Universität Konstanz

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Konstanzer Schriften in Mathematik

Nr. 327, Februar 2014

ISSN 1430-3558

# On a new class of Partial Integro-Differential Equations

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**Abstract**

We consider various initial-value problems for partial integro-differential equations of first order that are characterized by convolution-terms in the time-variable, where all factors depend on the solutions of the equations. The mathematical structure of such problems is based on problems for ordinary integro-differential equations that are used to describe certain glass-transition phenomena (see e.g. [12], [13], [19]). We start considering problems with kernels that are not depending on the space-variable and we will prove results concerning well-posedness and asymptotic behaviour. Afterwards, we will extend the results on problems with kernels that depend on the space-variable.

Keywords: integro-differential equations, well-posedness, asymptotic behaviour, convolution

## 1 Introduction

In [12], [13] and [14], initial-value-problems for ordinary integro-differential equations were studied, that are used to describe certain glass-transition-phenomena. The kernels of the convolution-terms of these problems are depending on the solutions of the equations, i.e. they are given by functions $k = F(\Phi)$ resp. $k = F(\Phi, \cdot)$. This is the main difference to integral equations as studied extensively in literature (e.g. equations of Volterra-type, see [8], [9], [11] or [17]) and to mainly considered integro-differential equations from [4], [5], [6] and [7]. In this paper we aim to treat problems for partial integro-differential equations that are of comparable structure, i.e. problems of the kind

$$\begin{aligned} u_t(t,x) + Au(t,x) + \int_0^t F(u)(t-s)u_t(s)ds &= 0, && (t,x) \in (0,\infty) \times G, \\ u(0,x) &= u_0(x), && x \in G, \\ u(t,x) &= 0, && (t,x) \in [0,\infty) \times \partial G, \end{aligned} \tag{1}$$

with a bounded domain $G \subseteq \mathbb{R}^n$, a kernel-function $F(u) : [0,\infty) \to \mathbb{R}$ that depends on $u$, $u : [0,\infty) \times G \to \mathbb{R}$, $u_0 : G \to \mathbb{R}$ and an elliptic operator

$A = \sum_{i,j=1}^n -\partial_i a_{ij}(\cdot)\partial_j + a(\cdot)$ $(a_{i,j}, a \in L^\infty(G))$, or

$$\begin{aligned} u_t(t,x) + Au(t,x) + \int_0^t F(u(t-s,x))u_t(s)ds &= 0, \qquad (t,x) \in (0,\infty)\times G, \\ u(0,x) &= u_0(x), \quad x \in G, \\ u(t,x) &= 0, \qquad (t,x) \in [0,\infty)\times \partial G, \end{aligned} \tag{2}$$

with a kernel-function $F : \mathbb{R} \to \mathbb{R}$ that is independent on $u$.
Partial integro-differential equations with at least one convolution-factor that is independent from $u$ got much attention in mathematical literature, e.g. in [21], [22], [23] and [24]. Integro-differential equations with similar nonlinearities were studied in [19], but in contrast to the problems (1) and (2), all considered equations have been of semilinear structure.
In this work, we aim at proving results concerning well-posedness and asymptotic behaviour for the problems (1) and (2). The restriction on Dirichlet-boundary conditions and on bounded domains is not mandatory, we will give some comments on more general cases at the end.
This work is based on the Ph.D. thesis [13].

# 2 Space-independent kernel-functions

In this chapter, we consider problem (1).

## 2.1 Preliminaries

Assume $a_{ij}, a \in L^\infty$ $(i,j = 1,\dots,n)$, $(a_{ij}(\cdot))_{ij}$ is symmetric and uniformly positive definite, i.e.

$$\exists p > 0\ \forall \xi \in \mathbb{R}^n\ \forall x \in G : \sum_{i,j=1}^n \xi_i a_{ij}(x)\xi_j \geq p|\xi|^2.$$

We consider the following bilinear form

$$B(u,v) := \sum_{i,j=1}^n \langle a_{ij}(\cdot)\partial_j u, \partial_i v\rangle + \langle a(\cdot)u, v\rangle, \quad u,v \in H_0^1(G),$$

where $\langle\cdot,\cdot\rangle$ denotes the $L^2$-scalar product. $B(\cdot,\cdot)$ requires to be strong coercive, i.e.

$$\exists q > 0\ \forall u \in H_0^1(G) : \Re B(u,u) \geq q\|u\|_{H^1}^2.$$

We define

$$D(A) := \{u \in H_0^1(G) | \exists f_u \in L^2(G)\ \forall v \in H_0^1(G) : B(u,v) = \langle f_u, v\rangle\}$$

and by this

$$\begin{aligned} A : D(A) &\to L^2(G) \\ u &\mapsto f_u. \end{aligned} \tag{3}$$

$A$ is a self-adjoint operator with positive spectrum $\sigma(A) \subseteq [q,\infty)$. Due to this, one has by using [3, Theorem 1.2.1 (p. 256)]

$$D(A^3) \subseteq D(A) \text{ is dense with respect to the graph-norm } \|\cdot\|_{D(A)}. \tag{4}$$

$u \in C^1([0,\infty), L^2(G)) \cap C^0([0,\infty), D(A))$ is called a solution of (1), iff

$$u_t + Au + F(u) * u_t \overset{L^2(G)}{=} 0, \quad u(0) \overset{D(A)}{=} u_0,$$

where $u_0 \in D(A)$, $F \in C^0(L^2(G), \mathbb{R})$ and

$$(F(u) * u_t)(t) = \int\limits_0^t F(u(t-s))u_t(s)ds, \quad t \in [0,\infty).$$

## 2.2 Linear problem

We consider the following related linear problem

$$\begin{aligned} &u \in C^0([0,\infty), D(A)) \cap C^1([0,\infty), L^2(G)) : \\ &u_t(t) + Au(t) + \int\limits_0^t m(t-s)u_t(s)ds = 0, \quad t \in [0,\infty), \\ &u(0) = u_0, \end{aligned} \tag{5}$$

where $m : [0,\infty) \to \mathbb{R}$.

**Theorem 1.** *If $m \in C^1([0,\infty), \mathbb{R})$ and $m(0) > -q$ then (5) has a unique solution $u \in C^1([0,\infty), D(A)) \cap C^2([0,\infty), L^2(G))$ for any given $u_0 \in D(A^2)$.*

*Proof.* Differentiation of (5) with respect to $t$ and variation of constants formula lead to

$$z_t + \tilde{A}z + \int\limits_0^t m'(t-s)z(s)ds = 0, \quad z(0) = z_0 := -Au_0, \tag{6}$$

where $z = u_t$ and $\tilde{A} = A + m(0)$. Problem (6) is equivalent to the following fixed-point problem

$$z = Tz, \quad Tz(t) := e^{-t\tilde{A}}z_0 - \int\limits_0^t e^{-(t-s)\tilde{A}} \int\limits_0^s m'(s-r)z(r)drds. \tag{7}$$

We define for any $N > 0$

$$\mathscr{X}_N := C^0([0,N], D(A)) \cap C^1([0,N], L^2(G)),$$

with the norm[1]

$$\|z\|_N := \max\left\{ \sup_{t\in[0,N]} \|z(t)\|_{D(A)}, \sup_{t\in[0,N]} \left\| \frac{d}{dt}z(t) \right\| \right\}$$

By using Banach fixed-point theorem, one proves the existence of a unique fixed-point $z_N \in \mathscr{X}_N$ for any $N > 0$. We define $z(t) := z_N(t)$ for $0 \le t \le N$. Due to the uniqueness of any $z_N$, $z : [0,\infty) \to D(A)$ is well-defined and fulfils $z \in \mathscr{X}_N$ for all $N > 0$. $u := u_0 + \int_0^t z(s)ds$ is the requested solution of (5). ☐

[1] $\|\cdot\|$ denotes the $L^2(G)$-norm.

One proves analoguously the following

**Corollary 2.** *If $m \in C^1([0,\infty),\mathbb{R})$ and $m(0) > -q$ then (5) has a unique solution $u \in C^1([0,\infty), D(A^2)) \cap C^2([0,\infty), D(A))$ for any given $u_0 \in D(A^3)$.*

**Lemma 3.** *Assume $m \in C^1([0,\infty),\mathbb{R})$ with $m(0) > -q$ and $|m'(t)| \le ke^{-c_1 t}$ for all $t \in [0,\infty)$, where $c_1 > \lambda_1$ with $\lambda_1 = q + m(0)$ and $k > 0$ such that*

$$k < \lambda_1(c_1 - \lambda_1).$$

*Furthermore, let $\lim\limits_{t\to\infty}(t) = 0$ and $u_0 \in D(A^2)$. Then the solution $u$ of (5) fulfils for all $t \in [0,\infty)$*

$$\|u_t(t)\| \le \|u_0\|_{D(A)} e^{\frac{k-\lambda_1(c_1-\lambda_1)}{c_1-\lambda_1}t}, \ \|u(t)\| \le \|u_0\|_{D(A)} \frac{\lambda_1 - c_1}{k - \lambda_1(c_1-\lambda_1)} e^{\frac{k-\lambda_1(c_1-\lambda_1)}{c_1-\lambda_1}t}.$$

*Proof.* We consider the fixed-point equation from the proof of Lemma 1

$$z(t) = e^{-t\tilde{A}} z_0 - \int\limits_0^t e^{-(t-s)\tilde{A}} \int\limits_0^s m'(s-r)z(r)drds.$$

We obtain

$$e^{\lambda_1 t}\|z(t)\| \le \|u_0\|_{D(A)} + k\int\limits_0^t \int\limits_r^t e^{(\lambda_1-c_1)(s-r)} ds e^{\lambda_1 r}\|z(r)\|dr.$$

Application of Gronwall's inequality finishes the proof. □

**Corollary 4.** *Let $u_0 \in D(A^3)$. The same asumptions from Lemma 3 lead to*

$$\|u_t(t)\|_{D(A)} \le \|Au_0\|_{D(A)} e^{\frac{k-\lambda_1(c_1-\lambda_1)}{c_1-\lambda_1}t} \text{ and}$$

$$\|u(t)\|_{D(A)} \le \|Au_0\|_{D(A)} \frac{\lambda_1 - c_1}{k - \lambda_1(c_1-\lambda_1)} e^{\frac{k-\lambda_1(c_1-\lambda_1)}{c_1-\lambda_1}t}.$$

## 2.3 Non-linear problem

We now aim to produce a self-mapping using the linear problem (5), so that its fixed-point solves the nonlinear problem (1).

At first, let $u_0 \in D(A^3)$. Furthermore, let $F \in C^0(L^2(G),\mathbb{R})$ locally Lip- schitz continuous and suppose $F \circ u \in C^1([0,\infty),\mathbb{R})$ for all $u \in C^1([0,\infty), L^2(G))$ and $F(u_0) > -q$.
Let $\lambda_1 := q + F(u_0)$, $c_1 > \lambda_1$ and $v_2, \beta > 0$.

(i) Let $k > 0$ such that $k < \lambda_1(c_1 - \lambda_1)$,

(ii) $\alpha_1, \alpha_2 > 0$ such that $(\alpha_1 + \alpha_2)\frac{k-\lambda_1(c_1-\lambda_1)}{c_1-\lambda_1} \le -c_1$,

(iii) $v_1 > 0$ such that $v_1 \|u_0\|_{D(A)}^{\alpha_1+\alpha_2} \left(\frac{\lambda_1-c_1}{k-\lambda_1(c_1-\lambda_1)}\right)^{\alpha_1} \le k$.

(iv) Furthermore, let $|F(u)| \le v_2\|u\|^\beta$, $|F(u_0)| \le \frac{k}{c_1}$,

(v) $\left|\frac{d}{dt}F(u(t))\right| \leq v_1\|u(t)\|^{\alpha_1}\|u_t(t)\|^{\alpha_2}$ for all $u \in C^1([0,\infty), L^2(G))$,

(vi) $\exists L_1 > 0\ \forall u_1, u_2 \in C^1([0,\infty), L^2(G))$ that satisfy

$$\|u_i(t)\| \leq \|u_0\|_{D(A)}\frac{\lambda_1 - c_1}{k - \lambda_1(c_1-\lambda_1)} \text{ and } \|u_{i\,t}(t)\| \leq \|u_0\|_{D(A)}$$

for all $t \in [0,\infty)$, $i = 1,2$ :

$$\left|\frac{d}{dt}F(u_1(t)) - \frac{d}{dt}F(u_2(t))\right| \leq (\|u_1(t) - u_2(t)\| + \|u_{1\,t}(t) - u_{2\,t}(t)\|)\,.$$

We consider

$$\mathscr{X} := \left\{u \in C^0([0,\infty), D(A)) \cap C^1([0,\infty), L^2(G)) : u, u_t \text{ are bounded}\right\}$$

together with the norm $\|u\|_{\mathscr{X}} := \max\left\{\sup_{t\in[0,\infty)}\|u(t)\|_{D(A)}, \sup_{t\in[0,\infty)}\|u_t(t)\|\right\}$ and the following subset

$$\mathscr{C} := \left\{ u \in \mathscr{X} \,\middle|\, u(0) \overset{D(A)}{=} u_0 : \begin{array}{l} \|u(t)\| \leq \|u_0\|_{D(A)}\frac{\lambda_1 - c_1}{k-\lambda_1(c_1-\lambda_1)}e^{\frac{k-\lambda_1(c_1-\lambda_1)}{c_1-\lambda_1}t}, \\ \|u_t(t)\| \leq \|u_0\|_{D(A)}e^{\frac{k-\lambda_1(c_1-\lambda_1)}{c_1-\lambda_1}t}, \\ \|u(t)\|_{D(A)} \leq \|Au_0\|_{D(A)}\frac{\lambda_1-c_1}{k-\lambda_1(c_1-\lambda_1)}e^{\frac{k-\lambda_1(c_1-\lambda_1)}{c_1-\lambda_1}t}. \end{array} \right\}$$

$\mathscr{C} \subseteq \mathscr{X}$ is bounded, closed, convex and due to (iv) not-empty. We define

$$\mathscr{T} : \mathscr{C} \to \mathscr{C},\ v \mapsto \mathscr{T}v := u_v,$$

where $u_v$ is the solution of (5) with kernel-function $m := F \circ v$. One has

$$|m'(t)| \overset{(ii),(iii),(v)}{\leq} ke^{-c_1 t}$$

and due to (iv)

$$|m(t)| \overset{t\to\infty}{\longrightarrow} 0.$$

With respect to (i), Lemma 3 and Corollary 4, one has proved that $\mathscr{T}$ is well-defined.
Using Gronwall's inequality and the Arzelà–Ascoli theorem, we easily prove the continuity and compactness of $\mathscr{T}$. By applying Schauder fixed-point theorem, we obtain the following

**Theorem 5.** *Assume $u_0 \in D(A^3)$, $F \in C^0(L^2(G), \mathbb{R})$ is locally Lipschitz-continuous and suppose $F \circ u \in C^1([0,\infty), \mathbb{R})$ for all $u \in C^1([0,\infty), L^2(G))$ and $F(u_0) > -q$. Furthermore, let $\lambda_1 = q + F(u_0)$, $c_1 > \lambda_1$, $v_2, \beta > 0$ and*

*(i) $k > 0$ such that $k < \lambda_1(c_1 - \lambda_1)$.*

*(ii) Let $\alpha_1, \alpha_2 > 0$ such that $(\alpha_1 + \alpha_2)\frac{k-\lambda_1(c_1-\lambda_1)}{c_1-\lambda_1} \leq -c_1$ and*

*(iii) $v_1 > 0$ such that $v_1\|u_0\|_{D(A)}^{\alpha_1+\alpha_2}\left(\frac{\lambda_1-c_1}{k-\lambda_1(c_1-\lambda_1)}\right)^{\alpha_1} \leq k$.*

*Furthermore, let*

*(iv)* $|F(u)| \le v_2\|u\|^{\beta}$, $|F(u_0)| \le \frac{k}{c_1}$,

*(v)* $\left|\frac{d}{dt}F(u(t))\right| \le v_1\|u(t)\|^{\alpha_1}\|u_t(t)\|^{\alpha_2}$ *for all* $u \in C^1([0,\infty), L^2(G))$ *and*

*(vi)* $\exists L_1 > 0\ \forall u_1, u_2 \in C^1([0,\infty), L^2(G))$ *that satisfy* $\|u_i(t)\| \le \|u_0\|_{D(A)}\frac{\lambda_1 - c_1}{k-\lambda_1(c_1-\lambda_1)}$ *and* $\|u_{i\,t}(t)\| \le \|u_0\|_{D(A)}$ *(*$i = 1,2$*):*

$$\left|\frac{d}{dt}F(u_1(t)) - \frac{d}{dt}F(u_2(t))\right| \le L_1\left(\|u_1(t)-u_2(t)\| + \|u_{1\,t}(t)-u_{2\,t}(t)\|\right).$$

*Then problem (1) has a unique solution* $u \in C^0([0,\infty), D(A)) \cap C^1([0,\infty), L^2(G))$ *that fulfils*

$$\|u(t)\| \le \|u_0\|_{D(A)}\frac{\lambda_1 - c_1}{k-\lambda_1(c_1-\lambda_1)}e^{\frac{k-\lambda_1(c_1-\lambda_1)}{c_1-\lambda_1}t},\ \ \|u_t(t)\| \le \|u_0\|_{D(A)}e^{\frac{k-\lambda_1(c_1-\lambda_1)}{c_1-\lambda_1}t}.$$

*Comment on the proof.* Uniqueness follows from a direct computation that uses Gronwall's inequality. □

The restriction $u_0 \in D(A^3)$ was necessary to obtain a self-mapping on $\mathscr{C}$ by using Corollary 4. We now aim to generalize the conditions of Theorem 5.

**Lemma 6.** *Assume* $u_0 \in D(A)$, $F \in C^0(L^2(G), \mathbb{R})$ *as given in Theorem 5 and let* $u_{01}, u_{02} \in D(A)$ *and* $u_1, u_2 \in \mathscr{X}$ *related solutions of (1) that satisfy*

$$\|u_i(t)\| \le M_1 \text{ and } \|u_{i\,t}(t)\| \le M_2$$

*for all* $t \in [0,\infty)$ *where* $M_1, M_2 > 0$. *Furthermore, suppose*

$$\exists L_1 = L_1(M_1, M_2, F) > 0\ \forall t \in [0,\infty):$$
$$\left|\frac{d}{dt}F(u_1(t)) - \frac{d}{dt}F(u_2(t))\right| \le L_1\left(\|u_1(t)-u_2(t)\|_{D(A)} + \|u_{1\,t}(t)-u_{2\,t}(t)\|\right).$$

*Then one has*

$$\forall N > 0\ \exists C = C(N, M_1, M_2, F) > 0 : \|u_1 - u_2\|_N \le C\|u_{01} - u_{02}\|_{D(A)}.$$

*Proof.* Let $p, N > 0$. Direct computations lead to

$$\begin{aligned}
e^{-pt}\|u_1(t)-u_2(t)\| \le&\|u_{01}-u_{02}\|_{D(A)} \\
&+ \frac{1}{p(p+q)}v_2M^{\beta}\sup_{t\in[0,N]} e^{-pt}\|u_{1\,t}(t)-u_{2\,t}(t)\| \\
&+ \frac{1}{p(p+q)}LM\sup_{t\in[0,N]} e^{-pt}\|u_1(t)-u_2(t)\|, \\
e^{-pt}\|u_{1\,t}(t)-u_{2\,t}(t)\| \le&\|u_{01}-u_{02}\|_{D(A)} \\
&+ \frac{1}{p(p+\lambda_1)}v_1M^{\alpha_1+\alpha_2}\sup_{t\in[0,N]} e^{-pt}\|u_{1\,t}(t)-u_{2\,t}(t)\| \\
&+ \frac{1}{p(p+\lambda_1)}L_1M\sup_{t\in[0,N]} e^{-pt}\|u_1(t)-u_2(t)\|_{D(A)} \\
&+ \frac{1}{p(p+\lambda_1)}L_1M\sup_{t\in[0,N]} e^{-pt}\|u_{1\,t}(t)-u_{2\,t}(t)\| \quad \text{and}
\end{aligned}$$

$$
\begin{aligned}
e^{-pt}\|Au_1(t) - Au_2(t)\| \leq & \|u_{01} - u_{02}\|_{D(A)} \\
& + \frac{1}{p(p+\lambda_1)} v_1 M^{\alpha_1+\alpha_2} \sup_{t\in[0,N]} e^{-pt}\|u_{1t}(t) - u_{2t}(t)\| \\
& + \frac{1}{p(p+\lambda_1)} L_1 M \sup_{t\in[0,N]} e^{-pt}\|u_1(t) - u_2(t)\|_{D(A)} \\
& + \frac{1}{p(p+\lambda_1)} L_1 M \sup_{t\in[0,N]} e^{-pt}\|u_{1t}(t) - u_{2t}(t)\| \\
& + \frac{1}{p} LM \sup_{t\in[0,N]} e^{-pt}\|u_1(t) - u_2(t)\|.
\end{aligned}
$$

Chosing $p > 0$ large enough, one obtains[2]

$$
\|u_1 - u_2\|_{N,p} \leq C\|u_{01} - u_{02}\|_{D(A)}
$$

with a constant $C > 0$. Due to the equivalence of the norms $\|\cdot\|_N$ and $\|\cdot\|_{N,p}$, the proof is finished. ☐

By the help of this lemma, on can prove the following

**Corollary 7.** *Assume $u_0 \in D(A)$, $F \in C^0(L^2(G), \mathbb{R})$ is locally Lipschitz-continuous and suppose $F \circ u \in C^1([0,\infty), \mathbb{R})$ for all $u \in C^1([0,\infty), L^2(G))$ and $F(u_0) > -q$. Furthermore, let $\lambda_1 = q + F(u_0)$, $c_1 > \lambda_1$, $v_2, \beta > 0$ and*

*(i) $k > 0$ such that $k < \lambda_1(c_1 - \lambda_1)$.*

*(ii) Let $\alpha_1, \alpha_2 > 0$ such that $(\alpha_1 + \alpha_2)\frac{k-\lambda_1(c_1-\lambda_1)}{c_1-\lambda_1} < -c_1$ and*

*(iii) $v_1 > 0$ such that $v_1\|u_0\|_{D(A)}^{\alpha_1+\alpha_2}\left(\frac{\lambda_1-c_1}{k-\lambda_1(c_1-\lambda_1)}\right)^{\alpha_1} < k$.*

*Furthermore, let*

*(iv) $|F(u)| \leq v_2\|u\|^\beta$, $|F(u_0)| \leq \frac{k}{c_1}$,*

*(v) $\left|\frac{d}{dt}F(u(t))\right| \leq v_1\|u(t)\|^{\alpha_1}\|u_t(t)\|^{\alpha_2}$ for all $u \in C^1([0,\infty), L^2(G))$ and*

*(vi) $\exists \varepsilon > 0\ \exists L_1 > 0\ \forall u_1, u_2 \in C^1([0,\infty), L^2(G))$ that satisfy $\|u_i(t)\| \leq \|u_0\|_{D(A)}\frac{\lambda_1-c_1}{k-\lambda_1(c_1-\lambda_1)} + \varepsilon$ and $\|u_{it}(t)\| \leq \|u_0\|_{D(A)} + \varepsilon$ $(i = 1, 2)$:*

$$
\left|\frac{d}{dt}F(u_1(t)) - \frac{d}{dt}F(u_2(t))\right| \leq L_1\left(\|u_1(t) - u_2(t)\| + \|u_{1t}(t) - u_{2t}(t)\|\right).
$$

*Then problem (1) has a unique solution $u \in C^0([0,\infty), D(A)) \cap C^1([0,\infty), L^2(G))$ that fulfils*

$$
\|u(t)\| \leq \|u_0\|_{D(A)}\frac{\lambda_1 - c_1}{k - \lambda_1(c_1-\lambda_1)}e^{\frac{k-\lambda_1(c_1-\lambda_1)}{c_1-\lambda_1}t} \text{ and } \|u_t(t)\| \leq \|u_0\|_{D(A)}e^{\frac{k-\lambda_1(c_1-\lambda_1)}{c_1-\lambda_1}t}.
$$

[2] We define the weighted norm $\|u\|_{N,p} := \max\left\{\sup_{t\in[0,N]} e^{-pt}\|u(t)\|_{D(A)}, \sup_{t\in[0,N]} e^{-pt}\|u_t(t)\|\right\}$ $(N, p > 0)$.

*Proof.* It follows from (4)

$$\exists (u_{0n})_{n\in\mathbb{N}} \subseteq D(A^3) : \|u_0 - u_{0n}\|_{D(A)} \overset{n\to\infty}{\longrightarrow} 0.$$

Due to the continuous dependence of $q+F(u)$ on $u \in L^2(G)$, there exists $n_0 \in \mathbb{N}$ such that problem (1) with kernel-function $F$ and initial-value $u_{0n}$ has a unique solution $u_n \in \mathscr{X}$ for all $n \geq n_0$, that satisfies

$$\begin{aligned} \|u_n(t)\| &\leq \|u_{0n}\|_{D(A)} \frac{\lambda_n - c_1}{k - \lambda_n(c_1 - \lambda_n)} e^{\frac{k-\lambda_n(c_1-\lambda_n)}{c_1-\lambda_n}t} \text{ and} \\ \|u_{nt}(t)\| &\leq \|u_{0n}\|_{D(A)} e^{\frac{k-\lambda_n(c_1-\lambda_n)}{c_1-\lambda_n}t} \text{ for all } t \in [0,\infty), \end{aligned} \tag{8}$$

where $\lambda_n := q + F(u_{0n})$ $(n \geq 2)$. We define $M_1 := \|u_0\|_{D(A)} \frac{\lambda_1 - c_1}{k-\lambda_1(c_1-\lambda_1)} + \varepsilon$ and $M_2 := \|u_0\|_{D(A)} + \varepsilon$. Then there exists $n_1 \geq n_0$ such that $\|u_n(t)\| \leq M_1$ and $\|u_{nt}(t)\| \leq M_2$ for all $n \geq n_1$ and $t \in [0,\infty)$. It follows from Lemma 6

$$\exists u \in \mathscr{X} \; \forall N > 0 : \|u - u_n\|_N \overset{n\to\infty}{\longrightarrow} 0.$$

Limit of (1) and (8) as $n \to \infty$ finally proves, that $u$ is the requested solution of (1) with kernel-function $F$ and initial-value $u_0$. □

## 2.4 Remark and example

**Remark 8.** *It is easy to extend the methods from the previous chapter on inhomogenious problems of the kind*

$$u_t(t) + Au(t) + \int_0^t F(u)(t-s)u_t(s)ds = f(t), \quad u(0) = u_0$$

*with* $f \in C^1([0,\infty), D(A^2))$*. To obatin a self-mapping as above, we will need the following conditions on* $F$

$$\|f'(t)\|_{D(A)} \leq Me^{-dt}, \quad \lim_{t\to\infty} \|f(t)\|_{D(A)} = 0,$$

*with* $M \geq 0$, $d > 0$ *and* $d(c_1 - \lambda_1) > k$ *in case of* $d < \lambda_1$*. The smallness-conditions will additionally depend on* $M$ *and* $d$*.*

**Example 9.** *We consider so-called radial-symmetric functions. Assume* $u_0 \in D(A)$ *and* $f \in C^1([0,\infty), \mathbb{R})$ *and suppose* $f(\|u_0\|) > -q$*, with* $f'$ *locally Lipschitz-continuous. Furthermore, let* $|f(x)| \leq v_2|x|^\beta$, $|f(\|u_0\|)| \leq \frac{k}{c_1}$ *and* $|f'(x)| \leq v_1|x|^{\alpha_1}$*. Then problem (1) with kernel-function* $F$*, defined by* $F(u) := f(\|u\|)$*, has a unique solution* $u \in \mathscr{X}$ *that decays exponentially.*

# 3 Space-dependent kernel-functions

In this chapter, we consider problem (2).

## 3.1 Preliminaries

We start proving some auxiliaries that will be used later. Assume $G \subseteq \mathbb{R}^n$ is open and bounded and has a $C^k$-boundary $\partial G$ where $k \in \mathbb{N}$ is chosen sufficiently large enough for the following results.

**V 1.** Let $k \in \mathbb{N}$, $a_{ij} \in C^{2k-1}(G)$ $(i,j = 1,\dots,n)$, $a \in C^{2k-2}(G)$ and $u \in D(A^k) \cap C^{2k}(G)$. Then one can easily prove by induction

$$A^k u = \sum_{\beta\in\mathbb{N}_0^n, |\beta|\le 2k} P_\beta(\partial^{\gamma_1} a_{ij}, \partial^{\gamma_2} a; i,j = 1,\dots,n, |\gamma_1| \le 2k-1, |\gamma_2| \le 2k-2) \cdot \partial^\beta u,$$

where $P_\beta$ are polynomials with degrees $\deg P_\beta \le k$.

**V 2.** *Ellipctic regularity:* Let $k \in \mathbb{N}$, $a_{ij}, a \in C^{2k-1}(\bar{G})$ $(i,j = 1,\dots,n)$. One can prove by using V 1 and [10, Theorem 5 (p. 340)]

$$\exists C_1, C_2 > 0 \ \forall u \in D(A^k) \cap C^\infty(G) : C_1 \|u\|_{H^{2k}(G)} \le \|u\|_{D(A^k)} \le C_2 \|u\|_{H^{2k}(G)}.$$

**V 3.** Assume $k \in \mathbb{N}$ satisfies $4k > n$. Furthermore, let $F \in C^{2k}(\mathbb{R},\mathbb{R})$ with $|F^{(i)}(x)| \le v_1 |x|^\alpha$ for $i = 0,\dots,2k$ and given $v_1, \alpha > 0$.
We consider the following improved formula from [18, proof of Lemma 4.7, p. 47][3]

$$\partial^\beta(F(u)) = \sum_{\mu=1}^{|\beta|} F^{(\mu)}(u) \sum_{\gamma\in\mathbb{N}_0^{|\beta|}, |\gamma|=\mu, \sum_{j=1}^{|\beta|} j\gamma_j = |\beta|} \ \sum_{p=1}^{(\mu+1)^{|\beta|-1}} C_{\mu,\gamma,p} \prod_{i=1}^{|\beta|} \prod_{l=1}^{\gamma_i} \partial^{\alpha^i_{l,p,\gamma}} u, \quad (9)$$

with $\beta \in \mathbb{N}_0^n$ such that $|\beta| \le 2k$, $C_{\mu,\gamma,p} \ge 0$ and $\alpha^i_{l,p,\gamma} \in \mathbb{N}_0^n$ with $\alpha^i_{l,p,\gamma} \le \beta$ and $|\alpha^i_{l,p,\gamma}| = i$. A similar calculation as [18, proof of Lemma 4.7] leads to

$$\|\partial^\beta(F(u))\| \le \sum_{\mu=1}^{|\beta|} \sum_{\gamma\in\mathbb{N}_0^{|\beta|}, |\gamma|=\mu, \sum_{j=1}^{|\beta|} j\gamma_j = |\beta|} \ \sum_{p=1}^{(\mu+1)^{|\beta|-1}} C_{\mu,\gamma,p} \|F^{(\mu)}(u)\|_\infty \prod_{i=1}^{|\beta|} \|u\|^{\gamma_i}_{W^{i,\frac{2|\beta|}{i}}}.$$

One obtains from Gagliardo-Nirenberg inequality ([18, Theorem 4.4])

$$\begin{aligned} &\|\partial^\beta(F(u))\| \\ &\le \sum_{\mu=1}^{|\beta|} \sum_{\gamma\in\mathbb{N}_0^{|\beta|}, |\gamma|=\mu, \sum_{j=1}^{|\beta|} j\gamma_j = |\beta|} \ \sum_{p=1}^{(\mu+1)^{|\beta|-1}} C_{\mu,\gamma,p} v_1 \|u\|^\alpha_\infty \prod_{i=1}^{|\beta|} C(i,|\beta|) \|u\|^{\frac{i\gamma_i}{|\beta|}}_{H^{|\beta|}} \|u\|_\infty^{\gamma_i - \frac{i\gamma_i}{|\beta|}} \end{aligned}$$

and from Sobolev embedding theorem and V 2

$$\|F(u)\|_{H^{2k}} \le C_5 v_1 \left\{ \begin{array}{ll} \|u\|^\alpha_{D(A^k)}, & \|u\|_{D(A^k)} \le 1 \\ \|u\|^{\alpha+2k}_{D(A^k)}, & \|u\|_{D(A^k)} > 1 \end{array} \right\},$$

[3] For a proof, see [13, Lemma 1.13].

where $C_5 = \left( \sum_{|\beta| \le 2k} C(C_0, C_1, |\beta|)^2 \right)^{\frac{1}{2}}$ with the Sobolev embedding constant $C_0$.

**V 4.** Assume $k$ and $F$ as in V 3 and $u \in D(A^k) \cap C^\infty(G)$.
Due to Sobolev embedding theorem and V 2, one has $u \in C^0(\bar{G})$ and from [18, Lemma 4.7] $F(u) \in H^{2k}(G)$. We consider the (unique) trace-operator $S : H^1(G) \to L^2(\partial G)$ from [10, Theorem 1 (p. 272)], that maps continuous functions on their restrictions on the boundary. Due to $u \in H_0^1(G)$, one has $Su = 0$ and it follows $u|_{\partial G} = 0$. One obtains $F(u)|_{\partial G} = 0$ from $F(0) = 0$ and due to [10, Theorem 2 (p. 273)] $F(u) \in H_0^1(g)$, resp. $F(u) \in D(A) \cap C^{2k}(\bar{G})$. In case of $F(u) \in D(A^m) \cap C^{2k}(\bar{G})$ for a fixed $m \in \{1, \dots, k-1\}$, application of V 1 and (9) lead to $S\left(A^m(F(u))\right) = 0$, i.e. $F(u) \in D(A^{m+1})$. It follows iteratively $F(u) \in D(A^k)$ and from V 3

$$\|F(u)\|_{D(A^k)} \le C_4 v_1 \left\{ \begin{array}{ll} \|u\|_{D(A^k)}^{\alpha}, & \|u\|_{D(A^k)} \le 1 \\ \|u\|_{D(A^k)}^{\alpha+2k}, & \|u\|_{D(A^k)} > 1 \end{array} \right\},$$

where $C_4 = C_2 C_5$.

**V 5.** Assume $k \in \mathbb{N}$ such that $4k > n$ and $F \in C^{2k}(\mathbb{R}, \mathbb{R})$ with $F^{(2k)}$ locally Lipschitz-continuous and $F^{(i)}(0) = 0$ $(i = 0, \dots, 2(k-1))$. Furthermore, let $u_1, u_2 \in D(A^k) \cap C^\infty(G)$ and $M > 0$ such that $\|u_i\|_{D(A)} \le M$ $(i = 1, 2)$. Using (9), one obtains

$$\begin{aligned}
&\partial^\beta (F(u_1) - F(u_2)) \quad = \\
&\sum_{\mu=1}^{|\beta|} \sum_{\gamma \in \mathbb{N}_0^{|\beta|}, |\gamma| = \mu, \sum_{i=1}^{|\beta|} i\gamma_i = |\beta|} \sum_{p=1}^{(\mu+1)^{|\beta|-1}} \left( F^{(\mu)}(u_1) - F^{(\mu)}(u_2) \right) C_{\mu,\gamma,p} \prod_{i=1}^{|\beta|} \prod_{l=1}^{\gamma_i} \partial^{\alpha^i_{l,p,\gamma}} u_1 \quad + \\
&\sum_{\mu=1}^{|\beta|} \sum_{\gamma \in \mathbb{N}_0^{|\beta|}, |\gamma| = \mu, \sum_{i=1}^{|\beta|} i\gamma_i = |\beta|} \sum_{p=1}^{(\mu+1)^{|\beta|-1}} F^{(\mu)}(u_2) C_{\mu,\gamma,p} \\
&\cdot \sum_{i=1}^{|\beta|} \left[ \begin{array}{r} \left( \sum_{l=1}^{\gamma_i} \left( \partial^{\alpha^i_{l,p,\gamma}} u_1 - \partial^{\alpha^i_{l,p,\gamma}} u_2 \right) \prod_{m=1}^{l-1} \partial^{\alpha^i_{m,p,\gamma}} u_2 \prod_{m=l+1}^{\gamma_i} \partial^{\alpha^i_{m,p,\gamma}} u_1 \right) \\ \cdot \prod_{m=1}^{i-1} \prod_{l=1}^{\gamma_m} \partial^{\alpha^m_{l,p,\gamma}} u_2 \prod_{m=i+1}^{|\beta|} \prod_{l=1}^{\gamma_m} \partial^{\alpha^m_{l,p,\gamma}} u_1 \end{array} \right].
\end{aligned}$$

One has for all $\mu \in \mathbb{N}_0^n$ with $|\mu| \le 2k$

$$\left\| F^{(\mu)}(u_1) - F^{(\mu)}(u_2) \right\|_\infty \le L_\mu \frac{C_0}{C_1} \|u_1 - u_2\|_{D(A^k)},$$

with Lipschitz-constants $L_\mu > 0$ of $F^{(\mu)}$ on the interval $\left[ -\frac{C_0}{C_1} M, \frac{C_0}{C_1} M \right]$. Additionally, one has by using Hölder's inequality

$$\begin{aligned}
&\left\| F^{(\mu)}(u_2)\left(\partial^{\alpha^i_{l,p,\gamma}}u_1 - \partial^{\alpha^i_{l,p,\gamma}}u_2\right)\prod_{m=1}^{l-1}\partial^{\alpha^i_{m,p,\gamma}}u_2 \prod_{m=l+1}^{\gamma_i}\partial^{\alpha^i_{m,p,\gamma}}u_1 \prod_{m=1}^{i-1}\prod_{l=1}^{\gamma_m}\partial^{\alpha^m_{l,p,\gamma}}u_2 \right.\\
&\qquad\left. \cdot \prod_{m=i+1}^{|\beta|}\prod_{l=1}^{\gamma_m}\partial^{\alpha^m_{l,p,\gamma}}u_1 \right\|\\
\leq\ & \left\|F^{(\mu)}(u_2)\right\|_\infty \left\|\partial^{\alpha^i_{l,p,\gamma}}u_1 - \partial^{\alpha^i_{l,p,\gamma}}u_2\right\|_{\frac{2|\beta|}{i}} \left\|\prod_{m=1}^{l-1}\partial^{\alpha^i_{m,p,\gamma}}u_2\right\|_{\frac{2|\beta|}{i(l-1)}}\\
&\cdot \left\|\prod_{m=l+1}^{\gamma_i}\partial^{\alpha^i_{m,p,\gamma}}u_1\right\|_{\frac{2|\beta|}{i(\gamma_i - l)}} \prod_{m=1}^{i-1}\left\|\prod_{l=1}^{\gamma_m}\partial^{\alpha^m_{l,p,\gamma}}u_2\right\|_{\frac{2|\beta|}{m\gamma_m}} \prod_{m=i+1}^{|\beta|}\left\|\prod_{l=1}^{\gamma_m}\partial^{\alpha^m_{l,p,\gamma}}u_1\right\|_{\frac{2|\beta|}{m\gamma_m}}
\end{aligned}$$

Application of Gagliardo-Nirenberg inequality, Sobolev embedding theorem and V 2 lead to

$$\left\|\partial^{\alpha^i_{l,p,\gamma}}u_1 - \partial^{\alpha^i_{l,p,\gamma}}u_2\right\|_{\frac{2|\beta|}{i}} \leq C(i,|\beta|)\frac{C_0^{1-\frac{i}{|\beta|}}}{C_1}\|u_1 - u_2\|_{D(A^k)}$$

and

$$\begin{aligned}
&\left\|\prod_{m=1}^{l-1}\partial^{\alpha^i_{m,p}}u_2\right\|_{\frac{2|\beta|}{i(l-1)}}, \left\|\prod_{m=l+1}^{\gamma_i}\partial^{\alpha^i_{m,p}}u_1\right\|_{\frac{2|\beta|}{i(\gamma_i-l)}}, \left\|\prod_{l=1}^{\gamma_m}\partial^{\alpha^m_{l,p}}u_2\right\|_{\frac{2|\beta|}{m\gamma_m}},\\
&\qquad \left\|\prod_{l=1}^{\gamma_m}\partial^{\alpha^m_{l,p}}u_1\right\|_{\frac{2|\beta|}{m\gamma_m}}, \left\|\prod_{i=1}^{|\beta|}\prod_{l=1}^{\gamma_i}\partial^{\alpha^i_{l,p}}u_1\right\| \leq C(|\beta|, C_0, C_1, M).
\end{aligned}$$

Altogether, one has proved

$$\begin{gathered}
\exists K = K(C_0, C_1, F, k, M) > 0\ \forall u_1, u_2 \in D(A^k)\cap C^\infty(G), \|u_i\|_{D(A^k)} \leq M\ (i = 1,2):\\
\left\|\partial^\beta\left(F(u_1) - F(u_2)\right)\right\| \leq K\|u_1 - u_2\|_{D(A^k)}.
\end{gathered}$$

It follows from V 2 and V 4

$$\begin{gathered}
\exists K = K(C_0, C_1, F, k, M) > 0\ \forall u_1, u_2 \in D(A^k)\cap C^\infty(G), \|u_i\|_{D(A^k)} \leq M\ (i = 1,2):\\
\|F(u_1) - F(u_2)\|_{D(A^k)} \leq K\|u_1 - u_2\|_{D(A^k)}.
\end{gathered}$$

**V 6.** Assume $k \in \mathbb{N}$ with $4k > n$, $u, v, w \in D(A^k)\cap C^\infty(G)$ and $F \in C^{(2k)}(\mathbb{R},\mathbb{R})$ such that $F^{(i)}(0) = 0$ $(i = 1,\ldots,2(k-1))$. Due to V 2 and V 3, one has $F(u) \in H^{2k}(G)\cap C^\infty(G)$. $H^{2k}(G)$ is a Banach algebra (see [1, Theorem 4.39]), i.e. $F(u)v, F(u)vw \in H^{2k}(G)\cap C^\infty(G)$. One can prove with similar methods as used in V 4: $F(u)v, F(u)vw \in D(A^k)\cap C^\infty(G)$.

**V 7.** We obtain from [3, Theorem 1.2.1 (p. 256)] for all $k \in \mathbb{N}$

$$\bigcap_{j\in\mathbb{N}_0} D(A^{k+j}) \subseteq D(A^k) \text{ is dense with respect to } \|\cdot\|_{D(A^k)}.$$

Due to V 2 and Sobolev embedding theorem, one has the density of $D(A^k)\cap C^\infty(G)$ in $D(A^k)$.

Application of V 7 on V 2–V 6 leads to the following

**Lemma 10.** *Assume $k \in \mathbb{N}$ with $4k > n$, $F \in C^{(2k)}(\mathbb{R},\mathbb{R})$ with $|F^{(i)}(x)| \leq v_1|x|^\alpha$ $(i = 0,\dots,2k,\ v_1,\alpha > 0)$ and $F^{(2k)}$ is locally Lipschitz-continuous. Furthermore, let $G \subseteq \mathbb{R}^n$ be open and bounded with a $C^{2k}$-boundary.*

*(i)*

$$\exists C_1, C_2 > 0\ \forall u \in D(A^k):\ u \in H^{2k}(G),$$
$$C_1\|u\|_{H^{2k}(G)} \leq \|u\|_{D(A^k)} \leq C_2\|u\|_{H^{2k}(G)}.$$

*(ii)*

$$\exists C_3 > 0\ \forall u, v, w \in D(A^k):\ F(u)v, F(u)vw \in D(A^k),$$
$$\|F(u)v\|_{D(A^k)} \leq C_3\|F(u)\|_{D(A^k)}\|v\|_{D(A^k)}.$$

*(iii)*

$$\exists C_4 > 0\ \forall u \in D(A^k):\ F(u) \in D(A^k),$$
$$\|F(u)\|_{D(A^k)} \leq C_4 v_1 \left\{ \begin{array}{ll} \|u\|^{\alpha}_{D(A^k)}, & \|u\|_{D(A^k)} \leq 1, \\ \|u\|^{\alpha+2k}_{D(A^k)}, & \|u\|_{D(A^k)} > 1. \end{array} \right\}$$

*(iv)*

$$\forall M > 0\ \exists K > 0\ \forall u_1, u_2 \in D(A^k) \text{ with } \|u_i\|_{D(A^k)} \leq M\ (i = 1, 2):$$
$$\|F(u_1) - F(u_2)\|_{D(A^k)} \leq K\|u_1 - u_2\|_{D(A^k)}.$$

**Remark 11.** *Most of the previous results will not need a $C^{2k}$-boundary of $G$. This regularity comes from [10, Theorem 5 (p. 340)] to prove V 2.*

## 3.2 Linear problem

Assume $k \in \mathbb{N}$ such that $4k > n$ and $G \subseteq \mathbb{R}^n$ is a bounded domain with a $C^{2k}$-boundary. Furthermore, let $q > 0$ such that $\sigma(A) \subseteq [q,\infty)$. We consider the following linear problem

$$\begin{aligned} & u \in C^0([0,\infty), D(A)) \cap C^1([0,\infty), L^2(G)) : \\ & u_t(t) + Au(t) + \int_0^t m(t-s)u_t(s)ds = 0, \quad t \in [0,\infty), \qquad (10) \\ & u(0) = u0 \in D(A), \end{aligned}$$

where $m : [0,\infty) \to L^2(G)$.

**Theorem 12.** *If $m \in C^1([0,\infty), D(A^k))$ with $m(0)(x) \geq -q + \varepsilon$ for a $\varepsilon > 0$ and for all $x \in G$, $u_0 \in D(A^{k+1})$ and $m_t(t)v \in D(A^k)$ for all $t \in [0,\infty)$ and $v \in D(A^k)$, then problem (2) has a unique solution $u \in C^1(0,\infty), D(A^k)) \cap C^2([0,\infty), D(A^{k-1}))$.*

*Proof.* We define for $t \in [0,\infty)$ and $u \in C^1([0,\infty), D(A^k))$

$$w(t) := \int_0^t m_t(t-s)u_t(s)ds.$$

By the help of Lemma 10 (ii), we easily see $w \in C^0([0,\infty), D(A^k))$. By this, the theorem can be proved analogously to Theorem 1 by replacing $\lambda_1$ by $\varepsilon$, $\|\cdot\|_{D(A)}$ by $\|\cdot\|_{D(A^k)}$ and $\|\cdot\|_{L^2(G)}$ by $\|\cdot\|_{D(A^{k-1})}$. □

**Lemma 13.** *Assume* $u_0 \in D(A^{k+1})$ *and* $m \in C^1([0,\infty), D(A^k))$ *as in Theorem 12. Suppose additionally* $\|m_t(t)\|_{D(A^k)} \leq \omega e^{-c_1 t}$ *and* $\lim_{t\to\infty} \|m(t)\|_{D(A^k)} = 0$ *where* $c_1 > \varepsilon$ *and* $\omega > 0$ *such that*

$$C_3\omega < \varepsilon(c_1 - \varepsilon) \quad \textit{and} \quad \frac{C_0}{C_1}\omega < \varepsilon(c_1 - \varepsilon).$$

*Then the solution* $u$ *of (2) fulfils*

$$\begin{aligned}
&\|u_t(t)\|_{D(A^k)} \leq \|Au_0\|_{D(A^k)} e^{\frac{C_3\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t},\\
&\|u(t)\|_{D(A^k)} \leq \|Au_0\|_{D(A^k)} \frac{\varepsilon - c_1}{C_3\omega - \varepsilon(c_1-\varepsilon)} e^{\frac{C_3\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t}\\
\textit{and}\quad &\|u_t(t)\| \leq \|u_0\|_{D(A)} e^{\frac{\frac{C_0}{C_1}\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t},\\
&\|u(t)\| \leq \|u_0\|_{D(A)} \frac{\varepsilon - c_1}{\frac{C_0}{C_1}\omega - \varepsilon(c_1-\varepsilon)} e^{\frac{\frac{C_0}{C_1}\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t}.
\end{aligned}$$

*Proof.* Differentiation of (2) with respect to $t$ leads to the operator $\tilde{A} := A + m(0,\cdot)$ with spectrum $\sigma(\tilde{A}) \subseteq [\varepsilon, \infty)$. Using Lemma 10 (i) and (ii), one can follow the same steps as in the proof of Lemma 3 to prove the requested results. □

**Remark 14.** *To obtain a comparable result to Corollary 4, one will need higher regularity for the kernel-function. But this is not sensible for the futher theory.*

## 3.3 Non-linear problem

Assume $k \in \mathbb{N}$ such that $4k > n$, $G \subseteq \mathbb{R}^n$ is a bounded domain with $C^{2k}$-boundary, $u_0 \in D(A^{k+1})$ and $F \in C^{2k+1}(\mathbb{R},\mathbb{R})$ with $F^{(2k+1)}$ locally Lipschitz-continuous and $F(u_0(x)) > -q + \varepsilon$ for a $\varepsilon > 0$ and for all $x \in G$.

*i)* Let $c_1 > \varepsilon$.

*ii)* Suppose $\omega > 0$ such that $C_3\omega < \varepsilon(c_1 - \varepsilon)$ and $\frac{C_0}{C_1}\omega < \varepsilon(c_1 - \varepsilon)$.

*iii)* Assume $\alpha > 0$ such that $(\alpha + 1)\frac{C_3\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon} \leq -c_1$.

*iv)* Let $v_1 > 0$ such that

$$\begin{aligned}
&v_1 C_3 C_4 \|Au_0\|^{\alpha+1}_{D(A^k)} \left(\frac{\varepsilon - c_1}{C_3\omega - \varepsilon(c_1-\varepsilon)}\right)^{\alpha} \leq \omega\\
\text{and}\quad &v_1 C_3 C_4 \|Au_0\|^{\alpha+2k+1}_{D(A^k)} \left(\frac{\varepsilon - c_1}{C_3\omega - \varepsilon(c_1-\varepsilon)}\right)^{\alpha+2k} \leq \omega.
\end{aligned}$$

Furthermore, suppose

*v)* $\left|F^{(i)}(x)\right| \leq v_1 |x|^{\alpha}$, $i = 0, \ldots, 2k+1$, $x \in \mathbb{R}$.

*vi)* $\|F(u_0)\|_{D(A^k)} \leq \frac{\omega}{c_1}$ for all $x \in G$.

We consider

$$\mathscr{X} := \left\{u \in C^1\left([0,\infty), D\left(A^k\right)\right) : \|u\|_{\mathscr{X}} < \infty\right\},$$

where $\|u\|_{\mathscr{X}} := \max\left\{\sup_{t\in[0,\infty)} \|u(t)\|_{D(A^k)}, \sup_{t\in[0,\infty)} \|u_t(t)\|_{D(A^k)}\right\}$ and

$$\mathscr{C} := \left\{ u \in \mathscr{X} \;\middle|\; \begin{array}{rcl} u(0) & \overset{D(A)}{=} & u_0, \\ \|u(t)\|_{D(A^k)} & \leq & \|Au_0\|_{D(A^k)} \frac{\varepsilon - c_1}{C_3\omega - \varepsilon(c_1-\varepsilon)} e^{\frac{C_3\omega-\varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t}, \\ \|u_t(t)\|_{D(A^k)} & \leq & \|Au_0\|_{D(A^k)} e^{\frac{C_3\omega-\varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t}, \\ \|u(t)\| & \leq & \|u_0\|_{D(A)} \frac{\varepsilon-c_1}{\frac{C_0}{C_1}\omega - \varepsilon(c_1-\varepsilon)} e^{\frac{\frac{C_0}{C_1}\omega-\varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t}, \\ \|u_t(t)\| & \leq & \|u_0\|_{D(A)} e^{\frac{\frac{C_0}{C_1}\omega-\varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t} \end{array} \right\}.$$

Due to (vi) and Lemma 13, $\mathscr{C}$ is not empty. We define

$$\mathscr{T} : \mathscr{C} \to \mathscr{C},\ v \mapsto \mathscr{T}v := u_v, \tag{11}$$

with the solution $u_v$ of (10) and the kernel-function $m := F \circ v$. We will prove that $\mathscr{T}$ is well-defined. For this, let $v \in \mathscr{C}$. Due to Lemma 10, one has $m \in C^1([0,\infty), D(A^k))$, $m'(t) = F'(v(t))v_t(t)$ in $D(A^k)$, $m(0)(x) = F(u_0(x)) > -q + \varepsilon$ for all $x \in G$ and due to (v), Lemma 10 (ii) and (iii)

$$\begin{aligned} & \|m_t(t)\|_{D(A^k)} \\ \leq\ & C_3C_4v_1\|Au_0\|_{D(A^k)}^{K(v(t))+1} \left(\frac{\varepsilon-c_1}{C_3\omega-\varepsilon(c_1-\varepsilon)}\right)^{K(v(t))} e^{(K(v(t))+1)\frac{C_3\omega-\varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t} \\ \overset{(iii),(iv)}{\leq}\ & \omega e^{-c_1 t}. \end{aligned}$$

Furthermore, one has using Lemma 10 (iii)

$$\begin{aligned} \|m(t)\|_{D(A^k)} & \leq & C_4v_1\|Au_0\|_{D(A^k)}^{\alpha} \left(\frac{\varepsilon-c_1}{C_3\omega-\varepsilon(c_1-\varepsilon)}\right)^{\alpha} e^{\alpha\frac{C_3\omega-\varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t} \\ & \overset{(i),(ii)}{\longrightarrow} & 0 \quad \text{for } t\to\infty. \end{aligned}$$

Application of Theorem 12 and Lemma 13 proves $\mathscr{T}v \in \mathscr{C}$.

**Remark 15.** *To obatin a fixed-point for $\mathscr{T}$, application of Schauder fixed-point theorem will not be successful (see Remark 14). Due to that, we will need a different ansatz.*

Let $u^0 \in \mathscr{C}$ and $u^n := \mathscr{T}u^{n-1} \in \mathscr{C}$ $(n \in \mathbb{N})$. We consider for $N > 0$ $\mathscr{X}_N := C^1([0,N], D(A^k))$ with the norm

$$\|u\|_N := \max\left\{\sup_{t\in[0,N]} \|u(t)\|_{D(A^k)}, \sup_{t\in[0,N]} \|u_t(t)\|_{D(A^k)}\right\},$$

resp. for $p > 0$

$$\|u\|_{N,p} := \max\left\{\sup_{t\in[0,N]} e^{-pt}\|u(t)\|_{D(A^k)}, \sup_{t\in[0,N]} e^{-pt}\|u_t(t)\|_{D(A^k)}\right\}.$$

By a straightforward calculation, we easily see that $(u^n)_{n\in\mathbb{N}} \subseteq \mathscr{X}_N$ is a Cauchy-sequence for all $N > 0$, i.e.

$$\forall N > 0\ \exists u_N \in \mathscr{X}_N : \|u^n - u_N\|_N \overset{n\to\infty}{\longrightarrow} 0.$$

We consider for all $t \in [0,\infty)$: $u(t) := u_N(t)$ if $t \leq N$. This defines an element $u \in \mathscr{C}$ that fulfils $\|u^n - u\|_N \overset{n\to\infty}{\longrightarrow} 0$ for all $N > 0$. Limit of

$$u_t^{n+1}(t) + Au^{n+1}(t) + \int_0^t F(u^n(t-s))u^{n+1}(s)ds = 0$$

as $n \to \infty$ proves $\mathscr{T}u = u$, i.e. $u$ is a solution of (2).

**Theorem 16.** *$u$ is the unique solution of (2) in $C^1([0,\infty), D(A^k))$.*

*Proof.* Let $N > 0$, $u_1, u_2 \in C^1([0,\infty), D(A^k))$ solutions of (2) with initial-values $u_1(0) = u_2(0) = u_0 \in D(A^k)$. One has

$$\exists M > 0\ \forall t \in [0,N]\ \forall x \in \bar{G} : |u_i(t)| \leq M,\ \ i = 1,2.$$

Let $L > 0$ be a Lipschitz-constant of $F$ on $[-M,M]$, $M_1 := \max_{x\in[-M,M]} |F'(x)|$ and $w := u_1 - u_2$ then one has for $t \in [0,N]$

$$\begin{aligned} &w_t + (A + F(u_0))w \\ &\quad + \int_0^t \left(F(u_1(t-s)) - F(u_2(t-s))\right) u_{1t}(s) - \left(\frac{d}{ds}F(u_2(t-s))\right) w(s)ds = 0. \end{aligned}$$

Multiplication with $w(t)$ in $L^2(G)$ leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 + \langle (A+F(u_0))w(t), w(t)\rangle \\ &\leq \frac{1}{2}\|w(t)\|^2 + \frac{1}{2}N\int_0^t L^2\frac{C_0}{C_1}\|w(t-s)\|^2 \max_{s\in[0,N]} \|u_{1t}(s)\|^2_{D(A^k)} \\ &\quad + \frac{C_0}{C_1}M_1 \max_{s\in[0,N]} \|u_{2t}(s)\|^2_{D(A^k)}\|w(s)\|^2 ds \\ &=: \frac{1}{2}\|w(t)\|^2 + C\int_0^t \|w(s)\|^2 ds. \end{aligned}$$

Gronwall's inequality proves $u_1 = u_2$. □

Altogether, we have proved the following

**Theorem 17.** *Assume $k \in \mathbb{N}$ such that $4k > n$, $G \subseteq \mathbb{R}^n$ is a bounded domain with $C^{2k}$-boundary and $A : D(A) \to L^2(G)$ the operator (3) with $a_{ij}, a \in C^{2k-1}(\overline{G})$ $(i,j = 1,\dots,n)$ and spectrum $\sigma(A) \subseteq [q,\infty)$ $(q > 0)$. Furthermore, let $u_0 \in D(A^{k+1})$, $F \in C^{2k+1}(\mathbb{R},\mathbb{R})$ with $F^{(2k+1)}$ locally Lipschitz-continuous and $F(u_0(x)) > -q + \varepsilon$ for a $\varepsilon > 0$ and for all $x \in G$.*

*i) Assume $c_1 > \varepsilon$.*

*ii) Let* $\omega > 0$ *such that* $C_3\omega < \varepsilon(c_1 - \varepsilon)$ *and* $\frac{C_0}{C_1}\omega < \varepsilon(c_1 - \varepsilon)$*.*

*iii) Suppose* $\alpha > 0$ *such that* $(\alpha+1)\frac{C_3\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon} \leq -c_1$*.*

*iv) Let* $v_1 > 0$ *such that*

$$v_1 C_3 C_4 \|Au_0\|_{D(A^k)}^{\alpha+1} \left(\frac{\varepsilon - c_1}{C_3\omega - \varepsilon(c_1-\varepsilon)}\right)^{\alpha} \leq \omega$$

$$\text{and} \quad v_1 C_3 C_4 \|Au_0\|_{D(A^k)}^{\alpha+2k+1} \left(\frac{\varepsilon - c_1}{C_3\omega - \varepsilon(c_1-\varepsilon)}\right)^{\alpha+2k} \leq \omega.$$

*In addition to that, suppose the following smallness-conditions on* $F$*:*

*v)* $\left|F^{(i)}(x)\right| \leq v_1|x|^{\alpha}$*,* $i = 0, \ldots, 2k+1$*,* $x \in \mathbb{R}$*,*

*vi)* $\|F(u_0)\|_{D(A^k)} \leq \frac{\omega}{c_1}$ *for all* $x \in G$*.*

*Then problem (2) has a unique solution* $u \in C^1\left([0,\infty), D\left(A^k\right)\right)$ *that fulfils*

$$\|u(t)\|_{D(A^k)} \leq \|Au_0\|_{D(A^k)} \frac{\varepsilon - c_1}{C_3\omega - \varepsilon(c_1-\varepsilon)} e^{\frac{C_3\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t},$$

$$\|u_t(t)\|_{D(A^k)} \leq \|Au_0\|_{D(A^k)} e^{\frac{C_3\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t}$$

$$\text{and} \quad \|u(t)\| \leq \|u_0\|_{D(A)} \frac{\varepsilon - c_1}{\frac{C_0}{C_1}\omega - \varepsilon(c_1-\varepsilon)} e^{\frac{\frac{C_0}{C_1}\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t},$$

$$\|u_t(t)\| \leq \|u_0\|_{D(A)} e^{\frac{\frac{C_0}{C_1}\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t}.$$

## 3.4 Example

**Corollary 18.** *Assume* $n \leq 3$ *and* $f \in C^3\left(\left[-\frac{C_0}{C_1}\|Au_0\|_{D(A)}\frac{2}{q}, \frac{C_0}{C_1}\|Au_0\|_{D(A)}\frac{2}{q}\right], \mathbb{R}\right)$ *four times differentiable in* $x = 0$ *with* $f'''$ *locally Lipschitz-continuous and* $f(0) = f'(0) = f''(0) = f'''(0) = 0$*. Then there exists a* $\kappa_0 > 0$ *such that for all* $\kappa \in (0, \kappa_0]$ *problem (2) for* $F = \kappa \cdot f$ *has a unique solution* $u \in C^1([0,\infty), D(A))$ *such that* $u$ *and* $u_t$ *decay exponentially with respect to* $\|\cdot\|_{D(A)}$ *and* $\|\cdot\|$*.*

*Proof.* We easily see that there exists a $M > 0$ such that $|f'''(x)| \leq M|x|$ for $x \in \left[-\frac{C_0}{C_1}\|Au_0\|_{D(A)}\frac{2}{q}, \frac{C_0}{C_1}\|Au_0\|_{D(A)}\frac{2}{q}\right]$. It follows $|f''(x)| \leq \frac{M}{2}|x|^2$, $|f'(x)| \leq \frac{M}{6}|x|^3$ and $|f(x)| \leq \frac{M}{24}|x|^4$. We set

$$\alpha := 1, c_1 := q, \varepsilon := \frac{c_1 + \frac{c_1}{\alpha+1}}{2} = \frac{3}{4}q \quad \text{and}$$

$$\omega := \min\left\{\frac{c_1 - \varepsilon}{C_3}\left[\varepsilon - \frac{c_1}{\alpha+1}\right], \frac{\varepsilon(c_1-\varepsilon)}{2C_3}, \frac{\varepsilon(C-1-\varepsilon)C_1}{2C_0}\right\}$$
$$= \min\left\{\frac{q^2}{16C_3}, \frac{3q^2}{24C_3}, \frac{3q^2C_1}{24C_0}\right\}.$$

Furthermore, let $v_1 > 0$ such that $v_1 \leq \frac{\frac{3}{4}\omega q^2 - 4C_3\omega^2}{qC_3C_4\|Au_0\|^2_{D(A)}}$ and $v_1 \leq \frac{\omega\left(\frac{3}{4}q - 4C_3\omega\right)^3}{C_3C_4\|Au_0\|^2_{D(A)}q^3}$. We chose $\kappa_0 > 0$ such that

$$\kappa_0 \frac{M}{6} \leq v_1, \quad \kappa_0 \frac{M}{6}\frac{C_0}{C_1}\|Au_0\|_{D(A)}\frac{2}{q} \leq v_1 \quad \text{and} \quad \kappa_0\|f(u_0)\|_{D(A)} \leq \frac{\omega}{c_1}.$$

The functions $F_\kappa := \kappa \cdot f$ $(\kappa \in [0, \kappa_0])$ fulfil the conditions of Theorem 17 that proves everything. □

## 3.5 Remarks

**Remark 19.** *Due to the boundedness of the solutions subject to Theorem 17, it is sufficient to formulate the smallness-conditions on the interval*

$$\left[-\frac{C_0}{C_1}\|Au_0\|_{D(A^k)}\frac{\varepsilon - c_1}{C_3k - \varepsilon(c_1 - \varepsilon)}, \frac{C_0}{C_1}\|Au_0\|_{D(A^k)}\frac{\varepsilon - c_1}{C_3k - \varepsilon(c_1 - \varepsilon)}\right].$$

**Remark 20.** *The methods presented in this chapter can easily be extended on inhomogenious problems of the kind*

$$u_t(t) + Au(t) + \int_0^t F(u(t-s))u_t(s)ds = f(t), \quad u(0) = u_0,$$

*where $f \in C^1([0,\infty), D(A^k))$. To obtain a self-mapping similar to (11), we will need further conditions as*

$$\|f'(t)\|_{D(A^k)} \leq Me^{-dt}, \quad \lim_{t\to\infty}\|f(t)\|_{D(A^k)} = 0,$$

*where $M \geq 0$ and $d > 0$ such that $d(c_1 - \varepsilon) > k$ if $d < \varepsilon$. In this case, the smallness-parameters on $F$ will additionally depend on $M$ and $d$.*

**Remark 21.** *The smallness-conditions in Theorem 17 are formulated for fixed initial-values $u_0$. Alternatively, one can understand them as smallness-conditons on $u_0$ for fixed kernels (i.e. fixed $\alpha$ and $v_1$).*

**Remark 22.** *Beside Dirichlet-boundary conditions, one can consider problems (1) and (2) for instance with Neumann-boundary conditions. For this, let $a_{ij}, a \in C^{2k-1}(\bar{G})$ $(i, j = 1, \ldots, n)$ and*

$$B(u, v) := \sum_{i,j=1}^n \langle a_{ij}(\cdot)\partial_j u, \partial_i v\rangle + \langle a(\cdot)u, v\rangle$$

*for $u, v \in H^1(G)$. We define*

$$D(A) := \left\{u \in H^1(G) \,\middle|\, \exists f_u \in L^2(G)\ \forall v \in H^1(G) : B(u, v) = \langle f_u, v\rangle\right\}$$

*and*

$$A : D(A) \to L^2(G), u \mapsto f_u.$$

*If $(a_{ij}(\cdot))_{ij}$ is symmetric and uniformly positive-definite and $a(x) > a_0$ for a given $a_0 > 0$ and for all $x \in G$, one obtains from Theorem A 10.3 from [2]*

$$\exists C > 0\ \forall u \in D\left(A^k\right) : u \in H^{2k}(G), \|u\|_{H^{2k}(G)} \leq C\|u\|_{D(A^k)}.$$

*A is self-adjouned with spectrum* $\sigma(A) \subseteq [a_0, \infty)$ *(see [15, Chapter 2.4]). Due to this, the preliminaries V 1–V 3 can be proved analogously. We sketch the proof of the other preliminaries for the case* $n \leq 3$. *Let* $F \in C^2(\mathbb{R}, \mathbb{R})$ *and* $u \in C^\infty(\bar{G}) \cap D(A)$. *One has*

$$\sum_{i,j=1}^{3} \nu_i a_{ij} \partial_j \left(F(u)\right) = F'(u) \sum_{i,j=1}^{3} \nu_i a_{ij} \partial_j u = 0,$$

*i.e.* $F(u) \in D(A)$. *By this, the proofs of the results V 4–V 7 can be transferred easily on the Neumann-case, as well as Theorem 17. To treat problems with more general operators, we refer the reader to [15], [16] and [20].*

# References


[1] R. A. Adams and J. J. F. Fournier. *Sobolev spaces.* Academic Press, Amsterdam, $2^{nd}$ edition, 2003.

[2] H. W. Alt. *Lineare Funktionalanalysis.* Springer, 2012.

[3] H. Amann. *Linear and Quasilinear Parabolic Problems: Volume I: Abstract Linear Theory.* Birkhäuser, Basel, 1995.

[4] J. A. D. Appleby, I. Gyõri, and D. W. Reynolds. Subexponential solutions of linear scalar integrodifferential equations with a delay. *Funct. Differ. Equ.*, 11(1-2):11–18, 2004.

[5] J. A. D. Appleby and D. W. Reynolds. On the non-exponential convergence of asymptotically stable solutions of linear scalar volterra integro-differential equations. *J. Integral Equations Appl.*, 14(2):109–118, 2002.

[6] J. A. D. Appleby and D. W. Reynolds. On necessary and sufficient conditions for exponential stability in linear Volterra integro–differential equations. *J. Integral Equations Appl.*, 16(3):221–240, 2004.

[7] J. A. D. Appleby and D. W. Reynolds. On the non-exponential decay to equilibrium of solutions of nonlinear scalar Volterra integro-differential equations. *Electron. J. Qual. Theory Differ. Equ.*, Proc. 7th Coll. QTDE(3):1–14, 2004.

[8] H. Brunner and Z. W. Yang. Blow-up behavior of Hammerstein-type Volterra integral equations. *J. Integral Equations Appl.*, 24(4):487, 2012.

[9] P. J. Bushell and W. Okrasinski. On the maximal interval of existence for solutions to some non-linear Volterra integral equations with convolution kernel. *Bull. London Math. Soc.*, 28(1):59–65, 1996.

[10] L. C. Evans. *Partial differential equations.* American Mathematical Society, Providence, RI, $2^{nd}$ edition, 2010.

[11] G. Gripenberg, S.-O. Londen, and O. Staffans. *Volterra Integral and Functional Equations.* Cambridge University Press, Cambridge [u.a.], 2009.

[12] W. Götze. The essentials of the mode-coupling theory for glassy dynamics. *Cond. Matter Phys.*, 1(4(16)):873–904, 1998.

[13] P. Kurth. Nichtlineare Integro-Differentialgleichungen aus der mathematischen Physik. *Ph.D. thesis, University of Konstanz*, 2013.

[14] P. Kurth. On a new class of integro-differential equations. *Konstanzer Schriften Math.*, (326), 2014.

[15] R. Leis. *Initial boundary value problems in mathematical physics.* Teubner, Stuttgart, 1986.

[16] J. L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications*, volume 1. Springer, Berlin, Heidelberg, New York, 1972.

[17] J. Prüss. *Evolutionary integral equations and applications.* Birckhäuser, Basel, Heidelberg [u.a.], 2012.

[18] R. Racke. *Lectures on Nonlinear Evolution Equations.* Vieweg, Braunschweig, 1992.

[19] M. Saal. Well-posedness and asymptotics of some nonlinear integro-differential equations. *J. Integr. Equat. Appl.*, 25(no. 1):103–141, 2013.

[20] H. Triebel. *Interpolation theory, function spaces, differential operators.* North-Holland, Amsterdam [u.a.], 1978.

[21] S.-T. Wu. Blow-up of solutions for an integro-differential equation with a nonlinear source. *Electron. J. Differ. Equ.*, 2006(45):1–9, 2006.

[22] S.-T. Wu. On global existence and blow-up of solutions for an integro-differential equation with strong damping. *Taiwanese journal of mathematics*, 10(4):979–1014, 2006.

[23] S.-T. Wu. General decay and blow-up of solutions for a viscoelastic equation with nonlinear boundary damping-source interactions. *Z. Angew. Math. Phy.*, 63:65–106, 2012.

[24] S.-T. Wu and C.-Y. Lin. Global nonexistence for an integro-differential equation. *Math. Meth. Appl. Sci.*, 35(1):72–83, 2012.