'}^{\ell_2} \varphi_j \mathfrak{B}_\delta(i, j, k)$ with $\ell_1 + \ell_2 = 2$. Here we may invoke Lemma 4.3 (b) in conjunction with (4.10) to find

$$|\varphi_k \partial_{\beta'}^{\ell_1} \varphi_i \partial_{\gamma'}^{\ell_2} \varphi_j \mathfrak{B}_\delta(i, j, k)| \le c\lambda.$$

Conversely, every summand in (4.5) on $\mathcal{O}_\lambda$ that contains some $\mathfrak{A}_{\delta,\kappa}(i, j, k)$, $\delta, \kappa \in \{\alpha, \beta, \gamma\}$ is of the form $\varphi_k \partial_{\beta'}^{\ell_1} \varphi_i \partial_{\gamma'}^{\ell_2} \varphi_j \mathfrak{A}_{\delta,\kappa}(i, j, k)$ with $\ell_1 + \ell_2 = 3$, and in this case Lemma 4.3 (a) in conjunction with (4.11) yields

$$|\varphi_k \partial_{\beta'}^{\ell_1} \varphi_i \partial_{\gamma'}^{\ell_2} \varphi_j \mathfrak{A}_{\delta,\kappa}(i, j, k)| \le c\lambda.$$

By the uniformly finite overlap of the cubes (cf. (W3)), this completes the proof. ∎

Lemma 4.5. *For every $\alpha \in \{1, 2, 3\}$, $T_\lambda(w_{\alpha 1}, w_{\alpha 2}, w_{\alpha 3})$ is solenoidal on $\mathcal{O}_\lambda$.*

The proof of this lemma relies on a slightly elaborate computation, mutually hinging on index permutations and the properties of the maps $\mathfrak{A}_{\alpha,\beta}$ and $\mathfrak{B}_\alpha$ as gathered in Lemma 4.2. For expository purposes, we thus accept Lemma 4.5 for the time being and refer the reader to Appendix A.1 for its proof.

4.5. Global divsym-freeness

As the last ingredient towards Proposition 4.1, we next address the regularity of $\operatorname{div}(T_\lambda w)$. Here, we do not assert that $T_\lambda w$ belongs to the Sobolev space $\mathrm{W}^{1,1}(\mathbb{R}^3;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$; this is so because $T_\lambda w$ is precisely constructed in a way such that the handling of the divergence is possible (cf. Lemma 4.6 below), whereas the control of the full gradients does not come up as a consequence of Lemma 4.3; in particular, there seems to be no reason for the series in (4.5) to converge in $\mathrm{W}^{1,1}_0(\mathbb{R}^3;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$. Note that if it did, we could directly infer from Lemma 4.5 that $\operatorname{div}(T_\lambda w) = 0$.

Lemma 4.6. *Let* $w \in (\mathrm{C}^\infty \cap \mathrm{L}^1)(\mathbb{R}^3;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *satisfy* $\operatorname{div}(w) = 0$ *and define* $T_\lambda w$ *for* $\lambda > 0$ *by* (4.5). *Then the distributional divergence of* $T_\lambda w$ *is an* $\mathbb{R}^3$*-valued regular distribution, that is,* $\operatorname{div}(T_\lambda w) \in \mathrm{L}^1(\mathbb{R}^3;\mathbb{R}^3)$.

Proof. We focus on the first column $(T_\lambda w)_1$ of $T_\lambda w$; the other columns are treated by analogous means. Let $\psi \in \mathrm{C}^\infty_c(\mathbb{R}^3)$. By a technical, yet elementary computation to be explained in detail in Appendix A.2, we have

$$\begin{aligned}\int_{\mathcal{O}_\lambda} (T_\lambda w)_1 \cdot \nabla\psi \,\mathrm{d}x = {} & 2\sum_{i,j,k} \int_{\mathcal{O}_\lambda} \varphi_k (\partial_2\varphi_j)(\partial_3\varphi_i)\mathfrak{B}_1(i,j,k)\partial_1\psi \,\mathrm{d}x \\ & + 2\sum_{i,j,k} \int_{\mathcal{O}_\lambda} \varphi_k (\partial_3\varphi_j)(\partial_1\varphi_i)\mathfrak{B}_1(i,j,k)\partial_2\psi \,\mathrm{d}x \\ & + 2\sum_{i,j,k} \int_{\mathcal{O}_\lambda} \varphi_k (\partial_1\varphi_j)(\partial_2\varphi_i)\mathfrak{B}_1(i,j,k)\partial_3\psi \,\mathrm{d}x \\ =: {} & \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{4.12}$$

We focus on term I first and consider the functions

$$\begin{aligned} v_{\mathrm{I},(1)}(y) &:= \sum_{i,j,k} v_{\mathrm{I}}^{ijk}(y) := \sum_{i,j,k} \varphi_k(\partial_2\varphi_j)(\partial_3\varphi_i)(\mathfrak{B}_1(i,j,k) - w_1(y)\cdot\nu_{ijk}), \\ w_{\mathrm{I}}(y) &:= \sum_{i,j,k} w_{\mathrm{I}}^{ijk}(y) := \sum_{i,j,k} \varphi_k(\partial_2\varphi_j)(\partial_3\varphi_i)(w_1(y)\cdot\nu_{ijk}). \end{aligned} \tag{4.13}$$

We claim that $v_{\mathrm{I},(1)} \in \mathrm{W}^{1,1}_0(\mathcal{O}_\lambda)$. Note that each summand belongs to $\mathrm{C}^\infty_c(\mathcal{O}_\lambda)$, and so it suffices to establish that the overall sum in (4.13) converges absolutely in $\mathrm{W}^{1,1}(\mathcal{O}_\lambda)$. We give bounds on the single summands: for $i,j,k \in \mathbb{N}$, note that whenever $y \in Q_i \cap Q_j \cap Q_k$, then

$$\begin{aligned} |\mathfrak{B}_1(i,j,k) - w_1(y)\cdot\nu_{ijk}| &\le \fint_{\langle x_i,x_j,x_k\rangle} |w_1(\xi) - w_1(y)|\,|\nu_{ijk}|\,\mathrm{d}^2\xi \\ &\le c\|\nabla w_1\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\ell(Q_k)^3 \end{aligned} \tag{4.14}$$

as a consequence of the usual Lipschitz estimate, $\mathrm{dist}(y, \langle x_i, x_j, x_k\rangle) \leq c\ell(Q_k)$ and $|\nu_{ijk}| \leq c\ell(Q_k)^2$ by (W4). Now, by (W4) and (P3), we consequently obtain by (4.14),

$$\begin{aligned}\|v_{\mathrm{I}}^{ijk}\|_{\mathrm{L}^1(Q_k)} &\leq c\ell(Q_k)^4\|\nabla w_1\|_{\mathrm{L}^\infty(\mathbb{R}^3)},\\ \|\nabla v_{\mathrm{I}}^{ijk}\|_{\mathrm{L}^1(Q_k)} &\leq c\ell(Q_k)^3\|\nabla w_1\|_{\mathrm{L}^\infty(\mathbb{R}^3)},\end{aligned}$$

so that, by the uniformly finite overlap of the cubes,

$$\begin{aligned}\sum_{i,j,k}\|v_{\mathrm{I}}^{ijk}\|_{\mathrm{W}^{1,1}(\mathcal{O}_\lambda)} &\leq c\sum_k(\ell(Q_k)^4+\ell(Q_k)^3)\|\nabla w_1\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\\ &\leq c(1+\mathcal{L}^3(\mathcal{O}_\lambda)^{\frac{1}{3}})\sum_k\ell(Q_k)^3\|\nabla w_1\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\\ &\leq c(1+\mathcal{L}^3(\mathcal{O}_\lambda)^{\frac{1}{3}})\mathcal{L}^3(\mathcal{O}_\lambda)\|\nabla w_1\|_{\mathrm{L}^\infty(\mathbb{R}^3)}<\infty.\end{aligned}$$

Hence, $v_{\mathrm{I},(1)}\in\mathrm{W}_0^{1,1}(\mathcal{O}_\lambda)$. Extend $v_{\mathrm{I},(1)}$ by zero to the entire $\mathbb{R}^3$ to obtain $v_{\mathrm{I},(2)}\in\mathrm{W}_0^{1,1}(\mathbb{R}^3)$. Then an integration by parts yields

$$\begin{aligned}\mathrm{I} &= 2\int_{\mathcal{O}_\lambda} v_{\mathrm{I},(1)}\partial_1\psi\,\mathrm{d}y + 2\int_{\mathcal{O}_\lambda} w_{\mathrm{I}}\partial_1\psi\,\mathrm{d}y\\ &= 2\int_{\mathbb{R}^3} v_{\mathrm{I},(2)}\partial_1\psi\,\mathrm{d}y + 2\int_{\mathcal{O}_\lambda} w_{\mathrm{I}}\partial_1\psi\,\mathrm{d}y\\ &\overset{v_{\mathrm{I},(2)}\in\mathrm{W}_0^{1,1}(\mathbb{R}^3)}{=} -2\int_{\mathbb{R}^3}(\partial_1 v_{\mathrm{I},(2)})\psi\,\mathrm{d}y + 2\int_{\mathcal{O}_\lambda} w_{\mathrm{I}}\partial_1\psi\,\mathrm{d}y =: \mathrm{I}_1+\mathrm{I}_2,\end{aligned}\tag{4.15}$$

and $\partial_1 v_{\mathrm{I},(2)}\in\mathrm{L}^1(\mathbb{R}^3)$. Regarding term I_2, observe that for all $y\in\mathbb{R}^3$,

$$\begin{aligned}-2\nu_{ijk} &= -(x_i-x_j)\times(x_k-x_j)\\ &= (y-x_j)\times(x_j-x_k)+(x_i-y)\times(y-x_k)\\ &\quad+(x_i-x_j)\times(x_j-y),\end{aligned}\tag{4.16}$$

which follows by direct computation using that $(x_j-y)\times(y-x_j)=0$. Working from the definition of w_{I} as in (4.13), we consequently find by (4.16),

$$\begin{aligned}\mathrm{I}_2 &= 2\int_{\mathcal{O}_\lambda} w_{\mathrm{I}}(y)\partial_1\psi\,\mathrm{d}y\\ &= 2\int_{\mathcal{O}_\lambda}\sum_{i,j,k}\varphi_k(\partial_2\varphi_j)(\partial_3\varphi_i)(w_1(y)\cdot\nu_{y,x_j,x_k})\partial_1\psi\,\mathrm{d}y \qquad (=0)\\ &\quad+2\int_{\mathcal{O}_\lambda}\sum_{i,j,k}\varphi_k(\partial_2\varphi_j)(\partial_3\varphi_i)(w_1(y)\cdot\nu_{x_i,y,x_k})\partial_1\psi\,\mathrm{d}y \qquad (=0)\\ &\quad+2\int_{\mathcal{O}_\lambda}\sum_{i,j}(\partial_2\varphi_j)(\partial_3\varphi_i)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi\,\mathrm{d}y =: \mathrm{I}_3,\end{aligned}$$

where we have used that $\sum_i \partial_3\varphi_i = 0$ on $\mathcal{O}_\lambda$ for the first, $\sum_j \partial_2\varphi_j = 0$ on $\mathcal{O}_\lambda$ for the second and $\sum_k \varphi_k = 1$ on $\mathcal{O}_\lambda$ for the final term. By a similar argument to above, the sum in the integrand of I_3 has an integrable majorant, whereby we may change the sum and the integral. Hence, integrating by parts with respect to ∂_2,

$$
\begin{aligned}
\mathrm{I}_3 = \mathrm{I}_3^1 := 2\sum_{ij}\int_{\mathcal{O}_\lambda} & \partial_2(\varphi_j(\partial_3\varphi_i)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi)\,\mathrm{d}y && (=T_1)\\
&-2\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_j(\partial_{23}\varphi_i)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi)\,\mathrm{d}y && (=T_2)\\
&-2\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_j(\partial_3\varphi_i)(\partial_2 w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi)\,\mathrm{d}y && (=T_3)\\
&-2\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_j(\partial_3\varphi_i)(w_1(y)\cdot\partial_2\nu_{x_i,x_j,y})\partial_1\psi)\,\mathrm{d}y && (=T_4)\\
&-2\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_j(\partial_3\varphi_i)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_{12}\psi)\,\mathrm{d}y && (=T_5),
\end{aligned}
$$

but on the other hand, now integrating by parts with respect to ∂_3,

$$
\begin{aligned}
\mathrm{I}_3 = \mathrm{I}_3^2 := 2\sum_{ij}\int_{\mathcal{O}_\lambda} & \partial_3(\varphi_i(\partial_2\varphi_j)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi)\,\mathrm{d}y && (=T_6)\\
&-2\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_i(\partial_{23}\varphi_j)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi)\,\mathrm{d}y && (=T_7)\\
&-2\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_i(\partial_2\varphi_j)(\partial_3 w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi)\,\mathrm{d}y && (=T_8)\\
&-2\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_i(\partial_2\varphi_j)(w_1(y)\cdot\partial_3\nu_{x_i,x_j,y})\partial_1\psi)\,\mathrm{d}y && (=T_9)\\
&-2\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_i(\partial_2\varphi_j)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_{13}\psi)\,\mathrm{d}y && (=T_{10}).
\end{aligned}
$$

We then have $\mathrm{I}_3 = \frac{1}{2}(\mathrm{I}_3^1 + \mathrm{I}_3^2)$. To proceed further, note that $T_1 = T_6 = 0$ by the fundamental theorem of calculus. Moreover, $\frac{1}{2}(T_2 + T_7) = 0$, which follows from permuting indices $i \leftrightarrow j$ in T_2 and using the antisymmetry property $\nu_{x_i,x_j,y} = -\nu_{x_j,x_i,y}$:

$$
\begin{aligned}
T_2 &= -2\sum_{ji}\int_{\mathcal{O}_\lambda}(\varphi_i(\partial_{23}\varphi_j)(w_1(y)\cdot\nu_{x_j,x_i,y})\partial_1\psi)\,\mathrm{d}y\\
&= 2\sum_{ji}\int_{\mathcal{O}_\lambda}(\varphi_i(\partial_{23}\varphi_j)(w_1(y)\cdot\nu_{x_i,x_j,y})\partial_1\psi)\,\mathrm{d}y = -T_7.
\end{aligned}
$$

To treat terms T_3 and T_8, define the smooth function $v_{I,(3)}\colon \mathcal{O}_\lambda \to \mathbb{R}$ by

$$v_{I,(3)} := \sum_{ij} (\varphi_j(\partial_3\varphi_i)(\partial_2 w_1(y)\cdot \nu_{x_i,x_j,y})) + (\varphi_i(\partial_2\varphi_j)(\partial_3 w_1(y)\cdot \nu_{x_i,x_j,y})). \quad (4.17)$$

By an argument similar to the one employed in (4.13)ff., we have $v_{I,(3)} \in W_0^{1,1}(\mathcal{O}_\lambda)$. More precisely, for all finite index sets $\mathcal{I}, \mathcal{J} \subset \mathbb{N}$, the functions

$$z_{\mathcal{I},\mathcal{J}} := \sum_{\substack{i\in\mathcal{I}\\ j\in\mathcal{J}}} z_{ij} := \sum_{\substack{i\in\mathcal{I}\\ j\in\mathcal{J}}} (\varphi_j(\partial_3\varphi_i)(\partial_2 w_1(y)\cdot \nu_{x_i,x_j,y})) + (\varphi_i(\partial_2\varphi_j)(\partial_3 w_1(y)\cdot \nu_{x_i,x_j,y}))$$

are finite sums of $C_c^\infty(\mathcal{O}_\lambda)$-functions. By the Leibniz rule in conjunction with (W2)–(W4) and (P3), we obtain

$$\begin{aligned}
\sum_{\substack{i\in\mathcal{I}\\ j\in\mathcal{J}}} \|z_{ij}\|_{W^{1,1}(\mathcal{O}_\lambda)} &= \sum_{\substack{i\in\mathcal{I}\\ j\in\mathcal{J}}} \|z_{ij}\|_{L^1(\mathcal{O}_\lambda)} + \|\nabla z_{ij}\|_{L^1(\mathcal{O}_\lambda)} \\
&\leq c \sum_{i\in\mathcal{I}} \ell(Q_i)^4 \|\nabla w_1\|_{L^\infty(\mathbb{R}^3)} \\
&\quad + c \sum_{i\in\mathcal{I}} \big(\ell(Q_i)^3 \|\nabla w_1\|_{L^\infty(\mathbb{R}^3)} + \ell(Q_i)^4 \|\nabla^2 w_1\|_{L^\infty(\mathbb{R}^3)}\big) \\
&\leq (1 + \mathcal{L}^3(\mathcal{O}_\lambda)^{\frac{1}{3}}) \|w_1\|_{W^{2,\infty}(\mathbb{R}^3)},
\end{aligned}$$

where c is a purely dimensional constant. Since the final term in the previous estimation is independent of $\mathcal{I}$ and $\mathcal{J}$, we conclude that the sum in (4.17) converges absolutely in the Banach space $W_0^{1,1}(\mathcal{O}_\lambda)$. Hence, in particular, it converges in $W_0^{1,1}(\mathcal{O}_\lambda)$ and so $v_{I,(3)} \in W_0^{1,1}(\mathcal{O}_\lambda)$.

Extending $v_{I,(3)}$ by zero to $v_{I,(4)} \in W_0^{1,1}(\mathbb{R}^3)$, we then obtain

$$\tfrac{1}{2}(T_3 + T_8) = \int_{\mathbb{R}^3} (\partial_1 v_{I,(4)})\psi \,\mathrm{d}y. \quad (4.18)$$

Since $I_3 = \frac{1}{2}(I_3^1 + I_3^2)$, the above arguments, permuting $i \leftrightarrow j$ in I_3^2 and (4.18) combine to

$$\begin{aligned}
I_3 = &-\frac{1}{2}\sum_{ij}\int_{\mathcal{O}_\lambda} (\varphi_j(\partial_3\varphi_i)(w_1(y)\cdot((x_i - x_j)\times e_2))\partial_1\psi)\,\mathrm{d}y \quad (= \tfrac{1}{2}T_4) \\
&+ \frac{1}{2}\sum_{ij}\int_{\mathcal{O}_\lambda} (\varphi_j(\partial_2\varphi_i)(w_1(y)\cdot((x_i - x_j)\times e_3))\partial_1\psi)\,\mathrm{d}y \quad (= \tfrac{1}{2}T_9) \\
&- \sum_{ij}\int_{\mathcal{O}_\lambda} (\varphi_j(\partial_3\varphi_i)(w_1(y)\cdot \nu_{x_i,x_j,y})\partial_{12}\psi)\,\mathrm{d}y \quad (= \tfrac{1}{2}T_5) \\
&+ \sum_{ij}\int_{\mathcal{O}_\lambda} (\varphi_j(\partial_2\varphi_i)(w_1(y)\cdot \nu_{x_i,x_j,y})\partial_{13}\psi)\,\mathrm{d}y \quad (= \tfrac{1}{2}T_{10}) \\
&+ \int_{\mathbb{R}^3} (\partial_1 v_{I,(4)})\psi\,\mathrm{d}y.
\end{aligned}$$

Next note that, expanding and using $\sum_i \varphi_i = 1$ as well as $\sum_i \partial_3\varphi_i = 0$ on $\mathcal{O}_\lambda$,

$$\begin{aligned}\frac{1}{2}T_4 &= -\frac{1}{2}\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_j(\partial_3\varphi_i)(w_1(y)\cdot((x_i-y)\times e_2))\partial_1\psi)\,\mathrm{d}y \\ &\quad -\frac{1}{2}\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_j(\partial_3\varphi_i)(w_1(y)\cdot((y-x_j)\times e_2))\partial_1\psi)\,\mathrm{d}y \quad (=0) \\ &= -\frac{1}{2}\sum_{i}\int_{\mathcal{O}_\lambda}((\partial_3\varphi_i)(w_1(y)\cdot((x_i-y)\times e_2))\partial_1\psi)\,\mathrm{d}y \\ &= \frac{1}{2}\sum_{i}\int_{\mathcal{O}_\lambda}(\varphi_i\partial_3 w_1(y)\cdot((x_i-y)\times e_2))\partial_1\psi)\,\mathrm{d}y \\ &\quad +\frac{1}{2}\int_{\mathcal{O}_\lambda}(w_1(y)\cdot(-e_3\times e_2)\partial_1\psi)\,\mathrm{d}y \\ &\quad +\frac{1}{2}\sum_{i}\int_{\mathcal{O}_\lambda}(\varphi_i w_1(y)\cdot((x_i-y)\times e_2)\partial_{13}\psi)\,\mathrm{d}y. \end{aligned} \tag{4.19}$$

By a similar argument to (4.17)ff., we use $w \in \mathrm{C}^\infty(\mathbb{R}^3;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ to see that the function

$$v_{\mathrm{I},(5)}(y) := -\frac{1}{2}\sum_i \varphi_i\partial_3 w_1(y)\cdot((x_i-y)\times e_2) \tag{4.20}$$

belongs to $\mathrm{W}^{1,1}_0(\mathcal{O}_\lambda)$, and hence, again denoting its trivial extension to $\mathbb{R}^3$ by $v_{\mathrm{I},(6)}$ and recalling that $e_2\times e_3 = e_1$,

$$\begin{aligned}\frac{1}{2}T_4 &= \int_{\mathbb{R}^3}(\partial_1 v_{\mathrm{I},(6)})\psi\,\mathrm{d}x + \frac{1}{2}\int_{\mathcal{O}_\lambda}(w_{11}(y)\partial_1\psi)\,\mathrm{d}y \\ &\quad +\frac{1}{2}\sum_i\int_{\mathcal{O}_\lambda}(\varphi_i w_1(y)\cdot((x_i-y)\times e_2)\partial_{13}\psi)\,\mathrm{d}y.\end{aligned} \tag{4.21}$$

We handle the term $\frac{1}{2}T_9$ in the same fashion (swapping the roles of the indices 2 and 3): introducing $v_{\mathrm{I},(7)} \in \mathrm{W}^{1,1}_0(\mathcal{O}_\lambda)$ as

$$v_{\mathrm{I},(7)}(y) := \frac{1}{2}\sum_i \varphi_i\partial_2 w_1(y)\cdot((x_i-y)\times e_3)$$

as a substitute for (4.20) and denoting its trivial extension to $\mathbb{R}^3$ by $v_{\mathrm{I},(8)}$, we arrive at

$$\begin{aligned}\frac{1}{2}T_9 &= \int_{\mathbb{R}^3}(\partial_1 v_{\mathrm{I},(8)})\psi\,\mathrm{d}x + \frac{1}{2}\int_{\mathcal{O}_\lambda}(w_{11}(y)\partial_1\psi)\,\mathrm{d}y \\ &\quad -\frac{1}{2}\sum_i\int_{\mathcal{O}_\lambda}(\varphi_i w_1(y)\cdot((x_i-y)\times e_3)\partial_{12}\psi)\,\mathrm{d}y.\end{aligned} \tag{4.22}$$

Working from (4.21) and (4.22), we then arrive at

$$\begin{aligned}\frac{1}{2}(T_4+T_9) &= \int_{\mathbb{R}^3} \partial_1(v_{\mathrm{I},(6)}+v_{\mathrm{I},(8)})\psi\,\mathrm{d}y + \int_{\mathcal{O}_\lambda}(w_{11}(y)\partial_1\psi)\,\mathrm{d}y\\ &\quad+\frac{1}{2}\sum_i\int_{\mathcal{O}_\lambda}(\varphi_i w_1(y)\cdot((x_i-y)\times e_2)\partial_{13}\psi)\,\mathrm{d}y\\ &\quad-\frac{1}{2}\sum_i\int_{\mathcal{O}_\lambda}(\varphi_i w_1(y)\cdot((x_i-y)\times e_3)\partial_{12}\psi)\,\mathrm{d}y. \end{aligned}\tag{4.23}$$

To summarise, by (4.12), (4.15) and (4.23), there exists $v_{\mathrm{I}}\in \mathrm{W}_0^{1,1}(\mathbb{R}^3)$, such that

$$\begin{aligned}\mathrm{I} &= \int_{\mathbb{R}^3}(\partial_1 v_{\mathrm{I}})\psi\,\mathrm{d}x + \int_{\mathcal{O}_\lambda}(w_{11}(y)\partial_1\psi)\,\mathrm{d}y\\ &\quad+\frac{1}{2}\sum_i\int_{\mathcal{O}_\lambda}(\varphi_i w_1(y)\cdot((x_i-y)\times e_2)\partial_{13}\psi)\,\mathrm{d}y\\ &\quad-\frac{1}{2}\sum_i\int_{\mathcal{O}_\lambda}(\varphi_i w_1(y)\cdot((x_i-y)\times e_3)\partial_{12}\psi)\,\mathrm{d}y\\ &\quad-\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_j(\partial_3\varphi_i)(w_1(y)\cdot\nu_{x_ix_jy})\partial_{12}\psi)\,\mathrm{d}y\\ &\quad+\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_j(\partial_2\varphi_i)(w_1(y)\cdot\nu_{x_ix_jy})\partial_{13}\psi)\,\mathrm{d}y. \end{aligned}\tag{4.24}$$

The same calculations with the coordinates $1\to 2\to 3\to 1$ permuted imply that there exist $v_{\mathrm{II}}, v_{\mathrm{III}}\in \mathrm{W}_0^{1,1}(\mathbb{R}^3)$ such that

$$\begin{aligned}\mathrm{II} &= \int_{\mathbb{R}^3}(\partial_2 v_{\mathrm{II}})\psi\,\mathrm{d}x + \int_{\mathcal{O}_\lambda}(w_{12}(y)\partial_2\psi)\,\mathrm{d}y\\ &\quad+\frac{1}{2}\sum_i\int_{\mathcal{O}_\lambda}(\varphi_i w_1(y)\cdot((x_i-y)\times e_3)\partial_{21}\psi)\,\mathrm{d}y\\ &\quad-\frac{1}{2}\sum_i\int_{\mathcal{O}_\lambda}(\varphi_i w_1(y)\cdot((x_i-y)\times e_1)\partial_{23}\psi)\,\mathrm{d}y\\ &\quad-\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_j(\partial_1\varphi_i)(w_1(y)\cdot\nu_{x_ix_jy})\partial_{23}\psi)\,\mathrm{d}y\\ &\quad+\sum_{ij}\int_{\mathcal{O}_\lambda}(\varphi_j(\partial_3\varphi_i)(w_1(y)\cdot\nu_{x_ix_jy})\partial_{21}\psi)\,\mathrm{d}y \end{aligned}\tag{4.25}$$

and

$$
\begin{aligned}
\mathrm{III} = &\int_{\mathbb{R}^3} (\partial_3 v_{\mathrm{III}})\psi \,\mathrm{d}x + \int_{\mathcal{O}_\lambda} (w_{13}(y)\partial_3\psi)\,\mathrm{d}y \\
&+ \frac{1}{2}\sum_i \int_{\mathcal{O}_\lambda} (\varphi_i w_1(y)\cdot((x_i - y)\times e_1)\partial_{32}\psi)\,\mathrm{d}y \\
&- \frac{1}{2}\sum_i \int_{\mathcal{O}_\lambda} (\varphi_i w_1(y)\cdot((x_i - y)\times e_2)\partial_{31}\psi)\,\mathrm{d}y \\
&- \sum_{ij} \int_{\mathcal{O}_\lambda} (\varphi_j(\partial_2\varphi_i)(w_1(y)\cdot \nu_{x_i x_j y})\partial_{31}\psi)\,\mathrm{d}y \\
&+ \sum_{ij} \int_{\mathcal{O}_\lambda} (\varphi_j(\partial_1\varphi_i)(w_1(y)\cdot \nu_{x_i x_j y})\partial_{32}\psi)\,\mathrm{d}y,
\end{aligned} \tag{4.26}
$$

and $\partial_1 v_{\mathrm{I}}$, $\partial_2 v_{\mathrm{II}}$, $\partial_3 v_{\mathrm{III}}$ all vanish outside $\mathcal{O}_\lambda$. Combining (4.24), (4.25) and (4.26), we get that there is $h \in \mathrm{L}^1(\mathcal{O}_\lambda)$, $h = \partial_1 v_{\mathrm{I}} + \partial_2 v_{\mathrm{II}} + \partial_3 v_{\mathrm{III}}$, such that

$$
\int_{\mathcal{O}_\lambda} (T_\lambda w)_1 \cdot \nabla\psi \,\mathrm{d}x = \int_{\mathcal{O}_\lambda} h\psi \,\mathrm{d}x + \int_{\mathcal{O}_\lambda} w_1 \cdot \nabla\psi \,\mathrm{d}x. \tag{4.27}
$$

Recall that w satisfies $\mathrm{div}(w) = 0$ and that $T_\lambda w = w$ on $\mathcal{O}_\lambda^{\complement}$. Therefore,

$$
\begin{aligned}
\int_{\mathbb{R}^3} (T_\lambda w)_1 \cdot \nabla\psi \,\mathrm{d}x &= \int_{\mathcal{O}_\lambda^{\complement}} (T_\lambda w)_1 \cdot \nabla\psi \,\mathrm{d}x + \int_{\mathcal{O}_\lambda} (T_\lambda w)_1 \cdot \nabla\psi \,\mathrm{d}x \\
&= \int_{\mathcal{O}_\lambda^{\complement}} w_1 \cdot \nabla\psi \,\mathrm{d}x + \int_{\mathcal{O}_\lambda} w_1 \cdot \nabla\psi \,\mathrm{d}x + \int_{\mathcal{O}_\lambda} h\psi \,\mathrm{d}x \\
&= \int_{\mathcal{O}_\lambda} h\psi \,\mathrm{d}x.
\end{aligned}
$$

Therefore, $\mathrm{div}((T_\lambda w)_1) \in \mathrm{L}^1(\mathbb{R}^3)$. Arguing in the exactly same way for the other columns, $\mathrm{div}(T_\lambda w) \in \mathrm{L}^1(\mathbb{R}^3;\mathbb{R}^3)$, and the proof is complete. ∎

As an immediate consequence of Lemmas 4.5 and 4.6, we obtain the following:

Corollary 4.7. *Let* $w \in (\mathrm{C}^\infty \cap \mathrm{L}^1)(\mathbb{R}^3;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ *satisfy* $\mathrm{div}(w) = 0$ *and define* $T_\lambda w$ *for* $\lambda > 0$ *by* (4.5). *Then for* $\mathcal{L}^1$*-almost every* $\lambda > 0$, $\mathrm{div}(T_\lambda w) = 0$ *in* $\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$.

Proof. Observe that on $\mathbb{R}^3 \setminus \partial\mathcal{O}_\lambda$ the function $T_\lambda w$ is strongly differentiable and, as w is (rowwise) solenoidal on $\mathbb{R}^3$ and $\mathrm{div}(T_\lambda w) = 0$ on $\mathcal{O}_\lambda$ (Lemma 4.5), $\mathrm{div}(T_\lambda w) = 0$ on the open set $\mathbb{R}^3 \setminus \partial\mathcal{O}_\lambda$. As $w \in \mathrm{C}^\infty$, $\mathcal{M}w \in \mathrm{C}(\mathbb{R}^3)$ and the set

$$
\{\lambda > 0 : \mathcal{L}^3(\partial\mathcal{O}_\lambda) \neq 0\} \subset \{\lambda > 0 : \mathcal{L}^3(\{\mathcal{M}w = \lambda\}) \neq 0\}
$$

is an $\mathcal{L}^1$-null set. Hence, for all λ not contained in this set, $\mathrm{div}(T_\lambda w) \in \mathrm{L}^1(\mathbb{R}^3;\mathbb{R}^3)$ and $\mathrm{div}(T_\lambda w) = 0\mathcal{L}^3$-a.e. Thus, for $\mathcal{L}^1$-almost every λ, $\mathrm{div}(T_\lambda w) = 0$ in $\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$. ∎

4.6. Strong stability and proof of Proposition 4.1

In view of Lemma 4.4 and Corollary 4.7, Proposition 4.1 will follow provided we can prove the strong stability (cf. Proposition 4.1 (a)). With this aim, we begin with the following lemma:

Lemma 4.8. *Then there exists a purely dimensional constant* $C > 0$ *such that, for each* $w \in \mathrm{L}^1(\mathbb{R}^3;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ *and each* $\lambda > 0$, *we have*

$$\mathcal{L}^3(\{\mathcal{M}w > \lambda\}) \leq \frac{C}{\lambda}\int_{\{|w|>\lambda/2\}} |w(x)|\,\mathrm{d}x.$$

The rough idea of the proof of this statement is to use the weak-$(1,1)$ estimate for the Hardy–Littlewood maximal operator $\mathcal{M}$ (cf. (2.1)) for the function h defined via

$$h(x) = \max\{0, |w(x)| - \lambda/2\}; \tag{4.28}$$

see Zhang [48] for the details. As an important consequence of Lemma 4.8 and the L^∞-bound of w_λ is the following:

Corollary 4.9. *Let* $w \in \mathrm{L}^1(\mathbb{R}^3;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ *satisfy* $\mathrm{div}(w) = 0$. *Moreover, for* $\lambda > 0$, *let* $w_\lambda := T_\lambda w$ *be as in* (4.5). *Then we have with a purely dimensional constant* $C > 0$

$$\|w - w_\lambda\|_{\mathrm{L}^1(\mathbb{R}^3)} \leq C\int_{\{|w|>\lambda/2\}} |w|\,\mathrm{d}x. \tag{4.29}$$

Proof. Recall that $\mathcal{O}_\lambda := \{\mathcal{M}w > \lambda\}$. By construction, $w = w_\lambda$ on $\mathcal{O}_\lambda^{\complement}$. Therefore,

$$\begin{aligned}\|w - w_\lambda\|_{\mathrm{L}^1(\mathbb{R}^3)} &\leq \int_{\mathcal{O}_\lambda} |w - w_\lambda|\,\mathrm{d}x \\ &\leq \int_{\mathcal{O}_\lambda} |w|\,\mathrm{d}x + \int_{\mathcal{O}_\lambda} |w_\lambda|\,\mathrm{d}x. \end{aligned}\tag{4.30}$$

On the one hand, Lemma 4.8 gives us

$$\begin{aligned}\int_{\mathcal{O}_\lambda} |w|\,\mathrm{d}x &\leq \lambda\mathcal{L}^3(\mathcal{O}_\lambda) + \int_{\{|w|>\lambda\}} |w|\,\mathrm{d}x \\ &\leq C\int_{\{|w|>\lambda/2\}} |w|\,\mathrm{d}x, \end{aligned}\tag{4.31}$$

and, on the other hand, using Lemmas 4.4 and 4.8,

$$\begin{aligned}\int_{\mathcal{O}_\lambda} |w_\lambda|\,\mathrm{d}x &\leq \|w_\lambda\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\mathcal{L}^3(\mathcal{O}_\lambda) \\ &\leq C\int_{\{|w|>\lambda/2\}} |w|\,\mathrm{d}x, \end{aligned}\tag{4.32}$$

$C > 0$ still being a purely dimensional constant. In view of (4.30), (4.31) and (4.32), we obtain (4.29), and this completes the proof. ∎

Proof of Proposition 4.1. Let $w \in (\mathrm{C}^\infty \cap \mathrm{L}^1)(\mathbb{R}^3;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ satisfy $\operatorname{div}(w) = 0$ and let $\lambda > 0$. Pick some $\tilde{\lambda} \in (\lambda, 2\lambda)$ such that $\mathcal{L}^3(\partial \mathcal{O}_{\tilde{\lambda}}) = 0$ and define $w_\lambda := T_{\tilde{\lambda}} w$ and $\mathcal{U}_\lambda := \mathcal{O}_{\tilde{\lambda}}$. Then

(a) $w = w_\lambda$ on $\mathcal{U}_\lambda^{\complement}$ by construction;

(b) Lemma 4.8 implies that

$$\mathcal{L}^3(\{w \neq w_\lambda\}) \leq \frac{c}{\tilde{\lambda}} \int_{\{|w|>\tilde{\lambda}/2\}} |w| \,\mathrm{d}x \leq \frac{c}{\lambda} \int_{\{|w|>\lambda/2\}} |w| \,\mathrm{d}x;$$

(c) $\operatorname{div}(w_\lambda) = 0$ in $\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$ by Corollary 4.7;

(d) $\|w_\lambda\|_{\mathrm{L}^\infty(\mathbb{R}^3)} \leq c\tilde{\lambda} \leq 2c\lambda$ by Lemma 4.4.

To summarise, w_λ satisfies all the required properties, and the proof is complete. ■

4.7. Proof of Theorem 1.2

We now establish Theorem 1.2, and hence let $\lambda > 0$ be given. Let $u \in \mathrm{L}^1(\mathbb{R}^3;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ satisfy $\operatorname{div}(u) = 0$ and pick a sequence $(w^j) \subset (\mathrm{C}^\infty \cap \mathrm{L}^1)(\mathbb{R}^3;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ such that $w^j \to u$ strongly in $\mathrm{L}^1(\mathbb{R}^3;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ as $j \to \infty$, still satisfying $\operatorname{div}(w^j) = 0$ for each $j \in \mathbb{N}$. Such a sequence can be constructed by convolution with smooth bumps.

For $\lambda > 0$, consider the truncation $w^j_{4\lambda}$ of w^j according to Proposition 4.1. Note that this sequence is uniformly bounded in L^∞ by $4c\lambda$. Therefore, a suitable, non-relabelled subsequence converges in the weak∗-sense to some u^λ in $\mathrm{L}^\infty(\mathbb{R}^3;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$. First of all,

$$\|u^\lambda\|_{\mathrm{L}^\infty(\mathbb{R}^3)} \leq \sup_{j\in\mathbb{N}} \|w^j_{4\lambda}\|_{\mathrm{L}^\infty(\mathbb{R}^3)} \leq 4c\lambda, \quad \operatorname{div}(u^\lambda) = 0.$$

We claim that $w^j_{4\lambda} \to u$ strongly in L^1 on the set $\{\mathcal{M}u \leq 2\lambda\}$ as $j \to \infty$, and hence $u^\lambda = u$ on $\{\mathcal{M}u \leq 2\lambda\}$. If this claim is proven, then Lemma 4.8 and Corollary 4.9 imply the small change and strong stability properties (b), (c) of Theorem 1.2. Therefore, u^λ will satisfy all properties displayed in Theorem 1.2 and thus finish the proof.

It remains to show the claim. Recall that the maximal function $\mathcal{M}$ is sublinear. Thus,

$$\{\mathcal{M}w^j > 4\lambda\} \setminus \{\mathcal{M}(w^j - u) > 2\lambda\} \subset \{\mathcal{M}u > 2\lambda\}. \tag{4.33}$$

Note that $\mathcal{L}^3(\{\mathcal{M}(w^j - u) > 2\lambda\})$ converges to zero as $j \to \infty$ since $w^j - u \to 0$ in L^1 and $\mathcal{M}$ is weak-$(1,1)$. After picking a suitable, non-relabelled subsequence of (w^j), we may suppose that $\|w^j - u\|_{\mathrm{L}^1(\mathbb{R}^3)} \leq 2^{-j}\lambda$ for all $j \in \mathbb{N}$ and hence

$$\mathcal{L}^3\{\mathcal{M}(w^j - u) > 2\lambda\} \leq C2^{-j} \quad \text{for all } j \in \mathbb{N}.$$

Therefore, for each $J \in \mathbb{N}$, the $\mathcal{L}^3$-measure of the set

$$E_J := \bigcup_{j>J} \{\mathcal{M}(w^j - u) > 2\lambda\}$$

can be bounded by $C2^{-J}$. Due to (4.33), we have $\{\mathcal{M}u \le 2\lambda\} \setminus E_J \subset \{\mathcal{M}w^j \le 4\lambda\}$ for $j > J$. Let us fix $J \in \mathbb{N}$ and bound the L^1-norm of $w^j_{4\lambda} - u$ on $\{\mathcal{M}u \le 2\lambda\}$ for $j > J$:

$$\begin{aligned}\int_{\{\mathcal{M}u\le 2\lambda\}} |w^j_{4\lambda} - u| \, dx &\le \int_{E_J} |w^j_{4\lambda} - u| \, dx + \int_{\{\mathcal{M}u\le 2\lambda\}\setminus E_J} |w^j_{4\lambda} - u| \, dx \\ &\le \int_{E_J} |w^j_{4\lambda}| + |u| \, dx + \int_{\{\mathcal{M}w^j\le 4\lambda\}} |w^j_{4\lambda} - u| \, dx \\ &\le C2^{-J}\lambda + \int_{E_J} |u| \, dx + \int_{\{\mathcal{M}w^j\le 4\lambda\}} |w^j - u| \, dx \\ &\le C2^{-J}\lambda + \int_{E_J} |u| \, dx + \|w^j - u\|_{L^1(\mathbb{R}^3)}.\end{aligned}$$

Letting $J \to \infty$ yields $w^j_{4\lambda} - u \to 0$ in $L^1(\{\mathcal{M}u \le 2\lambda\})$. As $(w^j_{4\lambda})$ weakly$*$ converges to u^λ in $L^\infty(\mathbb{R}^3, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$, we conclude that $u = u^\lambda$ on $\{\mathcal{M}u \le 2\lambda\}$, proving the claim. □

5. Proof of Theorem 1.1

The proof of Theorem 1.1 heavily depends on the validity of the truncation theorem, Theorem 1.2. In fact, Theorem 1.1 has been proven in a different setting, where the divergence is replaced by some other differential operator (e.g., [37, 49]). For the convenience of the reader, let us briefly present the argument here. First of all, note that the statement of Theorem 1.2 also holds if we consider functions $u \in L^1(\mathbb{T}_3; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ instead of functions defined on $\mathbb{R}^3$.

Proposition 5.1. *There exists $C > 0$ with the following property: for all $u \in L^1(\mathbb{T}_3; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ with* $\operatorname{div}(u) = 0$ *in $\mathcal{D}'(\mathbb{T}_3; \mathbb{R}^3)$ and $\lambda > 0$, there is $u_\lambda \in L^1(\mathbb{T}_3; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ satisfying*

(a) $\|u_\lambda\|_{L^\infty} \le C\lambda$ (L^∞-bound)*;*

(b) $\|u - u_\lambda\|_{L^1} \le C \int_{\{|u|>\lambda\}} |u| \, dx$ (strong stability)*;*

(c) $\mathcal{L}^3(\{u \ne u_\lambda\}) \le C\lambda^{-1} \int_{\{|u|>\lambda\}} |u| \, dx$ (small change)*;*

(d) $\operatorname{div}(u_\lambda) = 0$*, i.e., the differential constraint is still satisfied.*

To see this, one can either repeat the proof presented in Section 4 or write $u \in L^1(\mathbb{T}_3; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ as a $\mathbb{Z}^3$-periodic function on $\mathbb{R}^3$ and apply the obvious L^1_{loc}-version of Theorem 1.2.

Proof of Theorem 1.1. As $\mathcal{Q}_{\mathrm{sdqc}} f_1$ is a continuous symmetric div-quasiconvex function vanishing on K, all $y \in K^{(\infty)}$ are by definition also in $K^{(1)}$. It remains to show the other direction. Suppose that $\xi \in K^{(1)}$ and $(u_m) \subset L^1(\mathbb{T}_3; \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \cap \mathcal{T}$ is a test sequence with

$$0 = \mathcal{Q}_{\mathrm{sdqc}} f_1(\xi) = \lim_{m\to\infty} \int_{\mathbb{T}_3} f_1(\xi + u_m(x)) \, dx. \tag{5.1}$$

As K is a compact set, we find $R > 0$ with $K \subset \mathbb{B}_R(0)$ and $\xi \in \mathbb{B}_R(0)$. Thus, by (5.1),

$$\lim_{m\to\infty} \int_{\{|u_m|>3R\}} |u_m| \,\mathrm{d}x = 0. \tag{5.2}$$

Applying Proposition 5.1 gives a sequence $\tilde{v}_m \in \mathrm{L}^\infty(\mathbb{T}_3; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$, such that

(a) $\operatorname{div}(\tilde{v}_m) = 0$;

(b) $\|\tilde{v}_m - u_m\|_{\mathrm{L}^1(\mathbb{T}_3)} \to 0$ as $m \to \infty$;

(c) $\|\tilde{v}_m\|_{\mathrm{L}^\infty(\mathbb{T}_3)} \le CR$.

Mollification and subtracting the average gives a sequence $(v_m) \subset \mathrm{L}^\infty(\mathbb{T}_3; \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \cap \mathcal{T}$ also satisfying properties (a)–(c). Hence,

$$0 = \mathcal{Q}_{\mathrm{sdqc}} f_1(\xi) = \lim_{m\to\infty} \int_{\mathbb{T}_3} f_1(\xi + v_m(x)) \,\mathrm{d}x. \tag{5.3}$$

Now take a symmetric div-quasiconvex function $g \in \mathrm{C}(\mathbb{R}^{3\times 3}_{\mathrm{sym}})$. We may suppose that $\max g(K) = 0$ and, as $\max\{0, g\}$ is again symmetric div-quasiconvex, that $g \equiv 0$ on K. Using uniform boundedness of v_m we may estimate with $C > 0$ as in (c),

$$|g(\xi + v_m(x))| \le \sup_{\eta \in \mathbb{B}_{(2C+1)R}(0)} |g(\eta)| < \infty. \tag{5.4}$$

Due to (5.3), $\operatorname{dist}(\xi + v_m, K) \to 0$ in measure, and by passing to a non-relabelled subsequence, we may assume that $\operatorname{dist}(\xi + v_m, K) \to 0 \mathcal{L}^3$-a.e. As g is uniformly continuous on $\mathbb{B}_{(2C+1)R}(0)$, we get by (5.4) and dominated convergence,

$$g(\xi) \le \lim_{m\to\infty} \int_{\mathbb{T}_3} g(\xi + v_m(x)) \,\mathrm{d}x \le \int_{\mathbb{T}_3} \lim_{m\to\infty} g(\xi + v_m(x)) \,\mathrm{d}x = 0. \tag{5.5}$$

Therefore, $\xi \in K^{(\infty)}$. The proof is complete. ∎

Let us, for the sake of completeness, also discuss a proof of the statement $K^{(p)} = K^{(q)}$, $1 < p, q < \infty$, which can be easily adapted to general constant rank operators $\mathcal{A}$ of the form (2.3). To this end, recall that a Borel measurable function $F\colon \mathbb{R}^d \to \mathbb{R}$ is called $\mathcal{A}$-quasiconvex provided it satisfies (1.2) for all $\xi \in \mathbb{R}^d$ and $\varphi \in \mathcal{T}$, where $\mathcal{T} = \mathcal{T}_{\mathcal{A}}$ is now the set of all $\varphi \in \mathrm{C}^\infty(\mathbb{T}_n; \mathbb{R}^d)$ with zero mean and $\mathcal{A}\varphi = 0$. The $\mathcal{A}$-quasiconvexifications $\mathcal{Q}_{\mathcal{A}} f$ of functions f and, for non-empty, compact sets $K \subset \mathbb{R}^d$, the corresponding sets $K^{(p)}$ for $1 \le p \le \infty$ are defined as in (1.4), now systematically replacing the divsym-quasiconvexity by $\mathcal{A}$-quasiconvexity. In contrast to [10], we do not even need to use potentials, but can directly appeal to Lemma 2.2. Note that the construction of the projection $P_{\mathcal{A}}$ from Lemma 2.2 crucially relies on Fourier multipliers and hence is not applicable for $p = 1$ and $p = \infty$. Using this projection operator $P_{\mathcal{A}}$, we can prove the following statement.

Lemma 5.2. *Let $\mathcal{A}$ be a constant rank operator of the form (2.3) and let $K \subset \mathbb{R}^d$ be compact. Then, for $1 < p < q < \infty$, $K^{(p)} = K^{(q)}$.*

Proof. With slight abuse of notation, let $K \subset \mathbb{B}_R(0) := \{\eta \in \mathbb{R}^d : |\eta| < R\}$ and $y \in \mathbb{B}_R(0)$. On "$K^{(q)} \subset K^{(p)}$". Let $y \in K^{(q)}$ and let $(u_m) \subset \mathcal{T}_{\mathcal{A}}$ be a test sequence such that

$$0 = \mathcal{Q}_{\mathcal{A}} f_q(y) = \lim_{m\to\infty} \int_{\mathbb{T}_n} f_q(y + u_m(x))\,\mathrm{d}x.$$

As K is compact, (u_m) is bounded in $\mathrm{L}^q(\mathbb{T}_n;\mathbb{R}^d)$ and, as $q > p$, also bounded in $\mathrm{L}^p(\mathbb{T}_n;\mathbb{R}^d)$. Also note that for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that $f_p \leq \varepsilon + C_\varepsilon f_q$. Therefore,

$$\mathcal{Q}_{\mathrm{sdqc}} f_p(y) \leq \lim_{m\to\infty} \int_{\mathbb{T}_n} f_p(y + u_m(x))\,\mathrm{d}x \leq \lim_{m\to\infty} \int_{\mathbb{T}_n} \varepsilon + C_\varepsilon f_q(y + u_m(x))\,\mathrm{d}x \leq \varepsilon.$$

Thus, $y \in K^{(p)}$. The direction $K^{(p)} \subset K^{(q)}$ uses a similar, yet easier truncation statement than Theorem 1.1. Let $y \in K^{(p)}$ and let $(u_m) \subset \mathcal{T}_{\mathcal{A}}$ be a test sequence, such that

$$0 = \mathcal{Q}_{\mathrm{sdqc}} f_p(y) = \lim_{m\to\infty} \int_{\mathbb{T}_n} f_p(y + u_m(x))\,\mathrm{d}x.$$

Note that (u_m) is uniformly bounded in $\mathrm{L}^p(\mathbb{T}_n;\mathbb{R}^d)$ and that

$$\lim_{m\to\infty} \int_{\mathbb{T}_n} \mathrm{dist}^p(u_m(x), \mathbb{B}_{2R}(0))\,\mathrm{d}x = 0.$$

Write

$$\tilde{u}_m = \mathbb{1}_{\{|u_m| \leq 2R\}} u_m - \fint_{\mathbb{T}_n} \mathbb{1}_{\{|u_m| \leq 2R\}}(x) u_m(x)\,\mathrm{d}x$$

and define $v_m := P_{\mathcal{A}} \tilde{u}_m$ with the projection operator $P_{\mathcal{A}}$ from Lemma 2.2. Observe that

(a) $\mathcal{A} v_m = 0$ by Lemma 2.2 (a);

(b) $(\tilde{u}_m)$ is bounded in $\mathrm{L}^\infty(\mathbb{T}_n;\mathbb{R}^d)$ and q-equiintegrable. Since $1 < q < \infty$, the projection $P_{\mathcal{A}}\colon \mathrm{L}^q(\mathbb{T}_n;\mathbb{R}^d) \to \mathrm{L}^q(\mathbb{T}_n;\mathbb{R}^d)$ is bounded, (v_m) is bounded in $\mathrm{L}^q(\mathbb{T}_n;\mathbb{R}^d)$, q-equiintegrable by Lemma 2.2 (c) and, moreover, by Lemma 2.2 (b) and $1 < p < \infty$,

$$\begin{aligned}\|u_m - v_m\|_{\mathrm{L}^p(\mathbb{T}_n)} &\leq \|u_m - \tilde{u}_m\|_{\mathrm{L}^p(\mathbb{T}_n)} + \|\tilde{u}_m - v_m\|_{\mathrm{L}^p(\mathbb{T}_n)} \\ &\leq \|u_m - \tilde{u}_m\|_{\mathrm{L}^p(\mathbb{T}_n)} + C_{\mathcal{A},p}\|\mathcal{A}(\tilde{u}_m - u_m)\|_{\mathrm{W}^{-k,p}(\mathbb{T}_n)} \\ &\leq C_{\mathcal{A},p}\|u_m - \tilde{u}_m\|_{\mathrm{L}^p(\mathbb{T}_n)} \to 0.\end{aligned}$$

Hence, also

$$\lim_{m\to\infty} \int_{\mathbb{T}_n} f_p(y + v_m(x))\,\mathrm{d}x = 0.$$

We conclude that $f_q(y + v_m) \to 0$ in measure. Combining this with the L^q-boundedness and q-equiintegrability, we obtain

$$\lim_{m\to\infty} \int_{\mathbb{T}_n} f_q(y + v_m(x))\,\mathrm{d}x = 0.$$

Therefore, $y \in K^{(q)}$, concluding the proof. ∎

6. Potential truncations

In this concluding section we come back to the potential truncations alluded to in the introduction and discuss the limitations of this strategy in view of Theorems 1.1 and 1.2. Let $\mathcal{A}$ be a constant rank operator in the sense of Section 2.3. Recall that the potential truncation strategy, originally pursued in [8] for $\mathcal{A} = \operatorname{div}$, is to represent $u \in \mathrm{L}^p(\mathbb{T}_n;\mathbb{R}^d)$ with $\mathcal{A}u = 0$ and $\fint_{\mathbb{T}_n} u \,\mathrm{d}x = 0$ as $u = \mathbb{A}v$ for some potential $\mathbb{A}$ of order $l \in \mathbb{N}$ (cf. Lemma 2.1) and then performing a $\mathrm{W}^{l,p}$–$\mathrm{W}^{l,\infty}$-truncation on the potential v. We then write, with slight abuse of notation,[1] $v = \mathbb{A}^{-1}u$. Since it is of independent interest but also motivates the need for a different strategy for Theorem 1.2 for $p = 1$, we record the following proposition:

Proposition 6.1. *Let $\mathcal{A}$ be a constant rank differential operator of order $k \in \mathbb{N}$ and $\mathbb{A}$ be a potential of $\mathcal{A}$ of order $l \in \mathbb{N}$. Let $1 < p < \infty$. Then there exists a constant $C > 0$ such that the following hold: If $u \in \mathrm{L}^p(\mathbb{T}_n;\mathbb{R}^d) \cap \ker\mathcal{A}$ and $\lambda > 0$ then there exists $u_\lambda \in \mathrm{L}^\infty(\mathbb{T}_n;\mathbb{R}^d) \cap \ker\mathcal{A}$ satisfying the*

(a) L^∞*-bound:* $\|u_\lambda\|_{\mathrm{L}^\infty(\mathbb{T}_n)} \leq C\lambda$;

(b) weak stability:

$$\|u_\lambda - u\|^p_{\mathrm{L}^p(\mathbb{T}_n)} \leq C \int_{\{\sum_{j=0}^l |\nabla^j \circ \mathbb{A}^{-1}u| > \lambda\}} \sum_{j=0}^l |\nabla^j \circ \mathbb{A}^{-1}u|^p \,\mathrm{d}x;$$

(c) *small change:*

$$\mathcal{L}^n(\{u_\lambda \neq u\}) \leq \frac{C}{\lambda^p} \int_{\{\sum_{j=0}^l |\nabla^j \circ \mathbb{A}^{-1}u| > \lambda\}} \sum_{j=0}^l |\nabla^j \circ \mathbb{A}^{-1}u|^p \,\mathrm{d}x.$$

For simplicity, we state this result on $\mathbb{T}_n$; a version on $\mathbb{R}^n$ follows by analogous means.

Proof of Proposition 6.1. We start by outlining the $\mathrm{W}^{m,p}$–$\mathrm{W}^{m,\infty}$-truncation, which seems hard to trace in the literature; here, we choose a direct approach instead of appealing to McShane-type extensions. Let $m \in \mathbb{N}$. Then, for $v \in \mathrm{W}^{m,p}(\mathbb{T}_n;\mathbb{R}^d)$, let $\mathcal{O}_\lambda := \{\sum_{j=0}^m \mathcal{M}(\nabla^j v) > \lambda\}$. Since the sum of lower semicontinuous functions is lower semicontinuous, $\mathcal{O}_\lambda$ is open. We choose a Whitney decomposition $\mathcal{W} = (Q_j)$ of $\mathcal{O}_\lambda$ satisfying (W1)–(W4), and a partition of unity (φ_j) subject to $\mathcal{W}$ with (P1)–(P3). We note that the Whitney cover can be arranged in a way such that $\mathcal{L}^n(Q_j \cap Q_{j'}) \geq c\max\{\mathcal{L}^n(Q_j), \mathcal{L}^n(Q_{j'})\}$ holds for some $c = c(n) > 0$ and all $j, j' \in \mathbb{N}$ such that $Q_j \cap Q_{j'} \neq \emptyset$. For each $j \in \mathbb{N}$, we then denote by $\pi_j[v]$ the $(m-1)$th-order averaged Taylor polynomial of

[1] The notation $\mathbb{A}^{-1}$ is only symbolic as $\mathbb{A}$ might be non-invertible.

v over Q_j; cf. [29, Chap. 1.1.10]. In particular, we have the scaled version of Poincaré's inequality

$$\fint_{Q_j} |\partial^\alpha(w-\pi_j[w])|^q\,dx \le c(q,m,n)\ell(Q_j)^{q(m-|\alpha|)}\fint_{Q_j}|\nabla^m w|^q\,dx \tag{6.1}$$

for all $1\le q<\infty$, $w\in W^{m,q}(\mathbb{T}_n;\mathbb{R}^d)$ and $|\alpha|\le m$. We then put

$$v_\lambda := v-\sum_j \varphi_j(v-\pi_j[v]) = \begin{cases} v & \text{in } \mathcal{O}_\lambda^\complement, \\ \sum_j \varphi_j\pi_j[v] & \text{in } \mathcal{O}_\lambda. \end{cases} \tag{6.2}$$

Then $v_\lambda\in W^{m,p}(\mathbb{T}_n;\mathbb{R}^d)$, which can be seen as follows: on $\mathcal{O}_\lambda$, v_λ is a locally finite sum of C^∞-maps and hence of class C^∞ too. For an arbitrary $|\alpha|\le m$, (6.1) yields

$$\begin{aligned} &\sum_j \|\partial^\alpha(\varphi_j(v-\pi_j[v]))\|_{L^q(\mathcal{O}_\lambda)}^q \\ &\overset{(P3)}{\le} \sum_j\sum_{\beta+\gamma=\alpha}\frac{c(n,q)}{\ell(Q_j)^{q(|\beta|+|\gamma|)}}\ell(Q_j)^{q|\gamma|}\|\partial^\gamma(v-\pi_j[v])\|_{L^q(Q_j)}^q \\ &\le c(n,m,q)\sum_j \ell(Q_j)^{q(m-|\alpha|)}\|\nabla^m v\|_{L^q(Q_j)}^q \\ &\overset{(W3)}{\le} c(n,m,q)\mathcal{L}^n(\mathcal{O}_\lambda)^{\frac{q(m-|\alpha|)}{n}}\|\nabla^m v\|_{L^q(\mathcal{O}_\lambda)}^q. \end{aligned}$$

In conclusion, applying the previous inequality with $q=1$ on $(0,1)^n$, the series in (6.2) converges absolutely in $W_0^{m,1}((0,1)^n;\mathbb{R}^d)$ and hence $v_\lambda\in W^{m,1}(\mathbb{T}_n;\mathbb{R}^d)$; then applying the previous inequality with $q=p$ yields $v_\lambda\in W^{m,p}(\mathbb{T}_n;\mathbb{R}^d)$. Whenever $x\in Q_{j_0}$ for some $j_0\in\mathbb{N}$, (W2) implies that we may blow up Q_{j_0} by a fixed factor $c>0$ so that $cQ_{j_0}\cap\mathcal{O}_\lambda^\complement\neq\emptyset$. Fix some $z\in cQ_{j_0}\cap\mathcal{O}_\lambda^\complement$. Then, for some $c'=c'(n)>0$, $Q_{j_0}\subset B_{c'\ell(Q_{j_0})}(z)$ and so

$$\fint_{Q_{j_0}}|\partial^\alpha v|\,dx\le c(n)\fint_{B_{c'(n)\ell(Q_{j_0})}(z)}|\partial^\alpha v|\,dx\le c(n)\mathcal{M}(\nabla^{|\alpha|}v)(z)\le c(n)\lambda \tag{6.3}$$

for all $|\alpha|\le m$. Now let $Q_j\in\mathcal{W}$ be another cube with $Q_j\cap Q_{j_0}\neq\emptyset$; by (W3), there are only $\mathtt{N}=\mathtt{N}(n)<\infty$ such cubes. Since $\nabla^m\pi_{j_0}[v]=0$ and $\sum_j\varphi_j=1$ on $\mathcal{O}_\lambda$,

$$\begin{aligned} |\nabla^m v_\lambda(x)| &\le \Big|\sum_{j:Q_j\cap Q_{j_0}\neq\emptyset}\nabla^m(\varphi_j(\pi_j[v]-\pi_{j_0}[v]))(x)\Big| \\ &\overset{(P3)}{\le} c\sum_{\substack{j:Q_j\cap Q_{j_0}\neq\emptyset\\ |\alpha|+|\beta|=m}}\frac{1}{\ell(Q_j)^{|\alpha|}}\|\nabla^{|\beta|}(\pi_j[v]-\pi_{j_0}[v])\|_{L^\infty(Q_j\cap Q_{j_0})} \end{aligned}$$

$$
\begin{aligned}
&\overset{(*)}{\leq} c \sum_{\substack{j:Q_j\cap Q_{j_0}\neq\emptyset\\ |\alpha|+|\beta|=m}} \frac{1}{\ell(Q_j)^{|\alpha|}}\Big(\fint_{Q_j} |\nabla^{|\beta|}(\pi_j[v]-v)|\,\mathrm{d}x \\
&\qquad\qquad\qquad\qquad + \fint_{Q_{j_0}} |\nabla^{|\beta|}(v-\pi_{j_0}[v])|\,\mathrm{d}x\Big) \\
&\leq c \sum_{j:Q_j\cap Q_{j_0}\neq\emptyset} \fint_{Q_j} |\nabla^m v|\,\mathrm{d}x \quad \text{(by (6.1))} \\
&\leq c\lambda \quad \text{(by (6.3) and (W3))},
\end{aligned} \tag{6.4}
$$

where at $(*)$ we have used that on the polynomials of degree at most $(m-1)$ on cubes, all norms are equivalent (in particular, the L^1- and L^∞-norms), and scaling (recall that $\mathcal{L}^n(Q_j \cap Q_{j_0}) \geq c\max\{\mathcal{L}^n(Q_j), \mathcal{L}^n(Q_{j_0})\}$) whenever $Q_j \cap Q_{j_0} \neq \emptyset$, and (W3). Hence,

(i) $\|\nabla^m v\|_{\mathrm{L}^\infty(\mathbb{T}_n)} \leq c(m,n)\lambda$;

(ii) $\mathcal{L}^n(\{u \neq u_\lambda\}) \leq \frac{c(m,n,p)}{\lambda^p}\sum_{j=0}^m \|\nabla^j v\|^p_{\mathrm{L}^p(\mathbb{T}_n)}$.

We now let $u \in \mathrm{L}^p(\mathbb{T}_n;\mathbb{R}^d) \cap \ker\mathcal{A}$ satisfy $\int_{(0,1)} u\,\mathrm{d}x = 0$. Since $\mathbb{A}^{-1}$ has a Fourier symbol of class C^∞ off zero and is homogeneous of degree $(-l)$, $\nabla^l \circ \mathbb{A}^{-1}$ has a Fourier symbol of class C^∞ off zero and is homogeneous of degree zero. By Mihlin's theorem (cf. [41]), applicable because $1 < p < \infty$ and by Poincaré's inequality, we thus find that $\mathbb{A}^{-1}u \in \mathrm{W}^{l,p}(\mathbb{T}_n)$, together with $\|\mathbb{A}^{-1}u\|_{\mathrm{W}^{l,p}(\mathbb{T}_n)} \leq c\|u\|_{\mathrm{L}^p(\mathbb{T}_n)}$. We then perform a $\mathrm{W}^{l,p}$–$\mathrm{W}^{l,\infty}$-truncation on $v = \mathbb{A}^{-1}u$ as in the first part of the proof, yielding v_λ, and define $u_\lambda := \mathbb{A}v_\lambda$. By the properties gathered in the first part of the proof, we may employ Zhang's trick (see (4.28)ff.) to conclude (b) and (c) as well. The proof is complete. ∎

Remark 6.2 (Strong stability and $1 < p < \infty$ versus $p = 1$). It is clear from the above proof that the potential truncation only works fruitfully in the case $1 < p < \infty$ by the entering of Mihlin's theorem; indeed, the operator $\mathbb{A}^{-1}$ is defined via Fourier multipliers, and by Ornstein's non-inequality, we cannot conclude that $\mathbb{A}^{-1}u \in \mathrm{W}^{l,1}$ provided $u \in \mathrm{L}^1$. However, the potential truncations from Proposition 6.1 do *not satisfy the strong stability property* $\|u - u_\lambda\|^p_{\mathrm{L}^p(\mathbb{T}_n)} \leq C\int_{\{|u|>\lambda\}} |u|^p\,\mathrm{d}x$. The underlying reason is that $\nabla^l \circ \mathbb{A}^{-1}$ is a Fourier multiplication operator with symbol smooth off zero and homogeneous of degree zero; by Ornstein's non-inequality, we only have that $\nabla^l \circ \mathbb{A}^{-1}\colon \mathrm{L}^\infty \to \mathrm{BMO}$ in general, and here BMO *cannot* be replaced by L^∞. The potential truncation is performed on the sets where $\sum_{j=0}^l \mathcal{M}(\nabla^j \circ \mathbb{A}^{-1}u) > \lambda$. Thus, even if $u \in \mathrm{L}^\infty(\mathbb{T}_n;\mathbb{R}^d)$ is $\mathcal{A}$-free with $\|u\|_{\mathrm{L}^\infty(\mathbb{T}_n)} \leq \lambda$, the potential truncation might modify u regardless of $\lambda > 0$ and hence strong stability cannot be achieved. As established by Conti, Müller and Ortiz [10], in the case $1 < p < \infty$ this issue still can be circumvented to arrive at Lemma 5.2, but in the context of $p = 1$ the underlying techniques break down. In essence, this was the original motivation for the different proof displayed in Sections 3 and 4.

We conclude the paper with other possible approaches and extensions of Theorem 1.2.

Remark 6.3. As mentioned in the introduction, [6] constructs a divergence-free $W^{1,p}$–$W^{1,\infty}$-truncation. Here a Whitney-type truncation is performed first, leading to a non-divergence-free truncation. To arrive at a divergence-free truncation, the local divergence overshoots are then corrected by subtracting special solutions of suitable divergence equations. This is achieved by invoking the *Bogovskiĭ* operator [5], which selects specific solutions of the (heavily underdetermined) divergence equation $\operatorname{div}(Y) = f$ with $Y|_{\partial\Omega} = 0$ by

$$Y(x) = \operatorname{Bog}(f)(x) := \int_{\Omega} f(y)\frac{x-y}{|x-y|^n}\int_{|x-y|}^{\infty}\omega_R\Big(y + s\frac{x-y}{|x-y|}\Big)s^{n-1}\,\mathrm{d}s\,\mathrm{d}y, \quad x \in \Omega,$$

provided $\Omega \subset \mathbb{R}^n$ is star shaped with respect to a ball $B_R(x_0) \Subset \Omega$, f has integral zero over Ω and ω_R is a scaled cut-off relative to $B_R(x_0)$.

In our situation, the main drawback of the Bogovskiĭ operator is that if equations $\operatorname{div}(Y) = f$ for $f: (0,1)^n \to \mathbb{R}^n$ are considered, then the solution Y obtained by the row-wise application of the Bogovskiĭ operator does not necessarily take values in $\mathbb{R}^{n\times n}_{\mathrm{sym}}$; note that passing to the symmetric part Y^{sym} destroys the validity of the divergence equation. While this potentially could be repaired by passing to different solution operators, the method requires tools that are not fully clear to us in the present lower regularity context of Theorem 1.2. With our proof in Section 4 being tailored to divergence constraints, in principle it can be modified to yield divergence-free $W^{1,p}$–$W^{1,\infty}$-truncations as well. We will pursue this together with possible extensions of the approach in [6] elsewhere.

We finally comment on possible extensions of the strategy explained in Section 3 in the $\mathcal{A}$-free context. As discussed in Section 3, the key ingredients for the underlying construction are the availability of a $W^{\mathbb{A},1}$–$W^{\mathbb{A},\infty}$-truncation for a suitable operator $\mathbb{A}$ and the analogue of (3.2). Since for the class of $\mathbb{C}$-elliptic operators,[2] such truncations are available [4] (see [7,23,24] for similar strategies in view of trace and extension operators), this should then give truncations along the whole exact sequence starting with $\mathbb{A}$. As a consequence, we expect Theorem 1.2 to hold true for all operators with constant rank in $\mathbb{C}$:

Conjecture 6.4 (Theorem 1.2 for operators with constant rank in $\mathbb{C}$). Let

$$0 \to C^{\infty,0}(\mathbb{T}_n;\mathbb{R}^{d_0}) \xrightarrow{\mathcal{A}_1} C^{\infty,0}(\mathbb{T}_n;\mathbb{R}^{d_1}) \xrightarrow{\mathcal{A}_2} \cdots \xrightarrow{\mathcal{A}_k} C^{\infty,0}(\mathbb{T}_n;\mathbb{R}^{d_k}) \xrightarrow{\mathcal{A}_{k+1}} \cdots$$

be an exact sequence of differential operators with constant rank in $\mathbb{C}$, in particular, $\mathcal{A}_1$ being $\mathbb{C}$-elliptic. This is equivalent to

$$0 \to \mathbb{C}^{d_0} \xrightarrow{\mathcal{A}_1[\xi]} \mathbb{C}^{d_1} \xrightarrow{\mathcal{A}_2[\xi]} \mathbb{C}^{d_2} \xrightarrow{\mathcal{A}_3[\xi]} \cdots \xrightarrow{\mathcal{A}_k[\xi]} \mathbb{C}^{d_k} \xrightarrow{\mathcal{A}_{k+1}[\xi]} \cdots$$

being exact for all $\xi \in \mathbb{C}^n \setminus \{0\}$. Then for any differential operator $\mathcal{A}_k$ contained in this exact sequence there is $C_k > 0$, such that for $u \in L^1(\mathbb{T}_n;\mathbb{R}^{d_k})$ with $\mathcal{A}_k u = 0$ in $\mathcal{D}'(\mathbb{T}_n;\mathbb{R}^{d_{k+1}})$ and $\lambda > 0$, there is $u_\lambda \in L^1(\mathbb{R}^n;\mathbb{R}^{d_k})$ satisfying

[2] This means that $\mathbb{A}[\xi]$ has trivial nullspace for each $\xi \in \mathbb{C}^n \setminus \{0\}$; cf. Smith [40].

(a) $\|u_\lambda\|_{\mathrm{L}^\infty} \leq C\lambda$ (L^∞-bound);

(b) $\|u - u_\lambda\|_{\mathrm{L}^1} \leq C \int_{|u|>\lambda} |u| \,\mathrm{d}x$ (strong stability);

(c) $\mathcal{L}^n(\{u \neq u_\lambda\} \leq C\lambda^{-1} \int_{|u|>\lambda} |u| \,\mathrm{d}x$ (small change);

(d) $\mathcal{A}_k u_\lambda = 0$, i.e., the differential constraint is still satisfied.

If any differential operator $\mathcal{A}$ with constant rank over $\mathbb{C}$ is a part of such an exact sequence, this means that the $\mathcal{A}$-free truncation is possible for every such operator.

A. Computational details

In this appendix, we give the computational details for some of the identities used in the main part of the paper. We will need the following lemma:

Lemma A.1. *Let $a, b, c \in \mathbb{N}^3$ be multi-indices with $|a|, |b|, |c| \geq 1$ and $\alpha, \beta \in \{1, 2, 3\}$. Then on the set $\mathcal{O}_\lambda$ we have*

$$\sum_{ijk} \partial_a \varphi_k \partial_b \varphi_j \partial_c \varphi_i \mathfrak{B}_\alpha(i, j, k) = 0 \tag{A.1}$$

and

$$\sum_{ijk} \partial_a \varphi_k \partial_b \varphi_j \partial_c \varphi_i \mathfrak{A}_{\alpha,\beta}(i, j, k) = 0. \tag{A.2}$$

Proof. Recall from the definition of the φ_l that $\sum \varphi_l \equiv 1$ on $\mathcal{O}_\lambda$. We therefore have $\sum \partial_a \varphi_l = \sum \partial_b \varphi_l = \sum \partial_c \varphi_l = 0$. We can use this to get

$$\begin{aligned}\sum_{ijk} &\partial_a \varphi_k \partial_b \varphi_j \partial_c \varphi_i \mathfrak{B}_\alpha(i, j, k) \\ &= \sum_{ijkm} \partial_a \varphi_k \partial_b \varphi_j \partial_c \varphi_i \big(\mathfrak{B}_\alpha(i, j, k) - \mathfrak{B}_\alpha(m, j, k) - \mathfrak{B}_\alpha(i, m, k) - \mathfrak{B}_\alpha(i, j, m)\big).\end{aligned}$$

Now (A.1) follows from Lemma 4.2 (f); (A.2) can be shown completely analogously. ■

A.1. Proof of Lemma 4.5

We focus on the case $\alpha = 1$. Thus let $D := \operatorname{div}(T_\lambda w)_1$. To avoid notational overload we omit the arguments i, j and k of $\mathfrak{A}_{\alpha,\beta}(i, j, k)$ and $\mathfrak{B}_\alpha(i, j, k)$ in the following equation. Thus, all $\mathfrak{A}_{\alpha,\beta}$ and $\mathfrak{B}_\alpha$ implicitly depend on the summation indices. By the definition of $T_\lambda w$ on $\mathcal{O}_\lambda$, (4.5), we have

$$\begin{aligned}D = 6 &\sum_{ijk} \partial_1(\varphi_k \partial_2 \varphi_j \partial_3 \varphi_i)\mathfrak{B}_1 && (= T_1) \\ &+ 2 \sum_{ijk} \partial_1(\varphi_k(\partial_{33}\varphi_j \partial_2 \varphi_i - \partial_{23}\varphi_j \partial_3 \varphi_i))\mathfrak{A}_{3,1} && (= T_2)\end{aligned}$$

$$+ 2\sum_{ijk} \varphi_k(\partial_{33}\varphi_j\partial_2\varphi_i - \partial_{23}\varphi_j\partial_3\varphi_i)\partial_1\mathfrak{A}_{3,1} \qquad (= T_3)$$

$$+ 2\sum_{ijk} \partial_1(\varphi_k(\partial_{22}\varphi_j\partial_3\varphi_i - \partial_{23}\varphi_j\partial_2\varphi_i))\mathfrak{A}_{1,2} \qquad (= T_4)$$

$$+ 2\sum_{ijk} \varphi_k(\partial_{22}\varphi_j\partial_3\varphi_i - \partial_{32}\varphi_j\partial_2\varphi_i)\partial_1\mathfrak{A}_{1,2} \qquad (= T_5)$$

$$+ 3\sum_{ijk} \partial_2(\varphi_k\partial_3\varphi_j\partial_1\varphi_i)\mathfrak{B}_1 \qquad (= T_6)$$

$$+ 3\sum_{ijk} \partial_2(\varphi_k\partial_2\varphi_j\partial_3\varphi_i)\mathfrak{B}_2 \qquad (= T_7)$$

$$+ \sum_{ijk} \partial_2(\varphi_k(\partial_{23}\varphi_j\partial_3\varphi_i - \partial_{33}\varphi_j\partial_2\varphi_i))\mathfrak{A}_{2,3} \qquad (= T_8)$$

$$+ \sum_{ijk} \varphi_k(\partial_{23}\varphi_j\partial_3\varphi_i - \partial_{33}\varphi_j\partial_2\varphi_i)\partial_2\mathfrak{A}_{2,3} \qquad (= T_9)$$

$$+ \sum_{ijk} \partial_2(\varphi_k(\partial_{13}\varphi_j\partial_3\varphi_i - \partial_{33}\varphi_j\partial_1\varphi_i))\mathfrak{A}_{3,1} \qquad (= T_{10})$$

$$+ \sum_{ijk} \varphi_k(\partial_{13}\varphi_j\partial_3\varphi_i - \partial_{33}\varphi_j\partial_1\varphi_i)\partial_2\mathfrak{A}_{3,1} \qquad (= T_{11})$$

$$+ \sum_{ijk} \partial_2(\varphi_k(\partial_{13}\varphi_j\partial_2\varphi_i + \partial_{23}\varphi_j\partial_1\varphi_i - 2\partial_{12}\varphi_j\partial_3\varphi_i))\mathfrak{A}_{1,2} \qquad (= T_{12})$$

$$+ \sum_{ijk} (\varphi_k(\partial_{13}\varphi_j\partial_2\varphi_i + \partial_{23}\varphi_j\partial_1\varphi_i - 2\partial_{12}\varphi_j\partial_3\varphi_i))\partial_2\mathfrak{A}_{1,2} \qquad (= T_{13})$$

$$+ 3\sum_{ijk} \partial_3(\varphi_k\partial_2\varphi_j\partial_3\varphi_i)\mathfrak{B}_3 \qquad (= T_{14})$$

$$+ 3\sum_{ijk} \partial_3(\varphi_k\partial_1\varphi_j\partial_2\varphi_i)\mathfrak{B}_1 \qquad (= T_{15})$$

$$+ \sum_{ijk} \partial_3(\varphi_k(\partial_{12}\varphi_j\partial_2\varphi_i - \partial_{22}\varphi_j\partial_1\varphi_i))\mathfrak{A}_{1,2} \qquad (= T_{16})$$

$$+ \sum_{ijk} (\varphi_k(\partial_{12}\varphi_j\partial_2\varphi_i - \partial_{22}\varphi_j\partial_1\varphi_i))\partial_3\mathfrak{A}_{1,2} \qquad (= T_{17})$$

$$+ \sum_{ijk} \partial_3(\varphi_k(\partial_{23}\varphi_j\partial_2\varphi_i - \partial_{22}\varphi_j\partial_3\varphi_i))\mathfrak{A}_{2,3} \qquad (= T_{18})$$

$$+ \sum_{ijk} (\varphi_k(\partial_{23}\varphi_j\partial_2\varphi_i - \partial_{22}\varphi_j\partial_3\varphi_i))\partial_3\mathfrak{A}_{2,3} \qquad (= T_{19})$$

$$+ \sum_{ijk} \partial_3(\varphi_k(\partial_{23}\varphi_j\partial_1\varphi_i + \partial_{12}\varphi_j\partial_3\varphi_i - 2\partial_{13}\varphi_j\partial_2\varphi_i))\mathfrak{A}_{3,1} \qquad (= T_{20})$$

$$+\sum_{ijk}(\varphi_k(\partial_{23}\varphi_j\partial_1\varphi_i+\partial_{12}\varphi_j\partial_3\varphi_i-2\partial_{13}\varphi_j\partial_2\varphi_i))\partial_3\mathfrak{A}_{3,1} \quad (=T_{21})$$
$$=\sum_{ijk} f_{ijk}^{(1)}\mathfrak{B}_1+f_{ijk}^{(2)}\mathfrak{B}_2+f_{ijk}^{(3)}\mathfrak{B}_3$$
$$+f_{ijk}^{(1,2)}\mathfrak{A}_{1,2}+f_{ijk}^{(2,3)}\mathfrak{A}_{2,3}+f_{ijk}^{(3,1)}\mathfrak{A}_{3,1}$$
$$=:(*)$$

for suitable coefficient maps $f_{ijk}^{(\cdot)}$ or $f_{ijk}^{(\cdot,\cdot)}$, respectively. To achieve this grouping we use Lemma 4.2 (a) and (b) as well as the fact that $T_{11}=T_{17}=0$. In the following we will show that each of the six sums in $(*)$ vanishes individually. This is done by a very similar calculation every time.

On $f_{ijk}^{(1)}$. Here the coefficients are determined by terms T_1, T_6, T_{13}, T_{15} and T_{21}. Therefore,

$$\begin{aligned}f_{ijk}^{(1)} &= 6\partial_1\varphi_k\partial_2\varphi_j\partial_3\varphi_i+6\varphi_k\partial_{12}\varphi_j\partial_3\varphi_i+6\varphi_k\partial_2\varphi_j\partial_{13}\varphi_i+3\partial_2\varphi_k\partial_3\varphi_j\partial_1\varphi_i\\
&\quad+3\varphi_k\partial_{23}\varphi_j\partial_1\varphi_i+3\varphi_k\partial_3\varphi_j\partial_{12}\varphi_i+\varphi_k\partial_{13}\varphi_j\partial_2\varphi_i+\varphi_k\partial_{23}\varphi_j\partial_1\varphi_i\\
&\quad+(-2)\varphi_k\partial_{12}\varphi_j\partial_3\varphi_i+3\partial_3\varphi_k\partial_1\varphi_j\partial_2\varphi_i+3\varphi_k\partial_{13}\varphi_j\partial_2\varphi_i+3\varphi_k\partial_1\varphi_j\partial_{23}\varphi_i\\
&\quad+(-1)\varphi_k\partial_{23}\varphi_j\partial_1\varphi_i+(-1)\varphi_k\partial_{12}\varphi_j\partial_3\varphi_i+2\varphi_k\partial_{13}\varphi_j\partial_2\varphi_i\\
&=: P_1^{ijk}+\cdots+P_{15}^{ijk}.\end{aligned}$$

In the next step we group those of the P_l^{ijk} together that have the same structure apart from a permutation of the indices i, j and k. For example, we have

$$P_1^{ijk}=2P_4^{jki}=2P_{10}^{kij}.$$

We now group all the terms and then perform the corresponding index permutations:

$$\begin{aligned}\sum_{ijk} f_{ijk}^{(1)}\mathfrak{B}_1(i,j,k) &= \sum_{ijk}\big[(P_1^{ijk}+P_4^{ijk}+P_{10}^{ijk})+(P_2^{ijk}+P_6^{ijk}+P_9^{ijk}+P_{14}^{ijk})\\
&\qquad+(P_3^{ijk}+P_7^{ijk}+P_{11}^{ijk}+P_{15}^{ijk})\\
&\qquad+(P_5^{ijk}+P_8^{ijk}+P_{12}^{ijk}+P_{13}^{ijk})\big]\mathfrak{B}_1(i,j,k)\\
&=\sum_{ijk}P_1^{ijk}\big(\mathfrak{B}_1(i,j,k)+\tfrac12\mathfrak{B}_1(j,k,i)+\tfrac12\mathfrak{B}_1(k,i,j)\big)\\
&\quad+P_2^{ijk}\big(\mathfrak{B}_1(i,j,k)+\tfrac12\mathfrak{B}_1(j,i,k)-\tfrac13\mathfrak{B}_1(i,j,k)-\tfrac16\mathfrak{B}_1(i,j,k)\big)\\
&\quad+P_3^{ijk}\big(\mathfrak{B}_1(i,j,k)+\tfrac16\mathfrak{B}_1(j,i,k)+\tfrac12\mathfrak{B}_1(j,i,k)+\tfrac13\mathfrak{B}_1(j,i,k)\big)\\
&\quad+P_5^{ijk}\big(\mathfrak{B}_1(i,j,k)+\tfrac13\mathfrak{B}_1(i,j,k)+\mathfrak{B}_1(j,i,k)-\tfrac13\mathfrak{B}_1(i,j,k)\big)\\
&=2\sum_{ijk}P_1^{ijk}\mathfrak{B}_1(i,j,k)=:(**),\end{aligned}$$

where we used Lemma 4.2 (d) to get the last equality. Finally, Lemma A.1 implies that (∗∗) vanishes identically.

On $f^{(2)}_{ijk}$. For the corresponding coefficients, only terms T_5, T_7 and T_{19} matter here. Therefore,

$$\begin{aligned} f^{(2)}_{ijk} &= -2\varphi_k\partial_{22}\varphi_j\partial_3\varphi_i + 2\varphi_k\partial_{23}\varphi_j\partial_2\varphi_i + 3\partial_2\varphi_k\partial_2\varphi_j\partial_3\varphi_i + 3\varphi_k\partial_{22}\varphi_j\partial_3\varphi_i \\ &\quad + 3\varphi_k\partial_2\varphi_j\partial_{23}\varphi_i + \varphi_k\partial_{23}\varphi_j\partial_2\varphi_i + (-1)\varphi_k\partial_{22}\varphi_j\partial_3\varphi_i \\ &=: Q^{ijk}_1 + \cdots + Q^{ijk}_7. \end{aligned}$$

Grouping similar terms and permuting indices as above we get

$$\begin{aligned} \sum_{ijk} f^{(1)}_{ijk}\mathfrak{B}_2(i,j,k) &= \sum_{ijk}\big[(Q^{ijk}_1 + Q^{ijk}_4 + Q^{ijk}_7) \\ &\qquad + (Q^{ijk}_2 + Q^{ijk}_5 + Q^{ijk}_6) + Q^{ijk}_3\big]\mathfrak{B}_2(i,j,k) \\ &= \sum_{ijk} Q^{ijk}_1\big(\mathfrak{B}_2(i,j,k) - \tfrac{3}{2}\mathfrak{B}_2(i,j,k) + \tfrac{1}{2}\mathfrak{B}_2(i,j,k)\big) \\ &\qquad + Q^{ijk}_2\big(\mathfrak{B}_2(i,j,k) + \tfrac{3}{2}\mathfrak{B}_2(j,i,k) + \tfrac{1}{2}\mathfrak{B}_2(i,j,k)\big) \\ &\qquad + Q^{ijk}_3\mathfrak{B}_2(i,j,k) \\ &= \sum_{ijk} Q^{ijk}_3\mathfrak{B}_2(i,j,k) = 0, \end{aligned}$$

where we again used Lemma 4.2 (d) and in the last step Lemma A.1.

On $f^{(3)}_{ijk}$. Here, only terms T_3, T_9, T_{14} contribute to the corresponding coefficients. Thus,

$$\begin{aligned} f^{(3)}_{ijk} &= 2\varphi_k\partial_{33}\varphi_j\partial_2\varphi_i + (-2)\varphi_k\partial_{23}\varphi_j\partial_3\varphi_i + (-1)\varphi_k\partial_{23}\varphi_j\partial_3\varphi_i + \varphi_k\partial_{33}\varphi_j\partial_2\varphi_i \\ &\quad + 3\partial_3\varphi_k\partial_2\varphi_j\partial_3\varphi_i + 3\varphi_k\partial_{23}\varphi_j\partial_3\varphi_i + 3\varphi_k\partial_2\varphi_j\partial_{33}\varphi_i \\ &=: S^{ijk}_1 + \cdots + S^{ijk}_7. \end{aligned}$$

We thus get

$$\begin{aligned} &\sum_{ijk} f^{(3)}_{ijk}\mathfrak{B}_3(i,j,k) \\ &\quad = \sum_{ijk}\big[(S^{ijk}_1 + S^{ijk}_4 + S^{ijk}_7) + (S^{ijk}_2 + S^{ijk}_3 + S^{ijk}_6) + S^{ijk}_5\big]\mathfrak{B}_3(i,j,k) \\ &\quad = \sum_{ijk} S^{ijk}_1(\mathfrak{B}_3(i,j,k) + \tfrac{1}{2}\mathfrak{B}_3(i,j,k) + \tfrac{3}{2}\mathfrak{B}_3(j,i,k)) \\ &\qquad\quad + S^{ijk}_2(\mathfrak{B}_3(i,j,k) + \tfrac{1}{2}\mathfrak{B}_3(i,j,k) - \tfrac{3}{2}\mathfrak{B}_3(i,j,k)) + S^{ijk}_5\mathfrak{B}_3(i,j,k) \\ &\quad = \sum_{ijk} S^{ijk}_5\mathfrak{B}_3(i,j,k) = 0. \end{aligned}$$

On $f_{ijk}^{(1,2)}$. These coefficients are determined by T_4, T_{12} and T_{16}. In consequence,

$$\begin{aligned}
f_{ijk}^{(1,2)} &= 2\partial_1\varphi_k\partial_{22}\varphi_j\partial_3\varphi_i + 2\varphi_k\partial_{122}\varphi_j\partial_3\varphi_i + 2\varphi_k\partial_{22}\varphi_j\partial_{13}\varphi_i + (-2)\partial_1\varphi_k\partial_{23}\varphi_j\partial_2\varphi_i \\
&\quad + (-2)\varphi_k\partial_{123}\varphi_j\partial_2\varphi_i + (-2)\varphi_k\partial_{23}\varphi_j\partial_{12}\varphi_i + \partial_2\varphi_k\partial_{13}\varphi_j\partial_2\varphi_i + \varphi_k\partial_{123}\varphi_j\partial_2\varphi_i \\
&\quad + \varphi_k\partial_{13}\varphi_j\partial_{22}\varphi_i + \partial_2\varphi_k\partial_{23}\varphi_j\partial_1\varphi_i + \varphi_k\partial_{223}\varphi_j\partial_1\varphi_i + \varphi_k\partial_{23}\varphi_j\partial_{12}\varphi_i \\
&\quad + (-2)\partial_2\varphi_k\partial_{12}\varphi_j\partial_3\varphi_i + (-2)\varphi_k\partial_{122}\varphi_j\partial_3\varphi_i + (-2)\varphi_k\partial_{12}\varphi_j\partial_{23}\varphi_i \\
&\quad + \partial_3\varphi_k\partial_{12}\varphi_j\partial_2\varphi_i + \varphi_k\partial_{123}\varphi_j\partial_2\varphi_i + \varphi_k\partial_{12}\varphi_j\partial_{23}\varphi_i + (-1)\partial_3\varphi_k\partial_{22}\varphi_j\partial_1\varphi_i \\
&\quad + (-1)\varphi_k\partial_{223}\varphi_j\partial_1\varphi_i + (-1)\varphi_k\partial_{22}\varphi_j\partial_{13}\varphi_i \\
&=: U_1^{ijk} + \cdots + U_{21}^{ijk}.
\end{aligned}$$

Here we can first note that, by Lemma A.1, for each $l \in \{1, 4, 7, 10, 13, 16, 19\}$ the terms $U_l^{ijk}\mathfrak{A}_{1,2}(i,j,k)$ sum to zero. We thus have

$$\begin{aligned}
&\sum_{ijk} f_{ijk}^{(1,2)}\mathfrak{A}_{1,2}(i,j,k) \\
&\quad = \sum_{ijk}\big[(U_2^{ijk} + U_{14}^{ijk}) + (U_3^{ijk} + U_9^{ijk} + U_{21}^{ijk}) + (U_5^{ijk} + U_8^{ijk} + U_{17}^{ijk}) \\
&\qquad\qquad + (U_6^{ijk} + U_{12}^{ijk} + U_{15}^{ijk} + U_{18}^{ijk}) + (U_{11}^{ijk} + U_{20}^{ijk})\big]\mathfrak{A}_{1,2}(i,j,k) \\
&\quad = \sum_{ijk} U_2^{ijk}(\mathfrak{A}_{1,2}(i,j,k) - \mathfrak{A}_{1,2}(i,j,k)) \\
&\qquad\qquad + U_3^{ijk}(\mathfrak{A}_{1,2}(i,j,k) + \tfrac{1}{2}\mathfrak{A}_{1,2}(j,i,k) - \tfrac{1}{2}\mathfrak{A}_{1,2}(i,j,k)) \\
&\qquad\qquad + U_5^{ijk}(\mathfrak{A}_{1,2}(i,j,k) - \tfrac{1}{2}\mathfrak{A}_{1,2}(i,j,k) - \tfrac{1}{2}\mathfrak{A}_{1,2}(i,j,k)) \\
&\qquad\qquad + U_6^{ijk}(\mathfrak{A}_{1,2}(i,j,k) - \tfrac{1}{2}\mathfrak{A}_{1,2}(i,j,k) + \mathfrak{A}_{1,2}(j,i,k) - \tfrac{1}{2}\mathfrak{A}_{1,2}(j,i,k)) \\
&\qquad\qquad + U_{11}^{ijk}(\mathfrak{A}_{1,2}(i,j,k) - \mathfrak{A}_{1,2}(i,j,k)) = 0.
\end{aligned}$$

On $f_{ijk}^{(2,3)}$. Only the terms T_8 and T_{18} matter here. In particular,

$$\begin{aligned}
f_{ijk}^{(2,3)} &= \partial_2\varphi_k\partial_{23}\varphi_j\partial_3\varphi_i + (-1)\partial_2\varphi_k\partial_{33}\varphi_j\partial_2\varphi_i + \partial_3\varphi_k\partial_{23}\varphi_j\partial_2\varphi_i \\
&\quad + (-1)\partial_3\varphi_k\partial_{22}\varphi_j\partial_3\varphi_i + 2\varphi_k\partial_{23}\varphi_j\partial_{23}\varphi_i + (-1)\varphi_k\partial_{33}\varphi_j\partial_{22}\varphi_i \\
&\quad + (-1)\varphi_k\partial_{22}\varphi_j\partial_{33}\varphi_i \\
&=: V_1^{ijk} + \cdots + V_7^{ijk}.
\end{aligned}$$

We first note that the terms $V_l^{ijk}\mathfrak{A}_{2,3}(i,j,k)$ for $l \in \{1, 2, 3, 4\}$ all sum to zero (Lemma A.1). Consequently,

$$\sum_{ijk} f_{ijk}^{(2,3)}\mathfrak{A}_{2,3}(i,j,k) = \sum_{ijk}\big[(V_6^{ijk} + V_7^{ijk}) + V_5^{ijk}\big]\mathfrak{A}_{2,3}(i,j,k)$$

$$
\begin{aligned}
&= \sum_{ijk} V_6^{ijk}(\mathfrak{A}_{2,3}(i,j,k) + \mathfrak{A}_{2,3}(j,i,k)) + V_5^{ijk}\mathfrak{A}_{2,3}(i,j,k) \\
&= \sum_{ijk} V_5^{ijk}\mathfrak{A}_{2,3}(i,j,k).
\end{aligned}
$$

To see that the final term vanishes, we notice that $V_5^{ijk} = V_5^{jik}$ and thus

$$
\sum_{ijk} V_5^{ijk}\mathfrak{A}_{2,3}(i,j,k) = \sum_{ijk} V_5^{ijk}(\tfrac{1}{2}\mathfrak{A}_{2,3}(i,j,k) + \tfrac{1}{2}\mathfrak{A}_{2,3}(j,i,k)) = 0.
$$

On $f_{ijk}^{(3,1)}$. Here, only the terms T_2, T_{10} and T_{20} are relevant and therefore

$$
\begin{aligned}
f_{ijk}^{(3,1)} &= 2\partial_1\varphi_k\partial_{33}\varphi_j\partial_2\varphi_1 + (-2)2\partial_1\varphi_k\partial_{23}\varphi_j\partial_3\varphi_i + 2\varphi_k\partial_{133}\varphi_j\partial_2\varphi_i \\
&\quad + (-2)\varphi_k\partial_{123}\varphi_j\partial_3\varphi_i + 2\varphi_k\partial_{33}\varphi_j\partial_{12}\varphi_i + (-2)\varphi_k\partial_{23}\varphi_j\partial_{13}\varphi_i \\
&\quad + \partial_2\varphi_k\partial_{13}\varphi_j\partial_3\varphi_i + (-1)\partial_2\varphi_k\partial_{33}\varphi_j\partial_1\varphi_i + \varphi_k\partial_{123}\varphi_j\partial_3\varphi_i \\
&\quad + (-1)\varphi_k\partial_{233}\varphi_j\partial_1\varphi_i + \varphi_k\partial_{13}\varphi_j\partial_{23}\varphi_i + (-1)\varphi_k\partial_{33}\varphi_j\partial_{12}\varphi_i \\
&\quad + \partial_3\varphi_k\partial_{23}\varphi_j\partial_1\varphi_i + \partial_3\varphi_k\partial_{12}\varphi_j\partial_3\varphi_i + (-2)\partial_3\varphi_k\partial_{13}\varphi_j\partial_2\varphi_i + \varphi_k\partial_{233}\varphi_j\partial_1\varphi_i \\
&\quad + \varphi_k\partial_{123}\varphi_j\partial_3\varphi_i + (-2)\varphi_k\partial_{133}\varphi_j\partial_2\varphi_i + \varphi_k\partial_{23}\varphi_j\partial_{13}\varphi_i + \varphi_k\partial_{12}\varphi_j\partial_{33}\varphi_i \\
&\quad + (-2)\varphi_k\partial_{13}\varphi_j\partial_{23}\varphi_i \\
&=: W_1^{ijk} + \cdots + W_{21}^{ijk}.
\end{aligned}
$$

We first apply Lemma A.1 to see that we can ignore the terms corresponding to W_l^{ijk} for $l \in \{1, 2, 7, 8, 13, 14, 15\}$. For the remaining terms we calculate

$$
\begin{aligned}
&\sum_{ijk} f_{ijk}^{(3,1)}\mathfrak{A}_{3,1}(i,j,k) \\
&\quad = \sum_{ijk}\big[(W_3^{ijk} + W_{18}^{ijk}) + (W_4^{ijk} + W_9^{ijk} + W_{17}^{ijk}) + (W_5^{ijk} + W_{12}^{ijk} + W_{20}^{ijk}) \\
&\qquad\quad + (W_6^{ijk} + W_{11}^{ijk} + W_{19}^{ijk} + W_{21}^{ijk}) + (W_{10}^{ijk} + W_{16}^{ijk})\big]\mathfrak{A}_{3,1}(i,j,k) \\
&\quad = \sum_{ijk} W_3^{ijk}(\mathfrak{A}_{3,1}(i,j,k) - \mathfrak{A}_{3,1}(i,j,k)) \\
&\qquad\quad + W_4^{ijk}(\mathfrak{A}_{3,1}(i,j,k) - \tfrac{1}{2}\mathfrak{A}_{3,1}(i,j,k) - \tfrac{1}{2}\mathfrak{A}_{3,1}(i,j,k)) \\
&\qquad\quad + W_5^{ijk}(\mathfrak{A}_{3,1}(i,j,k) - \tfrac{1}{2}\mathfrak{A}_{3,1}(i,j,k) + \tfrac{1}{2}\mathfrak{A}_{3,1}(j,i,k)) \\
&\qquad\quad + W_6^{ijk}(\mathfrak{A}_{3,1}(i,j,k) - \tfrac{1}{2}\mathfrak{A}_{3,1}(j,i,k) - \tfrac{1}{2}\mathfrak{A}_{3,1}(i,j,k) + \mathfrak{A}_{3,1}(j,i,k)) \\
&\qquad\quad + W_{10}^{ijk}(\mathfrak{A}_{3,1}(i,j,k) - \mathfrak{A}_{3,1}(i,j,k)) \\
&\quad = 0.
\end{aligned}
$$

We thus have shown that $D = (*) = 0$, yielding that the truncation is solenoidal on $\mathcal{O}_\lambda$.

A.2. Proof of identity (4.12)

Let $\psi \in C_c^\infty(\mathbb{R}^3)$ be arbitrary. In order to obtain formula (4.12), we write

$$\begin{aligned}\int_{\mathcal{O}_\lambda} (T_\lambda w)_1 \cdot \nabla\psi \,\mathrm{d}x &= \int_{\mathcal{O}_\lambda} \mathbf{T}(\mathfrak{A}_{1,2}, \nabla\psi)\,\mathrm{d}x + \int_{\mathcal{O}_\lambda} \mathbf{T}(\mathfrak{A}_{2,3}, \nabla\psi)\,\mathrm{d}x \\ &\quad + \int_{\mathcal{O}_\lambda} \mathbf{T}(\mathfrak{A}_{3,1}, \nabla\psi)\,\mathrm{d}x + \int_{\mathcal{O}_\lambda} \mathbf{T}(\mathfrak{B}_1, \nabla\psi)\,\mathrm{d}x \\ &\quad + \int_{\mathcal{O}_\lambda} \mathbf{T}(\mathfrak{B}_2, \nabla\psi)\,\mathrm{d}x + \int_{\mathcal{O}_\lambda} \mathbf{T}(\mathfrak{B}_3, \nabla\psi)\,\mathrm{d}x \\ &=: \sum_{\ell=1}^{6} S_\ell,\end{aligned}$$

where we indicate, e.g., by $\mathbf{T}(\mathfrak{A}_{1,2}, \nabla\psi)$, that when writing out $w_1 \cdot \nabla\psi$ directly by means of (4.3) and (4.4), $\mathbf{T}(\mathfrak{A}_{1,2}, \nabla\psi)$ contains all appearances of $\mathfrak{A}_{1,2}(i,j,k)$, and analogously for the remaining terms. The underlying procedure of dealing with the different terms is analogous for the remaining columns w_2 and w_3, which is why we exclusively focus on w_1 but give all the details in this case.

In the following we will frequently interchange the triple sum $\sum_{ijk}$ and the integral over $\mathcal{O}_\lambda$, which allows us treat the single terms via integration by parts. This interchanging of sums and integrals is allowed since every sum $\sum_{ijk}(\ldots)$ has an integrable majorant, in turn being seen similarly to the reasoning that underlies the proof of Lemma 4.4.

We begin with S_1. This term is constituted by three parts S_1^1, S_1^2, S_1^3 given below, which stem from $w_{11}\partial_1\psi$, $w_{12}\partial_2\psi$ and $w_{13}\partial_3\psi$ (in this order). Here we have

$$\begin{aligned} S_1^1 &= 2\sum_{ijk} \int_{\mathcal{O}_\lambda} \varphi_k(\partial_{22}\varphi_j\,\partial_3\varphi_i - \partial_{23}\varphi_j\,\partial_2\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_1\psi\,\mathrm{d}x \\ &= -2\sum_{ijk} \int_{\mathcal{O}_\lambda} (\partial_2\varphi_j)(\partial_2\varphi_k\,\partial_3\varphi_i\,\mathfrak{A}_{1,2}(i,j,k)\partial_1\psi)\,\mathrm{d}x && (= T_1^1) \\ &\quad -2\sum_{ijk} \int_{\mathcal{O}_\lambda} (\partial_2\varphi_j)(\varphi_k\,\partial_{23}\varphi_i\,\mathfrak{A}_{1,2}(i,j,k)\partial_1\psi)\,\mathrm{d}x && (= T_2^1) \\ &\quad -2\sum_{ijk} \int_{\mathcal{O}_\lambda} (\partial_2\varphi_j)(\varphi_k\,\partial_3\varphi_i\,\partial_2\mathfrak{A}_{1,2}(i,j,k)\partial_1\psi)\,\mathrm{d}x && (= T_3^1) \\ &\quad -2\sum_{ijk} \int_{\mathcal{O}_\lambda} (\partial_2\varphi_j)(\varphi_k\,\partial_3\varphi_i\,\mathfrak{A}_{1,2}(i,j,k)\partial_{12}\psi)\,\mathrm{d}x && (= T_4^1) \\ &\quad -2\sum_{i,j,k} \int_{\mathcal{O}_\lambda} \varphi_k\,\partial_{23}\varphi_j\,\partial_2\varphi_i\,\mathfrak{A}_{1,2}(i,j,k)\partial_1\psi\,\mathrm{d}x && (= T_5^1). \end{aligned}$$

Permuting indices $j \leftrightarrow k$ and using the antisymmetry from Lemma 4.2 (c), we obtain

$$T_1^1 = -2\sum_{ijk} \int_{\mathcal{O}_\lambda} (\partial_2\varphi_j)(\partial_2\varphi_k\,\partial_3\varphi_i\,\mathfrak{A}_{1,2}(i,j,k)\partial_1\psi)\,\mathrm{d}x$$

$$= 2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\partial_2\varphi_k\partial_3\varphi_i\mathfrak{A}_{1,2}(i,k,j)\partial_1\psi)\,\mathrm{d}x$$
$$= 2\sum_{ikj}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\partial_2\varphi_k\partial_3\varphi_i\mathfrak{A}_{1,2}(i,k,j)\partial_1\psi)\,\mathrm{d}x$$
$$= -T_1^1, \tag{A.3}$$

and hence $T_1^1 = 0$. Equally, permuting $i \leftrightarrow j$, we find that $T_2^1 + T_5^1 = 0$. Therefore, using Lemma 4.2 (b) for T_3^1 and integrating by parts in term T_4^1 with respect to ∂_1,

$$S_1^1 = T_3^1 + T_4^1 = -2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_3\varphi_i\mathfrak{B}_1(i,j,k)\partial_1\psi)\,\mathrm{d}x \qquad (= T_6^1)$$
$$+ 2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_{12}\varphi_j)\varphi_k\partial_3\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x \qquad (= T_7^1)$$
$$+ 2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)\partial_1\varphi_k\partial_3\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x \qquad (= T_8^1)$$
$$+ 2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)\varphi_k\partial_{13}\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x \qquad (= T_9^1)$$
$$\overset{\text{Lem. 4.2(a)}}{=} 2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)\varphi_k\partial_3\varphi_i\mathfrak{B}_2(i,j,k)\partial_2\psi\,\mathrm{d}x \qquad (= T_{10}^1).$$

On the other hand,

$$S_1^2 = \sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k(\partial_{13}\varphi_j\partial_2\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x \qquad (= T_1^2)$$
$$+ \sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k(\partial_{23}\varphi_j\partial_1\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x \qquad (= T_2^2)$$
$$- \sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k(2\partial_{12}\varphi_j\partial_3\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x \qquad (= T_3^2).$$

We finally turn to S_1^3. Here we have

$$S_1^3 = \sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k(\partial_{12}\varphi_j\partial_2\varphi_i - \partial_{22}\varphi_j\partial_1\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_3\psi\,\mathrm{d}x$$
$$= -\sum_{ijk}\int_{\mathcal{O}_\lambda}\partial_1\varphi_j\partial_2\varphi_k\partial_2\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_3\psi\,\mathrm{d}x \qquad (= T_1^3)$$
$$- \sum_{ijk}\int_{\mathcal{O}_\lambda}\partial_1\varphi_j\varphi_k\partial_{22}\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_3\psi\,\mathrm{d}x \qquad (= T_2^3)$$

$$-\sum_{ijk}\int_{\mathcal{O}_\lambda} \partial_1\varphi_j\varphi_k\partial_2\varphi_i\partial_2\mathfrak{A}_{1,2}(i,j,k)\partial_3\psi\,\mathrm{d}x \qquad (= T_3^3)$$

$$-\sum_{ijk}\int_{\mathcal{O}_\lambda} \partial_1\varphi_j\varphi_k\partial_2\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_{23}\psi\,\mathrm{d}x \qquad (= T_4^3)$$

$$-\sum_{ijk}\int_{\mathcal{O}_\lambda} \varphi_k(\partial_{22}\varphi_j\partial_1\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_3\psi\,\mathrm{d}x \qquad (= T_5^3).$$

Again, T_1^3 vanishes by the same argument as for (A.3), $T_2^3 + T_5^3 = 0$ by permuting indices $i \leftrightarrow j$, and so we obtain, analogously to above,

$$\begin{aligned} S_1^3 = &-\sum_{ijk}\int_{\mathcal{O}_\lambda} \partial_1\varphi_j\varphi_k\partial_2\varphi_i\mathfrak{B}_1(i,j,k)\partial_3\psi\,\mathrm{d}x && (= T_6^3)\\ &+\sum_{ijk}\int_{\mathcal{O}_\lambda} \partial_{13}\varphi_j\varphi_k\partial_2\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x && (= T_7^3)\\ &+\sum_{ijk}\int_{\mathcal{O}_\lambda} \partial_1\varphi_j\partial_3\varphi_k\partial_2\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x && (= T_8^3)\\ &+\sum_{ijk}\int_{\mathcal{O}_\lambda} \partial_1\varphi_j\varphi_k\partial_{23}\varphi_i\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x && (= T_9^3)\\ &+\sum_{ijk}\int_{\mathcal{O}_\lambda} \partial_1\varphi_j\varphi_k\partial_2\varphi_i\underbrace{\partial_3\mathfrak{A}_{1,2}(i,j,k)}_{=0}\partial_2\psi\,\mathrm{d}x. \end{aligned}$$

Permuting indices $i \leftrightarrow j$ in T_1^2 and T_7^3 yields by virtue of the antisymmetry property of $\mathfrak{A}_{1,2}$ that $T_9^1 + T_1^2 + T_7^3 = 0$, and we directly find that $T_7^1 + T_3^2 = 0$. For terms T_8^1 and T_8^3, we permute indices $i \leftrightarrow j$ and $j \leftrightarrow k$ in term T_8^3 to obtain

$$T_8^1 + T_8^3 = 3\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_1\varphi_k)(\partial_2\varphi_j)(\partial_3\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x. \qquad \text{(A.4)}$$

For terms T_2^2 and T_9^3, we permute indices $i \leftrightarrow j$ in T_9^3 to obtain $T_2^2 + T_9^3 = 0$. Having left T_6^1 and T_6^3 untouched, we thus obtain

$$\begin{aligned} S_1 = &-2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_3\varphi_i\mathfrak{B}_1(i,j,k)\partial_1\psi)\,\mathrm{d}x && (= T_6^1)\\ &-\sum_{ijk}\int_{\mathcal{O}_\lambda} \partial_1\varphi_j\varphi_k\partial_2\varphi_i\mathfrak{B}_1(i,j,k)\partial_3\psi\,\mathrm{d}x && (= T_6^3)\\ &-2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)\varphi_k\partial_3\varphi_i\mathfrak{B}_2(i,j,k)\partial_2\psi\,\mathrm{d}x && (= T_{10}^1) \end{aligned}$$

$$+ 3\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_1\varphi_k)(\partial_2\varphi_j)(\partial_3\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi\,\mathrm{d}x \quad (= T_8^1 + T_8^3)$$
$$=: \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4'. \tag{A.5}$$

We now claim that $\mathbf{S}_4' = 0$. Let us first note that the overall sum in the definition of $\mathbf{S}_4'$ converges absolutely in $\mathrm{L}^1(\mathcal{O}_\lambda)$. This can be seen similarly to the proof of Lemma 4.4, and is a consequence of (P3), Lemma 4.3 (b) and $\mathcal{L}^3(\mathcal{O}_\lambda) < \infty$, together with the bound

$$\sum_{ijk}\int_{\mathcal{O}_\lambda}|(\partial_1\varphi_k)(\partial_2\varphi_j)(\partial_3\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi|\,\mathrm{d}x \leq c\lambda\|\nabla w_1\|_{\mathrm{L}^1(\mathbb{R}^3)}\mathcal{L}^3(\mathcal{O}_\lambda),$$

where $c = c(3) > 0$ is a constant only depending on the underlying space dimension $n = 3$. By Lemma A.1 we have

$$\sum_{ijk}(\partial_1\varphi_k)(\partial_2\varphi_j)(\partial_3\varphi_i)\mathfrak{A}_{1,2}(i,j,k)\partial_2\psi \equiv 0 \quad \text{pointwise in } \mathcal{O}_\lambda, \tag{A.6}$$

to be understood as the limit of the corresponding partial sums. Therefore,

$$\begin{aligned} S_1 &= -2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_3\varphi_i\mathfrak{B}_1(i,j,k)\partial_1\psi)\,\mathrm{d}x \quad && (= T_6^1)\\ &\quad -\sum_{ijk}\int_{\mathcal{O}_\lambda}\partial_1\varphi_j\varphi_k\partial_2\varphi_i\mathfrak{B}_1(i,j,k)\partial_3\psi\,\mathrm{d}x && (= T_6^3)\\ &\quad -2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)\varphi_k\partial_3\varphi_i\mathfrak{B}_2(i,j,k)\partial_2\psi\,\mathrm{d}x && (= T_{10}^1)\\ &=: \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3. \end{aligned} \tag{A.7}$$

We now turn to S_2. Our line of action is similar to that for dealing with S_1 and so, integrating by parts twice, we successively obtain

$$\begin{aligned} S_2 &= \sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k(\partial_{23}\varphi_j\partial_3\varphi_i - \partial_{33}\varphi_j\partial_2\varphi_i)\mathfrak{A}_{2,3}(i,j,k)\partial_2\psi\,\mathrm{d}x\\ &\quad + \sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k(\partial_{23}\varphi_j\partial_2\varphi_i - \partial_{22}\varphi_j\partial_3\varphi_i)\mathfrak{A}_{2,3}(i,j,k)\partial_3\psi\,\mathrm{d}x\\ &= \sum_{ijk}(-1)\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\partial_3\varphi_k\partial_3\varphi_i\mathfrak{A}_{2,3}(i,j,k)\partial_2\psi)\,\mathrm{d}x && (= T_1)\\ &\quad -\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_{33}\varphi_i\mathfrak{A}_{2,3}(i,j,k)\partial_2\psi)\,\mathrm{d}x && (= T_2)\\ &\quad -\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_3\varphi_i\partial_3\mathfrak{A}_{2,3}(i,j,k)\partial_2\psi)\,\mathrm{d}x && (= T_3) \end{aligned}$$

$$-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_3\varphi_i\mathfrak{A}_{2,3}(i,j,k)\partial_{23}\psi)\,\mathrm{d}x \qquad (=T_4)$$

$$-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\varphi_k\partial_{33}\varphi_j\partial_2\varphi_i)\mathfrak{A}_{2,3}(i,j,k)\partial_2\psi\,\mathrm{d}x \qquad (=T_5)$$

$$-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)(\partial_2\varphi_k\partial_2\varphi_i\mathfrak{A}_{2,3}(i,j,k)\partial_3\psi)\,\mathrm{d}x \qquad (=T_6)$$

$$-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)(\varphi_k\partial_{22}\varphi_i\mathfrak{A}_{2,3}(i,j,k)\partial_3\psi)\,\mathrm{d}x \qquad (=T_7)$$

$$-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)(\varphi_k\partial_2\varphi_i\partial_2\mathfrak{A}_{2,3}(i,j,k)\partial_3\psi)\,\mathrm{d}x \qquad (=T_8)$$

$$-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)(\varphi_k\partial_2\varphi_i\mathfrak{A}_{2,3}(i,j,k)\partial_{23}\psi)\,\mathrm{d}x \qquad (=T_9)$$

$$-\sum_{ijk}\int_{\mathcal{O}_\lambda}(\varphi_k\partial_{22}\varphi_j\partial_3\varphi_i)\mathfrak{A}_{2,3}(i,j,k)\partial_3\psi\,\mathrm{d}x \qquad (=T_{10}).$$

Terms T_1 and T_6 vanish by the same argument as in (A.3). Permuting indices $i \leftrightarrow j$, we then obtain $T_2 + T_5 = 0$, and in a similar manner we see that $T_7 + T_{10} = 0$ and $T_4 + T_9 = 0$. To conclude, we use Lemma 4.2 to obtain

$$\begin{aligned} S_2 = T_3 + T_8 &= -\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\varphi_j)(\varphi_k\partial_3\varphi_i\mathfrak{B}_2(i,j,k)\partial_2\psi)\,\mathrm{d}x \\ &\quad + \sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)(\varphi_k\partial_2\varphi_i\mathfrak{B}_3(i,j,k)\partial_3\psi)\,\mathrm{d}x \\ &=: \mathbf{S}_4 + \mathbf{S}_5. \end{aligned} \tag{A.8}$$

Term S_3 is given by

$$\begin{aligned} S_3 := {} & 2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_{33}\varphi_j\partial_2\varphi_i - \partial_{23}\varphi_i\partial_3\varphi_i)\mathfrak{A}_{3,1}(i,j,k)\partial_1\psi \\ & + \sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_{13}\varphi_j\partial_3\varphi_i - \partial_{33}\varphi_j\partial_1\varphi_i)\mathfrak{A}_{3,1}(i,j,k)\partial_2\psi \\ & + \sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_{23}\varphi_j\partial_1\varphi_i + \partial_{12}\varphi_j\partial_3\varphi_i - 2\partial_{13}\varphi_j\partial_2\varphi_i)\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi \\ =: {} & S_3^1 + S_3^2 + S_3^3. \end{aligned}$$

Terms S_3^1 and S_3^2 are treated as term S_1^1, where we now integrate by parts with respect to ∂_3 in S_3^1 or with respect to ∂_1 in S_3^2, respectively. Similarly to the computation underlying

S_1, this gives us

$$
\begin{aligned}
S_3 = {} & 2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)\varphi_k(\partial_2\varphi_i)\mathfrak{B}_1(i,j,k)\partial_1\psi && (= T_1') \\
& + 2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_{13}\varphi_j)\varphi_k\partial_2\varphi_i\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_2') \\
& + 2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)\partial_1\varphi_k\partial_2\varphi_i\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_3') \\
& + 2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)\varphi_k\partial_{12}\varphi_i\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_4') \\
& + 2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)\varphi_k\partial_2\varphi_i\mathfrak{B}_3(i,j,k)\partial_3\psi && (= T_5') \\
& + \sum_{ijk}\int_{\mathcal{O}_\lambda}\partial_1\varphi_j\varphi_k\partial_3\varphi_i\mathfrak{B}_1(i,j,k)\partial_2\psi\,\mathrm{d}x && (= T_6') \\
& + \sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_2\partial_1\varphi_j)\varphi_k\partial_3\varphi_i\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_7') \\
& + \sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_1\varphi_j)\partial_2\varphi_k\partial_3\varphi_i\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_8') \\
& + \sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_1\varphi_j)\varphi_k\partial_{23}\varphi_i\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_9') \\
& + \sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k(\partial_{23}\varphi_j\partial_1\varphi_i)\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_{10}') \\
& + \sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k(\partial_{12}\varphi_j\partial_3\varphi_i)\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_{11}') \\
& - 2\sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k\partial_{13}\varphi_j\partial_2\varphi_i\mathfrak{A}_{3,1}(i,j,k)\partial_3\psi && (= T_{12}').
\end{aligned}
$$

By an argument analogous to (A.5)ff., $T_3' = T_8' = 0$. Moreover, permuting indices yields as above $T_4' + T_7' + T_{11}' = 0$ and $T_9' + T_{10}' = 0$, whereas $T_2' + T_{12}' = 0$ follows directly. Therefore,

$$
\begin{aligned}
S_3 = {} & 2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)\varphi_k(\partial_2\varphi_i)\mathfrak{B}_1(i,j,k)\partial_1\psi \\
& + 2\sum_{ijk}\int_{\mathcal{O}_\lambda}(\partial_3\varphi_j)\varphi_k\partial_2\varphi_i\mathfrak{B}_3(i,j,k)\partial_3\psi
\end{aligned}
$$

$$+\sum_{ijk}\int_{\mathcal{O}_\lambda}\partial_1\varphi_j\varphi_k\partial_3\varphi_i\mathfrak{B}_1(i,j,k)\partial_2\psi\,\mathrm{d}x$$
$$=: \mathbf{S}_6+\mathbf{S}_7+\mathbf{S}_8 \tag{A.9}$$

Until now, we have only considered the contributions from $\mathfrak{A}_{1,2}$, $\mathfrak{A}_{3,1}$ and $\mathfrak{A}_{2,3}$. The contributions containing $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{B}_3$ then read as

$$\begin{aligned}S_4+S_5+S_6 &= 6\sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k\partial_2\varphi_j\partial_3\varphi_i\mathfrak{B}_1(i,j,k)\partial_1\psi\\ &\quad+3\sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k\partial_3\varphi_j\partial_1\varphi_i\mathfrak{B}_1(i,j,k)\partial_2\psi\\ &\quad+3\sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k\partial_1\varphi_j\partial_2\varphi_i\mathfrak{B}_1(i,j,k)\partial_3\psi\\ &\quad+3\sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k\partial_2\varphi_j\partial_3\varphi_i\mathfrak{B}_2(i,j,k)\partial_2\psi\\ &\quad+3\sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k\partial_2\varphi_j\partial_3\varphi_i\mathfrak{B}_3(i,j,k)\partial_3\psi\\ &= \mathbf{S}_9+\mathbf{S}_{10}+\mathbf{S}_{11}+\mathbf{S}_{12}+\mathbf{S}_{13}.\end{aligned}$$

Combining this with (A.7), (A.8) and (A.9), we may then build the overall sum $S_1+\cdots+S_6=\mathbf{S}_1+\cdots+\mathbf{S}_{13}$. Summing all terms, we note by an analogous permutation argument that $\mathbf{S}_3+\mathbf{S}_4+\mathbf{S}_{12}=0$, $\mathbf{S}_5+\mathbf{S}_7+\mathbf{S}_{13}=0$, and so

$$\begin{aligned}\int_{\mathcal{O}_\lambda}(T_\lambda w)_1\cdot\nabla\psi\,\mathrm{d}x &= 2\sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k\partial_2\varphi_j\partial_3\varphi_i\mathfrak{B}_1(i,j,k)\partial_1\psi\,\mathrm{d}x \qquad (\sim \mathbf{S}_1+\mathbf{S}_6+\mathbf{S}_9)\\ &\quad+2\sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k\partial_1\varphi_i\partial_3\varphi_j\mathfrak{B}_1(i,j,k)\partial_2\psi\,\mathrm{d}x \qquad (\sim \mathbf{S}_8+\mathbf{S}_{10})\\ &\quad+2\sum_{ijk}\int_{\mathcal{O}_\lambda}\varphi_k\partial_1\varphi_j\partial_2\varphi_i\mathfrak{B}_1(i,j,k)\partial_3\psi\,\mathrm{d}x \qquad (\sim \mathbf{S}_2+\mathbf{S}_{11}),\end{aligned}$$

where we use the symbol '$\sim$' to indicate where the single terms stem from. This is precisely (4.12), and so the proof is complete.

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