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AN ELLIPTIC BOUNDARY PROBLEM ACTING ON GENERALIZED SOBOLEV SPACES

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Abstract. We consider an elliptic boundary problem over a bounded region Ω in $\mathbb{R}^n$ and acting on the generalized Sobolev space $W_p^{0,\chi}(\Omega)$ for $1 < p < \infty$. We note that similar problems for Ω either a bounded region in $\mathbb{R}^n$ or a closed manifold acting on $W_2^{0,\chi}(\Omega)$, called Hörmander space, have been the subject of investigation by various authors. Then in this paper we will, under the assumption of parameter-ellipticity, establish results pertaining to the existence and uniqueness of solutions of the boundary problem. Furthermore, under the further assumption that the boundary conditions are null, we will establish results pertaining to the spectral properties of the Banach space operator induced by the boundary problem, and in particular, to the angular and asymptotic distribution of its eigenvalues.

1. introduction

In the latter half of the last century Hörmander [13, Chapter II] introduced a class of weight functions defined on $\mathbb{R}^n$, which he denoted by $\mathcal{K}$ (see Definition 2.1 below), and a Banach space $\mathcal{B}_{p,k}, k \in \mathcal{K}, 1 < p < \infty$, composed of tempered distributions u such that $\mathscr{F}u$ is a measurable function on $\mathbb{R}^n$ and $k\mathscr{F}u \in L_p(\mathbb{R}^n)$, where $\mathscr{F}$ denotes the Fourier transformation in $\mathbb{R}^n$. He then investigated various properties of this space as well as the regularity properties of solutions of partial differential equations acting on $\mathcal{B}_{p,k}$. We might mention at this point that the space $\mathcal{B}_{2,k}$, called Hörmander space, is of particular importance as it gives us a significant generalization of the classical Sobolev space based on $L_2(\mathbb{R}^n)$.

The work of Hörmander did stimulate significant interest and research at that time, but unlike Sobolev spaces, the Hörmander spaces were not widely applied to elliptic boundary problems and to elliptic operators acting over closed manifolds. However since the beginning of this century significant investigations have been devoted to these aforementioned problems (see for example [16], [17], and [8] as well as the book [18]). Indeed, in the references just cited the authors restrict themselves to the case $p = 2$ and to a certain subset of weight functions called interpolation parameters which ensures that every Hörmander space based on an interpolation parameter is actually an interpolation space obtained by interpolating between two Sobolev spaces. Thus in this way that authors obtain important results pertaining to elliptic boundary problems and to elliptic operators acting on such Hörmander spaces defined on closed manifolds.

Shortly after the appearance of the book [13] there appeared the paper of Volevich and Paneyakh [21] presenting, by means of an Hörmander type weight function,

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a generalization of Bessel-potential spaces for $1 < p < \infty$ and then described various properties of this space. This space, which they denote by H_p^μ, is precisely the space of tempered distributions u such that $\mathscr{F}^{-1}\,\mu\,\mathscr{F}u \in L_p(\mathbb{R}^n)$ for all μ belonging to a certain subset, denoted by $\mathcal{K}_0$, of non-vanishing functions in $C^\infty(\mathbb{R}^n)$ which, together with their inverses, belong to the Hörmander class of weight functions $\mathcal{K}$ and which are multipliers on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$, that is, as operators of multiplication, they map $\mathscr{S}(\mathbb{R}^n)$ into itself. By defining $\mu\, u(\phi) = u(\mu\,\phi)$ for $\mu \in \mathcal{K}_0, \phi \in \mathscr{S}(\mathbb{R}^n)$ and $u \in \mathscr{S}'(\mathbb{R}^n)$, the members of $\mathcal{K}_0$ also become multipliers on the space $\mathscr{S}'(\mathbb{R}^n)$. Lastly, let us mention that the spaces obtained by restricting of the members of H_p^μ to subsets of $\mathbb{R}^n$ are also discussed in [21].

We have mentioned above that $\mathcal{B}_{2,k}$ gives us a generalization of Sobolev spaces based on $L_2(\mathbb{R}^n)$. Motivated by the works cited above, our aim in this paper is to remove the restriction $p = 2$, and by fixing our attention upon a certain class of weight functions in $\mathcal{K}$, introduce our generalization of classical Sobolev space based on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, as well as on $L_p(G)$ for certain subsets G of $\mathbb{R}^n$. Then we will establish various results pertaining to the operator acting on our generalized Sobolev space induced by a parameter-elliptic boundary problem.

Accordingly, we will be concerned here with the boundary problem

$$A(x,D)u(x) - \lambda\, u(x) = f(x) \text{ for } x \in \Omega. \tag{1.1}$$

$$B_j(x,D)u(x) = g_j(x) \text{ for } x \in \Gamma, j = 1,\dots,m, \tag{1.2}$$

where Ω is a bounded region in $\mathbb{R}^n$, $n \geq 2$, with boundary Γ, $A(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha$ is a linear partial differential operator defined on Ω of order $2m$, and for $j = 1,\dots,m$, $B_j(x,D) = \sum_{|\alpha|\leq m_j} b_{j,\alpha}(x)D^\alpha$ is a linear partial differential operator defined on Γ of order $m_j < 2m$, while $\lambda \in \mathcal{L}$, where $\mathcal{L}$ is a closed sector in the complex plane with vertex at the origin. Our assumptions concerning the boundary problem (1.1), (1.2) will be made precise in Section 3.

In Section 2 we make precise our definition of the generalized Sobolev space over $\mathbb{R}^n$ and over certain subsets of $\mathbb{R}^n$. This is achieved by firstly defining the subsets $\mathcal{K}_0$ and $\mathcal{K}_1$ of the Hörmander class of weight functions $\mathcal{K}$ which will be used in this paper to define the generalized Sobolev spaces with which we will be concerned. Then for $\chi \in \mathcal{K}_0 \cup \mathcal{K}_1$ we introduce the space $H_p^\chi(\mathbb{R}^n)$, which is a generalization of $L_p(\mathbb{R}^n)$, and describe various properties of this space. And it is by means of $H_p^\chi(\mathbb{R}^n)$ that we are able to introduce the generalized Sobolev spaces $W_p^{k,\chi}(\mathbb{R}^n), W_p^{k,\chi}(\Omega)$ for $k \in \mathbb{N} \cup \{0\}$, and $W_p^{k-1/p,\chi}(\Gamma)$ for $k \in \mathbb{N}$ (see Definitions 2.1, 2.3, 2.8, 2.11, and 2.14).

In Section 3 we make precise our assumptions concerning the boundary problem (1.1), (1.2) and then use the results of Section 2 to establish our main result concerning the existence and uniqueness of solutions of this boundary problem (see Theorem 3.10 below).

Finally in Section 4 we let $A_{B,p}^\chi$ denote the Banach space operator, with domain $W_p^{2m,\chi}(\Omega)$, induced by the boundary problem (1.1), (1.2) under null boundary conditions. We then prove that $A_{B,p}^\chi$ has compact resolvent and various results are established concerning the completeness of its principal vectors in $W_p^{0,\chi}(\Omega)$ as well as the angular and asymptotic behaviour of its eigenvalues (see Theorems 4.4–4.6).

2. Generalized Sobolev space

In this section we are going to introduce our generalization of the classical Sobolev space and discuss some of its properties. To this end we need the following terminology.

Accordingly, we let $x = (x_1, \dots, x_n) = (x', x_n)$ denote a generic point in $\mathbb{R}^n$ and use the notation $D_j = -i\partial/\partial x_j, D = (D_1, \dots, D_n), D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = D'^{\alpha'} D_n^{\alpha_n}$, and $\xi^\alpha = \xi^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n}$ for $\xi = (\xi_1, \dots, \xi_n) = (\xi', \xi_n) \in \mathbb{R}^n$, where $\alpha = (\alpha_1, \dots, \alpha_n) = (\alpha', \alpha_n)$ is a multi-index whose length $\sum_{j=1}^n \alpha_j$ is denoted by $|\alpha|$. Differentiation with respect to another variable, say $y \in \mathbb{R}^n$, instead of x will be indicated by replacing D and D^α by D_y and D_y^α, respectively. We also let $\mathscr{S}(\mathbb{R}^n)$ denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ and let $\mathscr{S}'(\mathbb{R}^n)$ denote its dual, where in this paper it will always be supposed that $\mathscr{S}'(\mathbb{R}^n)$ is equipped with its weak-$*$-topology. In addition we let $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$ and $\langle\xi'\rangle = (1 + |\xi'|^2)^{1/2}$ for $\xi = (\xi', \xi_n) \in \mathbb{R}^n$, while for $1 < p < \infty$, $0 \le s < \infty$, and G an open set in $\mathbb{R}^n$, we let $W_p^s(G)$ denote the Sobolev space of order s related to $L_p(G)$ and denote the norm in this space by $\| \cdot \|_{s,p,G}$ (see [20, p.169, p.310, and Theorem 2.3.3, p.177]). Furthermore, we will use norms depending upon the parameter $\lambda \in \mathbb{C} \setminus \{ 0 \}$, namely for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{ 0 \}$ with $k \le 2m$, we let

$$|\!|\!| u |\!|\!|_{k,p,G} = \| u \|_{k,p,G} + |\lambda|^{k/2m} \| u \|_{0,p,G} \text{ for } u \in W_p^k(G).$$

We refer to [12] for details concerning parameter-dependent norms.

Assume for the moment that when $G \ne \mathbb{R}^n$ the boundary ∂G is of class C^{2m}. Then for $k \in \mathbb{N}$ with $k \le 2m$ the vectors $u \in W_p^k(G)$ have boundary values $v = u\big|_{\partial G}$ and we denote the space of these boundary values by $W_p^{k-1/p}(\partial G)$ and by $\| \cdot \|_{k-1/p,p,\partial G}$ the norm in this space, where $\| v \|_{k-1/p,p,\partial G} = \inf \| u \|_{k.p.G}$ for $v \in W_p^{k-1/p}(\partial G)$ and the infimum is taken over those $u \in W_p^k(G)$ for which $u\big|_{\partial G} = v$. In addition we will use norms depending upon the parameter $\lambda \in \mathbb{C} \setminus \{ 0 \}$, namely

$$|\!|\!| v |\!|\!|_{k-1/p,p,\partial G} = \| v \|_{k-1/p,p,\partial G} + |\lambda|^{(k-1/p)/2m} \| v \|_{0,p,\partial G} \text{ for } v \in W_p^{k-1/p}(\partial G),$$

where $\| \cdot \|_{0,p,\partial G}$ denotes the norm in $L_p(\partial G)$. Finally, we let $\mathbb{R}_\pm = \{ t \in \mathbb{R} \,\big|\, t \gtrless 0 \}$.

We are now going to define the generalized Sobolev spaces which will be considered in this paper. To this end we require the following definition.

Definition 2.1. Let $\mathcal{K}$ denote the class of real-valued measurable functions defined on $\mathbb{R}^n$ with values in $(0, \infty)$ such that for each member $\chi \in \mathcal{K}$ there exist positive constants $C_\chi^\dagger$ and $\ell_\chi^\dagger$ for which the inequality

$$\chi(\xi + \eta) \le C_\chi^\dagger \, (1 + |\xi|)^{\ell_\chi^\dagger} \chi(\eta) \text{ holds for every pair } \xi, \eta \in \mathbb{R}^n.$$

Note that

$$(C_\chi^\dagger)^{-1} \chi(0) \, (1 + |\xi|)^{-\ell_\chi^\dagger} \le \chi(\xi) \le C_\chi^\dagger \chi(0) \, (1 + |\xi|)^{\ell_\chi^\dagger} \text{ and also that } \chi^{-1} \in \mathcal{K}.$$

The class $\mathcal{K}$ is precisely the class of weight functions mentioned in Section 1 which was introduced by Hörmander in [13] and used there to define the Banach space $\mathcal{B}_{p,k}$. As mentioned in Section 1, Volevich and Paneyakh [21] defined the more restrictive class of weight functions $\mathcal{K}_0$ as the set of all smooth functions in $\mathcal{K}$ which are multipliers in $\mathscr{S}(\mathbb{R}^n)$ and whose inverses belong to $\mathcal{K}$, too. Our aim now is to use $\mathcal{K}$ in order to define for the case $p \le 2$ a less restrictive generalized Bessel-potential space than that considered in [21]. However for future considerations we will have to restrict ourselves to the subset $\mathcal{K}_1$ where

$$\mathcal{K}_1 = \left\{ \chi \in \mathcal{K} \middle| \, \chi \in C^{2n^+}(\mathbb{R}^n), |D^\alpha \chi(\xi)| \le C_\chi \langle \xi \rangle^{\ell_\chi} \text{ for } \xi \in \mathbb{R}^n \text{ and } |\alpha| \le 2n^+ \right\},$$

where C_χ and ℓ_χ denote positive constants and for $t \ge 0, t^+ = [t/2] + 1$, and $[t/2]$ denotes the integer part of $t/2$.

Remark 2.2. In order to avoid a proliferation of notation, we will also suppose that for $\chi \in \mathcal{K}_0$, $|D_\xi^\alpha \chi(\xi)| \le C_\chi \langle \xi \rangle^{\ell_\chi}$ for $|\alpha| \le 2n^+$.

We refer to [13, p.35] and [21] for examples of function in $\mathcal{K}_0$ and $\mathcal{K}_1$. Note that the following functions indicated there: (1) $\chi(\xi) = \langle \xi \rangle^t, t \in \mathbb{R}$, (2) $\chi(\xi) = \widetilde{P}(\xi) = \left(\sum_{|\alpha| \ge 0} |D_\xi^\alpha P(\xi)|^2\right)^{1/2}$, where P is a polynomial, and (3) $\chi(\xi) = \left(1 + \sum_{j=1}^n |\xi_j|^{2\ell_j}\right)^t$, where $t \in \mathbb{R}$ and the $\ell_j \in \overline{\mathbb{R}_+}$, all belong to $\mathcal{K}_0$ and $\mathcal{K}_1$. Note also that if $\chi \in \mathcal{K}_1$ (resp. $\mathcal{K}_0$), then so does χ^{-1}.

Definition 2.3. For $1 < p \le 2$ we henceforth suppose that $\chi \in \mathcal{K}_1$ and let

$$H_p^\chi(\mathbb{R}^n) = \left\{ u \in \mathscr{S}'(\mathbb{R}^n) \middle| \, \mathscr{F}u \text{ is a measurable function on } \mathbb{R}^n, \chi\mathscr{F}u \in \mathscr{S}'(\mathbb{R}^n) \text{ and } \mathscr{F}^{-1}\chi\mathscr{F}u \in L_p(\mathbb{R}^n) \right\},$$

while for $2 < p < \infty$ it will always be supposed that $\chi \in \mathcal{K}_0$, and in this case we let

$$H_p^\chi(\mathbb{R}^n) = \left\{ u \in \mathscr{S}'(\mathbb{R}^n) \,\middle|\, \mathscr{F}^{-1}\chi\mathscr{F}u \in L_p(\mathbb{R}^n) \right\}.$$

We then equip $H_p^\chi(\mathbb{R}^n)$ with the norm $\| u \|_{0,p,\mathbb{R}^n}^\chi = \| \mathscr{F}^{-1}\chi\mathscr{F}u \|_{0,p,\mathbb{R}^n}$ for $u \in H_p^\chi(\mathbb{R}^n)$.

We henceforth suppose that $1 < p < \infty$ and let $\hat{u} = \mathscr{F}u$ for $u \in \mathscr{S}'(\mathbb{R}^n)$.

Proposition 2.4. *$H_p^\chi(\mathbb{R}^n)$ is a Banach space.*

Proof. Since the proposition is proved in [21] for the case $p > 2$, we need only prove the proposition for the case $p \le 2$. Accordingly let $\{ u_j \}_{j \ge 1}$ denote a Cauchy sequence in $H_p^\chi(\mathbb{R}^n)$and put $v_j = \mathscr{F}^{-1}\chi\, \hat{u}_j$ for $j \ge 1$. Then $\{ v_j \}_{j \ge 1}$ is a Cauchy sequence in $L_p(\mathbb{R}^n)$, and hence converges in $L_p(\mathbb{R}^n)$ to some vector v. It now follows from the Hausdorff-Young theorem [9, p.6] that $\chi\hat{u}_j \to \hat{v}$ in $L_{p'}(\mathbb{R}^n)$, where $p' = p/(p-1)$, and hence also in $\mathscr{S}'(\mathbb{R}^n)$, as $j \to \infty$. Thus for $\phi \in \mathscr{S}(\mathbb{R}^n)$,

$$\left| \left(\hat{u}_j - \chi^{-1}\hat{v}\right)(\phi) \right| = \left| \int_{\mathbb{R}^n} \left(\hat{u}_j - \chi^{-1}\hat{v}\right) \phi \, dx \right| \le \|\chi\hat{u}_j - \hat{v}\|_{0,p',\mathbb{R}^n} \|\chi^{-1}\phi\|_{0,p,\mathbb{R}^n} \to 0$$

as $j \to \infty$.

Thus we have shown that $\hat{u}_j \to \chi^{-1}\hat{v}$ in $\mathscr{S}'(\mathbb{R}^n)$ as $j \to \infty$, and hence $u_j \to \mathscr{F}^{-1}\chi^{-1}\hat{u}$ in $\mathscr{S}'(\mathbb{R}^n)$ as $j \to \infty$, which completes the proof of the proposition. □

Proposition 2.5. *It is the case that $\mathscr{S}(\mathbb{R}^n) \subset H_p^\chi(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ in both the algebraic and topological sense. Furthermore, $\mathscr{S}(\mathbb{R}^n)$ is dense in $H_p^\chi(\mathbb{R}^n)$.*

Proof. Since the proposition is proved in [21] for the case $p > 2$, we need only prove the proposition for the case $p \le 2$. Accordingly, it follows from the proof of Proposition 2.4 that $H_p^\chi(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$. Turning now to $\mathscr{S}(\mathbb{R}^n)$, we have for $\phi \in \mathscr{S}(\mathbb{R}^n)$, $\mathscr{F}^{-1}\chi\hat{\phi} = \mathscr{F}^{-1}\chi(\xi)\langle\xi\rangle^{-\ell_\chi - n^+}\langle\xi\rangle^{\ell_\chi + n^+}\hat{\phi}$. Hence it follows from Mikhlin's multiplier theorem [20, p.166] that $\| \mathscr{F}^{-1}\chi\hat{\phi}\|_{0,p,\mathbb{R}^n} \le C_{\chi,p}\| \mathscr{F}^{-1}\langle\xi\rangle^{\ell_\chi + n^+}\hat{\phi}\|_{0,p,\mathbb{R}^n}$, where $C_{\chi,p}$ denotes a positive constant. But since $\langle\cdot\rangle^{\ell_\chi+n^+}$ is a multiplier on $\mathscr{S}(\mathbb{R}^n)$, we conclude that $\mathscr{F}^{-1}\langle\xi\rangle^{\ell_\chi+n^+}\hat{\phi}\in\mathscr{S}(\mathbb{R}^n)$, and hence $\|\mathscr{F}^{-1}\chi\hat{\phi}\|_{0,p,\mathbb{R}^n}<\infty$. Thus we conclude that $\mathscr{S}(\mathbb{R}^n)$ is a subspace of $H_p^\chi(\mathbb{R}^n)$.

Finally let f belong to the dual space of $H_p^\chi(\mathbb{R}^n)$. Then $\big|f(u)\big|\le C\|\mathscr{F}^{-1}\chi\hat{u}\|_{0,p,\mathbb{R}^n}$ for every $u\in H_p^\chi(\mathbb{R}^n)$, where C denotes a positive constant, Hence by the Hahn-Banach theorem there exists a $v\in L_{p'}(\mathbb{R}^n)$ such that $f(u)=\int_{\mathbb{R}^n}v\mathscr{F}^{-1}\chi\hat{u}\,dx$ for $u\in H_p^\chi(\mathbb{R}^n)$. This implies that if $\mathscr{S}(\mathbb{R}^n)$ is not dense in $H_p^\chi(\mathbb{R}^n)$, then there is a $v\neq 0$ in $L_{p'}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} v\,\mathscr{F}^{-1}\chi\hat{\phi}\,dx=0 \text{ for every } \phi\in\mathscr{S}(\mathbb{R}^n). \tag{2.1}$$

In order to make use of (2.1) to prove our assertion concerning density, we require some further information. To this end let us show that $\mathscr{F}^{-1}\chi\mathscr{F}$ maps $W_p^{2\ell^+}(\mathbb{R}^n)$ continuously into $L_p(\mathbb{R}^n)$, where $\ell=\max\{\,\ell_\chi+n^+,\ell_{\chi^{-1}}+n^+\,\}$. Indeed for $u\in W_p^{2\ell^+}(\mathbb{R}^n)$, we have $\|\mathscr{F}^{-1}\chi\,\mathscr{F}u\|_{0,p,\mathbb{R}^n}=\|\mathscr{F}^{-1}\chi\langle\cdot\rangle^{-2\ell^+}\mathscr{F}\Big((1-\Delta)^{\ell^+}u\Big)\|_{0,p,\mathbb{R}^n}$, where Δ denotes the Laplacian on $\mathbb{R}^n$, and hence the required result follows from Mikhlin's multiplier theorem.

Let us also show that $W_p^{4\ell^+}(\mathbb{R}^n)\subset \operatorname{ran}\mathscr{F}^{-1}\chi\,\mathscr{F}\Big(W_p^{2\ell^+}(\mathbb{R}^n)\Big)$, where ran denotes range. Indeed if $w\in W_p^{4\ell^+}(\mathbb{R}^n)$ and we let $u=\mathscr{F}^{-1}\chi^{-1}\mathscr{F}w$, then $\|u\|_{2\ell^+,p,\mathbb{R}^n}=\|\mathscr{F}^{-1}\chi^{-1}\langle\cdot\rangle^{-2\ell^+}\mathscr{F}\Big((1-\Delta)^{2\ell^+}w\Big)\|_{0,p,\mathbb{R}^n}$, and the required result follows from Mikhlin's multiplier theorem and the fact that $\mathscr{F}^{-1}\chi\,\mathscr{F}u=w$ (it is important to note that both $\hat{u}$ and $\hat{w}$ are in $L_{p'}(\mathbb{R}^n)$).

Next let $w\in W_p^{4\ell^+}(\mathbb{R}^n)$ and let $u\in W_p^{2\ell^+}(\mathbb{R}^n)$ such that $\mathscr{F}^{-1}\chi\,\mathscr{F}u=w$. Also let $\{\,\psi_j\,\}_{j\ge1}$ denote a sequence in $\mathscr{S}(\mathbb{R}^n)$ such that $\psi_j\to u$ in $W_p^{2\ell^+}(\mathbb{R}^n)$. Then for $j\ge1$, we have

$$\int_{\mathbb{R}^n} v\,w\,dx=\int_{\mathbb{R}^n} v\mathscr{F}^{-1}\chi\,\hat{\psi}_j dx+\int_{\mathbb{R}^n} v\mathscr{F}^{-1}\chi\langle\cdot\rangle^{-2\ell^+}\mathscr{F}(1-\Delta)^{\ell^+}(u-\psi_j)dx,$$

and hence in light of (2.1) and Mikhlin's multiplier theorem we have

$$\Big|\int_{\mathbb{R}^n} vw\,dx\Big|\le C\|v\|_{0,p',\mathbb{R}^n}\,\|u-\psi_j\|_{2\tilde{\ell}^+,p,\mathbb{R}^n},$$

where the constant C does not depend upon j. Thus we conclude that $\int_{\mathbb{R}^n}vw\,dx=0$ for every $w\in W_p^{4\ell^+}(\mathbb{R}^n)$. But since $v\in W_{p'}^{-4\ell^+}(\mathbb{R}^n)$, the dual space of $W_p^{4\ell^+}(\mathbb{R}^n)$ (see [20, Theorem 2.6, p.198]), we must have $v=0$, which is a contradiction, and this completes the proof of the proposition. □

Proposition 2.6. $H_p^\chi(\mathbb{R}^n)$ *is separable.*

Proof. In light of what was shown above, we see that the embeddings $\mathscr{S}(\mathbb{R}^n)\subset H_p^{\ell_\chi+n^+}(\mathbb{R}^n)\subset H_p^\chi(\mathbb{R}^n)$ hold, where $H_p^{\ell_\chi+n^+}(\mathbb{R}^n)$ denotes the Bessel-potential space of order $\ell_\chi+n^+$ based on $L_p(\mathbb{R}^n)$ (see [20, p.177]). Since $H_p^{\ell_\chi+n^+}(\mathbb{R}^n)$ is separable, the assertion of the proposition is an immediate consequence of Proposition 2.4. □

Under a further assumption on χ we also have the following result.

Proposition 2.7. *Suppose that* $u \in H_p^\chi(\mathbb{R}^n)$ *and* $\phi \in C_0^\infty(\mathbb{R}^n)$. *Suppose in addition* $\left|\xi^\alpha D_\xi^\alpha \left(\chi(\xi+\eta)\chi(\xi)^{-1}\right)\right| \leq c_\chi \langle\eta\rangle^{k_\chi}$ *for* $\eta \in \mathbb{R}^n$ *and* $|\alpha| \leq n^+$, *where* c_χ *and* k_χ *denote positive constants. Then* $\phi\, u \in H_p^\chi(\mathbb{R}^n)$ *and* $\|\phi\, u\|_{0,p,\mathbb{R}^n}^\chi \leq C_{p,\chi,\phi}\|u\|_{0,p,\mathbb{R}^n}^\chi$, *where the constant* $C_{p,\chi,\phi}$ *does not depend upon* λ *and* u.

Proof. For $p > 2$ the proposition is proved in [21], and hence we restrict ourselves to the case $p \leq 2$. Accordingly, it is clear that $\phi\, u \in \mathscr{S}'$, and hence $\mathscr{F}\phi\, u \in \mathscr{S}'$. We therefore have to show firstly that $\mathscr{F}\phi\, u$ is a measurable function on $\mathbb{R}^n$. Now observe that if we put $\check{f}(x) = f(-x)$, then

$$\mathscr{F}_{x\to\xi}\phi\, u = (2\pi)^n \mathscr{F}_{x\to\xi}^{-1}\check{\phi}\check{u} = (2\pi)^n \mathscr{F}_{x\to\xi}^{-1}\check{\phi} * \mathscr{F}_{x\to\xi}^{-1}\check{u}$$
$$= (2\pi)^{-n}\int_{\mathbb{R}^n}\hat{\phi}(\xi-\eta)\hat{u}(\eta)\,d\eta = (2\pi)^{-n}\int_{\mathbb{R}^n}\hat{\phi}(\xi-\eta)\chi(\eta)^{-1}\chi(\eta)\hat{u}(\eta)\,d\eta.$$

If we make use of the fact that $\chi\,\hat{u} \in L_{p'}(\mathbb{R}^n)$ and appeal to Definition 2.1, then it follows that $\mathscr{F}_{x\to\xi}\phi\, u \in L_p^{loc}(\mathbb{R}^n)$, and hence is measurable on $\mathbb{R}^n$. Furthermore, because of density, we need only complete the remainder of the proof under the assumption that $u \in \mathscr{S}(\mathbb{R}^n)$. Accordingly, it follows from Fubini's theorem and Minkowski's inequality that

$$\|\mathscr{F}_{\xi\to x}^{-1}\chi(\cdot)\mathscr{F}_{y\to\xi}\phi\, u\|_{0,p,\mathbb{R}^n}$$
$$\leq (2\pi)^{-n}\int_{\xi\in\mathbb{R}^n}\langle\xi\rangle^{k_\chi}|\hat{\phi}(\xi)|\|\mathscr{F}_{\eta\to x}^{-1}\langle\xi\rangle^{-k_\chi}\chi(\eta+\xi)\chi(\eta)^{-1}\chi(\eta)\hat{u}\|_{0,p,\mathbb{R}^n}\,d\xi,$$

and hence the assertion of the proposition is an immediate consequence of Mikhlin's multiplier theorem. □

We now turn to the definitions of the generalized Sobolev spaces which will be used in this paper, namely $W_p^{k,\chi}(\mathbb{R}^n), W_p^{k,\chi}(\Omega)$ for $k \in \mathbb{N}_0$, and $W_p^{k-1/p,\chi}(\Gamma), k \in \mathbb{N}$, with $k \leq 2m$ in all cases.

Definition 2.8. For $k \in \mathbb{N}_0$ let $W_p^{k,\chi}(\mathbb{R}^n) = H_p^{\langle\,\cdot\,\rangle^k\chi}(\mathbb{R}^n)$, and denote by $\|\cdot\|_{k,p,\mathbb{R}^n}^\chi$ the ordinary norm in this space (see Definition 2.3) and by $|\!|\!|\cdot|\!|\!|_{k,p,\mathbb{R}^n}^\chi = \|\cdot\|_{k,p,\mathbb{R}^n}^\chi + |\lambda|^{k/2m}\|\cdot\|_{0,p,\mathbb{R}^n}^\chi$ its parameter-dependent norm.

Note that $\langle\,\cdot\,\rangle^k\chi$ belongs to $\mathcal{K}_1$ if $p \leq 2$ and to $\mathcal{K}_0$ otherwise.

Remark 2.9. In the sequel it will always be supposed that all function spaces under consideration are equipped with their parameter-dependent norms unless otherwise stated. Furthermore, it is to be understood that when not stated explicitly, an isomorphism between any two such spaces is bounded in norm by a constant not dependent upon λ.

In the following proposition and in the proof of Proposition 2.12 below we suppose that for $k \in \mathbb{N}_0$, $W_p^k(\Omega)$ is equipped with its Bessel-potential space norm.

Proposition 2.10. *Let* $k \in \mathbb{N}_0$ *with* $k \leq 2m$. *Then the operator* $\mathscr{F}^{-1}\chi\mathscr{F}$ *maps* $W_p^{k,\chi}(\mathbb{R}^n)$ *isometrically and isomorphically onto* $W_p^k(\mathbb{R}^n)$, *and its norm as well as that of its inverse are bounded by a constant not dependent upon* λ.

Proof. Let $u \in W_p^{k,\chi}(\mathbb{R}^n)$ and let $v = \mathscr{F}^{-1}\chi\hat{u}$. Then $\|\mathscr{F}^{-1}\langle\,\cdot\,\rangle^k\mathscr{F}v\|_{0,p,\mathbb{R}^n} = \|\mathscr{F}^{-1}\langle\,\cdot\,\rangle^k\chi\hat{u}\|_{0,p,\mathbb{R}^n}$, and hence $v \in W_p^k(\mathbb{R}^n)$ (see [20, p. 177]).

Conversely, let $v \in W_p^k(\mathbb{R}^n)$ and let $u = \mathscr{F}^{-1}\chi^{-1}\hat{v}$. Then $\|\mathscr{F}^{-1}\langle\,\cdot\,\rangle^k\,\hat{u}\|_{0,p,\mathbb{R}^n} = \|\mathscr{F}^{-1}\langle\,\cdot\,\rangle^k\,\hat{v}\|_{0,p,\mathbb{R}^n}$, and hence $u \in W_p^{k,\chi}(\mathbb{R}^n)$.

In light of these results and the definitions of the parameter-dependent norms concerned, the proof of the proposition is complete. □

Let us now turn to the definitions of $W_p^{k,\chi}(\Omega)$ and $W_p^{k-1/p,\chi}(\Gamma)$ for those value of k cited above. Accordingly, let $\mathscr{D}'(\Omega)$ denote the space of distributions over Ω.

Definition 2.11. Let $W_p^{k,\chi}(\Omega) = \{ u \in \mathscr{D}'(\Omega)$ such that $u = \overline{u}\big|_\Omega$ for some $\overline{u} \in W_p^{k,\chi}(\mathbb{R}^n) \}$ and equip $W_p^{k,\chi}(\Omega)$ with the norm $\|u\|_{k,p,\Omega}^\chi = \inf \|\overline{u}\|_{k,p,\mathbb{R}^n}^\chi$, where the infimum is taken over all $\overline{u} \in W_p^{k,\chi}(\mathbb{R}^n)$ such that $u = \overline{u}\big|_\Omega$.

Note that if we let $r_{W_p^{k,\chi}(\mathbb{R}^n)\to W_p^{k,\chi}(\Omega)}$ denote the operator restricting the members of $W_p^{k,\chi}(\mathbb{R}^n)$ to Ω and $N_{k,p}^\Omega$ denote its kernel, then this operator induces a decomposition of $W_p^{k,\chi}(\mathbb{R}^n)$ into equivalent classes whereby any two distinct members of $W_p^{k,\chi}(\mathbb{R}^n)$, say $\overline{u}^1$ and $\overline{u}^2$ are said to be equivalent if $\overline{u}^1 - \overline{u}^2 \in N_{k,p}^\Omega$. Hence if we denote the induced quotient space by $W_p^{k,\chi}(\mathbb{R}^n)/N_{k,p}^\Omega$ and equip it with its quotient space norm, then it is clear that we can identify $W_p^{k,\chi}(\Omega)$ with $W_p^{k,\chi}(\mathbb{R}^n)/N_{k,p}^\Omega$ (in the sense that they are isometrically isomorphic to each other). We mention at this point that if N is a subspace of a linear vector space Y and $X = Y/N$ denotes the corresponding quotient space, then in the sequel we will use the notation $[u]$ to denote the member of X containing $u \in Y$ and $\|\cdot\|_X$ to denote the quotient space norm in X.

Next let $\mathcal{N}_{k,p}^\Omega = \{ \overline{v} \in W_p^k(\mathbb{R}^n)$ such that $\overline{v} = \mathscr{F}^{-1}\chi\mathscr{F}\overline{u}$ for $\overline{u} \in N_{k,p}^\Omega \}$ and define the space $W_p^k(\mathbb{R}^n)/\mathcal{N}_{k,p}^\Omega$ in an analogous fashion to the way we defined the space $W_p^{k,\chi}(\mathbb{R}^n)/N_{k,p}^\Omega$.

Proposition 2.12. *It is the case that $W_p^{k,\chi}(\Omega)$ is isometrically isomorphic to $W_p^k(\Omega)$.*

Proof. Since $W_p^k(\Omega)$ is isometrically isomorphic to $W_p^k(\mathbb{R}^n)/\mathcal{N}_{k,p}^\Omega$ (equipped with its quotient space norm), the proposition will be proved if we can show that $W_p^{k,\chi}(\mathbb{R}^n)/N_{k,p}^\Omega$ is isometrically isomorphic to $W_p^k(\mathbb{R}^n)/\mathcal{N}_{k.p}^\Omega$. Accordingly, let $[\overline{u}] \in W_p^{k,\chi}(\mathbb{R}^n)/N_{k,p}^\Omega$ and let $\{ \overline{u}_\ell \}_1^\infty$ be a sequence in $W_p^{k,\chi}(\mathbb{R}^n)$ such that $\overline{u} - \overline{u}_\ell \in N_{k,p}^\Omega$ and $\lim_{\ell\to\infty} \|\overline{u}_\ell\|_{k,p,\mathbb{R}^n}^\chi = \|[\overline{u}]\|\big|_{W_p^{k,\chi}(\mathbb{R}^n)/N_{k,p}^\Omega}$. Hence if we let $\overline{v} = \mathscr{F}^{-1}\chi\mathscr{F}\,\overline{u}$ and $\overline{v}_\ell = \mathscr{F}^{-1}\chi\mathscr{F}\,\overline{u}_\ell$, then it follows from Proposition 2.10 that $\|[\overline{u}]\|_{W_p^{k,\chi}(\mathbb{R}^n)/N_{k,p}^\Omega} = \lim_{\ell\to\infty} \|\overline{v}_\ell\|_{k,p,\mathbb{R}^n} \geq \|[\overline{v}]\|_{W_p^k(\mathbb{R}^n)/\mathcal{N}_{k,p}^\Omega}$. Since similar arguments show that $\|[\overline{u}]\|_{W_p^{k,\chi}(\mathbb{R}^n)/N_{k,p}^\Omega} \leq \|[\overline{v}]\|_{W_p^k(\mathbb{R}^n)/\mathcal{N}_{k,p}^\Omega}$, the proof of the proposition is complete. □

Next for $k \in \mathbb{N}, k \leq 2m$, let $\gamma_{k,p}^\dagger$ (resp. $\overline{\gamma}_{k,p}^\dagger$) denote the trace operator mapping $W_p^k(\Omega)$ (resp.$W_p^k(\mathbb{R}^n)$) onto $W_p^{k-1/p}(\Gamma)$ and let $\mathcal{N}_{k,p}$ (resp. $\overline{\mathcal{N}}_{k,p}$) denote its kernel. Then this operator induces a decomposition of $W_p^k(\Omega)$ (resp. $W_p^k(\mathbb{R}^n)$) into equivalent classes whereby any two distinct members of $W_p^k(\Omega)$ (resp. $W_p^k(\mathbb{R}^n)$), say u^1 and u^2 (resp. $\overline{u}^1$ and $\overline{u}^2$) are said to be equivalent if $u^1 - u^2 \in \mathcal{N}_{k,p}$ (resp. $\overline{u}^1 - \overline{u}^2 \in \overline{\mathcal{N}}_{k,p}$). Note that if we denote the induced quotient space by $W_p^k(\Omega)/\mathcal{N}_{k,p}$ (resp. $W_p^k(\mathbb{R}^n)/\overline{\mathcal{N}}_{k,p}$) and equip it with its quotient space norm, then it follows from [5, Propositions 2.2, 2.3] that $W_p^k(\Omega)/\mathcal{N}_{k,p}$ (resp $W_p^k(\mathbb{R}^n)/\overline{\mathcal{N}}_{k,p}$) is isomorphic to $W_p^{k-1/p}(\Gamma)$.

We now denote by $V_{k,p}$ (resp. $\overline{V}_{k,p}$) the operator mapping $W_p^k(\Omega)$ (resp. $W_p^k(\mathbb{R}^n)$) isometrically and isomorphically onto $W_p^{k,\chi}(\Omega)$ (resp. $W_p^{k,\chi}(\mathbb{R}^n)$) which is asserted in Proposition 2.12 (resp. Proposition 2.10) and put $N_{k,p} = V_{k,p}\mathcal{N}_{k,p}$ (resp. $\overline{N}_{k,p} = \overline{V}_{k,p}\overline{\mathcal{N}}_{k,p}$). Then the decomposition of $W_p^k(\Omega)$ (resp. $W_p^k(\mathbb{R}^n)$) into equivalent classes induces a decomposition of $W_p^{k,\chi}(\Omega)$ (resp. $W_p^{k,\chi}(\mathbb{R}^n)$) into equivalent classes whereby any two distinct members of $W_p^{k,\chi}(\Omega)$ (resp. $W_p^{k,\chi}(\mathbb{R}^n)$) , say u^1 and u^2 (resp. $\overline{u}^1$ and $\overline{u}^2$ are said to be equivalent if $u^1 - u^2 \in N_{k,p}$ (resp. $\overline{u}^1 - \overline{u}^2 \in \overline{N}_{k.p}$). We denote the induced equivalent space by $W_p^{k,\chi}(\Omega)/N_{k,p}$ (resp. $W_p^{k,\chi}(\mathbb{R}^n)/\overline{N}_{k,p}$) and equip it with its quotient space norm.

Proposition 2.13. *It is the case that the spaces* $W_p^{k,\chi}(\Omega)/N_{k,p}, W_p^{k,\chi}(\mathbb{R}^n)/\overline{N}_{k,p}$, *and* $W_p^{k-1/p}(\Gamma)$ *are isomorphic to each other.*

Proof. In light of what was shown above, it is clear that in order to prove the proposition we need only prove that $W_p^{k,\chi}(\Omega)/N_{k,p}$ (resp. $W_p^{k,\chi}(\mathbb{R}^n)/\overline{N}_{k,p}$) is isomorphic to $W_p^k(\Omega)/\mathcal{N}_{k,p}$ (resp.$W_p^k(\mathbb{R}^n)/\overline{\mathcal{N}}_{k.p}$). But for this, we can argue as we did in the proof of Proposition 2.12. □

Definition 2.14. We let $W_p^{k-1/p,\chi}(\Gamma) = W_p^{k,\chi}(\Omega)/N_{k,p}$ and denote the norm in this space by $\|\!| \cdot \|\!|_{k-1/p.p,\Gamma}^{\chi}$, where for $[u] \in W_p^{k-1/p,\chi}(\Gamma)$,

$$\|\!| [u] \|\!|_{k-1/p,p,\Gamma}^{\chi} = \|\!| [u] \|\!|_{W_p^{k,\chi}(\Omega)/N_{k,p}}.$$

Finally, in the sequel we denote by $\gamma_{k,p}^{\chi}$ the trace operator mapping $W_p^{k,\chi}(\Omega)$ onto $W_p^{k-1/p,\chi}(\Gamma) = W_p^{k,\chi}(\Omega)/N_{k,p}$. We shall also use the symbol $\gamma_{k,p}$ to denote the trace operator mapping $W_p^k(\Omega)$ onto $W_p^k(\Omega)/\mathcal{N}_{k,p}$.

3. The boundary problem (1.1), (1.2)

In this section we are going to use the results of Section 2 in order to establish our main results concerning the existence and uniqueness of solutions of the boundary problem (1.1), (1.2). To this end we require some further information.

Assumption 3.1. It will henceforth be supposed that
(1) the boundary Γ is of class $C^{2(m+(n-1)^+ + n^+)+1}$,
(2) $a_\alpha \in C^{2(m+(n-1)^+ + n^+)+1}(\overline{\Omega})$ for $|\alpha| \le 2m$, and
(3) $b_{j,\alpha} \in C^{2(m_j^\# + (n-1)^+ + n^+)+1}(\Gamma)$ for $|\alpha| \le m_j$, $j = 1, \dots, m$ where $m_j^\# = m - m_j/2$ if m_j is even and $m_j^\# = (2m - m_j)^+$ otherwise.

Remark 3.2. It follows from a standard extension procedure that there is no loss of generality in supposing henceforth that for each α and j, $a_\alpha \in C^{2(m+(n-1)^+ + n^+)+1}(\mathbb{R}^n)$, $b_{j,\alpha} \in C^{2(m_j^\# + (n-1)^+ + n^+)+1}(\mathbb{R}^n)$ and have compact support.

In the sequel we let $\mathring{A}(x, D)$ (resp., $\mathring{B}_j(x, D)$) denote the principal part of $A(x, D)$ (resp., $B_j(x, D)$, $j = 1, \dots, m$).

Definition 3.3. Let $\mathcal{L}$ be a closed sector in the complex plane with vertex at the origin. Then we say that the boundary problem (1.1), (1.2) is parameter-elliptic in $\mathcal{L}$ if the following two conditions are satisfied:
(1) $\mathring{A}(x, \xi) - \lambda \ne 0$ for $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$, and $\lambda \in \mathcal{L}$ if $|\xi| + |\lambda| \ne 0$;

(2) let x^0 be an arbitrary point in Γ. Assume that the boundary problem (1.1), (1.2) is rewritten in a local coordinate system associated with x^0 wherein $x^0 \to 0$ and $\nu \to e_n$, where ν denotes the interior normal to Γ at x^0 and $(e_1, \dots, e_n)$ denotes the standard basis in $\mathbb{R}^n$. Then the boundary problem on the half-line

$$\begin{aligned} \mathring{A}(0, \xi', D_n)v(t) - \lambda v(t) = 0 \quad & \text{for } t = x_n > 0, \\ \mathring{B}_j(0, \xi', D_n)v(t) = 0 \quad & \text{for } t = 0,\ j = 1, \dots, m, \\ v(t) \to 0 \quad & \text{as } t \to \infty \end{aligned}$$

has only the trivial solution for $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in \mathcal{L}$ if $|\xi'| + |\lambda| \neq 0$.

We denote by E the strong $\big(2(m + (n-1)^+ + n^+) + 1\big)$-extension operator mapping $W_p^{2(m+(n-1)^+ + n^+)+1}(\Omega)$ into $W_p^{2(m+(n-1)^+ + n^+)+1}(\mathbb{R}^n)$ (see [1, p. 83] for details) and for $v_{2m} \in W_p^{2m}(\Omega)$ let us put $v_{2m}^E = Ev_{2m}$ and $u_{2m}^E = \mathscr{F}^{-1}\chi^{-1}\mathscr{F}v_{2m}^E$. In the following proposition we denote transpose by $^\top$.

Proposition 3.4. *Suppose that the boundary problem* (1.1), (1.2) *is parameter-elliptic in $\mathcal{L}$. Suppose also that $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0 > 0$, $u \in W_p^{2m,\chi}(\Omega)$, and that f and $g = (g_1, \dots, g_m)^\top$ are defined by* (1.1) *and* (1.2)*, respectively. Then $f \in W_p^{0,\chi}(\Omega)$, $g_j \in W_p^{2m-m_j-\frac{1}{p},\chi}(\Gamma)$ for $j = 1, \dots, m$, and*

$$|||f|||_{0,p,\Omega}^{\chi} + \sum_{j=1}^{m} |||g_j|||_{2m-m_j-\frac{1}{p},p,\Gamma}^{\chi} \leq C|||u|||_{2m,p,\Omega}^{\chi},$$

where the constant C does not depend upon u and λ.

Remark 3.5. Proposition 3.4 requires some clarifications since we have not yet defined what we mean by $A(x, D)u$ and $B_j(x, D)u$ for $u \in W_p^{2m,\chi}(\Omega)$. Accordingly, in this paper we consider $A(x, D)$ (resp. $B_j(x, D)$, $j = 1, \dots, m$) as a pseudodifferential operator defined on $\mathbb{R}^n$ with a non-standard symbol $\sum_{|\alpha|\leq 2m} a_\alpha(x)\xi^\alpha$ (resp. $\sum_{|\alpha|\leq m_j} b_{j,\alpha}(x)\xi^\alpha$, $j = 1, \dots, m$). Then for $u \in W_p^{2m,\chi}(\mathbb{R}^n)$ we can appeal to Proposition 2.10 to show that $A(x, D)$ (resp. $B_j(x, D)$, $j = 1, \dots, m$) can be represented as a pseudodifferential operator defined on $\mathbb{R}^n$ acting on $v^E = \mathscr{F}^{-1}\chi\mathscr{F}u^E$. Thus it is by means of these pseudodifferential operators acting on classical Sobolev spaces and the results of Section 2 that enable us in the proof of the proposition to give meaning to the expressions $A(x, D)u = A(x, D)u^E|_\Omega$ (resp. $B_j(x, D)u = B_j(x, D)u^E|_\Omega$).

Proof of Proposition 3.4. In light of what was said in Section 2 we have $u = u_{2m} \in W_p^{2m,\chi}(\Omega)$ and we have to show firstly that $f = (A(x, D) - \lambda)u_{2m} \in W_p^{0,\chi}(\Omega)$, $B_j(x, D)u_{2m} \in W_p^{2m-m_j,\chi}(\Omega)$, $j = 1, \dots, m$, and then obtain estimates for $|||(A(x, D) - \lambda)u_{2m}|||_{0,p,\Omega}^{\chi}$ and for $|||[B_j(x, D)u_{2m}]|||_{2m-m_j-\frac{1}{p},p,\Gamma}^{\chi}$.

Accordingly, let us firstly fix our attention upon the operator $A(x, D) = \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha$ for $x \in \mathbb{R}^n$, and for a particular α obtain an estimate for

$$|||\mathscr{F}_{\xi\to x}^{-1}\chi(\xi)\mathscr{F}_{y\to\xi}a_\alpha(y)D_y^\alpha u_{2m}^E|||_{0,p,\mathbb{R}^n}.$$

To this end (see Remark 3.5) we consider the operator $a_\alpha(y)D_y^\alpha$ as a pseudodifferential operator defined on $\mathbb{R}^n$ with symbol $\sigma_\alpha(y, \xi) = a_\alpha(y)\xi^\alpha$. Then for our purposes we need to put $\sigma_\alpha(y, \xi)$ in x-form (see [11, p. 141]). To this end we can appeal to [11, p. 144] to show that in x-form $\sigma_\alpha(y,\xi)$ is given (as an oscillatory integral) by

$$\widetilde{\sigma}_\alpha(x,\xi) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iz\cdot\zeta}\sigma_\alpha(x-z,\xi-\zeta)dzd\zeta,$$

and hence by arguing as in [19, proof of Theorem 3.1, pp. 23-25] we have $\widetilde{\sigma}_\alpha(x,\xi) = a_\alpha(x)\xi^\alpha + \langle\xi\rangle^{2m-1}\sigma^1_\alpha(x,\xi)$, where

$$\sigma^1_\alpha(x,\xi) = \sum_{k=1}^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(x-z)\cdot\zeta}(1-\Delta_z)^{m+(n-1)^+ + n^+} D_{z,k}a_\alpha(z)\sigma^2_\alpha(\xi,\zeta)dzd\zeta,$$

$$\sigma^2_\alpha(\xi,\zeta) = \frac{i}{(2\pi)^n}\int_{t=0}^1 \langle\zeta\rangle^{-2(m+(n-1)^+ + n^+)}\langle\xi\rangle^{-(2m-1)}D_{\xi,k}(\xi-t\zeta)^\alpha dt,$$

and where $\cdot$ denotes the scalar product, Δ denotes the Laplacian over $\mathbb{R}^n$, $D_{z,k} = -i\frac{\partial}{\partial z_k}$, and $D_{\xi,k} = -i\frac{\partial}{\partial \xi_k}$. Thus we see that

$$\mathscr{F}^{-1}_{\xi\to x}\chi(\xi)\mathscr{F}_{y\to\xi}a_\alpha(y)D^\alpha_y u^E_{2m} = I^1_\alpha(x) + I^2_\alpha(x),$$

where $I^1_\alpha(x) = a_\alpha(x)D^\alpha v^E_{2m}(x)$, $I^2_\alpha(x) = \mathscr{F}^{-1}_{\xi\to x}\sigma^1_\alpha(x,\xi)\mathscr{F}_{y\to\xi}w(y)$, $w(y) = \mathscr{F}^{-1}_{\eta\to y}\langle\eta\rangle^{2m-1}\mathscr{F}_{z\to\eta}v^E_{2m}$, and $v^E_{2m} = \mathscr{F}^{-1}\chi\mathscr{F}u^E_{2m}$. It now follows that $\|I^1_\alpha(x)\|_{0,p,\mathbb{R}^n} \le C_1\|v^E_{2m}\|_{|\alpha|,p,\mathbb{R}^n}$, while it follows from a variant of Mikhlin's multiplier theorem (see [12, Theorem 1.6]) that $\|I^2_\alpha(x)\|_{0,p,\mathbb{R}^n} \le C_2\|v^E_{2m}\|_{2m-1,p,\mathbb{R}^n}$, where the constants C_j do not depend upon u_{2m}.

We conclude from these results that

$$\begin{aligned}\|(A(x,D)-\lambda)u_{2m}\|^\chi_{0,p,\Omega} &= \|(A(x,D)-\lambda)u^E_{2m}\big|_\Omega\|^\chi_{0,p,\Omega} \le \|(A(x,D)-\lambda)u^E_{2m}\|^\chi_{0,p,\mathbb{R}^n}\\ &\le C_3\|v^E_{2m}\|_{0,p,\mathbb{R}^n} \le C_4\|v_{2m}\|_{2m,p,\Omega} = C_4\|u_{2m}\|^\chi_{2m,p,\Omega},\end{aligned}$$

where the constants C_j do not depend upon u_{2m} and λ. This proves the assertion concerning f.

Suppose next that $1 \le j \le m$ and fix our attention upon the operator $B_j(x,D) = \sum_{|\alpha|\le m_j} b_{j,\alpha}(x)D^\alpha$. Then we are now going to show that $B_j(x,D)u^E_{2m} \in W^{2m-m_j,\chi}_p(\mathbb{R}^n)$ and obtain an estimate for its norm. To this end let us fix our attention upon the operator $b_{j,\alpha}(x)D^\alpha$ for a particular α. Then by arguing with $b_{j,\alpha}(x)D^\alpha$ as we did with $a_\alpha(x)D^\alpha$ above, we can show that

$$\mathscr{F}^{-1}_{\xi\to x}\chi(\xi)\mathscr{F}_{y\to\xi}b_{j,\alpha}(y)D^\alpha_y u^E_{2m} = I^1_{j,\alpha}(x) + I^2_{j,\alpha}(x),$$

where $I^1_{j,\alpha}(x) = b_{j,\alpha}(x)D^\alpha v_{2m}(x)$, $I^2_{j,\alpha}(x) = \mathscr{F}^{-1}_{\xi\to x}\langle\xi\rangle^{m_j-1}\sigma^1_{j,\alpha}(x,\xi)\mathscr{F}_{y\to\xi}w(y)$, $w(y) = \mathscr{F}^{-1}_{\eta\to y}\langle\eta\rangle^{m_j-1}\mathscr{F}_{z\to\eta}v^E_{2m}$,

$$\sigma^1_{j,\alpha}(x,\xi) = \sum_{k=1}^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(x-z)\cdot\zeta}(1-\Delta_z)^{m_j^\# + (n-1)^+ + n^+}D_{z,k}b_{j,\alpha}(z)\sigma^2_{j,\alpha}(\xi,\zeta)dzd\zeta,$$

and

$$\sigma^2_{j,\alpha}(\xi,\zeta) = \frac{i}{(2\pi)^n}\int_{t=0}^1 \langle\zeta\rangle^{-2(m_j^\# + (n-1)^+ + n^+)}\langle\xi\rangle^{-(m_j-1)}D_{\xi,k}(\xi-t\zeta)^\alpha dt.$$

It now follows that

$$\begin{aligned}\|I^1_{j,\alpha}(x)\|_{2m-m_j,p,\mathbb{R}^n} &\le C_5\|v^E_{2m}\|_{2m-m_j+|\alpha|,p,\mathbb{R}^n},\\ \|I^2_{j,\alpha}(x)\|_{2m-m_j,p,\mathbb{R}^n} &\le C_6\|v^E\|_{2m-1,p,\mathbb{R}^n},\end{aligned}$$

and hence that

$$\|B_j(x,D)u_{2m}^E\|^{\chi}_{2m-m_j,p,\mathbb{R}^n} \le C_7\|v_{2m}^E\|_{2m,p,\mathbb{R}^n} \le C_8\|u_{2m}\|^{\chi}_{2m,p,\Omega},$$

where the constants C_j do not depend upon u_{2m} and λ. On the other hand, we know from Section 2 and [5, Proposition 2.2] that $\|[B_j(x,D)u_{2m}]\|^{\chi}_{2m-m_j-1/p,p,\Gamma} \le C_9\|B_j(x,D)u_{2m}^E\|^{\chi}_{2m-m_j,p,\mathbb{R}^n}$, where the constant C_9 does not depend upon u and λ. In light of these results, the proof of the proposition is complete. □

A sort of converse to Proposition 3.4 is given by the following proposition.

Proposition 3.6. *Suppose that the boundary problem* (1.1), (1.2) *is parameter-elliptic in* $\mathcal{L}$. *Suppose also that* $u \in W_p^{2m,\chi}(\Omega)$ *and that* f *and* $g = (g_1,\dots,g_m)^\top$ *are defined by* (1.1) *and* (1.2), *respectively. Then* $f \in W_p^{0,\chi}(\Omega)$, $g_j \in W_p^{2m-m_j-1/p,\chi}(\Gamma)$ *for* $j = 1,\dots,m$, *and there exists a constant* $\lambda^0 = \lambda^0(p) > 0$ *such that for* $\lambda \in \mathcal{L}$ *with* $|\lambda| \ge \lambda^0$, *the a priori estimate*

$$\|u\|^{\chi}_{2m,p,\Omega} \le C\Big(\|f\|^{\chi}_{0,p,\Omega} + \sum_{j=1}^{m}\|g_j\|^{\chi}_{2m-m_j-1/p,p,\Gamma}\Big) \tag{3.1}$$

holds, where the constant C *does not depend upon* u *and* λ.

Proof. To begin with we assume that $\lambda \in \mathcal{L}$ with $|\lambda| \ge \lambda^\dagger$ for some $\lambda^\dagger > 0$. Then turning to the proof of Proposition 3.4, we know from that proof that we have $u = u_{2m} \in W_p^{2m,\chi}(\Omega)$, $f = (A(x,D)-\lambda)u_{2m} \in W_p^{0,\chi}(\Omega)$, and $g_j = [B_j(x,D)u_{2m}] \in W_p^{2m-m_j-1/p,\chi}(\Gamma)$. It was also shown there that with $v_{2m} = V_{2m,p}^{-1}u_{2m}$ (see the text preceding Proposition 2.13) we have

$$\mathscr{F}^{-1}_{\xi\to x}\chi(\xi)\mathscr{F}_{y\to\xi}\big(A(y,D_y)-\lambda\big)u_{2m}^E = \big(A(x,D)-\lambda\big)v_{2m}^E(x) + Qv_{2m}(x),$$
$$\mathscr{F}^{-1}_{\xi\to x}\chi(\xi)\mathscr{F}_{y\to\xi}B_j(y,D_y)u_{2m}^E = B_j(x,D)v_{2m}^E(x) + Q_jv_{2m}(x) \quad \text{for } j = 1,\dots,m,$$

where

$$Q = \mathscr{F}^{-1}_{\xi\to x}\sigma^1(x,\xi)\mathscr{F}_{y\to\xi}\widetilde{Q}, \qquad Q_j = \mathscr{F}^{-1}_{\xi\to x}\sigma^1_j(x,\xi)\mathscr{F}_{y\to\xi}\widetilde{Q}_j,$$
$$\sigma^1(x,\xi) = \sum_{|\alpha|\le 2m}\sigma^1_\alpha(x,\xi), \qquad \widetilde{Q}v_{2m} = \mathscr{F}_{\eta\to y}\langle\eta\rangle^{2m-1}\mathscr{F}_{z\to\eta}Ev_{2m},$$
$$\text{and } \sigma^1_j(x,\xi) = \sum_{|\alpha|\le m_j}\sigma^1_{j,\alpha}(x,\xi), \qquad \widetilde{Q}_jv_{2m} = \mathscr{F}^{-1}_{\eta\to y}\langle\eta\rangle^{m_j-1}\mathscr{F}_{z\to\eta}Ev_{2m}.$$

In addition, it follows from the proof of Proposition 3.4 and [5, Proposition 2.2] that

$$\|Qv_{2m}\|_{0,p,\mathbb{R}^n} + \sum_{j=1}^{m}\|Q_jv_{2m}\|_{2m-m_j,p,\mathbb{R}^n} \le C_1\|v_{2m}\|_{2m-1,p,\Omega}$$
$$\le C_2|\lambda|^{-1/(2m)}\|v\|_{2m,p,\Omega},$$

where the constants C_j do not depend upon u_{2m} and λ.

Next, let us put $u_0 = (A(x,D)-\lambda)u_{2m}$, $[u_{2m-m_j}] = [B_j(x,D)u_{2m}]$, $j = 1,\dots,m$, and denote by v_0 (resp. $[v_{2m-m_j}]$) the image of u_0 (resp. $[u_{2m-m_j}]$) under the isomorphic mapping of $W^{\chi}_{0,p}(\Omega)$ onto $W_{0,p}(\Omega)$ (resp. $W_p^{2m-m_j,\chi}(\Omega)/N_{2m-m_j,p}$ onto

$W_p^{2m-m_j}(\Omega)/\mathcal{N}_{2m-m_j,p})$. Let us also denote by $\mathcal{P}(x,D)-\lambda$ the operator mapping $W_p^{2m}(\Omega)$ into $W_p^0(\Omega)\times\prod_{j=1}^m W_p^{2m-m_j}(\Omega)/\mathcal{N}_{2m-m_j,p}$ defined by

$$\big(\mathcal{P}(x,D)-\lambda\big)v=\Big\{(A(x,D)-\lambda)v,\gamma_{2m-m_1,p}B_1(x,D)v,\dots,\gamma_{2m-m_m,p}B_m(x,D)v\Big\}$$

for $v\in W_p^{2m}(\Omega)$. Then writing $\mathcal{P}$ for the operator $\mathcal{P}(x,D)$ we have

$$(\mathcal{P}+\widetilde{\mathcal{P}}-\lambda)v_{2m}=(\mathcal{P}-\lambda)v_{2m}+\widetilde{\mathcal{P}}v_{2m}=\big\{v_0,[v_{2m-m_1}],\dots,[v_{2m-m_m}]\big\}, \tag{3.2}$$

where

$$\widetilde{\mathcal{P}}v_{2m}=\big\{Q_\Omega v_{2m},\gamma_{2m-m_1,p}(Q_{1,\Omega}v_{2m}),\dots,\gamma_{2m-m_m,p}(Q_{m,\Omega}v_{2m})\big\},$$

$Q_\Omega v_{2m}=Qv_{2m}\big|_\Omega$, and $Q_{j,\Omega}v_{2m}=Q_jv_{2m}\big|_\Omega$, $j=1,\dots,m$. It is important to observe from what was shown in the previous paragraphs and in the proof of Proposition 3.4 that

$$\|\widetilde{\mathcal{P}}v_{2m}\|_{W_p^{2m}(\Omega)\to W_p^0(\Omega)\times\prod_{j=1}^m W_p^{2m-m_j}(\Omega)/\mathcal{N}_{2m-m_j,p}}\le C_3|\lambda|^{-1/(2m)}\|v_{2m}\|_{2m,p,\Omega},$$

where the constant C_3 does not depend upon u_{2m} and λ. Furthermore, we know from [5, Theorem 2.1] that there exists a constant $\lambda_0=\lambda_0(p)>0$ such that the set $\{\lambda\in\mathcal{L}\big|\,|\lambda|\ge\lambda_0\}$ belongs to the resolvent set of $\mathcal{P}$. Hence if we suppose that λ belongs to this set and let $R(\lambda)$ denote the resolvent of $\mathcal{P}$, then the equation (3.2) can be written as

$$\big(I+\widetilde{\mathcal{P}}R(\lambda)\big)(\mathcal{P}-\lambda)v_{2m}=\big\{v_0,[v_{2m-m_1}],\dots,[v_{2m-m_m}]\big\}. \tag{3.3}$$

On the other hand we know from [5, Theorem 2.1] that

$$\|R(\lambda)\|_{\{L_p(\Omega),W_p^{2m-m_1}(\Omega)/\mathcal{N}_{2m-m_1,p},\dots,W_p^{2m-m_m}(\Omega)/\mathcal{N}_{2m-m_m,p}\}\to W_p^{2m}(\Omega)}\le C_4,$$

where the constant C_4 does not depend upon λ. Hence if we choose $\lambda^0\ge\lambda^\dagger$ large enough so that $|\lambda|^{-1/(2m)}C_3C_4\le\frac12$ for $\lambda\in\mathcal{L}$ with $|\lambda|\ge\lambda^0$, then for λ in this set $(I+\widetilde{\mathcal{P}}R(\lambda))$ is a bounded invertible operator on $L_p(\Omega)\times\prod_{j=1}^m W_p^{2m-m_j}(\Omega)/\mathcal{N}_{2m-m_j,p}$ and the norm of its inverse $(I+\widetilde{\mathcal{P}}R(\lambda))^{-1}$ is bounded by a constant not depending upon λ. Thus we can write the equation (3.3) in the form

$$(\mathcal{P}-\lambda)v_{2m}=\big(I+\widetilde{\mathcal{P}}R(\lambda)\big)^{-1}\big\{v_0,[v_{2m-m_1}],\dots,[v_{2m-m_m}]\big\}, \tag{3.4}$$

and hence it follows from (3.4) and [5, Theorem 2.1] that for $\lambda\in\mathcal{L}$ with $|\lambda|\ge\lambda^0$ we have

$$\|v_{2m}\|_{2m,p,\Omega}\le C_5\Big(\|v_0\|_{0,p,\Omega}+\sum_{j=1}^m\|[v_{2m-m_j}]\|_{2m-m_j-1/p,p,\Gamma}\Big), \tag{3.5}$$

where the constant C_5 does not depend upon u_{2m} and λ. The assertion of the proposition is an immediate consequence of this last result and the results of Section 2. □

We now turn to the question of necessity:

Proposition 3.7. *Suppose that for $\lambda\in\mathcal{L}$ with $|\lambda|\ge\lambda^0$ the a priori estimate* (3.1) *holds for every $u\in W_p^{2m,\chi}(\Omega)$, where $f=(A(x,D)-\lambda)u$, $g_j=[B_j(x,D)u]$ for $j=1,\dots,m$, and the constant C does not depend upon u and λ. Then the boundary problem* (1.1), (1.2) *is parameter-elliptic in $\mathcal{L}$.*

It is clear from the proofs of Proposition 3.4 and 3.6 that Proposition 3.7 will be proved if we can show that the following proposition holds.

Proposition 3.8. *Suppose that for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda^0$ the a priori estimate* (3.5) *holds for every $v_{2m} \in W_p^{2m}(\Omega)$, where $v_0 = (A(x,D) - \lambda + Q_\Omega)v_{2m}$, $v_{2m-m_j} = \gamma_{2m-m_j,p}(B_j(x,D) + Q_{j,\Omega})v_{2m}$ for $j = 1, \dots, m$, and the constant C_5 does not depend upon v_{2m} and λ. Then the boundary problem* (1.1), (1.2) *is parameter-elliptic in $\mathcal{L}$.*

In order to proof Proposition 3.8 we need the following lemma.

Lemma 3.9. *Let the hypotheses of Proposition 3.8 hold. Then:*

(1) For each point $x^0 \in \Omega$ there is a neighbourhood $U \subset \Omega$ of x^0 and a number $\lambda_1 > 0$ such that for every $v_{2m} \in W_p^{2m}(\Omega)$ with support contained in U the estimate

$$|\!|\!| v_{2m} |\!|\!|_{2m,p,\Omega} \leq c_1 \|(A(x^0,D) - \lambda)v_{2m}\|_{0,p,\Omega}$$

holds for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_1$, where the constant c_1 does not depend upon v_{2m} and λ.

(2) For each point $x^0 \in \Gamma$ there is a neighbourhood $U \subset \mathbb{R}^n$ of x^0 and a number $\lambda_2 > 0$ such that for every $v_{2m} \in W_p^{2m}(\Omega)$ with support contained in U the estimate

$$|\!|\!| v_{2m} |\!|\!|_{2m,p,\Omega} \leq$$
$$c_2\Big(\|(A(x^0,D) - \lambda)v_{2m}\|_{0,p,\Omega} + \sum_{j=1}^{m} |\!|\!| \widetilde{\gamma}_{2m-m_j,p}B_j(x^0,D)v_{2m} |\!|\!|_{2m-m_j-1/p,p,\Gamma}\Big)$$

holds for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_2$, where the constant c_2 does not depend upon v_{2m} and λ and $\widetilde{\gamma}_{2m-m_j,p}$ denotes the trace operator mapping $W_p^{2m-m_j}(\Omega)$ onto $W_p^{2m-m_j-1/p}(\Gamma)$ (see Proposition 2.13).

Proof. We know from the proofs of Proposition 3.4 and 3.6 that for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda^0$ and for $x^0 \in \Omega$ we have $\|Q_\Omega v_{2m}\|_{0,p,\Omega} \leq c_3|\lambda|^{-1/(2m)} |\!|\!| v_{2m} |\!|\!|_{2m,p,\Omega}$, while for $x^0 \in \Gamma$ we have

$$\|Q_\Omega v_{2m}\|_{0,p,\Omega} + \sum_{j=1}^{m} |\!|\!| \widetilde{\gamma}_{2m-m_j,p}Q_{j,\Omega}v_{2m} |\!|\!|_{2m-m_j-1/p,p,\Gamma} \leq c_3|\lambda|^{-1/(2m)} |\!|\!| v_{2m} |\!|\!|_{2m,p,\Omega},$$

where the constant c_3 does not depend upon v_{2m} and λ. Furthermore, for the same values of λ we can argue as in [7, proof of Lemma 4.2] and appeal to [5, Proposition 2.2] to show that for $x^0 \in \Omega$

$$\|(A(x,D) - A(x^0,D))v_{2m}\|_{0,p,\Omega} \leq c'd + c_4|\lambda|^{-1/(2m)} |\!|\!| v |\!|\!|_{2m,p,\Omega}, \tag{3.6}$$

while for $x^0 \in \Gamma$ we can likewise show that

$$\|(A(x,D) - A(x^0,D))v_{2m}\|_{0,p,\Omega}$$
$$+ \sum_{j=1}^{m} |\!|\!| \widetilde{\gamma}_{2m-m_j,p}(B_j(x,D) - B_j(x^0,D))v_{2m} |\!|\!|_{2m-m_j-1/p,p,\Gamma}$$

is bounded by the expression on the right side of (3.6), where d denotes the diameter of U and the constants c' and c_4 do not depend upon λ^0, v_{2m}, and λ. Hence by choosing d sufficiently small and λ sufficiently large, the assertion of the lemma follows immediately. □

Proof of Proposition 3.8. By appealing to Lemma 3.9 and [5, Proposition 2.2] we can argue as in [7, proof of Theorem 4.2] to establish the validity of the proposition. □

We now come to the main result of this section.

Theorem 3.10. *Suppose that the boundary problem* (1.1), (1.2) *is parameter-elliptic in* $\mathcal{L}$. *Then there exists a constant* $\lambda^0 = \lambda^0(p) > 0$ *such that for* $\lambda \in \mathcal{L}$ *with* $|\lambda| \geq \lambda^0$ *the boundary problem* (1.1), (1.2) *has a unique solution* $u \in W_p^{2m,\chi}(\Omega)$ *for every* $f \in W_p^{0,\chi}(\Omega)$ *and* $g = (g_1, \ldots, g_m)^\top$ *with* $g_j \in W_p^{2m-m_j-1/p,\chi}(\Gamma)$, *and the a priori estimate*

$$|||u|||^{\chi}_{2m,p,\Omega} \leq C\Big(|||f|||^{\chi}_{0,p,\Omega} + \sum_{j=1}^{m} |||g_j|||^{\chi}_{2m-m_j-1/p,p,\Gamma}\Big)$$

holds, where the constant C *does not depend upon* f, *the* g_j, *and* λ.

Proof. We set $u_0 := f$, and we know from Section 2 that $g_j \in [u_{2m-m_j}] \in W_p^{2m-m_j}(\Omega)/N_{2m-m_j,p}$ for some $u_{2m-m_j} \in W_p^{2m-m_j,\chi}(\Omega)$, $j = 1, \ldots, m$. Then referring to the proof of Proposition 3.6 for the terminology, let us now seek a solution of the equation

$$(\mathcal{P} - \lambda)u_{2m} = \big\{u_0, [u_{2m-m_1}], \ldots, [u_{2m-m_m}]\big\} \quad \text{for } u_{2m} \in W_p^{2m,\chi}(\Omega) \tag{3.7}$$

and for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda^0$, where λ^0 is the constant of Proposition 3.6. Accordingly, let v_{2m} (resp. v_0) denote the image of u_{2m} (resp. u_0) under the isomorphism mapping $W_p^{2m,\chi}(\Omega)$ onto $W_p^{2m}(\Omega)$ (resp. $W_p^{0,\chi}(\Omega)$ onto $W_p^{0}(\Omega)$) (see Proposition 2.12) and let $[v_{2m-m_j}]$ denote the image of $[u_{2m-m_j}]$ under the isomorphism mapping $W_p^{2m-m_j,\chi}(\Omega)/N_{2m-m_j,p}$ onto $W_p^{2m-m_j}(\Omega)/\mathcal{N}_{2m-m_j,p}$. Then we can argue as we did in the proof of Proposition 3.6 to show that the equation (3.7) has a solution u_{2m} if and only if the equation

$$(\mathcal{P} - \lambda)v_{2m} = (I + \widetilde{\mathcal{P}}R(\lambda))^{-1}\big\{v_0, [v_{2m-m_j}], \ldots, [v_{2m-m_m}]\big\} \tag{3.8}$$

has a solution $v_{2m} \in W_p^{2m}(\Omega)$, where all terms are defined in the proof of Proposition 3.6. Then again we can argue as we did in that proof to show indeed that the equation (3.8) has a unique solution v_{2m} such that the a priori estimate

$$|||v_{2m}|||_{2m,p,\Omega} \leq C\Big(\|v_0\|_{0,p,\Omega} + \sum_{j=1}^{m} |||[v_{2m-m_j}]|||_{2m-m_j-1/p,p,\Gamma}\Big)$$

holds, where the constant C does not depend upon v_0, the $[v_{2m-m_j}]$, and λ. All the assertions of the theorem follow immediately from these results and those of Section 2. □

4. Spectral Theory

In this section, we fix our attention upon the boundary problem (1.1), (1.2) under the assumption that in (1.2) the g_j are all zero, but with one exceptional case in Proposition 4.1 below. Then with $A^{\chi}_{B,p}$ denoting the Banach space operator induced by this boundary problem, with domain $D(A^{\chi}_{B,p}) \subset W_p^{2m,\chi}(\Omega)$, we are going to use the results of Sections 2 and 3 to show that $A^{\chi}_{B,p}$ has a compact resolvent and then derive various results pertaining to its spectral properties.

Accordingly, let $\mathcal{P}^\chi - \lambda$ denote the operator mapping $W_p^{2m,\chi}(\Omega)$ into $W_p^{0,\chi}(\Omega) \times \prod_{j=1}^m W_p^{2m-m_j,\chi}(\Omega)/N_{2m-m_j,p}$ defined by

$$(\mathcal{P}^\chi - \lambda)u = \left\{(A(x,D) - \lambda)u, \gamma^\chi_{2m-m_1,p} B_1(x,D)u, \dots, \gamma^\chi_{2m-m_m} B_m(x,D)u\right\}$$

for $u \in W_p^{2m,\chi}(\Omega)$. Also referring to the proof of Proposition 3.6 for terminology, let $\mathcal{P}$ (resp. $\mathcal{P} + \widetilde{\mathcal{P}}$) denote the operator that acts like $\mathcal{P}(x,D)$ (resp. $\mathcal{P}(x,D) + \widetilde{\mathcal{P}}$) with domain $W_p^{2m}(\Omega)$ and range in $L_p(\Omega) \times \prod_{j=1}^m W_p^{2m-m_j}(\Omega)/\mathcal{N}_{2m-m_j,p}$. Then we have shown in the proofs of Proposition 3.6 and Theorem 3.10 that the set $\left\{\lambda \in \mathcal{L} \,\middle|\, |\lambda| \geq \lambda^0\right\}$ is contained in the resolvent set of each of the operators $\mathcal{P}^\chi$, $\mathcal{P}$, and $\mathcal{P} + \widetilde{\mathcal{P}}$. Now turning for the moment to the general boundary problem (1.1), (1.2), that is, without the assumption that the g_j are all zero, we have the following result.

Proposition 4.1. *Let $\{f, g_1, \dots, g_m\} \in L_p(\Omega) \times \prod_{j=1}^m W_p^{2m-m_j}(\Omega)/\mathcal{N}_{2m-m_j,p}$, and for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda^0$ let v^1 (resp. v^2) denote the unique vector in $W_p^{2m}(\Omega)$ for which $(\mathcal{P} - \lambda)v^1 = \{f, g_1, \dots, g_m\}$ (resp. $(\mathcal{P} + \widetilde{\mathcal{P}} - \lambda)v^2 = \{f, g_1, \dots, g_m\}$. Then $\|v^1 - v^2\|_{2m,p,\Omega} \leq C|\lambda|^{-1/(2m)} \|v^2\|_{2m,p,\Omega}$, where the constant C does not depend upon f, the g_j, and λ.*

Proof. We have $(\mathcal{P} - \lambda)(v^1 - v^2) = -\{Q_\Omega v^2, [Q_{1,\Omega} v^2], \dots, [Q_{m,\Omega} v^2]\}$, and hence it follows from the proof of Proposition 3.6 (see also [5, Theorem 2.1]) that $\|v^1 - v^2\|_{2m,p,\Omega} \leq C|\lambda|^{-1/(2m)} \|v^2\|_{2m,p,\Omega}$, where the constant C is described above. This completes the proof of the Proposition. □

Next let $A^\chi_{B,p}$ denote the operator acting on $W_p^{0,\chi}(\Omega)$ induced by the restriction of the operator $\mathcal{P}^\chi$ to the set $\{u \in W_p^{2m,\chi}(\Omega) \,|\, [B_j u] = 0 \text{ for } j = 1, \dots, m\}$. Also let $A_{B,p}$ (resp. $\widetilde{A}_{B,p}$) denote the operator acting on $L_p(\Omega)$ induced by the restriction of the operator $\mathcal{P}$ (resp. $\mathcal{P} + \widetilde{\mathcal{P}}$) to the set $\{v \in W_p^{2m}(\Omega) \,|\, [B_j v] = 0,\ j = 1, \dots, m\}$ (resp. $\{v \in W_p^{2m}(\Omega) \,|\, [(B_j + Q_{j,\Omega})v] = 0 \text{ for } j = 1, \dots, m\}$). Then we know from the proofs of Proposition 3.6 and Theorem 3.10 that the set $\{\lambda \in \mathcal{L} \,|\, |\lambda| \geq \lambda^0\}$ belongs to the resolvent set of each of the operators $A^\chi_{B,p}$, $A_{B,p}$, and $\widetilde{A}_{B,p}$. Let $R^\chi_p(\lambda)$, $R_p(\lambda)$, and $\widetilde{R}_p(\lambda)$ denote the resolvents of $A^\chi_{B,p}$, $A_{B,p}$, and $\widetilde{A}_{B,p}$, respectively. Then we also know from the above proofs that for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda^0$

(4.1)

$$\|R^\chi_p(\lambda)\|_{W_p^{0,\chi}(\Omega) \to W_p^{2m,\chi}(\Omega)} + \|R_p(\lambda)\|_{L_p(\Omega) \to W_p^{2m}(\Omega)} + \|\widetilde{R}_p(\lambda)\|_{L_p(\Omega) \to W_p^{2m}(\Omega)} \leq C,$$

where C denotes a positive constant.

Proposition 4.2. *It is the case that $A^\chi_{B,p}$, $A_{B,p}$, and $\widetilde{A}_{B,p}$ have compact resolvents. Furthermore, μ is an eigenvalue of $A^\chi_{B,p}$ of algebraic multiplicity k if and only if μ is an eigenvalue of $\widetilde{A}_{B,p}$ of algebraic multiplicity k.*

Proof. Suppose that $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda^0$. Then it follows from (4.1), the fact that the embedding of $W_p^{2m}(\Omega)$ into $L_p(\Omega)$ is compact (see [1, p. 144]), and from the resolvent equation (see [14, p. 36]) that the first assertion of the proposition is true for $A_{B,p}$ and $\widetilde{A}_{B,p}$. That this is also true for the operator $A^\chi_{B,p}$ follows from the fact that $R^\chi_p(\lambda) = V_{2m,p} \circ \widetilde{R}_p(\lambda) \circ V_{0,p}^{-1}$, where we refer to the text preceding Proposition 2.13 for terminology.

Next we note that if λ_0 belongs to the resolvent set of $\widetilde{A}_{B,p}$, $T_p = (\widetilde{A}_{B,p} - \lambda_0)^{-1}$, and μ is an eigenvalue of $\widetilde{A}_{B,p}$, then $(\mu - \lambda_0)^{-1}$ is an eigenvalue of the compact operator T_p and the principal subspace of $\widetilde{A}_{B,p}$ corresponding to the eigenvalue μ has the same multiplicity as the principal subspace of T_p corresponding to the eigenvalue $(\mu - \lambda_0)^{-1}$. Thus the non-zero eigenvalues of $\widetilde{A}_{B,p}$ have finite algebraic multiplicities, and a similar remark holds for $A^{\chi}_{B,p}$.

Let $\overline{u}_{2m} \in D(A^{\chi}_{B,p})$, where $D(\cdot)$ denotes the domain, and for $\lambda \in \mathbb{C}$ let $(A^{\chi}_{B,p} - \lambda)\overline{u}_{2m} = \overline{u}_0 \in W^{0,\chi}_p(\Omega)$. Then we know that $\overline{v}_{2m} = V^{-1}_{2m,p}\overline{u}_{2m} \in D(\widetilde{A}_{B,p})$ and $(\widetilde{A}_{B,p} - \lambda)\overline{v}_{2m} = \overline{v}_0 = V^{-1}_{0,p}\overline{u}_0$. Hence if $\{u_j\}_{j=0}^{\ell-1}$ is a chain of length ℓ consisting of the eigenvector u_0 and the associated vectors u_j, $1 \le j \le \ell - 1$, corresponding to the eigenvalue μ of $A^{\chi}_{B,p}$, that is, $(A^{\chi}_{B,p} - \mu)u_0 = 0$, $(A^{\chi}_{B,p} - \mu)u_j = u_{j-1}$ for $j \ge 1$, and $(A^{\chi}_{B,p} - \mu)^{\ell} u_{\ell-1} = 0$, then it follows that $\{v_j\}_{j=0}^{\ell-1}$ is a chain of length ℓ consisting of the eigenvector v_0 and associated vectors v_j, $1 \le j \le \ell$, corresponding to the eigenvalue μ of $\widetilde{A}_{B,p}$, where $v_0 = V^{-1}_{2m,p}u_0$ and $v_j = V^{-1}_{2m,p}u_j$ for $j \ge 1$. Since a similar result holds if we interchange the roles of $A^{\chi}_{B,p}$ and $\widetilde{A}_{B,p}$, all the assertions of the proposition follow. □

As a consequence of Proposition 4.2, we are now able to present the main results of this section.

Theorem 4.3. *The eigenvalues as well as the principal vectors of $A^{\chi}_{B,p}$ corresponding to each such eigenvalue are the same for all p, $1 < p < \infty$.*

Proof. We know from [2] that the assertion is true when $A^{\chi}_{B,p}$ is replaced by$\widetilde{A}_{B,p}$. That the assertion is also true for $A^{\chi}_{B,p}$ follows from Proposition 4.2. □

Theorem 4.4. *Let $\{\mathcal{L}_k\}_{k=1}^{\ell}$ denote a family of distinct rays in the complex λ-plane which emanate from the origin and which divide the λ-plane into ℓ sectors. Suppose in addition that the boundary problem* (1.1), (1.2) *is parameter-elliptic along each of the rays $\mathcal{L}_k$ and that the angle between any two adjacent rays is less than $2m\pi/n$. Then $A^{\chi}_{B,p}$ has an infinite number of eigenvalues and the corresponding principal vectors are complete in $W^{0,\chi}_p(\Omega)$, $1 < p < \infty$.*

Proof. Since $D(A_{B,p})$ is dense in $L_p(\Omega)$, it follows from (3.5) and Proposition 4.1 that the same is true for $D(\widetilde{A}_{B,p})$. Hence we can argue as in [2, Proof of Theorem 3.2] to show that the theorem is true when $A^{\chi}_{B,p}$ is replaced by $\widetilde{A}_{B,p}$. That the assertion is true for $A^{\chi}_{B,p}$ follows from Propositions 2.12 and 4.2. □

In the following theorem we let $\mathcal{L}(\theta)$ denote the ray in the complex λ-plane emanating from the origin and making an angle θ with the positive real axis.

Theorem 4.5. *Suppose the boundary problem* (1.2), (1.2) *is parameter-elliptic along each of the rays $\mathcal{L}(\theta_1)$ and $\mathcal{L}(\theta_2)$, where $0 < \theta_2 - \theta_1 < \min\{2m\pi/n, 2\pi\}$, but not parameter-elliptic along the ray $\mathcal{L}(\theta_0)$, where $\theta_1 < \theta_0 < \theta_2$. Then the sector $\mathcal{L}^{\#}$ determined by the inequalities $\theta_1 \le \arg\lambda \le \theta_2$ contains an infinite number of eigenvalues of $A^{\chi}_{B,p}$.*

Proof. In light of Proposition 4.2 and Theorem 4.3, we need only prove the theorem for $\widetilde{A}_{B,p}$ in place of $A^{\chi}_{B,p}$ and for $p = 2$. Accordingly, we know from [2] that the assertion is certainly true for $A_{B,2}$ in place of $\widetilde{A}_{B,2}$. On the other hand if we suppose that the assertion is false for $\widetilde{A}_{B,2}$, then we also know from [2] that there exist positive constants $C^{\#}$ and $\lambda^{\#}$ such that the set $\mathcal{L}_1^{\#} = \{\lambda \in \mathcal{L}^{\#} \mid |\lambda| \geq \lambda^{\#}\}$ belongs to the resolvent set of $\widetilde{A}_{B,2}$ and for $\lambda \in \mathcal{L}_1^{\#}$ the inequality $|\lambda| \, \|\widetilde{R}_2(\lambda)\|_{L_2(\Omega)\to L_2(\Omega)} \leq C^{\#}$ holds. But since it follows from [5, Proposition 2.3] and Proposition 3.8 that the boundary problem (1.1), (1.2) is parameter-elliptic in $\mathcal{L}^{\#}$, we conclude from [5, Theorem 2.1] that there is a constant $\lambda_0 > 0$ such that the set $\{\lambda \in \mathcal{L}^{\#} \mid |\lambda| \geq \lambda_0\}$ belongs to the resolvent set of $A_{B,2}$ which is a contradiction. This completes the proof of the proposition. □

Let us now turn to the asymptotic behaviour of the eigenvalues of $A^{\chi}_{B,p}$. Accordingly for $0 < \theta < \pi$ let $\mathcal{L}_\theta$ denote the closed sector in the complex plane with vertex at the origin determined by the inequalities $\theta \leq |\arg \lambda| \leq \pi$. Then guided by future considerations we shall henceforth suppose that the sector $\mathcal{L}$ defined in the text following (1.2) coincides with $\mathcal{L}_\theta$ and that $\overline{\mathbb{R}_-}$ belongs to the resolvent set of $A^{\chi}_{B,p}$. We note that there is no loss of generality incurred by these assumptions since they can always be achieved by means of a rotation and a shift in the spectral parameter. Note also from Theorem 3.10 that there are at most a finite number of eigenvalues of $A^{\chi}_{B,p}$ contained in $\mathcal{L}_\theta$. Furthermore, we denote the eigenvalues of $A^{\chi}_{B,2}$ by $\{\lambda_j\}_{j\geq 1}$, where each eigenvalue is counted according to its algebraic multiplicity and arranged so that the $\{|\lambda_j|\}_{j\geq 1}$ form a non-decreasing sequence in $\mathbb{R}_+$. We note of course that the λ_j are the eigenvalues of $A^{\chi}_{B,p}$ and $\widetilde{A}_{B,p}$ for all p, $1 < p < \infty$.

For $t > 0$ let $N(t)$ denote the number of eigenvalues $\{\lambda_j\}_{j\geq 1}$ of $A^{\chi}_{B,p}$ for which $|\lambda_j| \leq t$.

Theorem 4.6. *Let the boundary problem* (1.1), (1.2) *be parameter-elliptic along every ray emanating from the origin in the complex plane except along* $\overline{\mathbb{R}_+}$. *Then*

$$N(t) = dt^{n/(2m)} + o(t^{n/(2m)}) \text{ as } t \to \infty, \text{ where } d = \frac{1}{(2\pi)^n} \int_\Omega dx \int_{\mathring{A}(x,\xi)<1} d\xi.$$

Before turning to the proof of Theorem 4.6, let us make the following observations. Firstly, under our assumptions we know that $\mathring{A}(x,\xi) \neq 0$ for $\xi \neq 0$. Secondly, it follows from Theorem 4.5 that $A^{\chi}_{B,p}$ has an infinite number of eigenvalues. Furthermore, it follows from Proposition 4.2 and Theorem 4.3 that we need only prove the theorem with $A^{\chi}_{B,p}$ replaced by $\widetilde{A}_{B,2}$. And in order to achieve this end we turn to the von Neumann-Schatten class of compact operators on $L_2(\Omega)$ (see [12, Chapters II and III]).

Let T be a compact operator on $L_2(\Omega)$. Then the non-zero eigenvalues $\{s_j(T)\}_{j\geq 1}$ of the non-negative operator $(T^*T)^{1/2}$, arranged so that $s_1(T) \geq s_2(T) \geq \ldots$, with each eigenvalue repeated according to its multiplicity, are called the singular values of T. For $0 < q < \infty$, we denote by $\mathscr{S}_q$ the class of compact operators T for which $\sum_{\ell\geq 1} s_\ell(T)^q < \infty$, and for $q \geq 1$ and $T \in \mathscr{S}_q$ we let $|T|_q = (\sum_{\ell \geq 1} s_\ell(T)^q)^{1/q}$. Note that $|\cdot|_q$ is a norm on $\mathscr{S}_q$ and with respect to this norm, $\mathscr{S}_q$ is a Banach space. Note also that if $q_1 < q_2$, then $\mathscr{S}_{q_1} \subset \mathscr{S}_{q_2}$. The class $\mathscr{S}_2$ are the Hilbert-Schmidt operators, that is the class of compact operators T which can be represented as an integral operator:

$$Tf(x) = \int_\Omega K(x,y) f(y) dy \quad \text{for } f \in L_2(\Omega), \tag{4.2}$$

where $K \in L_2(\Omega \times \Omega)$. The operators from $\mathscr{S}_1$ are the trace class operators, that is, they have the trace

$$\operatorname{tr} T = \sum_{j \geq 1} \lambda_j(T) \quad \text{for } T \in \mathscr{S}_1,$$

where $\{\lambda_j(T)\}_{j \geq 1}$ denote the non-zero eigenvalues of T, with each eigenvalue repeated according to its algebraic multiplicity and arranged so that their moduli form a non-increasing sequence in $\mathbb{R}_+$, and where the series converges absolutely. Furthermore, for $T \in \mathscr{S}_1$, we have

$$|\operatorname{tr} T| \leq |T|_1,$$

and T is an integral operator, with the kernel $K(x, y)$ in (4.2) being continuous in $\overline{\Omega} \times \overline{\Omega}$, and we also have $\operatorname{tr} T = \int_\Omega K(x, x)dx$. Note that if in Theorem 3.10 we suppose that $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda^0$, then it follows from that theorem and [5, Subsection 4.2] that for any $\epsilon > 0$ and $\ell \in \mathbb{N}$ we have

$$\widetilde{R}_2(\lambda) \in \mathscr{S}_{n(1+\epsilon)/(2m)} \quad \text{and } \widetilde{R}_2(\lambda)^\ell \in \mathscr{S}_{n(1+\epsilon)/(2m\ell)}. \tag{4.3}$$

Proposition 4.7. *If $2m > n$, then put $k = 1$, while if $2m \leq n$ let q denote the smallest even integer greater than $n/(2m)$ and put $k = q/2$. Then for $\lambda \in \mathcal{L}_\theta$ with $|\lambda| \geq \lambda^0$, $\widetilde{R}_2(\lambda)^k$ is a Hilbert-Schmidt operator such that in its integral representation its kernel $\widetilde{K}(x, y, \lambda)$ for $x \in \overline{\Omega}$, $\lambda \in \mathcal{L}_\theta$, the map $x \mapsto K(x, \cdot, \lambda)$, $\overline{\Omega} \to L_2(\Omega)$ is continuous for each $\lambda \in \mathcal{L}_\theta$ and*

$$\Big(\int_\Omega |\widetilde{K}(x, y, \lambda)|^2 dy \Big)^{1/2} \leq C_k |\lambda|^{\frac{n}{4m} - k}, \tag{4.4}$$

where the constant C_k does not depend upon x and λ.

Proof. To begin with, let us mention that the proposition has been proved in [3, Lemma 2.1, Theorem 5.1, and equation (7.7)] for the case $2m > n$ and in [5, Section 5] otherwise. However since we wish to refer to the proof of this proposition in the sequel, we shall give a brief outline of the proof given in [5]. Accordingly, we note from (4.3) that for $\lambda \in \mathcal{L}_\theta$ with $|\lambda| \geq \lambda^0$, $\widetilde{R}_2(\lambda)^k$ is a Hilbert-Schmidt operator on $L_2(\Omega)$ and its kernel is denoted by $\widetilde{K}(x, y, \lambda)$. We suppose henceforth that $\lambda \in \mathcal{L}_\theta$ with $|\lambda| \geq \lambda^0$. Then in order to prove the cited assertions, the following facts will be used: (1) if $2 < p < \infty$, then $\widetilde{R}_p(\lambda) = \widetilde{R}_2(\lambda)|_{L_p(\Omega)}$, and (2) if $1 < p < p_1$, $s \in \mathbb{N}$, and $0 < \tau < \frac{n}{s}(p^{-1} - p_1^{-1}) < 1$, then the embedding $W_p^s(\Omega) \to L_{p_1}(\Omega)$ is continuous and for $u \in W_p^s(\Omega)$ we have the estimate $\|u\|_{0,p_1,\Omega} \leq C_0 \|u\|_{s,p,\Omega}^\tau \|u\|_{0,p,\Omega}^{1-\tau}$, where the constant C_0 does not depend upon u. With these facts in mind, let us choose the numbers $\{p_j\}_{j=1}^k$ so that $2 = p_1 < p_2 < \ldots < p_k$, where $p_k > \frac{n}{2m}$ and $0 < \tau_j = \frac{n}{2m}(\frac{1}{p_j} - \frac{1}{p_{j+1}}) < 1$ for $j = 1, \ldots, k-1$.

Then we can write $\widetilde{R}_2(\lambda)$, considered as a mapping from $L_2(\Omega)$ into $W_2^{2m}(\Omega)$, as a product of operators $S_j \widetilde{R}_{p_j}(\lambda)$, $j = 1, \ldots, k-1$, where S_j is the embedding operator cited above mapping $W_p^{2m}(\Omega)$ into $L_{p_j}(\Omega)$:

$$L_{p_1}(\Omega) \overset{S_1 \widetilde{R}_{p_1}(\lambda)}{\longrightarrow} L_{p_2}(\Omega) \overset{S_2 \widetilde{R}_{p_2}(\lambda)}{\longrightarrow} \ldots$$
$$\ldots \longrightarrow L_{p_{k-1}}(\Omega) \overset{S_{k-1} \widetilde{R}_{p_{k-1}}(\lambda)}{\longrightarrow} L_{p_k}(\Omega) \overset{\widetilde{R}_{p_k}(\lambda)}{\longrightarrow} W_{p_k}^{2m}(\Omega).$$

It follows immediately from the embedding estimate cited above and (4.1) that

$$\|\widetilde{R}_2(\lambda)\|_{L_2(\Omega)\to W_p^{2m}(\Omega)} \le C|\lambda|^{\frac{n}{2m}-k},$$

where the constant C does not depend upon λ, and hence we can argue as in [3, Lemma 2.1] to establish all the assertions of the proposition. □

Proposition 4.8. *For $\lambda \in \mathcal{L}_\theta$ with $|\lambda| \ge \lambda^0$, $\widetilde{R}_2(\lambda)^q$ is an operator of trace class and*

$$\left|\operatorname{tr} \widetilde{R}_2(\lambda)^q\right| \le C|\lambda|^{\frac{n}{2m}-q},$$

where the constant C does not depend upon λ.

Proof. If we observe that $\widetilde{R}_2(\lambda)^q = \widetilde{R}_2(\lambda)^k \widetilde{R}_2(\lambda)^k$, then the assertion of the proposition is a immediate consequence of Proposition 4.7 and [4, Theorems 2.12, 2.18, and 2.19]. □

We now present a sharpening of Proposition 4.8.

Proposition 4.9. *It is the case that*

$$\operatorname{tr} \widetilde{R}_2(\lambda)^q = c_q(-\lambda)^{\frac{n}{2m}-q} + o\big(|\lambda|^{\frac{n}{2m}-q}\big) \text{ uniformly in } \mathcal{L}_\theta \text{ as } |\lambda| \to \infty,$$

where $c_q = \int_\Omega c_q(x)dx$, $c_q(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} \frac{d\xi}{(\mathring{A}(x,\xi)+1)^q}$, and where we assign to $\arg(-\lambda)$ its value in $[-\pi+\theta, \pi-\theta]$.

Proof. Supposing henceforth that $\lambda \in \mathcal{L}_\theta$ with $|\lambda| \ge \lambda^0$, we know from (4.1) and [5, Section 5] that $R_2(\lambda)^k$ is also a Hilbert-Schmidt operator, and if we let $K(x, y, \lambda)$ denote its associated kernel, then all the assertions of Proposition 4.7 hold in full force with $\widetilde{R}_2(\lambda)^k$ and $\widetilde{K}(x, y, \lambda)$ replaced by $R_2(\lambda)^k$ and $K(x, y, \lambda)$, respectively. Consequently Proposition 4.8 holds in full force with $\widetilde{R}_2(\lambda)$ replaced by $R_2(\lambda)$.

We are now going to obtain an estimate for $\operatorname{tr}(\widetilde{R}_2(\lambda)^q - R_2(\lambda)^q)$. To this end let us observe that with $q_1, q_2 \in \mathbb{N}_0$,

$$\begin{aligned}\widetilde{R}_2(\lambda)^q - R_2(\lambda)^q = \Big(\sum_{q_1+q_2=k-1} R_2(\lambda)^{q_1}(\widetilde{R}_2(\lambda) - R_2(\lambda))\widetilde{R}_2(\lambda)^{q_2}\Big)\widetilde{R}_2(\lambda)^k \\ + R_2(\lambda)^k\Big(\sum_{q_1+q_2=k-1} R_2(\lambda)^{q_1}(\widetilde{R}_2(\lambda) - R_2(\lambda))\widetilde{R}_2(\lambda)^{q_2}\Big).\end{aligned}$$

Hence if we fix our attention upon a fixed pair q_1, q_2 and appeal to Proposition 4.1 and [5, Section 5], then we can argue as we did in the proof of Proposition 4.7 to show that $R_2(\lambda)^{q_1}(\widetilde{R}_2(\lambda) - R_2(\lambda))\widetilde{R}_2(\lambda)^{q_2}$ is a Hilbert-Schmidt operator, and if we let $K^\dagger(x, y, \lambda)$ denote its associated kernel, then all the assertions of Proposition 4.7 with $\widetilde{R}_2(\lambda)$, $\widetilde{K}(x, y, \lambda)$, and $C_k|\lambda|^{\frac{n}{2m}-k}$ replaced by $R_2(\lambda)^{q_1}(\widetilde{R}_2(\lambda) - R_2(\lambda))\widetilde{R}_2(\lambda)^{q_2}$, $K^\dagger(x, y, \lambda)$, and $C_k|\lambda|^{-1/(2m)}|\lambda|^{n/(4m)-k}$, respectively. In light of this fact we can appeal to [5, Section 5] and argue as we did in the proof of Proposition 4.8 to show that

$$\operatorname{tr}\big(\widetilde{R}_2(\lambda)^q - R_2(\lambda)^q\big) \le C|\lambda|^{-\frac{1}{2m}}|\lambda|^{\frac{n}{2m}-q},$$

where the constant C does not depend upon λ. Since $\operatorname{tr}\widetilde{R}_2(\lambda)^q = \operatorname{tr}(R_2(\lambda)^q) + \operatorname{tr}(\widetilde{R}_2(\lambda)^q - R_2(\lambda)^q)$, the assertion of the proposition is an immediate consequence of the forgoing results and [5, Theorem 5.1]. □

Remark 4.10. Referring to Proposition 4.9 we note from [6] and [19, pp. 110-111] that c_q can also be written in the form

$$c_q = \frac{1}{n(2\pi)^n} b_{\frac{n}{2m},q} \int_\Omega dx \int_{|\eta|=1} \mathring{A}(x,\eta)^{-\frac{n}{2m}} d\eta,$$

as well as in the form

$$c_q = \frac{1}{(2\pi)^n} b_{\frac{n}{2m},q} \int_\Omega dx \int_{\mathring{A}(x,\xi)<1} d\xi,$$

where $b_{\frac{n}{2m},q} = \frac{n}{2m} B(\frac{n}{2m}, q - \frac{n}{2m})$ and $B(\cdot,\cdot)$ denotes the Beta function.

Proof of Theorem 4.6. As stated above we need only prove the theorem with $A^{\chi}_{B,p}$ replaced by $\widetilde{A}_{B,2}$. But the proof for $\widetilde{A}_{B,2}$ follows immediately from Proposition 4.9, Remark 4.10, and the arguments used in the proof of [5, Theorem 6.3]. □

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