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ASYMPTOTIC LIMITS IN MACROSCOPIC PLASMA MODELS

ANSGAR JÜNGEL*

Abstract. A model hierarchy of macroscopic equations for plasmas consisting of electrons and ions is presented. The model equations are derived from the transient Euler-Poisson system in the zero-relaxation-time, zero-electron-mass and quasineutral limits. These asymptotic limits are performed using entropy estimates and compactness arguments. The resulting limits equations are Euler systems with a nonlinear Poisson equation and nonlinear drift-diffusion equations.

Key words. Zero-relaxation-time limit, zero-electron-mass limit, quasineutral limit, entropy functional, weak solutions, compensated compactness, compactness by convexity, plasmas.

AMS(MOS) subject classifications. Primary 35K55, 35L60, 35B25, 82D10.

1. Introduction. For the description of physical phenomena in plasmas, fluid dynamical models like the hydrodynamic (or Euler-Poisson) equations are widely used. The numerical solution of the hydrodynamic models requires a lot of computing power and special algorithms. In some situations, however, the model equations can be approximated by simpler models in the sense that a small parameter appearing in the equations—for instance, the (scaled) momentum relaxation time, the electron mass, or the Debye length—is set equal to zero. Some of these formal limits are known by physicists, see, e.g., [2, 4, 13]. Then the natural question arises if the solution of the full model converges, as the parameter tends to zero, to a solution of the limit model. In this review we present recent results how these asymptotic limits can be made rigorously.

More specifically, we consider an unmagnetized plasma consisting of electrons with (scaled) mass m_e and charge $q_e = -1$ and of a single species of ions with mass m_i and charge $q_i = +1$. Denoting by $n_e = n_e(x,t)$, $j_e = j_e(x,t)$ (n_i, j_i, respectively) the scaled particle density and current density of the electrons (ions, respectively) and by $\phi = \phi(x,t)$ the electrostatic potential, these variables satisfy the following scaled Euler-Poisson system

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(HD-EI):

$$\partial_t n_\alpha + \operatorname{div} j_\alpha = 0, \tag{1.1}$$

$$m_\alpha \partial_t j_\alpha + m_\alpha \operatorname{div}\left(\frac{j_\alpha \otimes j_\alpha}{n_\alpha}\right) + \nabla p_\alpha(n_\alpha) = -q_\alpha n_\alpha \nabla\phi - m_\alpha \frac{j_\alpha}{\tau_\alpha}, \tag{1.2}$$

$$-\lambda^2 \Delta\phi = n_i - n_e, \tag{1.3}$$

where $\alpha = e, i$ and $(x,t) \in \mathbb{R}^d \times (0,\infty)$. Here, $j_\alpha \otimes j_\alpha$ denotes the tensor product with components $j_{\alpha k} j_{\alpha l}$ for $k, l = 1, \ldots, d$. The pressure functions are of the form

$$p_\alpha(n_\alpha) = a_\alpha n_\alpha^{\gamma_\alpha}, \quad n_\alpha \geq 0, \tag{1.4}$$

where $a_\alpha > 0$ and $\gamma_\alpha \geq 1$ are constants. The plasma is called *isothermal* if $\gamma_\alpha = 1$ and *adiabatic* if $\gamma_\alpha > 1$. The system (1.1)-(1.3) is complemented by initial conditions for n_α and j_α and by boundary conditions for ϕ ($\alpha = e, i$):

$$n_\alpha(x,0) = n_{\alpha 0}(x), \quad j_\alpha(x,0) = j_{\alpha 0}(x), \quad x \in \mathbb{R}^d, \tag{1.5}$$

$$\lim_{|x|\to\infty} \phi(x,t) = 0, \quad \text{a.e. } t > 0. \tag{1.6}$$

The homogeneous boundary condition for ϕ means that the plasma is in equilibrium at infinity.

The physical parameters are the (scaled) momentum relaxation time constants $\tau_e, \tau_i > 0$ of the electrons and ions, respectively, and the Debye length $\lambda > 0$. More precisely, these parameters are defined by

$$\tau_\alpha = \frac{\tau_\alpha^*}{\tau_0}, \quad \lambda = \sqrt{\frac{\varepsilon_0 k_B T_0}{q^2 L^2 N}}, \quad m_\alpha = \frac{m_\alpha^* v_0^2}{k_B T_0}, \tag{1.7}$$

where τ_α^* is the unscaled relaxation time, $\tau_0 = L/v_0$ a typical time scale, L a typical length, v_0 a typical velocity, ε_0 the permittivity constant, k_B the Planck constant, T_0 the ambient temperature, q the elementary charge, N a typical density and m_α^* the unscaled particle mass. We wish to perform rigorously the limits $\tau_\alpha \to 0$ for long time and small current density (i.e. we rescale $t \to t/\tau^2$ and $j_\alpha = \tau j_\alpha$ with $\tau = \tau_e = \tau_i \to 0$ and refer to this limit as the *zero-relaxation-time limit*) and the limits $m_e \to 0$ (*zero-electron-mass limit*) and $\lambda \to 0$ (*quasineutral limit*).

We now explain these asymptotic limits into more detail. Usually, the ions are heavy compared to the electrons, i.e. $m_i^* \gg m_e^*$. Therefore, if v_0^2 is equal to $k_B T_0/m_i^*$, we obtain (see (1.7))

$$m_i = 1, \quad m_e = \frac{m_e^*}{m_i^*} \ll 1.$$

Letting formally $m_e \to 0$ in Eq. (1.2) for $\alpha = e$, we get

$$0 = \nabla p_e(n_e) - n_e \nabla\phi = n_e \nabla(h_e(n_e) - \phi),$$

$$
\begin{array}{ccccccc}
\text{(QH-EI)} & \xleftarrow{\lambda\to 0} & \text{(HD-EI)} & \xrightarrow{m_e\to 0} & \text{(HD-I)} & \xrightarrow{\lambda\to 0} & \text{(QH-I)} \\
{}^{\prime}\tau_e=\tau_i\to 0^{\prime}\big\downarrow & & \big\downarrow{}^{\prime}\tau_e=\tau_i\to 0^{\prime} & & \big\downarrow{}^{\prime}\tau_i\to 0^{\prime} & & \big\downarrow{}^{\prime}\tau_i\to 0^{\prime} \\
\text{(QD-EI)} & \xleftarrow{\lambda\to 0} & \text{(DD-EI)} & \xrightarrow{m_e\to 0} & \text{(DD-I)} & \xrightarrow{\lambda\to 0} & \text{(QD-I)}
\end{array}
$$

FIG. 1. *A hierarchy of macroscopic plasma models. The limits $'\tau_\alpha \to 0'$ denote the zero-relaxation-time limits as explained in the text.*

where h_α ($\alpha = e, i$) is the enthalpy function defined by $h'_\alpha(s) = p'_\alpha(s)/s$ ($s > 0$), $h_\alpha(1) = 0$. Hence, if $n_e > 0$, we conclude $h_e(n_e) = \phi$ or, introducing the function $f_e = h_e^{-1}$, $n_e = f_e(\phi)$. The integration constant can be set equal to zero by choosing a reference point for the potential. Hence, the system (HD-EI) reduces in the zero-electron-mass limit to the model **(HD-I)** (see Fig. 1):

$$\partial_t n_i + \operatorname{div} j_i = 0, \tag{1.8}$$

$$m_i \partial_t j_i + m_i \operatorname{div}\left(\frac{j_i \otimes j_i}{n_i}\right) + \nabla p_i(n_i) = -n_i \nabla\phi - m_i \frac{J_i}{\tau_i}, \tag{1.9}$$

$$-\lambda^2 \Delta\phi = n_i - f_e(\phi). \tag{1.10}$$

Another set of equations is derived in the zero-relaxation-time limit in the models (HD-EI) and (HD-I). For this, introduce a scaling of time $s = t\tau$ (with $\tau = \tau_e = \tau_i$) and define

$$N_\alpha(x,s) = n_\alpha\Big(x, \frac{s}{\tau}\Big), \quad J_\alpha(x,s) = \frac{1}{\tau} j_\alpha\Big(x, \frac{s}{\tau}\Big), \tag{1.11}$$

$$\Phi(x,s) = \phi\Big(x, \frac{s}{\tau}\Big). \tag{1.12}$$

Setting again $t = s$, the model (HD-EI) becomes

$$\partial_t N_\alpha + \operatorname{div} J_\alpha = 0, \tag{1.13}$$

$$\tau^2 m_\alpha \partial_t J_\alpha + \tau^2 m_\alpha \operatorname{div}\left(\frac{J_\alpha \otimes J_\alpha}{N_\alpha}\right) + \nabla p_\alpha(N_\alpha) = -q_\alpha N_\alpha \nabla\Phi - m_\alpha J_\alpha, \tag{1.14}$$

$$-\lambda^2 \Delta\Phi = N_i - N_e. \tag{1.15}$$

Letting formally $\tau \to 0$, we obtain the drift-diffusion-model **(DD-EI)**:

$$m_\alpha \partial_t N_\alpha - \operatorname{div}(\nabla p_\alpha(N_\alpha) + q_\alpha N_\alpha \nabla\Phi) = 0, \tag{1.16}$$

$$-\lambda^2 \Delta\Phi = N_i - N_e, \tag{1.17}$$

where $\alpha = e, i$. Using the diffusion scaling (1.11)-(1.12) in the model (HD-I) and letting $\tau \to 0$, we obtain the model **(DD-I)**:

$$m_i \partial_t N_i - \operatorname{div}(\nabla p_i(N_i) + N_i \nabla \Phi) = 0, \tag{1.18}$$

$$-\lambda^2 \Delta \Phi = N_i - f_e(\Phi). \tag{1.19}$$

This model can also be derived from (DD-EI) in the limit $m_e \to 0$. Indeed, from (1.16) for $\alpha = e$ we obtain

$$0 = \operatorname{div}(\nabla p_e(N_e) - N_e \nabla \Phi) = \operatorname{div}(N_e \nabla (h_e(N_e) - \Phi)).$$

Usually, drift-diffusion equations are studied in bounded domains with mixed Dirichlet-Neumann boundary conditions (see Section 3). Hence, if $h_e(N_e) - \Phi = 0$ on a part of the domain boundary and $N_e > 0$ in the domain, we conclude $h_e(N_e) = \Phi$ or $N_e = f_e(\Phi)$ in the domain, which gives (1.19).

The quasineutral limit $\lambda \to 0$ in the model (HD-EI) implies $n \stackrel{\text{def}}{=} n_e = n_i$. Hence adding Eqs. (1.2) for $\alpha = e, i$, simplifying $\tau = \tau_e = \tau_i$ and introducing the center-of-mass current density $j = (m_e j_e + m_i j_i)/(m_e + m_i)$, we obtain the model **(QH-EI)**:

$$\partial_t n + \operatorname{div} j = 0, \quad \operatorname{div}(j_e - j_i) = 0, \tag{1.20}$$

$$\partial_t j + \operatorname{div}\left(\frac{j \otimes j}{n}\right) + \frac{m_e m_i}{(m_e + m_i)^2} \operatorname{div}\left(\frac{(j_e - j_i) \otimes (j_e - j_i)}{n}\right) + \frac{1}{m_e + m_i} \nabla(p_e(n) + p_i(n)) = -\frac{j}{\tau}. \tag{1.21}$$

Notice that the electric field $-\nabla \phi$ is eliminated in the equations. Under some assumptions, the above system can be simplified. Indeed, consider the equations in one space dimension in a bounded domain Ω with periodic boundary conditions and with no relaxation term, $\tau = +\infty$, and assume that the mean initial current densities for the electrons and ions are equal, $\int_\Omega j_e(x,0)dx = \int_\Omega j_i(x,0)dx$. Then from the second equation in (1.20) follows that $j_e - j_i$ is constant in space and from Eq. (1.21) we conclude after integration over Ω, that $j_\alpha(t)$ equals $\int_\Omega j_\alpha(x,0)dx/\operatorname{meas}(\Omega)$ and therefore, $j_e - j_i = 0$.

Performing the zero-relaxation-time limit in the above equations gives the diffusion model **(QD-EI)**:

$$\partial_t N - \frac{1}{m_e + m_i} \Delta(p_e(N) + p_i(N)) = 0, \tag{1.22}$$

where $N \stackrel{\text{def}}{=} N_e = N_i$. In the case of an adiabatic plasma (i.e. $p_\alpha(n) = a_\alpha n^{\gamma_\alpha}$ with $\gamma_\alpha > 1$ and $\alpha = e, i$), this equation is of degenerate type. This model is well known in plasma physics [4, p. 160]. Let us assume that the electrons and ions are described by the same density-pressure relation but with different diffusivities, which account for the different masses:

$$p_\alpha(s) = D_\alpha p(s), \quad \alpha = e, i.$$

Then Eq. (1.22) becomes

$$\partial_t N - D\Delta p(N) = 0 \quad \text{with} \quad D = \frac{D_e + D_i}{m_e + m_i}.$$

This means that the new diffusivity of the quasi-neutral plasma is given by the so-called *ambipolar* diffusion coefficient D [4, p. 160].

The model (QD-EI) can also be obtained directly from (DD-EI) after letting $\lambda \to 0$ and adding Eqs. (1.16) for $\alpha = e$ and $\alpha = i$.

The limit $\lambda \to 0$ in the system (HD-I) leads to $n_i = f_e(\phi)$ or $\phi = h_e(n_i)$ and the model **(QH-I)**

$$\partial_t n_i + \operatorname{div} j_i = 0, \tag{1.23}$$

$$m_i \partial_t j_i + m_i \operatorname{div}\left(\frac{j_i \otimes j_i}{n_i}\right) + \nabla(p_i(n_i) + p_e(n_i)) = -m_i \frac{j_i}{\tau_i}, \tag{1.24}$$

since $n_i \nabla\phi = n_i \nabla h_e(n_i) = \nabla p_e(n_i)$. These equations can be also found in the physical literature [2]. Performing the zero-relaxation-time limit in (1.23)-(1.24) we obtain the diffusion model **(QD-I)**:

$$\partial_t N_i - \frac{1}{m_i}\Delta(p_i(N_i) + p_e(N_i)) = 0. \tag{1.25}$$

Clearly, this model follows from (QD-EI) after letting $m_e \to 0$. Moreover, the quasineutral limit in (DD-I) also gives Eq. (1.25).

The (formal) limits of the above models are summarized in Fig. 1. Let us mention that there are other plasma models, e.g. magnetohydrodynamic and kinetic models [13] and nonlinear Poisson models [3], which can be derived from (DD-EI) or (DD-I) in the limit $m_e \to 0$ (see [28]).

In Section 2 we present mathematical results which make rigorously some of the above limits in the whole-space hydrodynamic models. Section 3 is devoted to the proofs of the asymptotic limits in the drift-diffusion equations in bounded domains. Finally, in Section 4, some open problems are mentioned.

2. Asymptotic limits in the hydrodynamic equations. In this section we prove rigorously the zero-relaxation-time, zero-electron-mass and quasineutral limits in the models (HD-EI), (HD-I), respectively.

2.1. Zero-relaxation-time limits. We consider the zero-relaxation-time limits in the hydrodynamic models (HD-EI) and (HD-I) in the one-dimensional case $d = 1$. We study only this case since an existence theory is available only for $d = 1$. Our main goal is to give rigorous proofs of the limits (HD-EI)→(DD-EI) and (HD-I)→(DD-I). Under the assumption (1.4) for $\gamma_\alpha > 1$ and

$$0 \le n_{\alpha_0},\ \frac{j_{\alpha_0}}{n_{\alpha_0}} \in L^\infty(\mathbb{R}) \quad \text{and} \quad n_{\alpha_0}(x) = 0 \quad \text{for } |x| \ge L, \tag{2.1}$$

where $\alpha = e, i$ and $L > 0$ is a given constant, there exists a global weak entropy solution $(N_\tau, J_\tau, \Phi_\tau)$ of the rescaled problem (HD-I)

$$\partial_t N_\tau + \partial_x J_\tau = 0, \tag{2.2}$$

$$\tau^2 m_i \left(\partial_t J_\tau + \partial_x \left(\frac{J_\tau^2}{N_\tau} \right) \right) + \partial_x p_i(N_\tau) = -N_\tau \partial_x \Phi_\tau - J_\tau, \tag{2.3}$$

$$-\partial_x^2 \Phi_\tau = N_\tau - f_e(\Phi_\tau), \tag{2.4}$$

for $x \in \mathbb{R}$, $t > 0$, with the initial and boundary conditions

$$N_\tau(\cdot, 0) = n_{i0}, \quad J_\tau(\cdot, 0) = \tau^{-1} j_{i0} \quad \text{in } \mathbb{R}, \tag{2.5}$$

$$\lim_{|x|\to\infty} \Phi_\tau(x,t) = 0, \quad \text{a.e. } t > 0. \tag{2.6}$$

This solution satisfies

$$0 \le N_\tau,\ \frac{J_\tau}{N_\tau} \in L^\infty(\mathbb{R} \times \mathbb{R}^+), \quad 0 \le \Phi_\tau \in L^\infty(0, T, W^{2,\infty}(\mathbb{R})),$$

where $T > 0$ [12, 38]. For the existence of weak entropy solutions $(N_{\alpha\tau}, J_{\alpha\tau}, \Phi_\tau)$ to (1.13)-(1.15) with initial and boundary conditions corresponding to (2.5)-(2.6), we refer to [7, 32, 39]. Then it holds:

THEOREM 2.1 ((HD-I)→(DD-I)). *Let the assumptions (1.4) for $\gamma_\alpha > 1$ ($\alpha = e, i$) and (2.1) hold. Then, as $\tau \to 0$, maybe passing to subsequences, $(N_\tau, J_\tau, \Phi_\tau)$ converges to (N, J, Φ), which is a solution of (1.18)-(1.19), in the following sense:*

$$\begin{aligned} N_\tau &\to N && \textit{strongly in } L^p_{\text{loc}}(\mathbb{R} \times (0,T)) \textit{ for any } p \in (1, \infty), \\ J_\tau &\rightharpoonup J && \textit{weakly in } L^2(\mathbb{R} \times (0,T)), \\ \tau^2 J_\tau^2 / N_\tau &\to 0 && \textit{strongly in } L^2(\mathbb{R} \times (0,T)), \\ \Phi_\tau &\to \Phi && \textit{strongly in } L^q_{\text{loc}}(\mathbb{R} \times (0,T)) \textit{ for any } q \in [\gamma_e/(\gamma_e - 1), \infty). \end{aligned}$$

The solution (N, J, Φ) satisfies

$$N \in L^\infty(0,T; L^p(\mathbb{R})),\ J \in L^2(\mathbb{R} \times (0,T)),\ \Phi \in L^\infty(0,T; W^{2,r}(\mathbb{R})),$$

where $p \in (1, \infty)$ and $r \in [\max(\gamma_e/(\gamma_e - 1), 2), \infty)$.

THEOREM 2.2 ((HD-EI)→(DD-EI)). *Let the assumptions of Theorem 2.1 hold. Then, as $\tau \to 0$, maybe passing to subsequences, $(N_{\alpha\tau}, J_{\alpha\tau}, \Phi_\tau)$ converges to $(N_\alpha, J_\alpha, \Phi)$ which is a solution of (1.16)-(1.17), in the sense* *of Theorem 2.1, where* N_τ, J_τ, N, J *are replaced by* $N_{\alpha\tau}, J_{\alpha\tau}, N_\alpha, J_\alpha$*, respectively.*

The proofs of the above theorems are based on high-energy (or entropy) estimates. More precisely, choosing special convex entropies in the sense of Lions-Perthame-Tadmor [30] and deriving the corresponding entropy inequalities, it is possible to prove that, for instance, $(N_\tau)_\tau$ is bounded in $L^\infty(0,T;L^p(\mathbb{R}))$ for any $p < \infty$ and $(J_\tau)_\tau$ is bounded in $L^2(\mathbb{R}\times(0,T))$. In order to get strong convergence of subsequences of $(N_\tau)_\tau$ and $(\Phi_\tau)_\tau$ we are using the div-curl lemma [42] and the Poisson equation. We refer to [26, 28] for the details and the proofs. Notice that not necessarily the whole sequence $(N_\tau, J_\tau, \Phi_\tau)$ converges since there is no general uniqueness result for the limiting problem (DD-I) or (DD-EI). (The uniqueness question of these models is addressed to in, e.g. [14, 24].)

The relaxation limit for *isothermal* plasmas has been carried out by Junca and Rascle in [20, 21]. They consider a plasma consisting only of ions. The proof of their result is based on an entropy inequality and the de la Vallée-Poisson lemma. Notice that here, L^1 weak convergence of the particle density is enough to pass to the limit in the linear pressure term.

The first mathematical relaxation limit for adiabatic plasmas has been shown by Marcati and Natalini [31, 36] assuming uniform L^∞ estimates. Let us mention some related results. In [5] the relaxation limit has been performed for adiabatic pressure and small smooth initial data. The same authors have also obtained a similar result for the so-called energy hydrodynamic model which includes an equation for the energy [6]. This result has been generalized by Ali et al. [1] for initial data that is a perturbation of the stationary solution of the thermal equilibrium state. A relaxation-time limit for a boundary-value problem has been performed in [19]. The steady-state problem has been addressed to in [33]. At our knowledge, no mathematical results are available for the relaxation limit (QH-EI)→(QD-EI). The limit (QH-I)→(QD-I), however, can be performed using the techniques of the proof of Theorem 2.1. In fact, the proof becomes easier since the electric term does not appear.

2.2. Zero-electron-mass limits. The limit $m_e \to 0$ in the hydrodynamic model is only solved in the one-dimensional case under additional assumptions [18].

THEOREM 2.3 ((HD-EI)→(HD-I)). *Let the assumptions (1.4) and (2.1) hold and set* $\delta = m_e$*. Let* $(n_{\alpha\delta}, j_{\alpha\delta}, E_\delta)$ *with* $E_\delta = -\partial_x\phi_\delta$ *be a weak entropy solution to (1.1)-(1.6). Assume furthermore that the sequences* $(n_{e\delta})_\delta$ *and* $(j_{e\delta}^2/n_{e\delta})_\delta$ *are bounded in* $L^{2\gamma_e}_{\text{loc}}(\mathbb{R}\times(0,T))$ *and* $L^2_{\text{loc}}(\mathbb{R}\times(0,T))$*, respectively. Then there is a subsequence of* $(n_{\alpha\delta}, j_{\alpha\delta}, E_\delta)$*, not relabeled,* *converging, as $\delta \to 0$, to (n_α, j_α, E) in the following sense:*

$$\begin{aligned} n_{i\delta} \to n_i,\ j_{i\delta} \to j_i &\quad \text{strongly in } L^p_{\rm loc}(\mathbb{R}\times(0,T)) \text{ for any } p\in[1,\infty), \\ n_{e\delta} \to n_e &\quad \text{strongly in } L^p_{\rm loc}(\mathbb{R}\times(0,T)) \text{ for any } p\in[1,\gamma_e+1], \\ j_{e\delta} \rightharpoonup j_e &\quad \text{weakly in } L^2(\mathbb{R}\times(0,T)) \\ E_\delta \rightharpoonup E &\quad \text{weakly* in } L^\infty(\mathbb{R}\times(0,T)), \end{aligned}$$

and (n_α, j_α, E) is a weak solution of (1.8)-(1.9) for $\alpha = i$ and

$$\lambda^2\partial_x E = n_i - n_e, \quad \partial_x p_e(n_e) = -n_e E \quad \text{in } \mathbb{R}\times(0,T). \tag{2.7}$$

Notice that if $n_e > 0$ in $\mathbb{R}\times(0,T)$ then we can eliminate n_e in Eq. (2.7) and obtain the nonlinear Poisson equation (1.10) with $E = -\partial_x\phi$. For the proof of Theorem 2.3, we use the div-curl lemma and the monotonicity of p_e to conclude the strong convergence of $(n_{e\delta})_\delta$. The weak convergence of $(j_{e\delta})_\delta$ is a consequence of our assumptions. Finally the strong convergence of the sequences $(n_{i\delta})_\delta$ and $(j_{i\delta})_\delta$ follows from the $H^{-1}_{\rm loc}$ compactness technique of [29].

2.3. Quasineutral limits. The quasineutral limit has been proved for strong solutions of (HD-I) or (HD-EI) locally in time [10, 11]. First we consider the limit in (HD-I).

Let the assumption (1.4) for $\gamma_e, \gamma_i \geq 1$ hold and let $m_i = 1$, $\tau_i = \infty$. Define the ion mean velocity by $u_i = j_i/n_i$ and let $(n_{i\lambda}, u_{i\lambda}, \phi_\lambda)$ be a solution of

$$\partial_t n_{i\lambda} + \partial_x(n_{i\lambda}u_{i\lambda}) = 0, \tag{2.8}$$

$$\partial_t u_{i\lambda} + u_{i\lambda}\partial_x u_{i\lambda} + \frac{1}{n_{i\lambda}}\partial_x p_i(n_{i\lambda}) = -\partial_x\phi_\lambda, \tag{2.9}$$

$$-\lambda^2\partial_x^2\phi_\lambda = n_{i\lambda} - f_e(\phi_\lambda) \quad \text{in } \mathbb{R}\times(0,\infty), \tag{2.10}$$

$$n_{i\lambda}(\cdot,0) = n_{i0},\ u_{i\lambda}(\cdot,0) = u_{i0} \quad \text{in } \mathbb{R}. \tag{2.11}$$

Assume that the plasma is uniform and electrically neutral at infinity, i.e. let ρ be a smooth, strictly positive function, constant outside of $[-1,1]$, and tending to $\rho_\pm$ as $x \to \pm\infty$, let $\lim_{|x|\to\infty}\phi(x,t) = 0$, a.e. $t > 0$ and let n_{i0}, u_{i0} satisfy

$$n_{i0} - \rho,\ u_{i0} \in H^s(\mathbb{R}), \quad n_{i0} \geq \underline{n} > 0 \quad \text{for some } s > \tfrac{3}{2} \text{ and } \underline{n} > 0.$$

If $\lambda = 0$ we obtain the following system:

$$\partial_t n_i + \partial_x(n_i u_i) = 0, \tag{2.12}$$

$$\partial_t u_i + u_i\partial_x u_i + \frac{1}{n_i}(p_e'(n_i) + p_i'(n_i))\partial_x n_i = 0. \tag{2.13}$$

THEOREM 2.4 ((HD-I)→(QH-I)). *Let the above assumptions hold and let* (n_i, u_i) *be a strong solution to (2.12)-(2.13) with initial conditions corresponding to (2.11), on the time interval* $[0, T]$, $T > 0$. *Then there exist solutions* $(n_{i\lambda},\ u_{i\lambda})$ *to (2.8)-(2.11) existing at least on* $[0, T]$. *Moreover, for any* $T' < T$ *there exist constants* $c > 0$, $s' < s$ *such that*

$$\|n_{i\lambda} - n_i\|_{L^\infty(0,T';H^{s'}(\mathbb{R}))} + \|u_{i\lambda} - u_i\|_{L^\infty(0,T';H^{s'}(\mathbb{R}))} \le c\lambda,$$

where $c > 0$ *is independent of* λ.

The existence of strong local-in-time solutions to the limit problem (2.12)-(2.13) follows from classical results (see [11]). In order to prove Theorem 2.4, the system (2.12)-(2.13) is written equivalently as a system for the functions $\bar{n}$ and $\bar{u}$ which are the first-order expansions of the solutions in powers of λ, i.e. $n_{i\lambda} = n_i + \lambda\bar{n}$ and $u_{i\lambda} = u_i + \lambda\bar{u}$. Then it is shown in [11], using pseudodifferential techniques and energy estimates, that $(\bar{n}, \bar{u})$ are bounded in $H^{s'}(\mathbb{R})$ uniformly in λ.

Let us consider now a plasma described by (HD-EI) in the one-dimensional torus $\mathbb{T}$ and assume the condition (1.4) for $\gamma_e = \gamma_i = 1$. Furthermore, let $m_e = m_i = 1$ and $\tau_e = \tau_i = \infty$. Define similarly as above the mean velocities $u_\alpha = j_\alpha/n_\alpha$, $\alpha = e, i$, and let $(n_{\alpha\lambda}, u_{\alpha\lambda}, \phi_\lambda)$ be a solution of $(\alpha = e, i)$

$$\partial_t n_{\alpha\lambda} + \partial_x(n_{\alpha\lambda}u_{\alpha\lambda}) = 0, \tag{2.14}$$

$$\partial_t(n_{\alpha\lambda}u_{\alpha\lambda}) + \partial_x(n_{\alpha\lambda}u_{\alpha\lambda}^2 + a_\alpha n_{\alpha\lambda}) = -q_\alpha n_{\alpha\lambda}\partial_x\phi_\lambda, \tag{2.15}$$

$$-\lambda^2\partial_x^2\phi_\lambda = n_{i\lambda} - n_{e\lambda} \quad \text{in } \mathbb{T}\times(0,\infty), \tag{2.16}$$

$$n_{\alpha\lambda}(\cdot, 0) = n_{\alpha 0},\ u_{\alpha\lambda}(\cdot, 0) = u_{\alpha 0} \quad \text{in } \mathbb{T}. \tag{2.17}$$

We assume that

$$n_{e0} = n_{i0}, \quad u_{e0} = u_{i0} \quad \text{in } \mathbb{T}. \tag{2.18}$$

In order to derive the limit problem, let $\lambda = 0$ in Eqs. (2.14)-(2.17) and set $n \stackrel{\text{def}}{=} n_e = n_i$. Then, by Eq. (2.14), $j \stackrel{\text{def}}{=} n_i u_i - n_e u_e = n(u_i - u_e)$ is constant independently of $x \in \mathbb{T}$. Integration of Eq. (2.15) for $\lambda = 0$ in the form

$$\partial_t u_\alpha + \partial_x(\tfrac{1}{2}u_\alpha^2 + a_\alpha \log n_\alpha) = -q_\alpha\partial_x\phi$$

over $\mathbb{T}$ yields

$$\int_{\mathbb{T}} u_\alpha(x,t)dx = \int_{\mathbb{T}} u_{\alpha 0}(x)dx.$$

Hence, by assumption (2.18),

$$j\int_{\mathbb{T}}\frac{dx}{n} = \int_{\mathbb{T}}(u_i - u_e)dx = \int_{\mathbb{T}}(u_{i0} - u_{e0})dx = 0,$$

which gives $j = 0$. Therefore, adding Eqs. (2.15) for $\alpha = e, i$ to eliminate ϕ and setting $u \overset{\text{def}}{=} (m_e u_e + m_i u_i)/(m_e + m_i)$, we obtain

$$\partial_t n + \partial_x (nu) = 0, \tag{2.19}$$

$$\partial_t (nu) + \partial_x (nu^2 + (a_e + a_i) n) = 0. \tag{2.20}$$

In [10] the following theorem is shown by using pseudodifferential techniques:

THEOREM 2.5 ((HD-EI)→(QH-EI)). *Let $n_{\alpha 0}$, $u_{\alpha 0} \in H^s(\mathbb{T})$ for sufficiently large $s > 0$ and assume (2.18). Then there exists $T > 0$ such that, for $\lambda > 0$ small enough, there are solutions $(n_{\alpha\lambda}, u_{\alpha\lambda})$ to (2.14)-(2.17) in $L^\infty(0, T; H^{s'}(\mathbb{T}))$ with $s' < s$ such that, as $\lambda \to 0$,*

$$n_{\alpha\lambda} \to n \quad \text{in } L^\infty(0, T; H^{s'}(\mathbb{T})),$$
$$u_{\alpha\lambda} \to u \quad \text{in } L^\infty(0, T; H^{s'}(\mathbb{T})),$$

and (n, u) solves Eqs. (2.19)-(2.20).

There are only a few further results in the mathematical literature. Gasser and Marcati [17] proved an interesting result for a *combined* relaxation and quasineutral limit in (HD-EI) for weak entropy solutions. Indeed, assuming the diffusion scaling $t \to t/\lambda$, $j_i \to \lambda j_i$ and the relation $\lambda = \tau_i^\beta$ with $0 < \beta < 2(\gamma_i - 2)/3(\gamma_i - 1)$ and $\gamma_e = \gamma_i > 2$, they have shown that the solution $(n_{\alpha\lambda}, j_{\alpha\lambda}, \lambda^2 \partial_x \phi_\lambda)$ of (HD-EI) converges in appropriate Lebesgue spaces to a solution of a pure diffusion model for the limit particle density. This model can be written equivalently as an inhomogenous Burgers equation for the limit function of $(\lambda^2 \partial_x \phi_\lambda)_\lambda$.

Furthermore, traveling wave solutions and jump relations in the quasineutral limit have been studied by Cordier et al. [8, 9]. Slemrod has proved the limit $\lambda \to 0$ in the one-dimensional *steady-state* hydrodynamic equations [41].

3. Asymptotic limits in drift-diffusion models. The limits $m_e \to 0$ and $\lambda \to 0$ in the drift-diffusion models are studied in bounded domains. For this, we assume that assumption (1.4) holds and that

$$\begin{aligned} &\Omega \subset \mathbb{R}^d \ (d \geq 1) \text{ is a bounded domain with Lipschitzian} \\ &\text{boundary } \partial\Omega = \Gamma_D \cup \Gamma_N,\ \Gamma_D \cap \Gamma_N = \emptyset,\ \text{meas}_{d-1}(\Gamma_D) > 0, \\ &\text{and } \Gamma_N \text{ is open in } \partial\Omega,\ T > 0. \end{aligned} \tag{3.1}$$

We consider the model (DD-EI) subject to the initial and boundary conditions

$$N_\alpha = N_{\alpha D},\ \Phi = \Phi_D \quad \text{on } \Gamma_D \times (0, T), \tag{3.2}$$

$$\nabla p_\alpha (N_\alpha) \cdot \nu = \nabla \Phi \cdot \nu = 0 \quad \text{on } \Gamma_N \times (0, T), \tag{3.3}$$

$$N_\alpha(\cdot, 0) = N_{\alpha 0} \quad \text{in } \Omega,\ \alpha = e, i, \tag{3.4}$$

where ν is the exterior unit normal vector to $\partial\Omega$. Analogous initial and boundary conditions are imposed for the model (DD-I). We assume furthermore that

$$\begin{aligned} &0 \le N_{\alpha D} \in C^0([0,T]; L^\infty(\Omega)) \cap H^1(\Omega \times (0,T)), \\ &0 \le N_{\alpha 0} \in L^\infty(\Omega), \quad \alpha = e, i, \\ &\Phi_D \in L^\infty(\Omega \times (0,T)) \cap H^1(0,T; H^1(\Omega)). \end{aligned} \tag{3.5}$$

Under these assumptions, there exists a global weak solution (N_α, Φ) to (1.16)-(1.17), (3.2)-(3.4), and a global weak solution (N_i, Φ) to (1.18)-(1.19), (3.2)-(3.4) for $\alpha = i$ [22, 23].

3.1. Zero-electron-mass limits. In order to perform the limit $m_e \to 0$ rigorously, we need additional assumptions. Assume that the initial and boundary data are compatible with the zero-electron-mass limit, i.e.

$$N_{eD} = f_e(\Phi_D) \quad \text{in } \Omega \times (0,T), \qquad N_{e0} = N_{i0} \quad \text{in } \Omega. \tag{3.6}$$

THEOREM 3.1 ((DD-EI)$\to$(DD-I)). *Let the assumptions (1.4) and (3.1), (3.5)-(3.6) hold and assume that $N_{\alpha D} \ge \underline{N} > 0$ on $\Gamma_D \times (0,T)$ and $N_{\alpha 0} \ge \underline{N} > 0$ in Ω, for some $\underline{N} > 0$, $\alpha = e, i$. Let $\delta = m_e$ and let $(N_{\alpha\delta}, \Phi_\delta)$ be a weak solution to (1.16)-(1.17), (3.2)-(3.4). Then there exists a subsequence, not relabeled, converging, as $\delta \to 0$, to a weak solution (N_i, Φ) to (1.18)-(1.19), (3.2)-(3.4) for $\alpha = i$, in the following sense:*

$$\begin{aligned} N_{e\delta} &\to f_e(\Phi) && \textit{strongly in } L^p(\Omega \times (0,T)) \textit{ for any } p \ge 1, \\ N_{i\delta} &\to N_i && \textit{strongly in } L^p(\Omega \times (0,T)) \textit{ for any } p \ge 1, \\ N_{i\delta} &\rightharpoonup N_i && \textit{weakly in } L^2(0,T; H^1(\Omega)), \\ \Phi_\delta &\to \Phi && \textit{strongly in } L^2(0,T; H^1(\Omega)). \end{aligned}$$

The result of this theorem remains valid for more general (monotone) pressure functions (see [25]).

The proof is based on entropy estimates and compactness arguments. Indeed, define the *entropy* (or *free energy*) of the model (DD-EI):

$$\begin{aligned} \eta(t) = \sum_{\alpha=e}^{i} \int_\Omega \left\{ \int_{N_{\alpha D}(t)}^{N_{\alpha\delta}(t)} \left(f_e^{-1}(s) - f_e^{-1}(N_{\alpha D}(t)) \right) ds \right\} dx \\ + \frac{\lambda^2}{2} \int_\Omega |\nabla(\Phi_\delta - \Phi_D)|^2 dx. \end{aligned}$$

Then the following entropy inequality holds for a.e. $t > 0$:

$$\eta(t) + \delta^{-1} \int_0^t \int_\Omega N_{e\delta} |\nabla(f_e^{-1}(N_{e\delta}) - \Phi_\delta)|^2 dx ds \le \eta(0) + C, \tag{3.7}$$

and $C > 0$ depends on $N_{i\delta}$ and Φ_D but not on δ (or λ). Here, we have used the conditions (3.6). By the minimum principle, $N_{e\delta} \geq \underline{N} > 0$ in $\Omega \times (0,T)$, for some $\underline{N} > 0$. Therefore, $f_e^{-1}(N_{e\delta}) - \Phi_\delta \to 0$ strongly in $L^2(0,T;H^1(\Omega))$ and $N_{e\delta} - f_e(\Phi_\delta) \to 0$ strongly in $L^2(\Omega \times (0,T))$. Using the maximum principle and the Poisson equation, it can be shown that $N_{e\delta}$ and Φ_δ converge *weakly* in some Lebesgue spaces to N_e and Φ, respectively. In order to identify N_e and $f_e(\Phi)$ we need the *strong* convergence of one of the sequences $(N_{e\delta})_\delta$ or $(\Phi_\delta)_\delta$. However, since we do not have an appropriate uniform bound for the time derivative $\partial_t N_{e\delta}$ in some space, we cannot conclude the strong compactness for $(N_{e\delta})_\delta$, like for $(N_{i\delta})_\delta$, by an application of Aubin's lemma [40]. Instead the strong compactness of $(\Phi_\delta)_\delta$ is shown by using the monotone Poisson equation (1.19) and an argument which is related to a compactness-by-convexity result [25].

3.2. Quasineutral limits. Recently, the limit $\lambda \to 0$ in the transient drift-diffusion equations has been studied by several authors in the plasma or semiconductor context. We present here the results of [27, 37]. The following theorem concerns the quasineutral limit in the model (DD-EI).

THEOREM 3.2 ((DD-EI)→(QD-EI)). *Let the hypotheses (1.4), (3.1) and (3.5) hold. Furthermore, let $N_{\alpha D} \geq \underline{N} > 0$ on $\Gamma_D \times (0,T)$ and $N_{\alpha 0} \geq \underline{N} > 0$ in Ω, for some $\underline{N} > 0$, $\alpha = e, i$, and let $N_{e0} = N_{i0}$ in Ω. Let $(N_{\alpha\lambda}, \Phi_\lambda)$ be a weak solution to (1.16)-(1.17), (3.2)-(3.4) and let $\omega \subset \Omega$ satisfy $\overline{\omega} \subset \Omega$. Then the sequence $(N_{\alpha\lambda}, \Phi_\lambda)$ converges, as $\lambda \to 0$, in the following sense:*

$$
\begin{aligned}
N_{\alpha\lambda} \to N \quad & \textit{strongly in } L^p(\omega \times (0,T)) \textit{ for any } p \geq 1, \\
N_{\alpha\lambda} \rightharpoonup N,\ \Phi_\lambda \rightharpoonup \Phi \quad & \textit{weakly in } L^2(0,T;H^1(\omega)),
\end{aligned}
$$

and N solves Eq. (1.22) and the initial condition $N(\cdot,0) = N_{e0} = N_{i0}$ in the sense of $H^{-1}(\omega)$. The limit Φ is a solution of

$$
\operatorname{div}(N\nabla\Phi) = \frac{1}{m_e + m_i}\Delta(m_e p_i(N) - m_i p_e(N)).
$$

Moreover, it holds, as $\lambda \to 0$,

$$
\begin{aligned}
\|N_{e\lambda} - N_{i\lambda}\|_{L^2(\Omega\times(0,T))} &= \mathcal{O}(\lambda^{1/2}), \\
\|N_{e\lambda} - N_{i\lambda}\|_{L^2(\omega\times(0,T))} &= \mathcal{O}(\lambda).
\end{aligned}
\tag{3.8}
$$

If in addition the condition $N_{eD} = N_{iD}$ on $\Gamma_D \times (0,T)$ holds, the above convergence results are valid for $\omega = \Omega$ and N satisfies the boundary condition $N = N_{eD}$ on $\Gamma_D \times (0,T)$.

For the quasineutral limit in the model (DD-I) we need additional assumptions since we need strong convergence of Φ_λ and it turns out that we have to estimate $\nabla\Phi_\lambda \cdot \nu$ on Γ_D:

$$
\partial\Omega \in C^{1,1},\ \overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset, \text{ and } \Phi_D \in L^2(0,T;H^2(\Omega)). \tag{3.9}
$$

This assumption ensures that $\Phi_\lambda \in L^2(0,T;H^2(\Omega))$ and hence, $\nabla\Phi_\lambda \cdot \nu \in L^2(\Gamma_D \times (0,T))$ [43].

THEOREM 3.3 ((DD-I)→(QD-I)). *Let the assumptions of Theorem 3.2 hold for $\alpha = i$ and let (3.9) hold. Let $(N_{i\lambda}, \Phi_\lambda)$ be a weak solution to (1.16)-(1.17), (3.2)-(3.4) for $\alpha = i$. Then $(N_{i\lambda}, \Phi_\lambda)$ converges, as $\lambda \to 0$, in the following sense:*

$$N_{i\lambda} \to N, \ \Phi_\lambda \to \Phi \quad \textit{strongly in } L^2(\Omega \times (0,T)),$$

and (N, Φ) solves (1.25) and $\Phi = f_e^{-1}(N)$ in $\Omega \times (0,T)$, and $N(\cdot, 0) = N_{i0}$ in the sense of $H^{-1}(\omega)$. Moreover, it holds, as $\lambda \to 0$,

$$\|N_{i\lambda} - f_e(\Phi_\lambda)\|_{L^2(\Omega\times(0,T))} = \mathcal{O}(\lambda^{1/2}), \quad \|\nabla\Phi_\lambda\|_{L^2(\Omega\times(0,T))} = \mathcal{O}(\lambda^{-1/2}),$$
$$\|N_{i\lambda} - f_e(\Phi_\lambda)\|_{L^2(\omega\times(0,T))} = \mathcal{O}(\lambda), \quad \|\nabla\Phi_\lambda\|_{L^2(\omega\times(0,T))} = \mathcal{O}(1).$$

The proofs of the above theorems are based on entropy estimates. Consider the model (DD-EI). Then an entropy inequality similar to (3.7) and an application of the maximum principles yields uniform bounds for the sequences $(f_\alpha^{-1}(N_{\alpha\lambda}) + q_\alpha\Phi_\lambda)_\lambda$ and $(\lambda\Phi_\lambda)_\lambda$ in $L^2(0,T;H^1(\Omega))$ and $L^\infty(0,T;H^1(\Omega))$, respectively. With these bounds, the Poisson equation and Aubin's lemma [40] we obtain the results of Theorem 3.2 [27]. (The estimate (3.8) is proved in [37].) The limit $\lambda \to 0$ can also be performed without the positivity assumption on the initial and boundary data if we assume that $p_e = p_i$ [27].

Concerning the proof of Theorem 3.3, entropy estimates yield uniform bounds for $f_i^{-1}(N_{i\lambda}) + \Phi_\lambda$ and $\lambda\Phi_\lambda$ in $L^2(0,T;H^1(\Omega))$ and $L^\infty(0,T;H^1(\Omega))$, respectively. Then, the strong convergence of Φ_λ (and $N_{i\lambda}$) in $L^2(\Omega \times (0,T))$ is shown by deriving a uniform estimate for $\lambda\nabla\Phi_\lambda \cdot \nu$ on Γ_D, using an idea of [3], and by employing compensated compactness and compactness-by-convexity tools [27]. Under the additional assumptions $p_i(s) = s$ and $N_{iD} = f_e(\Phi_D)$ the positivity condition on N_{eD} and N_{eD} can be removed.

The above theorems show that boundary layers may occur if no compatibility condition on the boundary data is imposed. One may ask which values the limit N takes on the boundary. In view of the uniform estimates for the quasi-Fermi potentials $f_\alpha^{-1}(N_{\alpha\lambda}) + q_\alpha\Phi_\lambda$ in $L^2(0,T;H^1(\Omega))$ it is not very difficult to show that, under the assumptions of Theorem 3.2,

$$f_e^{-1}(N) + f_i^{-1}(N) = f_e^{-1}(N_{eD}) + f_i^{-1}(N_{iD}) \quad \text{on } \Gamma_D \times (0,T).$$

For instance, if $f_e(s) = f_i(s) = \exp(s)$ (this corresponds to isothermal plasmas) then

$$N = \sqrt{N_{eD}N_{iD}} \quad \text{on } \Gamma_D \times (0,T).$$

The quasineutral limit in macroscopic plasma models has been first studied by Brézis et al. in [3]. In this paper the limit $\lambda \to 0$ is shown for the nonlinear Poisson equation, the ion density being fixed. Gasser et al. [15, 16] considered this limit in the semiconductor drift-diffusion equations (i.e. the Poisson equation contains a given function which models fixed background charges) with pure Neumann boundary conditions. Finally, for the one-dimensional stationary equations, asymptotic expansions of the solutions in powers of λ are derived in [34] and a boundary-layer analysis for semiconductor *pn*-junctions has been performed in [35].

4. Open problems. We present some open problems concerning the discussed asymptotic limits in the hydrodynamic and drift-diffusion plasma models:

- Prove the relaxation-time limit in the magnetohydrodynamic equations (see [13, p. 234]).
- Prove, if possible, the combined zero-relaxation-time and zero-electron-mass limits in the hydrodynamic equations (HD-EI).
- Prove the zero-electron-mass limit in the hydrodynamic equations (HD-EI)$\to$(HD-I) witout the uniform boundedness assumption of Theorem 2.3.
- Perform a boundary-layer analysis for the model (DD-EI) in the zero-electron-mass limit without the first assumption in (3.6).
- Prove the quasineutral limit in the hydrodynamic models for weak entropy solutions.
- Prove the quasineutral limit in the hydrodynamic models in bounded domains with appropriate (not periodic) boundary conditions.
- Prove the quasineutral limit (HD-EI)$\to$(QH-EI) without assuming (2.18).
- Prove the three asymptotic limits in the energy hydrodynamic equations.

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