Convex monotone semigroups on spaces of continuous functions

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0.1 Acknowledgements

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0.3 Introduction

Motivated by model uncertainty and stochastic control problems, this thesis aims to develop a systematic theory for strongly continuous convex monotone semigroups on spaces of continuous functions. The present approach is self-contained and does, in particular, not rely on the theory of viscosity solutions. Instead, we provide a comparison principle for semigroups related to HJB-type equations which uniquely determines the semigroup by its infinitesimal generator evaluated at smooth functions. While the statement itself resembles the classical analogue for strongly continuous linear semigroups, the proof requires the introduction of several novel analytical concepts such as the Lipschitz set and the Γ-generator. Furthermore, we provide general approximation schemes and stability results for nonlinear semigroups. While the focus of this thesis is on the development of a systematic theory for convex monotone semigroups, the abstract results are all motivated by applications and therefore providing conditions that can easily be verified is also a major contribution of this work. In the sequel, we briefly present the main results of this thesis in a rather informal and intuitive way and emphasize the links between nonlinear semigroups and various applications. Regarding technical details, formal notations and related literature, we mainly refer to the corresponding chapters.

At the beginning, our viewpoint on nonlinear semigroup was mainly motivated by model uncertainty. The classical theory of stochastic processes is based on the premise that, although the future can not be predicted, the probability that certain events occur can be modelled by a distribution. For example, in case of a one-dimensional symmetric random walk, for every time step, we have the possibility of either moving up or down. If we toss a coin to decide whether we move up or down, we can be fairly certain that both events occur with probability one half. By repeating this procedure, we can compute the probability for events like "moving up at least five times in the first ten steps" to derive a stochastic description for the whole stochastic process which is called *model in discrete time*. Furthermore, by considering an increasing number of decreasing time steps, one can derive the Brownian motion as a scaling limit of random walks which describes the random movement of a particle in *continuous time*. However, when modelling more complicated phenomena such as stock markets, it is not always clear whether the chosen model actually describes the reality. This is called *model uncertainty* and leads to nonlinear semigroups resulting from a worst case approach. For instance, if a stock market is modelled by a Brownian motion $(W_t)_{t\geq 0}$ with volatility $\sigma \geq 0$, the price of an discounted option with pay-off function f and maturity T at time $t \in [0,T]$ is given by

$$(S_\sigma(t)f)(x) := \mathbb{E}[f(x+\sigma W_t)],$$

where x is the current state of the discounted stock. Furthermore, the family $(S_\sigma(t))_{t\geq 0}$ of operators $S_\sigma(t)\colon \mathrm{C_b} \to \mathrm{C_b}$ forms a *linear semigroup*. In order to guarantee that this price is not to low, a conservative approach consists in taking the supremum over a set of parameters $\sigma \in \Sigma$, i.e.,

$$(I(t)f)(x) := \sup_{\sigma\in\Sigma} \mathbb{E}[f(x+\sigma W_t)] = \sup_{\sigma\in\Sigma} (S_\sigma(t)f)(x). \tag{0.1}$$

Hence, the price is based on the worst possible outcome among all considered models. While the family $(I(t))_{t\geq 0}$ of operators $I(t)\colon \mathrm{C_b} \to \mathrm{C_b}$ incorporates the aspect of model uncertainty, due to taking the supremum over a family of parameters, we loose the semigroup property. At this point, we recall that the Brownian motion can be obtained as a scaling limit of iterated coin flips describing the random outcome in one period. Transferred to the present example, this means that we want to construct a *nonlinear semigroup* $(S(t))_{t\geq 0}$ as the limit

$$S(t)f = \lim_{n\to\infty} \underbrace{\big(I(\tfrac{t}{n})\circ\ldots\circ I(\tfrac{t}{n})\big)}_{n \text{ times}} f$$

which gives the price of an option f depending on a stock market that is driven by a Brownian motion with uncertain volatility. The family $(S(t))_{t\geq 0}$ is the smallest semigroup dominating all linear models, i.e.,

$$S(t)f \geq S_\sigma(t)f \quad \text{for all } \sigma\in\Sigma.$$

Furthermore, the infinitesimal behaviour of the family $(I(t))_{t\geq 0}$ is given by

$$I'(0)f := \lim_{h\downarrow 0}\frac{I(h)f-f}{h} = \sup_{\sigma\in\Sigma}\frac{1}{2}\sigma^2 f''$$

for sufficiently smooth functions f. This property is transferred to the limit, i.e.,

$$Af := \lim_{h\downarrow 0}\frac{S(h)f-f}{h} = \sup_{\sigma\in\Sigma}\frac{1}{2}\sigma^2 f'' = \sup_{\sigma\in\Sigma} A_\sigma f,$$

where A_σ is the generator of the strongly continuous linear semigroup $(S_\sigma(t))_{t\geq 0}$. Hence, in order to state that $(S(t))_{t\geq 0}$ is the unique transition semigroup associated to a Brownian motion with uncertain drift and volatility, we need a comparison principle for strongly continuous convex monotone semigroups. Finally, we are interested in the question whether the semigroup $(S(t))_{t\geq 0}$ depends continuously on the set Σ, i.e., if $\Sigma_n\to\Sigma$ in the sense that $A_n f\to Af$ for all smooth functions f, we want to obtain $S_n(t)f\to S(t)f$ for all $(f,t)\in \mathrm{C_b}\times\mathbb{R}_+$. In the sequel, we briefly summarize the results of the Chapters 1–8.

In Chapter 1, we consider a family $(I(t))_{t\geq 0}$ of Lipschitz continuous mappings on a complete metric space X and show that the limit

$$S(t)x := \lim_{l\to\infty} I(h_{n_l})^{k^t_{n_l}} x \tag{0.2}$$

exists for all t in a countable dense set $\mathcal{T}\subset\mathbb{R}_+$ and all $x\in X$, where $k^t_n h_n\to t$ and $(n_l)_{l\in\mathbb{N}}$ is a suitable subsequence. Furthermore, the family $(S(t))_{t\in\mathcal{T}}$ can be extended to a strongly continuous nonlinear semigroup $(S(t))_{t\geq 0}$. The construction relies heavily on the so-called Lipschitz set and the key is to find conditions on the generating family $(I(t))_{t\geq 0}$ which can be transferred to the limit semigroup $(S(t))_{t\geq 0}$. Furthermore, under an additional condition, the infinitesimal generator of the semigroup is given by $Ax = I'(0)x$, whenever the limit $I'(0)x$ exists. For convex monotone operators defined on spaces of continuous function, the convergence in equation (0.2) can be verified by means of Arzéla–Ascoli's theorem and the statement about the generator is always valid. The abstract results are illustrated with several examples of nonlinear semigroups such as robustifications and perturbations of linear semigroups.

In Chapter 2, we provide a semigroup approach to viscous Hamilton–Jacobi equations. It turns out that exponential Orlicz hearts are suitable spaces to handle the (quadratic) non-linearity of the Hamiltonian and allow us to uniquely determine the semigroup via its infinitesimal generator. After establishing an abstract extension result for nonlinear semigroups on spaces of continuous functions, we represent the solution of the viscous Hamilton-Jacobi equation as a strongly continuous convex monotone semigroup on an exponential Orlicz heart. As a result, the solution depends continuously on the initial data. Furthermore, we determine the so-called symmetric Lipschitz set which is invariant under the semigroup. This automatically yields a priori estimates and regularity in Sobolev spaces. In particular, on the domain restricted to the symmetric Lipschitz set, the generator can be explicitly determined and linked with the viscous Hamilton–Jacobi equation.

In Chapter 3, we study convex monotone operators on spaces of continuous functions which are continuous w.r.t. the mixed topology and provide a short introduction to Γ-convergence. It turns out that, for convex monotone operators, continuity w.r.t. the mixed topology, continuity from above and upper semicontinuity w.r.t. Γ-convergence are equivalent. Furthermore, we are interested in extensions from spaces of continuous functions to spaces of upper semicontinuous functions and provide sufficient conditions which guarantee uniform continuity from above for bounded families of convex monotone operators. These results are frequently used in the subsequent chapters.

In Chapter 4, we show that strongly continuous convex monotone semigroups on spaces of continuous function can be uniquely determined by their Γ-generators defined on their upper Lipschitz sets. While the statement itself resembles the classical analogue from the linear semigroup theory, the proof is technically much more involved and relies heavily on the results of Chapter 3. Furthermore, under an additional assumption, we provide approximation results for the Γ-generator which are particular useful if we consider functions $f\colon \mathbb{R}^d \to \mathbb{R}$. In this case, we give explicit conditions which guarantee that strongly continuous convex monotone semigroups are already uniquely determined by their generators evaluated at smooth functions. This is a major improvement since the latter can typically computed in applications.

In Chapter 5, we pick up the ideas of Chapter 1 but the previously required norm convergence is replaced by convergence w.r.t. the mixed topology. Hence, the relative compactness which guarantees the convergence in equation (0.2) is imposed for a significantly weaker topology. Furthermore, we give explicit conditions on the one-step operators $(I(t))_{t\geq 0}$ that are transferred to the limit semigroup $(S(t))_{t\geq 0}$ such that the latter satisfies the comparison principle from Chapter 4. In particular, the limit does not depend on the choice of the convergent subsequence and we obtain

$$S(t)f = \lim_{n\to\infty} I(h_n)^{k_n^t} f \tag{0.3}$$

for all $t \geq 0$ and sequences $h_n k_n^t \to t$.

In Chapter 6, we illustrate the abstract results of the previous two chapters by showing that value functions of certain stochastic optimal control problems can be approximated using piecewise constant policies. Furthermore, the symmetric Lipschitz set provides a regularity result in Sobolev spaces and allows us to link the Γ-generator with distributional derivatives. We further investigate how these results can be extended to trace class Wiener processes with drift. In both examples, the one-step operators are defined similarly to equation (0.1). Therefore, we can interpret the value function also as transition semigroup corresponding to a Brownian motion with uncertain drift and volatility. The natural question arises whether we can consider not only parameter uncertainty but also non-parametric uncertainty, where we take the supremum over arbitrary distributions that are penalized according to their distance to a certain reference model. This leads to so-called Wasserstein perturbations of linear transition semigroups.

In Chapter 7, we investigate the structural link between approximation schemes in form of equation (0.3) and limit theorems for convex expectations. It turns out that, depending on whether we scale with $1/n$ or $1/\sqrt{n}$, normalized sums of iid samples convergence either to a maximal distribution or a G-distribution. The latter corresponds to a Brownian motion with uncertain volatility whose transition semigroup can also be obtained by starting with the family of one-step operators defined by equation (0.1). In contrast to previous works, our results are not restricted to sublinear expectations and therefore cover applications to model uncertainty as well as to large deviations. For instance, Cramér's theorem can be seen a LLN for the entropic risk measure.

In Chapter 8, we consider sequences $(S_n)_{n\in\mathbb{N}}$ of convex monotone semigroups and give explicit conditions under which convergence of their infinitesimal generators $(A_n)_{n\in\mathbb{N}}$ guarantees convergence of the semigroups $(S_n)_{n\in\mathbb{N}}$. The limit semigroup is first defined as $S(t)f := \lim_{l\to\infty} S_{n_l}(t)f$ for all (t,f) in a countable dense set and then extended to arbitrary (t,f). Furthermore, the generator generator of $(S(t))_{t\geq 0}$ is given by $Af = \lim_{n\to\infty} A_n f$ for smooth functions f. Since the semigroups $(S_n)_{n\in\mathbb{N}}$ satisfy the comparison principle from Chapter 4 which transfers to $(S(t))_{t\geq 0}$, the limit semigroup does not depend on the choice of the convergent subsequence $(n_l)_{l\in\mathbb{N}}$ and therefore satisfies $S(t)f = \lim_{n\to\infty} S_n(t)f$. The framework also allows for discretizations in time and space and extends the approximation schemes studied in Chapter 5 of the form $S(t)f = \lim_{n\to\infty} I_n^{k_n} f$, where $(I_n)_{n\in\mathbb{N}}$ is a family of one-step operators describing the dynamics on a discrete time scale of size $h_n > 0$ with $h_n \to 0$ and $k_n h_n \to t$. The abstract results are illustrated by a variety of applications including Euler schemes and Yosida-type approximations for upper envelopes of families of linear semigroups, stability results and finite-difference schemes for convex HJB equations, Freidlin–Wentzell-type results and Markov chain approximations for a class of stochastic optimal control problems and continuous-time Markov processes with uncertain transition probabilities.

Parts of this thesis can be found in [23–27].

0.4 Deutsche Zusammenfassung

Motiviert durch Modellunsicherheit und stochastische Kontrollprobleme zielt diese Arbeit darauf ab eine systematische Theorie für stark stetige konvexe monotone Halbgruppen auf Räumen stetiger Funktionen zu entwickeln. Der vorliegende Ansatz ist in sich geschlossen und stützt sich insbesondere nicht auf die Theorie der Viskositätslösungen. Stattdessen zeigen wir im Zusammenhang zu HJB-Gleichungen, dass nichtlineare Halbgruppen eindeutig durch ihren infinitesimalen Generator ausgewertet an glatten Funktionen bestimmt sind. Während die Aussage selbst dem klassischen Analogon für stark stetige lineare Halbgruppen ähnelt, erfordert der Beweis die Einführung mehrerer neuer analytischer Konzepte wie der Lipschitzmenge und den Γ-Generator. Darüber hinaus liefern wir allgemeine Approximationschemata und Stabilitätsergebnisse für nichtlineare Halbgruppen. Während der Schwerpunkt dieser Arbeit auf der Entwicklung einer systematischen Theorie für konvexe monotone Halbgruppen liegt, sind die abstrakten Ergebnisse alle durch Anwendungen motiviert und daher ist die Bereitstellung von Bedingungen, die leicht überprüft werden können, ebenfalls ein wichtiger Bestandteil dieser Arbeit. Im Folgenden werden wir kurz die Hauptergebnisse dieser Arbeit in einer eher informellen und intuitiven Weise präsentieren und die Verbindungen zwischen nichtlinearen Halbgruppen und verschiedenen Anwendungen betonen. Bezüglich technischer Details, formaler Notationen und verwandter Literatur verweisen wir hauptsächlich auf die entsprechenden Kapitel.

Zu Beginn war unsere Sicht auf nichtlineare Halbgruppen hauptsächlich durch Modellunsicherheit motiviert. Die klassische Theorie stochastischer Prozesse basiert auf der Prämisse, dass man, auch wenn die Zukunft nicht vorhersagbar ist, die Eintrittswahrscheinlichkeiten für bestimmte Ereignisse durch eine Verteilung modellieren kann. Im Falle einer eindimensionalen symmetrischen Irrfahrt hat man beispielsweise für jeden Schritt die Möglichkeit sich entweder nach oben oder nach unten zu bewegen. Wenn wir eine Münze werfen um zu entscheiden, ob wir uns nach oben oder nach unten bewegen, können wir ziemlich sicher sein, dass beide Ereignisse mit der Wahrscheinlichkeit einhalb eintreten. Wenn wir dieses Vorgehen wiederholen, können wir die Wahrscheinlichkeit für Ereignisse wie "Wir bewegen uns in den ersten zehn Schritten mindestens fünfmal nach oben" berechnen um eine stochastische Beschreibung für den gesamten Prozess herzuleiten, was wir als *Modell in diskreter Zeit* bezeichnen. Außerdem erhält man durch die Berücksichtigung einer zunehmenden Anzahl kleiner werdender Zeitschritte, die Brownsche Bewegung als Skalierungsgrenzwert von Irrfahrten, was die zufällige Bewegung eines Teilchens in *stetiger Zeit* beschreibt. Bei der Modellierung komplizierterer Phänomene wie beispielsweise Aktienmärkten ist jedoch nicht immer klar, ob das gewählte Modell tatsächlich die Realität beschreibt. Dies wird als *Modellunsicherheit* bezeichnet und führt aufgrund eines Worst-Case-Ansatzes zu nichtlinearen Halbgruppen. Wenn zum Beispiel ein Aktienmarkt durch eine Brownsche Bewegung $(W_t)_{t\geq 0}$ mit Volatilität $\sigma \geq 0$ modelliert wird, so ist der Preis einer diskontierten Option mit Auszahlungsfunktion f und Fälligkeitsdatum T zum Zeitpunkt $t \in [0,T]$ gegeben durch

$$(S_\sigma(t)f)(x) := \mathbb{E}[f(x+\sigma W_t)],$$

wobei x der aktuelle Zustand der diskontierten Aktie ist. Außerdem ist die Familie $(S_\sigma(t))_{t\geq 0}$ der Operatoren $S_\sigma(t)\colon \mathrm{C_b} \to \mathrm{C_b}$ eine *lineare Halbgruppe*. Um zu garantieren, dass dieser Preis nicht zu niedrig ist, besteht ein konservativer Ansatz darin, das Supremum über einen Menge von Parametern $\sigma \in \Sigma$ zu nehmen, d.h.

$$(I(t)f)(x) := \sup_{\sigma\in\Sigma} \mathbb{E}[f(x+\sigma W_t)] = \sup_{\sigma\in\Sigma} (S_\sigma(t)f)(x). \tag{0.4}$$

Der Preis basiert also auf dem schlechtest möglichen Ergebnis aller betrachteten Modelle. Während die Familie $(I(t))_{t\geq 0}$ der Operatoren $I(t)\colon \mathrm{C_b} \to \mathrm{C_b}$ den Aspekt der Modellunsicherheit berücksichtigt, verlieren wir die Halbgruppeneigenschaft, da wir das Supremum über eine Familie von Parametern nehmen. An dieser Stelle sei daran erinnert, dass die Brownsche Bewegung als Skalierungsgrenzwert von iterierten Münzwürfen erhalten werden kann, welche das zufällige Ergebnis in einer Periode beschreiben. Übertragen auf das vorliegende Beispiel bedeutet dies, dass wir eine *nichtlineare Halbgruppe* $(S(t))_{t\geq 0}$ als Grenzwert

$$S(t)f = \lim_{n\to\infty} \underbrace{\left(I(\tfrac{t}{n}) \circ \ldots \circ I(\tfrac{t}{n})\right)}_{n \text{ mal}} f$$

konstruieren wollen, was den Preis einer Option f in Abhängigkeit von einem Aktienmarkt angibt, der durch einer Brownschen Bewegung mit unsicherer Volatilität angetrieben wird. Die Familie $(S(t))_{t\geq 0}$ ist die kleinste Halbgruppe, die alle linearen Modelle dominiert, d.h.,

$$S(t)f \geq S_\sigma(t)f \quad \text{für alle } \sigma \in \Sigma.$$

Außerdem ist das infinitesimale Verhalten der Familie $(I(t))_{t\geq 0}$ gegeben durch

$$I'(0)f := \lim_{h\downarrow 0} \frac{I(h)f - f}{h} = \sup_{\sigma\in\Sigma} \frac{1}{2}\sigma^2 f''$$

für hinreichend glatte Funktionen f. Diese Eigenschaft wird auf den Grenzwert übertragen, d.h.

$$Af := \lim_{h\downarrow 0} \frac{S(h)f - f}{h} = \sup_{\sigma\in\Sigma} \frac{1}{2}\sigma^2 f'' = \sup_{\sigma\in\Sigma} A_\sigma f,$$

wobei A_σ der Generator der stark stetigen linearen Halbgruppe $(S_\sigma(t))_{t\geq 0}$ ist. Um also zu behaupten zu können, dass $(S(t))_{t\geq 0}$ die eindeutig bestimmte Übergangshalbgruppe einer Brownschen Bewegung mit unsicherem Drift und unsicherer Volatilität ist, benötigen wir ein Vergleichsprinzip für stark stetige konvexe monotone Halbgruppen. Abschließend interessieren wir uns für die Frage, ob die Halbgruppe $(S(t))_{t\geq 0}$ stetig von der Menge Σ abhängt, d.h. $\Sigma_n \to \Sigma$ im Sinne von $A_n f \to Af$ für alle glatte Funktionen f soll $S_n(t)f \to S(t)f$ für alle $(f,t) \in \mathrm{C_b} \times \mathbb{R}_+$ implizieren. Im Folgenden fassen wir die Ergebnisse der Kapitel 1–8 kurz zusammen.

In Kapitel 1 betrachten wir eine Familie $(I(t))_{t\geq 0}$ von Lipschitz stetigen Abbildungen auf einem vollständigen metrischen Raum X und zeigen, dass der Grenzwert

$$S(t)x := \lim_{l\to\infty} I(h_{n_l})^{k^t_{n_l}} x \tag{0.5}$$

für alle t in einer abzählbaren dichten Teilmenge $\mathcal{T} \subset \mathbb{R}_+$ und alle $x \in X$ existiert, wobei $k^t_n h_n \to t$ und $(n_l)_{l\in\mathbb{N}}$ eine geeignete Teilfolge ist. Außerdem kann die Familie

$(S(t))_{t\in\mathcal{T}}$ zu einer stark stetigen nichtlinearen Halbgruppe $(S(t))_{t\geq 0}$ erweitert werden. Die Konstruktion stützt sich stark auf die sogenannte Lipschitz-Menge und der Schlüssel ist es, Bedingungen für die erzeugende Familie $(I(t))_{t\geq 0}$ zu finden, welche auf die Grenzhalbgruppe $(S(t))_{t\geq 0}$ übertragen werden können. Des Weiteren ist unter einer zusätzlichen Annahme der infinitesimale Generator der Halbgruppe durch $Ax = I'(0)x$ gegeben, sofern der Grenzwert $I'(0)x$ existiert. Für konvexe monotone Operatoren, die auf Räumen stetiger Funktionen definiert sind, kann die Konvergenz in Gleichung (0.5) mit Hilfe des Satzes von Arzéla-Ascoli überprüft werden und die Aussage über den Generator ist immer gültig. Die abstrakten Ergebnisse werden anhand verschiendener Beispielen nichtlinearer Halbgruppen wie Robustifizierungen und Störungen von linearen Halbgruppen illustriert.

In Kapitel 2 verfolgen wir einen Halbgruppenansatz für viskose Hamilton–Jacobi Gleichungen. Es stellt sich heraus, dass exponentielle Orliczräume geeignet sind, um die in der Gleichung auftretende (quadratische) Nichtlinearität zu behandeln, und es uns erlauben, die Halbgruppe anhand ihres infinitesimalen Generator eindeutig zu bestimmen. Nach einem abstrakten Erweiterungsergebnis für nichtlineare Halbgruppen auf Räumen stetiger Funktionen, stellen wir die Lösung der viskosen Hamilton–Jacobi-Gleichung als stark stetige konvexe Halbgruppe auf einem exponentiellen Orliczraum dar. Die Lösung hängt stetig von den Anfangsdaten ab. Außerdem bestimmen wir die sogenannte symmetrische Lipschitzmenge, welche unter der Halbgruppe invariant ist. Daraus ergeben sich automatisch a priori Abschätzungen und Regularität in Sobolevräumen. Insbesondere können wir auf dem Schnitt der symmetrischen Lipschitzmenge mit dem Definitionsbereich des Generators letzteren explizit bestimmen und mit der viskosen Hamilton–Jacobi-Gleichung in Verbindung bringen.

In Kapitel 3 untersuchen wir konvexe monotone Operatoren auf Räumen stetiger Funktionen, welche bezüglich der gemischten Topologie stetig sind, und geben eine kurze Einführung in Γ-Konvergenz. Es zeigt sich, dass für konvexe monotone Operatoren Stetigkeit bezüglich der gemischten Topologie, Stetigkeit von oben und Oberhalbstetigkeit bezüglich Γ-Konvergenz äquivalent sind. Außerdem sind wir an Erweiterungen von Räumen stetiger Funktionen auf Räumen oberhalbstetiger Funktionen interessiert und geben hinreichende Bedingungen an, die gleichmäßige Stetigkeit von oben für beschränkte Familien konvexer monotoner Operatoren garantieren. Diese Ergebnisse werden in den folgenden Kapiteln häufig verwendet werden.

In Kapitel 4 zeigen wir, dass stark stetige konvexe monotone Halbgruppen auf Räumen stetiger Funkionen eindeutig durch die auf ihren oberen Lipschitzmengen definierten Γ-Generatoren bestimmt sind. Während die Aussage an sich dem klassischen Analogon aus der linearen Halbgruppentheorie ähnelt, ist der Beweis technisch deutlich aufwendiger und stützt sich stark auf die Ergebnisse aus Kapitel 3. Außerdem liefern wir unter einer zusätzlichen Annahme Approximationsergebnisse für den Γ-Generator, die besonders nützlich sind, wenn wir Funktionen $f\colon \mathbb{R}^d \to \mathbb{R}$ betrachten. In diesem Fall geben wir explizite Bedingungen an, welche garantieren, dass stark stetige konvexe monotone Halbgruppen bereits eindeutig durch die Auswertung ihrer Generatoren an glatten Funktionen bestimmt sind. Dies ist eine wesentliche Verbesserung, da letztere typischerweise in Anwendungen berechnet werden können.

In Kapitel 5 greifen wir die Ideen von Kapitel 1 auf, ersetzen jedoch die zuvor geforderte Normkonvergenz durch Konvergenz bezüglich der gemischten Topologie.

Dadurch wird die relative Kompaktheit, welche die Konvergenz in Gleichung (0.5) garantiert, nur noch für eine wesentlich schwächere Topologie vorausgesetzt. Außerdem geben wir explizite Bedingungen für die Ein-Schritt-Operatoren $(I(t))_{t\geq 0}$ an, welche im Grenzwert auf die Halbgruppe $(S(t))_{t\geq 0}$ übertragen werden, so dass letztere das Vergleichsprinzip aus Kapitel 4 erfüllt. Insbesondere hängt der Grenzwert nicht von der Wahl der konvergenten Teilfolge ab und wir erhalten

$$S(t)f = \lim_{n\to\infty} I(h_n)^{k_n^t} f \tag{0.6}$$

für alle $t \geq 0$ und Folgen $h_n k_n^t \to t$.

In Kapitel 6 illustrieren wir die abstrakten Ergebnisse der beiden vorangegangenen Kapitel, indem wir zeigen, dass Wertfunktionen bestimmter stochastischer optimaler Kontrollprobleme durch Verwendung stückweise konstanter Kontrollen approximiert werden können. Außerdem ergibt mit Hilfe der symmetrischen Lipschitzmenge ein Regularitätsergebnis in Sobolevräumen, welches uns erlaubt, den Γ-Generator mit distributionellen Ableitungen zu verbinden. Wir untersuchen weiter, wie diese Ergebnisse auf Wiener Prozesse der Spurklasse mit Drift erweitert werden können. Da die Ein-Schritt-Operatoren in beiden Beispielen ähnlich wie in Gleichung (0.4) definiert sind, können wir die Wertfunktion auch als Übergangshalbgruppe einer Brownschen Bewegung mit unsicherem Drift und unsicherer Volatilität auffassen. Es ergibt sich die natürliche Frage, ob wir nicht nur Parameterunsicherheit sondern auch nicht-parametrische Unsicherheit betrachten können, bei der wir das Supremum über beliebige Verteilungen nehmen, welche entsprechend ihrem Abstand zu einem bestimmten Referenzmodell penalisiert werden. Dies führt zu sogenannten Wasserstein-Störungen linearer Übergangshalbgruppen.

In Kapitel 7 untersuchen wir die strukturelle Verbindung zwischen Approximationsschemata in Form der Gleichung (0.6) und Grenzwertsätzen für konvexe Erwartungen. Je nachdem, ob wir mit $1/n$ oder mit $1/\sqrt{n}$ skalieren, konvergieren normalisierte Summen von iid-Stichproben entweder gegen eine Maximalverteilung oder eine G-Verteilung. Letztere entspricht einer Brownschen Bewegung mit unsicherer Volatilität, deren Übergangshalbgruppe man auch erhalten kann, indem man die durch Gleichung (0.4) definierten Ein-Schritt-Operatoren verwendet. Im Gegensatz zu früheren Arbeiten sind unsere Ergebnisse nicht auf sublineare Erwartungen beschränkt und decken daher sowohl Anwendungen auf Modellunsicherheit als auch auf die Theorie der großen Abweichungen ab. Beispielsweise kann Cramér's Theorem als Gesetz der großen Zahlen für das entropische Risikomaß angesehen werden.

In Kapitel 8 betrachten wir Folgen $(S_n)_{n\in\mathbb{N}}$ von konvexen monotonen Halbgruppen und geben explizite Bedingungen an unter denen die Konvergenz ihrer infinitesimalen Generatoren $(A_n)_{n\in\mathbb{N}}$ die Konvergenz der Halbgruppen $(S_n)_{n\in\mathbb{N}}$ garantiert. Die Halbgruppe im Grenzwert wird zunächst für alle (t, f) in einer abzählbaren dichten Menge als $S(t)f := \lim_{l\to\infty} S_{n_l}(t)f$ definiert und dann auf beliebige (t, f) erweitert. Der Generator von $(S(t))_{t\geq 0}$ ist für glatte Funktionen f gegeben durch $Af = \lim_{n\to\infty} A_n f$. Da die Halbgruppen $(S_n)_{n\in\mathbb{N}}$ das Vergleichsprinzip aus Kapitel 4 erfüllen, welches sich auf $(S(t))_{t\geq 0}$ überträgt, hängt die im Grenzwert erhaltene Halbgruppe nicht von der Wahl der der konvergenten Teilfolge $(n_l)_{l\in\mathbb{N}}$ ab und erfüllt daher $S(t)f = \lim_{n\to\infty} S_n(t)f$. Der gewählte Rahmen erlaubt auch Diskretisierungen in Zeit und Ort und erweitert die in Kapitel 5 untersuchten Approximationsschemata der Form $S(t)f = \lim_{n\to\infty} I_n^{k_n} f$, wobei $(I_n)_{n\in\mathbb{N}}$ eine Familie von Einschrittoperatoren ist, welche die Dynamik auf einer diskreten Zeitskala der Größe $h_n > 0$ mit $h_n \to 0$ und $k_n h_n \to t$ beschreiben. Die abstrakten Ergebnisse werden durch eine Vielzahl von Anwendungen illustriert, darunter Euler-Schemata und Yosida-Approximationen für obere Hüllen von Familien linearer Halbgruppen, Stabilitätsergebnisse und Finite-Differenzen-Schemata für konvexe HJB-Gleichungen, Freidlin–Wentzell-artige Ergebnisse und Markovketten-Approximationen für eine Klasse von stochastischen optimalen Kontrollproblemen und zeitstetige Markovprozesse mit unsicheren Übergangswahrscheinlichkeiten.

Teile dieser Arbeit können in [23–27] gefunden werden.

Chapter 1

The Chernoff approximation in complete metric spaces

1.1 Introduction

Let X be a complete metric space and $(I(t))_{t\geq 0}$ a family of Lipschitz continuous mappings $I(t)\colon X \to X$. We are interested in the question whether one can construct an associated semigroup $(S(t))_{t\geq 0}$ which is given by the limit

$$S(t)x := \lim_{n\to\infty} \big(I(\tfrac{t}{n})\big)^n x \quad \text{for all } (x,t) \in X \times \mathbb{R}_+. \tag{1.1}$$

Formulas of this type are called Chernoff approximation. In his monograph [44], Chernoff generalized his previous work [43] and the results by Trotter [159, 160]. Under suitable stability conditions on the approximating sequence $(I(t/n)^n x)_{n\in\mathbb{N}}$ and the assumption that $(I(t))_{t\geq 0}$ is strongly continuous, it was shown in [44, Theorem 2.5.3] that the family $(S(t))_{t>0}$ is a strongly continuous semigroup of Lipschitz continuous mappings. In this chapter, we provide a detailed study how a modification of the Chernoff approximation can be used for the construction of nonlinear semigroups. The key is to find properties of $(I(t))_{t\geq 0}$ which are preserved during the iteration and can therefore be transferred to the semigroup $(S(t))_{t\geq 0}$. Typically, the family $(I(t))_{t\geq 0}$ has a representation which allows for explicit verification of these properties. Compared to formula (1.1), we take the limit only for a convergent subsequence, i.e.,

$$S(t)x = \lim_{l\to\infty} I(h_{n_l})^{k^t_{n_l}} x \tag{1.2}$$

for sequences $(h_n)_{n\in\mathbb{N}}$ with $h_n \to 0$ and $(k^t_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ with $k^t_n h_n \to t$. Under suitable compactness and separability assumptions, we can choose a diagonal sequence for all (x,t) in a countable dense subset $\mathcal{D} \times \mathcal{T}$ of $X \times \mathbb{R}_+$. Hence, we have to extend the mapping $\mathcal{D} \times \mathcal{T} \to X,\ (x,t) \mapsto S(t)x$ to the closure $X \times \mathbb{R}_+$. To that end, we need sufficient Lipschitz continuity of $(I(t))_{t\geq 0}$ in the variables $(x,t) \in \mathcal{D} \times \mathcal{T}$ which is preserved during the construction. In several examples, the boundedness and Lipschitz assumptions on the family $(I(t))_{t\geq 0}$ are easily verified. The verification of the relative compactness of the sequence $(I(h_n)^{k^t_n} x)_{n\in\mathbb{N}}$ is more involved and depends on the choice of the space X. The described construction leads to a strongly continuous semigroup $(S(t))_{t\geq 0}$ of locally Lipschitz continuous operators $S(t)\colon X \to X$.

Another focus of this work is the relation between the local behaviour of $(I(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$. Let X be a Banach space. If $(I(t))_{t\geq 0}$ is a family of linear contractions, it was shown in [44, Theorem 3.7] that the infinitesimal generator of $(S(t))_{t\geq 0}$ is an extension of the derivative $I'(0)$. For the nonlinear case, under an additional condition, we show in Theorem 1.4.2 that

$$\lim_{t\downarrow 0}\left\|\frac{I(t)x-x}{t}-y\right\|=0 \quad \text{implies} \quad \lim_{t\downarrow 0}\left\|\frac{S(t)x-x}{t}-y\right\|=0.$$

For instance, this statement is always valid if $(I(t))_{t\geq 0}$ consists of convex monotone operators. We also discuss whether $(S(t))_{t\geq 0}$ represents the unique solution to the abstract Cauchy problem

$$\partial_t u(t)=Au(t) \quad \text{for all } t\geq 0, \quad u(0)=x,$$

where A denotes the generator of $(S(t))_{t\geq 0}$ and $x\in D(A)$. Unlike to the theory of linear semigroups, this is not immediately clear and depends strongly on additional properties of X. In particular, one has to verify $S(t)\colon D(A)\to D(A)$ for all $t\geq 0$ which is in general wrong, see [61]. In contrast, the invariance of the symmetric Lipschitz set, introduced in [61], does not depend on X but rather on convexity and monotonicity of $(S(t))_{t\geq 0}$. We provide sufficient conditions on $(S(t))_{t\geq 0}$ for the invariance and investigate how these conditions can be derived from $(I(t))_{t\geq 0}$. Furthermore, under additional conditions, the symmetric Lipschitz set can be completely described by $(I(t))_{t\geq 0}$ and can therefore explicitly determined in several examples, see Subsection 1.6.1. Finally, we want to emphasise that the results on the generator and the symmetric Lipschitz set do not require norm convergence of the approximating sequence as in equation (1.2).

The present approach is inspired by Nisio semigroups, see [138], where the sequence $(I(h_n)^{k_n^t}x)_{n\in\mathbb{N}}$ is non-decreasing. In this case, the limit in equation (1.2) is independent of the choice of the convergent subsequence, i.e., equation (1.1) and equation (1.2) are equivalent, see Subsection 1.2.2. Nisio semigroups are obtained from the family

$$I(t)x := \sup_{\lambda\in\Lambda} S_\lambda(t)x,$$

where $(S_\lambda)_{\lambda\in\Lambda}$ is a family of monotone linear semigroups. In case that each $(S_\lambda(t))_{t\geq 0}$ is the transition semigroup of a Markov process, the respective Nisio semigroup corresponds to a stochastic process under parameter uncertainty, see [60, 136]. Typical examples include Brownian motions with uncertain drift or diffusion, see [47, 142, 143], and Lévy processes with uncertain Lévy triplet, see [94, 98, 120, 137]. In contrast to Nisio semigroups, the construction based on norm convergence as in equation (1.2) does not rely on monotonicity and we do not require $(I(t))_{t\geq 0}$ to be the supremum over a family of monotone linear semigroups. This is, for instance, illustrated in Subsection 1.6.5 by means of reaction-diffusion equations, where the operators $I(t)$ are neither convex nor monotone.

In the literature, the Chernoff approximation is mainly used as a tool for finding approximative representations of semigroups or solutions of evolution equations for which the existence has already been established. In many applications the aim is to find an approximation of the form (1.1) to give explicit representations of semigroups, see, e.g., [36, 139, 153]. For a survey of Chernoff approximations of operator semigroups, we refer to [35], see also [93] for an overview on different kinds of approximations. A classical approach to nonlinear PDEs and the respective nonlinear semigroups is based on the theory of maximal monotone or m-accretive operators, see [6, 20, 29, 69, 104]. For HJB-type equations, as outlined in [61, 69, 75], it is rather delicate to verify the m-accreditivity. Furthermore, the theory of backward stochastic differential equations provides a powerful tool for the representation of second order quasi-linear equations, see, e.g., [57, 66] and [107, 154] for fully nonlinear equations.

1.2 Construction of nonlinear semigroups

Let (X,d) be a complete metric space. We denote by $B(x,r) := \{y \in X : d(x,y) \leq r\}$ the closed ball with radius $r \geq 0$ around $x \in X$, and by $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ the positive real numbers including zero. Let $(I(t))_{t\geq 0}$ be a family of operators $I(t)\colon X \to X$ satisfying the following boundedness and Lipschitz conditions.

Assumption 1.2.1. Suppose that $(I(t))_{t\geq 0}$ satisfies the following conditions:

(i) $I(0) = \mathrm{id}_X$.

(ii) There exists $x_0 \in X$ with

$$I(t)\colon B(x_0,r) \to B(x_0,\alpha(r,t)) \quad \text{for all } r,t \geq 0,$$

where $\alpha\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a function which is non-decreasing in the second argument and satisfies

$$\alpha(\alpha(r,s),t) \leq \alpha(r,s+t) \quad \text{for all } r,s,t \geq 0. \tag{1.3}$$

(iii) For every $r \geq 0$, there exists $\omega_r \geq 0$ with

$$d\big(I(t)x, I(t)y\big) \leq e^{t\omega_r} d(x,y) \quad \text{for all } t \in [0,1] \text{ and } x,y \in B(x_0,r).$$

Moreover, the mapping $\mathbb{R}_+ \to \mathbb{R}_+,\ r \mapsto w_r$ is non-decreasing.

Remark 1.2.2.

(i) Since α is non-decreasing in the second argument and $I(0) = \mathrm{id}_X$, we obtain $\alpha(r,t) \geq \alpha(r,0) \geq r$ for all $r,t \geq 0$.

(ii) Let $(I(t))_{t\geq 0}$ be a strongly continuous semigroup of bounded linear operators on a Banach space $(X, \|\cdot\|)$. Then, there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|I(t)x\| \leq Me^{\omega t}\|x\|$ for all $t \geq 0$ and $x \in X$, see [141]. If $(I(t))_{t\geq 0}$ is quasi-contractive, i.e. $M = 1$, Assumption 1.2.1(iii) is satisfied. In the nonlinear case, the exponent ω_r might depend on r. Since we are only interested in the short time behaviour of $(I(t))_{t\geq 0}$, we do not require this property for $t > 1$.

Definition 1.2.3. Let $(J(t))_{t\geq 0}$ be a family of operators $J(t)\colon X \to X$ with $J(0) = \mathrm{id}_X$. The Lipschitz set $\mathcal{L}^J$ consists of all $x \in X$ such that there exist $t_0 > 0$ and $c \geq 0$ with

$$d\big(J(t)x, x\big) \leq ct \quad \text{for all } t \in [0,t_0].$$

For convex semigroups, the Lipschitz set was introduced in [61]. The Lipschitz set allows us to establish strong continuity of the semigroup S, and will be used in Subsection 1.6.1 to prove a regularity result. Let $(h_n)_{n\in\mathbb{N}} \subset (0,1]$ be a sequence with $h_n \to 0$. For every $n \in \mathbb{N}$, $t \geq 0$ and $x \in X$, we define

$$I(\pi_n^t)x := I(h_n)^{k_n^t}x := \underbrace{\big(I(h_n)\circ \ldots \circ I(h_n)\big)}_{k_n^t \text{ times}}x,$$

where $k_n^t := \max\{k \in \mathbb{N}_0 \colon kh_n \leq t\}$ and $\pi_n^t := \{h_n, \ldots, k_n^t h_n\}$ denotes the corresponding equidistant partition with mesh size h_n. Furthermore, let $\mathcal{T} \subset \mathbb{R}_+$ be a countable dense set including zero.

Assumption 1.2.4. There exists a countable set $\mathcal{D} \subset \mathcal{L}^I$, which is dense in X, such that the sequence $(I(\pi_n^t)x)_{n\in\mathbb{N}}$ is relatively compact in X for all $(x,t) \in \mathcal{D}\times\mathcal{T}$.

The previous assumption implies the existence of a subsequence $(n_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ such that we can define $S(t)x := \lim_{l\to\infty} I(\pi_{n_l}^t)x$ for all $(x,t) \in \mathcal{D}\times\mathcal{T}$. Then, since the Lipschitz continuity of $(I(t))_{t\geq 0}$ in the variables $(x,t) \in \mathcal{D}\times\mathcal{T}$ from Assumption 1.2.1 and Definition 1.2.3 is preserved during the iteration and in the limit, we can extend the mapping $\mathcal{D}\times\mathcal{T} \to X,\ (x,t) \mapsto S(t)x$ to the closure $\overline{\mathcal{D}\times\mathcal{T}} = X \times \mathbb{R}_+$. This leads to the following main result.

Theorem 1.2.5. *Suppose that Assumption 1.2.1 and Assumption 1.2.4 are satisfied. Then, there exists a family $(S(t))_{t\geq 0}$ of operators $S(t)\colon X \to X$ with the following properties:*

(i) There exists a subsequence $(n_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ with

$$S(t)x = \lim_{l\to\infty} I\big(\pi_{n_l}^t\big)x \quad \text{for all } (x,t) \in X \times \mathcal{T}.$$

(ii) The family $(S(t))_{t\geq 0}$ forms a semigroup, i.e.,

$$S(0) = \mathrm{id}_X \quad \text{and} \quad S(s+t) = S(s)S(t) \quad \text{for all } s,t \geq 0.$$

(iii) For every $r, t \geq 0$,

$$d\big(x_0, S(t)x\big) \leq \alpha(r,t) \quad \text{for all } x \in B(x_0,r),$$
$$d\big(S(t)x, S(t)y\big) \leq e^{t\omega_{\alpha(r,t)}} d(x,y) \quad \text{for all } x,y \in B(x_0,r).$$

(iv) The semigroup $(S(t))_{t\geq 0}$ is strongly continuous, i.e., the mapping

$$\mathbb{R}_+ \to X,\ t \mapsto S(t)x$$

is continuous for all $x \in X$.

(v) For every $r \geq 0$ and $x \in B(x_0,r)\cap\mathcal{L}^I$, there exists $c \geq 0$ with

$$d\big(S(s)x, S(t)x\big) \leq ce^{T\omega_{\alpha(r,T)}}|s-t| \quad \text{for all } T \geq 0 \text{ and } s,t \in [0,T].$$

In the sequel, we call $(I(t))_{t\geq 0}$ a generating family of the semigroup $(S(t))_{t\geq 0}$, and $(S(t))_{t\geq 0}$ is referred to as an associated semigroup to the family $(I(t))_{t\geq 0}$.

Corollary 1.2.6. *It holds $\mathcal{L}^I \subset \mathcal{L}^S$ and $S(t)\colon \mathcal{L}^S \to \mathcal{L}^S$ for all $t \geq 0$.*

Proof. The inclusion $\mathcal{L}^I \subset \mathcal{L}^S$ follows from Theorem 1.2.5(v) and $S(0) = \mathrm{id}_X$. In order to show the invariance of $\mathcal{L}^S$, let $x \in \mathcal{L}^S$. Choose $t_0 > 0$ and $c \geq 0$ with

$$d\big(S(t)x, x\big) \leq ct \quad \text{for all } t \in [0, t_0]. \tag{1.4}$$

The semigroup property implies $S(s)S(t)x = S(s+t)x = S(t)S(s)x$ for all $s, t \geq 0$. Hence, we use Theorem 1.2.5(iii), inequality (1.4) and inequality (1.3) to obtain

$$\begin{aligned} d\big(S(s)S(t)x, S(t)x\big) = d\big(S(t)S(s)x, S(t)x\big) &\leq e^{t\omega_{\alpha(\alpha(r,s),t)}} d\big(S(s)x, x\big) \\ &\leq ce^{t\omega_{\alpha(r,s+t)}} s \end{aligned}$$

for all $s \in [0, t_0]$ and $t \geq 0$. □

1.2.1 Proof of Theorem 1.2.5

In the sequel, we establish a series of lemmas, which prove Theorem 1.2.5. We will always suppose that Assumption 1.2.1 and Assumption 1.2.4 are valid.

Lemma 1.2.7. *For every $n \in \mathbb{N}$ and $r, t \geq 0$,*

$$\begin{aligned} d\big(x_0, I(\pi_n^t)x\big) &\leq \alpha(r,t) \quad \text{for all } x \in B(x_0, r), \\ d\big(I(\pi_n^t)x, I(\pi_n^t)y\big) &\leq e^{t\omega_{\alpha(r,t)}} d(x,y) \quad \text{for all } x, y \in B(x_0, r). \end{aligned}$$

Proof. Let $n \in \mathbb{N}$. First, we show inductively that

$$I(h_n)^k\colon B(x_0, r) \to B(x_0, \alpha(r, kh_n)) \quad \text{for all } k \in \mathbb{N} \text{ and } r \geq 0. \tag{1.5}$$

For $k = 1$, the claim holds by Assumption 1.2.1(ii). Furthermore, if inequality (1.5) is valid for some $k \in \mathbb{N}$, it follows from Assumption 1.2.1(ii) that

$$I(h_n)^{k+1}x = I(h_n)^k I(h_n)x \in B\big(x_0, \alpha(\alpha(r, h_n), kh_n)\big) \subset B(x_0, \alpha(r, (k+1)h_n))$$

for all $r \geq 0$ and $x \in B(x_0, r)$.

Second, we show inductively that

$$d\big(I(h_n)^k x, I(h_n)^k y\big) \leq e^{kh_n\omega_{\alpha(r,kh_n)}} d(x,y) \tag{1.6}$$

for all $k \in \mathbb{N}$, $r \geq 0$ and $x, y \in B(x_0, r)$. For $k = 1$, the claim holds in view of Assumption 1.2.1(iii) and the estimate $\alpha(r, h_n) \geq r$. Furthermore, if inequality (1.6) is valid for some $k \in \mathbb{N}$, it follows from Assumption 1.2.1(ii) and (iii) that

$$\begin{aligned} d\big(I(h_n)^{k+1}x, I(h_n)^{k+1}y\big) &= d\big(I(h_n)^k I(h_n)x, I(h_n)^k I(h_n)y\big) \\ &\leq e^{kh_n\omega_{\alpha(\alpha(r,h_n),kh_n)}} d\big(I(h_n)x, I(h_n)y\big) \\ &\leq e^{kh_n\omega_{\alpha(r,(k+1)h_n)}} e^{h_n\omega_{\alpha(r,h_n)}} d(x,y) \\ &\leq e^{(k+1)h_n\omega_{\alpha(r,(k+1)h_n)}} d(x,y) \end{aligned}$$

for all $r \geq 0$ and $x, y \in B(x_0, r)$. □

Lemma 1.2.8. *Let $r \geq 0$ and $x \in B(x_0, r) \cap \mathcal{L}^I$. Choose $t_0 > 0$ and $c \geq 0$ such that*

$$d\big(I(t)x, x\big) \leq ct \quad \text{for all } t \in [0, t_0]. \tag{1.7}$$

Then, for every $n \in \mathbb{N}$ with $h_n \leq t_0$, $T \geq 0$ and $s, t \in [0, T]$,

$$d\big(I(\pi_n^s)x, I(\pi_n^t)x\big) \leq ce^{T\omega_{\alpha(r,T)}}(|s-t| + h_n).$$

Proof. Let $n \in \mathbb{N}$ with $h_n \leq t_0$. First, we show inductively that

$$d\big(I(h_n)^k x, x\big) \leq ce^{kh_n\omega_{\alpha(r,kh_n)}} kh_n \quad \text{for all } k \in \mathbb{N}. \tag{1.8}$$

For $k = 1$, the claim holds by inequality (1.7). Furthermore, if inequality (1.8) is valid for some fixed $k \in \mathbb{N}$, it follows from Lemma 1.2.7, inequality (1.7), inequality (1.3), and the non-decreasingness of the mappings $r \mapsto \omega_r$ and $t \mapsto \alpha(r, t)$ that

$$\begin{aligned} d\big(I(h_n)^{k+1}x, x\big) &\leq d\big(I(h_n)^k I(h_n)x, I(h_n)^k x\big) + d\big(I(h_n)^k x, x\big) \\ &\leq e^{kh_n\omega_{\alpha(\alpha(r,h_n),kh_n)}} d\big(I(h_n)x, x\big) + ce^{kh_n\omega_{\alpha(r,kh_n)}} kh_n \\ &\leq ce^{kh_n\omega_{\alpha(r,(k+1)h_n)}} e^{h_n\omega_{\alpha(r,h_n)}} h_n + ce^{kh_n\omega_{\alpha(r,kh_n)}} kh_n \\ &\leq ce^{(k+1)h_n\omega_{\alpha(r,(k+1)h_n)}} (k+1)h_n. \end{aligned}$$

Second, we show that

$$d\big(I(h_n)^k x, I(h_n)^l x\big) \leq ce^{kh_n\omega_{\alpha(r,kh_n)}} |k-l| h_n \quad \text{for all } k, l \in \mathbb{N}.$$

W.l.o.g., let $l \leq k$. It follows from Lemma 1.2.7, inequality (1.8) inequality (1.3) and the non-decreasingness of the mappings $r \mapsto \omega_r$ and $t \mapsto \alpha(r, t)$ that

$$\begin{aligned} d\big(I(h_n)^k x, I(h_n)^l x\big) &= d(I(h_n)^l I(h_n)^{k-l} x, I(h_n)^l x\big) \\ &\leq e^{lh_n\omega_{\alpha(\alpha(r,(k-l)h_n),lh_n)}} d\big(I(h_n)^{k-l}x, x\big) \\ &\leq ce^{lh_n\omega_{\alpha(r,kh_n)}} e^{(k-l)h_n\omega_{\alpha(r,(k-l)h_n)}} (k-l)h_n \\ &\leq ce^{kh_n\omega_{\alpha(r,kh_n)}} (k-l)h_n. \end{aligned}$$

Since $|k_n^s - k_n^t| h_n \leq |s-t| + h_n$ and the mappings $r \mapsto \omega_r$ and $t \mapsto \alpha(r, t)$ are non-decreasing, we obtain

$$d\big(I(\pi_n^s)x, I(\pi_n^t)x\big) \leq ce^{T\omega_{\alpha(r,T)}}(|s-t| + h_n)$$

for all $n \in \mathbb{N}$, $T \geq 0$ and $s, t \in [0, T]$. □

By using Assumption 1.2.4 and choosing a diagonal sequence for the countable set $\mathcal{D} \times \mathcal{T}$, there exists a subsequence $(n_l)_{l \in \mathbb{N}} \subset \mathbb{N}$ such that the limit

$$S(t)x := \lim_{l \to \infty} I\big(\pi_{n_l}^t\big)x \in X \tag{1.9}$$

exists for all $(x, t) \in \mathcal{D} \times \mathcal{T}$. Subsequently, we extend the family $(S(t))_{t \in \mathcal{T}}$ of operators $S(t)\colon \mathcal{D} \to X$ to a family $(S(t))_{t \geq 0}$ of operators $S(t)\colon X \to X$ and show that $(S(t))_{t \geq 0}$ is a strongly continuous semigroup.

Lemma 1.2.9. *The following statements are valid:*

(i) *For every $r \geq 0$ and $x \in B(x_0, r) \cap \mathcal{D}$, there exists $c \geq 0$ such that*

$$d\big(S(s)x, S(t)x\big) \leq ce^{T\omega_{\alpha(r,T)}}|s-t| \quad \text{for all } T \geq 0 \text{ and } s,t \in [0,T] \cap \mathcal{T}.$$

In particular, the mapping $S(\cdot)x\colon [0,T] \cap \mathcal{T} \to X$ has a unique continuous extension to $[0,T]$, which satisfies the previous inequality for all $s,t \in [0,T]$.

(ii) *For every $r, t \geq 0$,*

$$d\big(S(t)x, S(t)y\big) \leq e^{t\omega_{\alpha(r,t)}} d(x,y) \quad \text{for all } x,y \in B(x_0,r) \cap \mathcal{D}.$$

In particular, the mapping $S(t)\colon B(x_0,r) \cap \mathcal{D} \to X$ has a unique continuous extension to $B(x_0,r)$, which satisfies the previous inequality for all $x,y \in B(x_0,r)$.

Proof. First, let $r, T \geq 0$ and $x \in B(x_0,r) \cap \mathcal{D}$. Choose $t_0 > 0$ and $c \geq 0$ with

$$d\big(I(t)x, x\big) \leq ct \quad \text{for all } t \in [0,t_0].$$

For every $s,t \in [0,T] \cap \mathcal{T}$, equation (1.9) and Lemma 1.2.8 imply

$$d\big(S(s)x, S(t)x\big) = \lim_{l\to\infty} d\big(I(\pi_{n_l}^s)x, I(\pi_{n_l}^t)x\big) \leq ce^{T\omega_{\alpha(r,T)}}|s-t|$$

The existence and uniqueness of the extension follows, since $[0,T] \cap \mathcal{T} \subset [0,T]$ is dense and the mapping $S(\cdot)x\colon [0,T] \cap \mathcal{T} \to X$ is Lipschitz continuous.

Second, let $r \geq 0$ and $x,y \in B(x_0,r) \cap \mathcal{D}$. Equation (1.9) and Lemma 1.2.7 imply

$$d\big(S(t)x, S(t)y\big) = \lim_{l\to\infty} \big(I(\pi_{n_l}^t)x, I(\pi_{n_l}^t)y\big) \leq e^{t\omega_{\alpha(r,t)}} d(x,y) \quad \text{for all } t \in \mathcal{T}. \tag{1.10}$$

Now, let $t \geq 0$ be arbitrary and choose a sequence $(t_n)_{n\in\mathbb{N}} \subset [0,t] \cap \mathcal{T}$ with $t_n \to t$. We use part (i), inequality (1.10) and the non-decreasingness of α in the second argument to estimate

$$d\big(S(t)x, S(t)y\big) = \lim_{n\to\infty} d\big(S(t_n)x, S(t_n)y\big) \leq \sup_{n\in\mathbb{N}} e^{t_n\omega_{\alpha(r,t_n)}} d(x,y) \leq e^{t\omega_{\alpha(r,t)}} d(x,y).$$

The existence and uniqueness of the extension follows, since $B(x_0,r) \cap \mathcal{D} \subset B(x_0,r)$ is dense and the mapping $S(t)\colon B(x_0,r) \cap \mathcal{D} \to X$ is Lipschitz continuous. □

Lemma 1.2.10. *The mapping $S(\cdot)x\colon [0,\infty) \to X$ is continuous for all $x \in X$.*

Proof. Let $x \in X$, $t \geq 0$ and $\varepsilon > 0$. We define $r := d(x_0,x) + 1$, $T := t+1$ and choose $\delta_1 \in (0,1]$ with $2e^{T\omega_{\alpha(r,T)}}\delta_1 < \varepsilon/2$. Since $\mathcal{D} \subset X$ is dense, there exists $y \in B(x_0,\delta_1) \cap \mathcal{D}$. Moreover, by Lemma 1.2.9(i), there exists $c \geq 0$ with

$$d\big(S(s_1)y, S(s_2)y\big) \leq e^{T\omega_{\alpha(r,T)}}|s_1 - s_2| \quad \text{for all } s_1, s_2 \in [0,T].$$

Choose $\delta_2 \in (0,\delta_1]$ with $ce^{T\omega_{\alpha(r,T)}}\delta_2 < \varepsilon/2$. For every $s \geq 0$ with $|s-t| < \delta_2$, we obtain

$$\begin{aligned} d\big(S(s)x, S(t)x\big) &\leq d\big(S(s)x, S(s)y\big) + d\big(S(t)x, S(t)y\big) + d\big(S(s)y, S(t)y\big) \\ &\leq 2e^{T\omega_{\alpha(r,T)}} d(x,y) + ce^{T\omega_{\alpha(r,T)}}|s-t| \\ &\leq 2e^{T\omega_{\alpha(r,T)}}\delta_1 + ce^{T\omega_{\alpha(r,T)}}\delta_2 < \varepsilon. \end{aligned}$$

□

Lemma 1.2.11. *It holds* $S(t)x = \lim_{l\to\infty} I(\pi^t_{n_l})x$ *for all* $(x,t) \in X \times \mathcal{T}$. *In particular, we obtain* $S(t)\colon B(x_0,r) \to B(x_0,\alpha(r,t))$ *for all* $r,t \geq 0$.

Proof. First, let $x \in X$, $t \in \mathcal{T}$, $\varepsilon > 0$ and define $r := d(x_0,x)+1$. Choose $\delta > 0$ with $2e^{t\omega_{\alpha(r,t)}}\delta < \varepsilon$ and $y \in B(x,\delta) \cap \mathcal{D}$. Lemma 1.2.7 and Lemma 1.2.9(ii) imply

$$\begin{aligned} d\big(S(t)x, I(\pi^t_{n_l})x\big) &\leq d\big(S(t)x, S(t)y\big) + d\big(I(\pi^t_{n_l})x, I(\pi^t_{n_l})y\big) + d\big(S(t)y, I(\pi^t_{n_l})y\big) \\ &\leq 2e^{t\omega_{\alpha(r,t)}} d(x,y) + d\big(S(t)y, I(\pi^t_{n_l})y\big) \\ &< \varepsilon + d\big(S(t)y, I(\pi^t_{n_l})y\big). \end{aligned}$$

for all $l \in \mathbb{N}$ and thus equation (1.9) yields $\lim_{l\to\infty} d\big(S(t)x, I(\pi^t_{n_l})x\big) = 0$.

Second, let $r \geq 0$ and $x \in B(x_0,r)$. The first part and Lemma 1.2.7 imply

$$d\big(x_0, S(t)x\big) = \lim_{l\to\infty} d\big(x_0, I(\pi^t_{n_l})x\big) \leq \alpha(r,t) \quad \text{for all } t \in \mathcal{T}. \tag{1.11}$$

Now, let $t \geq 0$ be arbitrary and choose a sequence $(t_n)_{n\in\mathbb{N}} \subset [0,t] \cap \mathcal{T}$ with $t_n \to t$. It follows from Lemma 1.2.10, inequality (1.11) and the non-decreasingness of α in the second argument that

$$d\big(x_0, S(t)x\big) = \lim_{n\to\infty} d\big(x_0, S(t_n)x\big) \leq \sup_{n\in\mathbb{N}} \alpha(r,t_n) \leq \alpha(r,t).$$ □

Lemma 1.2.12. *It holds* $S(0) = \mathrm{id}_X$ *and* $S(s+t) = S(s)S(t)$ *for all* $s,t \geq 0$.

Proof. Assumption 1.2.1(i) and the previous construction guarantee that $S(0) = \mathrm{id}_X$. First, let $s,t \in \mathcal{T}$, $x \in \mathcal{D}$ and $r := d(x_0,x)$. Lemma 1.2.11 implies

$$\lim_{l\to\infty} d\big(S(s+t)x, I(\pi^{s+t}_{n_l})x\big) = 0.$$

It holds $k^{s+t}_n - k^s_n - k^t_n \in \{0,1\}$ for all $n \in \mathbb{N}$. In case that $k^{s+t}_n - k^s_n - k^t_n = 0$, we obtain $I(\pi^{s+t}_n) = I(\pi^s_n)I(\pi^t_n)$. Otherwise, we can use Lemma 1.2.7, inequality (1.3) and the non-decreasingness of the mappings $r \mapsto \omega_r$ and $t \mapsto \alpha(r,t)$ to estimate

$$\begin{aligned} d\big(I(\pi^{s+t}_n)x, I(\pi^s_n)I(\pi^t_n)x\big) &= d\big(I(h_n)^{k^t_n} I(h_n)^{k^s_n} I(h_n)x, I(h_n)^{k^t_n} I(h_n)^{k^s_n} x\big) \\ &\leq e^{s\omega_{\alpha(\alpha(\alpha(r,h_n),k^t_n h_n),k^s_n h_n)}} d\big(I(h_n)^{k^s_n} I(h_n)x, I(h_n)^{k^s_n} x\big) \\ &\leq e^{s\omega_{\alpha(r,s+t)}} e^{t\omega_{\alpha(\alpha(r,h_n),k^t_n h_n)}} d\big(I(h_n)x, x\big) \\ &\leq e^{(s+t)\omega_{\alpha(r,s+t)}} d\big(I(h_n)x, x\big). \end{aligned}$$

Since $x \in \mathcal{D} \subset \mathcal{L}^I$, the right-hand side converges to zero as $n \to \infty$. Furthermore, Lemma 1.2.7 and Lemma 1.2.11 imply

$$\begin{aligned} &d\big(S(s)S(t)x, I(\pi^s_{n_l})I(\pi^t_{n_l})x\big) \\ &\leq d\big(S(s)S(t)x, I(\pi^s_{n_l})S(t)x\big) + d\big(I(\pi^s_{n_l})S(t)x, I(\pi^s_{n_l})I(\pi^t_{n_l})x\big) \\ &\leq d\big(S(s)S(t)x, I(\pi^s_{n_l})S(t)x\big) + e^{s\omega_{\alpha(\alpha(r,t),s)}} d\big(S(t)x, I(\pi^t_{n_l})\big) \to 0 \quad \text{as } l \to \infty. \end{aligned}$$

For arbitrary $x \in X$, the claim follows by an approximation argument from Lemma 1.2.7 and Lemma 1.2.9(ii).

Second, let $s, t \geq 0$ and $x \in X$ be arbitrary. Choose sequences $(s_n)_{n\in\mathbb{N}} \subset [0, s] \cap \mathcal{T}$ and $(t_n)_{n\in\mathbb{N}} \subset [0, t] \cap \mathcal{T}$ with $s_n \to s$ and $t_n \to t$. We use the first part, Lemma 1.2.9(ii), Lemma 1.2.10 and Lemma 1.2.11 to obtain

$$\begin{aligned} d\big(S(s+t)x, S(s)S(t)x\big) &\leq d\big(S(s+t)x, S(s_n+t_n)x\big) + d\big(S(s)S(t)x, S(s_n)S(t)x\big) \\ &\quad + d\big(S(s_n)S(t)x, S(s_n)S(t_n)x\big) \\ &\leq d\big(S(s+t)x, S(s_n+t_n)x\big) + d\big(S(s)S(t)x, S(s_n)S(t)x\big) \\ &\quad + e^{(s+t)\omega_{\alpha(\alpha(r,s+t),s+t)}} d\big(S(t)x, S(t_n)x\big) \to 0. \end{aligned}$$ □

Lemma 1.2.13. *For every $r \geq 0$ and $x \in B(x_0, r) \cap \mathcal{L}^I$, there exists $c \geq 0$ with*

$$d\big(S(s)x, S(t)x\big) \leq c e^{T\omega_{\alpha(r,T)}} |s-t| \quad \text{for all } T \geq 0 \text{ and } s, t \in [0, T].$$

Proof. Fix $r, T \geq 0$, $x \in B(x_0, r) \cap \mathcal{L}^I$ and choose $t_0 > 0$ and $c \geq 0$ such that

$$d\big(I(t)x, x\big) \leq ct \quad \text{for all } t \in [0, t_0].$$

It follows from Lemma 1.2.11 and Lemma 1.2.8 that, for all $s, t \in [0, T] \cap \mathcal{T}$,

$$d\big(S(s)x, S(t)x\big) = \lim_{l\to\infty} d\big(I(\pi^s_{n_l})x, I(\pi^t_{n_l})x\big) \leq c e^{T\omega_{\alpha(r,T)}} |s-t|. \tag{1.12}$$

Now, let $s, t \in [0, T]$ be arbitrary and choose sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ in $[0, T] \cap \mathcal{T}$ with $s_n \to s$ and $t_n \to t$. We use Lemma 1.2.10 and inequality (1.12) to obtain

$$d\big(S(s)x, S(t)x\big) = \lim_{n\to\infty} d\big(S(s_n)x, S(t_n)x\big) \leq c e^{T\omega_{\alpha(r,T)}} |s-t|.$$ □

1.2.2 Discussion and comparison with Nisio semigroups

While most of the results of the previous subsection remain valid for arbitrary partitions, in order to guarantee the semigroup property we have to consider equidistant partitions. For every $t \geq 0$, we subsequently denote by P_t the set of all partitions $\pi = \{t_0, \ldots, t_n\}$ with $0 = t_0 < t_1 < \ldots < t_n = t$ and define the iterated operators

$$I(\pi) := I(t_1 - t_0) \cdots I(t_n - t_{n-1}).$$

For later reference, we state the following version of Lemma 1.2.7 and Lemma 1.2.8.

Remark 1.2.14. Let $(I(t))_{t\geq 0}$ be a family of operators which satisfy Assumption 1.2.1. Then, the following statements are valid:

(i) For every $r, t \geq 0$ and $\pi \in P_t$,

$$\begin{aligned} d\big(x_0, I(\pi)x\big) &\leq \alpha(r,t) \quad \text{for all } x \in B(x_0, r), \\ d\big(I(\pi)x, I(\pi)y\big) &\leq e^{t\omega_{\alpha(r,t)}} d(x,y) \quad \text{for all } x, y \in B(x_0, r). \end{aligned}$$

(ii) Let $r \geq 0$ and $x \in B(0, r) \cap \mathcal{L}^I$. Choose $t_0 > 0$ and $c \geq 0$ with

$$d\big(I(t)x, x\big) \leq ct \quad \text{for all } t \in [0, t_0].$$

Then, for every $t \geq 0$ and $\pi \in P_t$ with $|\pi| \leq t_0$,

$$d\big(I(\pi)x, x\big) \leq c e^{t\omega_{\alpha(r,t)}} t,$$

where $|\pi| := \max_{i=0,\ldots,n-1}(t_{i+1} - t_i)$ for $\pi = \{t_0, \ldots, t_n\}$ with $t_0 < \ldots < t_n$.

A priori the construction of an associated semigroup $(S(t))_{t\geq 0}$ to a generating family $(I(t))_{t\geq 0}$ depends on the choice of the partitions and the convergent subsequence. However, in some applications, it is possible that

$$S(t)x = \lim_{n\to\infty} I(\pi_n^t)x \quad \text{for all } (x,t)\in X\times\mathcal{T},$$

i.e., the convergence holds without choosing a subsequence. For instance, if the sequence $(I(\pi_n^t)x)_{n\in\mathbb{N}}$ is non-decreasing for all $(x,t)\in X\times\mathcal{T}$, we obtain

$$S(t)x = \sup_{l\in\mathbb{N}} I(\pi_{n_l}^t)x = \sup_{n\in\mathbb{N}} I(\pi_n^t)x \quad \text{for all } (x,t)\in X\times\mathcal{T}.$$

In particular, Nisio semigroups fall into this category, as we will see in the remainder of this subsection. Furthermore, if the associated semigroup is uniquely determined via its infinitesimal generator, the construction is independent of the choice of the partition and the convergent subsequence. For details, we refer to Subsection 1.4.2. For the following lemma and subsequent remark, let X be a Banach lattice. An operator $I(t)\colon X\to X$ is called

- monotone, if $I(t)x\leq I(t)y$ for all $x,y\in X$ with $x\leq y$,
- continuous from below, if $I(t)x=\sup_{n\in\mathbb{N}} I(t)x_n$ for every non-decreasing sequence $(x_n)_{n\in\mathbb{N}}\subset X$ such that $x:=\sup_{n\in\mathbb{N}} x_n\in X$ exists.

Assume that $\mathcal{T}_n\subset\mathcal{T}_{n+1}$ for all $n\in\mathbb{N}$, where $\mathcal{T}_n:=\{kh_n\colon k\in\mathbb{N}_0\}$.

Lemma 1.2.15. *Let $(I(t))_{t\geq 0}$ be a family of operators $I(t)\colon X\to X$ satisfying Assumption 1.2.1 and Assumption 1.2.4. Furthermore, let $(S(t))_{t\geq 0}$ be an associated semigroup as in Theorem 1.2.5. We impose the following additional conditions:*

(i) $I(t)$ is monotone and continuous from below for all $t\geq 0$.

(ii) $I(s+t)x\leq I(s)I(t)x$ for all $s,t\geq 0$ and $x\in X$.

(iii) The mapping $\mathbb{R}_+\to X,\ t\mapsto I(t)x$ is continuous for all $x\in X$.

(iv) The operator $T(t)\colon X\to X,\ x\mapsto \sup_{\pi\in P_t} I(\pi)x$ is well-defined for all $t\geq 0$.

Then, it holds $S(t)x=T(t)x$ for all $(x,t)\in X\times\mathbb{R}_+$. Furthermore,

$$S(t)x = \lim_{n\to\infty} I(\pi_n^t)x \quad \text{for all } (x,t)\in X\times\mathcal{T},$$

i.e., the convergence holds without choosing a subsequence.

Proof. By induction, it follows from condition (i) and (ii) that the sequence $(I(\pi_n^t)x)_{n\in\mathbb{N}}$ is non-decreasing for all $(x,t)\in X\times\mathcal{T}$. Moreover, by Theorem 1.2.5, there exists a subsequence $(n_l)_{l\in\mathbb{N}}\subset\mathbb{N}$ with $S(t)x=\lim_{l\to\infty} I(\pi_{n_l}^t)x$ for all $(x,t)\in X\times\mathcal{T}$. Since X is a Banach lattice, we obtain

$$I(t)x\leq S(t)x = \sup_{l\in\mathbb{N}} I(\pi_{n_l}^t)x = \sup_{n\in\mathbb{N}} I(\pi_n^t)x \leq T(t)x \quad \text{for all } (x,t)\in X\times\mathcal{T}.$$

Condition (iii) and strong continuity of $(S(t))_{t\geq 0}$ imply $I(t)x \leq S(t)x$ for all $t \geq 0$ and $x \in X$. We use the monotonicity of $I(s)$ and the semigroup property to conclude

$$I(s)I(t)x \leq I(s)S(t)x \leq S(s)S(t)x = S(s+t)x \quad \text{for all } s,t \geq 0 \text{ and } x \in X.$$

It follows inductively that $T(t)x \leq S(t)x$ for all $t \geq 0$ and $x \in X$ with equality for all $t \in \mathcal{T}$. It remains to show that the mapping

$$\mathbb{R}_+ \to X,\ t \mapsto T(t)x$$

is continuous for all $x \in X$. Condition (ii) guarantees that the set $\{I(\pi)\colon \pi \in P_t\}$ is directed upwards and, by assumption, the operator $I(t)$ is continuous from below for all $t \geq 0$. Hence, the family $(T(t))_{t\geq 0}$ forms a semigroup and Remark 1.2.14(ii) implies that the mapping $\mathbb{R}_+ \to X,\ t \mapsto T(t)x$ is locally Lipschitz continuous for all $x \in \mathcal{L}^I$. Since $\mathcal{L}^I \subset X$ is dense, it follows similarly to the proof of Lemma 1.2.10 from Remark 1.2.14(i) that the mapping $t \mapsto T(t)x$ is continuous for all $x \in X$. We obtain

$$S(t)x = T(t)x \quad \text{for all } t \geq 0 \text{ and } x \in X.$$

In particular, the limit in Theorem 1.2.5(i) does not depend on the choice of the convergent subsequence and therefore $S(t)x = \lim_{n\to\infty} I(\pi_n^t)x$ for all $(x,t) \in X \times \mathcal{T}$. □

Remark 1.2.16. Let $(S_\lambda)_{\lambda\in\Lambda}$ be a family of linear semigroups on X which satisfy the following conditions:

(i) $S_\lambda(t)$ is monotone and continuous from below for all $\lambda \in \Lambda$ and $t \geq 0$.

(ii) There exists $\omega \geq 0$ with $\|S_\lambda(t)x\| \leq e^{\omega t}\|x\|$ for all $\lambda \in \Lambda$, $t \geq 0$ and $x \in X$.

(iii) The operator $I(t)\colon X \to X,\ x \mapsto \sup_{\lambda\in\Lambda} S_\lambda(t)x$ is well-defined for all $t \geq 0$.

Moreover, we assume that, for every subset $Y \subset X$ such that the supremum $\sup Y \in X$ exists, it holds $\|\sup Y\| \leq \sup_{x\in Y}\|x\|$. For instance, the supremum norm has this property but not L^p-norms. For every $t \geq 0$ and $x, y \in X$, we use the assumption on the norm and condition (ii) to estimate

$$\|I(t)x\| \leq \sup_{\lambda\in\Lambda}\|S_\lambda(t)x\| \leq e^{\omega t}\|x\|,$$
$$\|I(t)x - I(t)y\| \leq \sup_{\lambda\in\Lambda}\|S_\lambda(t)x - S_\lambda(t)y\| \leq e^{\omega t}\|x-y\|.$$

Hence, Assumption 1.2.1 is satisfied. Furthermore, condition (i) implies that $(I(t))_{t\geq 0}$ satisfies the first two conditions of Lemma 1.2.15. In many examples, the forth condition of Lemma 1.2.15 follows from the assumptions that are already necessary for the construction of the semigroup $(S(t))_{t\geq 0}$, while the third condition requires a mild additional condition. If $(I(t))_{t\geq 0}$ satisfies the assumptions of Lemma 1.2.15, the associated semigroup $(S(t))_{t\geq 0}$ from Theorem 1.2.5 coincides with the family $(T(t))_{t\geq 0}$, defined by

$$T(t)x := \sup_{\pi\in P_t} I(\pi)x \quad \text{for all } (x,t) \in X \times \mathbb{R}_+.$$

It is also possible to weight the linear semigroups in the definition of $I(t)$ with a penalization term. This leads to semigroups which are convex rather than sublinear.

1.3 Relative compactness based on Arzéla-Ascoli's theorem

Let C be the space of all continuous functions $f\colon \mathbb{R}^d \to \mathbb{R}^m$, including the subsets C^∞, Lip and C_0 of all functions which are infinitely differentiable, Lipschitz continuous and vanish at infinity, respectively. Furthermore, let $\mathcal{L}^\infty$ be the space of all bounded (not necessarily measurable) functions $f\colon \mathbb{R}^d \to \mathbb{R}^m$ endowed with the supremum norm $\|f\|_\infty := \sup_{x\in\mathbb{R}^d} |f(x)|$, where $|\cdot|$ denotes the Euclidean norm. We define $C_b := C \cap \mathcal{L}^\infty$, $\mathrm{Lip}_b := \mathrm{Lip} \cap \mathcal{L}^\infty$ and $\mathrm{Lip}_0 := \mathrm{Lip} \cap C_0$. In addition, for every $c \geq 0$, we denote by $\mathrm{Lip}(r)$ the set of all c-Lipschitz continuous functions. For every $c \geq 0$, let

$$\mathrm{Lip}_b(c) := \{f \in \mathrm{Lip}_b\colon f \in \mathrm{Lip}(c),\ \|f\|_\infty \leq c\} \quad \text{and} \quad \mathrm{Lip}_0(c) = \mathrm{Lip}_b(c) \cap C_0.$$

1.3.1 Semigroups on C_0

We give sufficient conditions for a family of translation-invariant contractions, which will be illustrated in Section 1.6 with several examples, in order to guarantee that the assumptions of Section 1.2 are satisfied. We start with an application of Arzéla-Ascoli's theorem.

Lemma 1.3.1. *Let $(I(t))_{t\geq 0}$ be a family of operators $I(t)\colon C_0 \to C_0$ which satisfy Assumption 1.2.1(ii). Let $t \geq 0$ and $f \in C_0$ such that*

- $\{I(\pi_n^t)f\colon n \in \mathbb{N}\}$ *is equicontinuous,*
- $\lim_{|x|\to\infty} \sup_{n\in\mathbb{N}} |(I(\pi_n^t)f)(x)| = 0.$

Then, the sequence $(I(\pi_n^t)f)_{n\in\mathbb{N}}$ is relatively compact in C_0.

Proof. By assumption, the sequence $(I(\pi_n^t)_{n\in\mathbb{N}}$ is equicontinuous. Moreover, it follows from Lemma 1.2.7 that $(I(\pi_n^t)_{n\in\mathbb{N}}$ is bounded. Note that Lemma 1.2.7 is a consequence of Assumption 1.2.1(ii) and independent of the other assumptions in Section 1.2. By using Arzéla-Ascoli's theorem and choosing a diagonal sequence, we obtain a continuous function $g\colon \mathbb{R}^d \to \mathbb{R}^m$ such that $I(\pi_{n_l}^t)f \to g$ as $l \to \infty$ uniformly on compact subsets for a suitable subsequence. It remains to show

$$\lim_{l\to\infty} \|I(\pi_{n_l}^t)f - g\|_\infty = 0.$$

For every $\varepsilon > 0$, by assumption, there exists a compact set $K \subset \mathbb{R}^d$ with

$$\sup_{x\in K^c} \sup_{n\in\mathbb{N}} \big|(I(\pi_n^t)f)(x)\big| \leq \frac{\varepsilon}{2}.$$

This inequality is preserved in the limit, i.e., $\sup_{x\in K^c} |g(x)| \leq \varepsilon/2$. We obtain

$$\begin{aligned}\|I(\pi_{n_l}^t)f - g\|_\infty &= \|(I(\pi_{n_l}^t)f - g)\mathbb{1}_K\|_\infty + \|(I(\pi_{n_l}^t)f - g)\mathbb{1}_{K^c}\|_\infty \\ &\leq \|(I(\pi_{n_l}^t)f - g)\mathbb{1}_K\|_\infty + \varepsilon \to \varepsilon \quad \text{as } l \to \infty.\end{aligned}$$

Since C_0 is complete, it holds $g \in C_0$. □

An operator $I(t)\colon C_0 \to C_0$ is called translation-invariant, if

$$\big(I(t)f\big)(x) = \big(I(t)f_x\big)(0) \quad \text{for all } f \in C_0 \text{ and } x \in \mathbb{R}^d,$$

where $f_x\colon \mathbb{R}^d \to \mathbb{R}^m$, $y \mapsto f(x+y)$. Let $C_0^+ := \{f \in C_0\colon f \geq 0\}$.

Lemma 1.3.2. *Let $(I(t))_{t\geq 0}$ be a family of operators $I(t)\colon C_0 \to \mathcal{L}^\infty$ which satisfy the following conditions:*

(i) $I(0) = \mathrm{id}_{C_0}$.

(ii) $\|I(t)f\|_\infty \leq \|f\|_\infty$ *for all* $t \geq 0$ *and* $f \in C_0$.

(iii) $\|I(t)f - I(t)g\|_\infty \leq \|f-g\|_\infty$ *for all* $t \geq 0$ *and* $f, g \in C_0$.

(iv) $I(t)$ *is translation-invariant for all* $t \geq 0$.

(v) There exists a countable set $\mathcal{D} \subset \mathrm{Lip}_0 \cap \mathcal{L}^I$ *which is dense in* C_0.

(vi) For every $c \geq 0$*, there exists a family* $(T_c(t))_{t\geq 0}$ *of operators* $T_c(t)\colon C_0^+ \to C_0^+$ *such that*

- $|I(t)f| \leq T_c(t)|f|$ *for all* $f \in \mathrm{Lip}_0(c)$ *and* $t \geq 0$,
- $T_c(s)T_c(t)f \leq T_c(s+t)f$ *for all* $f \in C_0^+$ *and* $s, t \geq 0$,
- $T_c(t)$ *is monotone for all* $t \geq 0$.

Then, it holds $I(t)\colon C_0 \to C_0$ *and* $I(t)\colon \mathrm{Lip}_0(c) \to \mathrm{Lip}_0(c)$ *for all* $c, t \geq 0$. *Furthermore, the family* $(I(t))_{t\geq 0}$ *satisfies Assumption 1.2.1 and Assumption 1.2.4.*

Proof. First, we show that $I(t)\colon \mathrm{Lip}_0(c) \to \mathrm{Lip}_0(c)$ for all $c, t \geq 0$. Let $c, t \geq 0$ and $f \in \mathrm{Lip}_0(c)$. For every $x, y \in \mathbb{R}^d$, condition (iii) and (iv) imply

$$|(I(t)f)(x) - (I(t)f)(y)| = |(I(t)f_x - I(t)f_y)(0)| \leq \|f_x - f_y\|_\infty \leq c|x-y|$$

and thus condition (ii) guarantees $I(t)f \in \mathrm{Lip}_b(c)$. Moreover, condition (vi) yields

$$\lim_{|x|\to\infty} |(I(t)f)(x)| \leq \lim_{|x|\to\infty} \big(T_c(t)|f|\big)(x) = 0$$

and thus $I(t)f \in \mathrm{Lip}_0(c)$. We use $\overline{\mathrm{Lip}_0} = C_0$ as well as the fact that $I(t)\colon C_0 \to \mathcal{L}^\infty$ is Lipschitz continuous and $C_0 \subset \mathcal{L}^\infty$ is closed to conclude $I(t)\colon C_0 \to C_0$.

Second, due to the conditions (i)-(iii), Assumption 1.2.1 is satisfied. Furthermore, by condition (v), there exists a countable set $\mathcal{D} \subset \mathrm{Lip}_0 \cap \mathcal{L}^I$ which is dense in C_0. It remains to verify the assumptions from Lemma 1.3.1 for all $(f,t) \in \mathcal{D} \times \mathcal{T}$. To do so, let $(f,t) \in \mathcal{D} \times \mathcal{T}$ and choose $c \geq 0$ with $f \in \mathrm{Lip}_0(c)$. It follows inductively from $I(h_n)\colon \mathrm{Lip}_0(c) \to \mathrm{Lip}_0(c)$ that $I(\pi_n^t)f \in \mathrm{Lip}_0(c)$ for all $n \in \mathbb{N}$ which shows that the sequence $(I(\pi_n^t)f)_{n\in\mathbb{N}}$ is equicontinuous. In addition, we can use $I(h_n)f \in \mathrm{Lip}_0(c)$ and condition (vi) to estimate

$$|I(h_n)^2 f| \leq T_c(h_n)(|I(h_n)f|) \leq T_c(h_n)T_c(h_n)|f| \leq T_c(2h_n)|f|.$$

By induction, it follows that $|I(\pi_n^t)f| \leq T_c(t)|f|$ for all $n \in \mathbb{N}$ and thus condition (vi) implies

$$\lim_{|x|\to\infty} \sup_{n\in\mathbb{N}} |(I(\pi_n^t)f)(x)| \leq \lim_{|x|\to\infty} \big(T_c(t)|f|\big)(x) = 0.$$

Lemma 1.3.1 yields that Assumption 1.2.4 is satisfied. □

1.3.2 Closure of Lipschitz functions and weighted norms

To study examples which are not translation-invariant, the space $(C_0, \|\cdot\|_\infty)$ is not suitable. Hence, following the setting of Nendel and Röckner [136], we modify the supremum norm with a weight function κ. The verification of Assumption 1.2.4 becomes particularly simple, also in the translation-invariant case. Let $\kappa\colon \mathbb{R}^d \to (0,\infty)$ be a continuous function function vanishing at infinity. Let C_κ be the space of all continuous functions $f\colon \mathbb{R}^d \to \mathbb{R}^m$ with $\|f\kappa\|_\infty < \infty$ endowed with the norm $\|f\|_\kappa := \|f\kappa\|_\infty$. Since the mapping $C_\kappa \to C_b,\ f \mapsto f\kappa$ is a linear isometric order preserving isomorphism, the space C_κ is a Banach lattice. Defining UC_κ as the closure of Lip_b in C_κ, the space UC_κ is again a Banach lattice.

Lemma 1.3.3. *Let $(I(t))_{t\geq 0}$ be a family of operators $I(t)\colon \mathrm{UC}_\kappa \to \mathrm{UC}_\kappa$. Assume that there exists a function $\rho\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with*

- *$I(t)\colon \mathrm{Lip}_b(c) \to \mathrm{Lip}_b(\rho(c,t))$ for all $c, t \geq 0$,*
- *$\rho(\rho(c,s),t) \leq \rho(c, s+t)$ for all $c, s, t \geq 0$.*

Then, the sequence $(I(\pi_n^t)f)_{n\in\mathbb{N}}$ is relatively compact in UC_κ for all $f \in \mathrm{Lip}_b$ and $t \geq 0$.

Proof. First, we show that $\mathrm{Lip}_b(c) \subset \mathrm{UC}_\kappa$ is compact for all $c \geq 0$. Let $c \geq 0$ and $(f_n)_{n\in\mathbb{N}} \subset \mathrm{Lip}_b(c)$ be a sequence. By using Arzéla-Ascoli's theorem and choosing a diagonal sequence, we obtain a function $f \in C$ such that $f_{n_l} \to f$ as $l \to \infty$ uniformly on compact sets for a suitable subsequence. Since $\mathrm{Lip}_b(c)$ is closed under mere pointwise convergence, it holds $f \in \mathrm{Lip}_b(c)$. It remains to show $\|f_{n_l} - f\|_\kappa \to 0$. Let $\varepsilon > 0$ and choose a compact set $K \subset \mathbb{R}^d$ with $\sup_{x\in K^c} \kappa(x) < \varepsilon/2c$. We obtain

$$\begin{aligned}\|f_{n_l} - f\|_\kappa &= \|(f_{n_l} - f)\mathbb{1}_K\|_\kappa + \|(f_{n_l} - f)\mathbb{1}_{K^c}\|_\infty \\ &\leq \|(f_{n_l} - f)\mathbb{1}_K\|_\kappa + \varepsilon \to \varepsilon \quad \text{as } l \to \infty.\end{aligned}$$

Second, let $c, t \geq 0$ and $f \in \mathrm{Lip}_b(c)$. By induction, it follows from the assumptions on $(I(t))_{t\geq 0}$ and ρ that $I(\pi_n^t)f \in \mathrm{Lip}_b(\rho(c,t))$ for all $n \in \mathbb{N}$. The first part yields that the sequence $(I(\pi_n^t)f)_{n\in\mathbb{N}}$ is relatively compact in UC_κ. □

Let C_c^∞ be the space of all infinitely differentiable functions $f\colon \mathbb{R}^d \to \mathbb{R}^m$ with compact support.

Lemma 1.3.4. *Assume that κ is infinitely differentiable. Then, the space $C_c^\infty \subset \mathrm{UC}_\kappa$ is dense and the mapping $\varphi\colon \mathrm{UC}_\kappa \to C_0,\ f \mapsto f\kappa$ is an isomorphism.*

Proof. It follows from $\lim_{|x|\to\infty} \kappa(x) = 0$ and the continuity of κ that $\varphi(\mathrm{Lip}_b) \subset C_0$. We conclude that $\varphi(\mathrm{UC}_\kappa) \subset C_0$, since $\varphi\colon \mathrm{UC}_\kappa \to C_b$ is isometric and $\mathrm{Lip}_b \subset \mathrm{UC}_\kappa$ is dense. It remains to show $\varphi(\mathrm{UC}_\kappa) = C_0$. Let $f \in C_0$ and choose a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty$ with $\|f - f_n\|_\infty \to 0$. Since κ is smooth, it holds $f_n/\kappa \in C_c^\infty$ for all $n \in \mathbb{N}$. Furthermore, the sequence $(f_n/\kappa)_{n\in\mathbb{N}} \subset \mathrm{UC}_\kappa$ is a Cauchy sequence and the limit $g := \lim_{n\to\infty} f_n/\kappa \in \mathrm{UC}_\kappa$ exists because φ is isometric. We obtain

$$\varphi(g) = \lim_{n\to\infty} \varphi\left(\frac{f_n}{\kappa}\right) = \lim_{n\to\infty} f_n = f.$$

In addition, since $C_c^\infty \subset C_0$ is dense and φ is isometric, the set $\varphi^{-1}(C_c^\infty) \subset C_c^\infty \subset \mathrm{UC}_\kappa$ is also dense. □

1.4 The infinitesimal generator

Throughout this section, we assume that X is a Banach space with norm $\|\cdot\|$. We investigate the relation between the local behaviour of a generating family $(I(t))_{t\geq 0}$ and an associated semigroup $(S(t))_{t\geq 0}$. The technical condition (1.14) in Theorem 1.4.2 will be discussed in Subsection 1.4.1.

Assumption 1.4.1. Let $(I(t))_{t\geq 0}$ be a family of operators satisfying Assumption 1.2.1 with $x_0 := 0$ and let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup on X. In addition, we assume that

$$\left\|\frac{S(t)x-x}{t}-y\right\| \leq \sup_{n\in\mathbb{N}}\left\|\frac{I(\pi_n^t)x-x}{t}-y\right\| \quad \text{for all } t\in\mathcal{T}\backslash\{0\} \text{ and } x,y\in X. \tag{1.13}$$

Inequality (1.13) is clearly satisfied, if $(I(t))_{t\geq 0}$ satisfies Assumption 1.2.1 and Assumption 1.2.4 and $(S(t))_{t\geq 0}$ is an associated semigroup as in Theorem 1.2.5. However, requiring norm convergence $I(\pi_n^t)x \to S(t)x$ is an unnecessarily strong assumption for the next theorem. For instance, if X is a space of continuous functions endowed with the supremum or κ-norm, inequality (1.13) is satisfied if we have only pointwise convergence $I(\pi_{n_l}^t)f \to S(t)f$. In particular, Theorem 1.4.2 is applicable for Nisio semigroups.

Theorem 1.4.2. *Suppose that Assumption 1.4.1 is satisfied. Let $x,y\in X$ such that, for every $\varepsilon>0$, there exists $t_0>0$ with*

$$\left\|\frac{I(h_n)^k(x+h_ny)-I(h_n)^kx}{h_n}-y\right\| \leq \varepsilon \quad \textit{for all } k,n\in\mathbb{N} \textit{ with } kh_n\leq t_0. \tag{1.14}$$

Then,

$$\lim_{t\downarrow 0}\left\|\frac{I(t)x-x}{t}-y\right\|=0 \quad \textit{implies} \quad \lim_{t\downarrow 0}\left\|\frac{S(t)x-x}{t}-y\right\|=0. \tag{1.15}$$

Proof. Fix $\varepsilon>0$ and choose $r\geq 0$ with $x,y\in B(0,r)$. By assumption, there exists $t_0\in(0,1]$ with

$$\left\|\frac{I(t)x-x}{t}-y\right\| \leq \frac{\varepsilon}{2}e^{-\omega_{\alpha(2r,1)}} \quad \text{for all } t\in(0,t_0], \tag{1.16}$$

and

$$\left\|\frac{I(h_n)^k(x+h_ny)-I(h_n)^kx}{h_n}-y\right\| \leq \frac{\varepsilon}{2} \quad \text{for all } k,n\in\mathbb{N} \text{ with } kh_n\leq t_0. \tag{1.17}$$

First, we show inductively that

$$\left\|\frac{I(h_n)^kx-x}{kh_n}-y\right\| \leq \varepsilon \quad \text{for all } k,n\in\mathbb{N} \text{ with } kh_n\leq t_0. \tag{1.18}$$

For $k=1$, the claim holds by inequality (1.16). To prove the induction step, we assume for some fixed $k\in\mathbb{N}$ that

$$\left\|\frac{I(h_n)^kx-x}{kh_n}-y\right\| \leq \varepsilon \quad \text{for all } n\in\mathbb{N} \text{ with } kh_n\leq t_0. \tag{1.19}$$

Let $n \in \mathbb{N}$ with $(k+1)h_n \leq t_0$. It holds

$$\begin{aligned}
&\frac{I(h_n)^{k+1}x - x}{(k+1)h_n} - y \\
&= \frac{1}{k+1}\left(\frac{I(h_n)^k I(h_n)x - I(h_n)^k x}{h_n} - y\right) + \frac{k}{k+1}\left(\frac{I(h_n)^k x - x}{kh_n} - y\right).
\end{aligned} \tag{1.20}$$

The first term is further decomposed as

$$\begin{aligned}
&\frac{I(h_n)^k I(h_n)x - I(h_n)^k x}{h_n} - y \\
&= \frac{I(h_n)^k I(h_n)x - I(h_n)^k (x + h_n y)}{h_n} + \frac{I(h_n)^k (x + h_n y) - I(h_n)^k x}{h_n} - y.
\end{aligned} \tag{1.21}$$

We use Lemma 1.2.7 and inequality (1.16) to estimate

$$\begin{aligned}
\left\|\frac{I(h_n)^k I(h_n)x - I(h_n)^k (x + h_n y)}{h_n}\right\| &\leq e^{\omega_{\alpha(2r,1)}} \left\|\frac{I(h_n)x - x}{h_n} - y\right\| \\
&\leq e^{\omega_{\alpha(2r,1)}} \frac{\varepsilon}{2} e^{-\omega_{\alpha(2r,1)}} = \frac{\varepsilon}{2}.
\end{aligned} \tag{1.22}$$

Note that Lemma 1.2.7 relies only on Assumption 1.2.1, but not on Assumption 1.2.4. Combining inequality (1.17), equation (1.21) and inequality (1.22) yields

$$\left\|\frac{I(h_n)^k I(h_n)x - I(h_n)^k x}{h_n} - y\right\| \leq \varepsilon. \tag{1.23}$$

Furthermore, it follows from inequality (1.19), equation (1.20) and inequality (1.23) that

$$\begin{aligned}
&\left\|\frac{I(h_n)^{k+1}x - x}{(k+1)h_n} - y\right\| \\
&\leq \frac{1}{k+1}\left\|\frac{I(h_n)^k I(h_n)x - I(h_n)^k x}{h_n} - y\right\| + \frac{k}{k+1}\left\|\frac{I(h_n)^k x - x}{kh_n} - y\right\| \leq \varepsilon.
\end{aligned}$$

Second, we show that the right-hand side of equation (1.15) holds. Inequality (1.13) and inequality (1.18) imply

$$\left\|\frac{S(t)x - x}{t} - y\right\| \leq \sup_{n\in\mathbb{N}} \left\|\frac{I(\pi_n^t)x - x}{t} - y\right\| \leq \varepsilon \quad \text{for all } t \in (0, t_0] \cap \mathcal{T}.$$

Now, let $t \in (0, t_0]$ be arbitrary and choose a sequence $(t_n)_{n\in\mathbb{N}} \subset (0, t_0] \cap \mathcal{T}$ with $t_n \to t$. Since $(S(t))_{t\geq 0}$ is strongly continuous, we obtain

$$\left\|\frac{S(t)x - x}{t} - y\right\| = \lim_{n\to\infty} \left\|\frac{S(t_n)x - x}{t_n} - y\right\| \leq \varepsilon. \qquad \square$$

1.4.1 Condition (1.14)

If $I(t)$ is convex and monotone for all $t \geq 0$, inequality (1.14) is always satisfied. Furthermore, we will see in Subsection 1.6.5 an example, where $I(t)$ has none of these two properties.

Lemma 1.4.3. *Let X be a Banach lattice and $(I(t))_{t\geq 0}$ be a family of convex monotone operators $I(t)\colon X \to X$ satisfying Assumption 1.2.1 with $x_0 := 0$. Furthermore, let $\mathcal{L}^I \subset X$ be dense. Then, condition (1.14) is valid for all $x, y \in X$.*

Proof. We argue similar as in the proof of [136, Proposition 3.9]. Let $x, y \in X$ and $k, n \in \mathbb{N}$. Since the operator $I(h_n)^k$ is convex, the mapping

$$\mathbb{R} \to X,\ \lambda \mapsto I(h_n)^k(x+\lambda y) - I(h_n)^k x$$

is convex and maps zero to zero. This implies

$$\begin{aligned} -I(h_n)^k(x-y) + I(h_n)^k x - y &\leq \frac{I(h_n)^k(x+\lambda y) - I(h_n)^k x}{\lambda} - y \\ &\leq I(h_n)^k(x+y) - I(h_n)^k x - y \quad \text{for all } \lambda \in (0,1]. \end{aligned}$$

Hence, for $\lambda := h_n$, we obtain

$$\begin{aligned} &\left\| \frac{I(h_n)^k(x+h_n y) - I(h_n)^k x}{h_n} - y \right\| \\ &\leq \|I(h_n)^k(x+y) - (x+y)\| + \|I(h_n)^k(x-y) - (x-y)\| + \|I(h_n)^k x - x\|. \end{aligned}$$

It remains to show

$$\lim_{t\downarrow 0} \sup_{n\in\mathbb{N}} \|I(\pi_n^t)x - x\| = 0 \quad \text{for all } x \in X.$$

Let $x \in X$ and $\varepsilon > 0$. We define $r := \|x\| + 1$ and choose $\delta \in (0,1]$ with

$$(e^{\omega_{\alpha(r,1)}} + 1)\delta \leq \frac{\varepsilon}{2}. \tag{1.24}$$

Since $\mathcal{L}^I \subset X$ is dense, there exists $y \in B(x,\delta) \cap \mathcal{L}^I$. By Lemma 1.2.8, there exists $c \geq 0$ with

$$\|I(\pi_n^t)y - y\| \leq c e^{t\omega_{\alpha(r,t)}} t \quad \text{for all } t \geq 0.$$

Let $t_0 > 0$ such that

$$c e^{t_0 \omega_{\alpha(r,t_0)}} t_0 \leq \frac{\varepsilon}{2}. \tag{1.25}$$

We use Lemma 1.2.7, inequality (1.24), inequality (1.25) and the non-decreasingness of α in the second argument to conclude

$$\begin{aligned} \|I(\pi_n^t)x - x\| &\leq \|I(\pi_n^t)x - I(\pi_n^t)y\| + \|x-y\| + \|I(\pi_n^t)y - y\| \\ &\leq \left(e^{t\omega_{\alpha(r,t)}} + 1\right)\|x-y\| + c e^{t\omega_{\alpha(r,t)}} t \leq \varepsilon \end{aligned}$$

for all $t \in [0, t_0]$ and $n \in \mathbb{N}$. Note that Lemma 1.2.7 and Lemma 1.2.8 rely only on Assumption 1.2.1 but not Assumption 1.2.4. $\square$

In the linear case, the previous results remains valid without assuming the monotonicity, since

$$\frac{I(h_n)^k(x+h_ny)-I(h_n)^kx}{h_n}-y=I(h_n)^ky-y$$

for all $x,y\in X$ and $k,n\in\mathbb{N}$.

1.4.2 Invariance of the domain and uniqueness

Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup. Furthermore, we assume that, for every $r,T\geq 0$, there exists $c\geq 0$ with

$$\|S(t)x-S(t)y\|\leq c\|x-y\| \quad \text{for all } t\in[0,T] \text{ and } x,y\in B(0,r). \tag{1.26}$$

The local behaviour of the semigroup $(S(t))_{t\geq 0}$ is described by the infinitesimal generator

$$A\colon D(A)\to X,\ x\mapsto \lim_{h\downarrow 0}\frac{S(h)x-x}{h},$$

where the domain $D(A)$ consists of all $x\in X$ such that the previous limit exists.

Lemma 1.4.4. *For every $x\in D(A)$ and $t\geq 0$,*

$$S(t)x\in D(A) \quad\Longleftrightarrow\quad \lim_{h\downarrow 0}\frac{S(t)(x+hAx)-S(t)x}{h} \quad \text{exists.} \tag{1.27}$$

Proof. Fix $x\in D(A)$ and $t\geq 0$. For every $h>0$,

$$\frac{S(h)S(t)x-S(t)x}{h}=\frac{S(t)S(h)x-S(t)(x+hAx)}{h}+\frac{S(t)(x+hAx)-S(t)x}{h}.$$

It follows from inequality (1.26) and the definition of the generator that

$$\lim_{h\downarrow 0}\frac{S(t)S(h)x-S(t)(x+hAx)}{h}=0.$$

Hence, the equivalence (1.27) holds by definition of the generator. □

Remark 1.4.5. If X be a Banach lattice and $(S(t))_{t\geq 0}$ a semigroup of convex monotone operators, the quotient on the right hand side of (1.27) is non-decreasing in $h>0$ and bounded from below. Hence, if we further assume that the norm is order continuous, the limit on the right hand side of (1.27) exists and the domain is invariant. For details, we refer to [62]. Typical examples are L^p-spaces and Orlicz hearts, see [167], while spaces of continuous functions with the supremum or κ-norm lack this property. Thus, the domain of a nonlinear semigroup is in general not invariant, see [61] for a counter example. One possibility to overcome this problem is the extension of the semigroup to a space with order continuous norm, see [62] and Chapter 2.

The same arguments as in [62, Theorem 3.5] lead to the following uniqueness result.

Theorem 1.4.6. *Let $x \in X$ and $y\colon \mathbb{R}_+ \to X$ be a continuous function with $y(0) = x$ and $y(t) \in D(A)$ for all $t \geq 0$. Furthermore, we assume that*

$$\lim_{h \downarrow 0} \frac{y(t+h) - y(t)}{h} = Ay(t) \quad \textit{for all } t \geq 0,$$

where the existence of the limit is assumed. Then, it holds $y(t) = S(t)x$ for all $t \geq 0$.

Proof. Let $t \geq 0$ and define $g\colon [0,t] \to X,\ s \mapsto S(t-s)y(s)$. First, we show

$$\lim_{h \downarrow 0} \frac{g(s+h) - g(s)}{h} = 0 \quad \text{for all } s \in [0,t].$$

For every $s \in [0,t]$ and $h \in (0, t-s]$,

$$\frac{g(s+h) - g(s)}{h} = \frac{S(t-s-h)y(s+h) - S(t-s-h)S(h)y(s)}{h}.$$

By assumption, it holds

$$\frac{y(s+h) - S(h)y(s)}{h} = \frac{y(s+h) - y(s)}{h} - \frac{S(h)y(s) - y(s)}{h} \to 0 \quad \text{as } h \downarrow 0.$$

Hence, it follows from inequality (1.26) that

$$\lim_{h \downarrow 0} \frac{g(s+h) - g(s)}{h} = 0.$$

Second, we show that g is continuous. We have already established that g is right continuous, i.e., we only have to show left continuity. For every $s \in [0,t]$, continuity of the mapping $t \mapsto y(t)$, strong continuity of the semigroup $(S(t))_{t \geq 0}$ and inequality (1.26) imply $y(s) = \lim_{h \downarrow 0} S(h)y(s-h)$. Hence, it follows from inequality (1.26) that

$$g(s-h) - g(s) = S(t-s)S(h)y(s-h) - S(t-s)y(s) \to 0 \quad \text{as } h \downarrow 0.$$

Third, following the proof of [141, Lemma 1.1 in Section 2], one can show that every continuous function with vanishing right derivative is constant. In particular, we obtain $y(t) = g(t) = g(0) = S(t)x$. □

If the domain is invariant and dense, the semigroup $(S(t))_{t \geq 0}$ is uniquely determined through its generator. In general, we do not know whether the left derivative of the function y in the previous theorem exists and the abstract Cauchy problem

$$y'(t) = Ay(t) \quad \text{for all } t \geq 0, \quad y(0) = x,$$

has classical solution. However, we know that there exists at most one solution, and, if the solution exists, it depends locally Lipschitz continuous on the initial value x. If the norm is order continuous, the existence of a solution is also known, see [62].

1.5 Invariant symmetric Lipschitz sets

Throughout this section, let $(S(t))_{t\geq 0}$ and $(S^{\pm}(t))_{t\geq 0}$ be three semigroups on a Banach lattice X. For the choice $S^+(t) := S(t)$ and $S^-(t)x := -S(t)(-x)$, the set

$$\mathcal{L}^S_{\text{sym}} := \mathcal{L}^{S^+} \cap \mathcal{L}^{S^-} = \{x \in \mathcal{L}^S\colon\ -x \in \mathcal{L}^S\}$$

is called symmetric Lipschitz set of $(S(t))_{t\geq 0}$. In several examples, this set can be determined explicitly leading to regularity results, see Subsection 1.6.1 and Section 2.4. In contrast to the invariance of the domain, the invariance of the symmetric Lipschitz set does not depend on the underlying space, i.e., it is also valid for spaces of continuous function.

Theorem 1.5.1. *Assume that, for every $s, t \geq 0$ and $x \in X$,*

- $S^-(t)x \leq S(t)x \leq S^+(t)x$,
- $S(s)S^-(t)x \leq S^-(t)S(s)x$ *and* $S^+(s)S(t)x \leq S(t)S^+(s)x$.

Furthermore, for every $r, T \geq 0$, there exists $c \geq 0$ such that, for all $t \in [0, T]$ and $x, y \in B(0, r)$,

$$\|S(t)x - S(t)y\| \leq c\|x - y\| \quad \text{and} \quad \|S^{\pm}(t)x - S^{\pm}(t)y\| \leq c\|x - y\|. \tag{1.28}$$

Then, it holds $\mathcal{L}^{S^+} \cap \mathcal{L}^{S^-} \subset \mathcal{L}^S$ and $S(t)\colon \mathcal{L}^{S^+} \cap \mathcal{L}^{S^-} \to \mathcal{L}^{S^+} \cap \mathcal{L}^{S^-}$ for all $t \geq 0$.

Proof. First, we show $\mathcal{L}^{S^+} \cap \mathcal{L}^{S^-} \subset \mathcal{L}^S$. Let $x \in \mathcal{L}^{S^+} \cap \mathcal{L}^{S^-}$. By definition, there exist $t_0 > 0$ and $c \geq 0$ with

$$\|S^{\pm}(t)x - x\| \leq ct \quad \text{for all } t \in [0, t_0].$$

Furthermore, by assumption, it holds

$$S^-(t)x - x \leq S(t)x - x \leq S^+(t)x - x \quad \text{for all } t \geq 0.$$

We obtain the estimate

$$\|S(t)x - x\| \leq \max\left\{\|S^+(t)x - x\|, \|S^-(t)x - x\|\right\} \leq ct \quad \text{for all } t \in [0, t_0].$$

Second, we show $S(t)\colon \mathcal{L}^{S^+} \cap \mathcal{L}^{S^-} \to \mathcal{L}^{S^+} \cap \mathcal{L}^{S^-}$ for all $t \geq 0$. Let $x \in \mathcal{L}^{S^+} \cap \mathcal{L}^{S^-}$ and $t \geq 0$. By definition and the first part, there exist $t_0 > 0$ and $c \geq 0$ with

$$\|S(t)x - x\| \leq ct \quad \text{and} \quad \|S^{\pm}(t)x - x\| \leq ct \quad \text{for all } t \in [0, t_0]. \tag{1.29}$$

It follows from the assumptions of the theorem that, for all $s, t \geq 0$,

$$\begin{aligned} S(s)S^-(t)x - S(s)x &\leq S^-(t)S(s)x - S(s)x \leq S(t)S(s)x - S(s)x, \\ S(t)S(s)x - S(s)x &\leq S^+(t)S(s)x - S(s)x \leq S(s)S^+(t)x - S(s)x. \end{aligned}$$

We use inequality (1.28) and inequality (1.29) to estimate

$$\|S^-(t)S(s)x - S(s)x\| \leq \max\left\{\|S(s)S(t)x - S(s)x\|, \|S(s)S^-(t)x - S(s)x\|\right\}$$

$$\begin{aligned} &\leq c_1 \max\left\{\|S(t)x - x\|, \|S^-(t)x - x\|\right\} \leq cc_1 t, \\ \|S^+(t)S(s)x - S(s)x\| &\leq \max\left\{\|S(s)S(t)x - S(s)x\|, \|S(s)S^+(t)x - S(s)x\|\right\} \\ &\leq c_1 \max\left\{\|S(t)x - x\|, \|S^+(t)x - x\|\right\} \leq cc_1 t \end{aligned}$$

for all $s \in [0, t_0]$ and $t \geq 0$, where c_1 is a constant such that inequality (1.28) is valid with $T := t_0$ and $r := \max\{\|S(t)x\|, \|S^{\pm}(t)x\|\}$. □

The following conditions on the families $(I(t))_{t\geq 0}$ and $(I^{\pm}(t))_{t\geq 0}$ guarantees that the associated semigroups $(S(t))_{t\geq 0}$ and $(S^{\pm}(t))_{t\geq 0}$ satisfy the assumptions of Theorem 1.5.1. Furthermore, they allow us to study the relation between the Lipschitz sets of the generating families and the associated semigroups.

Assumption 1.5.2. Let $(I(t))_{t\geq 0}$ and $(I^{\pm}(t))_{t\geq 0}$ be three families of monotone operators on X satisfing Assumption 1.2.1 with $x_0 := 0$. For every $s, t \geq 0$ and $x \in X$, we suppose that

(i) $I^-(t)x \leq I(t)x \leq I^+(t)x$,

(ii) $I(s)I^-(t)x \leq I^-(t)I(s)x$ and $I^+(s)I(t)x \leq I(t)I^+(s)x$,

(iii) $I(s+t)x \leq I(s)I(t)$,

(iv) $I^+(t)$ is continuous from below.

In addition, we assume, for every $(x,t) \in X \times \mathbb{R}_+$,

$$S(t)x = \sup_{\pi \in P_t} I(\pi)x, \quad S^+(t)x = \sup_{\pi \in P_t} I^+(\pi)x, \quad \text{and} \quad S^-(t) = \inf_{\pi \in P_t} I^-(\pi)x.$$

Theorem 1.5.3. *Suppose that Assumption 1.5.2 is satisfied. Then,*

$$\mathcal{L}^{I^+} \cap \mathcal{L}^{I^-} = \mathcal{L}^{S^+} \cap \mathcal{L}^{S^-}.$$

Furthermore, it holds $S(t)\colon \mathcal{L}^{S^+} \cap \mathcal{L}^{S^-} \to \mathcal{L}^{S^+} \cap \mathcal{L}^{S^-}$ for all $t \geq 0$.

Proof. W.l.o.g., we assume that $(I(t))_{t\geq 0}$ and $(I^{\pm}(t))_{t\geq 0}$ satisfy Assumption 1.2.1 with the same function α and the same constant ω_r. First, we show that

$$\mathcal{L}^{I^+} \cap \mathcal{L}^{I^-} = \mathcal{L}^{S^+} \cap \mathcal{L}^{S^-}.$$

Let $x \in \mathcal{L}^{I^+} \cap \mathcal{L}^{I^-}$. By definition, there exist $t_0 > 0$ and $c \geq 0$ with

$$\|I^{\pm}(t)x - x\| \leq ct \quad \text{for all } t \in [0, t_0].$$

It follows from Assumption 1.5.2 and Remark 1.2.14(ii) that

$$\|S^{\pm}(t)x - x\| \leq \sup_{\pi \in P_t} \|I^{\pm}(\pi)x - x\| \leq ce^{t\omega_{\alpha(r,t)}} t \quad \text{for all } t \geq 0,$$

where $r := \|x\|$. On the other hand, let $x \in \mathcal{L}^{S^+} \cap \mathcal{L}^{S^-}$. By definition, there exist $t_0 > 0$ and $c \geq 0$ with

$$\|S^{\pm}(t)x - x\| \leq ct \quad \text{for all } t \in [0, t_0].$$

Assumption 1.5.2 implies

$$S^-(t)x - x \leq I^-(t)x - x \leq I^+(t)x - x \leq S^+(t)x - x,$$

and therefore

$$\|I^\pm(t)x - x\| \leq \max\left\{\|S^+(t)x - x\|, \|S^-(t)x - x\|\right\} \leq ct \quad \text{for all } t \in [0, t_0].$$

Second, we show that $S^-(t)x \leq S(t)x \leq S^+(t)x$ for all $t \geq 0$ and $x \in X$. It follows immediately from Assumption 1.5.2 that $S^-(t)x \leq I^-(t)x \leq I(t)x \leq S(t)x$. Furthermore, Assumption 1.5.2(i) and the monotonicity of $I(s)$ imply

$$I(s)I(t)x \leq I(s)I^+(t)x \leq I^+(s)I^+(t)x \quad \text{for all } s, t \geq 0 \text{ and } x \in X.$$

By Assumption 1.5.2 and induction, we obtain

$$S(t)x = \sup_{\pi \in P_t} I(\pi)x \leq \sup_{\pi \in P_t} I^+(\pi)x = S^+(t)x \quad \text{for all } (x, t) \in X \times \mathbb{R}_+.$$

Third, we show that $S(s)S^-(t)x \leq S^-(t)S(s)x$ and $S^+(s)S(t)x \leq S(t)S^+(s)x$ for all $s, t \geq 0$ and $x \in X$. We use Assumption 1.5.2(ii) and the monotonicity of $(I(t))_{t\geq 0}$ and $(I^-(t))_{t\geq 0}$ to estimate

$$I(s_1)I(s_2)I^-(t_1)I^-(t_2) \leq I^-(t_1)I^-(t_2)I(s_1)I(s_2)$$

for all $s_1, s_2, t_1, t_2 \geq 0$. It follows by induction that

$$I(\pi_s)I^-(\pi_t)x \leq I^-(\pi_t)I(\pi_s)x$$

for all $s, t \geq 0$, $\pi_s \in P_s$, $\pi_t \in P_t$ and $x \in X$. Assumption 1.5.2 and the monotonicity of $I(\pi_s)$ and $I^-(\pi_t)$ imply

$$\begin{aligned} S(s)S^-(t)x &= \sup_{\pi_s \in P_s} I(\pi_s)\Big(\inf_{\pi_t \in P_t} I^-(\pi_t)x\Big) \leq \sup_{\pi_s \in P_s} \inf_{\pi_t \in P_t} I(\pi_s)I^-(\pi_t)x \\ &\leq \sup_{\pi_s \in P_s} \inf_{\pi_t \in P_t} I^-(\pi_t)I(\pi_s)x \leq \inf_{\pi_t \in P_t} \sup_{\pi_s \in P_s} I^-(\pi_t)I(\pi_s)x \\ &\leq \inf_{\pi_t \in P_t} I^-(\pi_t)\Big(\sup_{\pi_s \in P_s} I(\pi_s)\Big) = S^-(t)S(s)x. \end{aligned}$$

Similarly, we use Assumption 1.5.2(ii) and the monotonicity of $(I(t))_{t\geq 0}$ and $(I^+(t))_{t\geq 0}$ to obtain

$$I^+(\pi_s)I(\pi_t)x \leq I(\pi_t)I^+(\pi_s)x$$

for all $s, t \geq 0$, $\pi_s \in P_s$, $\pi_t \in P_t$ and $x \in X$. Moreover, Assumption 1.5.2(iii) implies that the set $\{I(\pi) \colon \pi \in P_t\}$ is directed upwards for all $t \geq 0$ and Assumption 1.5.2(iv) ensures that $I^+(\pi)$ is continuous from below for all $s \geq 0$ and $\pi \in P_s$. We combine this with the monotonicity of $I(\pi_t)$ to conclude

$$\begin{aligned} S^+(s)S(t) &= \sup_{\pi_s \in P_s} I^+(\pi_s)\Big(\sup_{\pi_t \in P_t} I(\pi_t)x\Big) = \sup_{\pi_s \in P_s} \sup_{\pi_t \in P_t} I^+(\pi_s)I(\pi_t)x \\ &\leq \sup_{\pi_t \in P_t} \sup_{\pi_s \in P_s} I(\pi_t)I^+(\pi_s)x \leq \sup_{\pi_t \in P_t} I(\pi_t)\Big(\sup_{\pi_s \in P_s} I^+(\pi_s)x\Big) = S(t)S^+(s)x \end{aligned}$$

for all $s, t \geq 0$ and $x \in X$. Since inequality (1.28) follows from Assumption 1.2.1 and Remark 1.2.14(i), we can apply Theorem 1.5.1 and obtain

$$S(t) \colon \mathcal{L}^{S^+} \cap \mathcal{L}^{S^-} \to \mathcal{L}^{S^+} \cap \mathcal{L}^{S^-} \quad \text{for all } t \geq 0. \qquad \square$$

Let $(I(t))_{t\geq 0}$ be a family of operators $I(t)\colon X \to X$ satisfying Assumption 1.2.1 with $x_0 := 0$ and Assumption 1.2.4. If we define $I^+(t) := I(t)$ and $I^-(t)x := -I(t)(-x)$ for all $(x,t) \in X \times \mathbb{R}_+$, the families $(I^\pm(t))_{t\geq 0}$ satisfy Assumption 1.2.1 with $x_0 = 0$. Assumption 1.2.4 is, for instance, satisfied, if the set $\mathcal{D}$ can be chosen symmetric, i.e. $\mathcal{D} = \{-x\colon x \in \mathcal{D}\}$. Let $(S(t))_{t\geq 0}$ and $(S^\pm(t))_{t\geq 0}$ be associated semigroups as in Theorem 1.2.5. Then, by construction, it holds $S^+(t) = S(t)$ and $S^-(t)x = -S(t)(-x)$ for all $(x,t) \in X \times \mathbb{R}_+$. The condition $S(t)x = \sup_{\pi\in P_t} I(\pi)x$ has already been discussed in Subsection 1.2.2. In that case, we conclude

$$S^-(t)x = -S(t)(-x) = -\sup_{\pi\in P_t} I(\pi)(-x) = \inf_{\pi\in P_t} -I(\pi)(-x) = \inf_{\pi\in P_t} I^-(\pi)(x).$$

We will see in Subsection 1.6.1 that the verification of Assumption 1.5.2(i)-(iv) is straightforward, if $I(t)$ is a supremum over linear semigroups. Basically, condition (ii) is the consequence of interchanging a supremum with an infimum at the cost of an inequality. Furthermore, Theorem 1.5.3 remains valid without Assumption 1.5.2(iii) and (iv), if $I(t) = I^+(t)$ for all $t \geq 0$.

1.6 Examples

We illustrate the abstract results with several examples. First, we consider Nisio semigroups which have already been discussed in Subsection 1.2.2. This first type of examples is illustrated by a convex version of the g-expectation as well as a sublinear versions of the geometric Brownian motion and the G-expectation. Second, we start with a linear semigroup $(S_0(t))_{t\geq 0}$ and consider the generating family $(I(t))_{t\geq 0}$ given as the perturbation

$$I(t)x := S_0(t)x + \Psi(x)t \quad \text{for all } (x,t) \in X \times \mathbb{R}_+,$$

where $\Psi\colon X \to X$ is a Lipschitz continuous mapping. In particular, if we choose $X := \mathbb{R}^d$ and $S_0(t) := \mathrm{id}_{\mathbb{R}^d}$, for every $x \in \mathbb{R}^d$, the unique solution to the ODE

$$y'(t) = f(y(t)) \quad \text{for all } t \geq 0, \quad y(0) = x$$

is given by $y(t) := S(t)x$. The main example of this second type are reaction-diffusion equations, where the operators $I(t)$ are neither convex nor monotone. Recall that the results in Section 1.2 do not rely on these properties. The verification of condition (1.14) becomes more complicated, but is possible by proving a suitable recursion formula for the iterated operators $I(h_n)^k$. Both the abstract results and the examples presented in this paper are independent the established PDE-theory. For an alternative approach to nonlinear parabolic equations, we refer to [131], where short time existence is proved by a fixed-point argument. Furthermore, as long as the solution does not blow up, long time existence follows. In the semi-linear case, blow up in finite time is excluded, if the non-linearity is locally Lipschitz continuous and does not grow faster than linear. While the approach in [131] relies strongly on a priori estimates in suitable function spaces, we use stochastic representations for the generating family $(I(t))_{t\geq 0}$ and Itô calculus.

1.6.1 Convex g-expectation

In this subsection, we construct a semigroup corresponding to a Brownian motion with uncertain drift. The generator is a semi-linear second order differential operator, where the first-order non-linearity corresponds to the uncertainty in the drift. Let $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $L\colon \mathbb{R}^d \to [0,\infty]$ be a function satisfying

- $\min_{\lambda\in\mathbb{R}^d} L(\lambda) = 0$,
- $\lim_{|\lambda|\to\infty} \frac{L(\lambda)}{|\lambda|} = \infty$.

For every $t \geq 0$, $f \in C_0(\mathbb{R}^d;\mathbb{R})$ and $x \in \mathbb{R}^d$, we define

$$\big(I(t)f\big)(x) := \sup_{\lambda\in\mathbb{R}^d} \big(\mathbb{E}[f(x+W_t+\lambda t)] - L(\lambda)t\big),$$

where $\mathbb{E}[X]$ denotes the expectation of random variable $X\colon \Omega \to \mathbb{R}$. For every $\lambda \in \mathbb{R}^d$, we denote by $(S_\lambda(t))_{t\geq 0}$ the linear semigroup given by

$$\big(S_\lambda(t)f\big)(x) := \mathbb{E}[f(x+W_t+\lambda t)] \quad \text{for all } t \geq 0,\ f \in C_0 \text{ and } x \in \mathbb{R}^d.$$

Moreover, we can write $I(t)f = \sup_{\lambda\in\mathbb{R}^d} I_\lambda(t)f$ for all $t \geq 0$ and $f \in C_0$ by defining $I_\lambda(t)f := S_\lambda(t)f - L(\lambda)t$ for all $\lambda \in \mathbb{R}^d$. The first-order non-linearity will be described by the function

$$H\colon \mathbb{R}^d \to \mathbb{R},\ x \mapsto \sup_{y\in\mathbb{R}^d} \big(\langle x,y\rangle - L(y)\big),$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product on $\mathbb{R}^d$. Note that, in contrast to the results in [62], we are not restricted to non-linearities with at most linear growth. For example, power functions $H(x) := |x|^p$ with $p \geq 1$ are included in our setting.

Lemma 1.6.1. *For every $c \geq 0$, there exists a bounded set $\Lambda \subset \mathbb{R}^d$ with*

$$I(t)f = \sup_{\lambda\in\Lambda} I_\lambda(t)f \quad \text{for all } t \geq 0 \text{ and } f \in \mathrm{Lip}_0(c).$$

Proof. Let $c \geq 0$, $f \in \mathrm{Lip}_0(c)$ and $\lambda_0 \in \mathbb{R}^d$ with $L(\lambda_0) < \infty$ which exists by assumption. For every $\lambda \in \mathbb{R}^d$, $t \geq 0$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}\mathbb{E}[f(x+W_t+\lambda t)] - L(\lambda)t &\leq \mathbb{E}[f(x+W_t+\lambda_0 t)] + c|\lambda-\lambda_0|t - L(\lambda)t \\ &= \big(I_{\lambda_0}(t)f\big)(x) + \big(L(\lambda_0) + c|\lambda-\lambda_0| - L(\lambda)\big)t.\end{aligned}$$

Since the assumption on L implies $L(\lambda_0) + c|\lambda-\lambda_0| - L(\lambda) \to -\infty$ as $|\lambda| \to \infty$, the claim follows. □

Let C_0^2 be the space of all twice continuously differentiable functions $f \in C_0$ such that the first and second derivative vanish at infinity. For every $\lambda \in \mathbb{R}^d$, we denote by A_λ the generator of $(S_\lambda(t))_{t\geq 0}$. It follows from Itô's formula that $C_0^2 \subset D(A_\lambda)$ with

$$A_\lambda f = \frac{1}{2}\Delta f + \nabla_\lambda f \quad \text{for all } f \in C_0^2, \text{ where } \nabla_\lambda f := \langle\lambda, \nabla f\rangle.$$

Theorem 1.6.2. *The family $(I(t))_{t\geq 0}$ satisfies the assumptions of Lemma 1.3.2. Hence, Theorem 1.2.5 yields an associated semigroup $(S(t))_{t\geq 0}$ on C_0. Moreover, it holds $C_0^2 \subset D(A)$ with*

$$Af = \sup_{\lambda\in\mathbb{R}^d} A_\lambda f = \frac{1}{2}\Delta f + H(\nabla f) \quad \text{for all } f \in C_0^2.$$

Proof. First, we verify the conditions (i)-(iv) of Lemma 1.3.2.

(i) Clearly, $I(0) = \mathrm{id}_{C_0}$.

(ii) For every $t \geq 0$ and $f \in C_0$, the first assumption on L implies

$$-\|f\|_\infty = -\|f\|_\infty - \inf_{\lambda\in\mathbb{R}^d} L(\lambda)t \leq I(t)f \leq \sup_{\lambda\in\mathbb{R}^d} S_\lambda(t)f \leq \|f\|_\infty.$$

We obtain $\|I(t)f\|_\infty \leq \|f\|_\infty$.

(iii) For every $\lambda \in \mathbb{R}^d$, $t \geq 0$ and $f, g \in C_0$,

$$I_\lambda(t)f - I(t)g \leq I_\lambda(t)f - I_\lambda(t)g = S_\lambda(t)(f-g) \leq \|f-g\|_\infty.$$

By taking the supremum over $\lambda \in \mathbb{R}^d$ and changing the roles of f and g, we obtain $\|I(t)f - I(t)g\|_\infty \leq \|f-g\|_\infty$.

(iv) For every $t \geq 0$, $f \in C_0$ and $x \in \mathbb{R}^d$,

$$(I(t)f_x)(0) = \sup_{\lambda\in\mathbb{R}^d} \big(\mathbb{E}[f(x + (0 + W_t + \lambda t))] - L(\lambda)t\big) = (I(t)f)(x).$$

Second, we show that

$$\lim_{t\downarrow 0} \left\| \frac{I(t)f - f}{t} - \frac{1}{2}\Delta f - H(\nabla f) \right\|_\infty = 0 \quad \text{for all } f \in C_0^2. \tag{1.30}$$

Let $f \in C_0^2$. By Lemma 1.6.1, there exists $r \geq 0$ such that

$$I(t)f = \sup_{\lambda\in B(r)} I_\lambda(t)f \quad \text{for all } t \geq 0.$$

Moreover, the constant $r \geq 0$ can be chosen such that

$$H(\nabla f) = \sup_{\lambda\in B(r)} \big(\langle \lambda, \nabla f\rangle - L(\lambda)\big),$$

because $\|\nabla f\|_\infty < \infty$ and L growths faster than linear. Hence, it follows from Itô's formula that

$$\left\| \frac{I(t)f - f}{t} - \frac{1}{2}\Delta f - H(\nabla f) \right\|_\infty \leq \sup_{\lambda\in B(r)} \left\| \frac{S_\lambda(t)f - f}{t} - \frac{1}{2}\Delta f - \langle\lambda, \nabla f\rangle \right\|_\infty$$
$$\leq \sup_{\lambda\in B(r)} \frac{1}{t}\int_0^t \left(\|\nabla_\lambda f(\cdot + X_s^\lambda) - \nabla_\lambda f\|_\infty + \frac{1}{2}\|\Delta f(\cdot + X_s^\lambda) - \Delta f\|_\infty \right) \mathrm{d}s,$$

where $X_s^\lambda := W_s + \lambda s$ for all $\lambda \in \mathbb{R}^d$ and $s \geq 0$. Since $f \in C_0^2$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\Big(\|\nabla_\lambda f(\,\cdot\, + X_s^\lambda) - \nabla_\lambda f\|_\infty + \frac{1}{2}\|\Delta f(\,\cdot\, + X_s^\lambda) - \Delta f\|_\infty\Big)\, \mathbb{1}_{\{|X_s^\lambda|<\delta\}} < \varepsilon$$

for all $\lambda \in \mathbb{R}^d$ and $s \geq 0$. Furthermore, Chebyshev's inequality implies

$$\sup_{\lambda\in B(r)} \mathbb{P}\big(|X_s^\lambda| \geq \delta\big) \leq \sup_{\lambda\in B(r)} \frac{\mathbb{E}[|X_s^\lambda|^2]}{\delta^2} \leq \sup_{\lambda\in B(r)} \frac{2}{\delta^2}\big(\mathbb{E}[|W_s|^2] + |\lambda|^2 s^2\big) \to 0 \quad \text{as } s \downarrow 0.$$

Third, we verify the conditions (v) and (vi) of Lemma 1.3.2. It follows immediately from inequality (1.30) that $C_0^2 \subset \mathrm{Lip}_0 \cap \mathcal{L}^I$. In particular, we can choose a countable set $\mathcal{D} \subset \mathrm{Lip}_0 \cap \mathcal{L}^I$ which is dense in C_0. It remains to show condition (vi). Let $c \geq 0$. By Lemma 1.6.1, there exists $r \geq 0$ with

$$I(t)f = \sup_{\lambda\in B(r)} I_\lambda(t)f \quad \text{for all } t \geq 0 \text{ and } f \in \mathrm{Lip}_0(c).$$

We define

$$\big(T_c(t)f\big)(x) := e^{\frac{r^2 t}{2}}\, \mathbb{E}\left[f^2(x+W_t)\right]^{\frac{1}{2}} \quad \text{for all } t \geq 0,\ f \in C_0^+ \text{ and } x \in \mathbb{R}^d.$$

The family $(T_c(t))_{t\geq 0}$ is a monotone semigroup on C_0^+. It remains to show

$$|I(t)f| \leq T_c(t)|f| \quad \text{for all } f \in \mathrm{Lip}_0(c) \text{ and } t \geq 0.$$

Let $f \in \mathrm{Lip}_0(c)$ and $t \geq 0$. We use $W_t \sim \mathcal{N}(0, t\mathbb{1})$, where $\mathcal{N}(0, t\mathbb{1})$ denotes the normal distribution, and the formula for its moment generating function to estimate

$$\begin{aligned}
&|(S_\lambda(t)f)(x)| \leq \mathbb{E}[|f(x+W_t+\lambda t)|] \\
&= (2\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} |f(x+y+\lambda t)| \exp\left(-\tfrac{|y|^2}{2t}\right) \mathrm{d}y \\
&= (2\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} |f(x+y)| \exp\left(-\tfrac{|y-\lambda t|^2}{2t}\right) \mathrm{d}y \\
&= e^{-\frac{|\lambda|^2 t}{2}} \int_{\mathbb{R}^d} |f(x+y)| \exp\big(\langle\lambda, y\rangle\big)\, \mathcal{N}(0, t\mathbb{1})(\mathrm{d}y) \\
&\leq e^{-\frac{|\lambda|^2 t}{2}} \left(\int_{\mathbb{R}^d} f^2(x+y)\, \mathcal{N}(0, t\mathbb{1})(\mathrm{d}y)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \exp\big(2\langle\lambda, y\rangle\big)\, \mathcal{N}(0, t\mathbb{1})(\mathrm{d}y)\right)^{\frac{1}{2}} \\
&= e^{-\frac{|\lambda|^2 t}{2}}\, \mathbb{E}\left[f^2(x+W_t)\right]^{\frac{1}{2}} e^{|\lambda|^2 t} = e^{\frac{|\lambda|^2 t}{2}}\, \mathbb{E}\left[f^2(x+W_t)\right]^{\frac{1}{2}} \leq \big(T_c(t)|f|\big)(x)
\end{aligned}$$

for all $\lambda \in B(r)$. Taking the supremum yields

$$I(t)f \leq \sup_{\lambda\in B(r)} S_\lambda(t)f \leq T_c(t)|f|.$$

Furthermore, by assumption, there exists $\lambda_0 \in \mathbb{R}^d$ with $L(\lambda_0) = 0$. W.l.o.g. we can assume $r \geq |\lambda_0|$ and obtain $I(t)f \geq S_{\lambda_0}(t)f \geq -T_c(t)|f|$.

Fourth, by Theorem 1.2.5, there exists an associated semigroup $(S(t))_{t\geq 0}$ on C_0. In particular, Assumption 1.4.1 is satisfied. Since Lemma 1.4.3 guarantees that condition (1.14) is valid for all $f, g \in C_0$, Theorem 1.4.2 implies $C_0^2 \subset D(A)$ with

$$Af = \frac{1}{2}\Delta f + H(\nabla f) \quad \text{for all } f \in C_0^2. \qquad \square$$

Let L^∞ be the space of all bounded Borel measurable functions $f\colon \mathbb{R}^d \to \mathbb{R}$, where two of them are identified if they coincide Lebesgue almost everywhere. Moreover, we denote by $W^{1,\infty}$ the corresponding first order Sobolev space. For $f \in W^{1,\infty}$ we say that Δf exists in L^∞ if there exists a function $g \in L^\infty$ with

$$\int_{\mathbb{R}^d} g\varphi \, \mathrm{d}x = -\int_{\mathbb{R}^d} \langle \nabla f, \nabla \varphi \rangle \, \mathrm{d}x \quad \text{for all } \varphi \in C_c^\infty,$$

where C_c^∞ denotes the set of all infinitely differentiable functions $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ with compact support. In this case, since g is unique Lebesgue almost everywhere, we define $\Delta f := g$.

Theorem 1.6.3. *In addition to previous conditions, we assume that*

- $\sup_{\lambda \in B(r)\cap\{L<\infty\}} L(\lambda) < \infty$ *for all* $r > 0$,
- *there exists* $\varepsilon > 0$ *with* $\sup_{\{|\lambda|=\varepsilon\}} L(\lambda) < \infty$.

Then, it holds $S(t)\colon \mathcal{L}^S_{\mathrm{sym}} \to \mathcal{L}^S_{\mathrm{sym}}$ *for all* $t \geq 0$ *and*

$$\mathcal{L}^S_{\mathrm{sym}} = \mathcal{L}^I_{\mathrm{sym}} = \big\{ f \in W^{1,\infty} \cap C_0 \colon \Delta f \text{ exists in } L^\infty \big\}.$$

Proof. We define $I^+(t) := I(t)$ and $I^-(t)x := -I(t)(-x)$ for all $t \geq 0$ and $x \in X$. First, we verify the conditions (i)-(iii) of Assumption 1.5.2.

(i) It holds $I^-(t)f = -I(t)(-f) \leq I(t)f = I^+(t)f$ for all $(f,t) \in C_0 \times \mathbb{R}_+$ because $I(t)$ is convex with $I(t)0 = 0$.

(ii) Let $s, t \geq 0$ and $f \in C_0$. We use Fubini's theorem to conclude

$$\begin{aligned} I(s)I^-(t)f &= \sup_{\lambda_1 \in \mathbb{R}^d} \Big(S_{\lambda_1}(s)\Big(\inf_{\lambda_2 \in \mathbb{R}^d} \big(S_{\lambda_2}(t)f + L(\lambda_2)t\big)\Big) - L(\lambda_1)s\Big) \\ &\leq \sup_{\lambda_1 \in \mathbb{R}^d} \inf_{\lambda_2 \in \mathbb{R}^d} \big(S_{\lambda_1}(s)S_{\lambda_2}(t)f + L(\lambda_2)t - L(\lambda_1)s\big) \\ &\leq \inf_{\lambda_2 \in \mathbb{R}^d} \sup_{\lambda_1 \in \mathbb{R}^d} \Big(S_{\lambda_2}(t)\big(S_{\lambda_1}(s)f - L(\lambda_1)s\big) + L(\lambda_2)t\Big) \\ &\leq \inf_{\lambda_2 \in \mathbb{R}^d} \big(S_{\lambda_2}(t)I(s)f + L(\lambda_2)t\big) = I^-(t)I(s)f. \end{aligned}$$

(iii) For every $s, t \geq 0$ and $f \in C_0$,

$$\begin{aligned} I(s+t)f &= \sup_{\lambda \in \mathbb{R}^d} \big(S_\lambda(s+t)f - L(\lambda)(s+t)\big) \\ &= \sup_{\lambda \in \mathbb{R}^d} \big(S_\lambda(s)S_\lambda(t)f - L(\lambda)s - L(\lambda)t\big) \\ &= \sup_{\lambda \in \mathbb{R}^d} \Big(S_\lambda(s)\big(S_\lambda(t)f - L(\lambda)t\big) - L(\lambda)s\Big) \\ &\leq \sup_{\lambda \in \mathbb{R}^d} \big(S_\lambda(s)I(t)f - L(\lambda)s\big) = I(s)I(t)f. \end{aligned}$$

Second, we apply Lemma 1.2.15 to show that $S(t)f = \sup_{\pi\in P_t} I(\pi)f$ for all $t\geq 0$ and $f\in C_0$. Clearly, $I(t)$ is continuous from below for all $t\geq 0$. Next, we show that the mapping $I(\cdot)f$ is continuous for all $f\in C_0$: For every $f\in C_0^2$, it follows from Lemma 1.6.1 and Itô's formula that there exists $r\geq 0$ with

$$\begin{aligned}\|I(t)f-I(s)f\|_\infty &\leq \sup_{\lambda\in B(r)\cap\{L<\infty\}}\big(\|S_\lambda(t-s)f\|_\infty + L(\lambda)(t-s)\big)\\ &\leq \sup_{\lambda\in B(r)\cap\{L<\infty\}}\left(|\lambda|\cdot\|\nabla f\|_\infty + \frac{1}{2}\|\Delta f\|_\infty + L(\lambda)\right)(t-s)\end{aligned}$$

for all $0\leq s\leq t$. Moreover, the assumptions on L imply

$$\sup_{\lambda\in B(r)\cap\{L<\infty\}}\left(|\lambda|\cdot\|\nabla f\|_\infty + \frac{1}{2}\|\Delta f\|_\infty + L(\lambda)\right)<\infty.$$

Since $C_0^2\subset C_0$ is dense and $(I(t))_{t\geq 0}$ is a family of contractions, the mapping $I(\cdot)f$ is also continuous for arbitrary $f\in C_0$, see the proof of Lemma 1.2.10. It remains to show that, for every $t\geq 0$, the operator

$$T(t)\colon C_0\to C_0,\ f\mapsto \sup_{\pi\in P_t} I(\pi)f$$

is well-defined. Lemma 1.3.2 yields $\{I(\pi)\colon \pi\in P_t\}\subset \mathrm{Lip}_0(c)$, and thus $T(t)f\in\mathrm{Lip}_0(c)$ for all $c,t\geq 0$ and $f\in\mathrm{Lip}_0(c)$. Since $T(t)\colon C_0\to\mathcal{L}^\infty$ is continuous and $\mathrm{Lip}_0\subset C_0$ is dense, we obtain $T(t)\colon C_0\to C_0$ for all $t\geq 0$. Lemma 1.2.15 implies

$$S(t)x = T(t)x \quad \text{for all } (f,t)\in C_0\times\mathbb{R}_+.$$

Hence, Assumption 1.5.2 is satisfied and Theorem 1.5.3 yields $\mathcal{L}^S=\mathcal{L}^I$ and

$$S(t)\colon \mathcal{L}^S_{\mathrm{sym}}\to\mathcal{L}^S_{\mathrm{sym}} \quad \text{for all } t\geq 0.$$

Third, we show $\mathcal{L}^I_{\mathrm{sym}}\subset\{f\in W^{1,\infty}\cap C_0\colon \Delta f \text{ exists in } L^\infty\}$. Let $f\in\mathcal{L}^I_{\mathrm{sym}}$. By definition, there exist $t_0>0$ and $c\geq 0$ with

$$\|I(t)f-f\|_\infty\leq ct \quad\text{and}\quad \|I(t)(-f)+f\|_\infty\leq ct \quad \text{for all } t\in[0,t_0].$$

For every $\lambda\in\mathbb{R}^d$ and $t\in[0,t_0]$,

$$\begin{aligned}-\big(c+L(\lambda)\big)t &\leq -\big(I(t)(-f)+f+L(\lambda)t\big)\leq -\big(S_\lambda(t)(-f)+f\big)\\ &= S_\lambda(t)f-f\leq I(t)f-f+L(\lambda)t\leq \big(c+L(\lambda)\big)t,\end{aligned}$$

and therefore $\|S_\lambda(t)f-f\|_\infty\leq (c+L(\lambda))t$. Let $\eta\in C_c^\infty$ with $\eta\geq 0$, $\mathrm{supp}(\eta)\subset B(1)$ and $\int_{\mathbb{R}^d}\eta(x)\,\mathrm{d}x=1$. For every $n\in\mathbb{N}$ and $x\in\mathbb{R}^d$, we define $\eta_n(x):=n^d\eta(nx)$ and

$$f_n(x):=\big(f*\eta_n\big)(x)=\int_{\mathbb{R}^d} f(x-y)\eta_n(y)\,\mathrm{d}y.$$

Let $\lambda\in\mathbb{R}^d$ and $t\geq 0$. Fubini's theorem implies

$$\big|S_\lambda(t)f_n-f_n\big|(x)=\left|\mathbb{E}\left[\int_{\mathbb{R}^d} f(x+W_t+\lambda t-y)\eta_n(y)\,dy\right]-f_n(x)\right|$$

$$
\begin{aligned}
&= \left| \int_{\mathbb{R}^d} \mathbb{E}\big[f(x+W_t+\lambda t-y)\big]\eta_n(y)\,dy - f_n(x)\right| \\
&= \big|(S_\lambda(t)f-f)*\eta_n\big|(x) \le \big\|S_\lambda(t)f-f\big\|_\infty \le \big(c+L(\lambda)\big)t.
\end{aligned}
$$

Since $f_n \in C_0^2 \subset D(A_\lambda)$, we obtain

$$
\|A_\lambda f_n\|_\infty \le c + L(\lambda) \quad \text{for all } n\in\mathbb{N}. \tag{1.31}
$$

Moreover, for every $n\in\mathbb{N}$, we have the identities

$$
\Delta f_n = \frac{1}{2}\Delta f_n + \nabla_\lambda f_n + \frac{1}{2}\Delta f_n + \nabla_{-\lambda} f_n = A_\lambda f_n + A_{-\lambda} f_n, \tag{1.32}
$$

$$
2\nabla_\lambda f_n = \frac{1}{2}\Delta f_n + \nabla_\lambda f_n + \frac{1}{2}\Delta(-f_n) + \nabla_{-\lambda}(-f_n) = A_\lambda f_n + A_{-\lambda}(-f_n). \tag{1.33}
$$

It follows from inequality (1.31) and inequality (1.32) that $\sup_{n\in\mathbb{N}}\|\Delta f_n\|_\infty < \infty$, since the assumptions on L ensure the existence of $\lambda\in\mathbb{R}^d$ with $L(\pm\lambda)<\infty$. Furthermore, we use inequality (1.31), inequality (1.33) and the assumptions on L to estimate

$$
\begin{aligned}
\|\nabla f_n\|_\infty &= \frac{1}{\varepsilon}\sup_{\{|\lambda|=\varepsilon\}}\|\nabla_\lambda f_n\|_\infty \le \frac{1}{2\varepsilon}\sup_{\{|\lambda|=\varepsilon\}}\big(\|A_\lambda f_n\|_\infty + \|A_\lambda(-f_n)\|_\infty\big) \\
&\le \frac{1}{\varepsilon}\sup_{\{|\lambda|=\varepsilon\}}\big(c+L(\lambda)\big) < \infty \quad \text{for all } n\in\mathbb{N}.
\end{aligned}
$$

By Banach-Alaoglu's theorem, there exist functions $g, g_i \in L^\infty$ such that $\Delta f_{n_k}\to g$ and $\partial_i f_n \to g_i$ as $k\to\infty$ in the weak*-topology for a suitable subsequence. This implies $f\in\{h\in W^{1,\infty}\cap C_0\colon \Delta h \text{ exists in } L^\infty\}$ with $\Delta f = g$ and $\partial_i f = g_i$ for $i=1,\dots,d$.

Forth, we show $\{f\in W^{1,\infty}\cap C_0\colon \Delta f \text{ exists in } L^\infty\}\subset \mathcal{L}^I_{\text{sym}}$. Let $f\in W^{1,\infty}\cap C_0$ such that Δf exists in L^∞. By Lemma 1.6.1, there exists $r\ge 0$ with

$$
I(t)f = \sup_{\lambda\in B(r)} I_\lambda(t)f \quad \text{for all } t\ge 0.
$$

Let $t\ge 0$, $\lambda\in B(r)$ and $f_n := f*\eta_n$ for all $n\in\mathbb{N}$. We use Itô's formula, $\nabla f_n = (\nabla f)*\eta_n$ and $\Delta f_n = (\Delta f)*\eta_n$ to estimate

$$
\begin{aligned}
I_\lambda(t)f - f = \lim_{n\to\infty}\big(I_\lambda(t)f_n - f_n\big) &\le \sup_{n\in\mathbb{N}}\left(|\lambda|\cdot\|\nabla f_n\|_\infty + \frac{1}{2}\|\Delta f_n\|_\infty - L(\lambda)\right)t \\
&\le \left(r\cdot\|\nabla f\|_\infty + \frac{1}{2}\|\Delta f\|_\infty\right)t.
\end{aligned}
$$

Taking the supremum over $\lambda\in B(r)$ yields

$$
I(t)f - f \le \left(r\cdot\|\nabla f\|_\infty + \frac{1}{2}\|\Delta f\|_\infty\right)t \quad \text{for all } t\ge 0.
$$

For the lower bound, we choose $\lambda\in\mathbb{R}^d$ with $L(\lambda)=0$ and obtain

$$
I(t)f - f \ge S_\lambda(t)f - f \ge -\left((|\lambda|\cdot\|\nabla f\|_\infty + \frac{1}{2}\|\Delta f\|_\infty\right)t \quad \text{for all } t\ge 0.
$$

This shows $f\in\mathcal{L}^I$. Applying the same argument on $-f$ yields $f\in\mathcal{L}^I_{\text{sym}}$. □

Remark 1.6.4.

(i) For functions $f \in \mathcal{L}^S_{\text{sym}}$ the Laplacian and gradient are defined in the distributional sense and $\frac{1}{2}\Delta f + H(\nabla f) \in L^\infty$. By extending the semigroup $(S(t))_{t\geq 0}$ from C_0 to an exponential Orlicz heart, one can show that the Cauchy problem

$$\partial_t u(t) = \frac{1}{2}\Delta u(t) + H(\nabla u(t)) \quad \text{for all } t \geq 0, \quad u(0) = f,$$

has a unique classical solution, which is represented by the extended semigroup. Here, the initial value $f \in \mathcal{L}^S_{\text{sym}}$ is chosen such that the generator Af is defined w.r.t. to the Orlicz norm, which is weaker than the supremum norm. For details, we refer to Chapter 2. A sublinear version of this example was studied in [62].

(ii) The explicit description of the symmetric Lipschitz set in the previous theorem relies only on elementary estimates and Banach-Alaoglu's theorem. Nonetheless, by using the results in [131], we obtain

$$\mathcal{L}^S_{\text{sym}} = \left\{ f \in \bigcap\nolimits_{p\geq 1} W^{2,p}_{\text{loc}} \cap C_0 \colon \Delta f \in L^\infty \right\},$$

i.e., the symmetric Lipschitz set equals the domain of the Laplacian in C_0.

Proof. Let $\{f \in \bigcap_{p\geq 1} W^{2,p}_{\text{loc}} \cap C_0 \colon \Delta f \in L^\infty\}$. By [131, Theorem 3.1.7], it holds $f \in W^{1,\infty}$, and therefore $f \in \mathcal{L}^S_{\text{sym}}$. Now, let $f \in \mathcal{L}^I_{\text{sym}}$ and $f_n := f * \eta_n$ for all $n \in \mathbb{N}$. Fix $p > d$. By [131, Theorem 3.1.6], there exist $c \geq 0$ and $\varepsilon > 0$ such that, for all $n \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R}^d} \|D^2 f_n\|_{L^p(B(x,\varepsilon))} \leq c\big(\|f_n\|_\infty + \|\Delta f_n\|_\infty\big) \leq c\big(\|f\|_\infty + \|\Delta f\|_\infty.$$

This implies $\sup_{n\in\mathbb{N}} \|f_n\|_{W^{2,p}(B(0,r))} < \infty$ for all $r \geq 0$ and $n \in \mathbb{N}$. Taking the limit $n \to \infty$ yields $f \in W^{2,p}(B(0,r))$ for all $r \geq 0$, i.e., $f \in W^{2,p}_{\text{loc}}$. Since $W^{2,q}_{\text{loc}} \subset W^{2,p}_{\text{loc}}$ for all $p \leq q$, we obtain $f \in \bigcap_{p\geq 1} W^{2,p}_{\text{loc}}$. The last part of the claim follows from [131, Theorem 3.1.7]. □

1.6.2 Geometric Brownian motion

In this subsection, we construct a semigroup corresponding to a geometric Brownian motion with uncertain drift and volatility. Based on the Nisio approach, this example has been studied in [136], where the authors obtain a viscosity solution for the associated Cauchy problem. In contrast to the previous example, we have to weaken the supremum norm with a suitable weight function. Let $(W_t)_{t\geq 0}$ be a one-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $p > 1$. We choose

$$\kappa \colon \mathbb{R} \to (0,\infty), \ x \mapsto (1 + |x|^p)^{-1}.$$

Let $\Lambda \subset \mathbb{R} \times \mathbb{R}_+$ be a bounded set. For every $f \in \mathrm{UC}_\kappa(\mathbb{R}^d;\mathbb{R})$, $t \geq 0$ and $x \in \mathbb{R}$, we define

$$\big(I(t)f\big)(x) := \sup_{\lambda\in\Lambda} \mathbb{E}\big[f(X^{\lambda,x}_t)\big],$$

where $X_t^{\lambda,x} := xX_t^\lambda$ and $X_t^\lambda := \exp\big((\mu - \frac{\sigma^2}{2})t + \sigma W_t\big)$ for $\lambda := (\mu, \sigma^2)$. Furthermore, for every $\lambda \in \Lambda$, let $(S_\lambda(t))_{t\geq 0}$ be the linear semigroup given by

$$\big(S_\lambda(t)f\big)(x) := \mathbb{E}\big[f(X_t^{\lambda,x})\big] \quad \text{for all } t \geq 0,\ f \in \mathrm{UC}_\kappa \text{ and } x \in \mathbb{R}.$$

We start with two auxiliary lemmas.

Lemma 1.6.5. *For every $\lambda := (\mu, \sigma^2) \in \Lambda$, $t \geq 0$, $x \in \mathbb{R}$ and $f \in \mathrm{UC}_\kappa$,*

$$\mathbb{E}\big[|X_t^\lambda|\big] = e^{\mu t}, \quad \mathbb{E}\big[|X_t^\lambda|^p\big] \leq e^{\omega t} \quad \textit{and} \quad \mathbb{E}\big[|f(X_t^{\lambda,x})|\big] \leq \|f\|_\kappa\big(1 + |x|^p\big)e^{\omega t},$$

where $\omega := \sup_{\lambda\in\Lambda} p\big(\mu + \frac{(p-1)\sigma^2}{2}\big)^+ < \infty$.

Proof. Fix $t \geq 0$ and $\lambda \in \Lambda$. We use the moment generating function of the normal distribution to compute

$$\mathbb{E}\big[|X_t^\lambda|\big] = \exp\left(\left(\mu - \tfrac{\sigma^2}{2}\right)t\right)\mathbb{E}\big[\exp(\sigma W_t)\big] = \exp\left(\left(\mu - \tfrac{\sigma^2}{2}\right)t\right)\exp\left(\tfrac{\sigma^2 t}{2}\right) = e^{\mu t},$$

and

$$\mathbb{E}\big[|X_t^\lambda|^p\big] = \exp\left(p\left(\mu - \tfrac{\sigma^2}{2}\right)t\right)\mathbb{E}\big[\exp(p\sigma W_t)\big] = \exp\left(p\left(\mu + \tfrac{(p-1)\sigma^2}{2}\right)t\right) \leq e^{\omega t}.$$

Let $f \in \mathrm{UC}_\kappa$ and $x \in \mathbb{R}^d$. It follows from $|f(X_t^{\lambda,x})|\kappa(X_t^{\lambda,x}) \leq \|f\kappa\|_\infty = \|f\|_\kappa$ that $|f(X_t^{\lambda,x})| \leq \|f\|_\kappa(1 + |X_t^{\lambda,x}|^p)$. We use the previous estimate to conclude

$$\mathbb{E}\big[|f(X_t^{\lambda,x})|\big] \leq \|f\|_\kappa\Big(1 + |x|^p\mathbb{E}\big[|X_t^\lambda|^p\big]\Big) \leq \|f\|_\kappa\big(1 + |x|^p\big)e^{\omega t}. \qquad \square$$

Lemma 1.6.5 ensures that $S_\lambda(t)f\colon \mathbb{R} \to \mathbb{R}$ and $I(t)f\colon \mathbb{R} \to \mathbb{R}$ are well-defined functions for all $\lambda \in \Lambda$, $t \geq 0$ and $f \in \mathrm{UC}_\kappa$.

Lemma 1.6.6. *For every $q \in [1, \infty)$ and $\varepsilon > 0$,*

$$\lim_{t\downarrow 0}\sup_{\lambda\in\Lambda}\mathbb{P}\big(\big|(X_t^\lambda)^q - 1\big| \geq \varepsilon\big) = 0 \quad \textit{and} \quad \lim_{t\downarrow 0}\sup_{\lambda\in\Lambda}\mathbb{E}\big[\big|(X_t^\lambda)^q - 1\big|\big] = 0.$$

Proof. Let $q \in [1, \infty)$ and $\varepsilon > 0$. Choose $\delta > 0$ such that $|e^x - 1| < \varepsilon$ for all $x \in (-\delta, \delta)$. Then, for every $t \geq 0$ and $\lambda \in \Lambda$,

$$\mathbb{P}\big(\big|(X_t^\lambda)^q - 1\big| \geq \varepsilon\big) \leq \mathbb{P}\big(q\big|\big(\mu - \tfrac{\sigma^2}{2}\big)t + \sigma W_t\big| \geq \delta\big).$$

Let $c := \sup_{\lambda\in\Lambda}\max\{q\sigma, q|\mu - \sigma^2/2|\}$. Then, for every $t \in \big[0, \delta/2c\big]$ and $\lambda \in \Lambda$,

$$q\big|\big(\mu - \tfrac{\sigma^2}{2}\big)t + \sigma W_t\big| \geq \delta \quad \text{implies} \quad |W_t| \geq \tfrac{\delta}{2c} > 0.$$

We conclude

$$\sup_{\lambda\in\Lambda}\mathbb{P}\big(\big|(X_t^\lambda)^q - 1\big| \geq \varepsilon\big) \leq \mathbb{P}\big(|W_t| \geq \tfrac{\delta}{2c}\big) \quad \text{for all } t \in \big[0, \tfrac{\delta}{2c}\big].$$

Since the right hand side of the previous inequality converges to zero as $t \downarrow 0$, we obtain the first part of the claim. Furthermore, by Lemma 1.6.5, the set

$$\big\{\big|(X_t^\lambda)^q - 1\big| \colon t \in [0, 1], \lambda \in \Lambda\big\}$$

is bounded in $L^2(\mathbb{P})$, and therefore uniformly integrable. Hence, the second part of the claim follows from the first one, similar to the Vitali convergence theorem. $\square$

Let C_c^2 be the set of all twice continuously differentiable functions $f\colon \mathbb{R}\to\mathbb{R}$ with compact support.

Theorem 1.6.7. *The family $(I(t))_{t\geq 0}$ satisfies Assumption 1.2.1 and Assumption 1.2.4, i.e., Theorem 1.2.5 yields a semigroup $(S(t))_{t\geq 0}$ on UC_κ associated to $(I(t))_{t\geq 0}$. Furthermore, it holds $C_c^2\subset D(A)$ with*

$$(Af)(x)=\sup_{\lambda\in\Lambda}\left(\frac{1}{2}\sigma^2x^2f''(x)+\mu xf'(x)\right)\quad \text{for all } f\in C_c^2 \text{ and } x\in\mathbb{R}.$$

Proof. First, we show $I(t)\colon \mathrm{Lip}_b(c)\to\mathrm{Lip}_b(e^{\omega t}c)$ for all $c,t\geq 0$. Let $c,t\geq 0$ and $f\in\mathrm{Lip}_b(c)$. Lemma 1.6.5 implies

$$|\big(S_\lambda(t)f\big)(x)-(S_\lambda(t)f)(y)|\leq\mathbb{E}\big[\big|f(xX_t^\lambda)-f(yX_t^\lambda)\big|\big]\leq c|x-y|\mathbb{E}\big[\big|X_t^\lambda\big|\big]\leq ce^{\omega t}|x-y|$$

for all $\lambda\in\Lambda$ and $x,y\in\mathbb{R}$. Furthermore, it holds $\|I(t)f\|_\infty\leq\|f\|_\infty\leq e^{\omega t}c$.

Second, we verify Assumption 1.2.1.

(i) Clearly, $I(0)=\mathrm{id}_{\mathrm{UC}_\kappa}$.

(ii) Let $f\in\mathrm{UC}_\kappa$, $t\geq 0$, $\lambda\in\Lambda$ and $x\in\mathbb{R}$. It follows from Lemma 1.6.5 that

$$|S_\lambda(t)f(x)|\kappa(x)\leq\|f\|_\kappa(1+|x|^p)e^{\omega t}(1+|x|^p)^{-1}=e^{\omega t}\|f\|_\kappa.$$

Hence, we obtain $\|S_\lambda(t)f\|_\kappa\leq e^{\omega t}\|f\|_\kappa$ and therefore

$$\|I(t)f\|_\kappa\leq\sup_{\lambda\in\Lambda}\|S_\lambda(t)f\|_\kappa\leq e^{\omega t}\|f\|_\kappa.$$

(iii) For every $f,g\in\mathrm{UC}_\kappa$ and $t\geq 0$,

$$\|I(t)f-I(t)g\|_\kappa\leq\sup_{\lambda\in\Lambda}\|S_\lambda(t)(f-g)\|_\kappa\leq e^{\omega t}\|f-g\|_\kappa.$$

In particular, we obtain $I(t)\colon \mathrm{UC}_\kappa\to\mathrm{UC}_\kappa$, because $I(t)\colon \mathrm{Lip}_b\to\mathrm{Lip}_b$, $\mathrm{Lip}_b\subset\mathrm{UC}_\kappa$ is dense and $I(t)$ is Lipschitz continuous.

Third, we verify Assumption 1.2.4. Let $f\in C_c^2$ and $r\geq 0$ with $\mathrm{supp}(f)\subset[-r,r]$. Fix $\lambda\in\Lambda$, $t\geq 0$ and $x\in\mathbb{R}$. Itô's formula implies

$$(S_\lambda(t)f)(x)-f(x)=\mu\mathbb{E}\left[\int_0^t X_s^{\lambda,x}f'(X_s^{\lambda,x})\,ds\right]+\frac{\sigma^2}{2}\mathbb{E}\left[\int_0^t(X_s^{\lambda,x})^2f''(X_s^{\lambda,x})\,ds\right].$$

Hence, we can estimate

$$\|I(t)f-f\|_\kappa\leq\|I(t)f-f\|_\infty\leq\sup_{\lambda\in\Lambda}\|S_\lambda(t)f-f\|_\infty\leq ct,$$

where $c:=\sup_{\lambda\in\Lambda}\big(\mu r\|f'\|_\infty+\frac{\sigma^2}{2}r^2\|f''\|_\infty\big)$. Since $I(t)\colon\mathrm{Lip}_b(c)\to\mathrm{Lip}_b(e^{\omega t}c)$ for all $c,t\geq 0$, it follows from Lemma 1.3.3 that the sequence $(I(\pi_n^t)f)_{n\in\mathbb{N}}$ is relatively compact in UC_κ for all $f\in\mathrm{Lip}_b$ and $t\geq 0$. Furthermore, by Lemma 1.3.4, we can choose a countable set $\mathcal{D}\subset C_c^2$, which is dense in UC_κ. By Theorem 1.2.5 there exists a semigroup $(S(t))_{t\geq 0}$ on UC_κ associated to $(I(t))_{t\geq 0}$.

Forth, for every $f \in C_c^2$, we show that

$$\lim_{t\downarrow 0} \left\| \frac{I(t)f - f}{t} - g \right\|_\kappa = 0, \tag{1.34}$$

where $g := \sup_{\lambda\in\Lambda} g_\lambda$ and $g_\lambda(x) := \left(\frac{\sigma^2}{2}x^2 f''(x) + \mu x f'(x)\right)$ for all $\lambda \in \Lambda$ and $x \in \mathbb{R}$. It holds

$$\left\| \frac{I(t)f - f}{t} - g \right\|_\kappa \leq \sup_{\lambda\in\Lambda} \left\| \frac{S_\lambda(t)f - f}{t} - g_\lambda \right\|_\kappa \quad \text{for all } t > 0$$

and Itô's formula implies

$$\begin{aligned}\left| \frac{S_\lambda(t)f - f}{t} - g_\lambda f \right|(x) \leq \frac{|\mu|}{t} \int_0^t \mathbb{E}\big[\big|X_s^{\lambda,x} f'(X_s^{\lambda,x}) - x f'(x)\big|\big]\, ds \\ + \frac{\sigma^2}{2t} \int_0^t \mathbb{E}\big[\big|(X_s^{\lambda,x})^2 f''(X_s^{\lambda,x}) - x^2 f''(x)\big|\big]\, ds\end{aligned}$$

for all $\lambda \in \Lambda$, $t > 0$ and $x \in \mathbb{R}$. To estimate the first term on the right hand side, we show

$$\lim_{t\downarrow 0} \sup_{\lambda\in\Lambda} \sup_{x\in\mathbb{R}} \frac{|\mu|}{t} \int_0^t \mathbb{E}\big[\big|X_s^{\lambda,x} f'(X_s^{\lambda,x}) - x f'(x)\big|\big]\, \mathrm{d}s = 0. \tag{1.35}$$

Let $\varepsilon > 0$, $c := 1 + \sup_{\lambda\in\Lambda} |\mu|$, and choose $r \geq 0$ with $\operatorname{supp}(f) \subset [-r, r]$. Moreover, since f' is uniformly continuous, there exists $\delta > 0$ with

$$|f'(x) - f'(y)| < \frac{\varepsilon}{6cr} \quad \text{for all } x, y \in \mathbb{R} \text{ with } |x - y| < 2\delta r.$$

Fix $\lambda \in \Lambda$ and $x \in \mathbb{R}$. We distinguish two cases:

(i) Let $|x| \leq 2r$. Then, for every $s \geq 0$,

$$\begin{aligned}&\mathbb{E}\big[\big|X_s^{\lambda,x} f'(X_s^{\lambda,x}) - x f'(x)\big|\big] = |x|\mathbb{E}\big[\big|X_s^{\lambda} f'(X_s^{\lambda,x}) - f'(x)\big|\big] \\ &\leq 2r\mathbb{E}\big[|X_s^\lambda - 1| \cdot |f'(X_s^{\lambda,x})|\big] + 2r\mathbb{E}\big[|f'(X_s^{\lambda,x}) - f'(x)| \mathbb{1}_{\{|X_s^\lambda - 1| \geq \delta\}}\big] \\ &\quad + 2r\mathbb{E}\big[|f'(X_s^{\lambda,x}) - f'(x)| \mathbb{1}_{\{|X_s^\lambda - 1| < \delta\}}\big] \\ &\leq 2r\|f'\|_\infty \mathbb{E}[|X_s^\lambda - 1|] + 4r\|f'\|_\infty \mathbb{P}\big(|X_s^\lambda - 1| \geq \delta\big) \\ &\quad + 2r\mathbb{E}\big[|f'(X_s^{\lambda,x}) - f'(x)| \mathbb{1}_{\{|X_s^\lambda - 1| < \delta\}}\big].\end{aligned}$$

By Lemma 1.6.6, there exists $s_0 > 0$, independent of λ and x, such that

$$2r\|f'\|_\infty \mathbb{E}[|X_s^\lambda - 1|] \leq \frac{\varepsilon}{3c} \quad \text{and} \quad 4r\|f'\|_\infty \mathbb{P}(|X_s^\lambda - 1| \geq \delta) \leq \frac{\varepsilon}{3c} \tag{1.36}$$

for all $s \in [0, s_0]$. On the set $\{|X_s^\lambda - 1| < \delta\}$, it holds $|X_s^{\lambda,x} - x| = |x| \cdot |X_s^\lambda - 1| < 2r\delta$ and therefore $|f'(X_s^{\lambda,x}) - f'(x)| < \frac{\varepsilon}{6cr}$. This implies

$$2r\mathbb{E}\big[|f'(X_s^{\lambda,x}) - f'(x)| \mathbb{1}_{\{|X_s^\lambda - 1| < \delta\}}\big] \leq \frac{\varepsilon}{3c}. \tag{1.37}$$

We combine inequality (1.36) and inequality (1.37) to conclude

$$\sup_{s\in[0,s_0]} \sup_{\lambda\in\Lambda} \sup_{x\in[-2r,2r]} \mathbb{E}\big[\big|X_s^{\lambda,x} f'(X_s^{\lambda,x}) - x f'(x)\big|\big] \leq \frac{\varepsilon}{c}. \tag{1.38}$$

(ii) Let $|x| > 2r$. Then, for every $s \geq 0$,

$$\mathbb{E}\big[\big|X_s^{\lambda,x} f'(X_s^{\lambda,x}) - x f'(x)\big|\big] = \mathbb{E}\big[\big|X_s^{\lambda,x} f'(X_s^{\lambda,x})\big| \mathbb{1}_{\{|X_s^{\lambda,x}| \leq r\}}\big] \\ \leq r \|f'\|_\infty \mathbb{P}\big(|X_s^{\lambda,x}| \leq r\big),$$

because $\operatorname{supp}(f) \subset [-r, r]$. Furthermore, we use $|x| > 2r$ to estimate

$$\mathbb{P}(|X_s^{\lambda,x}| \leq r) = \mathbb{P}(|x| \cdot |X_s^\lambda| \leq r) \leq \mathbb{P}\left(|X^\lambda| < \tfrac{1}{2}\right).$$

By Lemma 1.6.6, there exists $s_1 \in (0, s_0]$, independent of λ and x, such that

$$r \|f'\|_\infty \mathbb{P}\left(|X_s^\lambda| < \tfrac{1}{2}\right) \leq \frac{\varepsilon}{c} \quad \text{for all } s \in [0, s_1].$$

We obtain

$$\sup_{s \in [0,s_1]} \sup_{\lambda \in \Lambda} \sup_{x \in [-2r,2r]^c} \mathbb{E}\big[\big|X_s^{\lambda,x} f'(X_s^{\lambda,x}) - x f'(x)\big|\big] \leq \frac{\varepsilon}{c}. \tag{1.39}$$

Combining inequality (1.38) and inequality (1.39) with the definition of c yields

$$\sup_{\lambda \in \Lambda} \sup_{x \in \mathbb{R}} \frac{|\mu|}{t} \int_0^t \mathbb{E}\big[\big|X_s^{\lambda,x} f'(X_s^{\lambda,x}) - x f'(x)\big|\big] \, \mathrm{d}s \leq \varepsilon \quad \text{for all } t \in [0, s_1].$$

By similar arguments, it follows from Lemma 1.6.6 with $q = 2$ that

$$\lim_{t \downarrow 0} \sup_{\lambda \in \Lambda} \sup_{x \in \mathbb{R}} \frac{\sigma^2}{2t} \int_0^t \mathbb{E}\big[\big|(X_s^{\lambda,x})^2 f''(X_s^{\lambda,x}) - x^2 f''(x)\big|\big] \, ds = 0. \tag{1.40}$$

As seen before, equation (1.34) is a consequence of inequality (1.35) and inequality (1.40).

Fifth, it follows from Theorem 1.2.5 that Assumption 1.4.1 is satisfied. In addition, by Lemma 1.4.3, condition (1.14) holds for all $f, g \in \mathrm{UC}_\kappa$. Hence, Theorem 1.4.2 implies $C_c^2 \subset D(A)$ with

$$(Af)(x) = \sup_{\lambda \in \Lambda} \left(\frac{1}{2}\sigma^2 x^2 f''(x) + \mu x f'(x)\right) \quad \text{for all } f \in C_c^2 \text{ and } x \in \mathbb{R}. \qquad \square$$

1.6.3 G-expectation

In this subsection, we construct a semigroup corresponding to a Brownian motion with uncertain drift and volatility, which is the so-called G-Brownian motion, see [142, 143]. Let $(W_t)_{t \geq 0}$ be a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We choose the weight function

$$\kappa \colon \mathbb{R}^d \to (0, \infty), \ x \mapsto (1 + |x|^2)^{-1}.$$

Let $\Lambda \subset \mathbb{R}^d \times \mathbb{S}_+^d$ be a bounded set, where $\mathbb{S}_+^d$ denotes the set of all symmetric positive semi-definite $d \times d$-matrices. For every $t \geq 0$, $f \in \mathrm{UC}_\kappa(\mathbb{R}^d; \mathbb{R})$ and $x \in \mathbb{R}^d$, we define

$$\big(I(t)f\big)(x) := \sup_{\lambda \in \Lambda} \mathbb{E}[f(x + \sigma W_t + \mu t)],$$

where $\lambda := (\mu, \sigma^2)$ and $\sigma := \sqrt{\sigma^2}$ denotes the matrix square root of σ^2. Moreover, for every $\lambda \in \Lambda$, let $(S_\lambda(t))_{t \geq 0}$ be the linear semigroup defined by

$$\big(S_\lambda(t)f\big)(x) := \mathbb{E}[f(x + \sigma W_t + \mu t)] \quad \text{for all } t \geq 0, \ f \in \mathrm{UC}_\kappa \text{ and } x \in \mathbb{R}^d.$$

Lemma 1.6.8. *For every $t \geq 0$, $\lambda \in \Lambda$, $f \in \mathrm{UC}_\kappa$, and $x \in \mathbb{R}^d$,*

$$\mathbb{E}[1 + |x + \sigma W_t + \mu t|^2]\kappa(x) \leq e^{\omega t} \quad \text{and} \quad \mathbb{E}[|f(x + \sigma W_t + \mu t)|] \leq \|f\|_\kappa (1 + |x|^2) e^{\omega t},$$

where $\omega := \sup_{\lambda \in \Lambda} \max\{1 + |\mu|^2 + |\sigma^2|, \sqrt{2}|\mu|\} < \infty$.

Proof. We use Young's inequality to estimate

$$\begin{aligned}
&\mathbb{E}[1 + |x + \sigma W_t + \mu t|^2]\kappa(x) \\
&= \mathbb{E}\left[1 + |x|^2 + 2\langle x, \sigma W_t\rangle + 2\langle x, \mu t\rangle + |\sigma W_t|^2 + 2\langle \sigma W_t, \mu t\rangle + |\mu|^2 t^2\right]\kappa(x) \\
&\leq (1 + |x|^2 + 2|x| \cdot |\mu| t + |\sigma|^2 t + |\mu|^2 t^2)\kappa(x) \\
&\leq (1 + |x|^2 + |x|^2 t + |\mu|^2 t + |\sigma|^2 t + |\mu|^2 t^2)\kappa(x) \\
&\leq (1 + (1 + |\mu|^2 + |\sigma|^2)t + |\mu|^2 t^2) \leq e^{\omega t}.
\end{aligned}$$

Here, the last inequality holds due to $1 + at + bt^2 \leq e^{\max\{a, \sqrt{2b}\}t}$ for all $a, b, t \geq 0$. The second part of the claim follows from the first one, see the proof of Lemma 1.6.5. ☐

Lemma 1.6.8 ensures that $S_\lambda(t)f \colon \mathbb{R}^d \to \mathbb{R}$ and $I(t) \colon \mathbb{R}^d \to \mathbb{R}$ are well-defined functions for all $\lambda \in \Lambda$, $t \geq 0$ and $f \in \mathrm{UC}_\kappa$. We denote by $\mathrm{tr}(a)$ the trace of a matrix $a \in \mathbb{R}^{d \times d}$. Recall that $\nabla_\mu f := \langle \mu, \nabla f\rangle$.

Theorem 1.6.9. *The family $(I(t))_{t\geq 0}$ satisfies Assumption 1.2.1 and Assumption 1.2.4, i.e., Theorem 1.2.5 yields a semigroup $(S(t))_{t\geq 0}$ on UC_κ associated to $(I(t))_{t\geq 0}$. Furthermore, it holds $C_c^2 \subset D(A)$ with*

$$Af = \sup_{\lambda \in \Lambda}\left(\frac{1}{2}\mathrm{tr}(\sigma^2 D^2 f) + \nabla_\mu f\right) \quad \text{for all } f \in C_c^2.$$

Proof. First, we show that $(I(t))_{t\geq 0}$ satisfies Assumption 1.2.1 and Assumption 1.2.4. Since $I(t)$ is translation-invariant and a contraction w.r.t. the supremum norm, we conclude $I(t) \colon \mathrm{Lip}_b(c) \to \mathrm{Lip}_b(c)$ for all $c, t \geq 0$. Furthermore, it follows from Lemma 1.6.8 that $\|I(t)f\|_\kappa \leq e^{\omega t}\|f\|_\kappa$ and $\|I(t)f - I(t)g\|_\kappa \leq e^{\omega t}\|f - g\|_\kappa$ for all $t \geq 0$ and $f, g \in \mathrm{UC}_\kappa$. In particular, we have $I(t) \colon \mathrm{UC}_\kappa \to \mathrm{UC}_\kappa$ for all $t \geq 0$. In addition, for every $f \in C_c^2$, Itô's formula implies

$$\|I(t)f - f\|_\kappa \leq ct \quad \text{for all } t \geq 0,$$

where $c := \sup_{in\Lambda}\left(|\mu| \cdot \|\nabla f\|_\infty + \frac{1}{2}|\sigma|^2 \|D^2 f\|_\infty\right) < \infty$.

Second, we show that

$$\lim_{t\downarrow 0}\left\|\frac{I(t)f - f}{t} - \sup_{\lambda \in \Lambda}\left(\frac{1}{2}\mathrm{tr}(\sigma^2 D^2 f) + \nabla_\mu f\right)\right\|_\infty = 0 \quad \text{for all } f \in C_c^2.$$

Let $f \in C_c^2$. We use Itô's formula to estimate

$$\begin{aligned}
&\left\|\frac{I(t)f - f}{t} - \sup_{\lambda \in \Lambda}\left(\frac{1}{2}\mathrm{tr}(\sigma^2 D^2 f) + \nabla_\mu f\right)\right\|_\infty \\
&\leq \sup_{\lambda \in \Lambda}\left\|\frac{S_\lambda(t)f - f}{t} - \frac{\mathrm{tr}(\sigma^2 D^2 f}{2}) - \nabla_\mu f\right\|_\infty
\end{aligned}$$

$$\leq \sup_{\lambda\in\Lambda} \frac{1}{t}\int_0^t \Big(\|\nabla_\mu f(\cdot + X_s^\lambda) - \nabla_\mu f\|_\infty + \frac{1}{2}\|\operatorname{tr}(\sigma^2 D^2 f)(\cdot + X_s^\lambda) - \operatorname{tr}(\sigma^2 D^2 f)\|_\infty\Big)\mathrm{d}s,$$

where $X_s^\lambda := \sigma W_t + \mu s$ for all $\lambda \in \mathbb{R}^d$ and $s \geq 0$. Since Λ is bounded and $f \in C_c^2$, the right hand side converges to zero as $t \downarrow 0$. It follows from $\|\cdot\|_\kappa \leq \|\cdot\|_\infty$, Theorem 1.4.2 and Lemma 1.4.3 that $C_c^2 \subset D(A)$ with

$$Af = \sup_{\lambda\in\Lambda}\left(\frac{1}{2}\operatorname{tr}(\sigma^2 D^2 f) + \nabla_\mu f\right) \quad \text{for all } f \in C_c^2. \qquad \square$$

1.6.4 Ordinary differential equations

In this subsection, we obtain the well-known existence and uniqueness result for ODEs with locally Lipschitz continuous data. Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be a function, which satisfies the following conditions:

- There exists $K \geq 0$ with $|f(x)| \leq K(1 + |x|)$ for all $x \in \mathbb{R}^d$.
- For every $r \geq 0$ there exists $L_r \geq 0$ with
$$|f(x) - f(y)| \leq L_r|x - y| \quad \text{for all } x, y \in B(0, r).$$

We define $I(t)x := x + tf(x)$ for all $t \geq 0$ and $x \in \mathbb{R}^d$.

Theorem 1.6.10. *The family $(I(t))_{t\geq 0}$ satisfies Assumption 1.2.1 and Assumption 1.2.4, i.e., Theorem 1.2.5 yields a semigroup $(S(t))_{t\geq 0}$ on $\mathbb{R}^d$ associated to $(I(t))_{t\geq 0}$. Furthermore, for every $x \in \mathbb{R}^d$ the Cauchy problem*

$$y'(t) = f(y(t)) \quad \textit{for all } t \geq 0, \quad y(0) = x,$$

has a unique classical solution given by $y(t) := S(t)x$.

Proof. First, we verify Assumption 1.2.1 and Assumption 1.2.4. It holds $|I(t)x| \leq \alpha(r, t)$ for all $r, t \geq 0$ and $x \in \mathbb{R}^d$, where

$$\alpha(r, t) := \begin{cases} e^{2Kt}, & r \leq 1, \\ e^{2Kt}r, & r > 1. \end{cases}$$

We have $|I(t)x - I(t)y| \leq e^{L_r t}|x - y|$ for all $r, t \geq 0$ and $x, y \in B(r)$. In addition, Lemma 1.2.7 implies that the sequence $(I(\pi_n^t)x)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ is bounded and therefore relatively compact for all $(x, t) \in \mathbb{R}^d \times \mathcal{T}$. Theorem 1.2.5 yields a semigroup $(S(t))_{t\geq 0}$ on $\mathbb{R}^d$ associated to $(I(t))_{t\geq 0}$.

Second, we note that

$$\frac{I(t)x - x}{t} - f(x) = 0 \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d.$$

Furthermore, one can show that condition (1.14) holds for all $x, y \in \mathbb{R}^d$, for details we refer to the proofs of Lemma 1.6.13 and Theorem 1.6.14 below. Hence, it follows from Theorem 1.4.2 that $D(A) = \mathbb{R}^d$ and $Ax = f(x)$ for all $x \in \mathbb{R}^d$. Let $x \in \mathbb{R}^d$ and define $y(t) := S(t)x$ for all $t \geq 0$. For every $t \geq 0$, the right-derivative of y exists and is given by $Ay(t) = f(y(t))$. Since the functions y and $f(y(\cdot))$ are continuous, it follows from [141, Corollary 1.2 in Section 2] that y is continuously differentiable. The uniqueness follows from Theorem 1.4.6. $\square$

1.6.5 Lipschitz perturbations

Throughout this subsection, we consider vector valued functions $f\colon \mathbb{R}^d \to \mathbb{R}^m$. We construct a semigroup corresponding to a perturbed linear semigroup. The nonlinear generator is the sum of the linear generator and a nonlinear zero-order coupling. The ODEs from the previous section are including in this setting, as well as reaction-diffusion equations, see Example 1.6.15. We endow $\mathbb{R}^m$ with the order $x \leq y$ if and only if $x_i \leq y_i$ for $i = 1, \dots, m$.

Assumption 1.6.11. Let $(S_0(t))_{t\geq 0}$ be a strongly continuous monotone linear semigroup on $C_0(\mathbb{R}^d;\mathbb{R}^m)$ which satisfies the following condition:

(i) There exists $\omega \geq 0$ with $\|S_0(t)f\|_\infty \leq e^{\omega t}\|f\|_\infty$ for all $t \geq 0$ and $f \in C_0$.

(ii) $\mathcal{L}^{S_0} \cap \mathrm{Lip}_0 \subset C_0$ is dense.

(iii) $S_0(t)\colon \mathrm{Lip}_0(c) \to \mathrm{Lip}_0(e^{\omega t}c)$ for all $c, t \geq 0$.

(iv) $\lim_{|x|\to\infty} \sup_{t\in[0,T]} |S_0(t)f)(x)| = 0$ for all $T \geq 0$ and $f \in C_0$.

Furthermore, let $\Psi\colon \mathbb{R}^m \to \mathbb{R}^m$ be a continuous function with $\Psi(0) = 0$ which satisfies the following conditions:

(v) There exists $K \geq 0$ with $|\Psi(x)| \leq K(1 + |x|)$ for all $x \in \mathbb{R}^m$.

(vi) For every $r \geq 0$ there exists $L_r \geq 0$ with

$$|\Psi(x) - \Psi(y)| \leq L_r|x - y| \quad \text{for all } x, y \in B(0, r).$$

W.l.o.g., we assume that the mapping $r \mapsto L_r$ is non-decreasing. We define

$$\big(I(t)f\big)(x) := \big(S_0(t)f\big)(x) + t\Psi(f(x)) \quad \text{for all } t \geq 0,\ f \in C_0 \text{ and } x \in \mathbb{R}^d.$$

Theorem 1.6.12. *The family $(I(t))_{t\geq 0}$ satisfies Assumption 1.2.1 and Assumption 1.2.4, i.e., Theorem 1.2.5 yields a semigroup $(S(t))_{t\geq 0}$ on C_0 associated to $(I(t))_{t\geq 0}$.*

Proof. First, we verify Assumption 1.2.1. It follows from $S_0(t)\colon C_0 \to C_0$ and the fact that Ψ is a continuous function with $\Psi(0) = 0$ that $I(t)\colon C_0 \to C_0$ for all $t \geq 0$. Clearly, it holds $I(0) = \mathrm{id}_{C_0}$. Let $t \geq 0$ and $f \in C_0$. If $\|f\|_\infty \leq 1$, we use Assumption 1.6.11(i) and (v) to estimate

$$\|I(t)f\|_\infty \leq e^{\omega t}\|f\|_\infty + tK(1 + \|f\|_\infty) \leq e^{\omega t} + 2Kt \leq e^{(\omega+2K)t}.$$

Similarly, if $\|f\|_\infty > 1$, we obtain the estimate $\|I(t)f\|_\infty \leq e^{(\omega+2K)t}\|f\|_\infty$. Hence, Assumption 1.2.1(i) is satisfied with

$$\alpha(r,t) := \begin{cases} e^{(\omega+2K)t}, & r \leq 1, \\ e^{(\omega+2K)t}r, & r > 1, \end{cases} \quad \text{for all } r, t \geq 0.$$

For every $r, t \geq 0$ and $f, g \in B(R)$, Assumption 1.6.11(i) and (vi) imply

$$\|I(t)f - I(t)g\|_\infty \leq e^{\omega t}\|f - g\|_\infty + tL_r\|f - g\|_\infty \leq e^{(\omega+L_r)t}\|f - g\|_\infty.$$

Second, we show that $\mathcal{L}^I = \mathcal{L}^{S_0}$. For every $t \geq 0$ and $f \in C_0$, it follows from $\Psi(0) = 0$ and Assumption 1.6.11(vi) that

$$\|I(t)f - S_0(t)f\|_\infty = t\|\psi(f)\|_\infty \leq tL_r, \quad \text{where} \quad r := \|f\|_\infty.$$

In particular, Assumption 1.6.11(ii) implies that $\mathcal{L}^I \cap \mathrm{Lip}_0 \subset C_0$ is dense. Hence, in order to verify Assumption 1.2.4, it suffices to show that the assumptions from Lemma 1.3.1 are satisfied for all $f \in \mathrm{Lip}_0$.

Third, we show that

$$I(t)\colon \mathrm{Lip}_0(c) \cap B(r) \to \mathrm{Lip}_0\left(e^{(\omega + L_r)t}c\right) \quad \text{for all } c, r, t \geq 0.$$

For every $c, r, t \geq 0$, $f \in \mathrm{Lip}_0(c) \cap B(r)$ and $x, y \in \mathbb{R}^d$, it follows from Assumption 1.6.11(iii) and (v) that

$$\begin{aligned}|(I(t)f)(x) - (I(t)f)(y)| &\leq |(S_0(t)f)(x) - (S_0(t)f)(y)| + t|\Psi(f(x)) - \Psi(f(y))| \\ &\leq \left(e^{\omega t} + L_r t\right)c|x - y| \leq e^{(\omega + L_r)t}c|x - y|.\end{aligned}$$

Let $c, t \geq 0$ and $f \in \mathrm{Lip}_0(c)$. By induction, it follows that $I(\pi_n^t)f \in \mathrm{Lip}_0(e^{(\omega+L_r)t}c)$ for all $n \in \mathbb{N}$, where $r := \alpha(c, t)$. In particular, the sequence $I(\pi_n^t)_{n\in\mathbb{N}}$ is equicontinuous.

Fourth, it remains to show that

$$\lim_{|x|\to\infty} \sup_{n\in\mathbb{N}} |(I(\pi_n^t)f)(x)| = 0 \quad \text{for all } f \in \mathrm{Lip}_0 \text{ and } t \geq 0.$$

To do so, for every $r, t \geq 0$, we define

$$J_r(t)\colon C_0 \to C_0,\ f \mapsto S_0(t)f + tL_r f.$$

Since $S_0(t)$ is monotone, it holds $J_r(t)f \leq J_r(t)g$ for all $r, t \geq 0$ and $f, g \in C_0$ with $0 \leq f \leq g$. Moreover, the mapping $r \mapsto J_r(t)f$ is non-decreasing for all $f \in C_0$ with $f \geq 0$, because we assumed the mapping $r \mapsto L_r$ to be non-decreasing. We show by induction that

$$\left|I(t)^k f\right| \leq \left(J_{\alpha(r,kt)}(t)\right)^k |f| \quad \text{for all } k \in \mathbb{N},\ r, t \geq 0 \text{ and } f \in B(r). \tag{1.41}$$

Let $r, t \geq 0$ and $f \in B(r)$. We use the monotonicity of $S_0(t)$, $\Psi(0) = 0$ and Assumption 1.6.11(vi) to estimate

$$|I(t)f| \leq |S_0(t)f| + t|\Psi(f)| \leq S_0(t)|f| + L_r t|f| = J_r(t)|f| \leq J_{\alpha(r,t)}(t)|f|.$$

For the induction step, we assume that inequality (1.41) holds for some fixed $k \in \mathbb{N}$. Let $r, t \geq 0$ and $f \in B(r)$. It follows from $I(t)f \in B(\alpha(r, t))$, inequality (1.41), inequality (1.3) and the monotonicity of J that

$$\begin{aligned}\left|I(t)^{k+1}f\right| &= \left|I(t)^k I(t)f\right| \leq \left(J_{\alpha(\alpha(r,t),kt)}(t)\right)^k |I(t)f| \\ &\leq \left(J_{\alpha(r,(k+1)t)}(t)\right)^k J_r(t)|f| \leq \left(J_{\alpha(r,(k+1)t)}(t)\right)^{k+1} |f|.\end{aligned}$$

Fifth, since $(S_0(t))_{t\geq 0}$ is a linear semigroup, the binomial theorem implies

$$J_r(t)^k = \sum_{l=0}^{k} \binom{k}{l} (L_r t)^l S_0((k-l)t) \quad \text{for all } r, t \geq 0 \text{ and } k \in \mathbb{N}. \tag{1.42}$$

Let $f \in C_0$, $t \geq 0$ and $\varepsilon > 0$. Inequality (1.41) yields

$$|I(\pi_n^t)f| \leq J_r(h_n)^{k_n^t}|f| \quad \text{for all } n \in \mathbb{N}, \tag{1.43}$$

where $r := \alpha(\|f\|_\infty, t)$. In addition, by Assumption 1.6.11(iv), there exists $R \geq 0$ with

$$\big|\big(S_0(s)|f|\big)(x)\big| \leq e^{-L_r t}\varepsilon \quad \text{for all } s \in [0,t] \text{ and } x \in B(R)^c. \tag{1.44}$$

Let $x \in B(R)^c$, $n \in \mathbb{N}$ and $k := k_n^t$. We use equation (1.42)-(1.44) and the binomial theorem to estimate

$$\begin{aligned}|I(\pi_n^t)f|(x) &\leq \big(J_r(h_n)^k|f|\big)(x) = \sum_{l=0}^{k}\binom{k}{l}(L_r h_n)^l\big(S_0\big((k-l)h_n\big)|f|\big)(x) \\ &\leq e^{-L_r t}\varepsilon\sum_{l=0}^{k}\binom{k}{l}(L_r h_n)^l = e^{-L_r t}\varepsilon(1+L_r h_n)^k \leq e^{-L_r t}\varepsilon e^{L_r k h_n} = \varepsilon.\end{aligned}$$

We obtain $\lim_{|x|\to\infty}\sup_{n\in\mathbb{N}}|(I(\pi_n^t)f)(x)| = 0$ for all $(f,t) \in C_0 \times \mathbb{R}_+$. □

To determine the generator of $(S(t))_{t\geq 0}$, we need the following recursion.

Lemma 1.6.13. *For every $f, g \in C_0$ and $k, n \in \mathbb{N}$,*

$$\begin{aligned}&I(h_n)^k f - I(h_n)^k g \\ &= S_0(kh_n)(f-g) + h_n\sum_{l=0}^{k-1} S_0\big((k-1-l)h_n\big)\left(\Psi\big(I(h_n)^l f\big) - \Psi\big(I(h_n)^l g\big)\right).\end{aligned}$$

Proof. Let $f, g \in C_0$ and $n \in \mathbb{N}$. We prove the claim by induction w.r.t. $k \in \mathbb{N}$. For $k = 1$, linearity of $S_0(h_n)$ implies

$$I(h_n)f - I(h_n)g = S_0(h_n)(f-g) + h_n\big(\Psi(f) - \Psi(g)\big).$$

For the induction step, we assume that the claim holds for some fixed $k \in \mathbb{N}$. Since $(S_0(t))_{t\geq 0}$ is a linear semigroup, we obtain

$$\begin{aligned}&I(h_n)^{k+1}f - I(h_n)^{k+1}g = I(h_n)^k I(h_n)f - I(h_n)^k I(h_n)g \\ &= S_0(kh_n)\big(I(h_n)f - I(h_n)g\big) \\ &\quad + h_n\sum_{l=0}^{k-1} S_0((k-1-l)h_n)\left(\Psi\big(I(h_n)^l I(h_n)f\big) - \Psi\big(I(h_n)^l I(h_n)g\big)\right) \\ &= S_0(kh_n)S_0(h_n)(f-g) + h_n S_0(kh_n)\big(\Psi(f) - \Psi(g)\big) \\ &\quad + h_n\sum_{l=0}^{k-1} S_0((k-(l+1)h_n)\left(\Psi\big(I(h_n)^{l+1}f\big) - \Psi\big(I(h_n)^{l+1}g\big)\right) \\ &= S_0((k+1)h_n)(f-g) + h_n S_0(kh_n)\left(\Psi(f) - \Psi(g)\right) \\ &\quad + h_n\sum_{l=1}^{k} S_0((k-l)h_n)\left(\Psi\big(I(h_n)^l f\big) - \Psi\big(I(h_n)^l g\big)\right) \\ &= S_0((k+1)h_n)(f-g) + h_n\sum_{l=0}^{k} S_0\big((k-l)h_n\big)\left(\Psi\big(I(h_n)^l f\big) - \Psi\big(I(h_n)^l g\big)\right).\end{aligned}$$

□

Theorem 1.6.14. *It holds $D(A_0) \subset D(A)$ with $Af = A_0 f + \psi(f)$ for all $f \in D(A_0)$, where A_0 denotes the generator of $(S_0(t))_{t\geq 0}$.*

Proof. For every $f \in D(A_0)$,

$$\frac{I(t)f - f}{t} - A_0 - \Psi(f) = \frac{S_0(t)f - f}{t} - A_0 \to 0 \quad \text{as } t \downarrow 0.$$

It remains to verify inequality (1.14). Let $f, g \in C_0$, $\varepsilon > 0$ $r := \max\{\|f\|_\infty, \|g\|_\infty\}$ and $c := \alpha(2r, t)$. Since $(S_0(t))_{t\geq 0}$ is strongly continuous, there exists $t_0 \in (0,1]$ with

$$\|S_0(t)g - g\|_\infty \leq \frac{\varepsilon}{2} \quad \text{and} \quad e^{(2\omega + L_c)t} L_c t \|g\|_\infty \leq \frac{\varepsilon}{2} \quad \text{for all } t \in [0, t_0].$$

Let $k, n \in \mathbb{N}$ with $kh_n \leq t_0$. We use Lemma 1.6.13, Assumption 1.6.11(i) and (vi), and Lemma 1.2.7 to estimate

$$\begin{aligned}
&\left\| \frac{I(h_n)^k(f + h_n g) - I(h_n)^k f}{h_n} - g \right\|_\infty \\
&\leq \|S_0(kh_n)g - g\|_\infty \\
&\quad + \sum_{l=0}^{k-1} \left\| S_0((k-1-l)h_n) \left(\Psi\big(I(h_n)^l (f + h_n g)\big) - \Psi\big(I(h_n)^l f\big)\right) \right\|_\infty \\
&\leq \frac{\varepsilon}{2} + \sum_{l=0}^{k-1} e^{\omega(k-1-l)h_n} L_c e^{(\omega + L_c) l h_n} \|h_n g\|_\infty \\
&\leq \frac{\varepsilon}{2} + kh_n e^{\omega(k-1)h_n} L_c e^{(\omega + L_c)kh_n} \|g\|_\infty \leq \varepsilon.
\end{aligned}$$

□

Example 1.6.15. Let $(W_t)_{t\geq 0}$ be a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define

$$(S_0(t)f)(x) := \mathbb{E}[f(x + W_t)] \quad \text{for all } t \geq 0, f \in C_0 \text{ and } x \in \mathbb{R}^d.$$

Then, the family $(S_0(t))_{t\geq 0}$ satisfies Assumption 1.6.11, and

$$\lim_{t \downarrow 0} \left\| \frac{S_0(t)f - f}{t} - \frac{1}{2}\Delta f \right\|_\infty = 0 \quad \text{for all } f \in C_0^2,$$

where the Laplacian is defined component-wise. Furthermore, let Ψ be a function which satisfies Assumption 1.6.11 and $(S(t))_{t\geq 0}$ an associated semigroup as in Theorem 1.6.12. Then, by Theorem 1.6.14, it holds $C_0^2 \subset D(A)$ with $Af = \frac{1}{2}\Delta f + \Psi(f)$ for all $f \in C_0^2$. Equations of the form $\partial_t f = \frac{1}{2}\Delta f + \Psi(f)$ are known as reaction diffusion systems, see, e.g. [131, Subsection 7.3].

Chapter 2

Extension to exponential Orlicz hearts

2.1 Introduction

In this chapter, we study the viscous Hamilton–Jacobi equations of the form

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + H(\nabla u(t,x)), & (t,x) \in (0,\infty)\times\mathbb{R}^d, \\ u(0,x) = f(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.1}$$

where $H\colon \mathbb{R}^d \to \mathbb{R}$ is a convex function with at most quadratic growth at infinity. In Section 1.6.1, we already constructed an associated semigroup on the space C_0 containing all continuous functions vanishing at infinity. However, in order to provide uniqueness, we have to extend the semigroup to a space with order continuous norm, see Remark 1.4.5. Recall that, in [62], Denk et al. have shown that convex semigroups on L^p-like spaces have properties which are familiar from the well-established theory of linear semigroups. In particular, the generator of a convex semigroup is a closed operator, its domain is invariant under the semigroup and the associated abstract Cauchy problem is classically well-posed. Since the differential operator on the right-hand side of equation (2.1) is convex, the aim of this chapter is to extend the the associated semigroup from Subsection 1.6.1 to a suitable Orlicz heart such that we can apply the results of [62]. If H is sublinear, the semigroup can be extended to L^p for all $p \in [1,\infty)$, see [62, Example 5.3]. As soon as H grows superlinear, the approach presented in [62] fails in L^p-spaces but it turns out that exponential Orlicz hearts are suitable. This choice is motivated by the fact that, in the particular case $H(x) := |x|^2/2$, an explicit solution of equation (2.1) is given by the formula

$$u(t,x) := \log\left((2\pi t)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\exp(f(x+y))e^{-\frac{|y|^2}{2t}}\,\mathrm{d}y\right). \tag{2.2}$$

For $H(x) = a|x|^p$ with $a > 0$, $p \geq 2$ and $f \in L^1_{\mathrm{loc}}$, it is shown in [19, Proposition 3.1] that if equation (2.1) has a classical solution $u \in C^{1,2}((0,\infty)\times\mathbb{R}^d;\mathbb{R})$ with $\lim_{t\downarrow 0} u(t) = f$ in L^1_{loc}, then $\exp(af) \in L^1_{\mathrm{loc}}$. Similar integrability assumptions on the initial data are used in [54, 76].

In Subsection 1.6.1, we constructed a semigroup $(S(t))_{t\geq 0}$ on C_0 associated to equation (2.1) as the limit

$$\lim_{l\to\infty} \|S(t)f - I(2^{-n_l}t)^{2^{n_l}}f\|_\infty = 0 \tag{2.3}$$

for a subsequence $(n_l)_{l\in\mathbb{N}} \subset \mathbb{N}$, where the one-step operators $(I(t))_{t\geq 0}$ are given by

$$\big(I(t)f\big)(x) := \sup_{\lambda\in\mathbb{R}^d} \left((2\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x+y+\lambda t) e^{-\frac{|y|^2}{2t}} \,\mathrm{d}y - L(\lambda)t \right)$$

for all $t \geq 0$, $f \in C_0$ and $x \in \mathbb{R}^d$. Here, we denote by L the convex conjugate of H. In order to extend the semigroup $(S(t))_{t\geq 0}$ from C_0 to a suitable exponential Orlicz heart M^Φ, the key idea is to find a dominating family $(T(t))_{t\geq 0}$ of bounded operators $T(t)\colon M^\Phi \to M^\Phi$ satisfying

$$|I(t)f| \leq T(t)|f| \quad \text{for all } t \geq 0 \text{ and } f \in C_0 \cap M^\Phi.$$

In Section 2.3 and Section 2.4, the operators $T(t)$ will be defined by an explicit formula similar to equation (2.2), where the exponential is replaced by a suitable Young function. From equation (2.3), we obtain

$$|S(t)f| \leq T(t)|f| \quad \text{for all } t \geq 0 \text{ and } f \in C_0 \cap M^\Phi.$$

In particular, the boundedness of $T(t)$ implies that $S(t)$ is a bounded convex operator and therefore locally Lipschitz continuous. Consequently, there exists a unique continuous extension $S(t)\colon M^\Phi \to M^\Phi$, see Theorem 2.3.6 and Theorem 2.4.5. To link the extended semigroup $(S(t))_{t\geq 0}$ with the viscous Hamilton–Jacobi equation, its generator A has to be identified with the differential operator on the right-hand side of equation (2.1). Determining the generator on the whole domain $D(A)$ seems to be rather difficult. Hence, we focus on the symmetric Lipschitz set $\mathcal{L}^S_{\mathrm{sym}}$ which is invariant under the semigroup and can be determined explicitly as the domain of the Laplacian in L^∞ restricted to $C_0 \cap M^\Phi$, see Theorem 1.6.2. For every $D(A) \cap \mathcal{L}^S_{\mathrm{sym}}$, we obtain

$$Af = \frac{1}{2}\Delta f + H(\nabla f)$$

where Δf and ∇f exist as regular distributions. This allows us to solve the Cauchy problem (2.1) and leads to estimates for the solution, see Theorem 2.4.8.

The approach in this chapter differs from the established PDE-theory and can be placed in context as follows. For $H(x) = |x|^q$ with $q > 0$ and bounded continuous initial data, existence and uniqueness of classical solutions in Hölder spaces have been established in [90]. In [19], the focus lies on the study of mild solutions for $H(x) = a|x|^q$ in L^p-spaces which are also classical solutions due to the theory of parabolic equations. The choice of the initial data depends on whether $q < 2$ or $q \geq 2$ and $a < 0$ or $a > 0$. Note that existence and uniqueness can both fail if $a > 0$, $q \geq 2$ or p is too small. As mentioned before, in [76], the authors assume that the initial value is exponentially integrable. More precisely, the existence of weak solutions in L^p-sapces on bounded domains $\Omega \subset \mathbb{R}^d$ is shown as well as the estimate

$$\int_\Omega e^{c|f(x)|} \,\mathrm{d}x < \infty \quad \Longrightarrow \quad \sup_{t\in[0,T]} \int_\Omega e^{c|u(t,x)|} \,\mathrm{d}x < \infty$$

for a suitable constant $c \geq 0$. We remark that the previous estimate is similar to our statement $S(t)\colon M^\Phi \to M^\Phi$. Equations with degenerate coercivity and quadratic gradient terms have been studied in [54]. For non-convex Hamiltonians H, existence and uniqueness of viscosity solutions have been investigated in [55]. In the latter article, it is also shown how one can obtain smooth solutions for very regular initial data from the classical theory presented in [128]. For long time behaviour and convergence to stationary solutions, we refer to [89, 156]. Concerning the regularity of solutions, we want to mention both Hölder and Lipschitz estimates for viscosity and weak solutions [3, 39, 45] as well as maximal L^p-regularity for functions defined on the torus [46]. From a stochastic point of view, the study of quadratic backward differential equations leads to PDEs with quadratic gradient terms, see [32, 48, 57, 110]. In particular, the solution of the PDE has a stochastic representation similar to equation (2.2) and the fundamental solution of the linear heat equation.

2.2 Preliminaries on Orlicz hearts

Let L^0 be the space of all Borel measurable functions $f\colon \mathbb{R}^d \to \mathbb{R}$, where two of them are identified if they coincide almost everywhere. We write $f \leq g$ if and only if $f(x) \leq g(x)$ for almost every $x \in \mathbb{R}^d$. Moreover, denoting by λ the Lebesgue measure, we define

$$\int_{\mathbb{R}^d} f \,\mathrm{d}\lambda := \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x.$$

Subsequently, we follow Rao and Ren [149]. Let $\Phi\colon \mathbb{R} \to \mathbb{R}_+$ be a Young function, i.e., Φ is convex, symmetric, $\Phi(0) = 0$ and $\lim_{|x|\to\infty} \Phi(x) = \infty$. In addition, we assume that $\Phi(x) > 0$ for all $x \neq 0$. The corresponding Orlicz heart is defined by

$$M^\Phi := \left\{ f \in L^0 \colon \int_{\mathbb{R}^d} \Phi\left(\frac{f}{m}\right) \mathrm{d}\lambda < \infty \text{ for all } m > 0 \right\}$$

and endowed with the Luxemburg norm

$$\|f\|_\Phi := \inf\left\{ m > 0 \colon \int_{\mathbb{R}^d} \Phi\left(\frac{f}{m}\right) \mathrm{d}\lambda \leq 1 \right\}.$$

We remark that $(M^\Phi, \|\cdot\|_\Phi, \leq)$ is a Banach lattice. For every sequence $(f_n)_{n\in\mathbb{N}} \subset M^\Phi$ and $f \in M^\Phi$ with

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \Phi\left(\frac{f - f_n}{m}\right) \mathrm{d}\lambda = 0 \quad \text{for all } m \in (0, 1], \tag{2.4}$$

it follows that $\lim_{n\to\infty} \|f - f_n\|_\Phi = 0$. In addition, we define

$$\|f\|_{\Phi,R} = \inf\left\{ m > 0 \colon \int_{\mathbb{R}^d} \Phi\left(\frac{f}{m}\right) \mathrm{d}\lambda \leq R \right\} \quad \text{for all } f \in M^\Phi \text{ and } R \geq 1.$$

For $f \neq 0$, the monotone convergence theorem implies that the infimum in the previous equation is attained at $\|f\|_{\Phi,R}$. Denote by $B_R(f, r) := \{g \in M^\Phi \colon \|f - g\|_{\Phi,R} \leq r\}$ the closed ball around $f \in M^\Phi$ with radius $r \geq 0$. Moreover, we define $B_\Phi(f, r) := B_1(f, r)$ and $B_R(r) := B_R(0, r)$.

Lemma 2.2.1. *It holds* $\|f\|_{\Phi,R} \leq \|f\|_\Phi \leq R\|f\|_{\Phi,R}$ *for all* $f \in M^\Phi$ *and* $R \geq 1$. *Furthermore, we have* $M^\Phi = \bigcup_{R\geq 1} B_R(r)$ *for all* $r > 0$.

Proof. W.l.o.g., we assume $f \neq 0$. By definition, it holds $\|f\|_{\Phi,R} \leq \|f\|_\Phi$. Since Φ is convex with $\Phi(0) = 0$, we obtain

$$\int_{\mathbb{R}^d} \Phi\left(\frac{f}{R\|f\|_{\Phi,R}}\right) \mathrm{d}\lambda \leq \frac{1}{R}\int_{\mathbb{R}^d} \Phi\left(\frac{f}{\|f\|_{\Phi,R}}\right) \mathrm{d}\lambda \leq 1$$

and therefore $\|f\|_\Phi \leq R\|f\|_{\Phi,R}$. Moreover, by definition, it holds $f \in B_R(r)$ for

$$R := 1 + \int_{\mathbb{R}^d} \Phi\left(\frac{f}{r}\right) \mathrm{d}\lambda < \infty. \qquad \square$$

Let C_c^∞ be the space of all infinitely differentiable functions $f\colon \mathbb{R}^d \to \mathbb{R}$ with compact support. In addition, we denote by $B_{\mathbb{R}^d}(r) := \{x \in \mathbb{R}^d \colon |x| \leq r\}$ the closed ball around zero with radius $r \geq 0$ w.r.t. the Euclidean norm.

Lemma 2.2.2. *The space* $C_c^\infty \subset M^\Phi$ *is dense. Hence, for every* $R \geq 1$, $r \geq 0$ *and* $f \in B_R(r)$, *there exists a sequence* $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty \cap B_R(r)$ *with* $\lim_{n\to\infty} \|f - f_n\|_{\Phi,R} = 0$.

Proof. First, we remark that, for every $n \in \mathbb{N}$, the function $f_n := f\mathbb{1}_{\{|f|\leq n\}\cap B(n)}$ is bounded and has compact support. It holds $|f_n| \leq |f|$ and therefore $f_n \in B_R(r)$ for all $n \in \mathbb{N}$. Furthermore, we use $f_n \to f$ almost everywhere and the dominated convergence theorem, see [149, Theorem 14 in Chapter 3.4], to conclude $\|f - f_n\|_{\Phi,R} \to 0$.

Second, we assume that f is bounded and has compact support. Let $\eta \in C_c^\infty$ with $\eta \geq 0$ and $\int_{\mathbb{R}^d} \eta \,\mathrm{d}\lambda = 1$. Define $\eta_n(x) := n^d\eta(nx)$ and $f_n := f * \eta_n$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$. For every $m > 0$ and $n \in \mathbb{N}$, we use Jensen's inequality, Fubini's theorem and the transformation theorem to estimate

$$\begin{aligned}
&\int_{\mathbb{R}^d} \Phi\left(\frac{f*\eta_n}{m}\right) \mathrm{d}\lambda = \int_{\mathbb{R}^d} \Phi\left(\int_{\mathbb{R}^d} \frac{f(x-y)}{m}\eta_n(y)\,\mathrm{d}y\right) \mathrm{d}x \\
&\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Phi\left(\frac{f(x-y)}{m}\right)\eta_n(y)\,\mathrm{d}y\,\mathrm{d}x = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \Phi\left(\frac{f(x-y)}{m}\right)\mathrm{d}x\right)\eta_n(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \Phi\left(\frac{f(x)}{m}\right)\mathrm{d}x\right)\eta_n(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \Phi\left(\frac{f}{m}\right)\mathrm{d}\lambda < \infty.
\end{aligned}$$

This shows $f * \eta_n \in B_R(r)$. Moreover, it holds $f_n \to f$ almost everywhere. Since f is bounded and has compact support, the dominated convergence theorem implies that equation (2.4) is satisfied and therefore $\|f - f_n\|_{\Phi,R} \to 0$. $\square$

2.3 Convex semigroups on Orlicz hearts

Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a convex strictly increasing twice continuously differentiable function with $\lim_{x\to\infty} \varphi(x) = \infty$, where $\mathbb{R}_+ := \{x \in \mathbb{R}\colon x \geq 0\}$. Hence, the mapping

$$\Phi\colon \mathbb{R} \to \mathbb{R}_+,\ x \mapsto \varphi(|x|) - \varphi(0)$$

is a Young function with $\Phi(x) > 0$ for all $x \neq 0$. Furthermore, let $(X_t)_{t\geq 0}$ be a Lévy process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$ and $X_0 = 0$ almost surely. In particular, the process $(X_t)_{t\geq 0}$ has independent stationary increments. For an overview on Lévy processes, we refer to [2] and [151]. Throughout this section, we make the following assumption.

Assumption 2.3.1. Suppose that Φ and X satisfy the following conditions:

(i) For every $t \geq 0$, the distribution $\mathbb{P}\circ X_t^{-1}$ is absolutely continuous w.r.t. the Lebesgue measure and the Radon-Nikodým derivative is essentially bounded.

(ii) For every $c \geq 0$, there exists $r \geq 0$ such that $\lim_{t\downarrow 0} \mathbb{P}(|X_t| \geq r)\Phi\big(\frac{c}{t}\big) = 0$.

The previous assumption is illustrated by the following example.

Example 2.3.2. Let $(X_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion. For every $t \geq 0$, we denote by $\mathcal{N}(0, t\mathbb{1})$ the d-dimensional normal distribution with mean zero and covariance matrix $t\mathbb{1}$, where $\mathbb{1} \in \mathbb{R}^{d\times d}$ is the identity matrix. Clearly, the distribution $\mathbb{P}\circ X_t^{-1} = \mathcal{N}(0, t\mathbb{1})$ satisfies condition (i) for all $t \geq 0$. Moreover, condition (ii) is, for instance, satisfied for the choices

$$\varphi(x) := x^p \ \text{ for } \ p \in [1,\infty), \quad \varphi(x) := e^x, \quad \varphi(x) := (bx-1)e^{bx} + 1 \ \text{ for } \ b \geq 0.$$

The last choice of φ will be used in Section 2.4, where we will also verify condition (ii).

2.3.1 Extension of convex semigroups

Denote by C_0 the space of all continuous functions $f\colon \mathbb{R}^d \to \mathbb{R}$ with $\lim_{|x|\to\infty} f(x) = 0$. We endow C_0 with the supremum norm $\|f\|_\infty := \sup_{x\in\mathbb{R}^d} |f(x)|$. In the sequel, let $(S(t))_{t\geq 0}$ be a strongly continuous convex semigroup on C_0, i.e., a family of convex operators $S(t)\colon C_0 \to C_0$ which satisfy

(i) $S(0) = \mathrm{id}_{C_0}$,

(ii) $S(s+t)f = S(s)S(t)f$ for all $s, t \geq 0$ and $f \in C_0$,

(iii) $\lim_{t\downarrow 0} \|S(t)f - f\|_\infty = 0$ for all $f \in C_0$.

Furthermore, the generator of $(S(t))_{t\geq 0}$ is defined by

$$A_\infty\colon D(A_\infty) \to C_0,\ f \mapsto \lim_{h\downarrow 0} \frac{S(h)f - f}{h},$$

where the domain $D(A_\infty)$ consists of all $f \in C_0$ such that the previous limit exists w.r.t. the supremum norm. In order to extend the semigroup $(S(t))_{t\geq 0}$ from C_0 to M^Φ, we assume that

$$|S(t)f| \leq T(t)|f| \quad \text{for all } t \geq 0 \text{ and } f \in C_0 \cap M^\Phi,$$

where the family $(T(t))_{t\geq 0}$ is defined by

$$(T(t)f)(x) := \varphi^{-1}\big(\mathbb{E}[\varphi(e^{at} f(x + X_t))]\big)$$

for all $t \geq 0$, $f \in M^\Phi_+$ and $x \in \mathbb{R}^d$. Here, the constant $a \geq 0$ is fixed and

$$\mathbb{E}[X] := \int_\Omega X(\omega)\,\mathbb{P}(\mathrm{d}\omega)$$

denotes the expectation of a random variable $X\colon \Omega \to \mathbb{R}$.

Remark 2.3.3. If $a = 0$, then $(T(t))_{t\geq 0}$ is a semigroup. In particular, if $\varphi(x) = e^x$ and $(X_t)_{t\geq 0}$ is a Brownian motion, we obtain the entropic semigroup

$$(T(t)f)(x) = \log\big(\mathbb{E}[\exp(f(x+X_t))]\big).$$

It follows from Itô's formula that, for sufficiently smooth f, the function $u(t) := T(t)f$ solves the Cauchy problem $\partial_t u = \frac{1}{2}\big(\Delta u + |\nabla u|^2\big)$ for all $t \geq 0$ and $u(0) = f$.

In order to establish some basic properties of the family $(T(t))_{t\geq 0}$, we need the following auxiliary result. In decision theory, the expression $-u''/u'$ is called the Arrow-Pratt measure of absolute risk-aversion of the utility function u.

Lemma 2.3.4. *Let $u, v\colon \mathbb{R}_+ \to \mathbb{R}_+$ be two strictly increasing twice continuously differentiable functions which satisfy*

$$\frac{u''(x)}{u'(x)} \leq \frac{v''(x)}{v'(x)} \quad \text{for all } x \in \mathbb{R}_+. \tag{2.5}$$

Then, it holds $u^{-1}\big(\mathbb{E}[u(X)]\big) \leq v^{-1}\big(\mathbb{E}[v(X)]\big)$ for all random variables $X\colon \Omega \to \mathbb{R}_+$. In particular, for every random variable $X\colon \Omega \to \mathbb{R}_+$ and $c \geq 1$,

$$c\varphi^{-1}\big(\mathbb{E}[\varphi(X)]\big) \leq \varphi^{-1}\big(\mathbb{E}[\varphi(cX)|\big).$$

Proof. We follow the proof of [80, Proposition 2.44]. Define $F := v \circ u^{-1}$. For every $x \in \mathbb{R}_+$ and $y := u^{-1}(x)$, we use $v'(y) > 0$ and inequality (2.5) to estimate

$$F''(x) = \frac{v'(y)}{u'(y)^2}\left(\frac{v''(y)}{v'(y)} - \frac{u''(y)}{u'(y)}\right) \geq 0.$$

Hence, the function F is convex and Jensen's inequality implies

$$u^{-1}\big(\mathbb{E}[u(X)]\big) = v^{-1}\Big(F\big(\mathbb{E}[u(X)]\big)\Big) \leq v^{-1}\big(\mathbb{E}[F(u(X))]\big) = v^{-1}\big(\mathbb{E}[v(X)]\big).$$

The second part of the statement follows from the first one by choosing $u(x) := \varphi(x)$ and $v(x) := \varphi(cx)$ for all $x \in \mathbb{R}_+$. □

Let C_c be space set of all continuous functions $f\colon \mathbb{R}^d \to \mathbb{R}$ with compact support. Define $M_+^\Phi := \{f \in M^\Phi\colon f \geq 0\}$ and $C_c^+ := \{f \in C_c\colon f \geq 0\}$.

Theorem 2.3.5. *The following statements are valid:*

(i) $T(t)\colon M_+^\Phi \to M_+^\Phi$ *for all* $t \geq 0$.

(ii) $\|T(t)f\|_{\Phi,R} \leq e^{at}\|f\|_{\Phi,R}$ *for all* $R \geq 1$ *and* $f \in M_+^\Phi \cap B_R(e^{-at})$.

(iii) $T(0) = \mathrm{id}_{M_+^\Phi}$ *and* $T(s)T(t)f \leq T(s+t)f$ *for all* $s, t \geq 0$ *and* $f \in M_+^\Phi$.

(iv) For every $f \in C_c^+$ *and* $m \in (0,1]$, *there exists* $r \geq 0$ *with*

$$\lim_{t\downarrow 0}\int_{B(r)^c} \Phi\left(\frac{T(t)f}{mt}\right)\mathrm{d}\lambda = 0. \tag{2.6}$$

Proof. First, we show that $T(t)\colon M_+^\Phi \to M_+^\Phi$ for all $t \geq 0$. For every $t \geq 0$ and $f \in M_+^\Phi$, it follows from $\varphi(x) = \Phi(x) + \varphi(0)$ and Assumption 2.3.1(i) that

$$\mathbb{E}[\varphi(e^{at} f(x + X_t))] = \int_{\mathbb{R}^d} \varphi(e^{at} f(x+y))\big(\mathbb{P} \circ X_t^{-1}\big)(\mathrm{d}y) < \infty \quad \text{for all } x \in \mathbb{R}^d.$$

Tonelli's theorem implies that $T(t)f\colon \mathbb{R}^d \to \mathbb{R}$ is a well-defined measurable function. For every $m \in (0,1]$, we use Lemma 2.3.4, Fubini's theorem and the transformation theorem to estimate

$$\begin{aligned}
\int_{\mathbb{R}^d} \Phi\left(\frac{T(t)f}{m}\right) \mathrm{d}\lambda &= \int_{\mathbb{R}^d} \varphi\left(\frac{1}{m}\varphi^{-1}\big(\mathbb{E}[\varphi(e^{at} f(x + X_t))]\big)\right) - \varphi(0)\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} \mathbb{E}\left[\varphi\left(\frac{e^{at} f(x + X_t)}{m}\right) - \varphi(0)\right] \mathrm{d}x \\
&= \mathbb{E}\left[\int_{\mathbb{R}^d} \Phi\left(\frac{e^{at} f(x + X_t)}{m}\right) \mathrm{d}x\right] \\
&= \int_{\mathbb{R}^d} \Phi\left(\frac{e^{at} f(x)}{m}\right) \mathrm{d}x < \infty. \qquad (2.7)
\end{aligned}$$

Moreover, for every $m \geq 1$, it follows from the monotonicity of Φ that

$$\int_{\mathbb{R}^d} \Phi\left(\frac{T(t)f}{m}\right) \mathrm{d}\lambda \leq \int_{\mathbb{R}^d} \Phi(T(t)f)\,\mathrm{d}\lambda = \int_{\mathbb{R}^d} \Phi(e^{at} f)\,\mathrm{d}\lambda < \infty.$$

Second, we show that $\|T(t)f\|_{\Phi,R} \leq e^{at}\|f\|_{\Phi,R}$ for all $R \geq 1$ and $f \in M_+^\Phi \cap B_R(e^{-at})$. W.l.o.g. we assume that $f \neq 0$. For $m := e^{at}\|f\|_{\Phi,R} \in (0,1]$, inequality (2.7) yields

$$\int_{\mathbb{R}^d} \Phi\left(\frac{T(t)f}{m}\right) \mathrm{d}\lambda \leq \int_{\mathbb{R}^d} \Phi\left(\frac{f}{\|f\|_{\Phi,R}}\right) \mathrm{d}\lambda \leq R.$$

We obtain $\|T(t)f\|_{\Phi,R} \leq e^{at}\|f\|_{\Phi,R}$.

Third, we show that $T(0)f = f$ and $T(s)T(t)f \leq T(s+t)f$ for all $s, t \geq 0$ and $f \in M_+^\Phi$. It follows from $X_0 = 0$ that $T(0)f = f$. In order to prove the second part, we define $\mathcal{F}_s := \sigma(X_u\colon u \in [0,s])$. Since $(X_t)_{t\geq 0}$ has independent stationary increments, the tower property and Lemma 2.3.4 imply

$$\begin{aligned}
(T(s)T(t)f)(x) &= \varphi^{-1}\big(\mathbb{E}[\varphi(e^{as}(T(t)f)(x + X_s))]\big) \\
&= \varphi^{-1}\Big(\mathbb{E}\Big[\varphi\big(e^{as}\varphi^{-1}\big(\mathbb{E}[\varphi(e^{at} f(y + X_t))]\big)\big)\big|_{y = x + X_s}\Big]\Big) \\
&\leq \varphi^{-1}\big(\mathbb{E}\big[\mathbb{E}[\varphi(e^{a(s+t)} f(x + X_s + X_{s+t} - X_s))|\mathcal{F}_s]\big]\big) \\
&= (T(s+t)f)(x).
\end{aligned}$$

Fourth, for every $f \in C_c^+$ and $m \in (0,1]$, we show that there exists $r \geq 0$ such that equation (2.6) is valid. Choose $r_0 \geq 0$ with $\operatorname{supp}(f) \subset B_{\mathbb{R}^d}(r_0)$ and define

$$c := \max\left\{\lambda\big(B_{\mathbb{R}^d}(r_0)\big), \frac{e^a\|f\|_\infty}{m}\right\}.$$

By Assumption 2.3.1(ii), there exists $r \geq \max\{c, r_0\}$ with

$$\lim_{t\downarrow 0} \mathbb{P}(|X_t| \geq r)\Phi\left(\frac{c}{t}\right) = 0.$$

We use inequality (2.7), $\operatorname{supp}(f) \subset B_{\mathbb{R}^d}(r_0) \subset B_{\mathbb{R}^d}(r)$, Fubini's theorem and the transformation theorem to estimate

$$\begin{aligned}
&\int_{B(2r)^c} \Phi\left(\frac{T(t)f}{mt}\right) \mathrm{d}\lambda \leq \int_{B(2r)^c} \mathbb{E}\left[\Phi\left(\frac{e^{at}f(x+X_t)}{mt}\right)\right] \mathrm{d}x \\
&= \int_{B(2r)^c} \mathbb{E}\left[\Phi\left(\frac{e^{at}f(x+X_t)}{mt}\right) \mathbb{1}_{\{|X_t|\geq r\}}\right] \mathrm{d}x \\
&= \mathbb{E}\left[\left(\int_{B(2r)^c} \Phi\left(\frac{e^{at}f(x+X_t)}{mt}\right) \mathrm{d}x\right) \mathbb{1}_{\{|X_t|\geq r\}}\right] \\
&\leq \mathbb{E}\left[\left(\int_{\mathbb{R}^d} \Phi\left(\frac{e^{at}f(x+X_t)}{mt}\right) dx\right) \mathbb{1}_{\{|X_t|\geq r\}}\right] = \mathbb{E}\left[\left(\int_{\mathbb{R}^d} \Phi\left(\frac{e^{at}f(x)}{mt}\right) \mathrm{d}x\right) \mathbb{1}_{\{|X_t|\geq r\}}\right] \\
&= \mathbb{P}(|X_t| \geq r) \int_{\mathbb{R}^d} \Phi\left(\frac{e^{at}f}{mt}\right) \mathrm{d}\lambda = \mathbb{P}(|X_t| \geq r) \int_{B(r_0)} \Phi\left(\frac{e^{at}f}{mt}\right) \mathrm{d}\lambda \\
&\leq c\mathbb{P}(|X_t| \geq r)\Phi\left(\frac{c}{t}\right) \to 0 \quad \text{as } t \downarrow 0. \qquad \square
\end{aligned}$$

The following theorem is the main result of this section.

Theorem 2.3.6. *Suppose that Φ and $(X_t)_{t\geq 0}$ satisfy Assumption 2.3.1. Furthermore, let $(S(t))_{t\geq 0}$ be a strongly continuous convex semigroup on C_0 such that*

$$|S(t)f| \leq T(t)|f| \quad \text{for all } t \geq 0 \text{ and } f \in C_0 \cap M^\Phi, \tag{2.8}$$

where the familiy $(T(t))_{t\geq 0}$ is defined by

$$(T(t)f)(x) := \varphi^{-1}\big(\mathbb{E}[\varphi(e^{at}f(x+X_t))]\big) \tag{2.9}$$

for all $t \geq 0$, $f \in M_+^\Phi$ and $x \in \mathbb{R}^d$. Then, the following statements are valid:

(i) It holds $S(t)\colon C_0 \cap M^\Phi \to C_0 \cap M^\Phi$ for all $t \geq 0$.

(ii) It holds $\|S(t)f\|_{\Phi,R} \leq e^{at}\|f\|_{\Phi,R}$ for all $t \geq 0$, $R \geq 1$, and $f \in C_0 \cap B_R(e^{-at})$.

(iii) For every $t \geq 0$, $R \geq 1$ and $f, g \in C_0 \cap B_R(\frac{e^{-at}}{3})$,

$$\|S(t)f - S(t)g\|_{\Phi,R} \leq 4e^{at}\|f-g\|_{\Phi,R}.$$

(iv) For every $f \in M^\Phi$ and $T \geq 0$, there exist $R \geq 1$ and $\delta > 0$ with

$$\sup_{t\in[0,T]} \|S(t)g_1 - S(t)g_2\|_\Phi \leq 4Re^{at}\|g_1 - g_2\|_\Phi \quad \text{for all } g_1, g_2 \in C_0 \cap B_\Phi(f,\delta).$$

Hence, for every $t \geq 0$, there exists a unique continuous extension $\tilde{S}(t)\colon M^\Phi \to M^\Phi$ such that the statements (ii)-(iv) are valid without the restriction to C_0. The family $(\tilde{S}(t))_{t\geq 0}$ of extended operators is a strongly continuous convex semigroup on M^Φ and, for every $f \in D(A_\infty) \cap C_c$ with $A_\infty f \in C_c$,

$$\lim_{h\downarrow 0} \left\| \frac{S(h)f - f}{h} - A_\infty f \right\|_\Phi = 0.$$

Proof. First, we show the statements (i)-(iv) are valid. For every $t \geq 0$ and $f \in C_0 \cap M^\Phi$, it follows from inequality (2.8) and Theorem 2.3.5(i) that

$$\int_{\mathbb{R}^d} \Phi\left(\frac{S(t)f}{m}\right) \mathrm{d}\lambda \leq \int_{\mathbb{R}^d} \Phi\left(\frac{T(t)|f|}{m}\right) \mathrm{d}\lambda < \infty \quad \text{for all } m > 0.$$

We obtain $S(t)\colon C_0 \cap M^\Phi \to C_0 \cap M^\Phi$ for all $t \geq 0$. Moreover, Theorem 2.3.5(ii) yields

$$\|S(t)f\|_{\Phi,R} \leq \|T(t)(|f|)\|_{\Phi,R} \leq e^{at}\|f\|_{\Phi,R}$$

for all $t \geq 0$, $R \geq 1$ and $f \in C_0 \cap B_R(e^{-at})$. In order to verify condition (iii), we define

$$S_f(t)\colon C_0 \to C_0,\ g \mapsto S(t)(f+g) - S(t)f$$

for fixed $t \geq 0$, $R \geq 1$ and $f \in C_0 \cap B_R(e^{-at}/3)$. The operator $S_f(t)$ is convex with $S_f(t)0 = 0$ and, by condition (ii), it holds

$$\|S_f(t)g\|_{\Phi,R} \leq \tfrac{4}{3} \quad \text{for all } g \in C_0 \cap B_R\left(\tfrac{2e^{-at}}{3}\right).$$

Hence, for every $g \in B_R(e^{-at}/3)$, it follows from [62, Lemma A.1] that

$$\|S(t)f - S(t)g\|_{\Phi,R} = \|S_f(t)(g-f)\|_{\Phi,R} \leq 4e^{at}\|f-g\|_{\Phi,R}.$$

Condition (iv) is an immediate consequence of condition (iii) and Lemma 2.2.1.

Second, we extend $(S(t))_{t\geq 0}$ from C_0 to M^Φ. To do so, let $t \geq 0$ and $f \in M^\Phi$. By Lemma 2.2.1 and Lemma 2.2.2, there exist $R \geq 1$ with $f \in B_R(e^{-at}/3)$ and a sequence $(f_n)_{n\in\mathbb{N}} \subset C_0 \cap B_R(e^{-at}/3)$ with $\|f - f_n\|_{\Phi,R} \to 0$. Due to condition (iii), the limit

$$\tilde{S}(t)f := \lim_{n\to\infty} S(t)f_n \in M^\Phi$$

exits and does not independent on the choice of the approximating sequence. Furthermore, the convex operator $\tilde{S}(t)$ is the unique continuous extension of $S(t)$ and the statements (ii)–(iv) are valid without the restriction to C_0. Next, we verify the semigroup property. Let $s, t \geq 0$ and $f \in M^\Phi$. Choose $R \geq 1$ with $\tilde{S}(t)f \in B_R(e^{-at}/4)$ and a sequence $(f_n)_{n\in\mathbb{N}} \subset C_0 \cap M^\Phi$ with $\|f - f_n\|_{\Phi,R} \to 0$. Since $\|\tilde{S}(t)f - S(t)f_n\|_{\Phi,R} \to 0$, we can assume, w.l.o.g., that $S(t)f_n \in B_R(e^{-at}/3)$ for all $n \in \mathbb{N}$. Property (iii) yields

$$\begin{aligned}
&\|\tilde{S}(s)\tilde{S}(t)f - \tilde{S}(s+t)f\|_{\Phi,R} \\
&\leq \|\tilde{S}(s)\tilde{S}(t)f - S(s)S(t)f_n\|_{\Phi,R} + \|S(s+t)f_n - \tilde{S}(s+t)f\|_{\Phi,R} \\
&\leq 4e^{as}\|\tilde{S}(t)f - S(t)f_n\|_{\Phi,R} + \|S(s+t)f_n - \tilde{S}(s+t)f\|_{\Phi,R} \to 0.
\end{aligned}$$

Third, we show that $(\tilde{S}(t))_{t\geq 0}$ is strongly continuous. To do so, we verify that

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^d} \Phi\left(\frac{S(t)f - f}{m}\right) \mathrm{d}\lambda = 0 \quad \text{for all } f \in C_c \text{ and } m \in (0,1].$$

Let $f \in C_c$ and $m \in (0,1]$. Choose $r \geq 0$ with $\operatorname{supp}(f) \subset B_{\mathbb{R}^d}(r)$ such that inequality (2.6) is valid for $|f| \in C_0$. It holds

$$\int_{\mathbb{R}^d} \Phi\left(\frac{S(t)f - f}{m}\right) \mathrm{d}\lambda = \int_{B(r)} \Phi\left(\frac{S(t)f - f}{m}\right) \mathrm{d}\lambda + \int_{B(r)^c} \Phi\left(\frac{S(t)f}{m}\right) \mathrm{d}\lambda.$$

Since $(S(t))_{t\geq 0}$ is strongly continuous w.r.t. the supremum norm, we obtain

$$\lim_{t\downarrow 0}\int_{B(r)}\Phi\left(\frac{S(t)f-f}{m}\right)\mathrm{d}\lambda = 0.$$

Furthermore, inequality (2.6) and inequality (2.8) imply

$$\int_{B(r)^c}\Phi\left(\frac{S(t)f}{m}\right)\mathrm{d}\lambda \leq \int_{B(r)^c}\Phi\left(\frac{T(t)|f|}{mt}\right)\mathrm{d}\lambda \to 0 \quad \text{as } t\downarrow 0.$$

We conclude that $\lim_{t\downarrow 0}\|S(t)f-f\|_\Phi = 0$ for all $f\in C_c$. For arbitrary $f\in M^\Phi$, the claim follows by an approximation argument due to Lemma 2.2.2 and property (iii). Finally, the statement about the generator can be shown using the same arguments. □

2.3.2 Convex semigroups on M^Φ

In this subsection, we give a short recap of the results in [62, Section 3] about convex semigroups on L^p-like spaces and their generators. In order to simplify the notation, the extension $(\tilde{S}(t))_{t\geq 0}$ of $(S(t))_{t\geq 0}$ from Theorem 2.3.6 is subsequently again denoted by $(S(t))_{t\geq 0}$. Recall that $(S(t))_{t\geq 0}$ is a strongly continuous convex semigroup on M^Φ which is locally uniformly Lipschitz continuous, i.e., for every $T\geq 0$ and $f\in M^\Phi$, there exist $c\geq 0$ and $\delta>0$ with

$$\sup_{t\in[0,T]}\|S(t)g_1-S(t)g_2\|_\Phi \leq c\|g_1-g_2\|_\Phi \quad \text{for all } g_1,g_2\in B_\Phi(f,\delta).$$

Furthermore, the generator of $(S(t))_{t\geq 0}$ is defined by

$$A\colon D(A)\to M^\Phi,\ f\mapsto \lim_{h\downarrow 0}\frac{S(h)f-f}{h},$$

where the domain $D(A)$ consists of all $f\in M^\Phi$ such that the previous limit exists w.r.t. the Luxembourg norm. The dominated convergence theorem, see [149, Theorem 14 in Chapter 3.4], guarantees that the Luxembourg norm is order continuous, i.e., for every net $(f_\alpha)_\alpha\subset M^\Phi$ with $f_\alpha\downarrow 0$, it holds $\|f_\alpha\|_\Phi\to 0$. Hence, by [133, Theorem 2.4.2], the Orlicz heart M^Φ is Dedekind σ-complete and therefore we can apply the results from [62, Section 3]. We remark that, in [62], the results are formulated for strongly continuous semigroups of bounded convex operators. However, it follows immediately from the corresponding proofs that locally uniform Lipschitz continuity is the crucial property rather than boundedness. A direct adaptation of the results in [62] to the present setting is summarized in the following theorem.

Theorem 2.3.7. *The following statements are valid:*

(i) It holds $S(t)\colon D(A)\to D(A)$ for all $t\geq 0$.

(ii) The mapping $\mathbb{R}_+\to M^\Phi$, $t\mapsto S(t)f$ is continuously differentiable with

$$\frac{d}{dt}(S(t)f) = \inf_{h>0}\frac{S(t)(f+hAf)-S(t)f}{h} = \sup_{h<0}\frac{S(t)(f+hAf)-S(t)f}{h}$$

for all $f\in D(A)$ and $t\geq 0$.

(iii) The operator A is closed, i.e., for every sequence $(f_n)_{n\in\mathbb{N}} \subset D(A)$ and $f, g \in M^\Phi$ with $f_n \to f$ and $Af_n \to g$, it holds $f \in D(A)$ with $Af = g$.

(iv) Let $v\colon \mathbb{R}_+ \to M^\Phi$ a continuous function with $v(t) \in D(A)$ and

$$Av(t) = \lim_{h\downarrow 0} \frac{v(t+h) - v(t)}{h} \quad \textit{for all } t \geq 0.$$

Then, it holds $v(t) = S(t)v(0)$ for all $t \geq 0$.

In particular, for every $f \in D(A)$, the abstract Cauchy problem

$$\partial_t u(t) = Au(t) \;\; \textit{for all } t \geq 0, \quad u(0) = f,$$

has a unique solution $u \in C^1([0,\infty); M^\Phi) \cap C([0,\infty); D(A))$ given by $u(t) := S(t)f$ for all $t \geq 0$. Moreover, the solution depends continuously on the initial data.

2.4 Viscous Hamilton–Jacobi equations

In this section, we apply the abstract results of Section 2.3 to the viscous Hamilton–Jacobi equation

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + H(\nabla u(t,x)), & (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) = f(x), & x \in \mathbb{R}^d. \end{cases} \tag{2.10}$$

Throughout this section, we make the following assumption, where $xy := \langle x, y\rangle$ denotes the Euclidean inner product on $\mathbb{R}^d$.

Assumption 2.4.1. The function $H\colon \mathbb{R}^d \to \mathbb{R}$ is convex and there exists $K \geq 0$ with

$$|H(x)| \leq K(|x| + |x|^2) \quad \text{for all } x \in \mathbb{R}^d.$$

Furthermore, there exists $r > 0$ such that the convex conjugate

$$L\colon \mathbb{R}^d \to [0,\infty],\; \lambda \mapsto \sup_{x\in\mathbb{R}^d} \big(\lambda x - H(x)\big)$$

satisfies $\sup_{|\lambda|=r} L(\lambda) < \infty$.

This section consists of two parts. First, we use the results from Chapter 1 and Section 2.3.1 in order to construct a strongly continuous convex monotone semigroup on a suitable Orlicz heart M^Φ. Then, on the symmetric Lipschitz set, we determine the generator explicitly and link the abstract Cauchy problem with equation (2.10).

2.4.1 Construction of the semigroup

Let $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For every $t \geq 0$, $f \in C_0$ and $x \in \mathbb{R}^d$, we define

$$(I(t)f)(x) := \sup_{\lambda\in\mathbb{R}^d} \big(\mathbb{E}[f(x + W_t + \lambda t)] - L(\lambda)t\big).$$

By Fenchel–Moreau's theorem, it holds $H(x) = \sup_{\lambda\in\mathbb{R}^d}(\lambda x - L(\lambda))$ for all $x \in \mathbb{R}^d$. Hence, the family $(I(t))_{t\geq 0}$ has the desired derivative at zero, i.e.,

$$\lim_{h\downarrow 0} \frac{I(h)f - f}{h} = \frac{1}{2}\Delta f + H(\nabla f) \quad \text{for all } f \in C_0^2,$$

where C_0^2 denotes the space of all twice differentiable functions $f \in C_0$ such that all partial derivatives up to order two are again in C_0. Let $(h_n)_{n\in\mathbb{N}} \subset (0,1]$ be a sequence with $h_n \to 0$. For every $n \in \mathbb{N}$, $t \geq 0$ and $f \in C_0$, we define

$$I(\pi_n^t)f := I(h_n)^{k_n^t} f := \underbrace{\big(I(h_n) \circ \ldots \circ I(h_n)\big)}_{k_n^t \text{ times}} f,$$

where $k_n^t := \max\{k \in \mathbb{N}_0 \colon kh_n \leq t\}$ and $\pi_n^t := \{h_n, \ldots, k_n^t h_n\}$ denotes the corresponding equidistant partition with mesh size h_n. Furthermore, let $\mathcal{T} \subset \mathbb{R}_+$ be a countable dense set including zero. As a direct application of Theorem 1.6.2, we obtain a semigroup $(S(t))_{t\geq 0}$ on C_0 associated to the family $(I(t))_{t\geq 0}$

Theorem 2.4.2. *There exist a family $(S(t))_{t\geq 0}$ of operators $S(t)\colon C_0 \to C_0$ and a subsequence $(n_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ such that, for every $t \in \mathcal{T}$ and $f \in C_0$,*

$$\lim_{l\to\infty} \|S(t)f - I(\pi_{n_l}^t)f\|_\infty = 0.$$

In addition, the following statements are valid:

(i) $S(0) = \mathrm{id}_{C_0}$ *and* $S(s)S(t) = S(s+t)$ *for all* $s, t \geq 0$.

(ii) $S(t)$ *is convex and monotone with* $S(t)0 = 0$ *for all* $t \geq 0$.

(iii) $\|S(t)f - S(t)g\|_\infty \leq \|f - g\|_\infty$ *for all* $t \geq 0$ *and* $f, g \in C_0$.

(iv) For every $f \in C_0$, *the mapping* $\mathbb{R}_+ \to C_0,\ t \mapsto S(t)f$ *is continuous.*

(v) For every $f \in C_0^2$,

$$\lim_{h\downarrow 0} \left\| \frac{S(h)f - f}{h} - \frac{1}{2}\Delta f - H(\nabla f) \right\|_\infty = 0.$$

Proof. We verify the assumptions of Subsection 1.6.1. By choosing $x := \frac{\lambda}{4K}$ in the definition of the convex conjugate for all $|\lambda| \geq 2K$, Assumption 2.4.1 implies

$$L(\lambda) \geq \frac{|\lambda|^2}{16K} \mathbb{1}_{B(2K)^c}(\lambda) \quad \text{for all } \lambda \in \mathbb{R}^d. \tag{2.11}$$

In particular, we obtain $\lim_{|\lambda|\to\infty} \frac{L(\lambda)}{|\lambda|} = \infty$. Furthermore, since H is sub-differentiable at zero with $H(0) = 0$, there exists $\lambda \in \mathbb{R}^d$ with $L(\lambda) = -H(0) = 0$. □

Next, we apply the results from Section 2.3 in order to extend the semigroup from C_0 to a suitable Orlicz heart M^Φ. Define

$$\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+,\ x \mapsto (bx-1)e^{bx} + 1, \quad \text{where} \quad b := 8K + 1.$$

The function φ is convex, strictly increasing and twice continuously differentiable with $\varphi(0) = 0$ and $\lim_{x\to\infty} \varphi(x) = \infty$. Moreover, the mapping

$$\Phi\colon \mathbb{R} \to \mathbb{R}_+,\ x \mapsto \varphi(|x|)$$

is a Young function satisfying $\Phi(x) > 0$ for all $x \neq 0$. The space M^Φ contains, in particular, all measurable locally bounded functions $f\colon \mathbb{R}^d \to \mathbb{R}$ such that there exists $p > d$ with

$$\lim_{|x|\to\infty} \frac{|f(x)|}{\log(1+|x|^{-p})} = 0.$$

Indeed, this follows from $\varphi(x) \le e^{2bx} - 1$ for all $x \ge 0$ and a straightforward computation. For every $t \ge 0$, $f \in M_+^\Phi$ and $x \in \mathbb{R}^d$, we define

$$(T(t)f)(x) := \varphi^{-1}\big(\mathbb{E}[\varphi(e^{2K^2t} f(x+W_t))]\big).$$

For the extension of $(S(t))_{t\ge 0}$, we need the following two auxiliary lemmas.

Lemma 2.4.3. *There exist $t_0 > 0$ and $r_0 \ge 0$ such that*

$$\mathbb{P}(|W_t| \ge r) \le te^{-\frac{r}{t}} \quad \textit{for all } t \in [0, t_0] \textit{ and } r \ge r_0.$$

Proof. Define $\mathbb{S}_u := \{x \in \mathbb{R}^d \colon |x| = u\}$ for all $u \ge 0$ and denote by $|\mathbb{S}_1|$ the area of the unit sphere. For every $t > 0$ and $r \ge 0$, using polar coordinates yields

$$\begin{aligned} \mathbb{P}(|W_t| \ge r) &= (2\pi t)^{-\frac{d}{2}} \int_{B(r)^c} e^{-\frac{|x|^2}{2t}}\,\mathrm{d}x = (2\pi t)^{-\frac{d}{2}} \int_r^\infty \int_{\mathbb{S}_u} e^{-\frac{|x|^2}{2t}}\,\mathrm{d}x\,\mathrm{d}u \\ &= (2\pi t)^{-\frac{d}{2}} |\mathbb{S}_1| \int_r^\infty u^{d-1} e^{-\frac{u^2}{2t}}\,\mathrm{d}u. \end{aligned} \tag{2.12}$$

Choose $r_0 \ge 1$ such that, for all $u \ge r_0$ and $t \in (0,1]$,

$$e^{-\frac{u^2}{2t}} u^{d-1} \le e^{-\frac{2u}{t}} e^{-\frac{u}{t}} u^{d-1} \le e^{-\frac{2u}{t}}. \tag{2.13}$$

In addition, due to $e^{-\frac{2u}{t}} \le e^{-\frac{1}{t}} e^{-\frac{u}{t}}$ for all $u \ge 1$ and $(2\pi t)^{-\frac{d}{2}} e^{-\frac{1}{t}} \to 0$ as $t \downarrow 0$, there exists $t_0 \in (0,1]$ with

$$|\mathbb{S}_1|(2\pi t)^{-\frac{d}{2}} e^{-\frac{2u}{t}} \le e^{-\frac{u}{t}} \quad \text{for all } t \in [0, t_0] \text{ and } u \ge 1. \tag{2.14}$$

We combine the equations (2.12)-(2.14) to obtain

$$\mathbb{P}(|W_t| \ge r) \le \int_r^\infty e^{-\frac{u}{t}}\,\mathrm{d}u = te^{-\frac{r}{t}} \quad \text{for all } t \in [0, t_0] \text{ and } r \ge r_0.$$ ☐

The following result, which is based on the Girsanov transformation, is similar to the estimates in [62, Example 5.3] and in the proof of Theorem 1.6.2.

Lemma 2.4.4. *Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a bounded measurable function and $p, q \in (1, \infty)$ such that ${}^1\!/_p + {}^1\!/_q = 1$. Then, for every $\lambda, x \in \mathbb{R}^d$ and $t \ge 0$,*

$$\mathbb{E}[|f(x+W_t+\lambda t)|] \le e^{\frac{(q-1)|\lambda|^2 t}{2}} \mathbb{E}[|f(x+W_t)|^p]^{\frac{1}{p}}.$$

Proof. We use $W_t \sim \mathcal{N}(0, t\mathbb{1})$ and the formula for the moment generating function of the normal distribution to estimate

$$\begin{aligned}
&\mathbb{E}[|f(x+W_t+\lambda t)|] \\
&= (2\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} |f(x+y+\lambda t)| \exp\left(-\frac{|y|^2}{2t}\right) \mathrm{d}y \\
&= (2\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} |f(x+y)| \exp\left(-\frac{|y-\lambda t|^2}{2t}\right) \mathrm{d}y \\
&= e^{-\frac{|\lambda|^2 t}{2}} \int_{\mathbb{R}^d} |f(x+y)| e^{\lambda y} \mathcal{N}(0, t\mathbb{1})(\mathrm{d}y) \\
&\leq e^{-\frac{|\lambda|^2 t}{2}} \left(\int_{\mathbb{R}^d} |f(x+y)|^p \mathcal{N}(0, t\mathbb{1})(\mathrm{d}y)\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} e^{q\lambda y} \mathcal{N}(0, t\mathbb{1})(\mathrm{d}y)\right)^{\frac{1}{q}} \\
&= e^{\frac{(q-1)|\lambda|^2 t}{2}} \mathbb{E}[|f(x+W_t)|^p]^{\frac{1}{p}}.
\end{aligned}$$

□

Theorem 2.4.5. *There exists a strongly continuous convex monotone locally uniformly Lipschitz continuous semigroup $(S(t))_{t\geq 0}$ on M^Φ with*

$$\lim_{h\downarrow 0} \left\| \frac{S(h)f - f}{h} - \frac{1}{2}\Delta f - H(\nabla f) \right\|_\Phi = 0 \quad \text{for all } f \in C_c^2,$$

where $\Phi(x) := (b|x|-1)e^{b|x|} + 1$ for all $x \in \mathbb{R}$. In addition, for every $f \in D(A)$, the unique solution $u \in C^1([0,\infty); M^\Phi) \cap C([0,\infty); D(A))$ of the abstract Cauchy problem

$$\partial_t u(t) = Au(t) \quad \text{for all } t \geq 0, \qquad u(0) = f,$$

is given by $u(t) := S(t)f$ for all $t \geq 0$.

Proof. Clearly, Assumption 2.3.1(i) is satisfied. Furthermore, choose $t_0 > 0$ and $r_0 \geq 0$ such that the statement of Lemma 2.4.3 is valid. Then, for $r := \max\{r_0, 2bc\}$,

$$\mathbb{P}(|W_t| \geq r)\Phi\left(\frac{c}{t}\right) \leq te^{-\frac{r}{t}} e^{\frac{2bc}{t}} = te^{\frac{2bc-r}{t}} \to 0 \quad \text{as } t \downarrow 0.$$

Next, we show that $|I(t)f| \leq T(t)|f|$ for all $t \geq 0$ and $f \in C_0 \cap M^\Phi$. To do so, let $t \geq 0$, $f \in C_0 \cap M^\Phi$ and $\lambda, x \in \mathbb{R}^d$. If $|\lambda| \leq 2K$, then applying Lemma 2.4.4 with $p = q = 2$ yields

$$\mathbb{E}[|f(x+W_t+\lambda t)|] \leq e^{2K^2 t} \mathbb{E}[f^2(x+W_t)]^{\frac{1}{2}}.$$

Moreover, the functions $u(x) := x^2$ and $v(x) := \varphi(x)$ satisfy

$$\frac{u''(x)}{u'(x)} = \frac{1}{x} \leq \frac{bx+1}{x} = \frac{v''(x)}{v'(x)}.$$

Thus, it follows from Lemma 2.3.4 that

$$\mathbb{E}[|f(x+W_t+\lambda t)|] - L(\lambda)t \leq \varphi^{-1}\big(\mathbb{E}[\varphi(e^{2K^2 t}|f(x+W_t)|)]\big) = \big(T(t)|f|\big)(x).$$

Now, let $|\lambda| > 2K$. We use Jensen's inequality, inequality (2.11) and Lemma 2.4.4 with $p = 8K+1$ and $q = 1 + {}^1\!/_{8K}$ to estimate

$$\mathbb{E}[|f(x+W_t+\lambda t)|] - L(\lambda)t \leq \log\big(\mathbb{E}[\exp(|f(x+W_t+\lambda t)|)]\big) - \frac{|\lambda|^2 t}{16K}$$

$$\begin{aligned}
&\le \log\left(e^{\frac{(q-1)|\lambda|^2 t}{2}}\mathbb{E}[\exp(p|f(x+W_t)|)]^{\frac{1}{p}}\right) - \frac{|\lambda|^2 t}{16K} \\
&= \frac{1}{p}\log\big(\mathbb{E}[\exp(p|f(x+W_t)|)]\big) + \left(\frac{q-1}{2} - \frac{1}{16K}\right)|\lambda|^2 t \\
&= \frac{1}{p}\log\big(\mathbb{E}[\exp(p|f(x+W_t)|)]\big). \qquad (2.15)
\end{aligned}$$

Due to $b = p$, the functions $u(x) := e^{px}$ and $v(x) := \varphi(x)$ satisfy

$$\frac{u''(x)}{u'(x)} = p \le \frac{bx+1}{x} = \frac{v''(x)}{v'(x)}.$$

Hence, Lemma 2.3.4 implies

$$\mathbb{E}[|f(x+W_t+\lambda t)|] - L(\lambda)t \le \big(T(t)|f|\big)(x).$$

Taking the supremum over all $\lambda \in \mathbb{R}^d$ yields $(I(t)f)(x) \le (T(t)|f|)(x)$. In addition, there exists $\lambda \in \mathbb{R}^d$ with $L(\lambda) = 0$ and therefore

$$(I(t)f)(x) \ge \mathbb{E}[f(x+W_t+\lambda t)] \ge -\big(T(t)|f|\big)(x).$$

We obtain $|I(t)f| \le T(t)|f|$ for all $t \ge 0$ and $f \in C_0 \cap M^\Phi$. Furthermore, it follows inductively from Theorem 2.3.5(iii) that $|I(\pi_n^t)f| \le T(t)|f|$ for all $t \in \mathcal{T}$, $n \in \mathbb{N}$ and $f \in C_0 \cap M^\Phi$. Hence, Theorem 2.4.2 implies $|S(t)f| \le T(t)|f|$ for all $t \in \mathcal{T}$. Let $t \ge 0$ be arbitrary and $(t_n)_{n\in\mathbb{N}} \subset \mathcal{T}$ be a sequence with $t_n \to t$. Theorem 2.4.2(iv) and the dominated convergence theorem yield

$$|S(t)f|(x) = \lim_{n\to\infty} |S(t_n)f|(x) \le \lim_{n\to\infty} \big(T(t_n)|f|\big)(x) = \big(T(t)|f|\big)(x) \quad \text{for all } x \in \mathbb{R}^d.$$

Now, the claim follows from Theorem 2.3.6 and Theorem 2.3.7. □

In view of the existing results on mild solutions, for $H(x) := a|x|^q$ with $q \in (1,2)$, one might expect that the previous theorem remains valid if we replace the exponential Orlicz heart by an L^p-space with p sufficiently large. However, due to the exponential transformation of the drift in Lemma 2.4.4, it is necessary to transform the multiplicative term $\exp((q-1)|\lambda|^2 t/2)$ into the additive term $(q-1)|\lambda|^2 t/2$ which can be cancelled by the penalization term $|\lambda|^2 t/16K$. Hence, even in the case that H grows only sub-quadratically, it is not possible to replace the exponential in inequality (2.15) by a function with mere polynomial growth. The latter would be necessary in order to extent $(S(t))_{t\ge 0}$ to a suitable L^p-space rather than an exponential Orlicz heart.

2.4.2 The symmetric Lipschitz set

Invariant (symmetric) Lipschitz sets have been first introduced in [61] and systematically studied in Section 1.5.

Definition 2.4.6. The Lipschitz set $\mathcal{L}^S$ consists of all $f \in C_0 \cap M^\Phi$ such that there exist $t_0 > 0$ and $c \ge 0$ with

$$\|S(t)f - f\|_\infty \le ct \quad \text{for all } t \in [0, t_0].$$

In addition, the symmetric Lipschitz set is defined as $\mathcal{L}^S_{\text{sym}} := \{f \in \mathcal{L}^S \colon -f \in \mathcal{L}^S\}$.

Similar to the domain of the generator, the symmetric Lipschitz set is invariant under the semigroup an can be, in contrast to $\mathcal{L}^S$ and $D(A)$, determined explicitly. Let L^∞ be the space of all bounded Borel measurable functions $f\colon \mathbb{R}^d \to \mathbb{R}$. For every $k \in \mathbb{N}$ and $p \in [1,\infty]$, we denote by $W^{k,p}$ the k-th order L^p-Sobolev space and by $W^{k,p}_{\mathrm{loc}}$ the respective local Sobolev space. For $f \in W^{1,\infty}$, we say that the Laplacian exists in L^∞ if there exists a function $g \in L^\infty$ with

$$\int_{\mathbb{R}^d} g\psi \,\mathrm{d}\lambda = -\int_{\mathbb{R}^d} \langle \nabla f, \nabla \psi\rangle \,\mathrm{d}\lambda \quad \text{for all } \psi \in C_c^\infty.$$

In this case, since g is unique almost everywhere, we define $\Delta f := g$.

Theorem 2.4.7. *It holds $S(t)\colon \mathcal{L}^S_{\mathrm{sym}} \to \mathcal{L}^S_{\mathrm{sym}}$ for all $t \geq 0$, where*

$$\mathcal{L}^S_{\mathrm{sym}} = \{f \in W^{1,\infty} \cap C_0 \cap M^\Phi\colon \Delta f \text{ exists in } L^\infty\}.$$

Moreover, for every $f \in \mathcal{L}^S_{\mathrm{sym}}$, there exists a constant $c \geq 0$, depending only on $\|\Delta f\|_\infty$ and $\|\nabla f\|_\infty$, such that $\|S(s)f - S(t)f\|_\infty \leq c|s-t|$ for all $s,t \geq 0$.

Proof. For the invariance and explicit characterization of the symmetric Lipschitz set, we refer to Theorem 1.6.2 and Remark 1.6.4. Moreover, it follows from the proof of Theorem 1.6.2 that there exists a constant $c \geq 0$, depending only on $\|\Delta f\|_\infty$ and $\|\nabla f\|_\infty$, such that

$$\|I(t)f - f\|_\infty \leq ct \quad \text{for all } t \geq 0.$$

We use Lemma 1.2.13 to conclude $\|S(t)f - S(s)f\|_\infty \leq c|s-t|$ for all $s,t \geq 0$. □

On the one hand, we know from Theorem 2.4.5 that the abstract Cauchy problem associated to the generator has a unique solution which is represented by the semigroup. On the other hand, we know from Theorem 2.4.7 that, for elements of the symmetric Lipschitz set, the differential operator on the right-hand side of equation (2.10) is well-defined. The natural questions whether this differential operator coincides with the generator can be answered in the affirmative.

Theorem 2.4.8. *Let $f \in D(A) \cap \mathcal{L}^S_{sym}$ and define $u(t) := S(t)f$ for all $t \geq 0$. Then,*

$$Au(t) = \frac{1}{2}\Delta u(t) + H(\nabla u(t)) \quad \textit{for all } t \geq 0,$$

where $u(t) \in \{g \in W^{1,\infty} \cap C_0 \cap M^\Phi\colon \Delta g \text{ exists in } L^\infty\}$. In particular, u is the unique solution of equation (2.10) in the class $C^1([0,\infty); M^\Phi) \cap C([0,\infty); D(A))$. Moreover, there exists a constant $c \geq 0$, depending only on $\|\Delta f\|_\infty$ and $\|\nabla f\|_\infty$, such that

$$\sup_{t\geq 0} \big(\|\partial_t u(t)\|_\infty + \|\Delta u(t)\|_\infty + \|\nabla u(t)\|_\infty\big) \leq c. \tag{2.16}$$

Proof. First, for every $g \in \mathcal{L}^S_{\mathrm{sym}}$ with compact support, we show that

$$g \in D(A) \quad \text{and} \quad Ag = \frac{1}{2}\Delta g + H(\nabla g).$$

Let $\eta \in C_c^\infty$ with $\eta \geq 0$ and $\int_{\mathbb{R}^d} \eta \,\mathrm{d}\lambda = 1$. Define $\eta_n(x) := n^d \eta(nx)$ and $g_n := g * \eta_n$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$. Theorem 2.4.5 implies $g_n \in D(A)$ and

$$Ag_n = \frac{1}{2}\Delta g_n + H(\nabla g_n) \quad \text{for all } n \in \mathbb{N}.$$

By Theorem 2.4.7, the functions ∇g and Δg are bounded and have compact support and thus the dominated convergence theorem, see [149, Theorem 14 in Chapter 3.4], yields

$$\lim_{n\to\infty} \left\| \frac{1}{2}\Delta g + H(\nabla g) - \frac{1}{2}\Delta g_n - H(\nabla g_n) \right\|_\Phi = 0.$$

Since Theorem 2.3.7(iii) guarantees that the generator A is closed, we obtain $g \in D(A)$ and $Ag = \frac{1}{2}\Delta g + H(\nabla g)$.

Second, we show that

$$\partial_t u(t) = \frac{1}{2}\Delta u(t) + H(\nabla u(t)) \quad \text{for all } t \geq 0.$$

Let $\zeta \in C_c^\infty$ with $0 \leq \zeta \leq 1$ and $\zeta(x) = 1$ for all $x \in B_{\mathbb{R}^d}(1)$. Define $\zeta_n(x) = \zeta(x/n)$ and $u_n(t) := \zeta_n u(t)$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ and $t \geq 0$. For every $t \geq 0$ and $n \in \mathbb{N}$, it follows from the explicit characterization of $\mathcal{L}^S_{\text{sym}}$ in Theorem 2.4.7 that $u_n(t) \in \mathcal{L}^S_{\text{sym}}$. We use that $u_n(t)$ has compact support and $\partial_t u_n(t) = \zeta_n \partial_t u(t)$ to obtain the pointwise equality

$$\partial_t u(t) = \lim_{n\to\infty} \partial_t u_n(t) = \lim_{n\to\infty} \left(\frac{1}{2}\Delta u_n(t) + H(\nabla u_n(t)) \right) = \frac{1}{2}\Delta u(t) + H(\nabla u(t))$$

for all $t \geq 0$ and $n \in \mathbb{N}$.

Third, we show inequality (2.16). By Theorem 2.4.7, there exists a constant $c \geq 0$, depending only on $\|\Delta f\|$ and $\|\nabla f\|_\infty$, such that

$$\max\left\{ \|S(t)f - f\|_\infty, \|S(t)(-f) + f\|_\infty \right\} \leq ct \quad \text{for all } t \geq 0.$$

Theorem 2.4.5 yields $u \in C^1([0,\infty); M^\Phi)$ and therefore $\|\partial_t u(t)\|_\infty \leq c$ for all $t \geq 0$. Furthermore, it follows from the proof of Theorem 1.5.1 that

$$\max\left\{ \|S(s)u(t) - u(t)\|_\infty, \|S(s)(-u(t)) + u(t)\|_\infty \right\} \leq 2cs \quad \text{for all } s \geq 0.$$

Hence, we can proceed similar to the proof of Theorem 1.6.3 in order to estimate the terms $\|\Delta u(t)\|_\infty$ and $\|\nabla u(t)\|_\infty$. □

In the proof of the previous theorem, we have shown that $\mathcal{L}^S_{\text{sym}} \cap C_c \subset D(A)$, but a complete characterization of $\mathcal{L}^S_{\text{sym}} \cap D(A)$ is beyond the scope of this paper. Moreover, it follows from [131, Theorem 3.1.7] that the symmetric Lipschitz set coincides with the domain of the Laplacian in L^∞ restricted to $C_0 \cap M^\Phi$. While the results in this article are independent of the established PDE-theory, by relying on the L^p-theory for the Laplacian presented in [131], we see that the solution u from Theorem 2.4.8 has in fact the additional regularity

$$u(t) \in \bigcap_{p\geq 1} W^{2,p}_{\text{loc}} \quad \text{for all } t \geq 0.$$

Moreover, for every $r \geq 0$, there exists a constant $c_r \geq 0$, depending only on c and r, such that $\|u(t)\|_{W^{2,p}(B(r))} \leq c_r$ for all $t \geq 0$. Indeed, this estimate follows immediately from inequality (2.16) and [131, Theorem 3.1.6].

Chapter 3

Convex monotone operators in the mixed topology and Γ-convergence

Throughout this chapter, let (X,d) be a complete separable metric space. Subsequently, for functions $f,g\colon X\to[-\infty,\infty)$ all order-related notations (sup, inf, max, min, lim sup, etc.) are understood w.r.t. the pointwise order $f\le g :\Leftrightarrow f(x)\le g(x)$ for all $x\in X$. In particular, we define $f\vee g:=\max\{f,g\}$, $f\wedge g:=\min\{f,g\}$, $f^+:=f\vee 0$ and $f^-:=-(f\wedge 0)$, where $f^-(x):=\infty$ if $f(x)=-\infty$. Furthermore, in order to relax the supremum norm and include unbounded functions with controlled growth behaviour at infinity, we fix a bounded continuous function $\kappa\colon X\to(0,\infty)$ and define

$$\|f\|_\kappa := \sup_{x\in X}|f(x)|\kappa(x)\in[0,\infty] \quad \text{for all } f\colon X\to[-\infty,\infty).$$

Let C_κ be the space of all continuous functions $f\colon X\to\mathbb{R}$ with $\|f\|_\kappa<\infty$ and U_κ be the set of all upper semicontinuous functions $f\colon X\to[-\infty,\infty)$ with $\|f^+\|_\kappa<\infty$. If $\kappa\equiv 1$, then $\|\cdot\|_\kappa$ is the usual supremum norm $\|\cdot\|_\infty$, C_κ coincides with the space C_b of all bounded continuous functions and U_κ is the set U_b of all upper semicontinuous functions which are bounded above by a real constant. Since the mapping

$$\mathrm{C}_\kappa\to\mathrm{C}_\mathrm{b},\ f\mapsto f\kappa$$

is an order-preserving linear isometric isomorphism, the space C_κ is a Banach lattice. Note that U_κ consists of all functions $f\colon X\to[-\infty,\infty)$ such that there exists a sequence $(f_n)_{n\in\mathbb{N}}\subset\mathrm{C}_\kappa$ that decreases pointwise to f. Indeed,

$$f(x)=\inf_{n\in\mathbb{N}}\sup_{y\in X}\frac{1}{\kappa(x)}\big(\max\{(f\kappa)(y),-n\}-n^2d(x,y)\big)$$

for all $f\in\mathrm{U}_\kappa$ and $x\in X$. We write $f_n\downarrow f$ if a sequence $(f_n)_{n\in\mathbb{N}}\subset\mathrm{U}_\kappa$ decreases pointwise to $f\in\mathrm{U}_\kappa$. A set $F\subset\mathrm{U}_\kappa$ is called bounded if $\sup_{f\in F}\|f\|_\kappa<\infty$ and bounded above if $\sup_{f\in F}\|f^+\|_\kappa<\infty$. Moreover, closed balls with radius $r\ge 0$ around zero are denoted by

$$B_{\mathrm{C}_\kappa}(r):=\{f\in\mathrm{C}_\kappa\colon \|f\|_\kappa\le r\} \quad\text{and}\quad B_{\mathrm{U}_\kappa}(r):=\{f\in\mathrm{U}_\kappa\colon \|f\|_\kappa\le r\}.$$

Instead of requiring norm convergence, we endow the space C_κ with the mixed topology between $\|\cdot\|_\kappa$ and the topology of uniform convergence on compacts sets which is the strongest locally convex topology on C_κ that coincides on $\|\cdot\|_\kappa$-bounded sets with the topology of uniform convergence on compact sets. It is well-known, see, e.g., [92, Proposition A.4], that a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa$ converges to $f \in \mathrm{C}_\kappa$ w.r.t. the mixed topology if and only if

$$\sup_{n\in\mathbb{N}} \|f_n\|_\kappa < \infty \quad \text{and} \quad \lim_{n\to\infty} \|f - f_n\|_{\infty,K} = 0$$

for all compact subsets $K \Subset X$, where $\|f\|_{\infty,K} := \sup_{x\in K} |f(x)|$. Similarly, for a family of functions $(f_s)_{s\geq 0} \subset \mathrm{C}_\kappa$ and $t \geq 0$, it holds $f_t = \lim_{s\to t} f_s$ if and only if $\sup_{|s-t|\leq\delta_0} \|f_s\|_\kappa < \infty$ for some $\delta_0 > 0$ and, for every $\varepsilon > 0$ and $K \Subset X$, there exists $\delta > 0$ with $\|f_s - f_t\|_{\infty,K} < \varepsilon$ for all $|s-t| \leq \delta$. Subsequently, if not stated otherwise, all limits in C_κ are understood w.r.t. the mixed topology and compact subsets are denoted by $K \Subset X$. We point out that the mixed topology is, in general, not metrizable but generated by the uncountable family of seminorms

$$p_{(a_n),(K_n)}(f) := \sup_{n\in\mathbb{N}} \sup_{x\in K_n} a_n |f(x)| \kappa(x),$$

where $(a_n)_{n\in\mathbb{N}} \subset (0,\infty)$ is a sequence with $a_n \to 0$ and $(K_n)_{n\in\mathbb{N}}$ is a sequence of compact sets $K_n \Subset X$. Moreover, the mixed topology has no neighbourhood of zero which is $\|\cdot\|_\kappa$-bounded or, equivalently, bounded in the mixed topology. Nevertheless, for monotone operators $\Phi\colon \mathrm{C}_\kappa \to \mathrm{C}_\kappa$, sequential continuity is equivalent to continuity and continuity is equivalent to continuity on $\|\cdot\|_\kappa$-bounded subsets, see [135]. We further remark that the mixed topology coincides with the Mackey topology of the dual pair $(\mathrm{C}_\kappa, \mathcal{M}_\kappa)$, where $\mathcal{M}_\kappa$ denotes the space of all countably additive signed Borel measures μ with

$$\int_X \frac{1}{\kappa(x)}\, |\mu|(\mathrm{d}x) < \infty,$$

where $|\mu|$ denotes the total variation measure of μ, see, e.g., [92]. In particular, the mixed topology belongs to the class of strict topologies, c.f. [83, 122, 152]. Finally, we remark that relative compactness can be characterized by means of Arzéla–Ascoli's theorem and dense subsets can be constructed by means of the Stone–Weierstraß theorem. For more details regarding strict and mixed topologies, we refer to [165, 166].

3.1 Γ-convergence

Following the works of Beer [16], Dal Maso [53] and Rockafellar and Wets [150], we gather some basics about Γ-convergence. We remark that, in [53] and [150], all results are formulated for extended real-valued lower semicontinuous functions. However, a function $f\colon X \to [-\infty,\infty)$ is upper semicontinuous if and only if $-f\colon X \to (-\infty,\infty]$ is lower semicontinuous, and all results immediately transfer to our setting.

Definition 3.1.1. For every sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{U}_\kappa$, which is bounded above, let

$$\Big(\Gamma\text{-}\limsup_{n\to\infty} f_n\Big)(x) := \sup\Big\{\limsup_{n\to\infty} f_n(x_n)\colon (x_n)_{n\in\mathbb{N}} \subset X \text{ with } x_n \to x\Big\} \in [-\infty,\infty)$$

for all $x \in X$. Moreover, we say that $f = \Gamma\text{-}\lim_{n\to\infty} f_n$ with $f \in \mathrm{U}_\kappa$ if, for every $x \in X$,

- $f(x) \geq \limsup_{n\to\infty} f_n(x_n)$ for every sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x$,
- $f(x) = \lim_{n\to\infty} f_n(x_n)$ for some sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x$.

For every $t \geq 0$ and family $(f_s)_{s\geq 0} \subset \mathrm{U}_\kappa$, which is bounded above, we define

$$\Gamma\text{-}\limsup_{s\to t} f_s := \sup\left\{ \Gamma\text{-}\limsup_{n\to\infty} f_{s_n} : s_n \to t \right\} \in \mathrm{U}_\kappa.$$

Furthermore, we say that $f = \Gamma\text{-}\lim_{s\to t} f_s$ with $f \in \mathrm{U}_\kappa$ if $f = \Gamma\text{-}\lim_{n\to\infty} f_{s_n}$ for all sequences $(s_n)_{n\in\mathbb{N}} \subset [0,\infty)$ with $s_n \to t$.

Lemma 3.1.2. *Let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{U}_\kappa$ and $(g_n)_{n\in\mathbb{N}} \subset \mathrm{U}_\kappa$ be bounded above and $f, g \in \mathrm{U}_\kappa$.*

(i) It holds $\Gamma\text{-}\limsup_{n\to\infty} f_n \in \mathrm{U}_\kappa$. Furthermore, $(f_n)_{n\in\mathbb{N}} \subset \mathrm{U}_\kappa$ has a Γ-convergent subsequence, i.e., $\Gamma\text{-}\lim_{k\to\infty} f_{n_k} \in \mathrm{U}_\kappa$ exists for a subsequence $(n_k)_{k\in\mathbb{N}}$.

(ii) We have $f = \Gamma\text{-}\lim_{n\to\infty} f_n$ if and only if every subsequence $(n_k)_{k\in\mathbb{N}}$ has another subsequence $(n_{k_l})_{l\in\mathbb{N}}$ with $f = \Gamma\text{-}\lim_{l\to\infty} f_{n_{k_l}}$.

(iii) If $f_n \downarrow f$, then $f = \Gamma\text{-}\lim_{n\to\infty} f_n$.

(iv) It holds $\Gamma\text{-}\limsup_{n\to\infty}(f_n+g_n) \leq \Gamma\text{-}\limsup_{n\to\infty} f_n + \Gamma\text{-}\limsup_{n\to\infty} g_n$. Moreover, we have $\Gamma\text{-}\limsup_{n\to\infty} f_n \leq \Gamma\text{-}\limsup_{n\to\infty} g_n$ if $f_n \leq g_n$ for all $n \in \mathbb{N}$.

(v) Assume that $f \in \mathrm{C}_\kappa$, $f_n \to f$ uniformly on compacts and $g = \Gamma\text{-}\lim_{n\to\infty} g_n$. Then, it holds $f + g = \Gamma\text{-}\lim_{n\to\infty}(f_n + g_n)$.

(vi) If $f = \Gamma\text{-}\limsup_{n\to\infty} f_n$ and $g \in \mathrm{C}_\kappa$, then $f \vee g = \Gamma\text{-}\limsup_{n\to\infty}(f_n \vee g)$.

(vii) It holds $(\Gamma\text{-}\limsup_{n\to\infty} f_n)(x) \geq \limsup_{n\to\infty} \sup_{y\in B(x,\delta_n)} f_n(y)$ for all $x \in X$ and $(\delta_n)_{n\in\mathbb{N}} \subset (0,\infty)$ with $\delta_n \to 0$.

Proof. Part (i) follows immediately from [53, Remark 4.11], [53, Theorem 4.16] and [53, Theorem 8.4]. Regarding part (ii) and (iii), we refer to [53, Proposition 8.3] and [53, Proposition 5.4]. art (iv) and (vii) are direct consequences of the definition of the Γ-limit superior. In order to show part (v), let $x \in X$ and $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence with $x_n \to x$ and $g(x) = \lim_{n\to\infty} g_n(x_n)$. For every $n \in \mathbb{N}$,

$$\big|(f+g)(x) - (f_n+g_n)(x_n)\big| \leq |f(x) - f(x_n)| + \sup_{y\in K} |f(y) - f_n(y)| + |g(x) - g_n(x_n)|,$$

where $K := \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact. Since $f_n \to f$ uniformly on compacts and f is continuous, the right-hand side converges to zero. We obtain $f + g \leq \Gamma\text{-}\lim_{n\to\infty}(f_n+g_n)$ and the reverse inequality follows from part (iv). It remains to show part (vi). The inequality $f \vee g \leq \Gamma\text{-}\limsup_{n\to\infty}(f_n \vee g)$ follows from part (iv). Let $(x_n)_{n\in\mathbb{N}} \subset X$ and $x \in X$ with $x_n \to x$. Continuity of g implies

$$\limsup_{n\to\infty}\, (f_n \vee g)(x) = \lim_{k\to\infty} (f_{n_k} \vee g)(x_k) = \Big(\limsup_{k\to\infty} f_{n_k}(x_{n_k})\Big) \vee g(x) \leq (f \vee g)(x),$$

where $(x_{n_k})_{k\in\mathbb{N}}$ is a suitable subsequence approximating the limit superior. □

The following geometric characterization of the Γ-limit superior is based on the work of Beer [16] and will be very useful to link Γ-upper semicontinuity with continuity from above. Let $f \in \mathrm{U}_\kappa$ and $\varepsilon > 0$. Following [16], the upper ε-parallel function to f is defined by

$$f^\varepsilon \colon X \to \mathbb{R},\ x \mapsto \frac{1}{\kappa(x)}\Big(\sup_{y\in B(x,\varepsilon)} \max\big\{f(y)\kappa(y), -\tfrac{1}{\varepsilon}\big\} + \varepsilon\Big),$$

where $B(x,\varepsilon) := \{y \in X \colon d(x,y) \le \varepsilon\}$ denotes the closed ball. If closed bounded sets in X are compact (e.g., if $X = \mathbb{R}^d$ or X is compact), one can show that f^ε is upper semicontinuous, cf. the proof of [16, Lemma 1.3]. For a general metric space (X,d), however, the upper semicontinuity of f^ε cannot be guaranteed. For this reason, we consider the upper semicontinuous envelope of f^ε defined by

$$\overline{f}^\varepsilon \colon X \to \mathbb{R},\ x \mapsto \limsup_{y\to x} f^\varepsilon(y) = \inf_{\delta>0} \sup_{y\in B(x,\delta)} f^\varepsilon(y).$$

Note that, for all $x \in X$,

$$\overline{f}^\varepsilon(x) = \sup\Big\{\limsup_{n\to\infty} f^\varepsilon(x_n) \colon (x_n)_{n\in\mathbb{N}} \subset X \text{ with } x_n \to x\Big\} = \Gamma\text{-}\limsup_{n\to\infty} f^\varepsilon,$$

where in the last expression f^ε stands for the constant sequence $(f^\varepsilon)_{n\in\mathbb{N}}$.

Lemma 3.1.3.

(i) For every $f \in \mathrm{U}_\kappa$ and $\varepsilon > 0$,

$$f^\varepsilon \le \overline{f}^\varepsilon \le \inf_{\varepsilon'>\varepsilon} f^{\varepsilon'} \quad \text{and} \quad -\tfrac{1}{\varepsilon} \le \overline{f}^\varepsilon\kappa \le \|f^+\|_\kappa + \varepsilon. \tag{3.1}$$

Furthermore, it holds $\overline{f}^\varepsilon \in \mathrm{U}_\kappa$ for all $\varepsilon > 0$ and $\overline{f}^\varepsilon \downarrow f$ as $\varepsilon \downarrow 0$.

(ii) Let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{U}_\kappa$ be bounded above and $f \in \mathrm{U}_\kappa$. Then, $\Gamma\text{-}\limsup_{n\to\infty} f_n \le f$ if and only if, for every $\varepsilon > 0$ and $K \Subset X$, there exists $n_0 \in \mathbb{N}$ with

$$f_n(x) \le \overline{f}^\varepsilon(x) \quad \text{for all } x \in K \text{ and } n \ge n_0. \tag{3.2}$$

Proof. First, we show inequality (3.1). Let $f \in \mathrm{U}_\kappa$, $\varepsilon > 0$ and $x \in X$. For every $\varepsilon' > \varepsilon$,

$$\begin{aligned}
f^\varepsilon(x) \le \overline{f}^\varepsilon(x) &\le \sup_{y\in B(x,\varepsilon'-\varepsilon)} f^\varepsilon(y) \\
&= \sup_{y\in B(x,\varepsilon'-\varepsilon)} \frac{1}{\kappa(y)}\Big(\sup_{z\in B(y,\varepsilon)} \max\big\{f(z)\kappa(z), -\tfrac{1}{\varepsilon}\big\} + \varepsilon\Big) \\
&\le \frac{c_{\varepsilon'}(x)}{\kappa(x)}\Big(\sup_{y\in B(x,\varepsilon'-\varepsilon)} \sup_{z\in B(y,\varepsilon)} \max\big\{f(z)\kappa(z), -\tfrac{1}{\varepsilon}\big\} + \varepsilon\Big) \\
&\le \frac{c_{\varepsilon'}(x)}{\kappa(x)}\Big(\sup_{z\in B(x,\varepsilon')} \max\big\{f(z)\kappa(z), -\tfrac{1}{\varepsilon'}\big\} + \varepsilon'\Big) = c_{\varepsilon'}(x) f^{\varepsilon'}(x),
\end{aligned}$$

where $c_{\varepsilon'}(x) := \sup_{y\in B(x,\varepsilon'-\varepsilon)} \frac{\kappa(x)}{\kappa(y)}$. Continuity of κ implies $c_{\varepsilon'}(x) \downarrow 1$ as $\varepsilon' \downarrow \varepsilon$. Hence, taking the infimum over $\varepsilon' > \varepsilon$ in the previous estimate yields the first part of inequality (3.1). Furthermore, we can estimate

$$\left(\overline{f}^{\varepsilon}\kappa\right)^{+}(x) \leq \inf_{\varepsilon'>\varepsilon} \left(f^{\varepsilon'}\kappa\right)^{+}(x) \leq \inf_{\varepsilon'>\varepsilon} \left(\sup_{y\in X} (f\kappa)^{+}(y) + \varepsilon'\right) = \|f^{+}\|_{\kappa} + \varepsilon.$$

In particular, we obtain $\overline{f}^{\varepsilon} \in \mathrm{U}_\kappa$, because $\overline{f}^{\varepsilon}$ is upper semicontinuous by definition.

Second, we show that $\overline{f}^{\delta} \downarrow f$ as $\delta \downarrow 0$. Due to inequality (3.1) it is sufficient to prove $f^{\delta} \downarrow f$ as $\delta \downarrow 0$. Since $f\kappa$ is upper semicontinuous, for every $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $(f\kappa)(y) \leq (f\kappa)(x) + \varepsilon$ for all $y \in B(x,\delta)$. We obtain

$$f(x) \leq f^{\delta}(x) = \frac{1}{\kappa(x)} \sup_{y\in B(x,\delta)} \max\left\{(f\kappa)(y), -\tfrac{1}{\delta}\right\} + \delta \leq \max\left\{f(x), -\tfrac{1}{\delta}\right\} + \delta + \varepsilon.$$

This implies $f(x) \leq \inf_{\delta>0} f^{\delta}(x) \leq f(x) + \varepsilon \downarrow f(x)$ as $\varepsilon \downarrow 0$.

Third, let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{U}_\kappa$ be bounded above and $f \in \mathrm{U}_\kappa$ with $\Gamma\text{-}\limsup_{n\to\infty} f_n \leq f$. We follow the proof of [16, Lemma 1.5] to verify inequality (3.2). Let $K \Subset X$ and $\varepsilon > 0$. Since $\Gamma\text{-}\limsup_{n\to\infty} f_n \leq f$ and $\kappa > 0$ is continuous, we obtain

$$\limsup_{n\to\infty} (f_n\kappa)(x_n) \leq (f\kappa)(x) \quad \text{for all } x \in K \text{ and } (x_n)_{n\in\mathbb{N}} \subset X \text{ with } x_n \to x.$$

Hence, for every $x \in K$, there exist $n_x \in \mathbb{N}$ and $r_x \in (0,\varepsilon)$ such that

$$(f_n\kappa)(y) \leq \max\left\{(f\kappa)(x), -\tfrac{1}{\varepsilon}\right\} + \varepsilon \quad \text{for all } n \geq n_x \text{ and } y \in B(x, r_x).$$

By compactness of K, we can choose $x_1, \ldots, x_k \in K$ with $K \subset \bigcup_{i=1}^{k} B(x_i, r_{x_i})$. Define $n_0 := n_{x_1} \vee \ldots \vee n_{x_k}$. Let $x \in K$ and $i \in \{1, \ldots, k\}$ with $d(x, x_i) < r_{x_i} < \varepsilon$. We obtain

$$f_n(x) \leq \frac{1}{\kappa(x)}\Big(\max\left\{(f\kappa)(x_i), -\tfrac{1}{\varepsilon}\right\} + \varepsilon\Big) \leq \overline{f}^{\varepsilon}(x) \quad \text{for all } n \geq n_0.$$

Fourth, let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{U}_\kappa$ be bounded above and $f \in \mathrm{U}_\kappa$ such that inequality (3.2) is valid. Let $x \in X$ and $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x$. Since $K := \{x_n \colon n \in \mathbb{N}\} \cup \{x\}$ is compact, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $f_n(x_n) \leq \overline{f}^{\varepsilon}(x_n)$ for all $n \geq n_0$. We obtain $\limsup_{n\to\infty} f_n(x_n) \leq \overline{f}^{\varepsilon}(x)$ for all $\varepsilon > 0$, because $\overline{f}^{\varepsilon}$ is upper semicontinuous. Hence, part (i) implies $\Gamma\text{-}\limsup_{n\to\infty} f_n \leq \inf_{\varepsilon>0} \overline{f}^{\varepsilon} = f$. □

We conclude this section by showing that the definition of $\overline{f}^{\varepsilon}$ simplifies if (X,d) satisfies an additional geometric property.

Lemma 3.1.4. *Assume that (X,d) has midpoints, i.e., for every $x, z \in X$ and $\lambda \in [0,1]$, there exists $y_\lambda \in X$ with*

$$d(x, y_\lambda) = \lambda d(x,z) \quad \textit{and} \quad d(y_\lambda, z) = (1-\lambda) d(x,z).$$

Then, it holds $\overline{f}^{\varepsilon} = \inf_{\varepsilon'>\varepsilon} f^{\varepsilon'}$ for all $f \in \mathrm{U}_\kappa$ and $\varepsilon > 0$.

Proof. Let $x \in X$ and $\varepsilon' > \varepsilon$. Since (X,d) has midpoints, for every $z \in B(x,\varepsilon')$ there exists $y \in X$ with $d(x,y) \le \varepsilon' - \varepsilon$ and $d(y,z) \le \varepsilon$. Hence, we can estimate

$$\begin{aligned} f^{\varepsilon'}(x) &= \frac{1}{\kappa(x)} \sup_{z\in B(x,\varepsilon')} \Big(\max\big\{ f(z)\kappa(z), -\tfrac{1}{\varepsilon'}\big\} + \varepsilon' \Big) \\ &= \frac{1}{\kappa(x)} \sup_{y\in B(x,\varepsilon'-\varepsilon)} \sup_{z\in B(y,\varepsilon)} \Big(\max\big\{ f(z)\kappa(z), -\tfrac{1}{\varepsilon'}\big\} + \varepsilon' \Big) \\ &\le c_{\varepsilon'}(x) \sup_{y\in B(x,\varepsilon'-\varepsilon)} \frac{1}{\kappa(y)} \sup_{z\in B(y,\varepsilon)} \Big(\max\big\{ f(z)\kappa(z), -\tfrac{1}{\varepsilon'}\big\} + \varepsilon' \Big), \end{aligned}$$

where $c_{\varepsilon'}(x) := \sup_{y\in B(x,\varepsilon'-\varepsilon)} \frac{\kappa(y)}{\kappa(x)}$. Continuity of κ implies $c_{\varepsilon'} \downarrow 1$ as $\varepsilon' \downarrow \varepsilon$. We obtain

$$\inf_{\varepsilon'>\varepsilon} f^{\varepsilon'}(x) \le \inf_{\varepsilon'>\varepsilon} \sup_{y\in B(x,\varepsilon'-\varepsilon)} f^{\varepsilon}(y) = \overline{f}^{\varepsilon}(x).$$

The reverse estimate follows from inequality (3.1). □

3.2 Convex monotone functionals and continuity from above

Denote by ca_κ^+ the set of all Borel measures $\mu\colon \mathcal{B}(X) \to [0,\infty]$ with $\int_X \frac{1}{\kappa}\,\mathrm{d}\mu < \infty$. Let $\varphi\colon \mathrm{C}_\kappa \to \mathbb{R}$ be a convex monotone functional with $\varphi(0) = 0$ and define

$$\varphi^*\colon \mathrm{ca}_\kappa^+ \to [0,\infty],\ \mu \mapsto \sup_{f\in \mathrm{C}_\kappa} \big(\mu f - \varphi(f)\big), \quad \text{where} \quad \mu f := \int_X f\,\mathrm{d}\mu.$$

Let B_κ be the space of all Borel measurable functions $f\colon X \to [-\infty,\infty)$ such that $\|f^+\|_\kappa < \infty$ and denote by $B_{\mathrm{B}_\kappa}(r) := \{f \in \mathrm{B}_\kappa \colon \|f\|_\kappa \le r\}$ the closed ball with radius $r \ge 0$ around zero. Based on the results from [12], we obtain the following extension and dual representation result.

Theorem 3.2.1. *Let $\varphi\colon \mathrm{C}_\kappa \to \mathbb{R}$ be a convex monotone functional with $\varphi(0) = 0$ which is continuous from above. Then, the following statements are valid:*

(i) For every $r \ge 0$, there exists a $\sigma(\mathrm{ca}_\kappa^+, \mathrm{C}_\kappa)$-compact convex set $M_r \subset \mathrm{ca}_\kappa^+$ with

$$\varphi(f) = \max_{\mu\in M_r} \big(\mu f - \varphi^*(\mu)\big) \quad \text{for all } f \in B_{\mathrm{C}_\kappa}(r).$$

Moreover, one can choose $M_r := \{\mu \in \mathrm{ca}_\kappa^+ \colon \varphi^*(\mu) \le \varphi(2r/\kappa) - 2\varphi(-r/\kappa)\}$.

(ii) Define $\varphi_1\colon \mathrm{U}_\kappa \to [-\infty,\infty)$, $f \mapsto \inf\{\varphi(g)\colon g \in \mathrm{C}_\kappa,\ g \ge f\}$. Then, the functional φ_1 is convex, monotone and the unique extension of φ which is continuous from above. In addition, φ_1 admits the dual representation

$$\varphi_1(f) = \max_{\mu\in M_r} \big(\mu f - \varphi^*(\mu)\big) \quad \text{for all } r \ge 0 \text{ and } f \in B_{\mathrm{U}_\kappa}(r).$$

(iii) Define $\varphi_2\colon \mathrm{B}_\kappa \to [-\infty,\infty)$, $f \mapsto \lim_{c\to\infty} \sup_{\mu\in \mathrm{ca}_\kappa^+} \big(\mu\big(\max\{f, -\frac{c}{\kappa}\}\big) - \varphi^*(\mu)\big)$. Then, the functional φ_2 is convex, monotone and an extension of φ. In addition, φ_2 admits the dual representation

$$\varphi_2(f) = \sup_{\mu\in M_r} \big(\mu f - \varphi^*(\mu)\big) \quad \text{for all } r \ge 0 \text{ and } f \in B_{\mathrm{B}_\kappa}(r),$$

In particular, for every $\varepsilon > 0$ and $r \ge 0$, there exists $K \Subset X$ with $\varphi_2\big(\frac{r}{\kappa}\mathbb{1}_{K^c}\big) < \varepsilon$.

Proof. First, we apply [12, Theorem 2.2] to obtain

$$\varphi(f) = \max_{\mu \in \mathrm{ca}_\kappa^+} \big(\mu f - \varphi^*(\mu)\big) \quad \text{for all } f \in \mathrm{C}_\kappa.$$

Let $r \geq 0$ and $f \in B_{\mathrm{C}_\kappa}(r)$. Choose $\mu \in \mathrm{ca}_\kappa^+$ with $\varphi(f) = \mu f - \varphi^*(\mu)$. It follows from the definition of φ^* and the monotonicity of φ that

$$\mu \tfrac{2r}{\kappa} - \varphi\big(\tfrac{2r}{\kappa}\big) \leq \varphi^*(\mu) = \mu f - \varphi(f) \leq \mu \tfrac{r}{\kappa} - \varphi\big(-\tfrac{r}{\kappa}\big).$$

We obtain $\mu \frac{r}{\kappa} \leq \varphi(\frac{2r}{\kappa}) - \varphi(-\frac{r}{\kappa})$ and therefore $\varphi^*(\mu) \leq \varphi\big(\frac{2r}{\kappa}\big) - 2\varphi\big(-\frac{r}{\kappa}\big)$. Hence,

$$\varphi(f) = \max_{\mu \in M_r} \big(\mu f - \varphi^*(\mu)\big) \quad \text{for all } f \in B_{\mathrm{C}_\kappa}(r),$$

where $M_r := \big\{\mu \in \mathrm{ca}_\kappa^+ \colon \varphi^*(\mu) \leq \varphi\big({}^{2r}/_{\kappa}\big) - 2\varphi\big(-{}^{r}/_{\kappa}\big)\big\}$. Moreover, the set M_r is convex and $\sigma(\mathrm{ca}_\kappa^+, \mathrm{C}_\kappa)$-compact, see [12, Theorem 2.2].

Second, by monotonicity of φ, the functional φ_1 is monotone and an extension of φ. We show that $\varphi_1(f) = \lim_{n\to\infty} \varphi(f_n)$ for all sequences $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa$ and $f \in \mathrm{U}_\kappa$ with $f_n \downarrow f$. By definition of the infimum, there exists a sequence $(g_k)_{k\in\mathbb{N}} \subset \mathrm{C}_\kappa$ such that $\varphi(g_k) \to \varphi_1(f)$ as $k \to \infty$. Let $g_n^k := f_n \vee g_k$ for all $k, n \in \mathbb{N}$. Since $g_n^k \downarrow g_k$ as $n \to \infty$, and φ is monotone and continuous from above, we obtain

$$\lim_{n\to\infty} \varphi(f_n) \leq \lim_{n\to\infty} \varphi(g_n^k) = \varphi(g_k) \quad \text{for all } k \in \mathbb{N}.$$

The monotonicity of φ_1 implies $\varphi_1(f) \leq \lim_{n\to\infty} \varphi(f_n) \leq \lim_{k\to\infty} \varphi(g_k) = \varphi_1(f)$. In particular, it follows that φ_1 is convex. Indeed, let $f, g \in \mathrm{U}_\kappa$ and $\lambda \in [0,1]$. Since $\mathrm{U}_\kappa = (\mathrm{C}_\kappa)_\delta$ and C_κ is directed downwards, there exist sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in C_κ with $f_n \downarrow f$ and $g_n \downarrow g$. We obtain

$$\begin{aligned}\varphi_1(\lambda f + (1-\lambda)g) &= \lim_{n\to\infty} \varphi(\lambda f_n + (1-\lambda)g_n) \leq \lim_{n\to\infty} \big(\lambda\varphi(f_n) + (1-\lambda)g_n\big) \\ &= \lambda\varphi_1(f) + (1-\lambda)\varphi_1(g).\end{aligned}$$

Third, we show that φ_1 is continuous from above. Let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{U}_\kappa$ and $f \in \mathrm{U}_\kappa$ with $f_n \downarrow f$. Since $\mathrm{U}_\kappa = (\mathrm{C}_\kappa)_\delta$, for every $k \in \mathbb{N}$, there exists a sequence $(f_k^n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa$ with $f_k^n \downarrow f_k$ as $n \to \infty$. Define $\tilde{f}_n := \min\{f_1^n, \ldots, f_n^n\} \in \mathrm{C}_\kappa$ for all $n \in \mathbb{N}$. It holds

$$\begin{aligned}\tilde{f}_{n+1} &= \min\{f_1^{n+1}, \ldots, f_n^{n+1}, f_{n+1}^{n+1}\} \leq \min\{f_1^n, \ldots, f_n^n\} = \tilde{f}_n \quad \text{for all } n \in \mathbb{N}, \\ f_n &= \min\{f_1, \ldots, f_n\} \leq \min\{f_1^n, \ldots, f_n^n\} = \tilde{f}_n \quad \text{for all } n \in \mathbb{N}, \\ \tilde{f}_n &= \min\{f_1^n, \ldots, f_k^n, \ldots, f_n^n\} \leq f_k^n \quad \text{for all } k, n \in \mathbb{N} \text{ with } k \leq n.\end{aligned}$$

We obtain $f = \lim_{n\to\infty} f_n \leq \lim_{n\to\infty} \tilde{f}_n \leq \lim_{n\to\infty} f_k^n = f_k$ for all $k \in \mathbb{N}$. Hence, it follows from the monotonicity of φ_1 and the second part of the proof that

$$\varphi_1(f) \leq \lim_{n\to\infty} \varphi_1(f_n) \leq \lim_{n\to\infty} \varphi(\tilde{f}_n) = \varphi_1(f).$$

We have shown that φ_1 is continuous from above and thus [12, Theorem 2.2] implies

$$\varphi_1(f) = \max_{\mu \in \mathrm{ca}_\kappa^+} \big(\mu f - \varphi^*(\mu)\big)$$

for all bounded $f \in \mathrm{U}_\kappa$. By the same arguments as in the first step, the maximum in the previous equation can be taken over the set M_r for all $r \geq 0$ and $f \in B_{\mathrm{U}_\kappa}(r)$. The uniqueness of φ_1 as extension, which is continuous from above, follows from $\mathrm{U}_\kappa = (\mathrm{C}_\kappa)_\delta$.

Fourth, the functional φ_2 is clearly convex and monotone. Since φ_1 is continuous from above, we obtain

$$\varphi_1(f) = \lim_{c\to\infty} \varphi_1\big(\max\big\{f, -\tfrac{c}{\kappa}\big\}\big) = \varphi_2(f) \quad \text{for all } f \in \mathrm{U}_\kappa.$$

Similar to the first part of this proof, it follows that the supremum in the definition of φ_2 can be taken over M_r' for all $r \geq 0$ and $f \in B_{\mathrm{B}_\kappa}(r)$. By [12, Theorem 2.2], the set M_r' is $\sigma(\mathrm{ca}_\kappa^+, \mathrm{C}_\kappa)$-compact and convex. The last statement follows from $\varphi^* \geq 0$ and the fact that, by Prokhorov's theorem, the set $\{\mu_\kappa \colon \mu \in M_r'\}$ is tight for all $r \geq 0$, where $\mu_\kappa(A) := \int_A \frac{1}{\kappa}\,\mathrm{d}\mu$ for all $A \in \mathcal{B}(X)$. □

Let $\varphi\colon \mathrm{C}_\kappa \to \mathbb{R}$ be a convex monotone functional with $\varphi(0) = 0$ which is continuous from above at zero, i.e., $\varphi(f_n) \downarrow 0$ for all $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa$ with $f_n \downarrow 0$. Then, it follows from the proof of [12, Theorem 2.2] that φ is continuous from above, i.e., $\varphi(f_n) \downarrow \varphi(f)$ for all $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa$ and $f \in \mathrm{C}_\kappa$ with $f_n \downarrow f$. However, if we replace C_κ by U_κ, this statement does not remain valid, because U_κ is not a vector space. The following result is crucial for the proof of the main result Theorem 4.2.8 below.

Theorem 3.2.2. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of functionals $\varphi_n\colon \mathrm{C}_\kappa \to \mathbb{R}$ which satisfy the following conditions:*

- *φ_n is convex and monotone with $\varphi_n(0) = 0$ for all $n \in \mathbb{N}$,*
- *$\sup_{n\in\mathbb{N}} \sup_{f\in B_{\mathrm{C}_\kappa}(r)} |\varphi_n(f)| < \infty$ for all $r \geq 0$,*
- *$\sup_{n\in\mathbb{N}} \varphi_n(f_k) \downarrow 0$ as $k \to \infty$ for all $(f_k)_{k\in\mathbb{N}} \subset \mathrm{C}_\kappa$ with $f_k \downarrow 0$.*

For every $n \in \mathbb{N}$, Theorem 3.2.1 yields that the functional $\varphi_n\colon \mathrm{C}_\kappa \to \mathbb{R}$ has a unique extension $\varphi_n\colon \mathrm{U}_\kappa \to [-\infty, \infty)$ which is continuous from above. Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be bounded sequences in U_κ. Then,

$$\limsup_{n\to\infty} \varphi_n(f_n + g_n) \leq \limsup_{n\to\infty} \varphi_n(f + g_n), \quad \textit{where} \quad f := \Gamma\text{-}\limsup_{n\to\infty} f_n.$$

Proof. First, we show that $\limsup_{n\to\infty} \varphi_n(f_n + g_n) \leq \limsup_{n\to\infty} \varphi_n(\overline{f}^\varepsilon + g_n)$ for all $\varepsilon \in (0, 1]$. Since the functionals $(\varphi_n)_{n\in\mathbb{N}}$ are uniformly bounded, the functional

$$\varphi\colon \mathrm{C}_\kappa \to \mathbb{R},\ f \mapsto \sup_{n\in\mathbb{N}} \varphi_n(f)$$

is well-defined, convex, monotone and satisfies $\varphi(0) = 0$. Furthermore, it follows from Theorem 3.2.1 and $\varphi^* \leq \varphi_n^*$ that

$$\varphi_n(f) = \max_{\mu\in M_r} \big(\mu f - \varphi_n^*(\mu)\big) \quad \text{for all } n \in \mathbb{N},\ r \geq 0 \text{ and } f \in B_{\mathrm{U}_\kappa}(r),$$

where $M_r := \{\mu \in \mathrm{ca}_\kappa^+ \colon \varphi^*(\mu) \leq \sup_{n\in\mathbb{N}} |\varphi_n({}^{2r}\!/\!_\kappa) - 2\varphi_n({}^{-r}\!/\!_\kappa)|\}$. In the sequel, we fix $r := \sup_{n\in\mathbb{N}}(\|f_n\|_\kappa + \|g_n\|_\kappa + 1)$ and $M := M_r$. Let $\varepsilon > 0$. Since φ is continuous from above, by [12, Theorem 2.2] and Prokhorov's theorem, the set $\{\mu_\kappa \colon \mu \in M\}$ is tight, where $\mu_\kappa(A) := \int_A \frac{1}{\kappa}\,\mathrm{d}\mu$ for all $A \in \mathcal{B}(X)$. Hence, there exists $K \Subset X$ with $\sup_{\mu\in M} \mu_\kappa(K^c)(r + 1/\varepsilon) < \varepsilon$. Moreover, by Lemma 3.1.3, we can choose $n_0 \in \mathbb{N}$ with

$$f_n(x) \le \overline{f}^\varepsilon(x) \quad \text{for all } x \in K \text{ and } n \ge n_0.$$

For every $n \ge n_0$, it follows from $\overline{f}^\varepsilon \ge -\frac{1}{\varepsilon\kappa}$ that

$$\begin{aligned}
\varphi_n(f_n + g_n) &= \max_{\mu\in M} \big(\mu(f_n + g_n) - \varphi_n^*(\mu)\big) \\
&\le \max_{\mu\in M} \Big(\mu\Big(\overline{f}^\varepsilon + g_n + \big(\tfrac{r+1/\varepsilon}{\kappa}\big)\mathbb{1}_{K^c}\Big) - \varphi_n^*(\mu)\Big) \\
&\le \max_{\mu\in M} \big(\mu(\overline{f}^\varepsilon + g_n) - \varphi_n^*(\mu)\big) + \varepsilon \le \varphi_n(\overline{f}^\varepsilon + g_n) + \varepsilon.
\end{aligned}$$

We obtain $\limsup_{n\to\infty} \varphi_n(f_n + g_n) \le \limsup_{n\to\infty} \varphi_n(\overline{f}^\varepsilon + g_n) + \varepsilon$ for all $\varepsilon > 0$.

Second, we show $\inf_{\varepsilon\in(0,1]} \limsup_{n\to\infty} \varphi_n(\overline{f}^\varepsilon + g_n) \le \limsup_{n\to\infty} \varphi_n(f + g_n)$. Using the dual representation from the first part, we obtain

$$\begin{aligned}
\inf_{\varepsilon\in(0,1]} \limsup_{n\to\infty} \varphi_n(\overline{f}^\varepsilon + g_n) &= \inf_{n\in\mathbb{N}} \inf_{\varepsilon\in(0,1]} \max_{\mu\in M} \sup_{k\ge n} \big(\mu(\overline{f}^\varepsilon + g_k) - \varphi_k^*(\mu)\big) \\
&= \inf_{n\in\mathbb{N}} \inf_{\varepsilon\in(0,1]} \max_{\mu\in M} \big(\mu\overline{f}^\varepsilon - \alpha_n(\mu)\big),
\end{aligned}$$

where $\alpha_n(\mu) := \inf_{k\ge n}(\varphi_k^*(\mu) - \mu g_k)$. For fixed $n \in \mathbb{N}$, we want to interchange the maximum over $\mu \in M$ with the infimum over $\varepsilon \in (0,1]$ by using [72, Theorem 2]. To do so, we have to replace α_n by a function which is convex and lower semicontinuous. It holds $\inf_{\mu\in M} \alpha_n(\mu) > -\infty$, because $\varphi_k^* \ge 0$ and $(g_k)_{k\in\mathbb{N}}$ is bounded. Hence, we can define $\overline{\alpha}_n\colon M \to \mathbb{R}$ as the lower semicontinuous convex hull of α_n, i.e., the supremum over all lower semicontinuous convex functions which are dominated by α_n. Fenchel–Moreau's theorem, [72, Theorem 2] and Lemma 3.1.3 imply

$$\begin{aligned}
&\inf_{n\in\mathbb{N}} \inf_{\varepsilon\in(0,1]} \max_{\mu\in M} \big(\mu\overline{f}^\varepsilon - \alpha_n(\mu)\big) = \inf_{n\in\mathbb{N}} \inf_{\varepsilon\in(0,1]} \max_{\mu\in M} \big(\mu\overline{f}^\varepsilon - \overline{\alpha}_n(\mu)\big) \\
&= \inf_{n\in\mathbb{N}} \max_{\mu\in M} \big(\mu f - \overline{\alpha}_n(\mu)\big) = \inf_{n\in\mathbb{N}} \max_{\mu\in M} \big(\mu f - \alpha_n(\mu)\big) = \limsup_{n\to\infty} \varphi_n(f + g_n). \qquad \square
\end{aligned}$$

The following result shows that, for a family of convex monotone functionals, uniform continuity from above is equivalent to uniformly equicontinuity on bounded sets w.r.t. the mixed topology. Similar results can be found in [135]. For later reference, we allow for a family $(X_i)_{i\in I}$ of closed subsets $X_i \subset X$ and functionals $\varphi_i\colon \mathrm{C}_\kappa(X_i) \to \mathbb{R}$, where $\mathrm{C}_\kappa(X_i)$ consists of all continuous functions $f\colon X_i \to \mathbb{R}$ with $\sup_{x\in X_i} |f(x)|\kappa(x) < \infty$. Due to the possibility of restricting functions, we consider $\mathrm{C}_\kappa(X)$ as a subset of $\mathrm{C}_\kappa(X_i)$ and define $\varphi_i(f) := \varphi_i(f|_{X_i})$ for all $f \in \mathrm{C}_\kappa(X)$. Moreover, we define $K_i := K \cap X_i$ for all $K \Subset X$ and $i \in I$.

Lemma 3.2.3. *Let $(X_i)_{\in I}$ be a family of closed sets $X_i \subset X$ and $(\varphi_i)_{i\in I}$ be a family of convex monotone functionals $\varphi_i\colon \mathrm{C}_\kappa(X_i) \to \mathbb{R}$ with $\varphi_i(0) = 0$ for all $i \in I$ and*

$$\sup_{i\in I} \sup_{f\in B_{\mathrm{C}_\kappa(X_i)}(r)} |\varphi_i(f)| < \infty \quad \textit{for all } r \ge 0.$$

Then the following two statements are equivalent:

(i) It holds $\sup_{i\in I}\varphi_i(f_n)\downarrow 0$ *for all sequences* $(f_n)_{n\in\mathbb{N}}\subset \mathrm{C}_\kappa(X)$ *with* $f_n\downarrow 0$.

(ii) For every $\varepsilon>0$ *and* $r\geq 0$*, there exist* $c\geq 0$ *and* $K\Subset X$ *with*

$$\sup_{i\in I}|\varphi_i(f)-\varphi_i(g)|\leq c\|f-g\|_{\infty,K_i}+\varepsilon$$

for all $i\in I$ *and* $f,g\in B_{\mathrm{C}_\kappa(X_i)}(r)$.

Proof. First, we assume that condition (i) is satisfied. Let $\varepsilon>0$ and $r\geq 0$. Since the functional $\varphi_i\colon \mathrm{C}_\kappa(X)\to\mathbb{R}$ is continuous from above, Theorem 3.2.1 implies

$$\varphi_i(f)=\max_{\mu\in M_i}\big(\mu f-\varphi_i^*(\mu)\big)\quad\text{for all } i\in I \text{ and } f\in B_{\mathrm{C}_\kappa(X)}(r), \tag{3.3}$$

where $M_i:=\{\mu\in\mathrm{ca}_\kappa^+(X)\colon \varphi_i^*(\mu)\leq\varphi_i(\frac{2r}{\kappa})-2\varphi_i(-\frac{r}{\kappa})\}$. Furthermore, it holds

$$\mu(X_i^c)=0\quad\text{for all } \mu\in M_i \text{ and } i\in I. \tag{3.4}$$

Indeed, by contradiction, we suppose that there exists $\mu\in M_i$ with $\mu(X_i^c)>0$. Then, due to Ulam's theorem, there exists $K\Subset X_i^c$ with $\mu(K)>0$. Moreover, by Urysohn's lemma, there exists a continuous function $f\colon X\to[0,1]$ with $f(x)=0$ for all $x\in X_i$ and $f(x)=1$ for all $x\in K$. We use $\varphi_i(\lambda f)=\varphi_i(\lambda f|_{X_i})=\varphi_i(0)=0$ to conclude

$$\varphi_i^*(\mu)\geq\sup_{\lambda\geq 0}\big(\mu(\lambda f)-\varphi_i(\lambda f)\big)\geq\sup_{\lambda\geq 0}\lambda\mu(K)=\infty.$$

This contradicts the fact that $\mu\in M_i$ guarantees $\varphi_i^*(\mu)<\infty$. Next, we show that

$$\varphi_i(f)=\max_{\mu\in M_i}\big(\mu f-\varphi_i^*(\mu)\big)\quad\text{for all } i\in I \text{ and } f\in B_{\mathrm{C}_\kappa(X_i)}(r), \tag{3.5}$$

where $\mu f=\int_{X_i}f\,\mathrm{d}\mu$ is well-defined due to equation (3.4). Let $i\in I$ and $f\in B_{\mathrm{C}_\kappa(X_i)}(r)$. Since $X_i\subset X$ is closed and $f\kappa\in\mathrm{C_b}(X_i)$, by Tietze's extension theorem, there exists a function $\tilde g\in\mathrm{C_b}(X)$ with $(f\kappa)(x)=\tilde g(x)$ for all $x\in X_i$ and $\|f\kappa\|_{\infty,X_i}=\|\tilde g\|_\infty$. Hence, the function $g:=\frac{1}{\kappa}\tilde g\in\mathrm{C}_\kappa(X)$ satisfies $f(x)=g(x)$ for all $x\in X_i$ and $\|f\|_{\kappa,X_i}=\|g\|_\kappa$. It follows from equation (3.3) and equation (3.4) that

$$\varphi_i(f)=\varphi_i(g)=\max_{\mu\in M_i}\big(\mu g-\varphi_i^*(\mu)\big)=\max_{\mu\in M_i}\big(\mu f-\varphi_i^*(\mu)\big).$$

Condition (i) guarantees that the convex monotone functional

$$\varphi\colon \mathrm{C}_\kappa(X)\to\mathbb{R},\ f\mapsto\sup_{i\in I}\varphi_i(f)$$

is continuous from above with $\varphi(0)=0$. It follows from [12, Theorem 2.2] that

$$M:=\Big\{\mu\in\mathrm{ca}_\kappa^+(X)\colon\varphi^*(\mu)\leq\sup_{i\in I}\Big(\varphi_i\big(\tfrac{2r}{\kappa}\big)-2\varphi_i\big(-\tfrac{r}{\kappa}\big)\Big)\Big\}$$

is $\sigma(\mathrm{ca}_\kappa^+(X),\mathrm{C}_\kappa(X))$-relatively compact and thus Prokhorov's theorem guarantees the existence of $K\Subset X$ with $\sup_{\mu\in M}\int_{K^c}\frac{1}{\kappa}\,\mathrm{d}\mu\leq\frac{\varepsilon}{2r}$. For every $i\in I$ and $f,g\in B_{\mathrm{C}_\kappa(X_i)}(r)$, we use equation (3.5) and $M_i\subset M$ to obtain

$$|\varphi_i(f)-\varphi_i(g)|\leq\sup_{\mu\in M_i}|\mu f-\mu g|$$

$$\begin{aligned}&\le \sup_{\mu\in M_i}\left(\int_K |f-g|\,\mathrm{d}\mu + \int_{K^c}|f-g|\,\mathrm{d}\mu\right)\\ &\le \sup_{\mu\in M_i}\left(\mu(K)\|f-g\|_{\infty,K_i} + \int_{K^c}\frac{2r}{\kappa}\,\mathrm{d}\mu\right)\\ &\le c\|f-g\|_{\infty,K_i}+\varepsilon,\end{aligned}$$

where $c := \sup_{\mu\in M}\mu(K) \le \varphi(1) + \sup_{\mu\in M}\varphi^*(\mu) < \infty$.

Second, we assume that condition (ii) is satisfied. Let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa(X)$ be a sequence with $f_n \downarrow 0$ and define $r := \|f_1\|_\kappa$. For every $\varepsilon > 0$, we can choose $c \ge 0$ and $K \Subset X$ with

$$\sup_{i\in I}\varphi_i(f_n) \le c\|f_n\|_{\infty,K} + \tfrac{\varepsilon}{2} \quad \text{for all } n\in\mathbb{N}.$$

Hence, by Dini's theorem, there exists $n_0 \in \mathbb{N}$ with $\sup_{i\in I}\varphi_i(f_n) \le \varepsilon$ for all $n \ge n_0$. □

3.3 Convex monotone operators in the mixed topology

In this section, we extend the results from the previous section on convex monotone functionals to convex monotone operators. At this point we would like to emphasize that the links between continuity from above, sequential continuity in the mixed topology and upper semicontinuity w.r.t. Γ-convergence are crucial in order to obtain the main results presented in Chapter 4. Since functions are ordered pointwise, an operator $S(t)\colon \mathrm{C}_\kappa \to \mathrm{U}_\kappa$ is convex and monotone if and only if the functionals

$$\mathrm{C}_\kappa \to [-\infty,\infty),\ f \mapsto (S(t)f)(x)$$

are convex and monotone for all $x \in X$. In addition, an operator $S(t)\colon \mathrm{C}_\kappa \to \mathrm{U}_\kappa$ is continuous from above if $S(t)f_n \downarrow S(t)f$ for all sequences $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa$ and $f \in \mathrm{C}_\kappa$ with $f_n \downarrow f$. If $S(t)$ is convex, then continuity from above is equivalent to the condition that $S(t)f_n \downarrow 0$ for all $f_n \downarrow 0$.

Assumption 3.3.1. Let $(S(t))_{t\ge 0}$ be a family of operators $S(t)\colon \mathrm{C}_\kappa \to \mathrm{U}_\kappa$ which satisfy the following conditions:

(i) $S(t)$ is convex and monotone and $S(t)0 = 0$ for all $t \ge 0$.

(ii) $S(t)$ is continuous from above for all $t \ge 0$.

(iii) It holds $\Gamma\text{-}\limsup_{s\to t} S(s)f \le S(t)f$ and $S(0)f = f$ for all $t \ge 0$ and $f \in \mathrm{C}_\kappa$.

(iv) It holds $\sup_{t\in[0,T]}\sup_{f\in B_{\mathrm{C}_\kappa}(r)}\|S(s)f\|_\kappa < \infty$ for all $r, t \ge 0$.

The following result is a consequence of Theorem 3.2.1.

Lemma 3.3.2. *Let $(S(t))_{t\ge 0}$ by a family of operators $S(t)\colon \mathrm{C}_\kappa \to \mathrm{U}_\kappa$ which satisfy Assumption 3.3.1. Then, the following statements are valid:*

(i) For every $t \ge 0$, there exists a unique extension $S(t)\colon \mathrm{U}_\kappa \to \mathrm{U}_\kappa$ which is continuous from above. The family of extended operators satisfies the conditions (i)-(iv)

from Assumption 3.3.1 with U_κ *instead of* C_κ*. In addition, for every* $x \in X$*, the functional*

$$\mathrm{U}_\kappa \to [-\infty, \infty), \; f \mapsto (S(t)f)(x)$$

can be extended to the space B_κ *such that, for every* $\varepsilon > 0$ *and* $c \geq 0$*, there exists* $K \Subset X$ *with* $\big(S(t)\big(\frac{c}{\kappa}\mathbb{1}_{K^c}\big)\big)(x) < \varepsilon$*.*

(ii) For every $T \geq 0$*,* $K \Subset X$ *and* $(f_n)_{n\in\mathbb{N}} \subset \mathrm{U}_\kappa$ *with* $f_n \downarrow 0$*,*

$$\sup_{(t,x)\in[0,T]\times K} \big(S(t)f_n\big)(x) \downarrow 0 \quad \text{as } n \to \infty.$$

Moreover, for every $c, T \geq 0$*,* $\varepsilon > 0$ *and* $K \Subset X$*, there exists* $K_1 \Subset X$ *with*

$$\sup_{(t,x)\in[0,T]\times K} \big(S(t)\big(\tfrac{c}{\kappa}\mathbb{1}_{K_1^c}\big)\big)(x) \leq \varepsilon.$$

Proof. First, we extend $(S(t))_{t\geq 0}$ from C_κ to U_κ. Let $t \geq 0$ and $x \in X$. It follows from Assumption 3.3.1(i), (ii) and (iv) that the functional

$$\varphi^{t,x} \colon \mathrm{C}_\kappa \to \mathbb{R}, \; f \mapsto (S(t)f)(x)$$

satisfies the assumptions of Theorem 3.2.1. Define $(S(t)f)(x) := \varphi_1^{t,x}(f)$ for all $f \in \mathrm{U}_\kappa$, where $\varphi_1^{t,x}$ denotes the extension of $\varphi^{t,x}$ from Theorem 3.2.1(ii). The family of extended operators satisfies the conditions (i), (ii) and (iv) with U_κ instead of C_κ and is the unique extension which is continuous from above. Next, we verify condition (iii), i.e.,

$$\Gamma\text{-}\limsup_{s\to t} S(s)f \leq S(t)f \quad \text{for all } t \geq 0 \text{ and } f \in \mathrm{U}_\kappa.$$

Fix $t \geq 0$, $f \in \mathrm{U}_\kappa$ and $x \in X$. Let $(t_n)_{n\in\mathbb{N}} \subset [0,\infty)$ and $(x_n)_{n\in\mathbb{N}} \subset X$ with $t_n \to t$ and $x_n \to x$. We use Theorem 3.2.1(ii) and Assumption 3.3.1(iii) to estimate

$$\begin{aligned}
\limsup_{n\to\infty} (S(t_n)f)(x_n) &= \inf_{n\in\mathbb{N}} \sup_{k\geq n} \inf \big\{ \big(S(t_k)g\big)(x_k) \colon g \in \mathrm{C}_\kappa, \, g \geq f \big\} \\
&\leq \inf \Big\{ \limsup_{n\to\infty} \big(S(t_n)g\big)(x_n) \colon g \in \mathrm{C}_\kappa, \, g \geq f \Big\} \\
&\leq \inf \big\{ \big(S(t)g\big)(x) \colon g \in \mathrm{C}_\kappa, \, g \geq f \big\} = (S(t)f)(x).
\end{aligned}$$

The statement about the extension to B_κ follows from Theorem 3.2.1(iii).

Second, let $T \geq 0$, $K \Subset X$ and $(f_n)_{n\in\mathbb{N}} \subset \mathrm{U}_\kappa$ be a sequence with $f_n \downarrow 0$. Since the first part of the lemma guarantees that the mappings

$$[0,T] \times K \to \mathbb{R}, \; (t,x) \mapsto \big(S(t)f_n\big)(x)$$

are upper semicontinuous for all $n \in \mathbb{N}$ and decrease pointwise to zero as $n \to \infty$, it follows from Dini's theorem that

$$\sup_{(t,x)\in[0,T]\times K} \big(S(t)f_n\big)(x) \downarrow 0 \quad \text{as } n \to \infty.$$

Now, let $c, T \geq 0$, $K \Subset X$ and $\varepsilon > 0$. For every $(t,x) \in [0,T] \times K$, we denote by

$$\varphi_2^{t,x} \colon \mathrm{B}_\kappa \to [-\infty, \infty)$$

the unique extension of $\varphi^{t,x}$ from Theorem 3.2.1(iii).

For every $(t,x)\in[0,T]\times K$ and $K_1 \Subset X$, Theorem 3.2.1(iii) implies

$$0\le \varphi_2^{t,x}\big(\tfrac{c}{\kappa}\mathbb{1}_{K_1^c}\big)\le \sup_{\mu\in M}\mu\big(\tfrac{c}{\kappa}\mathbb{1}_{K_1^c}\big),$$

where $M:=\bigcup_{(t,x)\in[0,T]\times K}\{\mu\in \mathrm{ca}_\kappa^+\colon (\varphi_2^{t,x})^*(\mu)\le \varphi_2^{t,x}(2c/\kappa)-2\varphi_2^{t,x}(-c/\kappa)\}$. Furthermore, by Dini's theorem, the functional

$$\varphi\colon \mathrm{C}_\kappa\to\mathbb{R},\ f\mapsto \sup_{(t,x)\in[0,T]\times K}\varphi_2^{t,x}(f)$$

is well-defined, convex, monotone and continuous from above. Hence, [12, Theorem 2.2] and condition (iv) imply that the set

$$M\subset\Big\{\mu\in\mathrm{ca}_\kappa^+\colon \varphi^*(\mu)\le \sup_{(t,x)\in[0,T]\times K}\big(\varphi_2^{t,x}\big(\tfrac{2c}{\kappa}\big)-2\varphi_2^{t,x}\big(-\tfrac{c}{\kappa}\big)\big)\Big\}$$

is $\sigma(\mathrm{ca}_\kappa^+,\mathrm{C}_\kappa)$-relatively compact. The claim follows from Prokhorov's theorem which guarantees that $\{\mu_\kappa\colon \mu\in M\}$ is tight, where $\mu_\kappa(A):=\int_A\frac{1}{\kappa}\,\mathrm{d}\mu$ for all $A\in\mathcal{B}(X)$. □

Lemma 3.3.3. *Let $(S(t))_{t\ge0}$ by a family of operators $S(t)\colon \mathrm{C}_\kappa\to\mathrm{U}_\kappa$ which satisfy Assumption 3.3.1. Let $(f_n)_{n\in\mathbb{N}}\subset\mathrm{U}_\kappa$ be bounded above and $(t_n)_{n\in\mathbb{N}}\subset[0,\infty)$ a convergent sequence. Define $f:=\Gamma\text{-}\limsup_{n\to\infty}f_n$ and $t:=\lim_{n\to\infty}t_n$. Then,*

$$\Gamma\text{-}\limsup_{n\to\infty}S(t_n)f_n\le S(t)f.$$

Proof. Let $\varepsilon\in(0,1]$ and $K\Subset X$. By Lemma 3.1.3, there exists $n_0\in\mathbb{N}$ with

$$f_n(x)\le\overline{f}^{\varepsilon}(x)\quad\text{for all } x\in K \text{ and } n\ge n_0.$$

We use the fact that $\overline{f}^{\varepsilon}\ge -1/\varepsilon\kappa$ and the monotonicity of $S(t_n)$ to obtain

$$S(t_n)f_n\le S(t_n)\big(\overline{f}^{\varepsilon}+\tfrac{c_\varepsilon}{\kappa}\mathbb{1}_{K^c}\big)\quad\text{for all } n\ge n_0,$$

where $c_\varepsilon:=\sup_{n\in\mathbb{N}}\|f_n^+\|_\kappa+1/\varepsilon<\infty$. For every $\lambda\in(0,1)$, the convexity of $S(t_n)$ yields

$$S(t_n)\big(\overline{f}^{\varepsilon}+\tfrac{c_\varepsilon}{\kappa}\mathbb{1}_{K^c}\big)\le\lambda S(t_n)\big(\tfrac{1}{\lambda}\overline{f}^{\varepsilon}\big)+(1-\lambda)S(t_n)\Big(\tfrac{c_\varepsilon}{\kappa(1-\lambda)}\mathbb{1}_{K^c}\Big).$$

Hence, it follows from Lemma 3.1.2(iv) and Lemma 3.3.2(i) that

$$\Gamma\text{-}\limsup_{n\to\infty}S(t_n)f_n\le\lambda S(t)\big(\tfrac{1}{\lambda}\overline{f}^{\varepsilon}\big)+(1-\lambda)\sup_{n\in\mathbb{N}}S(t_n)\Big(\tfrac{c_\varepsilon}{\kappa(1-\lambda)}\mathbb{1}_{K^c}\Big).$$

Since $K\Subset X$ is arbitrary, we can use Lemma 3.3.2(ii) to conclude that

$$\Gamma\text{-}\limsup_{n\to\infty}S(t_n)f_n\le\lambda S(t)\big(\tfrac{1}{\lambda}\overline{f}^{\varepsilon}\big).$$

Furthermore, the estimate $\overline{f}^{\varepsilon}\kappa\le\|f^+\|_\kappa+\varepsilon\le c_\varepsilon$ and the monotonicity of $S(t)$ yield

$$\Gamma\text{-}\limsup_{n\to\infty}S(t_n)f_n\le\lambda S(t)\big(\tfrac{1}{\lambda}\overline{f}^{\varepsilon}\big)\le\lambda S(t)\Big(\overline{f}^{\varepsilon}+\big(\tfrac{1}{\lambda}-1\big)\tfrac{c_\varepsilon}{\kappa}\Big).$$

Since $S(t)$ is continuous from above, the right-hand side converges to $S(t)\overline{f}^{\varepsilon}$ as $\lambda\uparrow1$. Thus, it follows from $\overline{f}^{\varepsilon}\downarrow f$ that

$$\Gamma\text{-}\limsup_{n\to\infty}S(t_n)f_n\le S(t)\overline{f}^{\varepsilon}\downarrow S(t)f\quad\text{as }\varepsilon\downarrow0.$$ □

For later reference, we state the following result in more generality. For a subset $Y \subset X$, we denote by $\mathrm{F}_\kappa(Y)$ the space of all functions $f\colon Y \to \mathbb{R}$ which satisfy $\|f\|_{\kappa,Y} := \sup_{x\in Y} |f(x)|\kappa(x) < \infty$.

Lemma 3.3.4. *Let $(X_i)_{i\in I}$ be a family of closed subsets $X_i \subset X$ and $(\Phi_i)_{i\in I}$ be a family of convex monotone operators $\Phi_i\colon \mathrm{C}_\kappa(X_i) \to \mathrm{F}_\kappa(X_i)$ with $\Phi_i(0) = 0$ for all $i \in I$ such that*

$$\sup_{i\in I} \sup_{f\in B_{\mathrm{C}_\kappa(X_i)}(r)} \|\Phi_i f\|_{\kappa,X_i} < \infty \quad \text{for all } r \geq 0.$$

Then, the following two statements are equivalent:

(i) For every $K \Subset \mathbb{R}^d$ and $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa(X)$ with $f_n \downarrow 0$,

$$\sup_{i\in I} \sup_{x\in K_i} (\Phi_i f_n)(x) \downarrow 0 \quad \text{as } n \to \infty.$$

(ii) For every $\varepsilon > 0$, $r \geq 0$ and $K \Subset \mathbb{R}^d$, there exist $c \geq 0$ and $K' \Subset \mathbb{R}^d$ with

$$\|\Phi_i f - \Phi_i g\|_{\infty,K_i} \leq c\|f - g\|_{\infty,K_i'} + \varepsilon$$

for all $i \in I$ and $f, g \in B_{\mathrm{C}_\kappa(X_i)}(r)$.

Proof. This follows from Lemma 3.2.3 with $\tilde{I} = \{(i, x)\colon i \in I,\, x \in K_i\}$, $\tilde{X}_{(i,x)} := X_i$ and $\tilde{\varphi}_{(i,x)}(f) := (\Phi_i f)(x)$ for all $(i, x) \in \tilde{I}$ and $f \in \mathrm{C}_\kappa(X)$. □

Corollary 3.3.5. *Let $(S(t))_{t\geq 0}$ be a family of operators $S(t)\colon \mathrm{C}_\kappa \to \mathrm{U}_\kappa$ which satisfy Assumption 3.3.1. Then, for every $\varepsilon > 0$, $r, T \geq 0$ and $K \Subset \mathbb{R}^d$, there exist $K' \Subset \mathbb{R}^d$ and $c \geq 0$ with*

$$\|S(t)f - S(t)g\|_{\infty,K} \leq c\|f - g\|_{\infty,K'} + \varepsilon$$

for all $t \in [0, T]$ and $f, g \in B_{\mathrm{C}_\kappa}(r)$. In particular, we obtain $S(t)f_n \to S(t)f$ for all $t \geq 0$ as well as $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa$ and $f \in \mathrm{C}_\kappa$ with $f_n \to f$.

Proof. Assumption 3.3.1(i) and (iv) and Lemma 3.3.2(ii) guarantee that Lemma 3.3.4 can be applied with $I := [0, T]$, $X_i := X$ and $\Phi_i f := S(t)f$ for all $i := t \in [0, T]$. □

3.4 Sufficient conditions for uniform continuity from above

In the previous section, we have have shown that, for families of convex monotone operators, uniform equicontinuity on bounded sets w.r.t. the mixed topology can be equivalently characterized by uniform continuity from above, see Lemma 3.3.4. In the sequel, we discuss sufficient conditions which imply that one the two equivalent statements is valid. These results will particularly useful in Chapter 7 and Chapter 8. For the next two lemmas, let $(X_i)_{i\in I}$ be a family of closed subsets $X_i \subset X$ and $(\Phi_i)_{i\in I}$ be a family of convex monotone operators $\Phi_i\colon \mathrm{C}_\kappa(X_i) \to \mathrm{F}_\kappa(X_i)$ with $\Phi_i(0) = 0$ and

$$\sup_{i\in I} \sup_{f\in B_{\mathrm{C}_\kappa(X_i)}(r)} \|\Phi_i(f)\|_\kappa < \infty \quad \text{for all } r \geq 0. \tag{3.6}$$

Recall that $\mathrm{F}_\kappa(X_i)$ consists of all functions $f\colon X_i \to \mathbb{R}$ with $\|f\|_\kappa < \infty$.

Lemma 3.4.1. *Let $\tilde{\kappa}\colon X \to (0,\infty)$ be another bounded continuous function such that, for every $\varepsilon > 0$, there exists $K \Subset X$ with $\sup_{x\in K^c} \frac{\tilde{\kappa}(x)}{\kappa(x)} \le \varepsilon$. Furthermore,*

$$\lim_{n\to\infty} \sup_{i\in I} \|\Phi_i f_n\|_{\tilde{\kappa}} = 0$$

for all $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa(X)$ with $\|f_n\|_{\tilde{\kappa}} \to 0$. Then, for every $\varepsilon > 0$, $r \ge 0$ and $K \Subset X$, there exist $c \ge 0$ and $K' \Subset X$ with

$$\|\Phi_i f - \Phi_i g\|_{\infty,K_i} \le c\|f-g\|_{\infty,K_i'} + \varepsilon \quad \textit{for all } i \in I \textit{ and } f,g \in B_{\mathrm{C}_\kappa(X_i)}(r).$$

Proof. It is sufficient to verify condition (i) from Lemma 3.3.4. To do so, let $K \Subset X$ and $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa(X)$ be a sequence with $f_n \downarrow 0$. Let $\varepsilon > 0$ and choose $\tilde{K} \Subset X$ with

$$\sup_{x\in\tilde{K}^c} \frac{\tilde{\kappa}(x)}{\kappa(x)} \le \varepsilon.$$

By Dini's theorem, there exists $n_0 \in \mathbb{N}$ with $f_n(x) \le \varepsilon$ for all $n \ge n_0$ and $x \in \tilde{K}$. Hence, for every $n \ge n_0$,

$$\|f_n\|_{\tilde{\kappa}} = \sup_{x\in\tilde{K}} |f_n(x)|\tilde{\kappa}(x) + \sup_{x\in\tilde{K}^c} |f_n(x)|\kappa(x)\frac{\tilde{\kappa}(x)}{\kappa(x)} \le \left(\sup_{x\in\tilde{K}} \tilde{\kappa}(x) + \|f_n\|_\kappa\right)\varepsilon.$$

We obtain $\|f_n\|_{\tilde{\kappa}} \to 0$ as $n \to \infty$ and thus $\sup_{i\in I} \|\Phi_i f_n\|_{\tilde{\kappa}} \to 0$. Since $\inf_{x\in K} \tilde{\kappa}(x) > 0$, this implies

$$\sup_{i\in I}\sup_{x\in K_i} (\Phi_i f_n)(x) \downarrow 0 \quad \text{as } n\to\infty.$$

Now, the claim follows from Lemma 3.3.4. □

Lemma 3.4.2. *Assume that, for every $\varepsilon > 0$, $r \ge 0$ and $K \Subset X$, there exist a family $(\zeta_x)_{x\in K}$ of continuous functions $\zeta_x\colon X \to \mathbb{R}$ and $\tilde{K} \Subset X$ with*

(i) $0 \le \zeta_x \le 1$ for all $x \in K$,

(ii) $\sup_{y\in\tilde{K}^c} \zeta_x(y) \le \varepsilon$ for all $x \in K$,

(iii) $\big(\Phi_i\big(\frac{r}{\kappa}(1-\zeta_x)\big)\big)(x) \le \varepsilon$ for all $i \in I$ and $x \in K_i$.

Then, for every $\varepsilon > 0$, $r \ge 0$ and $K \Subset X$, there exist $c \ge 0$ and $K' \Subset X$ with

$$\|\Phi_i f - \Phi_i g\|_{\infty,K_i} \le c\|f-g\|_{\infty,K_i'} + \varepsilon \quad \textit{for all } i \in I \textit{ and } f,g \in B_{\mathrm{C}_\kappa(X_i)}(r).$$

Proof. It is sufficient to verify condition (i) from Lemma 3.3.4. To do so, let $K \Subset X$ and $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa(X)$ be a sequence with $f_n \downarrow 0$. Let $\varepsilon \in (0,1]$ and $r := 2\|f_1\|_\kappa + 1$. It follows from condition (3.6) and $\inf_{x\in K} \kappa(x) > 0$ that

$$c := \sup_{i\in I}\sup_{x\in K_i} \big(\Phi_i\big(\tfrac{r}{\kappa}\big)\big)(x) + 1 < \infty.$$

By assumption, there exist a family $(\zeta_x)_{x\in K}$ of continuous functions $\zeta_x\colon X \to \mathbb{R}$ and $\tilde{K} \Subset X$ satisfying the conditions (i)-(iii) with ε/c. For every $i \in I$ and $x \in K_i$, the convexity and monotonicity of Φ_i, condition (i) and condition (iii) imply

$$(\Phi_i f_n)(x) \le \tfrac{1}{2}\big(\Phi_i(2f_n\zeta_x)\big)(x) + \tfrac{1}{2}\big(\Phi_i(2f_n(1-\zeta_x))\big)(x)$$

$$\leq \tfrac{1}{2}\big(\Phi_i(2f_n\zeta_x)\big)(x) + \tfrac{\varepsilon}{2}.$$

By condition (i), condition (ii) and Dini's theorem, there exists $n_0 \in \mathbb{N}$ with

$$2f_n\zeta_x \leq \tfrac{r\varepsilon}{c\kappa} \quad \text{for all } n \geq n_0 \text{ and } x \in K.$$

Hence, for every $i \in I$, $x \in K_i$ and $n \geq n_0$, we use the fact that Φ_i is convex and monotone with $\Phi_i(0) = 0$ to obtain

$$\big(\Phi_i(2f_n\zeta_x)\big)(x) \leq \big(\Phi_i\big(\tfrac{r\varepsilon}{c\kappa}\big)\big)(x) \leq \tfrac{\varepsilon}{c}\big(\Phi_i\tfrac{r}{\kappa}\big)(x) \leq \varepsilon.$$

Now, the claim follows from Lemma 3.3.4. □

The next corollary will be particularly useful in Chapter 7. Let $(I(t))_{t\geq 0}$ be a family of convex monotone operators $I(t)\colon \mathrm{C}_\kappa(X) \to \mathrm{C}_\kappa(X)$ such that there exists $\omega \geq 0$ with

$$\|I(t)f\|_\kappa \leq e^{\omega t}\|f\|_\kappa \quad \text{for all } t \geq 0 \text{ and } f \in \mathrm{C}_\kappa(X).$$

Denote by $\mathcal{P}_t$ the set of all partitions $\pi = \{t_0, \ldots, t_n\}$ with $0 = t_0 < t_1 < \ldots < t_n = t$ and define $I(\pi) := I(t_1 - t_0) \circ \ldots I(t_n - t_{n-1})$.

Corollary 3.4.3. *Let $\tilde{\kappa}\colon X \to (0,\infty)$ be another bounded continuous function such that, for every $\varepsilon > 0$, there exists $K \Subset \mathbb{R}^d$ with $\sup_{x\in K^c} \frac{\tilde{\kappa}(x)}{\kappa(x)} \leq \varepsilon$. Furthermore,*

$$\|I(t)f\|_{\tilde{\kappa}} \leq e^{\omega t}\|f\|_{\tilde{\kappa}} \tag{3.7}$$

for all $t \in [0,1]$ and $f \in \mathrm{C}_\kappa(X)$ with $\|f\|_{\tilde{\kappa}} \leq 1$. Then, for every $r, T \geq 0$, $\varepsilon > 0$ and $K \Subset X$, there exist $c \geq 0$ and $K' \Subset X$ with

$$\|I(\pi)f - I(\pi)g\|_{\infty,K} \leq c\|f - g\|_{\infty,K'} + \varepsilon$$

for all $t \in [0,T]$, $\pi \in \mathcal{P}_t$ and $f, g \in B_{\mathrm{C}_\kappa(X)}(r)$.

Proof. By induction, one can show that, for every $t \geq 0$ and $\pi \in \mathcal{P}_t$,

$$\|I(\pi)f\|_\kappa \leq e^{\omega t}\|f\|_\kappa \quad \text{for all } f \in \mathrm{C}_\kappa,$$
$$\|I(\pi)f\|_{\tilde{\kappa}} \leq e^{\omega t}\|f\|_{\tilde{\kappa}} \quad \text{for all } f \in \mathrm{C}_\kappa \text{ with } \|f\|_{\tilde{\kappa}} \leq e^{-\omega t}.$$

Hence, the claim follows from Lemma 3.4.1 with $I := \{(t,\pi)\colon t \in [0,T], \pi \in \mathcal{P}_t\}$, $X_{(t,\pi)} := X$ and $\Phi_{(t,\pi)} := I(\pi)$. □

In the sequel, we focus on the framework of Chapter 8. For the rest of this section, let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ and $(X_n)_{n\in\mathbb{N}}$ be sequences of subsets $\mathcal{T}_n \subset \mathbb{R}_+$ and $X_n \subset X$, respectively, such that X_n is closed for all $n \in \mathbb{N}$. Moreover, let $\{S_n(t)\colon n \in \mathbb{N}, t \in \mathcal{T}_n\}$ be a family of convex monotone operators $S_n(t)\colon \mathrm{C}_\kappa(X_n) \to \mathrm{F}_\kappa(X_n)$ with $S_n(t)0 = 0$ and

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]\cap\mathcal{T}_n} \sup_{f\in B_{\mathrm{C}_\kappa(X_n)}(r)} \|S_n(t)f\|_{\kappa,X_n} < \infty \quad \text{for all } r, T \geq 0. \tag{3.8}$$

Lemma 3.4.4. *The following two statements are equivalent:*

(i) *For every* $T \geq 0$, $K \Subset X$ *and* $(f_k)_{k\in\mathbb{N}} \subset \mathrm{C}_\kappa(X)$ *with* $f_k \downarrow 0$,

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]\cap\mathcal{T}_n} \sup_{x\in K_n} (S_n(t)f_k)(x) \downarrow 0 \quad \textit{as } k \to \infty.$$

(ii) *For every* $\varepsilon > 0$, $r, T \geq 0$ *and* $K \Subset X$, *there exist* $c \geq 0$ *and* $K' \Subset X$ *with*

$$\|S_n(t)f - S_n(t)g\|_{\infty,K_n} \leq c\|f - g\|_{\infty,K_n'} + \varepsilon$$

for all $n \in \mathbb{N}$, $t \in [0,T] \cap \mathcal{T}_n$ *and* $f, g \in B_{\mathrm{C}_\kappa(X_n)}(r)$.

Proof. Apply Lemma 3.3.4 with $I := \mathbb{N} \times [0,T]$, $X_{(n,t)} := X_n$ and $\Phi_{(n,t)} := S_n(t)$. □

The next lemma is, despite its simple nature, very useful in applications to Markovian transition semigroups under model uncertainty, since it typically reduces to bounding certain moments.

Lemma 3.4.5. *Let* $\tilde{\kappa}\colon X \to (0,\infty)$ *be another bounded continuous function such that, for every* $\varepsilon > 0$, *there exists* $K \Subset X$ *with*

$$\sup_{x\in K^c} \frac{\tilde{\kappa}(x)}{\kappa(x)} \leq \varepsilon.$$

Moreover, for every $T \geq 0$, *there exist* $\delta > 0$ *and* $c \geq 0$ *with*

$$\|S_n(t)f\|_{\tilde{\kappa},X_n} \leq c\|f\|_{\tilde{\kappa},X_n}$$

for all $n \in \mathbb{N}$, $t \in [0,T] \cap \mathcal{T}_n$ *and* $f \in B_{\mathrm{C}_\kappa(X_n)}(\delta)$. *Then, for every* $\varepsilon > 0$, $r, T \geq 0$ *and* $K \Subset X$, *there exist* $c \geq 0$ *and* $K' \Subset X$ *with*

$$\|S_n(t)f - S_n(t)g\|_{\infty,K_n} \leq c\|f - g\|_{\infty,K_n'} + \varepsilon$$

for all $n \in \mathbb{N}$, $t \in [0,T] \cap \mathcal{T}_n$ *and* $f, g \in B_{\mathrm{C}_\kappa(X_n)}(r)$.

Proof. Apply Lemma 3.4.1 with $I := \mathbb{N} \times [0,T]$, $X_{(n,t)} := X_n$ and $\Phi_{(n,t)} := S_n(t)$. □

The next lemma characterizes continuity from above by means of suitable cut-off functions. Remarkably, in many applications, the family $(\zeta_x)_{x\in K}$ can be constructed by scaling and shifting a single smooth functions with compact support, see the proof of Corollary 3.4.8 below. Furthermore, this criterion allows us to guarantee continuity from above by verifying sufficient conditions for the generators rather than the semigroups, see Corollary 3.4.9 and Corollary 3.4.10 below.

Lemma 3.4.6. *Assume that, for every* $\varepsilon > 0$, $r, T \geq 0$ *and* $K \Subset X$, *there exist a family* $(\zeta_x)_{x\in K}$ *of continuous functions* $\zeta_x\colon X \to \mathbb{R}$ *and* $\tilde{K} \Subset X$ *with*

(i) $0 \leq \zeta_x \leq 1$ *for all* $x \in K$,

(ii) $\sup_{y\in\tilde{K}^c} \zeta_x(y) \leq \varepsilon$ *for all* $x \in K$,

(iii) $\big(S_n(t)\big(\frac{r}{\kappa}(1-\zeta_x)\big)\big)(x) \leq \varepsilon$ *for all* $n \in \mathbb{N}$, $t \in [0,T] \cap \mathcal{T}_n$ *and* $x \in K_n$.

Then, for every $\varepsilon > 0$, $r, T \geq 0$ and $K \Subset X$, there exist $c \geq 0$ and $K' \Subset X$ with

$$\|S_n(t)f - S_n(t)g\|_{\infty,K_n} \leq c\|f-g\|_{\infty,K_n'} + \varepsilon$$

for all $n \in \mathbb{N}$, $t \in [0,T] \cap \mathcal{T}_n$ and $f, g \in B_{\mathrm{C}_\kappa(X_n)}(r)$.

Proof. Apply Lemma 3.4.2 with $I := \mathbb{N} \times [0,T]$, $X_{(n,t)} := X_n$ and $\Phi_{(n,t)} := S_n(t)$. □

Subsequently, we derive several corollaries to Lemma 3.4.6. To that end, let $s \pm t \in \mathcal{T}_n$ for all $n \in \mathbb{N}$ and $s, t \in \mathcal{T}_n$ with $s \geq t$. We assume that $S_n(t)\colon \mathrm{C}_\kappa(X_n) \to \mathrm{C}_\kappa(X_n)$,

$$S_n(0)f = f \quad \text{and} \quad S_n(s+t)f = S_n(s)S_n(t)f$$

for all $n \in \mathbb{N}$, $s, t \in \mathcal{T}_n$ and $f \in \mathrm{C}_\kappa(X_n)$. For every $r, T \geq 0$, there exists $c \geq 0$ with

$$\|S_n(t)f - S_n(t)g\|_\kappa \leq c\|f-g\|_\kappa \tag{3.9}$$

for all $n \in \mathbb{N}$, $t \in [0,T] \cap \mathcal{T}_n$ and $f, g \in B_{\mathrm{C}_\kappa(X_n)}(r)$.

Corollary 3.4.7. *Assume that, for every $\varepsilon > 0$, $r \geq 0$ and $K \Subset X$, there exist a family $(\zeta_x)_{x\in K}$ of continuous functions $\zeta_x\colon X \to \mathbb{R}$ and $\tilde{K} \Subset X$ such that*

(i) $0 \leq \zeta_x \leq 1$ and $\zeta_x(x) = 1$ for all $x \in K$,

(ii) $\sup_{y \in \tilde{K}^c} \zeta_x(y) \leq \varepsilon$ for all $x \in K$,

(iii) there exists $(t_n)_{n\in\mathbb{N}}$ with $t_n \in \mathcal{T}_n \backslash \{0\}$ and

$$\big\|(S_n(t)\big(\tfrac{r}{\kappa}(1-\zeta_x)\big) - \tfrac{r}{\kappa}(1-\zeta_x)\big\|_{\kappa,X_n} \leq \varepsilon t$$

for all $n \in \mathbb{N}$, $t \in [0,t_n] \cap \mathcal{T}_n$ and $x \in K_n$.

Then, for every $\varepsilon > 0$, $r, T \geq 0$ and $K \Subset X$, there exist $c \geq 0$ and $K' \Subset X$ with

$$\|S_n(t)f - S_n(t)g\|_{\infty,K_n} \leq c\|f-g\|_{\infty,K_n'} + \varepsilon$$

for all $n \in \mathbb{N}$, $t \in [0,T] \cap \mathcal{T}_n$ and $f, g \in B_{\mathrm{C}_\kappa(X_n)}(r)$.

Proof. Let $\varepsilon > 0$, $r, T \geq 0$, $K \Subset X$ and $(\zeta_x)_{x\in K}$ be a family of continuous functions $\zeta_x\colon X \to \mathbb{R}$ satisfying the conditions (i)-(iii) with ε/c. Let $n \in \mathbb{N}$ and $t \in [0,T] \cap \mathcal{T}_n$. Since $\mathcal{T}_n$ is closed under addition and subtraction, there exist $t_1, t_2 \in [0, t_0] \cap \mathcal{T}_n$ and $k \in \mathbb{N}$ with $t = kt_1 + t_2$. Let $x \in K$ and $f := \frac{r}{\kappa}(1-\zeta_x)$. We use the semigroup property, inequality (3.9) and condition (iii) to obtain

$$\begin{aligned}\|S_n(t)f - f\|_{\kappa,X_n} &\leq \|S_n(kt_1)S_n(t_2)f - S_n(kt_1)f\|_{\kappa,X_n} \\ &\quad + \sum_{i=0}^{k-1} \|S_n(it_1)S_n(t_1)f - S_n(it_1)f\|_{\kappa,X_n} \\ &\leq c\|S_n(t_2)f - f\|_{\kappa,X_n} + ck\|S_n(t_1)f - f\|_{\kappa,X_n} \leq \varepsilon t.\end{aligned}$$

Now, the claim follows from $f(x) = 0$, $\inf_{x\in K} \kappa(x) > 0$ and Lemma 3.4.6. □

The next result is particularly useful when the generators are differential operators. Furthermore, the condition imposed on κ is clearly satisfied if $\kappa \equiv 1$. For every infinitely differentiable function $f\colon \mathbb{R}^d \to \mathbb{R}$ and $i \in \mathbb{N}$, let

$$\|D^i f\|_\kappa := \sum_{|\alpha|=i} \|D^\alpha f\|_\kappa \in [0,\infty],$$

where $D^\alpha f := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_d} f$ and $|\alpha| := \sum_{j=1}^d \alpha_j$ for all $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. The space $\mathrm{C}_c^\infty(\mathbb{R}^d)$ consists of all infinitely differentiable functions $f\colon \mathbb{R}^d \to \mathbb{R}$ with compact support.

Corollary 3.4.8. *Let $X := \mathbb{R}^d$. Suppose that the function $1/\kappa$ is infinitely differentiable and, for every $\varepsilon > 0$, there exists $K \Subset \mathbb{R}^d$ with*

$$\sup_{x\in K^c} |(D^\alpha \tfrac{1}{\kappa})\kappa|(x) \leq \varepsilon \quad \textit{for all } 1 \leq |\alpha| \leq N.$$

Furthermore, there exist a sequence $(h_n)_{n\in\mathbb{N}}$ with $h_n \in \mathcal{T}_n \backslash \{0\}$, $N \in \mathbb{N}$ and a non-decreasing function $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\left\| \frac{S_n(h)f - f}{h} \right\|_{\kappa, X_n} \leq \rho\left(\sum_{i=1}^N \|D^i f\|_\kappa \right) \tag{3.10}$$

for all $n \in \mathbb{N}$, $h \in (0, h_n] \cap \mathcal{T}_n$ and $f := \frac{r}{\kappa}(1-\zeta)$ with $r \geq 0$ and $\zeta \in \mathrm{C}_c^\infty(\mathbb{R}^d)$. Then, for every $\varepsilon > 0$, $r, T \geq 0$ and $K \Subset \mathbb{R}^d$, there exist $c \geq 0$ and $K' \Subset \mathbb{R}^d$ with

$$\|S_n(t)f - S_n(t)g\|_{\infty, K_n} \leq c\|f-g\|_{\infty, K_n'} + \varepsilon$$

for all $n \in \mathbb{N}$, $t \in [0,T] \cap \mathcal{T}_n$ and $f, g \in B_{\mathrm{C}_\kappa(X_n)}(r)$.

Proof. Let $\varepsilon > 0$, $r \geq 0$ and $K \Subset \mathbb{R}^d$. Moreover, let $\xi\colon \mathbb{R}^d \to \mathbb{R}$ be infinitely differentiable with compact support, $0 \leq \xi \leq 1$ and $\xi(x) = 1$ for all $x \in B_{\mathbb{R}^d}(1)$. Choose $\delta_0 > 0$ with $\delta_0 r \leq \varepsilon/2$ and

$$\delta_0 r \sum_{\beta<\alpha} \binom{\alpha}{\beta} \|D^\beta \tfrac{1}{\kappa}\|_\kappa \|D^{\alpha-\beta}\xi\|_\infty \leq \frac{\varepsilon}{2} \quad \text{for all } 1 \leq |\alpha| \leq N. \tag{3.11}$$

Furthermore, there exists $\tilde{K} \Subset \mathbb{R}^d$ with $\sup_{x\in\tilde{K}^c} |(D^\alpha \frac{1}{\kappa})\kappa|(x) \leq \delta_0$ for all $1 \leq |\alpha| \leq N$. Let $\delta \in (0, \delta_0]$ with $\delta|y-x| \leq 1$ for all $(x,y) \in K \times \tilde{K}$ and define $\zeta_x(y) := \xi(\delta(y-x))$ for all $(x,y) \in K \times \mathbb{R}^d$. We verify the conditions (i)-(iii) from Corollary 3.4.7.

(i) Clearly, it holds $0 \leq \zeta_x \leq 1$ and $\zeta_x(x) = \xi(0) = 1$ for all $x \in K$.

(ii) Choose $R \geq 0$ with $\operatorname{supp}(\xi) \subset B_{\mathbb{R}^d}(R)$ and $K \subset B_{\mathbb{R}^d}(R)$. For every $x \in K$ and $y \in B_{\mathbb{R}^d}(R + R/\delta)^c$, it holds $\delta(y-x) \geq R$ and therefore $\zeta_x(y) = \xi(\delta(y-x)) = 0$.

(iii) For every $x \in K$ and $f := \frac{r}{\kappa}(1-\zeta_x)$, we use the product formula, inequality (3.11) and the fact that $(1-\zeta_x)(y) = 0$ for all $(x,y) \in K \times \tilde{K}$ to obtain

$$|D^\alpha f|\kappa \leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \big|D^\beta(\tfrac{r}{\kappa}) D^{\alpha-\beta}(1-\zeta_x)\big|\kappa$$

$$
\begin{aligned}
&= r \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \big|(D^\beta \tfrac{1}{\kappa})\kappa\big|\delta^{|\alpha-\beta|}|D^{\alpha-\beta}\xi| \\
&\leq \delta r \sum_{\beta<\alpha} \binom{\alpha}{\beta} \|D^\beta \tfrac{1}{\kappa}\|_\kappa \|D^{\alpha-\beta}\xi\|_\infty + r\big|(D^\alpha \tfrac{1}{\kappa})\kappa\big|(1-\zeta_x) \\
&\leq \sup_{y\in\tilde{K}^c} r\big|(D^\alpha \tfrac{1}{\kappa})\kappa\big|(y) + \tfrac{\varepsilon}{2} \leq \varepsilon.
\end{aligned}
$$

Combining the previous estimate with inequality (3.10) yields

$$
\left\| \frac{S_n(h)f - f}{h} \right\|_\kappa \leq \rho(c\varepsilon) \quad \text{for all } n\in\mathbb{N} \text{ and } h\in(0,h_n]\cap\mathcal{T}_n,
$$

where $c := \sum_{i=1}^N \#\{\alpha\in\mathbb{N}_0^d \colon |\alpha| = i\}$. Furthermore, it holds $\rho(c\varepsilon)\to 0$ as $\varepsilon\to 0$.

Now, the claim follows from Corollary 3.4.7. □

For discrete time $\mathcal{T}_n := \{kh_n \colon k\in\mathbb{N}_0\}$ with $h_n>0$, the semigroups S_n are given by $S_n(kh_n) := I_n^k = I_n\circ\cdots\circ I_n$ with $I_n := S_n(h_n)$ and the verification of condition (iii) in Corollary 3.4.7 and Corollary 3.4.8 only requires knowledge of the one-step operators I_n rather than the entire semigroups S_n. Furthermore, in many applications, the operators I_n have an explicit representation that is very useful in the verification of condition (iii). However, in the continuous-time case, explicit representations for the semigroups S_n are not available or rather complicated. Here, it is important to replace condition (iii) in the previous results by a corresponding condition on the infinitesimal generators. For the following two corollaries, we choose $\mathcal{T}_n := \mathbb{R}_+$ and assume that the mapping

$$
\mathbb{R}_+ \to \mathrm{C}_\kappa(X_n),\ t\mapsto S_n(t)f
$$

is continuous for all $n\in\mathbb{N}$ and $f\in\mathrm{C}_\kappa(X_n)$.

Corollary 3.4.9. *Assume that, for every $\varepsilon>0$, $r\geq 0$ and $K\Subset X$, there exist a family $(\zeta_x)_{x\in K}$ of continuous functions $\zeta_x\colon X\to\mathbb{R}$ and $\tilde{K}\Subset X$ such that*

(i) $0\leq\zeta_x\leq 1$ and $\zeta_x(x)=1$ for all $x\in K$,

(ii) $\sup_{y\in\tilde{K}^c}\zeta_x(y)\leq\varepsilon$ for all $x\in K$,

(iii) the function $f := \frac{r}{\kappa}(1-\zeta_x)$ satisfies $f\in D(A_n)$, $\|A_nf\|_\kappa\leq\varepsilon$ and

$$
\lim_{h\downarrow 0} \left\| \frac{S_n(h)f - f}{h} - A_nf \right\|_{\kappa,X_n} = 0 \quad \text{for all } n\in\mathbb{N} \text{ and } x\in K. \tag{3.12}
$$

Then, for every $\varepsilon>0$, $r,T\geq 0$ and $K\Subset X$, there exist $c\geq 0$ and $K'\Subset X$ with

$$
\|S_n(t)f - S_n(t)g\|_{\infty,K_n} \leq c\|f-g\|_{\infty,K_n'} + \varepsilon
$$

for all $n\in\mathbb{N}$, $t\in[0,T]$ and $f,g\in B_{\mathrm{C}_\kappa(X_n)}(r)$.

Proof. Let $\varepsilon > 0$, $r, T \geq 0$ and $K \Subset X$. We choose $c \geq 0$ such that inequality (3.9) is satisfied with $r + \varepsilon$ and a family $(\zeta_x)_{x \in K}$ of continuous functions $\zeta_x \colon X \to \mathbb{R}$ satisfying the conditions (i)-(iii) with ε/c. Let $x \in K$ and $f := \frac{r}{\kappa}(1 - \zeta_x)$. For every $n \in \mathbb{N}$ and $t \in [0, T]$, similar to inequality (8.28) and inequality (8.29), one can use inequality (3.9) and equation (3.12) to show that

$$(S_n(t)f - f)\kappa \leq \int_0^t \big(S_n(s)(f + A_n f) - S_n(s)f\big)\kappa \, \mathrm{d}s,$$
$$(S_n(t)f - f)\kappa \geq -\int_0^t \big(S_n(s)(f - A_n f) - S_n(s)f\big)\kappa \, \mathrm{d}s.$$

Hence, inequality (3.9) and condition (iii) imply

$$-\varepsilon t \leq -c\|A_n f\|_\kappa \leq \|S_n(t)f - f\|_\kappa \leq c\|A_n f\|_\kappa t \leq \varepsilon t.$$

Now, the claim follows from $f(x) = 0$, $\inf_{x \in K} \kappa(x) > 0$ and Lemma 3.4.6. □

Corollary 3.4.10. *Let $X := \mathbb{R}^d$. Suppose that the function $1/\kappa$ is infinitely differentiable and, for every $\varepsilon > 0$, there exists $K \Subset \mathbb{R}^d$ with*

$$\sup_{x \in K^c} |(D^\alpha \tfrac{1}{\kappa})\kappa|(x) \leq \varepsilon \quad \textit{for all } 1 \leq |\alpha| \leq N.$$

For every $r \geq 0$ and $\zeta \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$, the function $f := \frac{r}{\kappa}(1 - \zeta)$ satisfies $f \in D(A_n)$ and

$$\lim_{h \downarrow 0} \left\| \frac{S_n(h)f - f}{h} - A_n f \right\|_{\kappa, X_n} = 0 \quad \textit{for all } n \in \mathbb{N}.$$

Furthermore, there exist $N \in \mathbb{N}$ and a non-decreasing function $\rho \colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\|A_n f\|_{\kappa, X_n} \leq \rho\left(\sum_{i=1}^N \|D^i f\|_\kappa\right)$$

for all $n \in \mathbb{N}$ and $f := \frac{r}{\kappa}(1 - \zeta)$ with $r \geq 0$ and $\zeta \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$. Then, for every $\varepsilon > 0$, $r, T \geq 0$ and $K \Subset \mathbb{R}^d$, there exist $c \geq 0$ and $K' \Subset \mathbb{R}^d$ with

$$\|S_n(t)f - S_n(t)g\|_{\infty, K_n} \leq c\|f - g\|_{\infty, K_n'} + \varepsilon$$

for all $n \in \mathbb{N}$, $t \in [0, T] \cap \mathcal{T}_n$ and $f, g \in B_{\mathrm{C}_\kappa(X_n)}(r)$.

Proof. Using Corollary 3.4.9, one can proceed similar to the proof of Corollary 3.4.8. □

Remark 3.4.11. In many applications, it is straightforward to show that the operators $S_n(t) \colon \mathrm{C}_\kappa(X_n) \to \mathrm{F}_\kappa(X_n)$ are well-defined but verifying the continuity of $S_n(t)f$ is more complicated. However, due to the proofs of the corollaries 3.4.7–3.4.10, it is sufficient to require that

$$S_n(t)(\tfrac{r}{\kappa}(1 - \zeta_x)) \in \mathrm{C}_\kappa(X_n) \quad \text{and} \quad S_n(s + t)(\tfrac{r}{\kappa}(1 - \zeta_x)) = S_n(s)S_n(t)(\tfrac{r}{\kappa}(1 - \zeta_x)).$$

Furthermore, it is often straightforward to show that $S_n(t) \colon \mathrm{Lip_b}(X_n) \to \mathrm{Lip_b}(X_n)$. Hence, for every $f \in \mathrm{C}_\kappa(X_n)$, we can choose a sequene $(f_k)_{k \in \mathbb{N}} \subset \mathrm{Lip_b}(X_n)$ with $f_k \to f$ to obtain $S_n(t)f = \lim_{k \to \infty} S_n(t)f_k \in \mathrm{C}_\kappa(X_n)$ for all $n \in \mathbb{N}$ and $t \in \mathcal{T}_n$. Furthermore, in the particular case $\kappa \equiv 1$, one can usually choose $\frac{r}{\kappa}(1 - \zeta_x) \in \mathrm{Lip_b}(X)$.

3.5 A version of Arzéla–Ascoli's theorem

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of subsets $X_n \subset X$ such that, for every $x \in X$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in X_n$ and $x_n \to x$. A sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n\colon X_n \to \mathbb{R}$ is called bounded if $\sup_{n\in\mathbb{N}} \|f_n\|_{\kappa,X_n} < \infty$. Furthermore, the sequence is called uniformly equicontinuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ with

$$|f_n(x) - f_n(y)| < \varepsilon \quad \text{for all } n \in \mathbb{N} \text{ and } x, y \in X_n \text{ and } |x-y| < \delta.$$

Lemma 3.5.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of bounded uniformly equicontinuous functions $f_n\colon X_n \to \mathbb{R}$. Then, there exist $f \in \mathrm{C}_\kappa(X)$ and $(n_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ with*

$$\lim_{l\to\infty} \|f - f_{n_l}\|_{\infty,K_{n_l}} = 0$$

for all $K \Subset X$ with $K \cap X_{n_l} \neq 0$ for all $l \in \mathbb{N}$. Furthermore, it holds

$$f(x) = \lim_{l\to\infty} f_{n_l}(x_{n_l})$$

for all $x \in \mathbb{R}^d$ and sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \in X_n$ and $x_n \to x$.

Proof. Let $D \subset X$ be countable and dense. Moreover, for every $x \in D$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence with $x_n \in X_n$ and $x_n \to x$. We use the Bolzano–Weierstrass theorem and a diagonalization argument to choose a subsequence $(n_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ such that the limit

$$f(x) := \lim_{l\to\infty} f_{n_l}(x_{n_l}) \in \mathbb{R}$$

exists for all $x \in D$. This defines a uniformly continuous function $f\colon D \to \mathbb{R}$ satisfying $\sup_{x\in D} |f(x)|\kappa(x) < \infty$ which can be uniquely extended to a uniformly continuous function $f \in \mathrm{C}_\kappa(X)$. Moreover, for every $x \in X$ and sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in X_n$ and $x_n \to x$, it follows from the uniform equicontinuity that $f_{n_l}(x_{n_l}) \to f(x)$. Let $K \Subset X$ with $K \cap X_{n_l} \neq \emptyset$ for all $l \in \mathbb{N}$. We assume, by contradiction, that there exist $\varepsilon > 0$, a subsequence $(n_{l,1})_{l\in\mathbb{N}}$ of $(n_l)_{l\in\mathbb{N}}$ and $x_{n_{l,1}} \in K_{n_{l,1}}$ with

$$|f_{n_{l,1}}(x_{n_{l,1}}) - f(x_{n_{l,1}})| \geq \varepsilon \quad \text{for all } l \in \mathbb{N}.$$

Since K is compact, we can choose a further subsequence $(n_{l,2})_{l\in\mathbb{N}}$ of $(n_{l,1})_{l\in\mathbb{N}}$ and $x \in X$ with $x_{n_{l,2}} \to x$. It holds $f(x_{n_{l,2}}) \to f(x)$ and $f_{n_{l,2}}(x_{n_{l,2}}) \to f(x)$ which leads to a contradiction. We obtain $\lim_{l\to\infty} \|f - f_{n_l}\|_{\infty,K_{n_l}} = 0$. □

3.6 Basic convexity estimates

We collect some elementary results that will be frequently used in the following chapters.

Lemma 3.6.1. *Let $\mathcal{X}$ be a vector space and $\varphi\colon \mathcal{X} \to \mathbb{R}$ be a convex functional. Then,*

$$\varphi(x) - \varphi(y) \leq \lambda\left(\varphi\left(\frac{x-y}{\lambda} + y\right) - \varphi(y)\right) \quad \text{for all } x, y \in \mathcal{X} \text{ and } \lambda \in (0,1].$$

Proof. We use the convexity to estimate

$$\begin{aligned}\varphi(x)-\varphi(y) &= \varphi\left(\lambda\left(\frac{x-y}{\lambda}+y\right)+(1-\lambda)y\right)-\varphi(y)\\ &\le \lambda\varphi\left(\frac{x-y}{\lambda}+y\right)+(1-\lambda)\varphi(y)-\varphi(y)\\ &= \lambda\left(\varphi\left(\frac{x-y}{\lambda}+y\right)-\varphi(y)\right).\end{aligned}$$

□

Let F_κ^+ be the space of all functions $f\colon X\to[-\infty,\infty)$ with $\|f^+\|_\kappa<\infty$.

Lemma 3.6.2. *Let $\Phi\colon \mathrm{C}_\kappa\to\mathrm{F}_\kappa^+$ be a convex monotone operator with $\Phi(0)=0$. Then, the following statements are valid:*

(i) For every $r\ge 0$, there exists $c\ge 0$ with

$$\|\Phi f\|_\kappa\le c\|f\|_\kappa \quad \textit{for all } f\in B_{\mathrm{C}_\kappa}(r).$$

One can choose $c:=\frac{1}{r}\|\big(\Phi(\frac{r}{\kappa})\big)^+\|_\kappa$. In particular, the function Φf is real-valued, i.e., $\Phi f\colon X\to\mathbb{R}$ for all $f\in\mathrm{C}_\kappa$.

(ii) For every $r\ge 0$, there exists $c\ge 0$ with

$$\|\Phi f-\Phi g\|_\kappa\le c\|f-g\|_\kappa \quad \textit{for all } f,g\in B_{\mathrm{C}_\kappa}(r).$$

One can choose $c:=\frac{1}{r}\sup_{f'\in B_{\mathrm{C}_\kappa}(3r)}\|\Phi f'\|_\kappa<\infty$.

The previous statements remain valid if we replace C_κ by B_κ.

Proof. First, let $r>0$, $f\in B_{\mathrm{C}_\kappa}(r)$ and $\lambda:=\frac{\|f\|_\kappa}{r}$. We use the fact that Φ is convex and monotone with $\Phi(0)=0$ to estimate

$$\Phi(f)=\Phi\big(\lambda\tfrac{1}{\lambda}f+(1-\lambda)0\big)\le\lambda\Phi\big(\tfrac{1}{\lambda}f\big)\le\lambda\Phi(\tfrac{r}{\kappa})=\tfrac{\|f\|_\kappa}{r}\Phi(\tfrac{r}{\kappa}).$$

Moreover, it follows from the convexity of Φ and $\Phi(0)=0$ that

$$0=\Phi(0)=\Phi\big(\tfrac12 f+\tfrac12(-f)\big)\le\tfrac12\Phi(f)+\tfrac12\Phi(-f).$$

We conclude $(\Phi(\pm f))(x)>-\infty$ for all $x\in X$ and $-\Phi(-f)\le\Phi(f)$. Combining the previous estimates yields

$$-\tfrac{\|f\|_\kappa}{r}\Phi(\tfrac{r}{\kappa})\le-\Phi(-f)\le\Phi(f)\le\tfrac{\|f\|_\kappa}{r}\Phi(\tfrac{r}{\kappa}).$$

Hence, it holds $\|\Phi f\|_\kappa\le c\|f\|_\kappa$ with $c:=\frac{1}{r}\|\big(\Phi(\frac{r}{\kappa})\big)^+\|_\kappa<\infty$. For $r=0$, the claim follows from $\Phi(0)=0$.

Second, let $r\ge 0$ and $f,g\in B_{\mathrm{C}_\kappa}(r)$. We define

$$\Phi_f\colon \mathrm{C}_\kappa\to\mathrm{F}_\kappa,\ f'\mapsto\Phi(f+f')-\Phi(f) \quad \text{for all } f'\in\mathrm{C}_\kappa.$$

Note, that $\|\Phi(f)\|_\kappa<\infty$ by the first part and therefore $\Phi(f')\in\mathrm{F}_\kappa$ for all $f'\in\mathrm{C}_\kappa$. Furthermore, it follows from the first part that

$$\|\Phi(f)-\Phi(g)\|_\kappa=\big\|\Phi_f(f-g)\big\|_\kappa\le\frac{1}{2r}\big\|\big(\Phi_f(\tfrac{2r}{\kappa})\big)^+\big\|_\kappa\|f-g\|_\kappa\le c\|f-g\|_\kappa,$$

where $c:=\frac{1}{r}\sup_{f'\in B_{\mathrm{C}_\kappa}(3r)}\|\Phi(f')\|_\kappa<\infty$. □

Chapter 4

The Γ-generator and uniqueness

4.1 Introduction

In this chapter we address the question whether strongly continuous convex monotone semigroups can be uniquely determined via their infinitesimal generators. For a strongly continuous linear semigroup $(S(t))_{t\geq 0}$ on a Banach space $\mathcal{X}$ it is a classical result that $S(t)\colon D(A)\to D(A)$ for all $t\geq 0$ and that, for every $x\in D(A)$, the unique classical solution of the abstract Cauchy problem $\partial_t u(t)=Au(t)$ with initial condition $u(0)=x$ is given by $u(t)=S(t)x$ for all $t\geq 0$. Here, the domain $D(A)$ consist of all $x\in\mathcal{X}$ such that the limit $Ax:=\lim_{h\downarrow 0}\frac{S(h)x-x}{h}\in\mathcal{X}$ exists w.r.t. the norm $\|\cdot\|_{\mathcal{X}}$. For more details, we refer to Pazy [141] and Engel and Nagel [67]. Furthermore, these results can be extended to nonlinear semigroups which are generated by m-accretive or macximal monotone operators, see Barbu [6], Bénilan and Crandall [20], Brézis [29], Crandall and Liggett [50] and Kato [104]. While this approach closely resembles the theory of linear semigroups, the definition of the nonlinear resolvent typically requires the existence of a unique classical solution of a corresponding fully nonlinear elliptic PDE. As pointed out in Evans [69] and Feng and Kurtz [75], the necessary regularity of classical solutions is, in general, delicate. This observation was, among others, one of the motivations for the introduction of viscosity solutions, see Crandall et al. [49], Crandall and Lions [51] and Lions [130]. The key ideas in order to obtain uniqueness of viscosity solutions are local comparisons with a sufficiently large class of smooth test functions and regularizations by introducing additional viscosity terms. Based on monotone semigroups, which are defined on the space of bounded uniformly continuous functions and satisfy suitable regularity and locality assumptions, Alvarez et al. [1] provide an axiomatic foundation for viscosity solutions to fully nonlinear second-order PDEs. This approach was picked up later by Biton [21] for semigroups on more general spaces of continuous functions with a certain growth behaviour at infinity. While these works mainly focus on the existence and axiomatization of second-order differential operators through semigroups, the uniqueness of the associated semigroups in terms of their generator is not yet fully clarified, cf. the discussion in [21, Section 5]. We refer to Fleming and Soner [77, Chapter II.3] for a broad discussion on the relation between semigroups and viscosity solutions and to Yong and Zhou [168, Chapter 4] for an illustration of the interplay between the dynamic programming principle and viscosity solutions in a stochastic optimal control setting.

While viscosity solutions have been used in a variety of applications in the context of fully nonlinear PDEs, the question whether it is possible to develop a self-contained comparison principle for strongly continuous convex monotone semigroups, which resembles the classical analogue from the linear case, has received little attention. In order to uniquely characterize semigroups via their infinitesimal generator, it is crucial to show that the domain is invariant under the semigroup, i.e., $S(t)\colon D(A) \to D(A)$ for all $t \geq 0$. For linear semigroups, the proof of this statement is elementary but for nonlinear semigroups the invariance of the domain can fail, see, e.g., Crandall and Liggett [50, Section 4] and Denk et. al. [61, Example 5.2]. On the other hand, for strongly continuous convex monotone semigroups defined on spaces with order continuous norm such as L^p-spaces and Orlicz hearts, the domain is invariant under the semigroups and the unique classical of the of the abstract Cauchy problem $\partial_t u(t) = Au(t)$ with initial condition $u(0) = x$ is given by $u(t) = S(t)x$ for all $t \geq 0$, see Denk et. al. [62]. However, considering nonlinear semigroups on L^p-spaces or Orlicz hearts rather than spaces of continuous functions forces us to impose restrictive conditions on the nonlinear terms appearing in the generator, see Chapter 2. Since, for spaces of continuous functions, the domain is typically not invariant under the semigroup, we have to find an invariant set on which we can define a weaker notion of the infinitesimal generator that uniquely determines the semigroup. In Section 1.2, we used invariant Lipschitz sets in order to construct nonlinear semigroups and in Section 1.5 we studied invariant symmetric Lipschitz sets which provided the regularity result for viscous Hamilton–Jacobi equations presented in Section 2.4. Now, we introduce in addition the invariant upper Lipschitz set $\mathcal{L}^S_+$ on which we can define upper Γ-generator

$$A^+_\Gamma f := \Gamma\text{-}\limsup_{h\downarrow 0} \frac{S(h)f - f}{h} \quad \text{for all } f \in \mathcal{L}^S_+.$$

At this point we would like to emphasize that the function $A^+_\Gamma f$ is not continuous anymore but still upper upper semicontinuous. Hence, in order to define the term $S(t)A^+_\Gamma f$, we have to extend the operators $(S(t))_{t\geq 0}$ from continuous to upper semicontinuous functions. In Chapter 3, we investigated how this can be achieved for convex monotone operators which are continuous w.r.t. the mixed topology. Moreover, the equivalence between continuity from above, continuity w.r.t. the mixed topology and upper semi-continuity w.r.t. Γ-convergence for families of uniformly bounded convex monotone operators explains why Γ-convergence is a suitable topology for the definition of the generator in order to guarantee uniqueness. The choice of latter is a priori not clear since we could, for instance, also define the generator as a pointwise limit superior which does not coincide with the Γ-limit superior.

The first main result of this chapter is a comparison principle which states that strongly continuous convex monotone semigroups are minimal Γ-supersolutions of the abstract Cauchy problem $\partial_t u(t) = A^+_\Gamma u(t)$ with initial condition $u(0) = f$ for all functions $f \in \mathcal{L}^S_+$, see Theorem 4.2.8. In particular, strongly continuous convex monotone semigroups are uniquely determined by their upper Γ-generators defined on their upper Lipschitz sets, see Theorem 4.2.9. Under additional assumptions we further show that $A_\Gamma f = \Gamma\text{-}\lim_{n\to\infty} A_\Gamma f_n$ for suitable approximating sequences $(f_n)_{n\in\mathbb{N}}$ with $f_n \to f$ such that $(A^+_\Gamma f_n)_{n\in\mathbb{N}}$ is bounded above. Since Γ-converge is not symmetric, i.e., $f_n \to f$ does not imply $-f_n \to -f$, the upper bound $A_\Gamma f \leq \Gamma\text{-}\limsup_{n\to\infty} A_\Gamma f_n$

holds for arbitrary sequences while the equality $A_\Gamma f = \Gamma\text{-}\lim_{n\to\infty} A_\Gamma f_n$ requires a particular structure of the approximating sequence. The latter is, for instance, valid if the approximating sequence is constructed via convolutions with mollifiers or sup-inf-convolutions. We thus obtain an explicit description of the upper Γ-generator if, for smooth functions, the Γ-generator is given as a convex functional of certain partial derivatives. It was shown in Alvarez et al. [1] and Biton [21] that this is the case for typical fully nonlinear second-order PDEs. As second main result, in the case $X = \mathbb{R}^d$, we provide explicit conditions which guarantee that two semigroups have the same upper Lipschitz set. In particular, the comparison principle can already be applied if we only know that their generators evaluated at smooth functions coincide, see Theorem 4.4.6. This is a major improvement since, in contrast to the upper Lipschitz set and the upper Γ-generator, the generator evaluated at smooth functions can be computed explicitly in a variety of applications, see Chapter 6–8.

Finally, we remark that our notion of a strongly continuous convex monotone semigroup coincides with the one in Goldys et al. [92], where it is shown that the function $u(t) := S(t)f$ is a viscosity solution of the abstract Cauchy problem $\partial_t u = Au$ with initial condition $u(0) = f$. Hence, our approach is consistent with the theory of viscosity solutions. At this point we would also like to remark that the idea of weakening topological properties of the semigroup is already present in the literature. Goldys and Kocan [91], van Casteren [161], Kunze [122] and Kraaij [113] study linear semigroups in strict topologies, see also Kraaij [111] and Yosida [169] for semigroups in locally convex spaces. In addition, equicontinuity in the strict topology is suitable for stability results. Kraaij [114] provides convergence results for nonlinear semigroups based on the link between viscosity solutions to HJB equations and pseudo-resolvents and in [112] the author establishes Γ-convergence of functionals on path-spaces. Convergence of semigroups and their infinitesimal generators will be investigated in Chapter 8, where we also provide a detailed discussion of the related literature.

4.2 Comparison principle for convex monotone semigroups

Throughout this chapter, let (X, d) be a complete separable metric space. We fix a bounded continuous function $k\colon X \to \mathbb{R}$ and define

$$\|f\|_\kappa := \sup_{x\in X} |f(x)|\kappa(x) \in [0,\infty] \quad \text{for all } f\colon X \to [-\infty,\infty).$$

Recall that C_κ consists of all continuous functions $f\colon X \to \mathbb{R}$ with $\|f\|_\kappa < \infty$ and U_κ contains all upper semicontinuous functions $f\colon X \to [-\infty,\infty)$ with $\|f^+\|_\kappa < \infty$. Let $(S(t))_{t\geq 0}$ be a family of operators $S(t)\colon \mathrm{C}_\kappa \to \mathrm{U}_\kappa$. We do not assume that $(S(t))_{t\geq 0}$ satisfies the semigroup property on the whole space C_κ and that continuous functions are mapped to continuous functions. Indeed, it is sufficient to require these properties for functions in the upper Lipschitz set defined below.

Definition 4.2.1. The (infinitesimal) generator of $(S(t))_{t\geq 0}$ is defined by

$$A\colon D(A) \to \mathrm{C}_\kappa,\ f \mapsto \lim_{h\downarrow 0} \frac{S(h)f - f}{h},$$

where the domain $D(A)$ consists of all $f \in \mathrm{C}_\kappa$ such that the previous limit exists w.r.t. the mixed topology. Moreover, we require that $S(t)f \in \mathrm{C}_\kappa$ and $S(s+t)f = S(s)S(t)f$ for all $s, t \geq 0$ and $f \in D(A)$.

In contrast to the linear case, the domain is not invariant under the semigroup and this problem can not be avoided by restricting the semigroup and the generator to a subspace of C_κ. Hence, instead of considering the largest set on which the right derivative of the trajectories $t \mapsto S(t)f$ exists w.r.t. the mixed topology, we consider the largest set on which the trajectories are (upper) Lipschitz continuous. For these trajectories the right derivative can still be defined as a Γ-limit superior. This brings us to the following crucial definition.

Definition 4.2.2. The upper Lipschitz set $\mathcal{L}_+^S$ consists of all $f \in \mathrm{C}_\kappa$ such that

(i) $S(t)f \in \mathrm{C}_\kappa$ for all $t \geq 0$,

(ii) $S(s+t)f = S(s)S(t)f$ for all $s, t \geq 0$,

(iii) there exist $t_0 > 0$ and $c \geq 0$ with $\|(S(t)f - f)^+\|_\kappa \leq ch$ or all $t \in [0, t_0]$.

Furthermore, we define the upper Γ-generator by

$$A_\Gamma^+ f := \Gamma\text{-}\limsup_{h \downarrow 0} \frac{S(h)f - f}{h} \in \mathrm{U}_\kappa \quad \text{for all } f \in \mathcal{L}_+^S.$$

Lemma 3.1.2(v) implies $D(A) \subset \mathcal{L}_+^S$ and $Af = A_\Gamma^+ f$ for all $f \in D(A)$. Furthermore, under additional assumptions, one can show that the Γ-limit superior in the previous definition can be replaced by a Γ-limit, see Section 4.3 below. For the sake of completeness, we give the following definition but we will not use it in this section.

Definition 4.2.3. The Γ-generator is defined by

$$A_\Gamma \colon D(A_\Gamma) \to \mathrm{U}_\kappa, \; f \mapsto \Gamma\text{-}\lim_{h \downarrow 0} \frac{S(h)f - f}{f},$$

where the domain $D(A_\Gamma)$ consists of all $f \in \mathcal{L}_+^S$ such that the previous limit exists.

Subsequently, we impose the following conditions on the family $(S(t))_{t \geq 0}$ to guarantee that the upper Lipschitz set is invariant under the semigroup and that we can apply the results from Chapter 3 which are crucial for the proof of the comparison principle. Recall that, in contrast to Af, the function $A_\Gamma^+ f$ is not continuous anymore but still upper semicontinuous and might take the value $-\infty$. Hence, in order to define the term $S(t)A_\Gamma^+ f$, an extension of $S(t)$ from C_κ to U_κ is necessary. Moreover, the argumentation relies heavily on the fact that the extension is sequentially continuous w.r.t. the mixed topology and Γ-upper semicontinuous.

Assumption 4.2.4. The family $(S(t))_{t \geq 0}$ satisfies the following conditions:

(i) $S(t)$ is convex and monotone with $S(t)0 = 0$ for all $t \geq 0$.

(ii) $S(t)$ is continuous from above for all $t \geq 0$.

(iii) $\Gamma\text{-}\limsup_{s\to t} S(s)f \leq S(t)f$ and $S(0)f = f$ for all $t \geq 0$ and $f \in \mathrm{C}_\kappa$.

(iv) $\sup_{s\in[0,t]} \sup_{f\in B_{\mathrm{C}_\kappa}(0,r)} \|S(s)f\|_\kappa < \infty$ for all $r, t \geq 0$.

The following lemma shows that, in contrast to the domain, the upper Lipschitz set is invariant under the semigroup.

Lemma 4.2.5. *It holds* $S(t)\colon \mathcal{L}_+^S \to \mathcal{L}_+^S$ *for all* $t \geq 0$.

Proof. Let $f \in \mathcal{L}_+^S$. Choose $h_0 \in (0,1]$ and $c \geq 0$ with $\|(S(h)f - f)^+\|_\kappa \leq ch$ for all $h \in [0, h_0]$. We use $S(h)S(t)f = S(h+t) = S(t+h) = S(t)S(h)f$, Lemma 3.6.1, the monotonicity of $S(t)$ and Lemma 3.6.2 to estimate

$$\left(\frac{S(h)S(t)f - S(t)f}{h}\right)\kappa \leq \left(S(t)\left(f + \frac{(S(h)f-f)^+}{h}\right) - S(t)f\right)\kappa$$
$$\leq c'\left\|\frac{(S(h)f-f)^+}{h}\right\|_\kappa \leq cc' \quad \text{for all } h \in (0, h_0],$$

where $c' \geq 0$ is a constant independent of $h \in (0, h_0]$. □

We also recall the definition of the Lipschitz set from Chapter 1.

Definition 4.2.6. The Lipschitz set $\mathcal{L}^S$ consists of all $f \in \mathrm{C}_\kappa$ such that

(i) $S(t)f \in \mathrm{C}_\kappa$ for all $t \geq 0$,

(ii) $S(s+t)f = S(s)S(t)f$ for all $s, t \geq 0$,

(iii) there exist $t_0 > 0$ and $c \geq 0$ with $\|S(t)f - f\|_\kappa \leq ch$ or all $t \in [0, t_0]$.

Similar to the proof of Corollary 1.2.6, it follows from Lemma 3.6.2(ii) that

$$S(t)\colon \mathcal{L}^S \to \mathcal{L}^S \quad \text{for all } t \geq 0.$$

Before proceeding with the comparison principle, we briefly discuss some results about the Lipschitz set from the theory of linear semigroups relying on the reflexivity of the underlying Banach space, a property which C_κ does not have.

Remark 4.2.7. Let $(T(t))_{t\geq 0}$ be a be a strongly continuous semigroup of bounded linear operators on a Banach space $\mathcal{X}$. For simplicity, we assume that the growth bound of $(T(t))_{t\geq 0}$ is negative, i.e., there exist $c \geq 0$ and $\omega < 0$ with $\|T(t)x\| \leq ce^{\omega t}\|x\|$ for all $x \in \mathcal{X}$. Then, the set

$$F_1 := \left\{x \in \mathcal{X} \colon \sup_{h>0}\left\|\frac{T(h)x - x}{h}\right\| < \infty\right\}$$

is called the Favard space or the saturation class of $(T(t))_{t\geq 0}$, a notion coming from approximation theory, see, e.g., [37, Section 2.1] and [67, Section II.5.b]. Denoting by B the norm generator of $(T(t))_{t\geq 0}$, it is known that $F_1 = D(B)$ holds if $\mathcal{X}$ is reflexive, see [37, Theorem 2.1.2], and the generator uniquely determines the semigroup $(T(t))_{t\geq 0}$. If $(T(t))_{t\geq 0}$ is even holomorphic, then $F_1 = (\mathcal{X}, D(B))_{1,\infty}$, see [131, Proposition 2.2.2], where $(\cdot,\cdot)_{1,\infty}$ stands for the real interpolation functor. However, for non-reflexive $\mathcal{X}$, an explicit description of F_1 seems to be unknown in many cases.

The following theorem is the main result of this chapter and states that the minimal Γ-supersolution of the abstract Cauchy problem $\partial_t u(t) = A_\Gamma^+ u(t)$ with initial condition $u(0) = f$ is given by $u(t) = S(t)f$ for all $t \geq 0$ and $f \in \mathcal{L}_+^S$. For a further discussion on the concept of a Γ-supersolution, we refer to Remark 4.2.10 below. Note that we can not proof uniqueness for Γ-solutions, since Γ-converge is not symmetric, i.e., $f_n \to f$ does not imply $-f_n \to -f$. However, if $u(t) := T(t)f$ is given by another semigroup, we can reverse the roles of $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ to obtain uniqueness for strongly continuous convex monotone semigroups. In the next section we further investigate how the Γ-generator can be approximated and in the case $X = \mathbb{R}^d$ these results can be used to obtain that, under additional conditions, strongly continuous convex monotone semigroups are already uniquely determined by their generators evaluated at smooth functions.

Theorem 4.2.8. *Let $T_2, T_1 \geq 0$ with $T_1 \leq T_2$ and $f \in \mathcal{L}_+^S$. Let $u\colon [T_1, T_2] \to \mathcal{L}_+^S$ be a function with $S(T_1)f \leq u(T_1)$, $\sup_{t\in[T_1,T_2]} \|u(t)\|_\kappa < \infty$ and $\Gamma\text{-}\limsup_{s\to t} u(s) \leq u(t)$ for all $t \in [T_1, T_2]$. Suppose that, for every $t \in [T_1, T_2)$,*

$$\limsup_{h\downarrow 0} \Big\| \Big(\frac{u(t+h) - u(t)}{h} \Big)^- \Big\|_\kappa < \infty, \tag{4.1}$$

$$\Gamma\text{-}\limsup_{h\downarrow 0} \Big(A_\Gamma^+ u(t) - \frac{u(t+h) - u(t)}{h} \Big) \leq 0. \tag{4.2}$$

Then, it holds $S(t)f \leq u(t)$ for all $t \in [T_1, T_2]$.

Proof. It is sufficient to prove the result in the case $T_1 = 0$ and $S(0)f = u(0)$ since the general case follows immediately by considering the function

$$\tilde{u}\colon [0, T_2 - T_1] \to \mathcal{L}_+^S,\ t \mapsto u(t + T_1).$$

Indeed, suppose that result holds for $T_1 = 0$ and $S(0)f = u(0)$. The function $\tilde{u}$ also satisfies the conditions imposed on u and therefore we obtain $S(t)g \leq \tilde{u}(t)$ for all $t \in [0, T_2 - T_1]$, where $g := u(T_1)$. We use the semigroup property and the monotonicity of $S(t - T_1)$ to obtain

$$S(t)f = S(t - T_1)S(T_1)f \leq S(t - T_1)g \leq \tilde{u}(t - T_1) = u(t) \quad \text{for all } t \in [T_1, T_2].$$

Now, we prove the statement in the case $T_1 = 0$ and $S(0)f = u(0)$. To do, we fix $t \in [0, T]$ and show that the function

$$v\colon [0, t] \to \mathcal{L}_+^S,\ s \mapsto S(t - s)u(s)$$

satisfies $v(0) \leq v(s)$ for all $s \in [0, t]$. First, we show that

$$\liminf_{h\downarrow 0} \frac{v(s+h) - v(s)}{h} \geq 0 \quad \text{for all } s \in [0, t). \tag{4.3}$$

Let $s \in [0, t)$. Due to $u(s) \in \mathcal{L}_+^S$ and condition (4.1), there exists $h_0 > 0$ with

$$c := \sup_{h\in(0,h_0]} \max\Big\{ \Big\| \Big(\frac{S(h)u(s) - u(s)}{h} \Big)^+ \Big\|_\kappa, \Big\| \Big(\frac{u(s+h) - u(s)}{h} \Big)^- \Big\|_\kappa \Big\} < \infty. \tag{4.4}$$

For every $h \in (0, h_0]$, we define

$$f_h := \max\left\{\frac{S(h)u(s) - u(s)}{h}, -\frac{c}{\kappa}\right\} \quad \text{and} \quad g_h := \max\left\{-\frac{u(s+h) - u(s)}{h}, -\frac{c}{\kappa}\right\}.$$

It follows from Lemma 3.1.2(vi) that

$$f := \Gamma\text{-}\limsup_{h\downarrow 0} f_h = \max\left\{A_\Gamma^+ u(s), -\frac{c}{\kappa}\right\}.$$

Moreover, inequality (4.2), equation (4.4) and Lemma 3.1.2(vi) yield

$$\Gamma\text{-}\limsup_{h\downarrow 0} (f + g_h) \le 0. \tag{4.5}$$

Let $\varepsilon > 0$. Since u is bounded and by Assumption 4.2.4(iv), there exists $\lambda \in (0, 1]$ with

$$\sup_{a,b\in[0,t]} \lambda \|S(a)u(b)\|_\kappa < \varepsilon. \tag{4.6}$$

Lemma 3.6.1 and inequality (4.6) imply

$$\begin{aligned}
&- \liminf_{h\downarrow 0} \frac{v(s+h) - v(s)}{h} = \limsup_{h\downarrow 0} \frac{v(s) - v(s+h)}{h} \\
&= \limsup_{h\downarrow 0} \frac{S(t-s-h)S(h)u(s) - S(t-s-h)u(s+h)}{h} \\
&\le \limsup_{h\downarrow 0} \lambda \left(S(t-s-h)\left(\frac{S(h)u(s) - u(s+h)}{\lambda h} + u(s+h)\right) - S(t-s-h)u(s+h)\right) \\
&\le \limsup_{h\downarrow 0} \lambda S(t-s-h)\left(\frac{f_h + g_h}{\lambda} + u(s+h)\right) + \frac{\varepsilon}{\kappa}.
\end{aligned}$$

Furthermore, the boundedness of $(g_h)_{h\in(0,h_0]}$, $(f_h)_{h\in(0,h_0]}$ and u combined with Assumption 4.2.4(i) and (iv) and Lemma 3.3.2(ii) guarantee that we can apply Theorem 3.2.2 to estimate

$$\limsup_{h\downarrow 0} \lambda S(t-s-h)\left(\frac{f_h + g_h}{\lambda} + u(s+h)\right) \le \limsup_{h\downarrow 0} \lambda S(t-s-h)\left(\frac{f + g_h}{\lambda} + u(s+h)\right).$$

Lemma 3.3.3, inequality (4.5), Lemma 3.1.2(iv), $\Gamma\text{-}\limsup_{h\downarrow 0} u(s+h) \le u(s)$ and inequality (4.6) yield

$$\limsup_{h\downarrow 0} \lambda S(t-s-h)\left(\frac{f + g_h}{\lambda} + u(s+h)\right) \le \lambda S(t-s)u(s) \le \frac{\varepsilon}{\kappa}.$$

We combine the previous estimates and let $\varepsilon \downarrow 0$ to obtain inequality (4.3).

Second, we adapt the proof of [141, Lemma 1.1 in Chapter 2] to show $v(0) \le v(s)$ for all $s \in [0, t]$. Let $x \in X$ and $\varepsilon > 0$. Define $v(s, x) := (v(s))(x)$ and

$$v_\varepsilon(\cdot, x)\colon [0, t] \to \mathbb{R}, \ s \mapsto v(s, x) + \varepsilon s.$$

Moreover, let $s_0 := \sup\{s \in [0,t] \colon v_\varepsilon(0,x) \le v_\varepsilon(s,x)\}$. From $\Gamma\text{-}\limsup_{r\to s} u(r) \le u(s)$ and Lemma 3.3.3, we obtain that $v_\varepsilon(\cdot,x)$ is upper semicontinuous. In particular, it holds $v_\varepsilon(0,x) \le v_\varepsilon(s_0,x)$. By contradiction, we assume that $s_0 < t$. Let $(s_n)_{n\in\mathbb{N}} \subset (s_0,t]$ be a sequence with $s_n \downarrow s_0$. It follows from $v_\varepsilon(s_n,x) < v_\varepsilon(0,x) \le v_\varepsilon(s_0,x)$ and inequality (4.3) that

$$0 \ge \limsup_{n\to\infty} \frac{v_\varepsilon(s_n,x) - v_\varepsilon(s_0,x)}{s_n - s_0} = \limsup_{n\to\infty} \frac{v(s_n,x) - v(s_0,x)}{s_n - s_0} + \varepsilon \ge \varepsilon > 0.$$

This implies $v_\varepsilon(0,x) \le v_\varepsilon(t,x)$ and therefore $v(0,x) \le v(t,x)$ as $\varepsilon \downarrow 0$. In particular, we obtain $S(t)f = v(0) \le v(t) = u(t)$. □

Having a close look at the proof of the previous theorem, it seems natural to replace the conditions (4.1) and (4.2) by the assumption that

$$\limsup_{h\downarrow 0} \left\| \left(\frac{S(h)u(t) - u(t+h)}{h} \right)^+ \right\|_\kappa < \infty \quad \text{and} \quad \Gamma\text{-}\limsup_{h\downarrow 0} \frac{S(h)u(t) - u(t+h)}{h} \le 0.$$

Indeed, the previous theorem remains valid and the proof simplifies. In particular, we do not need Theorem 3.2.2. However, in applications, this assumption is not verifiable. Lemma 3.1.2(iv) implies that condition (4.2) is satisfied if

$$A_\Gamma^+ u(t) \le \Gamma\text{-}\liminf_{h\downarrow 0} \frac{u(t+h) - u(t)}{h} := -\Big(\Gamma\text{-}\limsup_{h\downarrow 0} -\frac{u(t+h) - u(t)}{h} \Big).$$

Furthermore, it follows from Lemma 3.1.3 that condition (4.2) is satisfied if

$$\lim_{h\downarrow 0} \Big(A_\Gamma^+ u(t) - \frac{u(t+h) - u(t)}{h} \Big)^+ = 0$$

w.r.t. the mixed topology. In case that $u(t) := T(t)f$ is given by another semigroup, we can reverse the roles of $(S(t))_{t\ge 0}$ and $(T(t))_{t\ge 0}$ in the following theorem to obtain that strongly continuous convex monotone semigroups are uniquely determined by their upper Γ-generators defined on the upper Lipschitz sets.

Theorem 4.2.9. *Let $(T(t))_{t\ge 0}$ be another family of operators $T(t)\colon \mathrm{C}_\kappa \to \mathrm{U}_\kappa$ satisfying Assumption 4.2.4 with Lipschitz set $\mathcal{L}^T$, generator B and upper Γ-generator B_Γ^+. Furthermore, let $\mathcal{D} \subset \mathcal{L}^T \cap \mathcal{L}_+^S$ be a set with $T(t)\colon \mathcal{D} \to \mathcal{D}$ for all $t \ge 0$ and*

$$A_\Gamma^+ f \le B_\Gamma^+ f \quad \text{for all } f \in \mathcal{D}. \tag{4.7}$$

Then, it holds $S(t)f \le T(t)f$ for all $t \ge 0$ and $f \in \mathcal{D} \cap D(B)$.

Proof. Let $f \in \mathcal{D} \cap D(B)$ and $u(t) := T(t)f$ for all $t \ge 0$. Assumption 4.2.4 and the invariance of $\mathcal{D} \subset \mathcal{L}_+^S$ imply that $u\colon \mathbb{R}_+ \to \mathcal{L}_+^S$ is a well-defined mapping with $u(0) = f$, $\sup_{s\in[0,t]} \|u(s)\|_\kappa < \infty$ and $\Gamma\text{-}\limsup_{s\to t} u(s) \le u(t)$ for all $t \ge 0$. Condition (4.1) is valid due to the invariance of $\mathcal{D} \subset \mathcal{L}^T$. It remains to verify condition (4.2). For every $t \ge 0$ and $h > 0$, we use $u(t) \in \mathcal{D}$ and inequality (4.7) to estimate

$$A_\Gamma^+ u(t) - \frac{u(t+h) - u(t)}{h} \le B_\Gamma^+ u(t) - \frac{u(t+h) - u(t)}{h}.$$

Let $(h_n)_{n\in\mathbb{N}} \subset (0,1]$ be a sequence with $h_n \downarrow 0$. For every $n \in \mathbb{N}$,

$$B_\Gamma^+ u(t) - \frac{u(t+h_n)-u(t)}{h_n} = B_\Gamma^+ u(t) - g_n + g_n - \frac{u(t+h_n)-u(t)}{h_n}, \tag{4.8}$$

where $g_n := \frac{1}{h_n}\big(T(t)(f + h_n B_\Gamma^+ f) - T(t)f\big)$. It follows from Lemma 3.6.1 that

$$\begin{aligned} &T(t)(f + h_n B_\Gamma^+ f) - T(t)\left(\frac{T(h_n)f - f}{h_n} - B_\Gamma^+ f + f + h_n B_\Gamma^+ f\right) \\ &\le g_n - \frac{u(t+h_n)-u(t)}{h_n} = \frac{T(t)(f + h_n B_\Gamma^+ f) - T(t)T(h_n)f}{h_n} \\ &\le T(t)\left(-\left(\frac{T(h_n)f - f}{h_n} - B_\Gamma^+ f\right) + T(h_n)f\right) - T(t)T(h_n)f. \end{aligned}$$

Combining the previous estimate with Corollary 3.3.5 yields

$$g_n - \frac{u(t+h_n)-u(t)}{h_n} \to 0.$$

Furthermore, since $T(t)$ is convex, the sequence $(g_n)_{n\in\mathbb{N}}$ is non-increasing. Hence, there exists a function $g \in \mathrm{U}_\kappa$ with $g_n \downarrow g$ and Lemma 3.1.2(iii) and (v) imply $g = B_\Gamma^+ u(t)$. It follows from inequality (4.8), the estimate $B_\Gamma^+ u(t) - g_n \le 0$ and Lemma 3.1.2(iv) that condition (4.2) is satisfied. Theorem 4.2.8 yields $S(t)f \le T(t)f$. □

We conclude this section with a short remark on the previously mention concept of a minimal Γ-supersolution.

Remark 4.2.10. Let $T \ge 0$ and $f \in \mathcal{L}_+^S$. A function $u\colon [0,T] \to \mathcal{L}_+^S$ can be seen as a Γ-supersolution of the equation

$$\partial_t u(t) = A_\Gamma^+ u(t) \quad \text{for all } t \in [0,T], \quad u(0) = f, \tag{4.9}$$

if u satisfies the conditions from Theorem 4.2.8. Let $T \ge 0$, $f \in D(A)$ and $u_0(t) := S(t)f$ for all $t \in [0,T]$. Then, similar to the proof of Theorem 4.2.9 that u_0 is the smallest Γ-supersolution of equation (4.9). In this way, we obtain a solution concept which is directly related to the concept of semigroups and their generators.

4.3 Approximation of the Γ-generator

Throughout this section, let $(S(t))_{t\ge 0}$ be a family of operators $S(t)\colon \mathrm{C}_\kappa \to \mathrm{U}_\kappa$ satisfying Assumption 4.2.4. In general, it is unknown whether the upper Γ-generator can be computed explicitly and how to show that two semigroups have the same upper Lipschitz set. Note that knowledge of the latter is necessary in order to apply the comparison principle stated in Theorem 4.2.9. However, in many applications, the generator can be computed explicitly for smooth functions. Furthermore, we recall that the generators of strongly continuous linear semigroups are closed. Here, we do not claim that A_Γ is closed, i.e., that the graph of A_Γ is a closed subset of $\mathrm{C}_\kappa \times \mathrm{U}_\kappa$. Indeed, while the inequality

$$A_\Gamma^+ f \le \Gamma\text{-}\limsup_{n\to\infty} A_\Gamma^+ f_n$$

is valid for arbitrary approximating sequences $f_n \to f$, the equality

$$A_\Gamma^+ f = \Gamma\text{-}\lim_{n\to\infty} A_\Gamma^+ f_n$$

only holds under additional assumptions on $(S(t))_{t\geq 0}$ and for particular choices of the approximating sequence $(f_n)_{n\in\mathbb{N}}$. In the sequel, we denote by $(S(t))_{t\geq 0}$ the family of extended operators $S(t)\colon \mathrm{U}_\kappa \to \mathrm{U}_\kappa$ which satisfies the conditions from Assumpton 4.2.4 with U_κ instead of C_κ and exists due to Lemma 3.3.2. For every $t\geq 0$, $f\in \mathrm{U}_\kappa$ and $x\in X$, we define the pointwise integral

$$\left(\int_0^t S(s)f\,\mathrm{d}s\right)(x) := \int_0^t \big(S(s)f\big)(x)\,\mathrm{d}s.$$

4.3.1 General approximation results

The proof of the upper bound is based on the following auxiliary estimate.

Lemma 4.3.1. *For every $r,T\geq 0$ and $\varepsilon>0$, there exists $\lambda_0\in(0,1]$ with*

$$S(t)f - f \leq \lambda \int_0^t S(s)\Big(\frac{1}{\lambda}A_\Gamma^+ f + f\Big)\mathrm{d}s + \frac{\varepsilon t}{\kappa}$$

for all $t\in[0,T]$, $f\in B_{\mathrm{C}_\kappa}(r)\cap \mathcal{L}_+^S$ and $\lambda\in(0,\lambda_0]$.

Proof. Fix $r,T\geq 0$ and $\varepsilon>0$. By Assumption 4.2.4(iv), there exists $\lambda_0\in(0,1]$ with

$$\sup_{t\in[0,T]}\ \sup_{f\in B_{\mathrm{C}_\kappa}(r)} \lambda_0\|S(t)f\|_\kappa \leq \varepsilon \quad\text{and}\quad \lambda_0 T\leq 1. \tag{4.10}$$

In the sequel, we fix $t\in[0,T]$, $f\in B_{\mathrm{C}_\kappa}(r)\cap\mathcal{L}_+^S$ and $\lambda\in(0,\lambda_0]$. Define $h_n := 2^{-n}t$ and $t_n^k := k2^{-n}t$ for all $k,n\in\mathbb{N}_0$. For every $n\in\mathbb{N}$, it follows from the semigroup property, inequality (4.10) and Lemma 3.6.1 that

$$\begin{aligned}
S(t)f - f &= \sum_{k=1}^{2^n}\big(S(t_n^k)f - S(t_n^{k-1})f\big) = \sum_{k=1}^{2^n}\big(S(t_n^{k-1})S(h_n)f - S(t_n^{k-1})f\big)\\
&\leq \lambda h_n \sum_{k=1}^{2^n}\left(S(t_n^{k-1})\left(\frac{S(h_n)f-f}{\lambda h_n} + f\right) - S(t_n^{k-1})f\right)\\
&\leq \lambda h_n \sum_{k=1}^{2^n} S(t_n^{k-1})\left(\frac{S(h_n)f-f}{\lambda h_n} + f\right) + \frac{\varepsilon t}{\kappa}\\
&= \lambda\int_0^t \sum_{k=1}^{2^n} S(t_n^{k-1})\left(\frac{S(h_n)f-f}{\lambda h_n} + f\right)\mathbb{1}_{[t_n^{k-1},t_n^k)}(s)\,\mathrm{d}s + \frac{\varepsilon t}{\kappa}.
\end{aligned}$$

We use Fatou's lemma and Lemma 3.3.3 to conclude that

$$S(t)f - f \leq \lambda\int_0^t \limsup_{n\to\infty}\sum_{k=1}^{n} S(t_n^{k-1})\left(\frac{S(h_n)f-f}{\lambda h_n} + f\right)\mathbb{1}_{[t_n^{k-1},t_n^k)}(s)\,\mathrm{d}s + \frac{\varepsilon t}{\kappa}$$

$$\leq \lambda \int_0^t S(s)\Big(\frac{1}{\lambda} A_\Gamma^+ f + f\Big) \mathrm{d}s + \frac{\varepsilon t}{\kappa}.$$

Note that the sequence inside the integral to which we apply Fatou's lemma is bounded from above, because of $f \in \mathcal{L}_+^S$ and Assumption 4.2.4(iv). □

Theorem 4.3.2. *Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{L}_+^S$ be a sequence and $f \in \mathrm{C}_\kappa$ with $f_n \to f$ such that $(A_\Gamma^+ f_n)_{n\in\mathbb{N}}$ is bounded above. Then,*

$$f \in \mathcal{L}_+^S \quad \text{and} \quad A_\Gamma^+ f \leq \Gamma\text{-}\limsup_{n\to\infty} A_\Gamma^+ f_n.$$

Proof. Let $\varepsilon > 0$ and $\lambda_0 \in (0,1]$ such that the statement of Lemma 4.3.1 is valid with $r := \sup_{n\in\mathbb{N}} \|f_n\|_\kappa$ and $t_0 := 1$. For every $h \in (0,1]$ and $\lambda \in (0,\lambda_0]$, it follows from Corollary 3.3.5, Fatou's lemma, Lemma 3.3.3 and Lemma 3.1.2(iv) that

$$\begin{aligned}
\frac{S(h)f - f}{h} &= \lim_{n\to\infty} \frac{S(h)f_n - f_n}{h} \\
&\leq \limsup_{n\to\infty} \frac{\lambda}{h} \int_0^h S(s)\Big(\frac{1}{\lambda} A_\Gamma^+ f_n + f_n\Big) \mathrm{d}s + \frac{\varepsilon}{\kappa} \\
&\leq \frac{\lambda}{h} \int_0^h \limsup_{n\to\infty} S(s)\Big(\frac{1}{\lambda} A_\Gamma^+ f_n + f_n\Big) \mathrm{d}s + \frac{\varepsilon}{\kappa} \\
&\leq \frac{\lambda}{h} \int_0^h S(s)\Big(\frac{1}{\lambda} g + f\Big) \mathrm{d}s + \frac{\varepsilon}{\kappa},
\end{aligned}$$

where $g := \Gamma\text{-}\limsup_{n\to\infty} A_\Gamma^+ f_n$. In particular, Assumption 4.2.4(iv) implies $f \in \mathcal{L}_+^S$. Moreover, Lemma 3.1.2(iv), Lemma 3.3.3 and $S(0) = \mathrm{id}_{\mathrm{U}_\kappa}$ yield

$$A_\Gamma^+ f \leq \Gamma\text{-}\limsup_{h\downarrow 0} \frac{\lambda}{h} \int_0^h S(s)\Big(\frac{1}{\lambda} g + f\Big) \mathrm{d}s + \frac{\varepsilon}{\kappa} \leq g + \lambda f + \frac{\varepsilon}{\kappa}.$$

Letting $\varepsilon, \lambda \downarrow 0$, we obtain $A_\Gamma^+ f \leq \Gamma\text{-}\limsup_{n\to\infty} A_\Gamma^+ f_n$. □

For particular choices of the approximating sequence $(f_n)_{n\in\mathbb{N}}$, we also obtain the reverse estimate and we can further conclude that $f \in D(A_\Gamma)$. In the next section, we discuss two possible constructions for sequences that satisfy the conditions required in the following theorem.

Theorem 4.3.3. *Let $(f_n)_{n\in\mathbb{N}} \subset D(A_\Gamma)$ be a bounded sequence and $f \in \mathrm{C}_\kappa$ with $f_n \to f$ such that $(A_\Gamma f_n)_{n\in\mathbb{N}}$ is bounded above. Suppose that, for every $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$, $h_0 > 0$ and a sequence $(r_n)_{n\in\mathbb{N}} \subset (0,\infty)$ with $r_n \to 0$ such that*

$$\left(\frac{S(h)f_n - f_n}{h}\right)(x) \leq \sup_{y\in B(x,r_n)} \left(\frac{S(h)f - f}{h}\right)(y) + \frac{\varepsilon}{\kappa(x)} \tag{4.11}$$

for all $h \in (0,h_0]$, $n \geq n_0$ and $x \in X$. Then, $f \in D(A_\Gamma)$ and $A_\Gamma f = \Gamma\text{-}\lim_{n\to\infty} A_\Gamma f_n$.

Proof. First, let $(h_m)_{m\in\mathbb{N}} \subset (0,\infty)$ be a sequence with $h_m \to 0$. By Theorem 4.3.2 and Lemma 3.1.2(i), there exists a subsequence, still denoted by $(h_m)_{m\in\mathbb{N}}$, such that

$$g_1 := \Gamma\text{-}\lim_{m\to\infty} \frac{S(h_m)f - f}{h_m} \in \mathrm{U}_\kappa \quad \text{exists.} \tag{4.12}$$

Moreover, since $(A_\Gamma f_n)_{n\in\mathbb{N}}$ is bounded above and due to Lemma 3.1.2(i), every subsequence $(f_{n_k})_{k\in\mathbb{N}}$ has a further subsequence $(f_{n_{k_l}})_{l\in\mathbb{N}}$ such that

$$g_2 := \Gamma\text{-}\lim_{l\to\infty} A_\Gamma f_{n_{k_l}} \in \mathrm{U}_\kappa \tag{4.13}$$

exists. To simplify the notation, we write $f_l := f_{n_{k_l}}$ for all $l \in \mathbb{N}$. Theorem 4.3.2 implies

$$g_1 \le A_\Gamma^+ f \le \Gamma\text{-}\lim_{l\to\infty} A_\Gamma f_l = g_2.$$

Second, we show that $g_1 \ge g_2$. To do so, let $x \in X$ and $\varepsilon > 0$. By definition of the Γ-limit, we can choose a sequence $(x_l)_{l\in\mathbb{N}} \subset X$ with $x_l \to x$ such that

$$\Big(\Gamma\text{-}\lim_{l\to\infty} A_\Gamma f_l\Big)(x) = \lim_{l\to\infty} A_\Gamma f_l(x_l).$$

In addition, there exist sequences $(m_l)_{l\in\mathbb{N}}$ and $(y_l)_{l\in\mathbb{N}}$ with $d(x_l, y_l) \to 0$ such that

$$A_\Gamma f_l(x_l) \le \left(\frac{S(h_{m_l})f_l - f_l}{h_{m_l}}\right)(y_l) + \varepsilon \quad \text{for all } l \in \mathbb{N}.$$

Choose $h_0 > 0$ and a sequence $(r_l)_{l\in\mathbb{N}} \subset (0,\infty)$ which satisfy condition (4.11). Then, for every $l \in \mathbb{N}$ with $n_{k_l} \ge n_0$ and $h_{m_l} \le h_0$,

$$\begin{aligned} A_\Gamma f_l(x_l) &\le \sup_{z\in B(y_l,r_l)} \left(\frac{S(h_{m_l})f - f}{h_{m_l}}\right)(z) + \frac{\varepsilon}{\kappa(y_l)} + \varepsilon \\ &\le \sup_{z\in B(x,\delta_l)} \left(\frac{S(h_{m_l})f - f}{h_{m_l}}\right)(z) + \sup_{l\in\mathbb{N}} \frac{\varepsilon}{\kappa(y_l)} + \varepsilon, \end{aligned}$$

where $\delta_l := d(x, y_l) + r_l \to 0$. Since $\kappa > 0$ is continuous, it holds $\inf_{l\in\mathbb{N}} \kappa(y_l) > 0$ and thus Lemma 3.1.2(vii) implies

$$g_2(x) = \Big(\Gamma\text{-}\lim_{l\to\infty} A_\Gamma f_l\Big)(x) = \lim_{l\to\infty} A_\Gamma f_l(x_l) \le \left(\Gamma\text{-}\lim_{l\to\infty} \frac{S(h_{m_l})f - f}{h_{m_l}}\right)(x) = g_1(x).$$

Third, we show that $f \in D(A_\Gamma)$ with $A_\Gamma f = \Gamma\text{-}\lim_{n\to\infty} A_\Gamma f_n$. From the first part, we know that every sequence $(h_m)_{m\in\mathbb{N}} \subset (0,\infty)$ with $h_m \to 0$ has a subsequence which satisfies equation (4.12). A priori the choice of the subsequence and the limit g_1 depend on the choice of the sequence $(h_m)_{m\in\mathbb{N}}$. However, it holds $g_1 = g_2$ and the function g_2 is independent of $(h_m)_{m\in\mathbb{N}}$. Hence, Lemma 3.1.2(ii) implies

$$g_1 = \Gamma\text{-}\lim_{h\downarrow 0} \frac{S(h)f - f}{h}$$

which means that $f \in D(A_\Gamma)$ with $A_\Gamma f = g_1$. Since the limit in equation (4.13) is also independent of the choice of the subsequence, we obtain $A_\Gamma f = \lim_{n\to\infty} A_\Gamma f_n$. □

4.3.2 Regularization by convolution

In this subsection, we study two particular constructions for approximating sequences $(f_n)_{n\in\mathbb{N}}$ satisfying condition (4.11).The first one, convolution with probability measures, works particularly well if $X = \mathbb{R}^d$ and the measure has a smooth density w.r.t. the Lebesgue measure. In this case, the sequence $(f_n)_{n\in\mathbb{N}}$ consists of smooth functions for which the generator can typically computed. This case will be studied in detail in Section 4.4 below. The second one, sup-inf-convolution, is restricted to first-order equations but can be applied in separable Hilbert spaces.

Convolution with probability measures

Let X be a separable Banach space and assume that

$$c_\kappa := \sup_{x\in X}\sup_{y\in B_X(1)} \frac{\kappa(x)}{\kappa(x-y)} < \infty, \tag{4.14}$$

where $B_X(1) := \{y \in X : \|x\|_X \le 1\}$. In the sequel, we fix a sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on the Borel σ-algebra $\mathcal{B}(X)$ concentrating around zero in the sense that

$$\mu_n\big(B_X(1/n)^c\big) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{4.15}$$

For every $n \in \mathbb{N}$, $f \in \mathrm{U}_\kappa$ and $x \in X$, we define the convolution by

$$(f * \mu_n)(x) := \int_X f(x-y)\mu_n(\mathrm{d}y) \in [-\infty, \infty),$$

where condition (4.14) guarantees that the integral is well-defined.

Lemma 4.3.4. *For every $n \in \mathbb{N}$, the mapping $\mathrm{U}_\kappa \to \mathrm{U}_\kappa$, $f \mapsto f * \mu_n$ is well defined and, for every sequence $(f_m)_{m\in\mathbb{N}} \subset \mathrm{U}_\kappa$ which is bounded above,*

$$\Gamma\text{-}\limsup_{m\to\infty}(f_m * \mu_n) \le \Big(\Gamma\text{-}\limsup_{m\to\infty} f_m\Big) * \mu_n.$$

Furthermore, it holds $\Gamma\text{-}\limsup_{n\to\infty}(f * \mu_n) \le f$ for all $f \in \mathrm{U}_\kappa$.

Proof. First, we show that $\mathrm{U}_\kappa \to \mathrm{U}_\kappa$, $f \mapsto f * \mu_n$ is well defined. Let $n \in \mathbb{N}$, $f \in \mathrm{U}_\kappa$ and $x \in X$. It follows from condition (4.14) and condition (4.15) that

$$(f * \mu_n)(x)\kappa(x) = \int_{B_X(1)} f(x-y)\kappa(x-y)\frac{\kappa(x)}{\kappa(x-y)}\,\mu_n(\mathrm{d}y) \le c_\kappa \|f^+\|_\kappa.$$

Moreover, for every sequence $(x_m)_{m\in\mathbb{N}} \subset X$ with $x_m \to x$, Fatou's lemma implies

$$\limsup_{m\to\infty}(f * \mu_n)(x_m) \le \int_X \limsup_{m\to\infty} f(x_m - y)\,\mu_n(\mathrm{d}y) \le \int_X f(x-y)\,\mu_n(\mathrm{d}y).$$

Second, we show that, for fixed $n \in \mathbb{N}$, the convolution is upper semicontinuous w.r.t. Γ-convergence. Let $(f_m)_{m\in\mathbb{N}} \subset \mathrm{U}_\kappa$ be bounded above, $x \in X$ and $(x_m)_{m\in\mathbb{N}} \subset X$ with $x_m \to x$. Since $\sup_{m\in\mathbb{N}}\|f_m^+\|_\kappa < \infty$ and $\nu(A) := \int_A \frac{1}{\kappa}\,\mathrm{d}\mu_n$ defines a finite Borel measure, we can apply Fatou's lemma to conclude that

$$\begin{aligned}\limsup_{m\to\infty}(f * \mu_n)(x_m) &= \limsup_{m\to\infty}\int_X f_m(x_m-y)\,\mu_n(\mathrm{d}y)\\ &\le \int_X \limsup_{m\to\infty} f_m(x_m-y)\,\mu_n(\mathrm{d}y) \le \int_X \Big(\Gamma\text{-}\limsup_{m\to\infty} f_m\Big)(x-y)\,\mu_n(\mathrm{d}y).\end{aligned}$$

Third, we show that $\Gamma\text{-}\limsup_{n\to\infty}(f * \mu_n) \le f$ for all $f \in \mathrm{U}_\kappa$. Let $x \in X$ and $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x$. Since f is upper semicontinuous, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $f(x_n - y) \le f(x) + \varepsilon$ for all $n \ge n_0$ and $y \in B_X(1/n)$. Hence,

$$(f * \mu_n)(x_n) = \int_{B_X(1/n)} f(x_n-y)\,\mu_n(\mathrm{d}y) \le f(x) + \varepsilon \quad \text{for all } n \ge n_0.$$

Letting $\varepsilon \downarrow 0$ yields $\limsup_{n\to\infty}(f * \mu_n)(x_n) \le f(x)$. □

For every $x \in X$, we define the shift operator $\tau_x \colon \mathrm{U}_\kappa \to \mathrm{U}_\kappa$ by

$$(\tau_x f)(y) := f(x+y) \quad \text{for all } y \in X.$$

Lemma 4.3.5. *Let $f \in \mathcal{L}^S_+$ such that $f_n := f * \mu_n \in D(A_\Gamma)$ for all $n \in \mathbb{N}$. Assume that, for every $\varepsilon > 0$, there exists $\delta, t_0 > 0$ with*

$$\|S(t)(\tau_x f) - \tau_x S(t) f\|_\kappa \le \varepsilon t \quad \text{for all } t \in [0, t_0] \text{ and } x \in B_X(\delta). \tag{4.16}$$

Then, it holds $f \in D(A_\Gamma)$ and $A_\Gamma f = \Gamma\text{-}\lim_{n\to\infty} A_\Gamma f_n$.

Proof. We verify the assumptions of Theorem 4.3.3. First, we show $f_n \to f$. Condition (4.14) yields $\|f_n^+\|_\kappa \le c_\kappa \|f^+\|_\kappa$ for all $n \in \mathbb{N}$. Moreover, for every $K \Subset X$, the continuity of f implies

$$\sup_{x\in K} |f_n(x) - f(x)| \le \sup_{x\in K} \int_{B_X(1/n)} |f(x-y) - f(x)| \, \mu_n(\mathrm{d}y) \to 0 \quad \text{as } n \to \infty.$$

Second, we verify condition (4.11). To do so, let $\varepsilon > 0$. By condition (4.16), there exist $h_0 > 0$ and $n_0 \in \mathbb{N}$ with

$$S(h)(\tau_{-y} f) \le \tau_{-y} S(h) f + \tfrac{\varepsilon h}{\kappa} \quad \text{for all } h \in [0, h_0] \text{ and } y \in B_X(1/n_0).$$

For every $h \in [0, h_0]$, $n \ge n_0$ and $x \in X$, we use Jensen's inequality and the monotonicity of $S(h)$ to estimate

$$\begin{aligned}
(S(h)f_n)(x) &= \left(S(h) \left(\int_{B_X(1/n)} (\tau_{-y} f)(\,\cdot\,) \, \mu_n(\mathrm{d}y) \right) \right)(x) \\
&\le \int_{B_X(1/n)} \big(S(h)(\tau_{-y} f)\big)(x) \, \mu_n(\mathrm{d}y) \le \int_{B_X(1/n)} \left(\big(\tau_{-y} S(h) f\big)(x) + \tfrac{\varepsilon h}{\kappa(x)} \right) \mu_n(\mathrm{d}y) \\
&= \left(S(h) f_n + \tfrac{\varepsilon h}{\kappa} \right)(x).
\end{aligned}$$

It follows from the linearity of the convolution and condition (4.15) that

$$\frac{S(h)f_n - f_n}{h} \le \left(\frac{S(h)f - f}{h} \right) * \mu_n + \frac{\varepsilon}{\kappa} \le \sup_{y \in B_X(\,\cdot\,, 1/n)} \left(\frac{S(h)f - f}{h} \right)(y) + \frac{\varepsilon}{\kappa} \tag{4.17}$$

for all $h \in (0, h_0]$ and $n \ge n_0$.

Third, we show that the sequence $(A_\Gamma f_n)_{n\in\mathbb{N}}$ is bounded above. By inequality (4.17) with $\varepsilon := 1$, Jensens's inequality and condition (4.14), there exists $n_0 \in \mathbb{N}$ with

$$\|(S(h)f_n - f_n)^+\|_\kappa \le \|(S(h)f - f)^+ * \mu_n\|_\kappa + h \le c\|(S(h)f - f)^+\|_\kappa + h$$

for all $h \in [0, 1]$ and $n \ge n_0$. Since $f \in \mathcal{L}^S_+$, we can choose $h_0 \in (0, 1]$ and $c' \ge 0$ with

$$\sup_{n\in\mathbb{N}} \|(S(h)f_n - f_n)^+\|_\kappa \le c\|(S(h)f - f)^+\|_\kappa + h \le (cc' + 1)h \quad \text{for all } h \in (0, h_0].$$

This shows that $(A_\Gamma^+ f_n)_{n\in\mathbb{N}}$ is bounded above. Now, Theorem 4.3.3 yields the claim. □

Regularization with sup-inf-convolution

Let X be a separable Hilbert space with norm $|\cdot|$. We fix $\kappa \equiv 1$ and denote by BUC the space of all bounded uniformly continuous functions $f\colon X \to \mathbb{R}$. For every $n \in \mathbb{N}$ and $f \in \mathrm{BUC}$, we define the sup-inf-convolution

$$\big(\theta_n f\big)(x) := \sup_{y\in X}\inf_{z\in X}\Big(f(z) + \tfrac{n}{2}|y-z|^2 - n|y-x|^2\Big). \tag{4.18}$$

It is shown in [129] that $\theta_n f \in \mathrm{Lip}_\mathrm{b}^1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\|\theta_n f - f\|_\infty = 0$ for all $f \in \mathrm{BUC}$, where $\mathrm{Lip}_\mathrm{b}^1$ denotes the space of all bounded Lipschitz continuous with bounded Lipschitz continuous first derivative. Define

$$B_{\mathrm{BUC}}(r) := \{f \in \mathrm{BUC}\colon \|f\|_\infty \le r\} \quad \text{for all } r \ge 0.$$

Lemma 4.3.6. *For every $r > 0$, $f, g \in B_{\mathrm{BUC}}(r)$, $n \in \mathbb{N}$ and $x \in X$,*

$$(\theta_n f - \theta_n g)(x) \le \sup_{y\in B(x,r_n)}(f-g)(y), \quad \textit{where} \quad r_n := (\sqrt{2}+2)\sqrt{\tfrac{r}{n}}.$$

Proof. Since the supremum in equation (4.18) can be restricted to the ball $B(x,\sqrt{2r/n})$ and the infimum to the ball $B(y,\sqrt{4r/n})$, we obtain

$$\begin{aligned}(\theta_n f - \theta_n g)(x) &\le \sup_{y\in B(x,\sqrt{2r/n})}\ \sup_{z\in B(y,\sqrt{4r/n})}\big(F(y,z) - G(y,z)\big)\\ &\le \sup_{y\in B(x,\sqrt{2r/n})}\ \sup_{z\in B(y,\sqrt{4r/n})}\big(f(z) - g(z)\big) = \sup_{z\in B(x,r_n)}\big(f(z)-g(z)\big),\end{aligned}$$

where $F(y,z) := f(z) + \frac{n}{2}|y-z|^2 - n|y-x|^2$ and $G(y,z) := g(z) + \frac{n}{2}|y-z|^2 - n|y-x|^2$. □

As an application of Theorem 4.3.3, we obtain the following result. Condition (i) is typically satisfied for first order equations, where we have $\mathrm{Lip}_\mathrm{b}^1 \subset D(A_\Gamma)$.

Lemma 4.3.7. *Assume that $S(t)\colon \mathrm{BUC} \to \mathrm{BUC}$ for all $t \ge 0$ and let $f \in \mathcal{L}_+^S \cap \mathrm{BUC}$ satisfy the following conditions:*

(i) $\theta_n f \in D(A_\Gamma)$ for all $n \in \mathbb{N}$.

(ii) $S(t)(\theta_n f) \le \theta_n S(t) f$ for all $n \in \mathbb{N}$ and $t \ge 0$.

Then, it holds $f \in D(A_\Gamma)$ and $A_\Gamma f = \Gamma\text{-}\lim_{n\to\infty} A_\Gamma(\theta_n f)$.

Proof. We verify the assumptions of Theorem 4.3.3. Define $f_n := \theta_n f$ for all $n \in \mathbb{N}$. By definition and [129], it holds $\sup_{n\in\mathbb{N}}\|f_n\|_\infty \le \|f\|_\infty$ and $\|\theta_n f - f\|_\infty \to 0$. Moreover, we use condition (ii), Assumption 4.2.4(iv) and Lemma 4.3.6 to choose a sequence $(r_n)_{n\in\mathbb{N}} \subset (0,\infty)$ with $r_n \to 0$ such that

$$S(h)f_n - f_n \le \theta_n S(h) f - f_n \le \sup_{y\in B(\cdot,r_n)}\big(S(h)f - f\big)(y) \quad \text{for all } h \in [0,1].$$

Hence, condition (4.11) is satisfied. Furthermore, since $f \in \mathcal{L}_+^S$, there exist $h_0 \in (0,1]$ and $c \ge 0$ with

$$\frac{S(h)f_n - f_n}{h} \le \sup_{y\in B(\cdot,r_n)}\frac{S(h)f - f}{h} \le c \quad \text{for all } h \in (0,h_0].$$

This shows that $(A_\Gamma^+ f_n)_{n\in\mathbb{N}}$ is bounded above and Theorem 4.3.3 yields the claim. □

4.4 The case $X = \mathbb{R}^d$

In this section, we provide explicit conditions which guarantee that two semigroups have the same upper Lipschitz set. In particular, we obtain that the abstract comparison principle for convex monotone semigroups from the previous section can already be applied if we only know that their generators evaluated at smooth functions coincide. This is a major improvement since, in many applications, the latter can be computed explicitly. In the sequel, we fix a bounded continuous function $\kappa\colon \mathbb{R}^d \to (0,\infty)$ which satisfies

$$c_\kappa := \sup_{x\in\mathbb{R}^d} \sup_{|y|\leq 1} \frac{\kappa(x)}{\kappa(x-y)} < \infty \tag{4.19}$$

and consider a family $(S(t))_{t\geq 0}$ of operators $S(t)\colon \mathrm{C}_\kappa(\mathbb{R}^d) \to \mathrm{C}_\kappa(\mathbb{R}^d)$. Moreover, we recall that $\mathrm{Lip_b}$ is the space of all bounded Lipschitz continuous functions $f\colon \mathbb{R}^d \to \mathbb{R}$ and, for every $r \geq 0$, the set $\mathrm{Lip_b}(r)$ consists of all functions $f\colon \mathbb{R}^d \to \mathbb{R}$ with

$$\max\left\{\sup_{x\in\mathbb{R}^d} |f(x)|,\, \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}\right\} \leq r.$$

The space $\mathrm{C}_\mathrm{b}^\infty$ contains all infinitely differentiable functions $f \in \mathbb{R}^d \to \mathbb{R}$ such that the partial derivatives of all orders are bounded and the shift operators are given by

$$(\tau_x f)(y) := f(x+y) \quad \text{for all } x,y\in\mathbb{R}^d \text{ and } f\colon \mathbb{R}^d \to \mathbb{R}.$$

The following definition summarises some basic terminology about nonlinear semigroups. Recall that limits (and thus continuity) in C_κ are understood w.r.t. the mixed topology rather than the norm $\|\cdot\|_\kappa$.

Definition 4.4.1. A family $(S(t))_{t\geq 0}$ of operators $S(t)\colon \mathrm{C}_\kappa \to \mathrm{C}_\kappa$ is called semigroup if $S(0) = \mathrm{id}_{\mathrm{C}_\kappa}$ and $S(s+t)f = S(s)S(t)f$ for all $s,t\geq 0$ and $f\in\mathrm{C}_\kappa$. The semigroup is called convex (monotone) if the mappings $\mathrm{C}_\kappa \to \mathbb{R},\ f\mapsto (S(t)f)(x)$ are convex (monotone) for all $t\geq 0$ and $x\in\mathbb{R}^d$. In addition, the semigroup is called strongly continuous if the mappings $\mathbb{R}_+ \to \mathrm{C}_\kappa,\ t\mapsto S(t)f$ are continuous for all $f\in\mathrm{C}_\kappa$. The generator of the semigroup is defined by

$$A\colon D(A) \to \mathrm{C}_\kappa,\ f\mapsto \lim_{h\downarrow 0} \frac{S(h)f-f}{h},$$

where the domain $D(A)$ consists of all $f\in\mathrm{C}_\kappa$ such that the previous limit exists.

4.4.1 The upper Lipschitz set

The proof that two semigroups, whose generators evaluated at smooth functions coincide, have the same upper Lipschitz set is based on the following estimate.

Lemma 4.4.2. *Let $(S(t))_{t\geq 0}$ be a convex semigroup on C_κ such that $S(t)$ is continuous from above for all $t\geq 0$. Moreover, for every $r,t\geq 0$, there exists $c\geq 0$ with*

$$\|S(s)f - S(s)g\|_\kappa \leq c\|f-g\|_\kappa \tag{4.20}$$

for all $s\in[0,t]$ and $f,g\in B_{\mathrm{C}_\kappa}(r)$. Then, for every $t\geq 0$ and $f\in D(A)$,

$$S(t)f - f \leq \int_0^t S(s)(f+Af) - S(s)f\,\mathrm{d}s.$$

Proof. For every $x \in \mathbb{R}^d$, it follows from $f \in D(A) \subset \mathcal{L}^S$ and inequality (4.20) that the mapping $\mathbb{R}_+ \to \mathbb{R},\ t \mapsto (S(t)f)(x)$ is locally Lipschitz continuous and thus differentiable almost everywhere. Hence, by Rademacher's theorem,

$$(S(t)f - f)(x) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(S(s)f)(x)\,\mathrm{d}s \quad \text{for all } t \geq 0.$$

For every $t \geq 0$ and $h \in (0,1]$, Lemma 3.6.1 implies

$$\begin{aligned}
&\frac{S(t)S(h)f - S(t)f}{h} - S(t)(f + Af) + S(t)f \\
&\leq S(t)\left(\frac{S(h)f - f}{h} + f\right) - S(t)(f + Af) \\
&\leq \frac{1}{2}S(t)\left(2\left(\frac{S(h)f - f}{h} - Af\right) + f + Af\right) - \frac{1}{2}S(t)(f + Af).
\end{aligned}$$

Since $S(t)$ is continuous, the right-hand side converges to zero as $h \downarrow 0$ which yields the claim. $\square$

The following assumption guarantees that two semigroups, whose generators evaluated at smooth functions coincide, have the same upper Lipschitz set.

Assumption 4.4.3. Let $(S(t))_{t\geq 0}$ be a convex monotone semigroup on C_κ satisfying $S(t)0 = 0$ and the following conditions:

(i) For every $r, T \geq 0$, there exists $c \geq 0$ with

$$\|S(t)f - S(t)g\|_\kappa \leq c\|f - g\|_\kappa \quad \text{for all } t \in [0,T] \text{ and } f, g \in B_{\mathrm{C}_\kappa}(r).$$

(ii) $S(t)$ is continuous from above for all $t \geq 0$.

(iii) For every $f \in \mathcal{L}^S_+ \cap \mathrm{Lip_b}$, there exist $L \geq 0$, $t_0 > 0$ and $\delta > 0$ with

$$\|S(t)(\tau_x f) - \tau_x S(t)f\|_\kappa \leq Lt$$

for all $t \in [0, t_0]$ and $x \in B_{\mathbb{R}^d}(\delta)$.

Theorem 4.4.4. *Let $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ be two semigroups satisfying Assumption 4.4.3 such that $\mathrm{C}_\mathrm{b}^\infty \subset D(A) \cap D(B)$ and*

$$(Af)^+ \leq (Bf)^+ \quad \textit{for all } f \in \mathrm{C}_\mathrm{b}^\infty,$$

where A and B denote the generators of $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$, respectively. Then,

$$\mathcal{L}^T_+ \cap \mathrm{Lip_b} \subset \mathcal{L}^S_+ \cap \mathrm{Lip_b}.$$

Proof. Let $f \in \mathcal{L}^T_+ \cap \mathrm{Lip_b}$ and choose $L \geq 0$, $t_0 > 0$ and $\delta > 0$ such that Assumption 4.4.3(iii) is valid. Moreover, let $\eta\colon \mathbb{R}^d \to \mathbb{R}_+$ be an infinitely differentiable function with $\operatorname{supp}(\eta) \subset B_{\mathbb{R}^d}(\delta)$ and $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$. For every $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$, we define $\eta_n(x) := n^d \eta(nx)$ and $f_n \in \mathrm{C}_\mathrm{b}^\infty$ by

$$f_n(x) := (f * \eta_n)(x) = \int_{\mathbb{R}^d} f(x - y)\eta_n(y)\,\mathrm{d}y.$$

For every $t \in [0, t_0]$, $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$, we use Jensen's inequality, Assumption 4.4.3(iii) and inequality (4.19) to estimate

$$\begin{aligned}(T(t)f_n - f_n)(x)\kappa(x) &\leq \int_{B(\delta)} \big(T(t)(\tau_{-y}f) - \tau_{-y}f\big)(x)\kappa(x)\eta_n(y)\,\mathrm{d}y \\ &\leq \int_{B(\delta)} (T(t)f - f)(x-y)\kappa(x)\eta_n(y)\,\mathrm{d}y + Lt \\ &\leq c_\kappa \int_{B(\delta)} (T(t)f - f)(x-y)\kappa(x-y)\eta_n(y)\,\mathrm{d}y + Lt \\ &\leq c_\kappa\big((T(t)f - f)^+\kappa\big) * \eta_n + Lt,\end{aligned}$$

where $B(\delta) := B_{\mathbb{R}^d}(\delta)$. Taking the supremum over $x \in \mathbb{R}^d$ yields

$$\|(T(t)f_n - f_n)^+\|_\kappa \leq c_\kappa \|(T(t)f - f)^+\|_\kappa + Lt.$$

Since $f \in \mathcal{L}_+^T$, there exist $c \geq 0$ and $t_1 \in (0, t_0]$ with

$$\|(T(t)f_n - f_n)^+\|_\kappa \leq (cc_\kappa + L)t \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [0, t_1].$$

Hence, we obtain $\|(Af_n)^+\|_\kappa \leq \|(Bf_n)^+\|_\kappa \leq cc_\kappa + L$ for all $n \in \mathbb{N}$. It follows from Lemma 4.4.2, the monotonicity of $S(s)$, Assumption 4.4.3(i) and inequality (4.19) that there exists $c' \geq 0$ with

$$\begin{aligned}(S(t)f_n - f_n)\kappa &\leq \int_0^t \big(S(s)(f_n + (Af_n)^+) - S(s)f_n\big)\kappa\,\mathrm{d}s \\ &\leq \int_0^t c'\|(Af_n)^+\|_\kappa \mathrm{d}s \leq c'\big(cc_\kappa + L\big)t\end{aligned}$$

for all $n \in \mathbb{N}$ and $t \in [0, t_1]$. Taking the limit $n \to \infty$ yields

$$\|(S(t)f - f)^+\|_\kappa \leq c'\big(cc_\kappa + L\big)t \quad \text{for all } t \in [0, t_1]$$

which shows that $\mathcal{L}_+^T \cap \mathrm{Lip_b} \subset \mathcal{L}_+^S \cap \mathrm{Lip_b}$. □

Assumption 4.4.5. Let $(S(t))_{t\geq 0}$ be a strongly continuous convex monotone semigroup on C_κ with $S(t)0 = 0$ such that the following conditions are satisfied:

(i) For every $r, T \geq 0$, there exists $c \geq 0$ with

$$\|S(t)f - S(t)g\|_\kappa \leq c\|f - g\|_\kappa \quad \text{for all } t \in [0, T] \text{ and } f, g \in B_{\mathrm{C}_\kappa}(r).$$

(ii) $S(t)$ is continuous from above for all $t \geq 0$.

(iii) For every $f \in \mathrm{Lip_b}$ and $\varepsilon > 0$, there exist $\delta, t_0 > 0$ with

$$\|S(t)(\tau_x f) - \tau_x S(t)f\|_\kappa \leq \varepsilon t$$

for all $t \in [0, t_0]$ and $x \in B_{\mathbb{R}^d}(\delta)$.

(iv) It holds $S(t)\colon \mathrm{Lip_b} \to \mathrm{Lip_b}$ for all $t \geq 0$.

Theorem 4.4.6. *Let $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ be two semigroups satisfying Assumption 4.4.5 such that $\mathrm{C}_\mathrm{b}^\infty \subset D(A)\cap D(B)$ and*

$$Af \leq Bf \quad \textit{for all } f\in \mathrm{C}_\mathrm{b}^\infty,$$

where A and B denote the generators of $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$, respectively. Then,

$$S(t)f \leq T(t)f \quad \textit{fort all } t\geq 0 \textit{ and } f\in \mathrm{C}_\kappa.$$

Proof. It follows from Theorem 4.4.4 that

$$\mathcal{L}_+^T \cap \mathrm{Lip_b} \subset \mathcal{L}_+^S \cap \mathrm{Lip_b} \tag{4.21}$$

and therefore Theorem 4.2.9 implies

$$S(t)f \leq T(t)f \quad \text{for all } (f,t)\in \mathrm{C}_\mathrm{b}^\infty \times \mathbb{R}_+. \tag{4.22}$$

Indeed, the semigroups $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ clearly satisfy Assumption 4.2.4 while equation (4.21) and Assumption 4.4.5(i) and (iv) imply $\mathcal{L}^T \cap \mathrm{Lip_b} \subset \mathcal{L}_+^S$ and

$$T(t)\colon \mathcal{L}^T \cap \mathrm{Lip_b} \to \mathcal{L}^T \cap \mathrm{Lip_b} \quad \text{for all } t\geq 0.$$

In addition, due to Assumption 4.4.5(i) and (iv), we can apply Theorem 4.3.2 and Lemma 4.3.5 to conclude that

$$A_\Gamma^+ f \leq B_\Gamma^+ f \quad \text{for all } \mathcal{L}^T \cap \mathrm{Lip_b}.$$

For instance, one can choose $f_n := f*\eta_n \in \mathrm{C}_\mathrm{b}^\infty$, where $\eta_n(x) := n^d\eta(nx)$ for an infinitely differentiable function $\eta\colon \mathbb{R}^d \to \mathbb{R}_+$ with $\operatorname{supp}(\eta)\subset B_{\mathbb{R}^d}(1)$ and $\int_{\mathbb{R}^d}\eta(x)\,\mathrm{d}x = 1$. We obtain $S(t)f\leq T(t)f$ for all $t\geq 0$ and $f\in \mathrm{C}_\mathrm{b}^\infty$. Since $\mathrm{C}_\mathrm{b}^\infty \subset \mathrm{C}_\kappa$ is dense, it follows from inequality (4.22), Assumption 4.4.5(ii) and Corollary 3.3.5 that

$$S(t)f \leq T(t)f \quad \text{fort all } t\geq 0 \text{ and } f\in \mathrm{C}_\kappa. \qquad \square$$

4.4.2 Link to distributional derivatives

The aim of this subsection is to provide conditions under which the Γ-generator can be identified with a convex differential operator, where the partial derivatives are supposed to exist as regular distributions. For that purpose, let $I\subset \mathbb{N}_0^d$ be an index set and $H\colon \mathbb{R}^I \to \mathbb{R}$ be a convex function. At this point we would like to emphasize that $A_\Gamma f$ is an upper semicontinuous function while $g := H((D^\alpha f)_{\alpha\in I})$ is only defined almost everywhere. Hence, in order to identify $A_\Gamma f$ with the upper semicontinuous hull $\overline{g}$, we have to choose a suitable representative in the equivalence class of g. We proceed as follows. For a locally integrable function $f\colon \mathbb{R}^d\to\mathbb{R}$, the set X_f contains all points $x\in\mathbb{R}^d$ such that the limit

$$\tilde{f}(x) := \lim_{r\downarrow 0} \fint_{B(x,r)} f(y)\,\mathrm{d}y$$

exists.[1] Note that the set X_f does, in contrast to the Lebesgue set of all $x \in \mathbb{R}^d$ with $\tilde{f}(x) = f(x)$, not depend on the choice of the representative in the equivalence class of f. Furthermore, by the Lebesgue differentiation theorem, the complement of the Lebesgue set has measure zero, see [157, Corollary 3.1.6]. In particular, we obtain $\lambda(X_f^c) = 0$ and therefore $X_f \subset \mathbb{R}^d$ is dense. Subsequently, we fix an infinitely differentiable function $\eta\colon \mathbb{R}^d \to \mathbb{R}_+$ with $\operatorname{supp}(\eta) \subset B_{\mathbb{R}^d}(1)$ and $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$. For every $n \in \mathbb{N}$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$, we define $\eta_n(x) := n^d \eta(nx)$ and $f_n \in \mathrm{C_b^\infty}$ by

$$f_n(x) := (f * \eta_n)(x) = \int_{\mathbb{R}^d} f(x-y)\eta_n(y)\,\mathrm{d}y.$$

Theorem 4.4.7. *Let $I \subset \mathbb{N}_0^d$ be an index set and $H\colon \mathbb{R}^I \to \mathbb{R}$ be a convex function. Furthermore, let $f \in \mathcal{L}_+^S$ such that, for every $\varepsilon > 0$, there exist $\delta, t_0 > 0$ with*

$$\|S(t)(\tau_x f) - \tau_x S(t) f\|_\kappa \le \varepsilon t \quad \text{for all } t \in [0, t_0] \text{ and } x \in B_{\mathbb{R}^d}(\delta).$$

For every $n \in \mathbb{N}$, we define $f_n := f * \eta_n$ and assume that

- *(i) $D^\alpha f$ and $D^\alpha f_n$ exist as regular distributions for all $\alpha \in I$,*
- *(ii) $H((D^\alpha f)_{\alpha\in I})$ and $H((D^\alpha f_n)_{\alpha \in I})$ are locally integrable,*
- *(iii) $f_n \in D(A_\Gamma)$ and $A_\Gamma f_n = H((D^\alpha f_n)_{n\in\mathbb{N}})$.*

Furthermore, we define the functions

$$\begin{aligned} g(x) &:= H\big((D^\alpha f(x))_{\alpha\in I}\big) \quad \text{for all } x \in \mathbb{R}^d, \\ \tilde{g}(x) &:= \lim_{r\downarrow 0} ⨍_{B(x,r)} g(y)\,\mathrm{d}y \quad \text{for all } x \in X_g, \\ \overline{g}(x) &:= \limsup_{y\in X_g, y\to x} \tilde{g}(y) \quad \text{for all } x \in \mathbb{R}^d. \end{aligned}$$

Then, it holds $f \in D(A_\Gamma)$ and $(A_\Gamma f)(x) = \overline{g}(x)$ for all $x \in \mathbb{R}^d$.

Proof. It follows from Lemma 4.3.5 that $f \in D(A_\Gamma)$ and $A_\Gamma f = \Gamma\text{-}\lim_{n\to\infty} A f_n$. First, we show $\overline{g} \le A_\Gamma f$. Define $F := (D^\alpha f)_{\alpha\in I}$ and $F_n := (D^\alpha f_n)_{\alpha\in I}$ for all $n \in \mathbb{N}$. It holds $F_n = F * \eta_n$ for all $n \in \mathbb{N}$, where the convolution of the vector valued-function F with η_n is understood componentwise. Hence, we can use condition (i) and [157, Theorem 3.2.1] to obtain $F_n \to F$ almost everywhere. The continuity of H and condition (iii) imply

$$g = H(F) = \lim_{n\to\infty} H(F_n) = \lim_{n\to\infty} A_\Gamma f_n \le \Gamma\text{-}\limsup_{n\to\infty} A_\Gamma f_n = A_\Gamma f$$

almost everywhere. This yields the estimate

$$⨍_{B(x,r)} g(y)\,\mathrm{d}y \le ⨍_{B(x,r)} (A_\Gamma f)(y)\,\mathrm{d}y \quad \text{for all } x \in \mathbb{R}^d \text{ and } r > 0.$$

[1] Denoting by λ the Lebesgue measure, the normalized integral is given by

$$⨍_{B(x,r)} f(y)\,\mathrm{d}y := \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f(y)\,\mathrm{d}y.$$

For every $x \in X_g$, it follows from the upper semicontinuity of $A_\Gamma f$ that

$$\tilde{g}(x) = \lim_{r \downarrow 0} \fint_{B(x,r)} g(y)\,\mathrm{d}y \le \limsup_{r \downarrow 0} \fint_{B(x,r)} (A_\Gamma f)(y)\,\mathrm{d}y \le (A_\Gamma f)(x).$$

In particular, $\tilde{g}$ is bounded above and therefore $\overline{g}(x) < \infty$ for all $x \in \mathbb{R}^d$. We use again that $A_\Gamma f$ is upper semicontinuous in order to conclude that

$$\overline{g}(x) = \limsup_{y \in X_g, y \to x} \tilde{g}(y) \le \limsup_{y \in X_g, y \to x} (A_\Gamma f)(y) \le (A_\Gamma f)(x) \quad \text{for all } x \in \mathbb{R}^d.$$

Second, we show $A_\Gamma f \le \overline{g}$. Let $x \in \mathbb{R}^d$ and $\varepsilon > 0$. Since $A_\Gamma f = \Gamma\text{-}\lim_{n\to\infty} A_\Gamma f_n$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ with $x_n \to x$ and $(A_\Gamma f_n)(x_n) \to (A_\Gamma f)(x)$. In addition, since $\overline{g}$ is upper semicontinuous, there exists $\delta > 0$ with $\overline{g}(y) < \overline{g}(x) + \varepsilon$ for all $y \in B(x,\delta)$. Choose $n_0 \in \mathbb{N}$ with $B(x_n, 1/n) \subset B(x,\delta)$ for all $n \ge n_0$. Since g is locally integrable, it holds $H(F) = g = \tilde{g} \le \overline{g}$ almost everywhere. It follows from the previous considerations and Jensen's inequality that

$$\begin{aligned}
\big(A_\Gamma f_n\big)(x_n) = H\big(F_n(x_n)\big) &= H\left(\int_{B(1/n)} F(x_n - y)\eta_n(y)\,\mathrm{d}y\right) \\
&\le \int_{B(1/n)} H\big(F(x_n - y)\big)\eta_n(y)\,\mathrm{d}y \\
&\le \int_{B(1/n)} \overline{g}(x_n - y)\eta_n(y)\,\mathrm{d}y \le \overline{g}(x) + \varepsilon.
\end{aligned}$$

We obtain $(A_\Gamma f)(x) = \lim_{n\to\infty} A_\Gamma f_n(x_n) \le \overline{g}(x)$ for all $x \in \mathbb{R}^d$. □

In several examples, the set $\mathcal{L}^S_{\mathrm{sym}} \cap \mathrm{Lip_b}$ is invariant under the semigroup $(S(t))_{t\ge 0}$ and can be represented explicitly by means of Sobolev spaces. In addition, one can show that $\mathrm{C}^\infty_\mathrm{b} \subset D(A_\Gamma)$ and $A_\Gamma f = H((D^\alpha f)_{\alpha\in I})$ for all $f \in \mathrm{C}^\infty_\mathrm{b}$. Hence, for every $f \in \mathcal{L}^S_{\mathrm{sym}} \cap \mathrm{Lip_b}$, the previous theorem yields that $u(t) := S(t)f$ solves the equation

$$\Gamma\text{-}\lim_{h\downarrow 0} \frac{u(t+h) - u(t)}{h} = H\big((D^\alpha u(t))_{\alpha\in I}\big) \quad \text{for all } t \ge 0.$$

Although these ideas can be carried out by rather elementary calculations, as it is shown in Subsection 6.1, we want to discuss a general result from [21] that locality of the generator implies the existence of a function H with $Af = H((D^\alpha f)_{\alpha\in I})$ for sufficiently smooth f.

Remark 4.4.8. Let $\kappa \equiv 1$ and suppose that $S(t)\colon \mathrm{BUC} \to \mathrm{BUC}$ for all $t \ge 0$, $S(s+t)f = S(s)S(t)f$ for all $s,t \ge 0$ and $f \in \mathrm{BUC}$ and that the mapping $t \mapsto S(t)f$ is continuous w.r.t. the supremum norm for all $f \in \mathrm{BUC}$. Recall that BUC contains all bounded uniformly continuous functions $f\colon \mathbb{R}^d \to \mathbb{R}$. Furthermore, suppose that $\mathrm{C}^\infty_\mathrm{b} \subset D(A)$ and that there exists convex function $H\colon \mathbb{R}^I \to \mathbb{R}$ with

$$Af = H((D^\alpha f)_{\alpha\in I}) \quad \text{for all } f \in \mathrm{C}^\infty_\mathrm{b}, \tag{4.23}$$

where $I := \{\alpha \in \mathbb{N}_0^d \colon |\alpha| \le 2\}$. Then, A is local in sense that, for fixed $x \in \mathbb{R}^d$, it holds $Af(x) = Ag(x)$ if $f, g \in \mathrm{C}^\infty_\mathrm{b}$ coincide on an open neighbourhood of x. On the other hand, it was shown in [21] that locality and a certain regularity of the semigroup already imply the existence of a convex function H satisfying equation (4.23).

To formulate the conditions from [21], we denote by C^∞ the space of all infinitely differentiable functions $f\colon \mathbb{R}^d \to \mathbb{R}$ and by $\mathrm{C}_\mathrm{c}^\infty$ the subset of all $f \in C^\infty$ with compact support. For every sequence $r := (r_n)_{n\in\mathbb{N}} \subset [0,\infty)$, let Q_r be the set of all $f \in \mathrm{C}_\mathrm{c}^\infty$ with $\|D^\alpha f\|_\infty \le r_n$ for all $n \in \mathbb{N}$ and $|\alpha| \le n$. The semigroup $(S(t))_{t\ge 0}$ is called regular if the following holds. Let $(r_n)_{n\in\mathbb{N}} \subset [0,\infty)$, $f \in \mathrm{BUC} \cap C^\infty$ and $K \subset \mathbb{R}^d$ compact. Then, for every $T \ge 0$ and $\varepsilon > 0$, there exists $\delta_0 > 0$ such that, for all $t \in [0,T]$, $\delta \in (0,\delta_0]$, $g \in Q_r$ and $x \in K$

$$\big|\big(S(t)(f+\delta g)\big)(x) - \big(S(t)f\big)(x) - \delta g(x)\big| \le \varepsilon t.$$

Moreover, the semigroup $(S(t))_{t\ge 0}$ is called local if the following holds. Let $x \in \mathbb{R}^d$ and $f, g \in \mathrm{BUC} \cap C^\infty$ which coincide on an open neighbourhood of x. Then, for every $\varepsilon > 0$, there exists $h_0 > 0$ with

$$|S(h)f - S(h)g|(x) < \varepsilon h \quad \text{for all } h \in [0,h_0].$$

In particular, this condition implies that $Af(x) = Ag(x)$ if $f, g \in D(A)$.

In the sequel, let $(S(t))_{t\ge 0}$ be regular and local. Then, by [21, Theorem 3.1], there exists a continuous function $H\colon \mathbb{R}^d \times \mathbb{R}^I \to \mathbb{R}$ with

$$Af(x) = H\big(x, (D^\alpha f(x))_{\alpha\in I}\big) \quad \text{for all } f \in \mathrm{C}_\mathrm{b}^\infty \text{ and } x \in \mathbb{R}^d.$$

It follows from the proof of [21, Theorem 3.1] that convexity of $(S(t))_{t\ge 0}$ implies convexity of the mapping $H(x,\cdot\,)\colon \mathbb{R}^I \to \mathbb{R}$ for all $x \in \mathbb{R}^d$. Furthermore, if $(S(t))_{t\ge 0}$ is translation invariant, i.e., $S(t)(\tau_x f) = \tau_x S(t)f$, then H does not depend on $x \in \mathbb{R}^d$, see [21, Proposition 4.1].

Chapter 5

Chernoff-type approximations in the mixed topology

In Chapter 1, we already studied iterative approximation schemes for semigroups which resemble the ideas of Chernoff's original work, see [43, 44], that generalizes the Trotter–Kato product formula for linear semigroups, see [105, 160]. Here, we pick up the ideas from Chapter 1, but there are some major improvements which will be discussed in the following. We start with a generating family $(I(t))_{t\geq 0}$ of operators $I(t)\colon \mathrm{C}_\kappa \to \mathrm{C}_\kappa$ which do not form a semigroup but have, for smooth functions, the derivative

$$I'(0)f := \lim_{h\downarrow 0} \frac{I(h)f - f}{h} \in \mathrm{C}_\kappa.$$

In order to obtain a corresponding semigroup $(S(t))_{t\geq 0}$ on C_κ with prescribed generator $Af = I'(0)f$, we iterate the operators $(I(t))_{t\geq 0}$ over a sequence of equidistant partitions with mesh size tending to zero and define the semigroup as the limit

$$S(t)f := \lim_{n\to\infty} I\big(\tfrac{t}{n}\big)^n f \in \mathrm{C}_\kappa. \tag{5.1}$$

Recall that, in case of the Trotter–Kato formula, the choice $I(t) := e^{tA_1}e^{tA_2}$ leads to a semigroup with generator $A = A_1 + A_2$. There are two major questions to address.

First, we need a suitable topology for the convergence in equation (5.1). The norm convergence required in Chapter 1 restricts the results to functions which (after multiplication with a weight function) vanish at infinity. Furthermore, we only obtain convergence for a subsequence and it is a priori not clear whether semigroups constructed in this way satisfy the comparison principle from Chapter 4. In the particular case of Nisio semigroups, see [60, 136, 138], the condition

$$I(s+t)f \leq I(s)I(t)f \quad \text{for all } s,t\geq 0 \text{ and } f\in \mathrm{C}_\kappa$$

guarantees that the sequence $(I(2^{-n}t)^{2^n}f)_{n\in\mathbb{N}}$ is non-decreasing. Hence, the semigroup can be defined as a pointwise monotone limit

$$S(t)f = \lim_{n\to\infty} I(2^{-n})^{2^n} f = \sup_{n\in\mathbb{N}} I(2^{-n}t)^{2^n} f$$

without choosing a subsequence. Here, typical examples are of the form

$$I(t)f := \sup_{\lambda\in\Lambda} \big(S_\lambda(t)f - \varphi(\lambda)t\big)$$

for a family of linear semigroups $(S_\lambda(t))_{t\geq 0}$. In some applications it is also possible to define the semigroup as the limit of a decreasing sequence, see [14, 85], where the authors study semigroups corresponding to Markov processes with uncertain transition probabilities. However, apart from certain applications where the one-step operators $(I(t))_{t\geq 0}$ have a particular structure, requiring monotone converge or convergence w.r.t. the supremum norm keeps us from considering general approximation schemes. At this point we would like to mention that, by Dini's theorem, monotone convergence implies convergence w.r.t. the mixed topology. In addition, compactness in the mixed topology can be characterized by means of Arzéla-Ascoli's theorem as stated in Lemma 3.5.1, i.e., it is sufficient to require equicontinuity of the sequence $(I(t/n)^n f)_{n\in\mathbb{N}}$ rather than monotonicity. The latter property will, for instance, never be satisfied in the application of Chernoff-type approximations to limit theorems for convex expectations which will be presented in Chapter 7.

Second, since the convergence in equation (5.1) is based on relative compactness, we obtain a priori only convergence for a subsequence, i.e.,

$$S(t)f := \lim_{l\to\infty} I\left(\tfrac{t}{n_l}\right)^{n_l} f. \tag{5.2}$$

Following the arguments in Chapter 1, one can only show that the convergent subsequence can be chosen independent of $f \in \mathrm{C}_\kappa$ and $t \in \mathcal{T}$, where $\mathcal{T} \subset [0,\infty)$ is countable and dense, see Theorem 5.1.2. If the semigroup $(S(t))_{t\geq 0}$ was uniquely determined by its infinitesimal generator $Af = I'(0)f$, one could argue that the limit in equation (5.2) does not depend on the choice of the convergent subsequence and thus obtain the desired convergence in equation (5.1). At this point we would like to emphasize that this can seemingly only be achieved by verifying a comparison principle. In [92], the authors show that the function $u(t) := S(t)f$ is a viscosity solution of the abstract Cauchy problem $\partial_t u = Au$ with initial condition $u(0) = f$ but uniqueness has to be verified on a case-by-case basis. In addition, for general state spaces, it is currently an open question how to guarantee that two semigroups, which are obtained from choosing different convergent subsequences in equation (5.2), have the same upper Lipschitz set and the same upper Γ-generator. The latter is required for the comparison principle stated in Theorem 4.2.9. However, in the case $X = \mathbb{R}^d$, we can provide explicit conditions on the generating family $(I(t))_{t\geq 0}$ which are transferred to the semigroup $(S(t))_{t\geq 0}$ and guarantee that Assumption 4.4.5 is valid. Since the comparison principle stated in Theorem 5.2.3 only requires knowledge of the generator evaluated at smooth functions, which given by $Af = I'(0)f$, we obtain the desired convergence in equation (5.1) and the semigroup $(S(t))_{t\geq 0}$ is uniquely determined by the infinitesimal behaviour of the generating family $(I(t))_{t\geq 0}$.

5.1 Generating families

Let $(I(t))_{t\geq 0}$ be a family of operators $I(t)\colon \mathrm{C}_\kappa \to \mathrm{C}_\kappa$ and $(h_n)_{n\in\mathbb{N}} \subset (0,\infty)$ be a sequence with $h_n \to 0$. For every $n \in \mathbb{N}$, $t \geq 0$ and $f \in \mathrm{C}_\kappa$, we define

$$I(\pi_n^t)f := I(h_n)^{k_n^t} f = \underbrace{\big(I(h_n) \circ \ldots \circ I(h_n)\big)}_{k_n^t \text{ times}} f,$$

where $k_n^t := \max\{k \in \mathbb{N}_0 \colon kh_n \leq t\}$ and $\pi_n^t := \{0, h_n, \dots, k_n^t h_n\}$ denotes the corresponding equidistant partition with mesh size h_n. Recall that the Lipschitz set $\mathcal{L}^I$ consists of all $f \in \mathrm{C}_\kappa$ such that there exist $t_0 > 0$ and $c \geq 0$ with

$$\|I(t)f - f\|_\kappa \leq ct \quad \text{for all } t \in [0, t_0].$$

Moreover, for every $f \in \mathrm{C}_\kappa$ such that the following limit exists, we define the derivative of the mapping $t \mapsto I(t)f$ at zero by

$$I'(0)f := \lim_{h \downarrow 0} \frac{I(h)f - f}{h} \in \mathrm{C}_\kappa.$$

Since convergence w.r.t. the mixed topology requires boundedness w.r.t. the norm, the existence of the limit $I'(0)f$ implies $f \in \mathcal{L}^I$.

Assumption 5.1.1. Suppose that $(I(t))_{t \geq 0}$ satisfies the following conditions:

(i) $I(0) = \mathrm{id}_{\mathrm{C}_\kappa}$.

(ii) $I(t)$ is convex and monotone with $I(t)0 = 0$ for all $t \geq 0$.

(iii) There exists a function $\alpha \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, which is non-decreasing in the second argument, such that, for every $r, s, t \geq 0$,

$$I(t) \colon B_{\mathrm{C}_\kappa}(r) \to B_{\mathrm{C}_\kappa}(\alpha(r, t)) \quad \text{and} \quad \alpha(\alpha(r, s), t) \leq \alpha(r, s + t).$$

(iv) For every $r \geq 0$, there exists $\omega_r \geq 0$ with

$$\|I(t)f - I(t)g\|_\kappa \leq e^{t\omega_r} \|f - g\|_\kappa \quad \text{for all } t \geq 0 \text{ and } f, g \in B_{\mathrm{C}_\kappa}(r).$$

Moreover, the mapping $r \mapsto \omega_r$ is non-decreasing.

(v) There exists a countable set $\mathcal{D} \subset \mathcal{L}^I$ such that the sequence $(I(\pi_n^t)f)_{n \in \mathbb{N}}$ is uniformly equicontinuous for all $(f, t) \in \mathcal{D} \times \mathcal{T}$. Moreover, for every $f \in \mathrm{C}_\kappa$, there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{D}$ with $\|f_n\|_\kappa \leq \|f\|_\kappa$ for all $n \in \mathbb{N}$ and $f_n \to f$.

(vi) For every $\varepsilon > 0$, $r, T \geq 0$ and $K \Subset X$, there exist $K' \Subset X$ and $c \geq 0$ with

$$\|I(\pi_n^t)f - I(\pi_n^t)g\|_{\infty, K} \leq c\|f - g\|_{\infty, K'} + \varepsilon$$

for all $t \in [0, T]$ and $f, g \in B_{\mathrm{C}_\kappa}(r)$.

In view of Lemma 3.3.4, condition (vi) is equivalent to the assumption that, for every $T \geq 0$, $K \Subset X$ and $(f_n)_{n \in \mathbb{N}} \subset \mathrm{C}_\kappa$ with $f_n \downarrow 0$,

$$\sup_{(t,x) \in [0,T] \times K} \sup_{n \in \mathbb{N}} \big(I(\pi_n^t) f_k\big)(x) \downarrow 0 \quad \text{as } k \to \infty.$$

Furthermore, in Section 3.4, we discussed sufficient conditions on the one-step operators $I(t)$ which guarantee that the iterated operators $I(\pi_n^t)$ satisfy condition (vi), see, e.g., Corollary 3.4.3, Corollary 3.4.7 and Corollary 3.4.8.

Theorem 5.1.2. *Suppose that Assumption 5.1.1 is satisfied and let $\mathcal{T} \subset \mathbb{R}_+$ be a countable dense set including zero. Then, there exist a strongly continuous convex monotone semigroup $(S(t))_{t\geq 0}$ on C_κ with $S(t)0 = 0$ and a subsequence $(n_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ such that*

$$S(t)f = \lim_{l\to\infty} I(\pi^t_{n_l})f \quad \text{for all } (f,t) \in \mathrm{C}_\kappa \times \mathcal{T}. \tag{5.3}$$

Furthermore, the following statements are valid:

(i) It holds $f \in D(A)$ and $Af = I'(0)f$ for all $f \in \mathrm{C}_\kappa$ such that $I'(0)f \in \mathrm{C}_\kappa$ exists.

(ii) For every $r, t \geq 0$ and $f, g \in B_{\mathrm{C}_\kappa}(r)$,

$$\|S(t)f\|_\kappa \leq \alpha(r,t) \quad \text{and} \quad \|S(t)f - S(t)g\|_\kappa \leq e^{t\omega_{\alpha(r,t)}}\|f-g\|_\kappa.$$

(iii) For every $\varepsilon > 0$, $r, T \geq 0$ and $K \Subset X$, there exist $K' \Subset X$ and $c \geq 0$ with

$$\|S(t)f - S(t)g\|_{\infty,K} \leq c\|f-g\|_{\infty,K'} + \varepsilon$$

for all $t \in [0,T]$ and $f, g \in B_{\mathrm{C}_\kappa}(r)$.

(iv) It holds $\mathcal{L}^I \subset \mathcal{L}^S$ and $S(t)\colon \mathcal{L}^S \to \mathcal{L}^S$ for all $t \geq 0$.

Proof. First, we show that there exist a family $(S(t))_{t\geq 0}$ of convex monotone operators $S(t)\colon \mathrm{C}_\kappa \to \mathrm{C}_\kappa$ and a subsequence $(n_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ with

$$S(t)f = \lim_{l\to\infty} I(\pi^t_{n_l})f \quad \text{for all } (f,t) \in \mathcal{D} \times \mathcal{T}. \tag{5.4}$$

Assumption 5.1.1(iii) and (iv) and Lemma 1.2.7 imply

$$I(\pi^t_n)f \in B_{\mathrm{C}_\kappa}(\alpha(r,t)) \quad \text{and} \quad \|I(\pi^t_n)f - I(\pi^t_n)g\|_\kappa \leq e^{t\omega_{\alpha(r,t)}}\|f-g\|_\kappa \tag{5.5}$$

for all $n \in \mathbb{N}$, $r, t \geq 0$ and $f, g \in B_{\mathrm{C}_\kappa}(r)$. Hence, we can use Assumption 5.1.1(v), Lemma 3.5.1 and a diagonalization argument to choose a subsequence $(n_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ such that the limit

$$S(t)f := \lim_{l\to\infty} I(\pi^t_{n_l})f \in \mathrm{C}_\kappa$$

exists for all $(f,t) \in \mathcal{D} \times \mathcal{T}$. Moreover, for every $f \in \mathcal{D}$, Lemma 1.2.8 guarantees the existence of $c \geq 0$ and $t_0 > 0$ with

$$\|I(\pi^s_n)f - I(\pi^t_n)f\|_\kappa \leq ce^{T\omega_{\alpha(r,T)}}(|s-t| + h_n) \tag{5.6}$$

for all $T \geq 0$, $s, t \in [0,T]$ and $n \in \mathbb{N}$ with $h_n \leq t_0$. Hence, for every $f \in \mathcal{D}$, Lemma 1.2.9 implies that the mapping $\mathcal{T} \to \mathrm{C}_\kappa$, $t \mapsto S(t)f$ has an extension to $\mathbb{R}_+$ which satisfies

$$\|S(s)f - S(t)f\|_\kappa \leq ce^{T\omega_{\alpha(r,T)}}|s-t| \quad \text{for all } T \geq 0 \text{ and } s, t \in [0,T]. \tag{5.7}$$

By construction, the operators $S(t)$ are convex and monotone with $S(t)0 = 0$ for all $t \geq 0$. Moreover, statement (ii) is valid for all $(f,t) \in \mathcal{D} \times \mathbb{R}_+$.

Second, we extend $(S(t))_{t\geq 0}$ from $\mathcal{D}$ to C_κ. For every $\varepsilon > 0$, $r, T \geq 0$ and $K \Subset X$, it follows from Assumption 5.1.1(vi) and equation (5.4) that there exist $K' \Subset X$ and $c \geq 0$ with

$$\|S(t)f - S(t)g\|_{\infty,K} \leq c\|f-g\|_{\infty,K'} + \varepsilon \tag{5.8}$$

for all $t \in [0,T]$ and $f,g \in B_{\mathrm{C}_\kappa}(r) \cap \mathcal{D}$. For every $f \in B_{\mathrm{C}_\kappa}(r)$, Assumption 5.1.1(v) yields a sequence $(f_n)_{n\in\mathbb{N}} \subset B_{\mathrm{C}_\kappa}(r) \cap \mathcal{D}$ with $\|f_n\|_\kappa \le \|f\|_\kappa$ for all $n \in \mathbb{N}$ and $f_n \to f$. Inequality (5.5) and inequality (5.8) guarantee that the limit

$$S(t)f := \lim_{n\to\infty} S(t)f_n \in \mathrm{C}_\kappa$$

exists and is independent of the choice of the approximating sequence $(f_n)_{n\in\mathbb{N}}$. Moreover, the statements (ii) and (iii) are valid for arbitrary functions in C_κ and $\mathcal{L}^I \subset \mathcal{L}^S$ can be derived similarly to previously shown inclusion $\mathcal{D} \subset \mathcal{L}^S$. Next, we verify equation (5.3). Let $(f,t) \in \mathrm{C}_\kappa \times \mathcal{T}$, $\varepsilon > 0$ and $K \Subset X$. Define $r := \|f\|_\kappa$ and choose $K' \Subset X$ and $c \ge 0$ such that Assumption 5.1.1(vi) and inequality (5.8) are valid for arbitrary $f,g \in B_{\mathrm{C}_\kappa}(r)$. Since Assumption 5.1.1(v) yields $g \in B_{\mathrm{C}_\kappa}(r) \cap \mathcal{D}$ with $\|f-g\|_{\infty,K'} < \varepsilon$, we obtain

$$\begin{aligned}\|S(t)f - I(\pi_{n_l}^t)f\|_{\infty,K} &\le \|S(t)f - S(t)g\|_{\infty,K} + \|S(t)g - I(\pi_{n_l}^t)g\|_{\infty,K} \\ &\quad + \|I(\pi_{n_l}^t)g - I(\pi_{n_l}^t)f)\|_{\infty,K} \\ &\le 2c\|f-g\|_{\infty,K'} + 2\varepsilon + \|S(t)g - I(\pi_{n_l}^t)g\|_{\infty,K} \\ &\le 2(c+1)\varepsilon + \|S(t)g - I(\pi_{n_l}^t)g\|_{\infty,K}.\end{aligned}$$

Hence, it follows from equation (5.4) that

$$\lim_{l\to\infty} \|S(t)f - I(\pi_{n_l}^t)f\|_{\infty,K} = 0 \quad \text{for all } (f,t) \in \mathrm{C}_\kappa \times \mathcal{T}. \tag{5.9}$$

In addition, for every $t \ge 0$, $f \in \mathrm{C}_\kappa$, $\varepsilon > 0$ and $K \Subset X$, Assumption 5.1.1(v) and equation (5.8) guarantee that there exists $g \in \mathcal{D}$ with

$$\|S(s)f - S(t)f\|_{\infty,K} \le \|S(s)g - S(t)g\|_{\infty,K} + \varepsilon \quad \text{for all } s \in [0,t+1].$$

Hence, it follows from inequality (5.7) that $\lim_{s\to t} \|S(s)f - S(t)f\|_{\infty,K} = 0$.

Third, we show that $(S(t))_{t\ge 0}$ forms a semigroup. Clearly, we have $S(0) = \mathrm{id}_{\mathrm{C}_\kappa}$. Let $r \ge 0$, $s,t \in \mathcal{T}$ and $f \in B_{\mathrm{C}_\kappa}(r) \cap \mathcal{D}$. For every $n \in \mathbb{N}$,

$$\begin{aligned}&S(s+t)f - S(s)S(t)f \\ &= \big(S(s+t)f - I(\pi_n^{s+t})f\big) + \big(I(\pi_n^{s+t})f - I(\pi_n^s)I(\pi_n^t)f\big) \\ &\quad + \big(I(\pi_n^s)I(\pi_n^t)f - I(\pi_n^s)S(t)f\big) + \big(I(\pi_n^s)S(t)f - S(s)S(t)f\big).\end{aligned}$$

It follows from equation (5.9) that the first and last term on the right-hand side convergence to zero for the subsequence $(n_l)_{l\in\mathbb{N}}$. Since $f \in \mathcal{D} \subset \mathcal{L}^I$, there exists $c \ge 0$ such that we can use inequality (5.5) and $k_n^{s+t} - k_n^s - k_n^t \in \{0,1\}$ to obtain

$$\|I(\pi_n^{s+t})f - I(\pi_n^s)I(\pi_n^t)f\|_\kappa \le e^{(s+t)\omega_{\alpha(r,s+t)}}\|I(h_n)f - f\|_\kappa \to 0 \quad \text{as } n \to \infty.$$

Let $\varepsilon > 0$ and $K \Subset X$. By $I(\pi_n^t)f, S(t)f \in B_{\mathrm{C}_\kappa}(\alpha(r,t))$ and Assumption 5.1.1(vi), there exist $K' \Subset X$ and $c \ge 0$ with

$$\|I(\pi_n^s)I(\pi_n^t)f - I(\pi_n^s)S(t)f\|_{\infty,K} \le c\|I(\pi_n^t)f - S(t)f\|_{\infty,K'} + \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Equation (5.9) guarantees $S(s+t)f - S(s)S(t)f = 0$ for all $s,t \in \mathcal{T}$ and $f \in \mathcal{D}$ while Assumption 5.1.1(v), Assumption 5.1.1(vi) and equation (5.8) imply

$$S(s+t)f = S(s)S(t)f \quad \text{for all } s,t \in \mathcal{T} \text{ and } f \in \mathrm{C}_\kappa.$$

In order to extend the previous equation to arbitrary times $s,t\geq 0$, we choose sequences $(s_n)_{n\in\mathbb{N}}\subset[0,s]\cap\mathcal{T}$ and $(t_n)_{n\in\mathbb{N}}\subset[0,t]\cap\mathcal{T}$ with $s_n\to s$ and $t_n\to t$. For every $n\in\mathbb{N}$,

$$\begin{aligned}S(s+t)f-S(s)S(t)f&=\big(S(s+t)f-S(s_n+t_n)f\big)+\big(S(s_n)S(t_n)f-S(s_n)S(t)f\big)\\&\quad+\big(S(s_n)S(t)f-S(s)S(t)f\big).\end{aligned}$$

It follows from equation (5.8) and the strong continuity of $(S(t))_{t\geq 0}$ that the terms on the right-hand side converge to zero as $n\to\infty$. In addition, for every $f\in\mathcal{L}^S$ and $t\geq 0$, there exist $c,r\geq 0$ and $h_0>0$ with

$$\begin{aligned}\|S(h)S(t)f-S(t)f\|_\kappa&=\|S(t)S(h)f-S(t)f\|_\kappa\\&\leq e^{t\omega_{\alpha(r,t+h)}}\|S(h)f-f\|_\kappa\leq ce^{t\omega_{\alpha(r,t+h)}}h\end{aligned}$$

for all $h\in[0,h_0]$ which shows that $S(t)\colon\mathcal{L}^S\to\mathcal{L}^S$ for all $t\geq 0$.

Fourth, we show that $f\in D(A)$ and $Af=I'(0)f$ for all $f\in\mathrm{C}_\kappa$ such that

$$I'(0)f=\lim_{h\downarrow 0}\frac{I(h)f-f}{h}\in\mathrm{C}_\kappa$$

exists. Let $\varepsilon>0$ and $K\Subset X$. Define $g:=I'(0)f$ and

$$r:=\sup_{n\in\mathbb{N}}\max\left\{I(h_n)f,\,f+h_ng,\,\frac{I(h_n)f-f}{h_n}-g\right\}<\infty.$$

By Assumption 5.1.1(vi), there exist $K'\Subset X$ and $c\geq 0$ with

$$\|I(h_n)^kf_1-I(h_n)^kf_2\|_{\infty,K}\leq c\|f_1-f_2\|_{\infty,K'}+\frac{\varepsilon}{4}\tag{5.10}$$

for all $k,n\in\mathbb{N}$ with $kh_n\leq 1$ and $f_1,f_2\in B_{\mathrm{C}_\kappa}(2r)$. W.l.o.g, we assume that $K\subset K'$. Since Assumption 5.1.1(v) implies that $\mathcal{L}^I\subset\mathrm{C}_\kappa$ is dense, it follows from Assumption 5.1.1(vi) and similar arguments as in the proof of Lemma 1.4.3 that there exists $t_0\in(0,1]$ with

$$\left\|\frac{I(h_n)^k(f+h_ng)-I(h_n)^kf}{h_n}-g\right\|_{\infty,K}\leq\frac{\varepsilon}{2}\tag{5.11}$$

for all $k,n\in\mathbb{N}$ with $kh_n\leq t_0$. Moreover, by assumption, we can suppose that

$$\left\|\frac{I(h)f-f}{h}-g\right\|_{\infty,K'}\leq\frac{\varepsilon}{4c}\quad\text{for all }h\in(0,t_0].\tag{5.12}$$

By induction, we show that, for every $k,n\in\mathbb{N}$ with $kh_n\leq t_0$,

$$\left\|\frac{I(h_n)^kf-f}{kh_n}-g\right\|_{\infty,K}\leq\varepsilon.\tag{5.13}$$

For $k=1$, the previous inequality holds due to inequality (5.12). Moreover, for every $k\in\mathbb{N}$, Lemma 3.6.1 implies

$$-\left(I(h_n)^k\left(g-\frac{I(h_n)f-f}{h_n}+I(h_n)f\right)-I(h_n)^kI(h_n)f\right)$$

$$
\begin{aligned}
&\le \frac{I(h_n)^k I(h_n)f - I(h_n)^k(f + h_n g)}{h_n} \\
&\le I(h_n)^k \left(\frac{I(h_n)f - f}{h_n} - g + f + h_n g \right) - I(h_n)^k(f + h_n g).
\end{aligned}
$$

It follows from inequality (5.10) and inequality (5.12) that

$$
\left\| \frac{I(h_n)^k I(h_n)f - I(h_n)^k(f + h_n g)}{h_n} \right\|_{\infty,K} \le c \left\| \frac{I(h_n)f - f}{h_n} - g \right\|_{\infty,K'} + \frac{\varepsilon}{4} \le \frac{\varepsilon}{2}.
$$

If inequality (5.13) is valid for a fixed $k \in \mathbb{N}$, we can use inequality (5.11) to conclude

$$
\begin{aligned}
\left\| \frac{I(h_n)^{k+1}f - f}{(k+1)h_n} - g \right\|_{\infty,K} &\le \frac{1}{k+1} \left\| \frac{I(h_n)^k I(h_n)f - I(h_n)^k(f + h_n g)}{h_n} \right\|_{\infty,K} \\
&\quad + \frac{1}{k+1} \left\| \frac{I(h_n)^k(f + h_n g) - I(h_n)^k f}{h_n} - g \right\|_{\infty,K} \\
&\quad + \frac{k}{k+1} \left\| \frac{I(h_n)^k f - f}{k h_n} - g \right\|_{\infty,K} \\
&\le \frac{1}{k+1} \cdot \frac{\varepsilon}{2} + \frac{1}{k+1} \cdot \frac{\varepsilon}{2} + \frac{k}{k+1}\varepsilon = \varepsilon.
\end{aligned}
$$

For every $t \in (0, t_0] \cap \mathcal{T}$, equation (5.9) and inequality (5.13) yield

$$
\left\| \frac{S(t)f - f}{t} - g \right\|_{\infty,K} = \lim_{l \to \infty} \left\| \frac{I(\pi_{n_l}^t)f - f}{k_{n_l}^t h_{n_l}} - g \right\|_{\infty,K} \le \varepsilon.
$$

Since $(S(t))_{t \ge 0}$ is strongly continuous, the previous estimate remains valid for arbitrary times $t \in (0, t_0]$ which shows that

$$
\lim_{h \downarrow 0} \left\| \frac{S(h)f - f}{h} - g \right\|_{\infty,K} = 0. \qquad \square
$$

5.2 The case $X = \mathbb{R}^d$

In this section, we focus on the case $X := \mathbb{R}^d$ and suppose that

$$
c_\kappa := \sup_{x \in \mathbb{R}^d} \sup_{|y| \le 1} \frac{\kappa(x)}{\kappa(x - y)} < \infty \tag{5.14}
$$

Let $(I(t))_{t \ge 0}$ be a family of operators $I(t)\colon \mathrm{C}_\kappa \to \mathrm{C}_\kappa$ and $(h_n)_{n \in \mathbb{N}} \subset (0, \infty)$ be a sequence with $h_n \to 0$. For every $n \in \mathbb{N}$, $t \ge 0$ and $f \in \mathrm{C}_\kappa$, we define

$$
I(\pi_n^t)f := I(h_n)^{k_n^t} f = \underbrace{\big(I(h_n) \circ \ldots \circ I(h_n)\big)}_{k_n^t \text{ times}} f,
$$

where $k_n^t := \max\{k \in \mathbb{N}_0 \colon k h_n \le t\}$ and $\pi_n^t := \{0, h_n, \ldots, k_n^t h_n\}$. The next assumption guarantees that the limit semigroup satisfies the comparison principle in form of

Theorem 4.4.6. In particular, since the limit does not depend on the choice of the convergent subsequence, the Chernoff approximation converges without choosing a subsequence. We need the following approximation result, where $\mathrm{C}_\mathrm{c}^\infty$ consists of all infinitely differentiable functions $f\colon \mathbb{R}^d \to \mathbb{R}$ with compact support.

Lemma 5.2.1. *There exists a countable set $\mathcal{D} \subset \mathrm{C}_\mathrm{c}^\infty$ such that, for every $f \in \mathrm{C}_\kappa$, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ with $\|f_n\|_\kappa \le \|f\|_\kappa$ for all $n \in \mathbb{N}$ and $f_n \to f$.*

Proof. For every $n \in \mathbb{N}$, there exists a countable set $\mathcal{D}_n \subset \mathrm{C}_\mathrm{c}^\infty$ of functions with support in $B_{\mathbb{R}^d}(n)$ such that, for every continuous function $f\colon \mathbb{R}^d \to \mathbb{R}$ with $\operatorname{supp}(f) \subset B_{\mathbb{R}^d}(n)$, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}_n$ with $\|f - f_n\|_\kappa \to 0$. For instance, one can mollify functions which are piece-wise constant on cubes of the form $[a_1, b_1) \times \ldots \times [a_d, b_d)$ with $a_i, b_i \in \mathbb{Q}$. Now, let $f \in \mathrm{C}_\kappa$ be arbitrary. For every $n \in \mathbb{N}$, we define

$$\tilde{f}_n := \left(1 - \tfrac{1}{n}\right) f\zeta_n,$$

where $\zeta_n \in \mathrm{C}_\mathrm{c}^\infty$ satisfies $0 \le \zeta_n \le 1$, $\zeta \equiv 1$ on $B_{\mathbb{R}^d}(n-1)$ and $\zeta_n \equiv 0$ on $B_{\mathbb{R}^d}(n)$. Choose $f_n \in \mathcal{D}_n$ with $\|\tilde{f}_n - f_n\|_\kappa \le 1/n$. Then, it holds $\|f_n\|_\kappa \le \|f\|_\kappa$ and $f_n \to f$. Hence, the set $\mathcal{D} := \bigcup_{n\in\mathbb{N}} \mathcal{D}_n$ satisfies the claim. □

Assumption 5.2.2. Suppose that the following statements are valid:

(i) $I(0) = \mathrm{id}_{\mathrm{C}_\kappa}$.

(ii) $I(t)$ is convex and monotone with $I(t)0 = 0$ for all $t \ge 0$.

(iii) There exists $\omega \ge 0$ with

$$\|I(t)f - I(t)g\|_\kappa \le e^{\omega t}\|f - g\|_\kappa \quad \text{for all } t \ge 0 \text{ and } f, g \in \mathrm{C}_\kappa.$$

(iv) There exist $t_0 > 0$, $\delta \in (0, 1]$ and $L \ge 0$ with

$$\|I(t)(\tau_x f) - \tau_x I(t)f\|_\kappa \le Lrt|x|$$

for all $t \in [0, t_0]$, $x \in B_{\mathbb{R}^d}(\delta)$, $r \ge 0$ and $f \in \mathrm{Lip}_\mathrm{b}(r)$.

(v) The derivative $I'(0)f \in \mathrm{C}_\kappa$ exists for all $f \in \mathrm{C}_\mathrm{b}^\infty$.

(vi) For every $\varepsilon > 0$, $r, T \ge 0$ and $K \Subset X$, there exist $K' \Subset X$ and $c \ge 0$ with

$$\|I(\pi_n^t)f - I(\pi_n^t)g\|_{\infty,K} \le c\|f - g\|_{\infty,K'} + \varepsilon$$

for all $t \in [0, T]$ and $f, g \in B_{\mathrm{C}_\kappa}(r)$.

(vii) It holds $I(t)\colon \mathrm{Lip}_\mathrm{b}(r) \to \mathrm{Lip}_\mathrm{b}(e^{\omega t}r)$ for all $r, t \ge 0$.

Recalling that $I'(0)f = \lim_{h\downarrow 0} \frac{I(h)f - f}{h}$, it follows from condition (v) that $\mathrm{C}_\mathrm{b}^\infty \subset \mathcal{L}^I$. In particular, $\mathcal{L}^I$ is dense in C_κ.

Theorem 5.2.3. *Suppose that $(I(t))_{t\ge 0}$ satisfies Assumption 5.2.2. Then, there exist a strongly continuous convex monotone semigroup $(S(t))_{t\ge 0}$ on C_κ with $S(t)0 = 0$ with*

$$S(t)f = \lim_{n\to\infty} I(\pi_n^t)f \quad \text{for all } (f, t) \in \mathrm{C}_\kappa \times \mathbb{R}_+. \tag{5.15}$$

Furthermore, the following statements are valid:

(i) *It holds $f \in D(A)$ and $Af = I'(0)f$ for all $f \in \mathrm{C}_\kappa$ such that $I'(0)f \in \mathrm{C}_\kappa$ exists. In particular, this is valid for all $f \in \mathrm{C}_\mathrm{b}^\infty$.*

(ii) *It holds $\|S(t)f - S(t)g\|_\kappa \le e^{\omega t}\|f-g\|_\kappa$ for all $t \ge 0$ and $f, g \in \mathrm{C}_\kappa$.*

(iii) *For every $\varepsilon > 0$, $r, T \ge 0$ and $K \Subset X$, there exist $K' \Subset X$ and $c \ge 0$ with*

$$\|S(t)f - S(t)g\|_{\infty,K} \le c\|f-g\|_{\infty,K'} + \varepsilon$$

for all $t \in [0,T]$ and $f, g \in B_{\mathrm{C}_\kappa}(r)$.

(iv) *It holds $\mathcal{L}^I \subset \mathcal{L}^S$ and $\mathcal{L}^I_+ \subset \mathcal{L}^S_+$. Moreover, for every $t \ge 0$,*

$$S(t)\colon \mathcal{L}^S \to \mathcal{L}^S \quad \textit{and} \quad S(t)\colon \mathcal{L}^S_+ \to \mathcal{L}^S_+.$$

(v) *For every $r, t \ge 0$, $f \in \mathrm{Lip_b}(r)$ and $x \in B_{\mathbb{R}^d}(\delta)$,*

$$\|S(t)(\tau_x f) - \tau_x S(t)f\|_\kappa \le Le^{\omega t} rt|x|.$$

Furthermore, it holds $S(t)\colon \mathrm{Lip_b}(r) \to \mathrm{Lip_b}(e^{\omega t}r)$ for all $r, t \ge 0$.

In addition, the semigroup $(S(t))_{t\ge 0}$ satisfies the conditions of Assumption 4.4.5 and does therefore not depend on the choice of approximating sequence $(h_n)_{n\in\mathbb{N}}$.

Proof. First, we verify Assumption 5.1.1. The conditions (i)–(iv) and (vi) are satisfied due to the respective conditions of Assumption 5.2.2. In addition, it follows from Assumption 5.2.2(vii) that $I(\pi_n^t)\colon \mathrm{Lip_b}(r) \to \mathrm{Lip_b}(e^{\omega t}r)$ for all $r, t \ge 0$ and $n \in \mathbb{N}$. In particular, the sequence $(I(\pi_n^t)f)_{n\in\mathbb{N}}$ is uniformly Lipschitz continuous for all $t \ge 0$ and $f \in \mathrm{Lip_b}$. Lemma 5.2.1 and Assumption 5.2.2(v) guarantee the existence of a countable set $\mathcal{D} \subset \mathrm{C}_\mathrm{c}^\infty$ which satisfies Assumption 5.1.1(v). Let $\mathcal{T} \subset \mathbb{R}_+$ be a countable dense set including zero. By Theorem 5.1.2, there exist a strongly continuous convex monotone semigroup on C_κ with $S(t)0 = 0$ and a subsequence $(n_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ such that

$$S(t)f = \lim_{l\to\infty} I(\pi_{n_l}^t)f \quad \text{for all } (f,t) \in \mathrm{C}_\kappa \times \mathcal{T}. \tag{5.16}$$

The statements (i)–(iii) are valid and it holds $\mathcal{L}^I \subset \mathcal{L}^S$. Furthermore,

$$S(t)\colon \mathcal{L}^S \to \mathcal{L}^S \quad \text{and} \quad S(t)\colon \mathrm{Lip_b}(r) \to \mathrm{Lip_b}(e^{\omega t}r) \quad \text{for all } t \ge 0.$$

Second, we show that $\mathcal{L}^I_+ \subset \mathcal{L}^S_+$ and $S(t)\colon \mathcal{L}^S_+ \to \mathcal{L}^S_+$ for all $t \ge 0$. For every $f \in \mathcal{L}^I_+$, there exist $c \ge 0$ and $t_0 \in (0,1]$ with

$$\|(I(t)f - f)^+\|_\kappa \le ct \quad \text{for all } t \in [0,t_0].$$

By induction, we show that, for every $k \in \mathbb{N}$ and $n \in \mathbb{N}$ with $h_n \le t_0$,

$$\|(I(h_n)^k f - f)^+\|_\kappa \le ckh_n e^{\omega k h_n}. \tag{5.17}$$

For $k = 1$, the previous estimate is clearly satisfied. Moreover, Lemma 3.6.1 and the monotonicity of $I(h_n)^k$ yield

$$I(h_n)^{k+1}f - f = I(h_n)^k I(h_n)f - I(h_n)^k f + I(h_n)^k f - f$$

$$\leq h_n \left(I(h_n)^k \left(\frac{(I(h_n)f - f)^+}{h_n} + f \right) - I(h_n)f \right) + I(h_n)^k f - f.$$

Hence, if inequality (5.17) is valid for a fixed $k \in \mathbb{N}$, we can use Assumption 5.2.2(iii) and Lemma 1.2.7 to obtain

$$\begin{aligned} \|(I(h_n)^{k+1}f - f)^+\|_\kappa &\leq e^{\omega k h_n} \|(I(h_n)f - f)^+\|_\kappa + \|(I(h_n)^k f - f)^+\|_\kappa \\ &\leq c h_n e^{\omega k h_n} + c k h_n e^{\omega k h_n} \leq c(k+1) h_n e^{\omega (k+1) h_n}. \end{aligned}$$

Equation (5.16) and the strong continuity of $(S(t))_{t\geq 0}$ imply

$$\|(S(t)f - f)^+\|_\kappa \leq c t e^{\omega t} \quad \text{for all } t \geq 0.$$

Now, let $f \in \mathcal{L}_+^S$ and $t \geq 0$. By definition, there exists $h_0 > 0$ with

$$c := \sup_{h \in (0, h_0]} \left\| \frac{(S(h)f - f)^+}{h} \right\|_\kappa < \infty.$$

We use the semigroup property, Lemma 3.6.1, the monotonicity of $S(t)$ and statement (ii) to obtain

$$\begin{aligned} \left(\frac{S(h)S(t)f - S(t)f}{h} \right) \kappa &\leq \left(S(t) \left(f + \frac{(S(h)f - f)^+}{h} \right) - S(t)f \right) \kappa \\ &\leq e^{\omega t} \left\| \frac{(S(h)f - f)^+}{h} \right\|_\kappa \leq c e^{\omega t} \quad \text{for all } h \in (0, h_0]. \end{aligned}$$

Third, for every $k, n \in \mathbb{N}$, $r \geq 0$, $f \in \mathrm{Lip_b}(r)$ and $x \in B_{\mathbb{R}^d}(\delta)$, we show that

$$\|I(2^{-n})^k(\tau_x f) - \tau_x I(2^{-n})^k f\|_\kappa \leq L e^{\omega k h_n} r k 2^{-n} |x|.$$

Let $k, n \in \mathbb{N}$, $r \geq 0$, $f \in \mathrm{Lip_b}(r)$ and $x \in B_{\mathbb{R}^d}(\delta)$. We use $I(h_n)^{l-1} f \in \mathrm{Lip_b}(e^{\omega(l-1)h_n} r)$, Lemma 1.2.7 and Assumption 5.2.2(iv) to obtain

$$\begin{aligned} &\|\tau_x I(h_n)^k f - I(h_n)(\tau_x f)\|_\kappa \\ &\leq \sum_{l=1}^k \|I(h_n)^{k-l}(\tau_x I(h_n)^l) - I(h_n)^{k-l} I(h_n)(\tau_x I(h_n)^{l-1} f)\|_\kappa \\ &\leq \sum_{l=1}^k e^{\omega(k-l)h_n} \|\tau_x I(h_n) I(h_n)^{l-1} f - I(h_n)(\tau_x I(h_n)^{l-1} f)\|_\kappa \\ &\leq \sum_{l=1}^k e^{\omega(k-l)h_n} L e^{\omega(l-1)h_n} r h_n |x| \leq L e^{\omega k h_n} r k h_n |x|. \end{aligned}$$

Equation (5.16) and the strong continuity of $(S(t))_{t\geq 0}$ imply

$$\|\tau_x S(t) f - S(t)(\tau_x f)\|_\kappa \leq L e^{\omega t} r t |x|$$

for all $r, t \geq 0$, $f \in \mathrm{Lip_b}(r)$ and $x \in B_{\mathbb{R}^d}(\delta)$.

Fourth, we verify equation (5.15). To that end, we show that the limit in equation (5.16) does not depend on the choice of the convergent subsequence. For every subsequence $(\tilde{n}_k)_{k\in\mathbb{N}} \subset \mathbb{N}$, there exist a further subsequence $(\tilde{n}_{k_l})_{l\in\mathbb{N}}$ and a strongly continuous convex monotone semigroup $(\tilde{S}(t))_{t\geq 0}$ on C_κ with

$$\tilde{S}(t)f = \lim_{l\to\infty} I(\pi^t_{\tilde{n}_{k_l}})f \quad \text{for all } (f,t) \in \mathrm{C}_\kappa \times \mathcal{T}$$

such that the statements (i)-(v) are valid. Hence, Theorem 4.4.6 yields

$$S(t)f = \tilde{S}(t)f \quad \text{for all } t \geq 0 \text{ and } f \in \mathrm{C}_\kappa.$$

Since every subsequence has a further subsequence which converges to a limit which is independent of the choice of the subsequence, we conclude

$$S(t)f = \lim_{n\to\infty} I(\pi^t_n)f \quad \text{for all } (f,t) \in \mathrm{C}_\kappa \times \mathcal{T}.$$

In order to show that the convergence in the previous equation holds for arbitrary points in time, let $t \geq 0$ and define $\tilde{\mathcal{T}} := \mathcal{T} \cup \{t\}$. Analogously to the previous arguments, we obtain a semigroup $(\tilde{S}(t))_{t\geq 0}$ with

$$S(t)f = \tilde{S}(t)f = \lim_{n\to\infty} I(\pi^t_n)f \quad \text{for all } (f,t) \in \mathrm{C}_\kappa \times \tilde{\mathcal{T}}.$$

Since $t \geq 0$ was arbitrary, we obtain that equation (5.15) is valid. □

In particular, the semigroup $(S(t))_{t\geq 0}$ is uniquely determined by the derivative $I'(0)f$ for smooth functions $f \in \mathrm{C}^\infty_\mathrm{b}$.

Corollary 5.2.4. *Let $(I(t))_{t\geq 0}$ and $(J(t))_{t\geq 0}$ be two families of operators satisfying Assumption 5.2.2 with $I'(0)f \leq J'(0)f$ for all $f \in \mathrm{C}^\infty_\mathrm{b}$. Then,*

$$S(t)f \leq T(t)f \quad \textit{for all } t \geq 0 \textit{ and } f \in \mathrm{C}_\kappa,$$

where $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ are the semigroups associated to $(I(t))_{t\geq 0}$ and $(J(t))_{t\geq 0}$, respectively.

Proof. This follows immediately from Theorem 5.2.3 and Theorem 4.4.6. □

Chapter 6

Stochastic optimal control problems and Markovian transition semigroups under model uncertainty

6.1 Control problems and upper semigroup envelopes

In this subsection, we show that value functions of a class of stochastic optimal control problems can be approximated by iterating a sequence of related static optimization problems over increasingly finer partitions. The latter corresponds to the construction of so-called upper semigroup envelopes, see Nisio [138] and Nendel and Röckner [136], as well as to approximations using piece-wise constant policies, see Krylov [116]. For every $t \geq 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$, we consider the value function

$$(T(t)f)(x) := \sup_{(a,b)\in\mathcal{A}} \left(\mathbb{E}\Big[f\Big(x + \int_0^t \sqrt{a_s}\,\mathrm{d}W_s + \int_0^t b_s\,\mathrm{d}s\Big)\Big] - \mathbb{E}\Big[\int_0^t L(a_s, b_s)\,\mathrm{d}s\Big]\right)$$

of a dynamic stochastic control problem with finite time horizon, where $(W_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions and $\mathcal{A}$ consists of all predictable processes $(a, b)\colon \Omega \times \mathbb{R}_+ \to \mathbb{S}_+^d \times \mathbb{R}^d$ with

$$\mathbb{E}\left[\int_0^t |a_s| + |b_s|\,\mathrm{d}s\right] < \infty \quad \text{for all } t \geq 0.$$

Here, we denote by $\mathbb{S}_+^d$ the set of all symmetric positive semidefinite $d \times d$-matrices and, in order to apply Itô's isometry, we endow $\mathbb{R}^{d\times d}$ with the Frobenius norm

$$|a| := \sqrt{\sum_{i,j=1}^d |a_{ij}|^2} \quad \text{for all } a \in \mathbb{R}^{d\times d}.$$

The function $L\colon \mathbb{S}_+^d \times \mathbb{R}^d \to [0, \infty]$ is measurable and satisfies the following conditions.

Assumption 6.1.1. The following statements are valid:

(i) There exists $(a^*, b^*) \in \mathbb{S}_+^d \times \mathbb{R}^d$ with $L(a^*, b^*) = 0$.

(ii) The function L grows superlinearly, i.e., $\lim_{|a|+|b|\to\infty} \frac{L(a,b)}{|a|+|b|} = \infty$.

For every $t \geq 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$, we define the value function

$$(I(t)f)(x) := \sup_{(a,b)\in\mathbb{S}^d_+\times\mathbb{R}^d} \big(\mathbb{E}[f(x+\sqrt{a}W_t + bt)] - L(a,b)t\big)$$

of the related static optimal control problem, where the controls a deterministic. Let $(h_n)_{n\in\mathbb{N}} \subset (0,\infty)$ be a sequence with $h_n \to 0$ and define $I(\pi_n^t) := I(h_n)^{k_n^t}$ for all $t \geq 0$ and $k, n \in \mathbb{N}$, where $k_n^t := \max\{k \in \mathbb{N}_0 \colon kh_n \leq t\}$ and $\pi_n^t := \{0, h_n, \ldots, k_n^t h_n\}$. Subsequently, we use the following notations:

$$\begin{aligned}
X_t^{a,b} &:= \int_0^t \sqrt{a_s}\,\mathrm{d}W_s + \int_0^t b_s\,\mathrm{d}s \quad \text{for all } (a,b)\in\mathcal{A} \text{ and } t\geq 0,\\
c_L &:= \sup_{(a,b)\in\mathbb{S}^d_+\times\mathbb{R}^d} \frac{|a|+|b|}{1+L(a,b)},\\
L^*(c) &:= \sup_{(a,b)\in\mathbb{S}^d_+\times\mathbb{R}^d} \big(c(|a|+|b|) - L(a,b)\big) \quad \text{for all } c\geq 0,\\
B_{a,b}f &:= \frac{1}{2}\operatorname{tr}(aD^2f) + \langle b, \nabla f\rangle \quad \text{for all } f\in\mathrm{C}^2_\mathrm{b} \text{ and } (a,b)\in\mathbb{S}^d_+\times\mathbb{R}^d,
\end{aligned}$$

where C^2_b denotes the space of all bounded twice differentiable continuously differentiable functions $f\colon \mathbb{R}^d \to \mathbb{R}$ such that the partial derivative sup to order two are all bounded. Assumption 6.1.1(ii) implies $c_L < \infty$ and $L^*(c) \to 0$ as $c \downarrow 0$. In order to show that the family $(T(t))_{t\geq 0}$ is a strongly continuous semigroup satisfying Assumption 4.4.5, we need the following estimates.

Lemma 6.1.2. *The following statements are valid:*

(i) For every $c, t \geq 0$, $r > 0$ and $(a,b) \in \mathcal{A}$,

$$c\mathbb{P}(|X_t^{a,b}| \geq r) - \mathbb{E}\left[\int_0^t L(a_s,b_s)\,\mathrm{d}s\right] \leq \frac{c}{r} + L^*\Big(\frac{c}{r}\Big)t.$$

In particular, $T(t)$ is continuous from above for all $t \geq 0$.

(ii) For every $r, t \geq 0$, $\delta \in (0,1]$, $x \in \mathbb{R}^d$, $(a,b) \in \mathcal{A}$ and $f \in \mathrm{Lip_b}(r)$,

$$\mathbb{E}\left[\int_0^t f(x+X_s^{a,b})\,\mathrm{d}s\right] \leq \big(f(x)+r\delta\big)t + \frac{2rc_L}{\delta^2}\mathbb{E}\left[\int_0^t\int_0^s 1 + L(a_u,b_u)\,\mathrm{d}u\,\mathrm{d}s\right].$$

(iii) Let $t \geq 0$, $x \in \mathbb{R}^d$, $f \in \mathrm{C}^2_\mathrm{b}$ and $(a,b) \in \mathcal{A}$ with

$$(T(t)f)(x) \leq t + \mathbb{E}\big[f(x+X_t^{a,b})\big] - \mathbb{E}\left[\int_0^t L(a_s,b_s)\,\mathrm{d}s\right].$$

Then, for $c_f := 2\big(1+\|B_{a^*,b^*}f\|_\infty\big) + L^*\big(\|D^2f\|_\infty \vee 2\|\nabla f\|_\infty\big)$, it holds

$$\mathbb{E}\left[\int_0^t L(a_s,b_s)\,\mathrm{d}s\right] \leq c_f t.$$

Proof. First, for every $c, t \geq 0$, $r > 0$ and $(a, b) \in \mathcal{A}$, we show that

$$c\mathbb{P}(|X_t^{a,b}| \geq r) - \mathbb{E}\left[\int_0^t L(a_s, b_s)\,\mathrm{d}s\right] \leq \frac{c}{r} + L^*\Big(\frac{c}{r}\Big)t.$$

Chebyshev's inequality, Jensen's inequality and Itô's isometry imply

$$\begin{aligned}
\mathbb{P}(|X_t^{a,b}| \geq r) &\leq \frac{\mathbb{E}[|X_t^{a,b}|]}{r} \leq \frac{1}{r}\left(\mathbb{E}\left[\left|\int_0^t \sqrt{a_s}\,\mathrm{d}W_s\right|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[\int_0^t |b_s|\,\mathrm{d}s\right]\right) \\
&= \frac{1}{r}\left(\mathbb{E}\left[\int_0^t |a_s|\,\mathrm{d}s\right]^{\frac{1}{2}} + \mathbb{E}\left[\int_0^t |b_s|\,\mathrm{d}s\right]\right) \leq \frac{1}{r}\left(1 + \mathbb{E}\left[\int_0^t |a_s| + |b_s|\,\mathrm{d}s\right]\right).
\end{aligned}$$

Using the definition of L^*, we conclude

$$c\mathbb{P}(|X_t^{a,b}| \geq r) - \mathbb{E}\left[\int_0^t L(a_s, b_s)\ ds\right] \leq \frac{c}{r} + L^*\Big(\frac{c}{r}\Big)t.$$

Second, we show that $T(t)$ is continuous from above for all $t \geq 0$. Let $t \geq 0$, $x \in \mathbb{R}^d$ and $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C_b}$ be a sequence with $f_n \downarrow 0$. For every $r > 0$ and $(a, b) \in \mathcal{A}$, we use the first part to estimate

$$\begin{aligned}
&\mathbb{E}\big[f_n(x + X_t^{a,b})\big] - \mathbb{E}\left[\int_0^t L(a_s, b_s)\,\mathrm{d}s\right] \\
&= \mathbb{E}\left[f_n(x + X_t^{a,b})\mathbb{1}_{\{|X_t^{a,b}|<r\}}\right] + \mathbb{E}\left[f_n(x + X_t^{a,b})\mathbb{1}_{\{|X_t^{a,b}|\geq r\}}\right] - \mathbb{E}\left[\int_0^t L(a_s, b_s)\,\mathrm{d}s\right] \\
&\leq \sup_{y\in B(x,r)} f_n(y) + \|f_1\|_\infty \cdot \mathbb{P}(|X_t^{a,b}| \geq r) - \mathbb{E}\left[\int_0^t L(a_s, b_s)\,\mathrm{d}s\right] \\
&\leq \sup_{y\in B(x,r)} f_n(y) + \frac{\|f_1\|_\infty}{r} + L^*\left(\frac{\|f_1\|_\infty}{r}\right)t.
\end{aligned}$$

It follows from $L^*(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ and Dini's theorem that $(T(t)f_n)(x) \downarrow 0$.

Third, for every $r, t \geq 0$, $\delta \in (0, 1]$, $x \in \mathbb{R}^d$, $(a, b) \in \mathcal{A}$ and $f \in \mathrm{Lip_b}(r)$, we show that

$$\mathbb{E}\left[\int_0^t f(x + X_s^{a,b})\,\mathrm{d}s\right] \leq \big(f(x) + r\delta\big)t + \frac{rc_L}{\delta^2}\mathbb{E}\left[\int_0^t\int_0^s 1 + L(a_u, b_u)\,\mathrm{d}u\,\mathrm{d}s\right].$$

We use Chebyshev's inequality, Itô's isometry and the definition of c_L to estimate

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t f(x + X_s^{a,b})\,\mathrm{d}s\right] \\
&\leq \mathbb{E}\left[\int_0^t \Big(f(x + X_s^{a,0}) + r\int_0^s |b_u|\,\mathrm{d}u\Big)\mathrm{d}s\right] \\
&= \mathbb{E}\left[\int_0^t \Big(f(x + X_s^{a,0})\mathbb{1}_{\{|X_s^{a,0}|<\delta\}} + f(x + X_s^{a,0})\mathbb{1}_{\{|X_s^{a,0}|\geq\delta\}} + r\int_0^s |b_u|\,\mathrm{d}u\Big)\mathrm{d}s\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \big(f(x) + r\delta\big)t + 2r\mathbb{E}\left[\int_0^t \Big(\mathbb{P}(|X_s^{a,0}| \geq \delta) + \int_0^t |b_u|\,\mathrm{d}u\Big)\mathrm{d}s\right] \\
&\leq \big(f(x) + r\delta\big)t + \frac{2r}{\delta^2}\mathbb{E}\left[\int_0^t \int_0^s |a_u| + |b_u|\,\mathrm{d}u\,\mathrm{d}s\right] \\
&\leq \big(f(x) + r\delta\big)t + \frac{2rc_L}{\delta^2}\mathbb{E}\left[\int_0^t \int_0^s 1 + L(a_u, b_u)\,\mathrm{d}u\,\mathrm{d}s\right].
\end{aligned}
$$

Fourth, let $t \geq 0$, $x \in \mathbb{R}^d$, $f \in \mathrm{C}_\mathrm{b}^2$ and $(a, b) \in \mathcal{A}$ with

$$
(T(t)f)(x) \leq t + \mathbb{E}\big[f(x + X_t^{a,b})\big] - \mathbb{E}\left[\int_0^t L(a_s, b_s)\,\mathrm{d}s\right].
$$

Assumption 6.1.1(i) implies $\mathbb{E}[f(x + X_t^{a^*,b^*})] \leq (T(t)f)(x)$. Define

$$
c_f := 2\big(1 + \|B_{a^*,b^*}f\|_\infty\big) + L^*\big(\|D^2 f\|_\infty \vee 2\|\nabla f\|_\infty\big).
$$

We use Itô's formula and the definition of L^* to estimate

$$
\begin{aligned}
&2\mathbb{E}\left[\int_0^t L(a_s, b_s)\,\mathrm{d}s\right] \leq 2t + 2\mathbb{E}\big[f(x + X_t^{a,b})\big] - 2\mathbb{E}\big[f(x + X_t^{a^*,b^*})\big] \\
&\leq 2t + 2\mathbb{E}\left[\int_0^t B_{a_s,b_s} f(x + X_s^{a,b})\,\mathrm{d}s\right] - 2\mathbb{E}\left[\int_0^t B_{a^*,b^*} f(x + X_s^{a^*,b^*})\,\mathrm{d}s\right] \\
&\leq 2\big(1 + \|B_{a^*,b^*}f\|_\infty\big)t + \mathbb{E}\left[\int_0^t |a_s| \cdot \|D^2 f\|_\infty + 2|b_s| \cdot \|\nabla f\|_\infty\,\mathrm{d}s\right] \\
&\leq \Big(2\big(1 + \|B_{a^*,b^*}f\|_\infty\big) + L^*\big(\|D^2 f\|_\infty \vee 2\|\nabla f\|_\infty\big)\Big)t + \mathbb{E}\left[\int_0^t L(a_s, b_s)\,\mathrm{d}s\right] \\
&= c_f t + \mathbb{E}\left[\int_0^t L(a_s, b_s)\,\mathrm{d}s\right].
\end{aligned}
$$

Rearranging the previous inequality yields

$$
\mathbb{E}\left[\int_0^t L(a_s, b_s)\,\mathrm{d}s\right] \leq c_f t. \qquad \square
$$

Let L^∞ be the space of all bounded measurable functions $f\colon \mathbb{R}^d \to \mathbb{R}$ and denote by $W^{1,\infty}$ the corresponding first order Sobolev space. For $f \in W^{1,\infty}$ and $a \in \mathbb{S}_+^d$, we say that $\Delta_a f$ exists in L^∞ if and only if there exists a function $g \in L^\infty$ with

$$
\int_{\mathbb{R}^d} g\varphi\,\mathrm{d}x = -\int_{\mathbb{R}^d} \big\langle \sqrt{a}\nabla f, \sqrt{a}\nabla\varphi \big\rangle\,\mathrm{d}x
$$

for all infinitely differentiable functions $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ with compact support. In this case, since g is unique almost everywhere, we define $\Delta_a f := g$. Clearly, it holds

$$
\Delta_a f = \sum_{i,j=1}^n a_{ij}\partial_{ij} f = \operatorname{tr}(aD^2 f) \quad \text{for all } f \in \mathrm{C}_\mathrm{b}^2.
$$

Let $\nabla_b f := \langle b, \nabla f\rangle$ for all $b \in \mathbb{R}^d$ and $f \in W^{1,\infty}$. Furthermore, the set $\mathbb{S}_L$ consists of all $a \in \mathbb{S}_+^d$ such that there exists $b \in \mathbb{R}^d$ with $L(a, b) < \infty$ and $L(a, -b) < \infty$.

Theorem 6.1.3. *The family $(T(t))_{t\geq 0}$ is a strongly continuous convex monotone semigroup on $\mathrm{C_b}$ satisfying Assumption 4.4.5. Furthermore, it holds $\mathrm{C_b^\infty} \subset D(B)$ and*

$$Bf = \sup_{(a,b)\in\mathbb{S}_+^d\times\mathbb{R}^d} \Big(\frac{1}{2}\Delta_a f + \nabla_b f - L(a,b)\Big) \quad \textit{for all } f\in \mathrm{C_b^\infty},$$

where B denotes the generator of $(T(t))_{t\geq 0}$.

Proof. Clearly, the operators $T(t)\colon \mathrm{C_b}\to \mathrm{F_b}$ are convex and monotone with $T(t)0=0$, where $\mathrm{F_b}$ consists of all bounded functions $f\colon \mathbb{R}^d\to\mathbb{R}$. Moreover,

$$\|T(t)f - T(t)g\|_\infty \leq \|f-g\|_\infty \quad \text{for all } f,g\in \mathrm{C_b}$$

and $T(t)\colon \mathrm{Lip_b}(r)\to \mathrm{Lip_b}(r)$ for all $r,t\geq 0$. Since Lemma 6.1.2(i) guarantees that $T(t)$ is continuous from above, Lemma 3.3.4 implies $T(t)\colon \mathrm{C_b}\to\mathrm{C_b}$ for all $t\geq 0$. Indeed, for every $f\in\mathrm{C_b}$, there exists a sequence $(f_n)_{n\in\mathbb{N}}\subset \mathrm{Lip_b}$ with $f_n\to f$ and thus $T(t)f=\lim_{n\to\infty}T(t)f_n\in\mathrm{C_b}$. We have $\tau_x T(t)f = T(t)(\tau_x f)$ for all $t\geq 0$, $f\in\mathrm{C_b}$ and $x\in\mathbb{R}^d$. The dynamic programming principle, see, e.g., Fabbri et al. [70, Theorem 2.24] or Pham [147, Theorem 3.3.1], yields that $T(s+t)f = T(s)T(t)f$ for all $s,t\geq 0$ and $f\in\mathrm{Lip_b}$. Again, we can use Lemma 3.3.4 and an approximation argument to conclude that the semigroup property is valid for arbitrary functions $f\in\mathrm{C_b}$. Next, we show that $\mathrm{C_b^\infty}\subset D(B)$ and

$$Bf = \sup_{(a,b)\in\mathbb{S}_+^d\times\mathbb{R}^d} \Big(\frac{1}{2}\Delta_a f + \nabla_b f - L(a,b)\Big) \quad \text{for all } f\in \mathrm{C_b^\infty}.$$

Let $f\in\mathrm{C_b^\infty}$, $\varepsilon\in(0,1]$, $t>0$, $x\in\mathbb{R}^d$ and $(a,b)\in\mathcal{A}$ with

$$(T(t)f)(x) \leq \frac{\varepsilon t}{2} + \mathbb{E}\big[f(x+X_t^{a,b})\big] - \mathbb{E}\left[\int_0^t L(a_s,b_s)\,\mathrm{d}s\right].$$

Moreover, due to Assumption 6.1.1(ii), there exists $r_1\geq 0$ with

$$g := \sup_{(a,b)\in\mathbb{S}_+^d\times\mathbb{R}^d} \Big(\frac{1}{2}\Delta_a f + \nabla_b f - L(a,b)\Big) \in \mathrm{Lip_b}(r_1).$$

For $\delta_1 := \varepsilon/2r_1$, we use Itô's formula and Lemma 6.1.2(ii) and (iii) to obtain

$$\begin{aligned}
\left(\frac{T(t)f-f}{t}\right)(x) &\leq \frac{\varepsilon}{2} + \mathbb{E}\left[\frac{1}{t}\int_0^t B_{a_s,b_s}f(x+X_s^{a,b}) - L(a_s,b_s)\,\mathrm{d}s\right] \\
&\leq \frac{\varepsilon}{2} + \mathbb{E}\left[\frac{1}{t}\int_0^t g(x+X_s^{a,b})\,\mathrm{d}s\right] \\
&\leq g(x) + \varepsilon + \frac{2r_1 c_L}{\delta_1^2}\mathbb{E}\left[\frac{1}{t}\int_0^t\int_0^s 1 + L(a_u,b_u)\,\mathrm{d}u\,\mathrm{d}s\right] \\
&\leq g(x) + \varepsilon + \frac{2r_1 c_L}{\delta_1^2}\frac{(1+c_f)t}{2}.
\end{aligned} \tag{6.1}$$

In view of Assumption 6.1.1(ii), there exists $r_2 \geq r_1$ with $|a^*| + |b^*| \leq r_2$,

$$g = \sup_{|a|+b|\leq r_2} \Big(\frac{1}{2}\Delta_a f + \nabla_b f - L(a,b)\Big),$$
$$I(t)f = \sup_{|a|+b|\leq r_2} \big(\mathbb{E}[f(\,\cdot\, + \sqrt{a}W_t + bt)] - L(a,b)t\big) \quad \text{for all } t \geq 0.$$

Furthermore, we can choose $r_3 \geq r_2$ with $B_{a,b}f \in \mathrm{Lip_b}(r_3)$ for all $|a| + |b| \leq r_2$. Hence, similarly to the proof of Lemma 6.1.2(ii), it follows from Itô's formula that

$$\begin{aligned}\frac{T(t)f - f}{t} \geq \frac{I(t)f - f}{t} &= \sup_{|a|+|b|\leq r_2} \left(\mathbb{E}\left[\int_0^t B_{a,b}(\,\cdot\, + X_s^{a,b})\,\mathrm{d}s\right] - L(a,b)\right) \\ &\geq g - \varepsilon - \sup_{|a|+|b|\leq r_2} \left(\frac{r_3 c_L}{\delta_3^2}(1 + L(a,b)) - B_{a,b}f\right) t \end{aligned} \tag{6.2}$$

for all $t > 0$, where $\delta_3 := {}^{\varepsilon}\!/_{r_3}$. Since inequality (6.1) and inequality (6.2) do not depend on the choice of $t \geq 0$ and $x \in \mathbb{R}^d$, we obtain

$$\lim_{h\downarrow 0} \left\| \frac{T(h)f - f}{h} - g \right\|_\infty = 0. \qquad \square$$

Recall that the symmetric Lipschitz set $\mathcal{L}^S_{\mathrm{sym}}$ consists of all $f \in \mathrm{C_b}$ such that there exist $t_0 > 0$ and $c \geq 0$ with

$$\|S(t)f - f\|_\infty \leq ct \quad \text{and} \quad \|S(t)(-f) + f\|_\infty \leq ct \quad \text{for all } t \in [0, t_0].$$

Theorem 6.1.4. *There exists a strongly continuous convex monotone semigroup* $(S(t))_{t\geq 0}$ *on* $\mathrm{C_b}$ *given by*

$$S(t)f = \lim_{n\to\infty} I(\pi_n^t)f \quad \textit{for all } (f,t) \in \mathrm{C_b} \times \mathbb{R}_+$$

such that $\mathrm{C_b^2} \subset D(A)$ *and*

$$Af = \sup_{(a,b)\in\mathbb{S}_+^d\times\mathbb{R}^d} \Big(\frac{1}{2}\Delta_a f + \nabla_b f - L(a,b)\Big) \quad \textit{for all } f \in \mathrm{C_b^2}.$$

It holds $S(t)\colon \mathcal{L}^S_{\mathrm{sym}} \cap \mathrm{Lip_b} \to \mathcal{L}^S_{\mathrm{sym}} \cap \mathrm{Lip_b}$ *for all* $t \geq 0$ *and, if* $\mathbb{S}_L \neq \emptyset$,

$$\mathcal{L}^S_{\mathrm{sym}} \cap \mathrm{Lip_b} = \Big\{ f \in \bigcap_{a\in\mathbb{S}_L} D(\Delta_a) \cap W^{1,\infty} \colon \sup_{(a,b)\in\mathbb{S}_L\times\mathbb{R}^d} \big(\|B_{a,b}f\|_\infty - L(a,b)\big) < \infty \Big\}.$$

Furthermore, the semigroup $(S(t))_{t\geq 0}$ *satisfies Assumption 4.4.5 and therefore*

$$S(t)f = T(t)f \quad \textit{for all } t \geq 0 \textit{ and } f \in \mathrm{C_b}.$$

Proof. First, we verify Assumption 5.2.2. The operators $I(t)\colon \mathrm{C_b} \to \mathrm{F_b}$ are convex and monotone with $I(t)0 = 0$. Moreover,

$$\|I(t)f - I(t)g\|_\infty \leq \|f - g\|_\infty \quad \text{for all } f, g \in \mathrm{C_b}$$

and $I(t)\colon \mathrm{Lip}_\mathrm{b}(r)\to \mathrm{Lip}_\mathrm{b}(r)$ for all $r,t\geq 0$. It follows from $I(t)f\leq T(t)f$ that $I(t)$ is continuous from above and thus we use Lemma 3.3.4 and an approximation argument to conclude $I(t)\colon \mathrm{C}_\mathrm{b}\to \mathrm{C}_\mathrm{b}$ for all $t\geq 0$. In addition, for every $T\geq 0$, $K\Subset\mathbb{R}^d$ and sequence $(f_n)_{n\in\mathbb{N}}\subset \mathrm{C}_\mathrm{b}$ with $f_n\downarrow 0$, Dini's theorem implies

$$0\leq \sup_{(t,x)\in[0,T]\times K}\sup_{n\in\mathbb{N}}\big(I(\pi_n^t)f_k\big)(x)\leq \sup_{(t,x)\in[0,T]\times K}(T(t)f_k)(x)\downarrow 0 \quad \text{as } k\to\infty.$$

It holds $\tau_x I(t)f=I(t)(\tau_x f)$ all $t\geq 0$, $f\in \mathrm{C}_\mathrm{b}$ and $x\in\mathbb{R}^d$. We show that $\mathrm{C}_\mathrm{b}^2\subset\mathcal{L}^I$ and

$$I'(0)f=\sup_{(a,b)\in\mathbb{S}_+^d\times\mathbb{R}^d}\Big(\frac{1}{2}\Delta_a f+\nabla_b f-L(a,b)\Big)\quad\text{for all } f\in \mathrm{C}_\mathrm{b}^2.$$

Let $f\in \mathrm{C}_\mathrm{b}^2$. Assumption 6.1.1(ii) and Itô's formula guarantee $\|I(t)f-f\|_\infty\leq ct$ for all $t\geq 0$, where

$$c:=\sup_{(a,b)\in\mathbb{S}_+^d\times\mathbb{R}^d}\Big(\frac{1}{2}|a|\cdot\|D^2 f\|_\infty+|b|\cdot\|\nabla f\|_\infty-L(a,b)\Big)<\infty.$$

Let $\varepsilon>0$ and $K\Subset\mathbb{R}^d$. Assumption 6.1.1(ii) and Itô's formula yield $r\geq 0$ with

$$\begin{aligned}&\left\|\frac{I(h)f-f}{h}-\sup_{(a,b)\in\mathbb{S}_+^d\times\mathbb{R}^d}\Big(\frac{1}{2}\Delta_a f+\nabla_b f-L(a,b)\Big)\right\|_\infty\\ &\leq \sup_{|a|+|b|\leq r}\left\|\frac{S_{a,b}(h)f-f}{h}-\frac{1}{2}\Delta_a f-\nabla_b f\right\|_\infty\\ &\leq \sup_{|a|+|b|\leq r}\int_0^h\Big(\|\nabla_b f(\,\cdot\,+X_s^{a,b})-\nabla_b f\|_\infty+\frac{1}{2}\|\Delta_a f(\,\cdot\,+X_s^{a,b})-\Delta_a f\|_\infty\Big)\,\mathrm{d}s\end{aligned}$$

for all $h>0$, where $(S_{a,b}(h)f)(x):=\mathbb{E}[f(x+aW_h+bh)]$. Choose $\delta>0$ with

$$\Big(\|\nabla_b f(\,\cdot\,+X_s^{a,b})-\nabla_b f\|_\infty+\frac{1}{2}\|\Delta_a f(\,\cdot\,+X_s^{a,b})-\Delta_a f\|_\infty\Big)\mathbb{1}_{\{|X_s^{a,b}|<\delta\}}<\varepsilon$$

for all $|a|+|b|\leq r$, $s\geq 0$ and $x\in K$. Furthermore, Chebyshev's inequality implies

$$\sup_{|a|+|b|\leq r}\mathbb{P}\big(|X_s^{a,b}|\geq\delta\big)\leq \sup_{|a|+|b|\leq r}\frac{2}{\delta^2}\big(\mathbb{E}[|\sqrt{a}W_s|^2]+|b|^2 s^2\big)\to 0\quad\text{as } s\downarrow 0.$$

We obtain

$$\lim_{h\downarrow 0}\frac{I(h)f-f}{h}=\sup_{(a,b)\in\mathbb{S}_+^d\times\mathbb{R}^d}\Big(\frac{1}{2}\Delta_a f+\nabla_b f-L(a,b)\Big)\quad\text{for all } f\in \mathrm{C}_\mathrm{b}^2.$$

Now, the first part of the claim follows from Theorem 5.2.3 and Theorem 4.4.6 implies

$$S(t)f=T(t)f\quad\text{for all } t\geq 0 \text{ and } f\in \mathrm{C}_\mathrm{b}.$$

Second, we show that $S(t)\colon \mathcal{L}^S_{\mathrm{sym}}\cap\mathrm{Lip}_\mathrm{b}\to\mathcal{L}^S_{\mathrm{sym}}\cap\mathrm{Lip}_\mathrm{b}$ for all $t\geq 0$ and

$$\mathcal{L}^S_{\mathrm{sym}}\cap\mathrm{Lip}_\mathrm{b}=\Big\{f\in\bigcap_{a\in\mathbb{S}_L}D(\Delta_a)\cap W^{1,\infty}\colon \sup_{(a,b)\in\mathbb{S}_L\times\mathbb{R}^d}\big(\|B_{a,b}f\|_\infty-L(a,b)\big)<\infty\Big\}.$$

It is straightforward to show that Assumption 1.5.2 is satisfied with $I^+(t)f := I(t)f$ and $I^-(t)f := -I(t)(-f)$. Moreover, the estimate

$$-S(t)(-f) \leq -I(t)(-f) \leq I(t)f \leq S(t)f \quad \text{for all } t \geq 0 \text{ and } f \in \mathrm{C_b}$$

is sufficient for Theorem 1.5.3 to be valid. We obtain $\mathcal{L}^S_{\mathrm{sym}} = \mathcal{L}^I_{\mathrm{sym}}$ and

$$S(t)\colon \mathcal{L}^S_{\mathrm{sym}} \to \mathcal{L}^S_{\mathrm{sym}} \quad \text{for all } t \geq 0.$$

Let $f \in \mathcal{L}^I_{\mathrm{sym}} \cap \mathrm{Lip_b}$. We choose $c \geq 0$ and $t_0 > 0$ with

$$\|I(t)f - f\|_\infty \leq ct \quad \text{and} \quad \|I(t)(-f) + f\|_\infty \leq ct \quad \text{for all } t \in [0, t_0].$$

Let $(a,b) \in \mathbb{S}^d_+ \times \mathbb{R}^d$ with $L(a, \pm b) < \infty$ and $t \in [0, t_0]$. It holds

$$\begin{aligned} -\big(c + L(a,b)\big)t &\leq -\big(I(t)(-f) + f + L(a,b)t\big) \leq -\big(S_{a,b}(t)(-f) + f\big) \\ &= S_{a,b}(t)f - f \leq I(t)f - f + L(a,b)t \leq \big(c + L(a,b)\big)t \end{aligned}$$

and thus $\|S_{a,b}(t)f - f\|_\infty \leq (c + L(a,b))t$. Let $\eta\colon \mathbb{R}^d \to \mathbb{R}_+$ be infinitely differentiable with $\mathrm{supp}(\eta) \subset B_{\mathbb{R}^d}(1)$ and $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$. Define $\eta_n(x) := n^d \eta(nx)$ and

$$f_n(x) := (f * \eta_n)(x) = \int_{\mathbb{R}^d} f(x-y)\eta_n(y)\,\mathrm{d}y \quad \text{for all } n \in \mathbb{N} \text{ and } x \in \mathbb{R}^d.$$

For every $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$, Fubini's theorem implies

$$\begin{aligned} \big|S_{a,b}(t)f_n - f_n\big|(x) &= \left|\mathbb{E}\left[\int_{\mathbb{R}^d} f(x + \sqrt{a}W_t + bt - y)\eta_n(y)\,dy\right] - f_n(x)\right| \\ &= \left|\int_{\mathbb{R}^d} \mathbb{E}\big[f(x + \sqrt{a}W_t + bt - y)\big]\eta_n(y)\,dy - f_n(x)\right| \\ &= \big|(S_{a,b}(t)f - f) * \eta_n\big|(x) \leq \big\|S_{a,b}(t)f - f\big\|_\infty \leq \big(c + L(a,b)\big)t. \end{aligned}$$

It follows from $f_n \in \mathrm{C}^\infty_\mathrm{b} \subset D(B_{a,b})$ that

$$\|B_{a,b}f_n\|_\infty \leq c + L(a,b) \quad \text{for all } n \in \mathbb{N}. \tag{6.3}$$

In addition, for every $n \in \mathbb{N}$,

$$\Delta_a f_n = \frac{1}{2}\Delta_a f_n + \nabla_b f_n + \frac{1}{2}\Delta_a f_n + \nabla_{-b} f_n = B_{a,b}f_n + B_{a,-b}f_n. \tag{6.4}$$

Inequality (6.3) and equation (6.4) imply

$$\sup_{n\in\mathbb{N}} \|\Delta_a f_n\|_\infty \leq \|B_{a,b}f_n\|_\infty + \|B_{a,-b}f_n\|_\infty \leq 2c + L(a,b) + L(a,-b).$$

By Banach-Alaoglu's theorem, there exists $g \in L^\infty$ such that $\Delta_a f_{n_k} \to g$ in the weak*-topology for a suitable subsequence. Moreover, it follows from $f \in \mathrm{Lip_b} = W^{1,\infty}$ that $\sqrt{a}^T \nabla f_n = (\sqrt{a}^T \nabla f) * \eta_n \to \sqrt{a}^T \nabla f$ and thus $f \in D(\Delta_a)$ with $\Delta_a f = g$. Since the supremum norm is lower semicontinuous w.r.t. the weak*-topology, inequality (6.3) yields $\|B_{a,b}f\|_\infty \leq c + L(a,b)$. We obtain

$$f \in \bigcap_{a\in\mathbb{S}_L} D(\Delta_a) \cap W^{1,\infty} \quad \text{and} \quad \sup_{(a,b)\in\mathbb{S}_L\times\mathbb{R}^d} \big(\|B_{a,b}f\|_\infty - L(a,b)\big) < \infty.$$

Now, let $f \in \bigcap_{a\in\mathbb{S}_L} D(\Delta_a) \cap W^{1,\infty}$ and assume that

$$c := \sup_{(a,b)\in\mathbb{S}_L\times\mathbb{R}^d} \big(\|B_{a,b}f\|_\infty - L(a,b)\big) < \infty.$$

Let $t \geq 0$, $(a,b) \in \mathbb{S}_L \times \mathbb{R}^d$ and define $f_n := f * \eta_n$ for all $n \in \mathbb{N}$. It follows from Itô's formula and $B_{a,b}f_n = (B_{a,b}f) * \eta_n$ that

$$\begin{aligned} S_{a,b}(t)f - f - L(a,b)t &= \lim_{n\to\infty} \big(S_{a,b}(t)f_n - f_n - L(a,b)t\big) \\ &\leq \sup_{n\in\mathbb{N}} \big(\|B_{a,b}f_n\|_\infty - L(a,b)\big)t \\ &\leq \big(\|B_{a,b}\|_\infty - L(a,b)\big)t \leq ct. \end{aligned}$$

This implies $I(t)f - f \leq ct$. For the lower bound, we use $L(a^*,b^*) = 0$ to estimate

$$I(t)f - f \geq S_{a^*,b^*}(t)f - f \geq -\|B_{a^*,b^*}f\|_\infty t.$$

We obtain $f \in \mathcal{L}^I$ and applying the previous estimate on $-f$ yields $f \in \mathcal{L}^I_{\text{sym}}$. □

We do not assume that the matrices $a \in \mathbb{S}_L$ are positive definite which is a common assumption for parabolic PDEs. Moreover, in the completely degenerate case $\mathbb{S}_L = \{0\}$, we obtain $\mathcal{L}^S_{\text{sym}} \cap \text{Lip}_\text{b} = \text{Lip}_\text{b}$ and $(S(t))_{t\geq 0}$ is a shift semigroup corresponding to a first order PDE. On the other hand, if there exists a positive definite matrix $a \in \mathbb{S}_L$, then it follows from the proof of [131, Theorem 3.1.7] that

$$\mathcal{L}^S_{\text{sym}} \cap \text{Lip}_\text{b} \subset D(\Delta) = \bigcap_{p\geq 1} \{f \in W^{2,p}_{\text{loc}} \cap \text{C}_\text{b} \colon \Delta f \in L^\infty\}.$$

In particular, if the function

$$H \colon \mathbb{R}^{d\times d} \times \mathbb{R}^d, (x,y) \mapsto \sup_{(a,b)\in\mathbb{S}^d_+\times\mathbb{R}^d} \Big(\frac{1}{2}\sum_{i,j=1}^d a_{ij}x_{ij} + \sum_{i=1}^d b_i y_i\Big)$$

has at most polynomial growth, we can apply Theorem 4.4.7 to link the generator with distributional derivatives. The generator A can still be degenerate, since positive definiteness is not required for all $a \in \mathbb{S}_L$. We also remark that, if there exist $a \in \mathbb{S}^d_+$ and $\varepsilon > 0$ with

$$\sup_{\{b\in\mathbb{R}^d \colon |b|=\varepsilon\}} L(a,b) < \infty,$$

one can show $\mathcal{L}^S_{\text{sym}} \subset \text{Lip}_\text{b}$. For details, we refer to the proof of Theorem 1.6.3. However, without this additional assumption on L, the intersection with Lip_b is necessary. To our knowledge, an explicit description of $\mathcal{L}^S_{\text{sym}}$ seems to be unknown even in the linear case $A = \Delta$, where $\mathcal{L}^S_{\text{sym}}$ coincides with the Favard space which has been discussed in

Remark 4.2.7. Furthermore, if the sequence $(h_n)_{n\in\mathbb{N}}$ generates a sequence of partitions $\mathcal{T}_n := \{kh_n : k \in \mathbb{N}_0\}$ with $\mathcal{T}_n \subset \mathcal{T}_{n+1}$ for all $n \in \mathbb{N}$, one can show that the sequence $(I(\pi_n^t)f)_{n\in\mathbb{N}}$ is non-decreasing and therefore the semigroup can be obtained as monotone limit

$$S(t)f = \lim_{n\to\infty} I(\pi_n^t)f = \sup_{n\in\mathbb{N}} I(\pi_n^t)f \quad \text{for all } t \geq 0 \text{ and } f \in \mathrm{C_b}.$$

Hence, our result covers the Nisio construction as a particular case.

6.2 Trace class Wiener processes with drift

Let X be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. We denote by $\mathscr{S}_1(X)$ the space of all trace class operators endowed with the trace class norm $\|\cdot\|_{\mathscr{S}_1(X)}$. Recall that BUC consists of all bounded uniformly continuous functions $f \colon X \to \mathbb{R}$. Throughout this section, we make the following assumptions.

Assumption 6.2.1. Let $B \times \mathcal{Q} \subset X \times \mathscr{S}_1(X)$ satisfy the following conditions:

(i) Q is selfadjoint and positive semidefinite for all $Q \in \mathcal{Q}$.

(ii) $Q_1Q_2 = Q_2Q_1$ for all $Q_1, Q_2 \in \mathcal{Q}$.

(iii) $B \times \mathcal{Q}$ is bounded.

By the previous assumption and [170, Corollary 3.2.5], there exists a joint orthonormal basis of eigenvectors $(e_k)_{k\in\mathbb{N}} \subset X$, i.e., for every $Q \in \mathcal{Q}$ and $k \in \mathbb{N}$, there exists $\mu_{Q,k} \geq 0$ with $Qe_k = \mu_{Q,k}e_k$. For every $Q \in \mathcal{Q}$ and $x \in X$,

$$Qx = \sum_{k\in\mathbb{N}} \mu_{Q,k} \langle x, e_k\rangle e_k.$$

Hence, every $Q \in \mathcal{Q}$ can be identified with the sequence $\mu_Q = (\mu_{Q,k})_{k\in\mathbb{N}} \in \ell^1$ satisfying $\operatorname{tr}(Q) = \|Q\|_{\mathscr{S}_1(X)} = \|\mu_Q\|_{\ell^1}$. Assumption 6.2.1(iii) implies that $\Lambda := B \times \mu(\mathcal{Q}) \subset X \times \ell^1$ is bounded, where $\mu(\mathcal{Q}) := \{\mu_Q : Q \in \mathcal{Q}\}$. Let $(\xi^k)_{k\in\mathbb{N}}$ be a sequence of independent one-dimensional standard Brownian motions on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. For every $\lambda := (b, \mu) \in \Lambda$,

$$(S_\lambda(t)f)(x) := \mathbb{E}\left[f\left(x + tb + \sum_{k\in\mathbb{N}} \sqrt{\mu_k}\xi_t^k e_k\right)\right]$$

defines the unique linear semigroup $(S_\lambda(t))_{t\geq 0}$ associated to the Q-Wiener process with drift b and $\mu = \mu_Q$. Let

$$I(t)f := \sup_{\lambda\in\Lambda} S_\lambda(t)f \quad \text{for all } t \geq 0 \text{ and } f \in \mathrm{BUC}.$$

Moreover, for every $t \geq 0$, $f \in \mathrm{BUC}$ and $x \in X$, we define

$$(T(t)f)(x) := \sup_{(b,\mu)\in\mathcal{A}} \mathbb{E}\left[f\left(x + \int_0^t b_s\,\mathrm{d}s + \sum_{k\in\mathbb{N}} \Big(\int_0^t \sqrt{\mu_s^k}\,\mathrm{d}\xi_s^k\Big) e_k\right)\right],$$

where $\mathcal{A}$ consists of all predictable processes

$$(b,\mu)\colon \Omega\times[0,\infty)\to\Lambda,\ (\omega,t)\mapsto (b_t(\omega),(\mu_t^k(\omega))_{k\in\mathbb{N}}).$$

Let $(h_n)_{n\in\mathbb{N}}\subset(0,\infty)$ be a sequence with $h_n\to 0$ and $\mathcal{T}_n\subset\mathcal{T}_{n+1}$ for all $n\in\mathbb{N}$, where $\mathcal{T}_n:=\{kh_n\colon k\in\mathbb{N}_0\}$. We define the iterated operators $I(\pi_n^t):=I(h_n)^{k_n^t}$ for all $t\geq 0$ and $k,n\in\mathbb{N}$, where $k_n^t:=\max\{k\in\mathbb{N}_0\colon kh_n\leq t\}$ and $\pi_n^t:=\{0,h_n,\dots,k_n^t h_n\}$. One can show that the sequence $(I(\pi_n^t)f)_{n\in\mathbb{N}}$ is non-decreasing for all $t\geq 0$ and $f\in\mathrm{BUC}$. Denote by $\mathrm{Lip}_\mathrm{b}^2$ the space of all twice Fréchet differentiable functions $f\colon X\to\mathbb{R}$ with bounded Lipschitz continuous derivatives up to order two.

Theorem 6.2.2. *There exists a strongly continuous convex monotone semigroup* $(S(t))_{t\geq 0}$ *on* BUC *with* $S(t)0=0$ *given by*

$$S(t)f=\lim_{n\to\infty}I(\pi_n^t)f=\sup_{n\in\mathbb{N}}I(\pi_n^t)f \quad \textit{for all } f\in\mathrm{BUC} \textit{ and } t\geq 0$$

such that $\mathrm{Lip}_\mathrm{b}^2\subset D(A)$ *and*

$$Af=\sup_{\lambda\in\Lambda}\Big(\frac{1}{2}\operatorname{tr}(QD^2f)+\langle b,\nabla f\rangle\Big) \quad \textit{for all } f\in\mathrm{Lip}_\mathrm{b}^2.$$

Furthermore, it holds $S(t)f=T(t)f$ *for all* $t\geq 0$ *and* $f\in\mathrm{BUC}$ *such that there exist a compact linear operator* $K\colon X\to X$ *and* $g\in\mathrm{BUC}$ *with* $f(x)=g(Kx)$ *for all* $x\in X$.

Proof. Since $\sup_{(b,\mu)\in\Lambda}\big(|b|+\|\mu\|_{\ell^1}\big)<\infty$, we can apply the results from [136, Section 2 and Example 7.2] which guarantee the existence strongly continuous convex monotone semigroup $(S(t))_{t\geq 0}$ on BUC with $I(\pi_n^t)f\uparrow S(t)f$ for all $t\geq 0$ and $f\in\mathrm{BUC}$ such that $\mathrm{Lip}_\mathrm{b}^2\subset D(A)$ and

$$Af=\sup_{\lambda\in\Lambda}\Big(\frac{1}{2}\operatorname{tr}(QD^2f)+\langle b,\nabla f\rangle\Big) \quad \text{for all } f\in\mathrm{Lip}_\mathrm{b}^2.$$

Dini's theorem implies $I(\pi_n^t)f\to S(t)f$ uniformly on compacts and the inequality $S(t)f\leq T(t)f$ holds by definition for all $t\geq 0$ and $f\in\mathrm{BUC}$.

First, we show that $S(t)f=T(t)f$ for all $t\geq 0$ and $f\in\mathrm{BUC}$ depending only on finitely many coordinates, i.e., there exists $n\in\mathbb{N}$ such that

$$f(x)=f\left(\sum_{k=1}^n\langle x,e_k\rangle e_k\right) \quad \text{for all } x\in X.$$

Let $n\in\mathbb{N}$ and $X_n:=\operatorname{span}\{e_1,\dots,e_n\}\subset X$. For every $t\geq 0$, $f\in\mathrm{C_b}(X_n)$ and $x\in X_n$, we define

$$(T_n(t)f)(x):=\sup_{(b,\mu)\in\mathcal{A}_n}\mathbb{E}\left[f\left(x+\int_0^t b_s\,\mathrm{d}s+\sum_{k=1}^n\Big(\int_0^t\sqrt{\mu_s^k}\,\mathrm{d}\xi_s^k\Big)e_k\right)\right],$$

where $\mathcal{A}_n$ denotes the set of all predictable processes $(b,\mu)\colon\Omega\times[0,\infty)\to\Lambda_n$ with

$$\Lambda_n:=\Big\{\big((\langle b,e_k\rangle)_{k=1,\dots,n},(\mu_k)_{k=1,\dots,n}\big)\colon (b,\mu)\in\Lambda\Big\}\subset\mathbb{R}^n\times\mathbb{R}_+^n.$$

In addition, for every $t \geq 0$, $f_n \in \mathrm{C_b}(X_n)$ and $x \in X_n$, we define

$$\begin{aligned}(S_{\lambda,n}(t)f)(x) &:= \mathbb{E}\left[f\left(x + tb + \sum_{k=1}^{n} \sqrt{\mu_k}\xi_t^k e_k\right)\right] \quad \text{for all } \lambda := (b,\mu) \in \Lambda_n,\\ (I_n(t)f)(x) &:= \sup_{\lambda\in\Lambda_n} (S_{\lambda,n}(t)f)(x).\end{aligned}$$

Since Assumption 6.1.1 is satisfied with $L := \infty \cdot \mathbb{1}_{\Lambda_n^c}$, Theorem 6.1.4 guarantees the existence of a strongly continuous convex monotone semigroup $(S_n(t))_{t\geq 0}$ on $\mathrm{C_b}(X_n)$, which is given by

$$S_n(t)f = \lim_{k\to\infty} I_n(\pi_k^t)f \quad \text{for all } (f,t) \in \mathrm{C_b}(X_n) \times \mathbb{R}_+,$$

such that $S_n(t)f = T_n(t)f$ for all $t \geq 0$ and $f \in \mathrm{C_b}(X_n)$. Let $f \in \mathrm{BUC}$ such that there exists $\tilde{f} \in \mathrm{BUC}(X_n)$ with

$$f(x) = \tilde{f}\left(\sum_{k=1}^{n} \langle x, e_k\rangle e_k\right) \quad \text{for all } x \in X.$$

From the construction of $(S(t))_{t\geq 0}$ and $(S_n(t))_{t\geq 0}$ and the definition of $(T(t))_{t\geq 0}$ and $(T_n(t))_{t\geq 0}$, we obtain

$$S(t)f = S_n(t)\tilde{f} = T_n(t)\tilde{f} = T(t)f \quad \text{for all } t \geq 0.$$

Second, we show that $S(t)f = T(t)f$ for all $t \geq 0$ and all $f \in \mathrm{BUC}$ such that there exist a compact linear operator $K\colon X \to X$ and $g \in \mathrm{BUC}$ with $f(x) = g(Kx)$ for all $x \in X$. Since the finite rank operators are dense in the space of all compact linear operators w.r.t. the operator norm $\|\cdot\|_{L(X)}$, there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of finite rank operators with $\|K - K_n\|_{L(X)} \to 0$. Let $f_n(x) := g(K_n x)$ for all $x \in X$ and $n \in \mathbb{N}$. It holds $\|f_n\|_\infty \leq \|g\|_\infty$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \sup_{x\in B_X(r)} |f(x) - f_n(x)| = 0 \quad \text{for all } r \geq 0.$$

Moreover, the boundedness of Λ yields

$$c := \sup_{(b,\mu)\in\mathcal{A}} \mathbb{E}\left[\left|\int_0^t b_s\,\mathrm{d}s + \sum_{k=1}^{\infty}\Big(\int_0^t \sqrt{\mu_s^k}\,\mathrm{d}\xi_s^k\Big)e_k\right|\right] < \infty.$$

For every $r > 0$ and $x \in X$, we use Chebyshev's inequality to estimate

$$\begin{aligned}&\big(S(t)|f - f_n|\big)(x)\\ &\leq \sup_{y\in B_X(r)} |f(y) - f_n(y)|\\ &\quad + 2\|g\|_\infty \sup_{(b,\mu)\in\mathcal{A}} \mathbb{P}\left(\left|x + \int_0^t b_s\,\mathrm{d}s + \sum_{k=1}^{\infty}\Big(\int_0^t \sqrt{\mu_s^k}\,\mathrm{d}\xi_s^k\Big)e_k\right| > r\right)\\ &\leq \sup_{y\in B_X(r)} |f(y) - f_n(y)| + \frac{2\|g\|_\infty}{r} \sup_{(b,\mu)\in\mathcal{A}} \mathbb{E}\left[\left|x + \int_0^t b_s\,\mathrm{d}s + \sum_{k=1}^{\infty}\Big(\int_0^t \sqrt{\mu_s^k}\,\mathrm{d}\xi_s^k\Big)e_k\right|\right]\end{aligned}$$

$$\leq \sup_{y\in B_X(r)} |f(y) - f_n(y)| + \frac{2\|g\|_\infty(|x|+c)}{r}.$$

By choosing first $r > 0$ and then $n \in \mathbb{N}$ sufficiently large, we obtain

$$\begin{aligned}&\max\{|S(t)f - S(t)f_n|(x), |T(t)f - T(t)f_n|(x)\}\\ &\leq \max\{(S(t)|f - f_n|)(x), (T(t)|f - f_n|)(x)\} = (S(t)|f - f_n|)(x) \to 0.\end{aligned}$$

Hence, we can use the first part to conclude $S(t)f = T(t)f$. □

6.3 Wasserstein perturbations of linear transition semigroups

Let $p \in (1, \infty)$ and denote by $\mathcal{P}_p$ the set of all probability measures on the Borel-σ-algebra $\mathcal{B}(\mathbb{R}^d)$ with finite p-th moment. Let $(\mu_t)_{t\geq 0} \subset \mathcal{P}_p$ and $(\psi_t)_{t\geq 0}$ be a family of functions $\psi_t \colon \mathbb{R}^d \to \mathbb{R}^d$. Following the setting in [14, 85], for every $t \geq 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$, we define a reference semigroup by

$$(R(t)f)(x) := \int_{\mathbb{R}^d} f(\psi_t(x) + y)\,\mathrm{d}\mu_t(y).$$

Typical examples include Koopman semigroups and transition semigroups of Lévy and Ornstein–Uhlenbeck processes, see [85]. Denote by $\mathcal{L}^R$ the Lipschitz set of R and by $\mathrm{C_c^\infty}$ the the space of all infinitely differentiable functions $f \colon \mathbb{R}^d \to \mathbb{R}$ with compact support.

Assumption 6.3.1. Suppose that $(R(t))_{t\geq 0}$ is a semigroup. Furthermore, let $(\mu_t)_{t\geq 0}$ and $(\psi_t)_{t\geq 0}$ satisfy the following conditions:

(i) $\lim_{t\downarrow 0} \int_{\mathbb{R}^d} |y|^p\,\mathrm{d}\mu_t(y) = 0$.

(ii) There exist $r > 0$ and $c \geq 0$ such that $\mu_t(B_{\mathbb{R}^d}(r)^c) \leq ct$ for all $t \in [0, 1]$.

(iii) $\psi_t(0) = 0$ for all $t \geq 0$.

(iv) There exists $L \geq 0$ such that, for every $x, y \in \mathbb{R}^d$ and $t \in [0, 1]$,

$$|\psi_t(x) - \psi_t(y) - (x - y)| \leq Lt|x - y|.$$

(v) For every $f \in \mathrm{C_b^\infty} \cap \mathcal{L}^R$, the limit

$$R'(0)f := \lim_{h\downarrow 0} \frac{R(h)f - f}{h} \in \mathrm{C_b}$$

exists. Furthermore, it holds $\mathrm{C_c^\infty} \subset \mathcal{L}^R$.

For every $x, y \in \mathbb{R}^d$ and $t \in [0, 1]$, the conditions (iii) and (iv) imply

$$|\psi_t(x) - x| \leq Lt|x| \quad \text{and} \quad |\psi_t(x) - \psi_t(y)| \leq e^{Lt}|x - y|. \tag{6.5}$$

In the sequel, we consider a perturbation of the linear transition semigroup $(R(t))_{t\geq 0}$, where we take the supremum over all transition probabilities which are sufficiently close to the reference measure μ_t. In Subsection 6.1, we considered a Brownian motion with uncertain drift and diffusion, where the uncertainty was parametrized by a finite-dimensional parameter space. Here, the non-parametric uncertainty is instead given by an infinite-dimensional ball of probability measures. To that end, we endow $\mathcal{P}_p$ with the p-Wasserstein distance

$$W_p(\mu,\nu) := \left(\inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |y-z|^p \,\mathrm{d}\pi(y,z) \right)^{\frac{1}{p}}$$

where $\Pi(\mu,\nu)$ consists of all probability measures on $\mathcal{B}(\mathbb{R}^d\times\mathbb{R}^d)$ with first marginal μ and second marginal ν. Let $\varphi\colon \mathbb{R}_+\to[0,\infty]$ be a convex lower semicontinuous function with $\varphi(0)=0$ and $\varphi(v)>0$ for some $v>0$. Furthermore,we assume that that the mapping $[0,\infty)\to[0,\infty],\ v\mapsto\varphi\big(v^{1/p}\big)$ is convex. The previous assumptions guarantee that

$$\varphi^*(w) := \sup_{v\geq 0}\big(vw-\varphi(v)\big) < \infty \quad \text{for all } w\geq 0.$$

For every $t\geq 0$, $f\in\mathrm{C_b}$ and $x\in\mathbb{R}^d$, we define

$$(I(t)f)(x) := \sup_{\nu\in\mathcal{P}_p}\left(\int_{\mathbb{R}^d} f(\psi_t(x)+z)\,\mathrm{d}\nu(z) - \varphi_t\big(\mathcal{W}_p(\mu_t,\nu)\big)\right),$$

where $\varphi_t\colon[0,\infty)\to[0,\infty]$ denotes the rescaled function

$$\varphi_t(v) := \begin{cases} t\varphi\big(\frac{v}{t}\big), & t>0,\ v\geq 0,\\ 0, & t=v=0,\\ +\infty, & t=0,\ v\neq 0.\end{cases}$$

Let $(h_n)_{n\in\mathbb{N}}\subset(0,\infty)$ be a sequence with $h_n\to 0$ and $\mathcal{T}_n\subset\mathcal{T}_{n+1}$ for all $n\in\mathbb{N}$, where $\mathcal{T}_n:=\{kh_n\colon k\in\mathbb{N}_0\}$. We define the iterated operators $I(\pi_n^t):=I(h_n)^{k_n^t}$ for all $t\geq 0$ and $k,n\in\mathbb{N}$, where $k_n^t:=\max\{k\in\mathbb{N}_0\colon kh_n\leq t\}$ and $\pi_n^t:=\{0,h_n,\ldots,k_n^th_n\}$. One can show that the sequence $(I(\pi_n^t)f)_{n\in\mathbb{N}}$ is non-increasing for all $t\geq 0$ and $f\in\mathrm{C_b}$. The following result is a consequence of [85, Theorem 3.13].

Theorem 6.3.2. *There exists a strongly continuous convex monotone semigroup $(S(t))_{t\geq 0}$ on $\mathrm{C_b}$ with $S(t)0=0$ given by*

$$S(t)f = \lim_{n\to\infty} I(\pi_n^t)f = \inf_{n\in\mathbb{N}} I(\pi_n^t)f \quad \textit{for all } (f,t)\in\mathrm{C_b}\times\mathbb{R}_+$$

such that $\mathrm{C_b^\infty}\cap\mathcal{L}^R\subset D(A)$ and

$$Af = R'(0)f + \varphi^*(|\nabla f|) \quad \textit{for all } f\in\mathrm{C_b^\infty}\cap\mathcal{L}^R.$$

Furthermore, the semigroup $(S(t))_{t\geq 0}$ satisfies Assumption 4.2.4, condition (4.16) is valid for all $f\in\mathrm{Lip_b}$ and it holds $S(t)\colon\mathrm{Lip_b}\to\mathrm{Lip_b}$ for all $t\geq 0$.

Proof. By [85, Theorem 3.13 and Remark 3.14], there exists a strongly continuous convex monotone semigroup $(S(t))_{t\geq 0}$ on $\mathrm{C_b}$ with $S(t)0=0$ given by

$$S(t)f = \lim_{n\to\infty} I(\pi_n^t)f = \inf_{n\in\mathbb{N}} I(\pi_n^t)f \quad \text{for all } (f,t)\in\mathrm{C_b}\times\mathbb{R}_+$$

such that $\mathrm{C}_\mathrm{b}^\infty \cap \mathcal{L}^R \subset D(A)$ and

$$Af = R'(0)f + \varphi^*(|\nabla f|) \quad \text{for all } f \in \mathrm{C}_\mathrm{b}^\infty \cap \mathcal{L}^R.$$

Dini's theorem implies $I(\pi_n^t)f \to S(t)f$ uniformly on compacts and Assumption 4.2.4 is satisfied. Next, we show that

- $I(t)\colon B_{\mathrm{C_b}}(r) \to B_{\mathrm{C_b}}(r)$ for all $r, t \geq 0$,
- $\|I(t)f - I(t)g\|_\infty \leq \|f - g\|_\infty$ for all $t \geq 0$ and $f, g \in \mathrm{C_b}$,
- $I(t)\colon \mathrm{Lip_b}(r) \to \mathrm{Lip_b}(e^{Lt}r)$ for all $r, t \geq 0$,
- $\|\tau_x(I(t)f) - I(t)(\tau_x f)\|_\infty \leq Lrt|x|$ for all $r, t \geq 0$, $f \in \mathrm{Lip_b}(r)$ and $x \in \mathbb{R}^d$,

where $L \geq 0$ is the same constant as the one in Assumption 6.3.1(iv). Then, it follows from the proof of Theorem 5.2.3 that condition (4.16) is valid for all $f \in \mathrm{Lip_b}$. The first two statements follow immediately from the definition of $I(t)$. For every $r, t \geq 0$, $f \in \mathrm{Lip_b}(r)$ and $x, y \in \mathbb{R}^d$, inequality (6.5) implies

$$\begin{aligned} |(I(t)f)(x) - (I(t)f)(y)| &\leq \sup_{\nu \in \mathcal{P}_p} \int_{\mathbb{R}^d} |f(\psi_t(x) + z) - f(\psi_t(y) + z)| \,\mathrm{d}\nu(z) \\ &\leq e^{Lt} r|x - y|. \end{aligned} \tag{6.6}$$

Furthermore, we use Assumption 6.3.1(iv) to estimate

$$\begin{aligned} |(\tau_x(I(t)f) - I(t)(\tau_x f)|(y) &\leq \sup_{\nu \in \mathcal{P}_p} \int_{\mathbb{R}^d} |f(\psi_t(x+y) + z) - f(\psi_t(y) + x + z)| \,\mathrm{d}\nu(z) \\ &\leq r|\psi_t(x+y) - \psi_t(y) - x| \leq Lrt|x|. \end{aligned}$$

□

In addition to the Wasserstein perturbation, we consider a perturbation which is parametrized only by drifts $b \in \mathbb{R}^d$. For every $t \geq 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$, we define

$$(J(t)f)(x) := \sup_{b \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(\psi_t(x) + y + b) \,\mathrm{d}\mu_t(y) - \varphi_t(|b|t) \right).$$

We remark that $\mathcal{W}_p(\mu_t, \nu_b) = |b|t$ for all $b \in \mathbb{R}^d$ and $t \geq 0$, where

$$\nu_b(A) := \int_{\mathbb{R}^d} \mathbb{1}_A(b + y) \,\mathrm{d}\mu_t(y) \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^d).$$

Furthermore, the previous definition of $J(t)$ is consistent with the one in Subsection 6.1 if we fix the diffusion matrix a and choose $\mu_t := \mathbb{P} \circ (\sqrt{a} W_t)^{-1}$ and $L(b) := \varphi(|b|)$. However, the framework of this section does not cover diffusion uncertainty.

Theorem 6.3.3. *There exists a strongly continuous convex monotone semigroup $(T(t))_{t \geq 0}$ on $\mathrm{C_b}$ with $T(t)0 = 0$ given by*

$$T(t)f = \lim_{n \to \infty} J(\pi_n^t)f \quad \textit{for all } (f, t) \in \mathrm{C_b} \times \mathbb{R}_+ \tag{6.7}$$

such that $\mathrm{C}_\mathrm{b}^\infty \cap \mathcal{L}^R \subset D(B)$ *and*

$$Bf = R'(0)f + \varphi^*(|\nabla f|) \quad \textit{for all } f \in \mathrm{C}_\mathrm{b}^\infty \cap \mathcal{L}^R,$$

where B *denotes the generator of* $(T(t))_{t\geq 0}$. *The semigroup* $(T(t))_{t\geq 0}$ *satisfies Assumption 4.2.4, condition (4.16) is valid for all* $f \in \mathrm{Lip_b}$ *and it holds* $T(t)\colon \mathrm{Lip_b} \to \mathrm{Lip_b}$ *for all* $t \geq 0$. *In particular,*

$$S(t)f = T(t)f \quad \textit{for all } t \geq 0 \textit{ and } f \in \mathrm{C_b}.$$

Proof. First, we verify Assumption 5.2.2, where condition (v) is satisfied with $\mathrm{C}_\mathrm{b}^\infty \cap \mathcal{L}^R$ instead of $\mathrm{C}_\mathrm{b}^\infty$. Clearly, the operators $J(t)\colon \mathrm{C_b} \to \mathrm{F_b}$ are convex and monotone with $J(t)0 = 0$. Moreover,

$$\|J(t)f - J(t)g\|_\infty \leq \|f - g\|_\infty \quad \text{for all } f, g \in \mathrm{C_b}$$

and $J(t)\colon \mathrm{Lip_b}(r) \to \mathrm{Lip_b}(r)$ for all $r, t \geq 0$. It follows from $J(t)f \leq S(t)f$ that $J(t)$ is continuous from above and thus we can use Lemma 3.3.4 and an approximation argument to conclude $J(t)\colon \mathrm{C_b} \to \mathrm{C_b}$ for all $t \geq 0$. Moreover, for every $T \geq 0$, $K \Subset \mathbb{R}^d$ and sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C_b}$ with $f_n \downarrow 0$, we can use Dini's theorem to obtain

$$0 \leq \sup_{(t,x)\in[0,T]\times K} \sup_{n\in\mathbb{N}} \big(J(\pi_n^t)f_k\big)(x) \leq \sup_{(t,x)\in[0,T]\times K} (S(t)f_k)(x) \downarrow 0 \quad \text{as } k \to \infty.$$

Similar to the proof of Theorem 6.3.2, one can show that $J(t)\colon \mathrm{Lip_b}(r) \to \mathrm{Lip_b}(e^{Lt}r)$ and $\|\tau_x(J(t)f) - J(t)(\tau_x f)\|_\infty \leq Lrt|x|$ for all $r, t \geq 0$, $f \in \mathrm{Lip_b}(r)$ and $x \in \mathbb{R}^d$. Next, we show that $\mathrm{C}_\mathrm{b}^\infty \cap \mathcal{L}^R \subset \mathcal{L}^J$ and

$$J'(0)f = R'(0)f + \varphi^*(|\nabla f|) \quad \text{for all } f \in \mathrm{C}_\mathrm{b}^\infty \cap \mathcal{L}^R.$$

For every $r, t \geq 0$ and $f \in \mathrm{Lip_b}(r)$, it follows from [85, Equation (3.7)] that

$$0 \leq J(t)f - R(t)f \leq I(t)f - R(t)f \leq \varphi^*(r)t \tag{6.8}$$

and therefore $\mathrm{C}_\mathrm{b}^\infty \cap \mathcal{L}^R \subset \mathcal{L}^J$. Let $f \in \mathrm{C}_\mathrm{b}^\infty \cap \mathcal{L}^R$. Theorem 6.3.2 yields

$$\frac{J(h)f - f}{h} \leq \frac{S(h)f - f}{h} \to R'(0)f + \varphi^*(|\nabla f|) \quad \text{as } h \downarrow 0.$$

Moreover, for every $h > 0$ and $b, x, y \in \mathbb{R}^d$, Taylor's formula implies

$$|f(\psi_h(x) + y + bh) - f(\psi_h(x) + y) - \langle \nabla f(\psi_h(x) + y), bh\rangle| \leq \|D^2 f\|_\infty |b|^2 h^2.$$

As seen in the proof of [85, Lemma 3.6], it holds $R(h)g \to g$ as $h \downarrow 0$ for all $g \in \mathrm{Lip_b}$ and therefore

$$\begin{aligned}
&\frac{J(h)f - f}{h} = \frac{J(h)f - R(h)f}{h} + \frac{R(h)f - f}{h} \\
&\geq \frac{R(h)f - f}{h} + \int_{\mathbb{R}^d} \langle \nabla f(\psi_h(\,\cdot\,) + y, b\rangle \mathrm{d}\mu_h(y) - \varphi(|b|) - \|D^2 f\|_\infty |b|^2 h \\
&= \frac{R(h)f - f}{h} + R(h)\big(\langle \nabla f, b\rangle\big) - \varphi(|b|) - \|D^2 f\|_\infty |b|^2 h
\end{aligned}$$

$$\to R'(0)f + \langle \nabla f, b\rangle - \varphi(|b|)$$

as $h \downarrow 0$ for all $b \in \mathbb{R}^d$. Taking the supremum over $b \in \mathbb{R}^d$ yields

$$\lim_{h\downarrow 0} \frac{J(h)f - f}{h} = R'(0)f + \varphi^*(|\nabla f|).$$

Second, similarly to the proof of Theorem 5.2.3 one can show that Assumption 5.1.1 is satisfied. Hence, by Theorem 5.1.2, there exist a strongly continuous convex monotone semigroup $(T(t))_{t\geq 0}$ on $\mathrm{C_b}$ with $T(t)0 = 0$ and a subsequence $(n_l)_{l\in\mathbb{N}}$ such that

$$T(t)f = \lim_{l\to\infty} J(\pi^t_{n_l})f \quad \text{for all } (f,t) \in \mathrm{C_b} \times \mathcal{T} \tag{6.9}$$

for a countable dense set $\mathcal{T} \subset \mathbb{R}_+$ including zero. Furthermore, it holds $\mathrm{C}^\infty_\mathrm{b} \cap \mathcal{L}^R \subset D(B)$ and $Bf = R'(0)f + \varphi^*(|\nabla f|)$ for all $f \in \mathrm{C}^\infty_\mathrm{b} \cap \mathcal{L}^R$, where B denotes the generator of $(T(t))_{t\geq 0}$. Similarly to the proof of Theorem 5.2.3 one also show that condition (4.16) is valid for all $f \in \mathrm{Lip_b}$ and that $T(t)\colon \mathrm{Lip_b} \to \mathrm{Lip_b}$ for all $t \geq 0$. Since inequality (6.8) implies

$$0 \leq T(t)f - R(t)f \leq \varphi^*(r)t$$

for all $r, t \geq 0$ and $f \in \mathrm{Lip_b}(r)$, we obtain that the set

$$\mathcal{D} := \mathcal{L}^R \cap \mathrm{Lip_b} = \mathcal{L}^T \cap \mathrm{Lip_b}$$

does not depend on the choice of the convergence subsequence in equation (6.9) and satisfies $T(t)\colon \mathcal{D} \to \mathcal{D}$ for all $t \geq 0$. It remains to show that, for every $f \in \mathcal{D}$, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}^\infty_\mathrm{b} \cap \mathcal{L}^R$ with $f_n \to f$ and $B_\Gamma f = \Gamma\text{-}\lim_{n\to\infty} Bf_n$. To that end, let $f \in \mathcal{D}$ and $\eta\colon \mathbb{R}^d \to \mathbb{R}_+$ be infinitely differentiable with $\operatorname{supp}(\eta) \subset B_{\mathbb{R}^d}(1)$, and $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$. Define $\eta_n(x) := n^d\eta(nx)$ and $f_n := f * \eta_n \in \mathrm{C}^\infty_\mathrm{b}$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$. For every $t \geq 0$, $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$, Fubini's theorem and Assumption 6.3.1(iv) imply

$$\begin{aligned} &|R(t)f_n - (R(t)f) * \eta_n|(x) \\ &\leq \int_{B(1)} \left(\int_{\mathbb{R}^d} |f(\psi_t(x) + y - z) - f(\psi_t(x - z) + y)|\,\mathrm{d}\mu_t(y) \right) \eta_n(z)\,\mathrm{d}z \leq Lrt. \end{aligned}$$

We obtain $\|R(t)f_n - f_n\|_\infty \leq \|R(t)f - f\|_\infty + Lrt$ and thus $f_n \in \mathcal{L}^R$. Since Lemma 4.3.5 yields $B_\Gamma f = \Gamma\text{-}\lim_{n\to\infty} Bf_n$, similarly to the proof of Theorem 5.2.3, we can use Theorem 4.2.9 to obtain that equation (6.7) is valid.

Third, we show that $S(t)f = T(t)f$ for all $t \geq 0$ and $f \in \mathrm{C_b}$. It follows from inequality (6.8) that

$$0 \leq T(t)f - R(t)f \leq S(t)f - R(t)f \leq \varphi^*(r)t$$

for all $r, t \geq 0$ and $f \in \mathrm{Lip_b}(r)$ and therefore

$$\mathcal{D} := \mathcal{L}^R \cap \mathrm{Lip_b} = \mathcal{L}^S \cap \mathrm{Lip_b} = \mathcal{L}^T \cap \mathrm{Lip_b}.$$

Since $S(t)\colon \mathcal{D} \to \mathcal{D}$ and $T(t)\colon \mathcal{D} \to \mathcal{D}$ for all $t \geq 0$, we can use Lemma 4.3.5 and Theorem 4.2.9 to conclude that both semigroups coincide. □

In the particular case that $(S(t))_{t\geq 0}$ can be represented by the entropic risk measure, as a byproduct of Theorem 6.3.2, we recover that μ_t satisfies the Talagrand T_2 inequality, see [163, Chapter 22] and [158]. Denote by $\mathcal{N}(0,t\mathbb{1})$ the d-dimensional normal distribution with mean zero and covariance matrix $t\mathbb{1}$, where $\mathbb{1}\in\mathbb{R}^{d\times d}$ is the identity matrix.

Corollary 6.3.4. *It holds $\mathcal{W}_2(\nu,\mu_t)\leq\sqrt{2tH(\nu|\mu_t)}$ for all $t\geq 0$ and $\nu\in\mathbb{P}_2$, where $H(\nu|\mu_t)$ denotes the relative entropy of ν w.r.t. $\mu_t:=\mathcal{N}(0,t)$.*

Proof. Choose $\psi_t:=\mathrm{id}_{\mathbb{R}^d}$, $\mu_t:=\mathcal{N}(0,t\mathbb{1})$ and $\varphi(v):={}^{v^2}\!/2$ for all $t,v\geq 0$. Moreover, let $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion on a complete filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ satisfying the usual conditions. We show that

$$(S(t)f)(x)=(\tilde{S}(t)f)(x):=\frac{1}{2}\log\big(\mathbb{E}[\exp(2f(x+W_t))]\big)$$

for all $t\geq 0$, $f\in\mathrm{C_b}$ and $x\in\mathbb{R}^d$. Since the reference semigroup

$$(R(t)f)(x)=\mathbb{E}[f(x+W_t)]$$

satisfies $\mathrm{C}_\mathrm{b}^\infty\subset\mathcal{L}^R$, it follows from the proof of Theorem 6.3.2 that $(S(t))_{t\geq 0}$ satisfies Assumption 4.4.5. Moreover, by straightforward computations, one can show that the same is valid for $(\tilde{S}(t))_{t\geq 0}$ and that $\mathrm{C}_\mathrm{b}^\infty\subset D(\tilde{A})$ with

$$\tilde{A}f=\frac{1}{2}\big(\Delta f+|\nabla f|^2\big)\quad\text{for all } f\in\mathrm{C}_\mathrm{b}^\infty.$$

Theorem 4.4.6 implies $S(t)f=\tilde{S}(t)f$ for all $t\geq 0$ and $f\in\mathrm{C_b}$. Since Theorem 6.3.2 guarantees $S(t)f\leq I(t)f$ for all $t\geq 0$ and $f\in\mathrm{C_b}$, we can apply Fenchel–Moreau's theorem to obtain

$$\begin{aligned}\frac{\mathcal{W}_2(\nu,\mu_t)^2}{2t}&=\sup_{f\in\mathrm{C_b}}\left(\int_{\mathbb{R}^d}f\,\mathrm{d}\nu-(I(t)f)(0)\right)\\&\leq\sup_{f\in\mathrm{C_b}}\left(\int_{\mathbb{R}^d}f\,\mathrm{d}\nu-(\tilde{S}(t)f)(0)\right)=H(\nu|\mu_t).\end{aligned}$$

□

Chapter 7

Limit theorems for convex expectations

7.1 Introduction

It turns out that there is a structural link between Chernoff-type approximation results for convex monotone semigroups and LLN and CLT type results for convex expectations. The following diagram depicts this link for the classical CLT: for every iid sequence $(\xi_n)_{n\in\mathbb{N}}$ with finite second moments,

$$\begin{array}{ccc} \mathbb{E}\left[f\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n \xi_i\right)\right] & = & (I(\frac{1}{n})^n)(0) \\ \text{CLT}\Big\downarrow & & \Big\downarrow\text{Chernoff} \\ \int_{\mathbb{R}^d} f(y)\,\mathcal{N}(0,1)(\mathrm{d}y) & = & (S(1)f)(0), \end{array}$$

where $(I(t)f)(x) := \mathbb{E}[f(x+\sqrt{t}\xi_1]$ and $(S(t)f)(x) := \int_{\mathbb{R}^d} f(x+\sqrt{t}y)\,\mathcal{N}(0,1)(\mathrm{d}y)$ denotes the heat semigroup for all $t \geq 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$. In the linear case, it is natural to use the convergence on the left-hand side to obtain the convergence on the right-hand side which means that the CLT yields an approximation scheme for the heat semigroup. However, in the nonlinear case, we can not rely on characteristic functions to prove the convergence on the left-hand side and therefore we will use Theorem 5.2.3 to show the reverse implication instead. To be precise: we will replace the linear expectation $\mathbb{E}[\,\cdot\,]$ by a convex expectation $\mathcal{E}[\,\cdot\,]$ and use the Chernoff approximation to obtain

$$(S(t)f)(x) = \lim_{n\to\infty} \frac{t}{n}\mathcal{E}\left[\frac{n}{t} f\left(x + \sqrt{\frac{t}{n}}\sum_{i=1}^n \xi_i\right)\right]$$

for all $t > 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$. Furthermore, the generator of $(S(t))_{t\geq 0}$ is given by

$$(Af)(x) = \mathcal{E}\left[\frac{1}{2}\xi_1^T D^2 f(x)\xi_1\right] \quad \text{for all } f \in \mathrm{C_b^2} \text{ and } x \in \mathbb{R}^d$$

and uniquely characterizes the semigroup, i.e., the limit is G-distributed with

$$G\colon \mathbb{S}^d \to \mathbb{R},\; a \mapsto \mathcal{E}\left[\frac{1}{2}\xi_1^T a\xi_1\right].$$

Here, the set $\mathbb{S}^d$ consists of all symmetric $d\times d$-matrices. In the sublinear case, Peng [143] introduced the G-distribution by

$$F_G\colon \mathrm{Lip_b} \to \mathbb{R},\ f \mapsto u^f(1,0),$$

where u^f denotes the unique viscosity solution of the fully nonlinear PDE

$$\partial_t u(t,x) = G(D^2 u(x)) \quad \text{for all } (t,x) \in (0,\infty)\times\mathbb{R}^d$$

with initial condition $u(0,x) = f(x)$ for all $x \in \mathbb{R}^d$. In addition, the function G is given by $G(a) := \frac{1}{2}\sup_{\lambda\in\Lambda} \mathrm{tr}(\lambda\lambda^T a)$ for a bounded closed non-empty set $\Lambda \subset \mathbb{R}^{d\times d}$. Since [92, Theorem 6.2] implies $F_G(f) = (S(1)f)(0)$ for all $f \in \mathrm{Lip_b}$, our notation of the G-distribution is therefore consistent with one in [143]. At this point we would also like to mention the existence of CLT-type results for sublinear expectations [144], extensions to Lévy processes [15,97] and results about convergence rates [96,101,119,155]. Note that the proofs given in [144, 146] rely heavily on a deep interior estimate of a fully nonlinear PDE while the approach in [145] based on tightness and weak compactness requires an additional moment condition. In contrast to previous works, which are stated for sublinear expectations, Theorem 7.4.1 is, to our knowledge, the first result in the convex case. Moreover, the proof resembles the probabilistic approach in [145], but covers the sublinear case under the more natural moment condition $\lim_{c\to\infty}\mathcal{E}[(|\xi_1|^2 - c)^+] = 0$ from [146].

If we scale the sum of iid samples with ${}^1\!/\!{}_n$ instead of ${}^1\!/\!\sqrt{n}$, we obtain LLN-type results as well. In the language of semigroups this means that

$$(S(t)f)(x) = \lim_{n\to\infty} \frac{t}{n}\mathcal{E}\left[\frac{n}{t} f\left(x + \frac{t}{n}\sum_{i=1}^n \xi_i\right)\right]$$

for all $t > 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$, where the generator is given by $(Af)(x) = \mathcal{E}[\nabla f(x)\xi_1]$ for all $f \in \mathrm{C}^1_\mathrm{b}$ and $x \in \mathbb{R}^d$. Moreover, it follows from Theorem 7.3.4 that the limit is maximally distributed, i.e.,

$$(S(1)f)(0) = \sup_{x\in\mathbb{R}^d} \big(f(x) - \varphi(y)\big) \quad \text{for all } f \in \mathrm{C_b}.$$

In the sublinear case, Peng [146] introduced the maximal distribution by

$$F_\Lambda\colon \mathrm{Lip_b} \to \mathbb{R},\ f \mapsto \sup_{\lambda\in\Lambda} f(\lambda)$$

for a bounded closed non-empty set $\Lambda \subset \mathbb{R}^d$ and the previously mentioned references contain both CLT and LLN-type results. We want to point out two major differences to the framework of this article. First, the main result Theorem 7.3.2 is not restricted to sums of iid samples. Instead, we consider a sequence $(\psi_n)_{n\in\mathbb{N}}$ of recursively defined functions and are interested in the limit behaviour of

$$X_n^{t,x} := \psi_n\big(\tfrac{t}{n}, x, \xi_1, \ldots, \xi_n\big).$$

This includes, for instance, the case that the sample ξ_{n+1} is randomly shifted by a function depending on the average of the previous samples $\xi_1, \ldots, \xi_n$. Second, working with convex rather than sublinear expectations enables us to obtain large deviation results as well, e.g., Cramér's theorem can be seen as LLN for the entropic risk measure. Moreover, based on the weak convergence approach by Dupuis and Ellis [63], Lacker [127] has previously worked in a similar setting with convex expectations and established a non-exponential extension of Sanov's theorem leading to polynomial convergence rates that only require the existence of finite p-moments. Large deviations results based on other classes of risk measures have recently also been studied by several other authors, see, e.g., [5, 65, 78, 123]. Finally, we also want to mention related LLN-type results for capacities [132] and for coherent lower previsions [56].

7.2 Convex expectation spaces

Nonlinear expectations were introduced by Peng, see [146] for a detailed discussion, and are a closely related to several other concepts. For instance, sublinear expectations are called upper expectation in robust statistics [102], upper coherent prevision in the theory of imprecise probabilities [164] and coherent risk measure [4] in mathematical finance. Moreover, a convex expectation coincides (up to the sign) with the notion of a convex risk measure [79, 84]. In the sequel, we mainly follow [146, Chapter 1] up to some direct transfers from the sublinear to the convex case.

Definition 7.2.1. Let Ω be a set and $\mathcal{H}$ a linear space of functions $X\colon \Omega \to \mathbb{R}$ with $c \in \mathcal{H}$ and $|X| \in \mathcal{H}$ for all $c \in \mathbb{R}$ and $X \in \mathcal{H}$.[1] A convex expectation is a functional $\mathcal{E}\colon \mathcal{H} \to \mathbb{R}$ with

- $\mathcal{E}[c] = c$ for all $c \in \mathbb{R}$,
- $\mathcal{E}[X] \le \mathcal{E}[Y]$ for all $X, Y \in \mathcal{H}$ with $X \le Y$,
- $\mathcal{E}[\lambda X + (1-\lambda)Y] \le \lambda\mathcal{E}[X] + (1-\lambda)\mathcal{E}[Y]$ for all $X, Y \in \mathcal{H}$ and $\lambda \in [0,1]$.

The triplet $(\Omega, \mathcal{H}, \mathcal{E})$ is called a convex expectation space. Furthermore, we say that $\mathcal{E}$ is continuous from above if $\mathcal{E}[X_n] \downarrow 0$ for all $(X_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ with $X_n \downarrow 0$. If $\mathcal{E}[\lambda X] = \lambda\mathcal{E}[X]$ for all $\lambda \ge 0$ and $X \in \mathcal{H}$ we say that $\mathcal{E}$ is sublinear and call $(\Omega, \mathcal{H}, \mathcal{E})$ a sublinear expectation space.

If $\mathcal{E}$ is continuous from above it follows from the convexity that $\mathcal{E}[X_n] \downarrow \mathcal{E}[X]$ for all $(X_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ and $X \in \mathcal{H}$ with $X_n \downarrow X$. Moreover, in the previous definition all inequalities and the monotone convergence are understood pointwise. We collect some elementary properties of convex expectations.

Lemma 7.2.2. *For a convex expectation space $(\Omega, \mathcal{H}, \mathcal{E})$ the following statements hold:*

(i) $\mathcal{E}[X + c] = \mathcal{E}[X] + c$ *for all* $X \in \mathcal{H}$ *and* $c \in \mathbb{R}$.

(ii) $|\mathcal{E}[X] - \mathcal{E}[Y]| \le \|X - Y\|_\infty$ *for all bounded* $X, Y \in \mathcal{H}$.

(iii) $\mathcal{E}[\lambda X] \le \lambda\mathcal{E}[X]$ *for all* $\lambda \in [0,1]$ *and* $X \in \mathcal{H}$.

(iv) $-\mathcal{E}[-X] \le \mathcal{E}[X]$ *for all* $X \in \mathcal{H}$.

[1] We identity c with the constant function $c\mathbb{1}_\Omega$.

(v) $|\mathcal{E}[X]| \leq \mathcal{E}[|X|]$ *for all* $X \in \mathcal{H}$.

(vi) Let $X \in \mathcal{H}$ *with* $\mathcal{E}[aX] = 0$ *for all* $a \in \mathbb{R}$. *Then, it holds*

$$\mathcal{E}[X+Y] = \mathcal{E}[Y] \quad \text{for all } Y \in \mathcal{H}.$$

Proof.

(i) For every $X \in \mathcal{H}$, $c \in \mathbb{R}$ and $\lambda \in (0,1)$, we use that $\mathcal{E}$ is convex and preserves constants to estimate

$$\mathcal{E}[X+c] \leq \lambda \mathcal{E}\left[\frac{X}{\lambda}\right] + (1-\lambda)\mathcal{E}\left[\frac{c}{1-\lambda}\right] = \lambda \mathcal{E}\left[\frac{X}{\lambda}\right] + c$$

and

$$\mathcal{E}[X] = \mathcal{E}[X+c-c] \leq \lambda \mathcal{E}\left[\frac{X+c}{\lambda}\right] + (1-\lambda)\mathcal{E}\left[-\frac{c}{1-\lambda}\right] = \lambda \mathcal{E}\left[\frac{X+c}{\lambda}\right] - c.$$

Since the real-valued mapping $\lambda \mapsto \mathcal{E}[\lambda X]$ is convex and therefore continuous, we obtain in the limit $\lambda \to 1$ that $\mathcal{E}[X+c] = \mathcal{E}[X] + c$.

(ii) It follows from the monotonicity and part (i) that

$$\mathcal{E}[X] \leq \mathcal{E}[Y + \|X-Y\|_\infty] = \mathcal{E}[Y] + \|X-Y\|_\infty.$$

Changing the roles of X and Y yields the claim.

(iii) For every $\lambda \in [0,1]$ and $X \in \mathcal{H}$,

$$\mathcal{E}[\lambda X] = \mathcal{E}[\lambda X + (1-\lambda)0] \leq \lambda \mathcal{E}[X] + (1-\lambda)\mathcal{E}[0] = \lambda \mathcal{E}[X].$$

(iv) It holds $0 = \mathcal{E}[0] = \mathcal{E}[\frac{1}{2}X + \frac{1}{2}(-X)] \leq \frac{1}{2}\mathcal{E}[X] + \frac{1}{2}\mathcal{E}[-X]$.

(v) We use the monotonicity and part (iv) to estimate

$$-\mathcal{E}[|X|] \leq -\mathcal{E}[-X] \leq \mathcal{E}[X] \leq \mathcal{E}[|X|].$$

(vi) For every $\lambda \in (0,1]$, it follows from Lemma 3.6.1 that

$$\begin{aligned}\mathcal{E}[X+Y] - \mathcal{E}[Y] &\leq \lambda \mathcal{E}\left[\frac{X}{\lambda} + Y\right] - \lambda \mathcal{E}[Y] \\ &\leq \frac{\lambda}{2}\mathcal{E}\left[\frac{2X}{\lambda}\right] + \frac{\lambda}{2}\mathcal{E}[2Y] - \lambda \mathcal{E}[Y] \\ &= \frac{\lambda}{2}\mathcal{E}[2Y] - \lambda \mathcal{E}[Y] \to 0 \quad \text{as } \lambda \downarrow 0.\end{aligned}$$

Similarly one can show that

$$\begin{aligned}\mathcal{E}[X+Y] - \mathcal{E}[Y] &\geq -\frac{\lambda}{2}\mathcal{E}\left[-\frac{2X}{\lambda}\right] - \frac{\lambda}{2}\mathcal{E}[2(X+Y)] + \lambda \mathcal{E}[X+Y] \\ &= \frac{\lambda}{2}\mathcal{E}[2(X+Y)] + \lambda \mathcal{E}[X+Y] \to 0 \quad \text{as } \lambda \downarrow 0.\end{aligned}$$

□

Definition 7.2.3. Let $(\Omega, \mathcal{H}, \mathcal{E})$ be a convex expectation space with $f(X_1, \ldots, X_n) \in \mathcal{H}$ for all $n \in \mathbb{N}$, $f \in \mathrm{C_b}(\mathbb{R}^n)$ and $X \in \mathcal{H}^n$. Let $m, n \in \mathbb{N}$, $X \in \mathcal{H}^m$ and $Y \in \mathcal{H}^n$.

(i) The distribution of X is given by the functional

$$F_X\colon \mathrm{C_b}(\mathbb{R}^m) \to \mathbb{R},\ f \mapsto \mathcal{E}[f(X)].$$

(ii) We say that X and Y are identically distributed if $m = n$ and

$$\mathcal{E}[f(X)] = \mathcal{E}[f(Y)] \quad \text{for all } f \in \mathrm{Lip_b}(\mathbb{R}^m).$$

(iii) We say that Y is independent of X if

$$\mathcal{E}[f(X,Y)] = \mathcal{E}\left[\mathcal{E}[f(x,Y)]\big|_{x=X}\right] \quad \text{for all } f \in \mathrm{Lip_b}(\mathbb{R}^m \times \mathbb{R}^n).$$

In general, the statements “Y is independent of X” and “X is independent of Y” are not equivalent, see [146, Example 1.3.15] for a counterexample.

Definition 7.2.4. Let $(\Omega, \mathcal{H}, \mathcal{E})$ be a convex expectation space and $(X_n)_{n\in\mathbb{N}} \subset \mathcal{H}^d$ be a sequence of random vectors for some $d \in \mathbb{N}$.

(i) We say that $(X_n)_{n\in\mathbb{N}}$ is independent and identically distributed (iid) if X_m and X_n have the same distribution and X_{n+1} is independent of $(X_1, \ldots, X_n)$ for all $m, n \in \mathbb{N}$.

(ii) We say that $(X_n)_{n\in\mathbb{N}}$ converges in distribution if the sequence $(\mathcal{E}[f(X_n)])_{n\in\mathbb{N}}$ converges for all $f \in \mathrm{Lip_b}(\mathbb{R}^d)$.

Let $\Omega_\infty := (\mathbb{R}^d)^{\mathbb{N}}$. For every $n \in \mathbb{N}$, we define $\pi_n\colon \Omega_\infty \to (\mathbb{R}^d)^n,\ x \mapsto (x_1, \ldots, x_n)$ and $\mathcal{H}_n := \{f \circ \pi_n \colon f \in \mathrm{Lip_b}((\mathbb{R}^d)^n)\}$. Furthermore, let $\mathcal{H}_\infty := \bigcup_{n\in\mathbb{N}} \mathcal{H}_n$. For a convex expectation $\mathcal{E}$ on $(\mathbb{R}^d, \mathrm{Lip_b}(\mathbb{R}^d))$, we define recursively a sequence of convex expectations $\mathcal{E}_n\colon \mathrm{Lip_b}((\mathbb{R}^d)^n) \to \mathbb{R}$ by $\mathcal{E}_1 := \mathcal{E}$ and

$$\mathcal{E}_{n+1}[f] := \mathcal{E}_n\big[\mathbb{E}[f(x,Y)]\big|_{x=X}\big],$$

where $X := \mathrm{id}_{(\mathbb{R}^d)^n}$ and $Y := \mathrm{id}_{\mathbb{R}^d}$. Note that the function $x \mapsto \mathcal{E}[f(x,Y)]$ is Lipschitz continuous, since Lemma 7.2.2(ii) implies

$$|\mathcal{E}[f(x,\cdot)] - \mathcal{E}[f(y,\cdot)]| \le \|f(x,\cdot) - f(y,\cdot)\|_\infty \le r|x-y|$$

for all $r \ge 0$, $f \in \mathrm{Lip_b}(r)$ and $x, y \in \mathbb{R}^d$. Furthermore, let

$$\mathcal{E}_\infty\colon \mathcal{H}_\infty \to \mathbb{R},\ f \circ \pi_n \mapsto \mathcal{E}_n[f].$$

The next result is a direct transfer of [146, Proposition 1.3.17] from the sublinear to the convex case.

Lemma 7.2.5. *Let $\mathcal{E}$ be a convex expectation on $(\mathbb{R}^d, \mathrm{Lip_b}(\mathbb{R}^d))$ and define $(\Omega_\infty, \mathcal{H}_\infty, \mathcal{E}_\infty)$ as above. Furthermore, let*

$$\xi_n\colon \Omega_\infty \to \mathbb{R},\ x \mapsto x_n \quad \textit{for all } n \in \mathbb{N}.$$

Then, the sequence $(\xi_n)_{n\in\mathbb{N}}$ is iid and satisfies $\mathcal{E}_\infty[f(\xi_n)] = \mathcal{E}[f]$ for all $f \in \mathrm{Lip_b}(\mathbb{R}^d)$.

The next result is a direct application of [59, Theorem 4.6]. Let $\mathcal{B}^{\mathbb{N}} := \bigotimes_{n\in\mathbb{N}} \mathcal{B}(\mathbb{R})$ be the product-σ-algebra, where $\mathcal{B}(\mathbb{R})$ denotes the Borel-σ-algebra, and define $\mathcal{L}^\infty(\Omega_\infty)$ as the space of all bounded $\mathcal{B}^{\mathbb{N}}$-measurable functions $f\colon \Omega_\infty \to \mathbb{R}$.

Theorem 7.2.6 (Kolmogorov)**.** *Let $\mathcal{E}$ be a convex expectation on $(\mathbb{R}^d, \mathrm{C_b}(\mathbb{R}^d))$ which is continuous from above and define $(\Omega_\infty, \mathcal{H}_\infty, \mathcal{E}_\infty)$ as before. Then, there exists a unique convex expectation $\bar{\mathcal{E}}_\infty\colon \mathcal{L}^\infty(\Omega_\infty) \to \mathbb{R}$ with $\bar{\mathcal{E}}_\infty[f] = \mathcal{E}_\infty[f]$ for all $f \in \mathcal{H}_\infty$ which is continuous from below on $\mathcal{L}^\infty(\Omega_\infty)$ and continuous from above on*

$$(H_\infty)_\delta := \{f \in \mathcal{L}^\infty(\Omega_\infty)\colon \text{there exists } (f_n)_{n\in\mathbb{N}} \subset \mathcal{H}_\infty \text{ with } f_n \downarrow f\}.$$

Moreover, the sequence $(\xi_n)_{n\in\mathbb{N}}$ defined by

$$\xi_n\colon \Omega_\infty \to \mathbb{R},\ x \mapsto x_n \quad \text{for all } n \in \mathbb{N}$$

is iid and satisfies $\mathcal{E}_\infty[f(\xi_n)] = \mathcal{E}[f]$ for all $f \in \mathrm{C_b}(\mathbb{R}^d)$.

While the property that a random vector has mean zero w.r.t. a convex expectation might seem quite restrictive, this can always be achieved by a simple transformation of the convex expectation if the random vector has non-negative mean. If the convex expectation is defined as a supremum over a set of probability measures, the transformed expectation will be given by a supremum over a smaller set of measures which satisfy an additional constraint, see Subsection 7.4.2.

Lemma 7.2.7. *Let $(\Omega, \mathcal{H}, \mathcal{E})$ be a convex expectation space and $\xi \in \mathcal{H}^d$ with $\mathcal{E}[a\xi] \geq 0$ for all $a \in \mathbb{R}^d$. Then,*

$$\tilde{\mathcal{E}}\colon \mathcal{H} \to \mathbb{R},\ X \mapsto \inf_{a\in\mathbb{R}^d} \mathcal{E}[X + a\xi]$$

is a convex expectation with $\tilde{\mathcal{E}}[a\xi] = 0$ for all $a \in \mathbb{R}^d$. If $\mathcal{E}$ is continuous from above, then the same is valid for $\tilde{\mathcal{E}}$. Moreover, if $\mathcal{E}[a\xi] = 0$ for all $a \in \mathbb{R}^d$, then $\mathcal{E} = \tilde{\mathcal{E}}$.

Proof. Clearly, the functional $\tilde{\mathcal{E}}$ is monotone and satisfies $\tilde{\mathcal{E}}[c] = c$ for all $c \in \mathbb{R}$. For every $X, Y \in \mathcal{H}$, $\lambda \in [0,1]$ and $a, b \in \mathbb{R}^d$,

$$\begin{aligned}\tilde{\mathcal{E}}[\lambda X + (1-\lambda)Y] &\leq \mathcal{E}[\lambda X + (1-\lambda)Y + (\lambda a + (1-\lambda)b)\xi] \\ &\leq \lambda\mathcal{E}[X + a\xi] + (1-\lambda)\mathcal{E}[Y + b\xi].\end{aligned}$$

Taking the infimum over $a, b \in \mathbb{R}^d$ yields

$$\tilde{\mathcal{E}}[\lambda X + (1-\lambda)Y] \leq \lambda\tilde{\mathcal{E}}[X] + (1-\lambda)\tilde{\mathcal{E}}[Y].$$

Since $\mathcal{E}[a\xi + b\xi] = \mathcal{E}[(a+b)\xi] \geq 0$ for all $a, b \in \mathbb{R}^d$ with equality for $b = -a$, we obtain $\tilde{\mathcal{E}}[a\xi] = 0$ for all $a \in \mathbb{R}^d$. Assume that $\mathcal{E}$ is continuous from above and let $(X_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ with $X_n \downarrow 0$. Then,

$$\inf_{n\in\mathbb{N}} \tilde{\mathcal{E}}[X_n] = \inf_{n\in\mathbb{N}} \inf_{a\in\mathbb{R}^d} \mathcal{E}[X_n + a\xi] = \inf_{a\in\mathbb{R}^d} \inf_{n\in\mathbb{N}} \mathcal{E}[X_n + a\xi] = \inf_{a\in\mathbb{R}^d} \mathcal{E}[a\xi] = 0$$

which shows that $\tilde{\mathcal{E}}$ is continuous from above. If $\mathcal{E}[a\xi] = 0$ for all $a \in \mathbb{R}^d$, it follows from Lemma 7.2.2(vi) that $\tilde{\mathcal{E}}[X] = \inf_{a\in\mathbb{R}^d} \mathcal{E}[X + a\xi] = \mathcal{E}[X]$ for all $X \in \mathcal{H}$. □

7.3 First order scaling limits

Throughout this section, we choose the weight function

$$\kappa\colon \mathbb{R}^d \to (0,\infty),\ x \mapsto (1+|x|)^{-1}.$$

Let $(\xi_n)_{n\in\mathbb{N}} \subset \mathcal{H}^d$ be an iid sequence of random vectors on a convex expectation space $(\Omega, \mathcal{H}, \bar{\mathcal{E}})$ with finite first moments and let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of functions $\psi_n\colon \mathbb{R}_+ \times \mathbb{R}^d \times (\mathbb{R}^d)^n \to \mathbb{R}^d$ which will be specified below. In addition, for every $t \geq 0$ and $x \in \mathbb{R}^d$, we define a sequence $(X_n^{t,x})_{n\in\mathbb{N}}$ of random vectors by

$$X_n^{t,x} := \psi_n\big(\tfrac{t}{n}, x, \xi_1, \ldots, \xi_n\big)$$

whose behaviour in the limit we are interested in. For $\psi_n(t,x,y_1,\ldots,y_n) := x + t\sum_{i=1}^n y_i$ we end up with an averaged sum of iid samples, i.e.,

$$X_n^{t,x} = x + \frac{t}{n}\sum_{i=1}^n \xi_i \quad \text{for all } n \in \mathbb{N}.$$

Since we are only interested in weak convergence, we mainly consider the distributions of the vectors X_n. In this framework, requiring finite first moments means that the distribution of ξ_1 is well-defined for continuous functions with at most linear growth at infinity, i.e.,

$$F_{\xi_1}\colon \mathrm{C}_\kappa \to \mathbb{R},\ f \mapsto \bar{\mathcal{E}}[f(\xi_1)].$$

This functional is subsequently denoted by $\mathcal{E}[\,\cdot\,]$ and supposed to be continuous from above in order to guarantee that Assumption 5.2.2(vi) is valid. In the linear case, continuity from above is always given since $\lim_{c\to\infty} \mathbb{E}[|\xi_1| \mathbb{1}_{\{|\xi_1|\geq c\}}] = 0$. Furthermore, continuity from above on C_κ is equivalent to the uniform integrability condition $\lim_{c\to\infty} \bar{\mathcal{E}}[(|\xi_1| - c)^+] = 0$ from [146]. We define $I(0) := \mathrm{id}_{\mathrm{C}_\kappa}$ and

$$(I(t)f)(x) := t\bar{\mathcal{E}}\left[\frac{1}{t} f\big(\psi_1(t,x,\xi_1)\big)\right]$$

for all $t > 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$. The aim of this section is to show that

$$S(t)f = \lim_{n\to\infty} I\big(\tfrac{t}{n}\big)f \quad \text{for all } (f,t) \in \mathrm{C_b} \times \mathbb{R}_+,$$

where $(S(t))_{t\geq 0}$ is a the unique strongly continuous convex monotone semigroup with generator $(Af)(x) = \mathcal{E}[\nabla f(x)\xi]$ for all $f \in \mathrm{C}_\mathrm{b}^1$, $x \in \mathbb{R}^d$ and $\xi := \mathrm{id}_{\mathbb{R}^d}$. Furthermore, the independence of $(\xi_n)_{n\in\mathbb{N}}$ implies

$$\big(I\big(\tfrac{t}{n}\big)^n f\big)(x) = \frac{t}{n}\bar{\mathcal{E}}\left[\frac{t}{n} f(X_n^{t,x})\right]$$

and the limit semigroup can be identified with a maximal distribution. Hence, the previous result about convergence of operators can be formulated equivalently as a result about convergence of random variables. In Subsection 7.3.1, we prove this result for first order scaling limits as well as some immediate consequences. Motivated by Cramér's theorem, we then show that similar convergence rates can be extended beyond the case of averaged sums of iid samples, see Subsection 7.3.2. Moreover, we use a clever estimate from [127] to obtain polynomial convergence rates without requiring exponential moments. Finally, we consider convex expectations that are given as a supremum of probability measures which are penalized according to their Wasserstein distance to a reference model, see Subsection 7.3.3.

7.3.1 LLN for convex expectations and large deviations

Let $\mathcal{E}\colon \mathrm{C}_\kappa \to \mathbb{R}$ be a convex expectation which is continuous from above, i.e.,

- $\mathcal{E}[c] = c$ for all $c \in \mathbb{R}$,
- $\mathcal{E}[f] \le \mathcal{E}[g]$ for all $f, g \in \mathrm{C}_\kappa$ with $f \le g$,
- $\mathcal{E}[\lambda f + (1-\lambda)g] \le \lambda\mathcal{E}[f] + (1-\lambda)\mathcal{E}[g]$ for all $f, g \in \mathrm{C}_\kappa$ and $\lambda \in [0,1]$,
- $\mathcal{E}[f_n] \downarrow 0$ for all $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa$ with $f_n \downarrow 0$.

By Theorem 3.2.1, there exists a convex monotone extension $\mathcal{E}\colon \mathrm{B}_\kappa \to \mathbb{R}$ such that, for every $\varepsilon > 0$ and $c \ge 0$, there exists $K \Subset \mathbb{R}^d$ with

$$\mathcal{E}\left[\tfrac{c}{\kappa}\mathbb{1}_{K^c}\right] < \varepsilon. \tag{7.1}$$

Recall that B_κ consists of all Borel measurable functions $f\colon \mathbb{R}^d \to \mathbb{R}$ with $\|f\|_\kappa < \infty$. Furthermore, it follows from the convexity of $\mathbb{E}$ that $\mathcal{E}[f_n] \downarrow \mathcal{E}[f]$ for all $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa$ and $f \in \mathrm{C}_\kappa$ with $f_n \downarrow f$.

Assumption 7.3.1. Let $\psi\colon \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function which satisfies the following conditions:

(i) There exists $L \ge 0$ such that, for every $t \ge 0$ and $x, y, z \in \mathbb{R}^d$,

$$|x + \psi(t, y, z) - \psi(t, x + y, z)| \le Lt|x|.$$

Furthermore, it holds $|\psi(t, x, y) - x| \le L(1 + |y|)t$ for all $t \ge 0$ and $x, y \in \mathbb{R}^d$.

(ii) There exists a continuous function $\psi_0\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ with

$$\lim_{h\downarrow 0} \sup_{x,y\in K} \left|\frac{\psi(h, x, y) - x}{h} - \psi_0(x, y)\right| = 0 \quad \text{for all } K \Subset \mathbb{R}^d.$$

We define $I(0) := \mathrm{id}_{\mathrm{C}_\mathrm{b}}$ and, for every $t > 0$, $f \in \mathrm{C}_\mathrm{b}$ and $x \in \mathbb{R}^d$,

$$(I(t)f)(x) := t\mathcal{E}\left[\frac{1}{t} f\big(\psi(t, x, \cdot)\big)\right]$$

Let C^1_b consist of all bounded continuously differentiable functions $f\colon \mathbb{R}^d \to \mathbb{R}$ such that all partial derivatives are bounded and denote by $xy := \langle x, y\rangle$ the Euclidean inner product on $\mathbb{R}^d$. At this point we would like to emphasize that C_κ appears in this section only to guarantee that $\mathcal{E}[\,\cdot\,]$ is defined and continuous from above on functions with at most linear growth at infinity. However, we consider $(I(t))_{t\ge 0}$ and $(S(t))_{t\ge 0}$ as operator families on C_b. In particular, the definition of the (upper) Lipschitz sets and the statements of Theorem 5.2.3 are understood w.r.t. the supremum norm $\|\cdot\|_\infty$ rather than $\|\cdot\|_\kappa$. Furthermore, the convergence in equation (7.2) and the definition of the generator Af are understood as convergence in $\mathrm{C_b}$ w.r.t. the corresponding mixed topology. We define a sequence $(\psi_n)_{n\in\mathbb{N}}$ of functions $\psi_n\colon \mathbb{R}_+\times\mathbb{R}^d\times(\mathbb{R}^d)^n\to\mathbb{R}^d$ recursively by $\psi_1:=\psi$ and

$$\psi_{n+1}(t,x,y_1,\ldots,y_{n+1}) := \psi\big(t,\psi_n(t,x,y_1,\ldots,y_n),y_{n+1}\big) \quad \text{for all } n\in\mathbb{N}.$$

The following theorem is the main result of this section.

Theorem 7.3.2. *Suppose that Assumption 7.3.1 is satisfied. Then, there exists a strongly continuous convex monotone semigroup* $(S(t))_{t\geq 0}$ *on* $\mathrm{C_b}$ *given by*

$$S(t)f = \lim_{n\to\infty} I\big(\tfrac{t}{n}\big)^n f \quad \text{for all } t\geq 0 \text{ and } f\in\mathrm{C_b} \tag{7.2}$$

such that Assumption 4.4.5 is valid. It holds $\mathrm{C_b^1}\subset D(A)$ *and*

$$(Af)(x) = \mathcal{E}[\nabla f(x)\psi_0(x,\cdot)] \quad \text{for all } f\in\mathrm{C_b^1} \text{ and } x\in\mathbb{R}^d.$$

In addition, for every convex expectation space $(\Omega,\mathcal{H},\bar{\mathcal{E}})$ *and iid sequence* $(\xi_n)_{n\in\mathbb{N}}\subset\mathcal{H}^d$ *with* $\bar{\mathcal{E}}[f(\xi_n)]=\mathcal{E}[f]$ *for all* $f\in\mathrm{C_b}$,

$$(S(t)f)(x) = \lim_{n\to\infty}\frac{t}{n}\bar{\mathcal{E}}\left[\frac{n}{t}f(X_n^{t,x})\right] \tag{7.3}$$

for all $t>0$, $f\in\mathrm{C_b}$ *and* $x\in\mathbb{R}^d$, *where* $X_n^{t,x}:=\psi_n\big(\tfrac{t}{n},x,\xi_1,\ldots,\xi_n\big)$.

Proof. First, we verify the conditions (i)–(iv), (vi) and (vii) from Assumption 5.2.2. Lemma 7.2.2(ii) guarantees that the conditions (i)-(iii) are satisfied with $\omega=0$. For every $t\in(0,1]$, $r\geq 0$, $f\in\mathrm{Lip_b}(r)$ and $x,y\in\mathbb{R}^d$, it follows from Lemma 7.2.2(ii) and Assumption 7.3.1(i) that

$$\begin{aligned}|I(t)(\tau_x f)-\tau_x I(t)f|(y) &= t\left|\mathcal{E}\left[\frac{1}{t}f(x+\psi(t,y,\cdot))\right]-\mathcal{E}\left[\frac{1}{t}f(\psi(t,x+y,\cdot))\right]\right|\\ &\leq \|f(x+\psi(t,y,\cdot))-f(\psi(t,x+y,\cdot))\|_\infty\\ &\leq r\|x+\psi(t,y,\cdot)-\psi(t,x+y,\cdot)\|_\infty \leq Lrt|x| \end{aligned}\tag{7.4}$$

which shows that condition (iv) is satisfied. For every $f\in\mathrm{C_b}$ with $\|f\|_\kappa\leq 1$, $t\in(0,1]$ and $x\in\mathbb{R}^d$, we use Lemma 7.2.2(iii) and (v) and Assumption 7.3.1(i) to estimate

$$\begin{aligned}|(I(t)f)(x)| &\leq t\mathcal{E}\left[\frac{1}{t}|f(\psi(t,x,\cdot))|\frac{\kappa(\psi(t,x,\cdot))}{\kappa(\psi(t,x,\cdot))}\right] \leq t\|f\|_\kappa\mathcal{E}\left[\frac{1}{t}\big(1+|\psi(t,x,\cdot)|\big)\right]\\ &\leq \|f\|_\kappa\big(1+|x|+t\mathcal{E}[L/\kappa]\big) \leq e^{ct}\|f\|_\kappa(1+|x|),\end{aligned}$$

where $c:=\mathcal{E}[L/\kappa]$. This shows that $\|I(t)f\|_\kappa\leq e^{ct}\|f\|_\kappa$ and thus Corollary 3.4.3 yields that condition (vi) is satisfied. Regarding condition (vii), inequality (7.4) implies

$$\begin{aligned}|(I(t)f)(x+y)-(I(t)f)(x)| &\leq \|\tau_y I(t)f - I(t)(\tau_y f)\|_\infty + \|I(t)(\tau_y f)-I(t)f\|_\infty\\ &\leq Lrt|y| + \|\tau_y f - f\|_\infty \leq e^{Lt}r|y|\end{aligned}$$

for all $r, t \geq 0$, $f \in \mathrm{Lip_b}(r)$ and $x, y \in \mathbb{R}^d$ and therefore $I(t)\colon \mathrm{Lip_b}(r) \to \mathrm{Lip_b}(e^{Lt}r)$. Moreover, it follows from Assumption 7.3.1(i) that

$$|f(\psi(t,x,y)) - f(x)| \leq r|\psi(t,x,y) - x| \leq Lr(1+|y|)t$$

for all $r, t \geq 0$, $f \in \mathrm{Lip_b}(r)$ and $x, y \in \mathbb{R}^d$. We use Lemma 7.2.2(v) to obtain

$$|(I(t)f - f)(x)| \leq t\mathcal{E}\left[\frac{|f(\psi(t,x,\cdot) - f(x)|}{t}\right] \leq \mathcal{E}\left[\frac{Lr}{\kappa}\right] t$$

and therefore $\mathrm{Lip_b} \subset \mathcal{L}^I$.

Second, we show that $(I'(0)f)(x) = \mathcal{E}[\nabla f(x)\psi_0(x,\cdot)]$ for all $f \in \mathrm{C}^1_\mathrm{b}$ and $x \in \mathbb{R}^d$. For every $h > 0$, $f \in \mathrm{C}^1_\mathrm{b}$, $x \in \mathbb{R}^d$ and $\lambda \in (0,1]$, we use Lemma 3.6.1 to estimate

$$\begin{aligned}
&\left(\frac{I(h)f - f}{h}\right)(x) - \mathcal{E}[\nabla f(x)\psi_0(x,\cdot)] \\
&= \mathcal{E}\left[\left(\frac{\psi(h,x,\cdot) - x}{h}\right)\int_0^1 \nabla f(x_s)\,\mathrm{d}s\right] - \mathcal{E}[\nabla f(x)\psi_0(x,\cdot)] \\
&\leq \lambda\mathcal{E}\left[\frac{1}{\lambda}\left(\left(\frac{\psi(h,x,\cdot) - x}{h}\right)\int_0^1 \nabla f(x_s)\,\mathrm{d}s - \nabla f(x)\psi_0(x,\cdot)\right) + \nabla f(x)\psi_0(x,\cdot)\right] \\
&\quad - \lambda\mathcal{E}[\nabla f(x)\psi_0(x,\cdot)],
\end{aligned}$$

where $x_s := x + s(\psi(h,x,\cdot) - x)$. Furthermore, Jensen's inequality implies

$$\begin{aligned}
&\lambda\mathcal{E}\left[\frac{1}{\lambda}\left(\left(\frac{\psi(h,x,\cdot) - x}{h}\right)\int_0^1 \nabla f(x_s)\,\mathrm{d}s - \nabla f(x)\psi_0(x,\cdot)\right) + \nabla f(x)\psi_0(x,\cdot)\right] \\
&\leq \frac{\lambda}{4}\mathcal{E}\left[\frac{4}{\lambda}\left(\left(\frac{\psi(h,x,\cdot) - x}{h}\right)\int_0^1 (\nabla f(x_s) - \nabla f(x))\,\mathrm{d}s\right)\mathbb{1}_{B(r)}(\cdot)\right] \\
&\quad + \frac{\lambda}{4}\mathcal{E}\left[\frac{4}{\lambda}\left(\left(\frac{\psi(h,x,\cdot) - x}{h}\right)\int_0^1 (\nabla f(x_s) - \nabla f(x))\,\mathrm{d}s\right)\mathbb{1}_{B(r)^c}(\cdot)\right] \\
&\quad + \frac{\lambda}{4}\mathcal{E}\left[\frac{4}{\lambda}\left(\frac{\psi(h,x,\cdot) - x}{h} - \psi_0(x,\cdot)\right)\nabla f(x)\right] + \frac{\lambda}{4}\mathcal{E}[4\nabla f(x)\psi_0(x,\cdot)]
\end{aligned}$$

for all $r \geq 0$ and $B(r) := B_{\mathbb{R}^d}(r)$. Let $\varepsilon > 0$ and $K \Subset \mathbb{R}^d$. Due to Assumption 7.3.1(i) and (ii), there exists $\lambda \in (0,1]$ with

$$\sup_{x\in\mathbb{R}^d} \lambda\mathcal{E}[4\|\nabla f\|_\infty|\psi_0(x,\cdot)|] \leq \sup_{x\in\mathbb{R}^d} \lambda\mathcal{E}\left[\frac{4L\|\nabla f\|_\infty}{\kappa}\right] \leq \frac{\varepsilon}{2}.$$

Furthermore, in view of Assumption 7.3.1(i) and (ii) and inequality (7.1), there exist $K_1 \Subset \mathbb{R}^d$ and $h_0 \in (0,1]$ with

$$\begin{aligned}
&\frac{\lambda}{4}\mathcal{E}\left[\frac{4}{\lambda}\left(\frac{\psi(h,x,\cdot) - x}{h} - \psi_0(x,\cdot)\right)\nabla f(x)\right] \\
&\leq \frac{\lambda}{4}\mathcal{E}\left[\frac{4}{\lambda}\left|\frac{\psi(h,x,\cdot) - x}{h} - \psi_0(x,\cdot)\right| \cdot \|\nabla f\|_\infty\right]
\end{aligned}$$

$$\leq \frac{\lambda}{8}\mathcal{E}\left[\frac{8\|\nabla f\|_\infty}{\lambda}\left|\frac{\psi(h,x,\cdot)-x}{h}-\psi_0(x,\cdot)\right|\mathbb{1}_{K_1}(\cdot)\right]+\frac{\lambda}{8}\mathcal{E}\left[\frac{16L\|\nabla f\|_\infty}{\lambda\kappa}\mathbb{1}_{K_1^c}(\cdot)\right]\leq\frac{\varepsilon}{8}$$

for all $x \in K$ and $h \in (0, h_0]$. By Assumption 7.3.1(i) and inequality (7.1), there exist $r \geq 0$ with

$$\frac{\lambda}{4}\mathcal{E}\left[\frac{4}{\lambda}\left(\left(\frac{\psi(h,x,\cdot)-x}{h}\right)\int_0^1(\nabla f(x_s)-\nabla f(x))\,\mathrm{d}s\right)\mathbb{1}_{B(r)^c}(\cdot)\right]$$
$$\leq \frac{\lambda}{4}\mathcal{E}\left[\frac{8L\|\nabla f\|_\infty}{\lambda\kappa}\mathbb{1}_{B(r)^c}(\cdot)\right]\leq\frac{\varepsilon}{8}.$$

for all $x \in \mathbb{R}^d$ and $h \in (0, h_0]$. Since $K \Subset \mathbb{R}^d$ is compact, there exists $\delta > 0$ with

$$|\nabla f(x+y)-\nabla f(x)| < \frac{\varepsilon}{8L(r+1)} \quad \text{for all } x \in K \text{ and } y \in B_{\mathbb{R}^d}(\delta).$$

Hence, by Assumption 7.3.1(i), there exist $h_1 \in (0, h_0]$ with

$$\frac{\lambda}{4}\mathcal{E}\left[\frac{4}{\lambda}\left(\left(\frac{\psi(h,x,\cdot)-x}{h}\right)\int_0^1(\nabla f(x_s)-\nabla f(x))\,\mathrm{d}s\right)\mathbb{1}_{B(n)}(\cdot)\right]$$
$$\leq \frac{\lambda}{4}\mathcal{E}\left[\frac{4L(1+r)}{\lambda}\frac{\varepsilon}{8L(1+r)}\right]=\frac{\varepsilon}{8}$$

for all $x \in K$ and $h \in (0, h_1]$. It follows from the previous estimates that

$$\left(\frac{I(h)f-f}{h}\right)(x)-\mathcal{E}[\nabla f(x)\psi_0(x,\cdot)]\leq\varepsilon$$

for all $x \in K$ and $h \in (0, h_1]$. Regarding the lower bound, Lemma 3.6.1 yields

$$\mathcal{E}\left[\left(\frac{\psi(h,x,\cdot)-x}{h}\right)\int_0^1\nabla f(x_s)\,\mathrm{d}s\right]-\mathcal{E}[\nabla f(x)\psi_0(x,\cdot)]$$
$$\geq -\lambda\mathcal{E}\left[\frac{\nabla f(x)\psi_0(x,\cdot)-y_s}{\lambda}+y_s\right]+\lambda\mathcal{E}[y_s]$$

for all $h > 0$, $x \in \mathbb{R}^d$ and $\lambda \in (0, 1]$, where

$$y_s := \left(\frac{\psi(h,x,\cdot)-x}{h}\right)\int_0^1\nabla f(x_s)\,\mathrm{d}s.$$

Furthermore, we can use Jensen's inequality to estimate

$$\lambda\mathcal{E}\left[\frac{\nabla f(x)\psi_0(x,\cdot)-y_s}{\lambda}+y_s\right]$$
$$\leq \frac{\lambda}{4}\mathcal{E}\left[\frac{4}{\lambda}\left(\psi_0(x,\cdot)-\frac{\psi(h,x,\cdot)-x}{h}\right)\int_0^1\nabla f(x_s)\,\mathrm{d}s\right]$$
$$+\frac{\lambda}{4}\mathcal{E}\left[\frac{4}{\lambda}\left(\psi_0(x,\cdot)\int_0^1(\nabla f(x)-\nabla f(x_s))\,\mathrm{d}s\right)\mathbb{1}_{B(r)}(\cdot)\right]$$

$$+\frac{\lambda}{4}\mathcal{E}\left[\frac{4}{\lambda}\left(\psi_0(x,\cdot)\int_0^1 \big(\nabla f(x)-\nabla f(x_s)\big)\,\mathrm{d}s\right)\mathbb{1}_{B(r)^c}(\cdot)\right]$$
$$+\frac{\lambda}{4}\mathcal{E}\left[4\left(\frac{\psi(h,x,\cdot)-x}{h}\right)\int_0^1 \nabla f(x_s)\,\mathrm{d}s\right].$$

for all $r\geq 0$. Analogously to the upper bound, one can estimate the terms on the right-hand side of the previous inequality to obtain

$$\left(\frac{I(h)f-f}{h}\right)(x)-\mathcal{E}[\nabla f(x)\psi_0(x,\cdot)]\geq -\varepsilon$$

for all $x\in K$ and sufficiently small $h>0$. This shows

$$\lim_{h\downarrow 0}\sup_{x\in K}\left|\left(\frac{I(h)f-f}{h}\right)(x)-\mathcal{E}[\nabla f(x)\psi_0(x,\cdot)]\right|=0 \quad \text{for all } K\Subset\mathbb{R}^d.$$

We obtain that $I'(0)f\in\mathrm{C_b}$ exists and is given by

$$(I'(0)f)(x)=\mathcal{E}[\nabla f(x)\psi_0(x,\cdot)] \quad \text{for all } x\in\mathbb{R}^d.$$

Now, the first part of the claim follows from Theorem 5.2.3.

Third, we verify equation (7.3). Let $t\geq 0$, $n\in\mathbb{N}$ and define $h:={}^t\!/_n$. We show by induction that

$$(I(h)^k f)(x)=h\bar{\mathcal{E}}\left[\frac{1}{h}f\big(\psi_k(h,x,\xi_1,\dots,\xi_k)\big)\right]$$

for all $f\in\mathrm{C_b}$, $x\in\mathbb{R}^d$ and $k\in\mathbb{N}$. For $k=1$, we have

$$(I(h)f)(x)=h\mathcal{E}\left[\frac{1}{h}f\big(\psi(h,x,\cdot)\big)\right]=h\bar{\mathcal{E}}\left[\frac{1}{h}f\big(\psi_1(h,x,\xi_1)\big)\right].$$

For the induction step, we use that ξ_{k+1} is independent of $(\xi_1,\dots,\xi_k)$ and has the same distribution as ξ_1 to obtain

$$\begin{aligned}(I(h)^{k+1}f)(x)&=(I(h)^k I(h)f)(x)=h\bar{\mathcal{E}}\left[\frac{1}{h}(I(h)f)\big(\psi_k(h,x,\xi_1,\dots,\xi_k)\big)\right]\\ &=h\bar{\mathcal{E}}\left[\bar{\mathcal{E}}\left[\frac{1}{h}f\big(\psi(h,\psi_k(h,x,y_1,\dots,y_k),\xi_{k+1})\big)\right]\Big|_{(y_1,\dots,y_k)=(\xi_1,\dots,\xi_k)}\right]\\ &=h\bar{\mathcal{E}}\left[\frac{1}{h}f\big(\psi(h,\psi_k(h,x,\xi_1,\dots,\xi_k),\xi_{k+1})\big)\right]\\ &=h\bar{\mathcal{E}}\left[\frac{1}{h}f\big(\psi_{k+1}(h,x,\xi_1,\dots,\xi_k,\xi_{k+1})\big)\right].\end{aligned}$$

□

We remark that the existence of $(\Omega,\mathcal{H},\bar{\mathcal{E}})$ and $(\xi_n)_{n\in\mathbb{N}}\subset\mathcal{H}^d$ as in the previous theorem is always guaranteed by a nonlinear version of Kolmogorov's extension theorem, see Theorem 7.2.6. To give an explicit example for the sequence $(\psi_n)_{n\in\mathbb{N}}$, we consider the case that the sample ξ_{n+1} is randomly shifted by a function depending on the average of the previous samples $\xi_1,\dots,\xi_n$. Let $\varphi_0\in\mathrm{Lip_b}(L)$ for some $L\geq 0$ and choose $\psi(t,x,y):=x+\varphi_0(x)t+ty$. Defining $X_n:=\psi_n(\frac{1}{n},0,\xi_1,\dots,\xi_n)$ as before, it holds

$$X_{n+1}=\psi_n\big(\tfrac{1}{n+1},0,\xi_1,\dots,\xi_n\big)+\tfrac{1}{n+1}\varphi_0\big(\psi_n\big(\tfrac{1}{n+1},0,\xi_1,\dots,\xi_n\big)\big)+\tfrac{1}{n+1}\xi_{n+1}$$

for all $n \in \mathbb{N}$. Hence, for sufficiently large n, the random variable X_{n+1} is approximately given by the average of the samples $\xi_1, \dots, \xi_{n+1}$ and an additional shift φ_0 depending on the average of the samples $\xi_1, \dots, \xi_n$.

Corollary 7.3.3. *Let $\psi(t,x,y) := x + \varphi(t,x) + ty$ for all $t \geq 0$ and $x, y \in \mathbb{R}^d$, where $\varphi\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function such that there exists $L \geq 0$ with*

$$\varphi(t,\cdot) \in \mathrm{Lip_b}(Lt) \quad \text{for all } t \geq 0.$$

Furthermore, we assume that the limit $\varphi_0 := \lim_{h\downarrow 0} \frac{\varphi(h,\cdot)}{h} \in \mathrm{C_b}$ exists. Then, there exists a strongly continuous convex monotone semigroup $(S(t))_{t\geq 0}$ on $\mathrm{C_b}$ given by

$$S(t)f = \lim_{n\to\infty} I\big(\tfrac{t}{n}\big)^n f \quad \text{for all } t \geq 0 \text{ and } f \in \mathrm{C_b}$$

such that Assumption 4.4.5 is valid. It holds $\mathrm{C_b^1} \subset D(A)$ and

$$(Af)(x) = \mathcal{E}[\nabla f(x)\xi] + \varphi_0(x)\nabla f(x) \quad \text{for all } f \in \mathrm{C_b^1} \text{ and } x \in \mathbb{R}^d,$$

where $\xi := \mathrm{id}_{\mathbb{R}^d}$. Moreover, for every convex expectation space $(\Omega, \mathcal{H}, \bar{\mathcal{E}})$ and iid sequence $(\xi_n)_{n\in\mathbb{N}} \subset \mathcal{H}^d$ with $\bar{\mathcal{E}}[f(\xi_n)] = \mathcal{E}[f]$ for all $f \in \mathrm{C_b}$,

$$(S(t)f)(x) = \lim_{n\to\infty} \frac{t}{n}\bar{\mathcal{E}}\left[\frac{n}{t} f(X_n^{t,x})\right]$$

for all $t > 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$, where $X_n^{t,x} := \psi_n\big(\frac{t}{n}, x, \xi_1, \dots, \xi_n\big)$.

Proof. It is straightforward to show that Assumption 7.3.1 is satisfied with

$$\psi_0(x,y) := \varphi_0(x) + y \quad \text{for all } x, y \in \mathbb{R}^d$$

and therefore the claim follows from Theorem 7.3.2. □

For non-perturbed averaged sums of iid samples, the semigroup can be represented explicitly by the Hopf–Lax formula.

Theorem 7.3.4. *Let $\psi(t,x,y) := x + ty$ for all $t \geq 0$ and $x, y \in \mathbb{R}^d$. Then,*

$$(S(t)f)(x) = \sup_{y\in\mathbb{R}^d} \big(f(x+ty) - \varphi(y)t\big) \quad \text{for all } t \geq 0, f \in \mathrm{C_b} \text{ and } x \in \mathbb{R}^d,$$

where $(S(t))_{t\geq 0}$ denotes the from Theorem 7.3.2 and

$$\varphi(y) := \sup_{z\in\mathbb{R}^d} (yz - \mathcal{E}[z\xi]) \quad \text{for } \xi := \mathrm{id}_{\mathbb{R}^d}.$$

It holds $\mathrm{C_b^1} \subset D(A)$ and $(Af)(x) = \mathcal{E}[\nabla f(x)\xi]$ for all $f \in \mathrm{C_b^1}$ and $x \in \mathbb{R}^d$. Moreover, for every convex expectation space $(\Omega, \mathcal{H}, \bar{\mathcal{E}})$ and iid sequence $(\xi_n)_{n\in\mathbb{N}} \subset \mathcal{H}^d$ with $\bar{\mathcal{E}}[f(\xi_n)] = \mathcal{E}[f]$ for all $f \in \mathrm{C_b}$,

$$\lim_{n\to\infty} \frac{1}{n}\bar{\mathcal{E}}\left[nf\left(\frac{1}{n}\sum_{i=1}^n \xi_i\right)\right] = \sup_{y\in\mathbb{R}^d} \big(f(y) - \varphi(y)\big) \quad \text{for all } f \in \mathrm{C_b}.$$

Proof. For every $t \geq 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$, we define

$$(T(t)f)(x) := \sup_{y \in \mathbb{R}^d} \big(f(x+ty) - \varphi(y)t \big)$$

and show that $(T(t))_{t\geq 0}$ satisfies Assumption 4.4.5. The convexity of the mapping $\mathbb{R}^d \to \mathbb{R}$, $x \mapsto \mathbb{E}[x\xi]$ and Fenchel–Moreau's theorem yield

$$\mathcal{E}[x\xi] = \sup_{y \in \mathbb{R}^d} \big(xy - \varphi(y) \big) \quad \text{for all } x \in \mathbb{R}^d$$

and therefore $\inf_{x\in\mathbb{R}^d} \varphi(x) = -\mathcal{E}[0] = 0$. For every $t \geq 0$ and $f \in \mathrm{C_b}$, it follows from $\lim_{|x|\to\infty} \varphi(x) = \infty$ that there exists $K \Subset \mathbb{R}^d$ with

$$(T(t)f)(x) = \sup_{y \in K} \big(f(x+ty) - \varphi(y)t \big) \quad \text{for all } x \in \mathbb{R}^d.$$

Since f is uniformly continuous on compacts, we obtain $T(t)f \in \mathrm{C_b}$. In addition, for every $t \geq 0$, $x \in \mathbb{R}^d$ and $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C_b}$ with $f_n \downarrow 0$, there exists $K \Subset \mathbb{R}^d$ with

$$(T(t)f_n)(x) = \sup_{y \in K} \big(f_n(x+ty) - \varphi(y)t \big) \quad \text{for all } n \in \mathbb{N}$$

such that Dini's theorem implies $(T(t)f_n)(x) \downarrow 0$ as $n \to \infty$. By definition, the operators $T(t)\colon \mathrm{C_b} \to \mathrm{C_b}$ are convex and monotone with $T(t)0 = 0$, where the latter follows from $\inf_{x\in\mathbb{R}^d} \varphi(x) = 0$. Moreover,

$$\|T(t)f - T(t)g\|_\infty \leq \|f-g\|_\infty \quad \text{for all } t \geq 0 \text{ and } f, g \in \mathrm{C_b}.$$

It holds $T(t)(\tau_x f) = \tau_x T(t)f$ and $T(t)\colon \mathrm{Lip_b}(r) \to \mathrm{Lip_b}(r)$ for all $r, t \geq 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$. For every $r \geq 0$ and $f \in \mathrm{Lip_b}(r)$, we use

$$f(x+ty) - \varphi(y)t \leq f(x) + \big(r|y| - \varphi(y) \big)t$$

and $\lim_{|x|\to\infty} \frac{\varphi(x)}{|x|} = \infty$ to choose $K \Subset \mathbb{R}^d$ with

$$(T(t)f)(x) = \sup_{y \in K} \big(f(x+ty) - \varphi(y)t \big) \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}^d. \tag{7.5}$$

This shows that $0 \leq T(t)f - f \leq \sup_{y\in K} rt|y|$ for all $t \geq 0$ and thus $\mathrm{Lip_b} \subset \mathcal{L}^T$. Now, let $f \in \mathrm{C_b^1}$ and choose $K \Subset \mathbb{R}^d$ such that equation (7.5) is valid. For every $x \in \mathbb{R}^d$, we use Fenchel–Moreau's theorem to estimate

$$\begin{aligned} \left| \frac{(T(t)f - f)(x)}{t} - \mathcal{E}[\nabla f(x)\xi] \right| &\leq \sup_{y\in K} \left| \frac{f(x+ty) - f(x)}{t} - \nabla f(x)y \right| \\ &\leq \sup_{y \in K} \frac{1}{t} \int_0^t |\nabla f(x+sy) - \nabla f(x)| \cdot |y| \, \mathrm{d}s. \end{aligned}$$

Since ∇f is uniformly continuous on compacts, we obtain $f \in D(B)$ and

$$(Bf)(x) = \mathcal{E}[\nabla f(x)\xi] \quad \text{for all } x \in \mathbb{R}^d,$$

where B denotes the generator of $(T(t))_{t\geq 0}$. Next, we show that $T(s+t)f = T(s)T(t)f$ for all $s,t \geq 0$ and $f \in \mathrm{C_b}$. Let $s,t \geq 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$. Then,

$$\begin{aligned}(T(s+t)f)(x) &= \sup_{y\in\mathbb{R}^d} \big(f(x+sy+ty) - \varphi(y)s - \varphi(y)t\big) \\ &\leq \sup_{y\in\mathbb{R}^d}\sup_{z\in\mathbb{R}^d} \big(f(x+sy+tz) - \varphi(y)s - \varphi(z)t\big) = (T(s)T(t)f)(x).\end{aligned}$$

Furthermore, due to $\lim_{|x|\to\infty}\varphi(x) = \infty$, there exist $y_s, y_t \in \mathbb{R}^d$ with

$$(T(s)T(t)f)(x) = (T(t)f)(x+sy_s) - \varphi(y_s)s = f(x+sy_s+ty_t) - \varphi(y_s)s - \varphi(y_t)t.$$

For $y_{s+t} := \frac{s}{s+t}y_s + \frac{t}{s+t}y_t$, it follows from the convexity of φ that

$$\varphi(y_{s+t})(s+t) = \varphi\left(\frac{s}{s+t}y_s + \frac{t}{s+t}y_t\right)(s+t) \leq \varphi(y_s)s + t\varphi(y_t)t.$$

We obtain

$$\begin{aligned}(T(s)T(t)f)(x) &= f(x+(s+t)y_{s+t}) - \varphi(y_s)s - \varphi(y_t)t \\ &\leq f(x+(s+t)y_{s+t}) - \varphi(y_{s+t})(s+t) \leq (T(s+t)f)(x).\end{aligned}$$

Theorem 4.4.6 implies $S(t)f = T(t)f$ for all $t \geq 0$ and $f \in \mathrm{C_b}$. Furthermore,

$$\psi_n(t,x,y_1,\ldots,y_n) = x + t\sum_{i=1}^n y_i$$

for all $n \in \mathbb{N}$, $t \geq 0$ and $x, y_1, \ldots, y_n \in \mathbb{R}^d$ and thus the second part of the claim follows from Theorem 7.3.2. □

The previous theorem extends Peng's results from the sublinear to the convex case. Indeed, if $\mathcal{E}[\,\cdot\,]$ is a sublinear expectation, Theorem 7.3.4 implies

$$\lim_{n\to\infty} \bar{\mathcal{E}}\left[f\left(\frac{1}{n}\sum_{i=1}^n \xi_i\right)\right] = \sup_{\{x\in\mathbb{R}^d\colon \varphi(x)=0\}} f(x) \quad \text{for all } f \in \mathrm{C_b}.$$

Hence, the averaged sums convergence weakly to a maximal distribution as in [144, 146] and the limit in Theorem 7.3.4 can be seen as a convex version thereof. Furthermore, as an immediate consequence of the previous result, we obtain Cramér's theorem as LLN for the entropic risk measure. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\xi_n)_{n\in\mathbb{N}}$ be an iid sequence of random vectors $\xi_n\colon \Omega \to \mathbb{R}^d$ with $\mathbb{E}[e^{c|\xi_1|}] < \infty$ for all $c \geq 0$. Let

$$\Lambda\colon \mathbb{R}^d \to \mathbb{R},\ x \mapsto \log\big(\mathbb{E}[e^{x\xi_1}]\big)$$

be the logarithmic moment generating function and denote by

$$\Lambda^*\colon \mathbb{R}^d \to \mathbb{R},\ x \mapsto \sup_{y\in\mathbb{R}^d}\big(xy - \Lambda(y)\big)$$

its convex conjugate. Moreover, we define $X_n := \frac{1}{n}\sum_{i=1}^n \xi_i$ for all $n \in \mathbb{N}$.

Corollary 7.3.5 (Cramér)**.** *For every* $f \in \mathrm{C_b}$,

$$\lim_{n\to\infty} \frac{1}{n} \log\big(\mathbb{E}\big[e^{nf(X_n)}\big]\big) = \sup_{x\in\mathbb{R}^d} \big(f(x) - \Lambda^*(x)\big).$$

Proof. Clearly, the functional $\mathcal{E}\colon \mathrm{C}_\kappa \to \mathbb{R}$, $f \mapsto \log(\mathbb{E}[e^{f(\xi_1)}])$ is convex and monotone with $\mathcal{E}[c] = c$ for all $c \in \mathbb{R}$. Since the dominated convergence theorem guarantees that $\mathcal{E}$ is continuous from above, the claim follows from Theorem 7.3.4. □

7.3.2 Upper bounds and asymptotic convergence rates

Throughout this subsection, let $\mathcal{E}\colon \mathrm{C}_\kappa \to \mathbb{R}$ be a convex expectation which is continuous from above and $\psi\colon \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be a function satisfying Assumption 7.3.1. For $\psi(t,x,y) := x + ty$, the limit semigroup can explicitly be represented by the Hopf–Lax formula, see Theorem 7.3.4. In general, such a formula is not available but we can still provide explicit (upper) bounds and resulting asymptotic convergence rates. To that end, let $H_\pm\colon \mathbb{R}^d \to \mathbb{R}$ be convex functions with $H_\pm(0) = 0$ and

$$H_-(y) \le \mathcal{E}[\psi_0(x,\cdot)y] \le H_+(y) \quad \text{for all } x, y \in \mathbb{R}^d.$$

At least an upper bound always exists, since we can choose

$$H_+(y) := \sup_{x\in\mathbb{R}^d} \mathcal{E}[\psi_0(x,\cdot)y] \le \mathcal{E}[L(1+|\cdot|)y].$$

Moreover, under the assumptions of Corollary 7.3.3 we can choose

$$H_+(y) := \mathcal{E}[y\xi] + \sup_{x\in\mathbb{R}^d} \varphi_0(x)y \le \mathcal{E}[y\xi] + L|y|, \quad \text{where } \xi := \mathrm{id}_{\mathbb{R}^d}.$$

For the following lemma, let $(S(t))_{t\ge 0}$ be the semigroup from Theorem 7.3.2 given by

$$S(t)f = \lim_{n\to\infty} I\big(\tfrac{t}{n}\big)^n f \quad \text{for all } t \ge 0 \text{ and } f \in \mathrm{C_b},$$

where $I(0) := \mathrm{id}_{\mathrm{C_b}}$ and, for every $t > 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$,

$$(I(t)f)(x) := t\mathcal{E}\left[\frac{1}{t} f(\psi(t,x,\cdot))\right].$$

Lemma 7.3.6. *It holds* $S_-(t)f \le S(t)f \le S_+(t)f$ *for all* $t \ge 0$ *and* $f \in \mathrm{C_b}$, *where*

$$(S_\pm(t)f)(x) := \sup_{y\in\mathbb{R}^d} \big(f(x+ty) - H_\pm^*(y)t\big)$$

and $H_\pm^*(y) := \sup_{z\in\mathbb{R}^d}(yz - H_\pm(z))$.

Proof. As seen during the proof of Theorem 7.3.4, the strongly continuous convex monotone semigroups $(S_\pm(t))_{t\ge 0}$ satisfy Assumption 4.4.5 as well as $\mathrm{C}_\mathrm{b}^1 \subset D(A_\pm)$ and

$$(A_\pm f)(x) = H_\pm(\nabla f(x)) \quad \text{for all } f \in \mathrm{C}_\mathrm{b}^1 \text{ and } x \in \mathbb{R}^d.$$

In particular, it holds $A_- f \le Af \le A_+ f$ for all $f \in \mathrm{C}_\mathrm{b}^1$ and thus Theorem 4.4.6 implies

$$S_-(t)f \le S(t)f \le S_+(t)f \quad \text{for all } t \ge 0 \text{ and } f \in \mathrm{C_b}.$$

□

We illustrate the previous result for the entropic risk measure which asymptotically yields exponential convergence rates in the situation of Corollary 7.3.3, where we considered averaged sums of perturbed iid samples. Recall that the perturbation of the ξ_{n+1} consisted of a random shift depending on the average of the previous samples $\xi_1, \dots, \xi_n$. It turns out that the well-known convergence rate from the case of unperturbed iid samples is reduced according to the size of the shift, see Theorem 7.3.7. Furthermore, if we only require that $(\xi_n)_{n\in\mathbb{N}}$ has finite p-th moments instead of finite exponential moments, we still obtain polynomial convergence rates as in [127], see Theorem 7.3.8. For the following two theorems, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\xi_n)_{n\in\mathbb{N}}$ be an iid sequence of random vectors $\xi_n\colon \Omega \to \mathbb{R}^d$. In addition, let $\varphi\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be a function such that there exist $L \geq 0$ with

$$\varphi(t,\cdot) \in \mathrm{Lip_b}(Lt) \quad \text{for all } t \geq 0$$

and assume that $\varphi_0 := \lim_{h\downarrow 0} \varphi(h,\cdot)/h \in \mathrm{C_b}$ exists. For every $t \geq 0$ and $x, y \in \mathbb{R}^d$, we define $\psi(t,x,y) := x + \varphi(t,x) + ty$. Moreover, let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of functions $\psi_n\colon \mathbb{R}_+ \times \mathbb{R}^d \times (\mathbb{R}^d)^n \to \mathbb{R}^d$ which are recursively defined by $\psi_1 := \psi$ and

$$\psi_{n+1}(t,x,y_1,\dots,y_{n+1}) := \psi\big(t, \psi_n(t,x,y_1,\dots,y_n), y_{n+1}\big).$$

Finally, let $X_n := \psi_n(\frac{1}{n}, 0, \xi_1, \dots, \xi_n)$ for all $n \in \mathbb{N}$.

Theorem 7.3.7. *Assume that $\mathbb{E}[e^{c|\xi_1|}] < \infty$ for all $c \geq 0$. Denote by*

$$\Lambda\colon \mathbb{R}^d \to \mathbb{R},\ x \mapsto \log\big(\mathbb{E}[e^{x\xi_1}]\big)$$

the logarithmic moment generating function and by

$$\Lambda^*\colon \mathbb{R}^d \to \mathbb{R},\ x \mapsto \sup_{y\in\mathbb{R}^d}\big(xy - \Lambda(y)\big)$$

its convex conjugate. Then,

$$\lim_{n\to\infty} \frac{1}{n}\log\big(\mathbb{E}[e^{nf(X_n)}]\big) \leq \sup_{x\in\mathbb{R}^d}\big(f(x) - H_+^*(x)\big) \quad \text{for all } f \in \mathrm{C_b},$$

where $H_+^*(x) := \inf_{y\in B(L)} \Lambda^*(x+y)$. Furthermore,

$$\limsup_{n\to\infty} \frac{1}{n}\log(\mathbb{P}(X_n \in A)) \leq -\inf_{x\in A_L} \Lambda^*(x)$$

for all closed sets $A \subset \mathbb{R}^d$, where $A_L := \{x+y\colon x \in A,\ y \in B_{\mathbb{R}^d}(L)\}$.

Proof. Applying Corollary 7.3.3 with $\mathcal{E}[f] := \log(\mathbb{E}[e^{f(\xi_1)}])$ yields a semigroup $(S(t))_{t\geq 0}$ with generator $(Af)(x) = \Lambda(\nabla f(x)) + \varphi_0(x)\nabla f(x)$ for all $f \in \mathrm{C}_\mathrm{b}^1$ and $x \in \mathbb{R}^d$. Moreover, for every $t \geq 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$, we define $H_+(x) := \Lambda(x) + L|x|$ and

$$(S_+(t)f)(x) := \sup_{y\in\mathbb{R}^d}\big(f(x+ty) - H_+^*(y)t\big),$$

where $H_+^*(y) := \sup_{z\in\mathbb{R}^d}(yz - H_+(z))$. Lemma 7.3.6 implies $S(t)f \leq S_+(t)f$ for all $t \geq 0$ and $f \in \mathrm{C_b}$. Moreover, by Fenchel–Moreau's theorem, the function

$$f\colon \mathbb{R}^d \to \mathbb{R},\ x \mapsto \inf_{x=y+z}\big(\Lambda^*(y) + \infty\mathbb{1}_{B_{\mathbb{R}^d}(L)^c}(z)\big)$$

satisfies $f^*(x) = \Lambda(x) + L|x| = H_+(x)$ and thus

$$H_+^*(x) = f(x) = \inf_{y \in B_{\mathbb{R}^d}(L)} \Lambda^*(x+y) \quad \text{for all } x \in \mathbb{R}^d.$$

Let $f := -\infty \mathbb{1}_{A^c} \in \mathrm{U_b}$ for a closed subset $A \subset \mathbb{R}^d$ and $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C_b}$ be a sequence with $f_n \downarrow f$, where $\mathrm{U_b}$ consists of all upper semicontinuous functions $g\colon \mathbb{R}^d \to [-\infty,\infty)$ with $\|g^+\|_\infty < \infty$. Due to Dini's theorem and $\lim_{|x|\to\infty} H_+^*(x) = \infty$, the functionals

$$\Phi\colon \mathrm{U_b} \to [-\infty,\infty),\ g \mapsto \sup_{x\in\mathbb{R}^d} \big(f(x) - H_+^*(x)\big),$$
$$\Phi_n\colon \mathrm{U_b} \to [-\infty,\infty),\ g \mapsto \frac{1}{n}\log\big(\mathbb{E}\big[e^{nf(X_n)}\big]\big) \quad \text{for all } n \in \mathbb{N}$$

are continuous from above. Hence, by changing a supremum with an infimum at the cost of an inequality, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\log(\mathbb{P}(X_n \in A)) &= \limsup_{n\to\infty} \Phi_n(f) = \limsup_{n\to\infty} \inf_{k\in\mathbb{N}} \Phi_n(f_k)\\ &\le \inf_{k\in\mathbb{N}} \inf_{n\in\mathbb{N}} \sup_{l\ge n} \Phi_l(f_k) = \inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \Phi_n(f_k)\\ &\le \inf_{k\in\mathbb{N}} \Phi(f_k) = \Phi(f) = -\inf_{x\in A_L} \Lambda^*(x).\end{aligned}$$ □

Replacing the entropic risk measure by the short fall risk measure leads to polynomial rather than exponential convergence rates.

Theorem 7.3.8. *Assume that* $\mathbb{E}[|\xi_1|^p] < \infty$ *for some* $p \in (1,\infty)$. *Define*

$$\Lambda\colon \mathbb{R}^d \to \mathbb{R},\ x \mapsto \inf\{m \in \mathbb{R}\colon \mathbb{E}[((1 + x\xi_1 - m)^+)^p] \le 1\}$$

and the convex conjugate

$$\Lambda^*\colon \mathbb{R}^d \to \mathbb{R},\ x \mapsto \sup_{y\in\mathbb{R}^d} (xy - \Lambda(y)).$$

Then, for every closed subset $A \subset \mathbb{R}^d$,

$$\limsup_{n\to\infty} n^{p-1}\mathbb{P}(X_n \in A) \le \left(\inf_{x\in A_L} \Lambda^*(x)\right)^{-p},$$

where $A_L := \{x + y\colon x \in A,\ y \in B_{\mathbb{R}^d}(L)\}$.

Proof. We consider the short fall risk measure

$$\mathcal{E}\colon \mathrm{C}_\kappa \to \mathbb{R},\ f \mapsto \inf\{m \in \mathbb{R}\colon \mathbb{E}[((1 + f(\xi_1) - m)^+)^p] \le 1\}$$

which is a convex expectation and continuous from above, see [80, Chapter 4.9]. Hence, Corollary 7.3.3 yields a corresponding semigroup $(S(t))_{t\ge0}$ with generator

$$(Af)(x) = \Lambda(\nabla f(x)) + \varphi_0(x)\nabla f(x) \quad \text{for all } f \in \mathrm{C}_\mathrm{b}^1 \text{ and } x \in \mathbb{R}^d.$$

Similarly to the proof of Theorem 7.3.7, by using that the functionals $\Phi(f) := (S(1)f)(0)$ and $\Phi_n(f) := (I(\frac{1}{n})^n f)(0)$ are continuous from above on $\mathrm{U_b}$, one can show that

$$\limsup_{n\to\infty} \big(I\big(\tfrac{1}{n}\big)^n f\big)(0) \le -\inf_{x\in A_L} \Lambda^*(x),$$

where $f := -\infty \mathbb{1}_{A^c}$ and $(I(t)f)(x) := t\mathcal{E}[\frac{1}{t} f(x+t\xi_1)]$. It remains to show that

$$-n^{-\frac{p-1}{p}} \mathbb{P}(X_n \in A)^{-\frac{1}{p}} \le \big(I\big(\tfrac{1}{n}\big)^n f\big)(0) \quad \text{for all } n \in \mathbb{N}.$$

To do so, we apply [127, Lemma 4.2]. Let $\mathcal{P}$ be the set of all probability measures on the Borel-σ-algebra $\mathcal{B}(\mathbb{R}^d)$. Fenchel–Moreau's theorem implies

$$\mathcal{E}[f] = \sup_{\nu\in\mathcal{P}} \left(\int_{\mathbb{R}^d} f \,\mathrm{d}\nu - \alpha(\nu) \right) \quad \text{for all } f \in \mathrm{C_b},$$

where $\alpha(\nu) := \sup_{f\in \mathrm{C_b}} \big(\int_{\mathbb{R}^d} f\,\mathrm{d}\nu - \mathcal{E}[f] \big)$. By induction, one can show that

$$\big(I\big(\tfrac{1}{n}\big)^n f\big)(0) = \sup_{\nu\in\mathcal{P}^n} \left(\int_{(\mathbb{R}^d)^n} f\big(\psi_n\big(\tfrac{1}{n}, 0, x_1, \ldots, x_n\big)\big) \nu(\mathrm{d}x_1, \ldots, \mathrm{d}x_n) - \frac{1}{n}\alpha_n(\nu) \right)$$

for all $n \in \mathbb{N}$ and $f \in \mathrm{C_b}$, where $\mathcal{P}^n$ consists of all probability measures on the Borel-σ-algebra $\mathcal{B}((\mathbb{R}^d)^n)$. Moreover, the functions $\alpha_n \colon \mathcal{P}^n \to [0,\infty]$ are defined by

$$\alpha_n(\nu) := \int_{(\mathbb{R}^d)^n} \sum_{i=1}^n \alpha(\nu_{i-1,i}(x_1, \ldots, x_{i-1}))\, \nu(\mathrm{d}x_1, \ldots, \mathrm{d}x_n),$$

where the kernels $\nu_{i-1,i}$ are determined by the disintegration

$$\nu(\mathrm{d}x_1, \ldots, \mathrm{d}x_n) = \nu_{0,1}(\mathrm{d}x_1) \prod_{i=2}^n \nu_{i-1,i}(x_1, \ldots, x_{i-1})(\mathrm{d}x_i).$$

It follows from [127, Lemma 4.2] that

$$\alpha_n(\nu) \le n^{\frac{1}{p}} \left\| \frac{\mathrm{d}\nu}{\mathrm{d}\mu^n} \right\|_{L^q(\mu^n)} \quad \text{for all } \nu \ll \mu,$$

where $\mu := \mathbb{P} \circ \xi_1^{-1}$ and $\mu^n := \mu \otimes \ldots \otimes \mu$ denotes the n-fold product measure. Now, the claim follows similarly to the proof of [127, Theorem 1.2]. □

7.3.3 Uncertain samples in Wasserstein spaces

As an illustration of the abstract results, we consider convex expectations that are defined as a supremum over a set of probability measures which are weighted according to their Wasserstein distance to a fixed reference model. This type of uncertainty has previously been studied in a framework with nonlinear semigroups corresponding to Markov processes with uncertain transition probabilities, see [14, 85] and Section 6.3. Let $p \in (1,\infty)$ and denote by $\mathcal{P}_p$ the p-Wasserstein space consisting of all probability measures on the Borel-σ-algebra $\mathcal{B}(\mathbb{R}^d)$ with finite p-th moment. We endow $\mathcal{P}_p$ with the p-Wasserstein distance

$$\mathcal{W}_p(\mu,\nu) := \left(\inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^{\frac{1}{p}}\,\pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{p}},$$

where $\Pi(\mu,\nu)$ consists of all probability measures on $\mathcal{B}(\mathbb{R}^d\times\mathbb{R}^d)$ with first marginal μ and second marginal ν. Furthermore, let $\varphi\colon \mathbb{R}_+\to[0,\infty]$ be a function with $\varphi(0)=0$ and $\lim_{c\to\infty}\varphi(c)/c=\infty$. In the sequel, we fix $\mu\in\mathcal{P}_p$ and define

$$\mathcal{E}\colon \mathrm{C}_\kappa\to\mathbb{R},\; f\mapsto \sup_{\nu\in\mathcal{P}_p}\left(\int_{\mathbb{R}^d} f(x)\,\nu(\mathrm{d}x) - \varphi(\mathcal{W}_p(\mu,\nu))\right).$$

In the case $\varphi := \infty\mathbb{1}_{[0,r]}$, the functional $\mathcal{E}$ is defined as supremum over all measures in a Wasserstein ball with radius $r\geq 0$ around the reference model μ. Moreover, for every $t>0$, $f\in\mathrm{C_b}$ and $x\in\mathbb{R}^d$,

$$\begin{aligned} t\mathcal{E}\left[\frac{1}{t}f(x+t\xi)\right] &= \sup_{\nu\in\mathcal{P}_p}\left(\int_{\mathbb{R}^d} f(x+ty)\,\nu(\mathrm{d}y) - t\varphi(\mathcal{W}_p(\mu,\nu))\right)\\ &= \sup_{\nu\in\mathcal{P}_p}\left(\int_{\mathbb{R}^d} f(x+y)\,\nu_t(\mathrm{d}y) - t\varphi\left(\frac{\mathcal{W}_p(\mu_t,\nu_t)}{t}\right)\right)\\ &= \sup_{\nu\in\mathcal{P}_p}\left(\int_{\mathbb{R}^d} f(x+y)\,\nu(\mathrm{d}y) - t\varphi\left(\frac{\mathcal{W}_p(\mu_t,\nu)}{t}\right)\right), \end{aligned}$$

where $\xi := \mathrm{id}_{\mathbb{R}^d}$, $\nu_t := \nu\circ(t\xi)^{-1}$ and $\mu_t := \mu\circ(t\xi)^{-1}$. Hence, due to the expression in the last line, the definition

$$(I(t)f)(x) := t\mathcal{E}\left[\frac{1}{t}f(x+t\xi)\right]$$

is consistent with the scaling of the penalization function φ in [14,85] and Section 6.3. The proof of Theorem 7.3.10 below requires the following tightness result for Wasserstein balls.

Lemma 7.3.9. *Let $p,q\in[1,\infty)$ with $p>q$, $\mu\in\mathcal{P}_p$, $R\geq 0$ and*

$$M := \{\nu\in\mathcal{P}_p\colon \mathcal{W}_p(\mu,\nu)\leq R\}.$$

Then, for every $\varepsilon>0$, there exists $r\geq 0$ with

$$\sup_{\nu\in M}\int_{B(r)^c} |x|^q\,\nu(\mathrm{d}x)\leq\varepsilon.$$

Proof. Let $\varepsilon>0$ and choose $r_1\geq 0$ with

$$2^{q-1}\int_{B(r_1)^c}|x|^q\,\mu(\mathrm{d}x)\leq\frac{\varepsilon}{4}\quad\text{and}\quad 2^{q-1}R^{\frac{q}{p}}\mu\big(B(r_1)^c\big)^{\frac{p-q}{p}}\leq\frac{\varepsilon}{4}. \tag{7.6}$$

Let $\nu\in M$ and choose an optimal coupling $\pi\in\Pi(\mu,\nu)$, i.e.,

$$\mathcal{W}_p(\mu,\nu) = \left(\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^p\,\pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{p}}.$$

It follows from Hölder's inequality and inequality (7.6) that

$$\begin{aligned}
&\int_{B(r_1)^c\times B(r_1)^c} |x-y|^q\,\pi(\mathrm{d}x,\mathrm{d}y)\\
&\le \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p\,\pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{q}{p}} \pi\big(B(r_1)^c\times B(r_1)^c\big)^{\frac{p-q}{p}}\\
&\le R^{\frac{q}{p}}\pi\big(B(r_1)^c\times\mathbb{R}^d\big)^{\frac{p-q}{p}} = R^{\frac{q}{p}}\mu\big(B(r_1)^c\big)^{\frac{p-q}{p}} \le \frac{\varepsilon}{4}.
\end{aligned}\tag{7.7}$$

Moreover, for every $x\in B(r_1)$,

$$\frac{|y|^q}{|x-y|^p} \le \frac{|y|^q}{(|y|-|x|)^p} \le \frac{|y|^q}{(|y|-r_1)^p} \to 0 \quad \text{as } |y|\to\infty.$$

Hence, we can choose $r_2\ge r_1$ with

$$\frac{|y|^q}{(|x-y|)^p} \le \frac{\varepsilon}{2R^p} \quad \text{for all } x\in B(r_1) \text{ and } y\in B(r_2)^c. \tag{7.8}$$

It follows from inequality (7.6)-(7.8) that

$$\begin{aligned}
&\int_{B(r_2)^c} |y|^q\,\nu(\mathrm{d}y) = \int_{B(r_1)\times B(r_2)^c} |y|^q\,\pi(\mathrm{d}x,\mathrm{d}y) + \int_{B(r_1)^c\times B(r_2)^c} |y|^q\,\pi(\mathrm{d}x,\mathrm{d}y)\\
&\le \int_{B(r_1)\times B(r_2)^c} \frac{|y|^q}{|x-y|^p}\cdot|x-y|^p\,\pi(\mathrm{d}x,\mathrm{d}y)\\
&\quad + 2^{q-1}\int_{B(r_1)^c\times B(r_2)^c} |x|^q + |x-y|^q\,\pi(\mathrm{d}x,\mathrm{d}y)\\
&\le \frac{\varepsilon}{2R^p}\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p\,\pi(\mathrm{d}x,\mathrm{d}y) + 2^{q-1}\int_{B(r_1)^c} |x|^q\,\mu(\mathrm{d}x)\\
&\quad + 2^{q-1}\int_{B(r_1)^c\times B(r_1)^c} |x-y|^q\,\pi(\mathrm{d}x,\mathrm{d}y)\\
&\le \frac{\varepsilon}{2R^p}\mathcal{W}_p(\mu,\nu)^p + \frac{\varepsilon}{2} \le \varepsilon.
\end{aligned}$$

□

Theorem 7.3.10. *The functional $\mathcal{E}$ is a convex expectation which is continuous from above. Hence, there exists a strongly continuous convex monotone semigroup $(S(t))_{t\ge 0}$ on $\mathrm{C_b}$ given by*

$$S(t)f = \lim_{n\to\infty} I\big(\tfrac{t}{n}\big)^n f \quad \textit{for all } t\ge 0 \textit{ and } f\in\mathrm{C_b}$$

such that Assumption 4.4.5 is valid. It holds $\mathrm{C_b^1}\subset D(A)$ and

$$(Af)(x) = \sup_{c\ge 0}\big(c|\nabla f(x)| - \varphi(c)\big) + m\nabla f(x) \quad \textit{for all } f\in\mathrm{C_b^1} \textit{ and } x\in\mathbb{R}^d,$$

where $m := \int_{\mathbb{R}^d} y\,\mu(\mathrm{d}y)$. In addition, for every convex expectation space $(\Omega,\mathcal{H},\bar{\mathcal{E}})$ and iid sequence $(\xi_n)_{n\in\mathbb{N}}\subset\mathcal{H}^d$ with $\bar{\mathcal{E}}[f(\xi_n)] = \mathcal{E}[f]$ for all $f\in\mathrm{C_b}$,

$$\lim_{n\to\infty}\frac{1}{n}\bar{\mathcal{E}}\left[nf\left(\frac{1}{n}\sum_{i=1}^n \xi_i\right)\right] = \sup_{y\in\mathbb{R}^d}\big(f(y+m) - \varphi(|y|)\big) \quad \textit{for all } f\in\mathrm{C_b}.$$

Proof. First, we show that $\mathcal{E}$ is a convex expectation which is continuous from above. Let $f \in \mathrm{C}_\kappa$ and choose $c \geq 0$ with $|f(x)| \leq c(1+|x|)$ for all $x \in \mathbb{R}^d$. We use

$$\left| \int_{\mathbb{R}^d} |x| \, \mu(\mathrm{d}x) - \int_{\mathbb{R}^d} |y| \, \nu(\mathrm{d}y) \right| \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \big| |x| - |y| \big| \, \pi(\mathrm{d}x, \mathrm{d}y) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| \, \pi(\mathrm{d}x, \mathrm{d}y)$$

for all $\pi \in \Pi(\mu,\nu)$, Hölder's inequality and $\lim_{c\to\infty} \frac{\varphi(c)}{c} = \infty$ to estimate

$$\begin{aligned}
\mathcal{E}[|f|] &\leq \sup_{\nu \in \mathcal{P}_p} \left(\int_{\mathbb{R}^d} c(1+|x|) \, \nu(\mathrm{d}x) - \varphi(\mathcal{W}_p(\mu,\nu)) \right) \\
&= \sup_{\nu \in \mathcal{P}_p} \left(\int_{\mathbb{R}^d} c(1+|x|) \, \mu(\mathrm{d}x) + \int_{\mathbb{R}^d} c|x| \, \nu(\mathrm{d}x) - \int_{\mathbb{R}^d} c|x| \, \mu(\mathrm{d}x) - \varphi(\mathcal{W}_p(\mu,\nu)) \right) \\
&\leq \int_{\mathbb{R}^d} c(1+|x|) \, \mu(\mathrm{d}x) + \sup_{\nu \in \mathcal{P}_p} \left(c\mathcal{W}_1(\mu,\nu) - \varphi(\mathcal{W}_p(\mu,\nu)) \right) \\
&\leq \int_{\mathbb{R}^d} c(1+|x|) \, \mu(\mathrm{d}x) + \sup_{\nu \in \mathcal{P}_p} \left(c\mathcal{W}_p(\mu,\nu) - \varphi(\mathcal{W}_p(\mu,\nu)) \right) < \infty.
\end{aligned}$$

It follows from $\varphi(0) = 0$ that $\mathcal{E}[c] = c$ for all $c \in \mathbb{R}$. Moreover, $\mathcal{E}$ is clearly convex and monotone. Let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa$ with $f_n \downarrow 0$ and choose $c \geq 0$ with $f_1 \leq c(1+|x|)$. Then,

$$\int_{\mathbb{R}^d} f_n(x) \, \nu(\mathrm{d}x) - \varphi(\mathcal{W}_p(\mu,\nu)) \leq \int_{\mathbb{R}^d} c(1+|x|) \, \mu(\mathrm{d}x) + c\mathcal{W}_p(\mu,\nu) - \varphi(\mathcal{W}_p(\mu,\nu))$$

for all $\nu \in \mathcal{P}_p$. Since $\lim_{c\to\infty} \frac{\varphi(c)}{c} = \infty$, there exists $R \geq 0$ with

$$\mathcal{E}[f_n] = \sup_{\nu \in M} \left(\int_{\mathbb{R}^d} f_n(x) \, \nu(\mathrm{d}x) - \varphi(\mathcal{W}_p(\mu,\nu)) \right) \leq \sup_{\nu \in M} \int_{\mathbb{R}^d} f_n(x) \, \nu(\mathrm{d}x),$$

where $M := \{\nu \in \mathcal{P}_p \colon \mathcal{W}_p(\mu,\nu) \leq R\}$. For every $\varepsilon > 0$, Lemma 7.3.9 implies that there exists $r \geq 0$ with

$$\sup_{\nu \in M} \int_{B(r)^c} c(1+|x|) \, \nu(\mathrm{d}x) \leq \frac{\varepsilon}{2}.$$

Moreover, we can use Dini's theorem to choose $n_0 \in \mathbb{N}$ with

$$\int_{\mathbb{R}^d} f_n(x) \, \nu(\mathrm{d}x) \leq \int_{B(r)} f_n(x) \, \nu(\mathrm{d}x) + \int_{B(r)^c} c(1+|x|) \, \nu(\mathrm{d}x) \leq \varepsilon$$

for all $n \geq n_0$ and $\nu \in M$. We obtain $\mathcal{E}[f_n] \downarrow 0$ as $n \to \infty$. Now, Theorem 7.3.2 yields the existence of the semigroup $(S(t))_{t\geq 0}$.

Second, for every $x \in \mathbb{R}^d$, we show that

$$\mathcal{E}[x\xi] = \sup_{c \geq 0} \big(c|x| - \varphi(c) \big) + \int_{\mathbb{R}^d} xy \, \mu(\mathrm{d}y).$$

W.l.o.g., let $x \neq 0$. For every $c \geq 0$, we choose $\nu := \mu * \delta_{\frac{cx}{|x|}}$. Then,

$$\int_{\mathbb{R}^d} xy \, \nu(\mathrm{d}y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} x(y+z) \, \mu(\mathrm{d}y) \delta_{\frac{cx}{|x|}}(\mathrm{d}z) = c|x| + \int_{\mathbb{R}^d} xy \, \mu(\mathrm{d}y).$$

We take the supremum over $c \geq 0$ and use $\mathcal{W}_p(\mu,\nu) = c$ to conclude

$$\mathcal{E}[x\xi] \geq \sup_{c\geq 0} \big(c|x| - \varphi(c)\big) + \int_{\mathbb{R}^d} xy\,\mu(\mathrm{d}y).$$

For every $\nu \in \mathcal{P}_p$, it follows from

$$\left|\int_{\mathbb{R}^d} xy\,\mu(\mathrm{d}y) - \int_{\mathbb{R}^d} xz\,\nu(\mathrm{d}z)\right| \leq |x| \int_{\mathbb{R}^d\times\mathbb{R}^d} |y-z|\,\pi(\mathrm{d}y,\mathrm{d}z)$$

for all $\pi \in \Pi(\mu,\nu)$ and Hölder's inequality that

$$\begin{aligned}\int_{\mathbb{R}^d} xy\,\nu(\mathrm{d}y) - \varphi(\mathcal{W}_p(\mu,\nu)) &= \int_{\mathbb{R}^d} xy\,\mu(\mathrm{d}y) + \int_{\mathbb{R}^d} xy\,\nu(\mathrm{d}y) - \int_{\mathbb{R}^d} xy\,\mu(\mathrm{d}y) - \varphi(c)\\ &\leq \int_{\mathbb{R}^d} xy\,\mu(\mathrm{d}y) + \mathcal{W}_1(\mu,\nu)|x| - \varphi(c)\\ &\leq \int_{\mathbb{R}^d} xy\,\mu(\mathrm{d}y) + c|x| - \varphi(c),\end{aligned}$$

where $c := \mathcal{W}_p(\mu,\nu)$. Taking the supremum over $\nu \in \mathcal{P}_p$ yields

$$\mathcal{E}[x\xi] \leq \sup_{c\geq 0} \big(c|x| - \varphi(c)\big) + \int_{\mathbb{R}^d} xy\,\mu(\mathrm{d}y).$$

In particular, for every $f \in \mathrm{C}_\mathrm{b}^1$ and $x \in \mathbb{R}^d$, we obtain from Theorem 7.3.4 that

$$(Af)(x) = \mathcal{E}[\nabla f(x)\xi] = \sup_{c\geq 0} \big(c|\nabla f(x)| - \varphi(c)\big) + m\nabla f(x),$$

where $m := \int_{\mathbb{R}^d} y\,\mu(\mathrm{d}y)$. Moreover, for every $x \in \mathbb{R}^d$,

$$\mathcal{E}[x\xi] = \sup_{y\in\mathbb{R}^d} \big(xy - \psi(y)\big), \quad \text{where } \psi(y) := \varphi(|y-m|).$$

Hence, Fenchel–Moreau's theorem and Theorem 7.3.4 imply

$$(S(t)f)(x) = \sup_{y\in\mathbb{R}^d} \big(f(x + t(m+y)) - \varphi(|y|)\big)$$

for all $t \geq 0$, $f \in \mathrm{C}_\mathrm{b}$ and $x \in \mathbb{R}^d$. We apply again Theorem 7.3.4 to obtain the last part of the statement. □

We conclude this subsection by showing that the semigroup (and thus the distribution) which we obtain in the limit is the same whether we define the convex expectation as the supremum over an uncertainty set of measures or a set of parameters in $\mathbb{R}^d$. The latter corresponds to shifting the reference measure μ in all possible deterministic directions. We define

$$\tilde{\mathcal{E}}\colon \mathrm{C}_\kappa \to \mathbb{R},\; f \mapsto \sup_{\lambda\in\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x+\lambda)\,\mu(\mathrm{d}x) - \varphi(|\lambda|)\right),$$

$J(0) := \mathrm{id}_{\mathrm{C}_\mathrm{b}}$ and, for every $t > 0$, $f \in \mathrm{C}_\mathrm{b}$ and $x \in \mathbb{R}^d$,

$$(J(t)f)(x) := t\tilde{\mathcal{E}}\left[\frac{1}{t} f(x + t\xi)\right].$$

Corollary 7.3.11. *Denoting by* $(S(t))_{t\geq 0}$ *the semigroup from Theorem 7.3.10, we have*

$$S(t)f = \lim_{n\to\infty} J\big(\tfrac{t}{n}\big)^n f \quad \textit{for all } t \geq 0 \textit{ and } f \in \mathrm{C_b}.$$

Proof. It follows from $\mathcal{W}_p(\mu,\nu) = |\lambda|$ for $\nu := \mu * \delta_\lambda$ with $\lambda \in \mathbb{R}^d$ that $\tilde{\mathcal{E}}[|f|] \leq \mathcal{E}[|f|]$ for all $f \in \mathrm{C}_\kappa$. Hence, $\tilde{\mathcal{E}}\colon \mathrm{C}_\kappa \to \mathbb{R}$ is a well-defined convex expectation which is continuous from above. By Theorem 7.3.2 and Theorem 7.3.4, there exists a semigroup $(T(t))_{t\geq 0}$ on C_κ with

$$T(t)f = \lim_{n\to\infty} J\big(\tfrac{t}{n}\big)^n f \quad \text{for all } t \geq 0 \text{ and } f \in \mathrm{C_b}$$

and generator $(Bf)(x) = \tilde{\mathcal{E}}[\nabla f(x)\xi]$ for all $f \in \mathrm{C}_\mathrm{b}^1$ and $x \in \mathbb{R}^d$. By Theorem 7.3.10 and a straightforward computation, for every $f \in \mathrm{C}_\mathrm{b}^1$ and $x \in \mathbb{R}^d$,

$$(Af)(x) = \sup_{c\geq 0} \big(c|\nabla f(x)| - \varphi(c)\big) + m\nabla f(x) = (Bf)(x).$$

Theorem 4.4.6 implies $S(t)f = T(t)f$ for all $t \geq 0$ and $f \in \mathrm{C_b}$. □

7.4 Second order scaling limits

Throughout this section, we choose the weight function

$$\kappa\colon \mathbb{R}^d \to (0,\infty),\ x \mapsto (1+|x|^2)^{-1}.$$

Let $(\xi_n)_{n\in\mathbb{N}} \subset \mathcal{H}^d$ be an iid sequence on a convex expectation space $(\Omega, \mathcal{H}, \bar{\mathcal{E}})$ with finite second moments and $\bar{\mathcal{E}}[a\xi_1] = 0$ for all $a \in \mathbb{R}^d$, i.e., we have no mean uncertainty. For the sake of readability, we do not introduce a sequence $(\psi_n)_{n\in\mathbb{N}}$ of recursively defined functions as in Section 7.3 but only study the limit behaviour of the sequence

$$X_n^{t,x} := x + \sqrt{\frac{t}{n}}\sum_{i=1}^n \xi_i$$

which corresponds to choosing $\psi(t,x,y) := x + \sqrt{t}y$. Again, we are interested in the behaviour of the distributions of X_n and requiring finite second moments means that the distribution of ξ_1 is well-defined for continuous functions with at most quadratic growth at infinity, i.e.,

$$F_{\xi_1}\colon \mathrm{C}_\kappa \to \mathbb{R},\ f \mapsto \bar{\mathcal{E}}[f(\xi_1)].$$

Denoting this functional by $\mathcal{E}[\,\cdot\,]$, we remark that continuity from above on C_κ is equivalent to the uniform integrability condition $\lim_{c\to\infty} \bar{\mathcal{E}}[(|\xi_1|^2 - c)^+] = 0$ from [146]. We define $I(0) := \mathrm{id}_{\mathrm{C_b}}$ and, for every $t > 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$,

$$(I(t)f)(x) := t\bar{\mathcal{E}}\left[\frac{1}{t}f(x+\sqrt{t}\xi_1)\right].$$

The aim of this section is to show that

$$S(t)f = \lim_{n\to\infty} \big(I\big(\tfrac{t}{n}\big)^n f \quad \text{for all } (f,t) \in \mathrm{C_b} \times \mathbb{R}_+.$$

where $(S(t))_{t\geq 0}$ is a the unique strongly continuous convex monotone semigroup with generator $(Af)(x) = \mathcal{E}[\frac{1}{2}\xi^T D^2 f(x)\xi]$ for all $f \in \mathrm{C}_\mathrm{b}^2$, $x \in \mathbb{R}^d$ and $\xi := \mathrm{id}_{\mathbb{R}^d}$. We prove this result in Subsection 7.4.1 analogously to Theorem 7.3.2. In Subsection 7.4.2, we illustrate the abstract results again by considering convex expectations that are defined as a supremum over a set of probability measures which are penalized according to their distance to a reference model. The necessary modifications of the setting will be explained in full detail. We would also like to mention that this approach allows for a generalization of the previous results from [14, 85] and Chaper 6.3 about Markov processes with uncertain transition probabilities, which were restricted to first order perturbations, to the second order case. The possibility of this extension was already conjectured in [14] but whether the original construction of the semigroup based on monotone convergence can be transferred remains an open question.

7.4.1 CLT for convex expectations

Let $\mathcal{E}\colon \mathrm{C}_\kappa \to \mathbb{R}$ be a convex expectation which is continuous from above. By Theorem 3.2.1, there exists a convex monotone extension $\mathcal{E}\colon \mathrm{B}_\kappa \to \mathbb{R}$ such that, for every $\varepsilon > 0$ and $c \geq 0$, there exists $K \Subset \mathbb{R}^d$ with

$$\mathcal{E}\left[\tfrac{c}{\kappa}\mathbb{1}_{K^c}\right] < \varepsilon. \tag{7.9}$$

We define $I(0) := \mathrm{id}_{\mathrm{C}_\mathrm{b}}$ and, for every $t > 0$, $f \in \mathrm{C}_\mathrm{b}$ and $x \in \mathbb{R}^d$,

$$(I(t)f)(x) := t\mathcal{E}\left[\frac{1}{t}f(x+\sqrt{t}\xi)\right], \quad \text{where} \quad \xi := \mathrm{id}_{\mathbb{R}^d}.$$

Let C_b^2 consists of all twice continuously differentiable functions $f\colon \mathbb{R}^d \to \mathbb{R}$ such that all partial derivatives up to order two are bounded.

Theorem 7.4.1. *Assume that $\mathcal{E}[a\xi] = 0$ for all $a \in \mathbb{R}^d$. Then, there exists a strongly continuous convex monotone semigroup $(S(t))_{t\geq 0}$ on C_b given by*

$$S(t)f = \lim_{n\to\infty} I\left(\tfrac{t}{n}\right)^n f \quad \text{for all } t \geq 0 \text{ and } f \in \mathrm{C}_\mathrm{b} \tag{7.10}$$

such that Assumption 4.4.5 is valid. It holds $\mathrm{C}_\mathrm{b}^2 \subset D(A)$ and

$$(Af)(x) = \mathcal{E}\left[\frac{1}{2}\xi^T D^2 f(x)\xi\right] \quad \text{for all } f \in \mathrm{C}_\mathrm{b}^2 \text{ and } x \in \mathbb{R}^d.$$

In addition, for every convex expectation space $(\Omega, \mathcal{H}, \bar{\mathcal{E}})$ and iid sequence $(\xi_n)_{n\in\mathbb{N}} \subset \mathcal{H}^d$ with $\bar{\mathcal{E}}[f(\xi_n)] = \mathcal{E}[f]$ for all $f \in \mathrm{C}_\mathrm{b}$,

$$(S(t)f)(x) = \lim_{n\to\infty} \frac{t}{n}\bar{\mathcal{E}}\left[\frac{t}{n}f\left(x + \sqrt{\frac{t}{n}}\sum_{i=1}^n \xi_i\right)\right] \tag{7.11}$$

for all $t > 0$, $f \in \mathrm{C}_\mathrm{b}$ and $x \in \mathbb{R}^d$.

Proof. It follows from Lemma 7.2.2(ii) that the conditions (i)–(iv) and (vii) from Assumption 5.2.2 are satisfied. Similar to the proof of Theorem 7.3.2, one can show that $\|I(t)f)\|_\kappa \leq e^{ct}\|f\|_\kappa$ for all $f \in \mathrm{C_b}$ with $\|f\|_\kappa \leq 1$ and $t \in (0,1]$, where $c := \frac{1}{2}\mathcal{E}[2|\xi|^2]$. Hence, Corollary 3.4.3 yields that condition (vi) is satisfied. Next, we show that

$$(I'(0)f)(x) = \mathcal{E}\left[\frac{1}{2}\xi^T D^2 f(x)\xi\right] \quad \text{for all } f \in \mathrm{C}_\mathrm{b}^2 \text{ and } x \in \mathbb{R}^d.$$

Let $f \in \mathrm{C}_\mathrm{b}^2$. For every $h > 0$ and $x \in \mathbb{R}^d$, Taylor's formula implies

$$f(x+\sqrt{h}\xi) = f(x) + \nabla f(x)\sqrt{h}\xi + \int_0^1\int_0^1 sh\xi^T D^2 f(x+us\sqrt{h}\xi)\xi \,\mathrm{d}u\,\mathrm{d}s.$$

Hence, it follows from Lemma 7.2.2(vi) and Lemma 3.6.1 that

$$\begin{aligned}
&\left(\frac{I(h)f-f}{h}\right)(x) - \mathcal{E}\left[\frac{1}{2}\xi^T D^2 f(x)\xi\right] \\
&= \mathcal{E}\left[\frac{1}{\sqrt{h}}\nabla f(x)\xi + \int_0^1\int_0^1 s\xi^T D^2 f(x+us\sqrt{h}\xi)\xi\,\mathrm{d}u\,\mathrm{d}s\right] - \mathcal{E}\left[\frac{1}{2}\xi^T D^2 f(x)\xi\right] \\
&\leq \lambda\mathcal{E}\left[\frac{1}{\lambda}\left(\int_0^1\int_0^1 s\xi^T D^2 f(x+us\sqrt{h}\xi)\xi\,\mathrm{d}u\,\mathrm{d}s - \frac{1}{2}\xi^T D^2 f(x)\xi\right) + \frac{1}{2}\xi^T D^2 f(x)\xi\right] \\
&\quad - \lambda\mathcal{E}\left[\frac{1}{2}\xi^T D^2 f(x)\xi\right] \\
&\leq \frac{\lambda}{2}\mathcal{E}\left[\frac{2}{\lambda}\left(\int_0^1\int_0^1 s\xi^T D^2 f(x+us\sqrt{h}\xi)\xi\,\mathrm{d}u\,\mathrm{d}s - \frac{1}{2}\xi^T D^2 f(x)\xi\right)\right] \\
&\quad + \frac{\lambda}{2}\mathcal{E}\left[\xi^T D^2 f(x)\xi\right] - \lambda\mathcal{E}\left[\frac{1}{2}\xi^T D^2 f(x)\xi\right]
\end{aligned}$$

for all $h > 0$, $\lambda \in (0,1]$ and $x \in \mathbb{R}^d$. Let $\varepsilon > 0$ and $\lambda \in (0,1]$ with $\lambda\mathcal{E}[\|D^2 f\|_\infty|\xi|^2] < \frac{\varepsilon}{2}$. Lemma 7.2.2(iii) and (v) imply

$$\frac{\lambda}{2}\mathcal{E}\left[\xi^T D^2 f(x)\xi\right] - \lambda\mathcal{E}\left[\frac{1}{2}\xi^T D^2 f(x)\xi\right] \leq \frac{\varepsilon}{2} \quad \text{for all } x \in \mathbb{R}^d.$$

Furthermore, by inequality (7.9), there exists $r \geq 0$ with

$$\begin{aligned}
&\frac{\lambda}{2}\mathcal{E}\left[\frac{2}{\lambda}\left(\int_0^1\int_0^1 s\xi^T D^2 f(x+us\sqrt{h}\xi)\xi\,\mathrm{d}u\,\mathrm{d}s - \frac{1}{2}\xi^T D^2 f(x)\xi\right)\mathbb{1}_{B(r)^c}(\xi)\right] \\
&\leq \frac{\lambda}{2}\mathcal{E}\left[\frac{2}{\lambda}\left(\int_0^1 s\,\mathrm{d}s + \frac{1}{2}\right)\|D^2 f\|_\infty|\xi|^2\mathbb{1}_{B(r)^c}(\xi)\right] \leq \frac{\varepsilon}{4} \quad \text{for all } x \in \mathbb{R}^d.
\end{aligned}$$

Let $K \Subset \mathbb{R}^d$. Since $D^2 f$ is uniformly continuous on compacts, there exists $\delta > 0$ with

$$|D^2 f(x+y) - D^2 f(x)| < \frac{\varepsilon}{2r^2} \quad \text{for all } x \in K \text{ and } y \in B_{\mathbb{R}^d}(\delta).$$

For every $h \in (0, \delta^2/r^2]$ and $x \in K$, we obtain

$$\frac{\lambda}{2}\mathcal{E}\left[\frac{2}{\lambda}\left(\int_0^1\int_0^1 s\xi^T D^2 f(x+us\sqrt{h}\xi)\xi\,\mathrm{d}u\,\mathrm{d}s - \frac{1}{2}\xi^T D^2 f(x)\xi\right)\mathbb{1}_{B(r)}(\xi)\right]$$

$$= \frac{\lambda}{2}\mathcal{E}\left[\frac{2}{\lambda}\left(\int_0^1\int_0^1 s\xi^T\big(D^2f(x+us\sqrt{h}\xi)-D^2f(x)\big)\xi\,\mathrm{d}u\,\mathrm{d}s\right)\mathbb{1}_{B(r)}(\xi)\right]$$
$$\leq \frac{\lambda}{2}\mathcal{E}\left[\frac{2}{\lambda}\left(\int_0^1 s|\xi|^2\frac{\varepsilon}{2r^2}\,\mathrm{d}s\right)\mathbb{1}_{B(r)}(\xi)\right]\leq\frac{\varepsilon}{4}.$$

Hence, for every $\varepsilon>0$ and $K\Subset\mathbb{R}^d$, there exists $h_0>0$ with

$$\left(\frac{I(h)f-f}{h}\right)(x)-\mathcal{E}\left[\frac{1}{2}\xi^TD^2f(x)\xi\right]\leq\varepsilon$$

for all $x\in K$ and $h\in(0,h_0]$. The lower bound follows by similar arguments. Furthermore,

$$\|I(t)f-f\|_\infty\leq\mathcal{E}\left[\frac{1}{2}\|D^2f\|_\infty|\xi|^2\right]t\quad\text{for all } t\geq 0.$$

This shows that Assumption 5.2.2(v) is satisfied and thus the first part of the claim follows from Theorem 5.2.3 while the second part follows similarly to the proof of Theorem 7.3.2. □

If we weaken the condition $\mathcal{E}[a\xi]=0$ for all $a\in\mathbb{R}^d$ by merely requiring

$$\mathcal{E}[a\xi]\geq 0\quad\text{for all } a\in\mathbb{R}^d, \tag{7.12}$$

one can still apply the previous result on the transformed expectation

$$\tilde{\mathcal{E}}\colon\mathcal{H}\to\mathbb{R},\ X\mapsto\inf_{a\in\mathbb{R}^d}\mathcal{E}[X+a\xi]. \tag{7.13}$$

In the particular case $\mathcal{E}[f]=\sup_{\mu\in M}\int_{\mathbb{R}^d}f(x)\,\mu(\mathrm{d}x)$ for a set M of probability measures on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$, condition (7.12) is satisfied if $\int_{\mathbb{R}^d}x\,\mu(\mathrm{d}x)=0$ for some $\mu\in M$.

Corollary 7.4.2. *Assume that $\mathcal{E}[a\xi]\geq 0$ for all $a\in\mathbb{R}^d$. Define $I(0):=\mathrm{id}_{\mathrm{C_b}}$ and*

$$(I(t)f)(x):=\inf_{a\in\mathbb{R}^d}t\mathcal{E}\left[\frac{1}{t}f(x+\sqrt{t}\xi)+a\xi\right]$$

for all $t>0$, $f\in\mathrm{C_b}$ and $x\in\mathbb{R}^d$. Then, there exists a strongly continuous convex monotone semigroup $(S(t))_{t\geq0}$ on $\mathrm{C_b}$ given by

$$S(t)f=\lim_{n\to\infty}I\left(\tfrac{t}{n}\right)^nf\quad\text{for all } t\geq0 \text{ and } f\in\mathrm{C_b}$$

such that Assumption 4.4.5 is valid. It holds $\mathrm{C_b^2}\subset D(A)$ and

$$(Af)(x)=\inf_{a\in\mathbb{R}^d}\mathcal{E}\left[\frac{1}{2}\big(\xi^TD^2f(x)\xi+a\xi\big)\right]\quad\text{for all } f\in\mathrm{C_b^2} \text{ and } x\in\mathbb{R}^d.$$

Proof. By Lemma 7.2.7, the convex expectation $\tilde{\mathcal{E}}$ defined by equation (7.13) satisfies $\tilde{\mathcal{E}}[a\xi]=0$ for all $a\in\mathbb{R}^d$. Hence, we can apply Theorem 7.4.1 to obtain the claim. □

7.4.2 Uncertain samples in Wasserstein spaces

Similar to Subsection 7.3.3, we consider convex expectations that are defined as a supremum over a set of probability measures, which are weighted according to their distance to a reference model, but some natural modifications of the setting are necessary. To guarantee that the convex expectation is continuous from above on functions with at most quadratic growth at infinity, let $p > 2$ and $\varphi\colon [0,\infty] \to [0,\infty]$ be a non-decreasing function with $\varphi(0) = 0$, $\varphi(\infty) = \infty$ and $\lim_{c\to\infty} \varphi(c)/c^2 = \infty$. Moreover, we fix a reference measure $\mu \in \mathcal{P}_p$ with mean zero, i.e., $\int_{\mathbb{R}^d} x\,\mu(\mathrm{d}x) = 0$. In view of Lemma 7.2.7 it seems natural to define

$$\tilde{\mathcal{E}}\colon \mathrm{C}_\kappa \to \mathbb{R},\ f \mapsto \inf_{a\in\mathbb{R}^d} \sup_{\nu\in\mathcal{P}_p} \left(\int_{\mathbb{R}^d} f(x) + ax\,\nu(\mathrm{d}x) - \varphi(\mathcal{W}_p(\mu,\nu)) \right).$$

Using [72, Theorem 2] to interchange the supremum with the infimum, one can show that

$$\tilde{\mathcal{E}}[f] = \sup_{\nu\in\mathcal{P}_p^0} \left(\int_{\mathbb{R}^d} f(x)\,\nu(\mathrm{d}x) - \varphi(\mathcal{W}_p(\mu,\nu)) \right) \quad \text{for all } f \in \mathrm{C}_\kappa,$$

where

$$\mathcal{P}_p^0 := \left\{ \nu \in \mathcal{P}_p\colon \int_{\mathbb{R}^d} x\,\nu(\mathrm{d}x) = 0 \right\}.$$

Furthermore, Theorem 7.4.1 yields a corresponding semigroup $(S(t))_{t\geq 0}$ with generator

$$(Af)(x) = \mathcal{E}\left[\frac{1}{2}\xi^T D^2 f(x)\xi\right] \quad \text{for all } f \in \mathrm{C}_\mathrm{b}^2 \text{ and } x \in \mathbb{R}^d.$$

However, if we want to give an explicit formula for the generator, i.e., the generator should be a given as a supremum over a set of parameters in $\mathbb{R}^d$ rather than a set of measures, it seems necessary to replace the Wasserstein distance $\mathcal{W}_p(\mu,\nu)$ by a transport cost which is given as the infimum over a smaller set of couplings. One possible natural choice are martingale couplings [17, 18]. We call $\pi \in \Pi(\mu,\nu)$ a martingale coupling between μ and ν if there exist random variables X, Y on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\pi = \mathbb{P} \circ (X,Y)^{-1}$ and $\mathbb{E}[Y|X] = X$. Equivalently, we could require that

$$\int_{\mathbb{R}^d} f(x)(y-x)\,\pi(\mathrm{d}x,\mathrm{d}y) = 0 \quad \text{for all } f \in \mathrm{C}_\mathrm{b}. \tag{7.14}$$

Denoting by $\Pi_M(\mu,\nu)$ the set of of all martingale couplings between μ and ν, it follows from Strassen's theorem that $\Pi_M(\mu,\nu) \neq \emptyset$ if and only if $\mu \preccurlyeq \nu$ in convex order. Moreover, we define the corresponding transport cost by

$$C_M(\mu,\nu) := \left(\inf_{\pi\in\Pi_M(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p\,\pi(\mathrm{d}x,\mathrm{d}y) \right)^{\frac{1}{p}},$$

where $\inf\emptyset = \infty$. The convex expectation is then defined by

$$\mathcal{E}\colon \mathrm{C}_\kappa \to \mathbb{R},\ f \mapsto \sup_{\nu\in\mathcal{P}_p^0} \left(\int_{\mathbb{R}^d} f(x)\,\nu(\mathrm{d}x) - \varphi(C_M(\mu,\nu)) \right).$$

As before, we define $I(0) := \mathrm{id}_{\mathrm{C_b}}$ and, for every $t > 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$,

$$(I(t)f)(x) := t\mathcal{E}\left[\frac{1}{t} f(x + \sqrt{t}\xi)\right], \quad \text{where } \xi := \mathrm{id}_{\mathbb{R}^d}.$$

Similar to Subsection 7.3.3, one can show that

$$(I(t)f)(x) = \sup_{\nu \in \mathcal{P}_p^0} \left(\int_{\mathbb{R}^d} f(x+y)\,\nu(\mathrm{d}y) - t\varphi\left(\frac{C_M(\mu_t, \nu)}{\sqrt{t}}\right)\right)$$

for all $t > 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$, where $\mu_t := \mu \circ (\sqrt{t}\xi)^{-1}$. The proof of the next theorem is omitted since it is almost identical with the one of Theorem 7.4.4 below. Furthermore, due to Theorem 4.4.6, the semigroups from Theorem 7.4.3 and Theorem 7.4.4 coincide, see the discussion following Theorem 7.4.4.

Theorem 7.4.3. *The functional $\mathcal{E}$ is a convex expectation which is continuous from above and satisfies $\mathcal{E}[a\xi] = 0$ for all $a \in \mathbb{R}^d$. Hence, there exists a strongly continuous convex monotone semigroup $(S(t))_{t\geq 0}$ on $\mathrm{C_b}$ given by*

$$S(t)f = \lim_{n\to\infty} I\big(\tfrac{t}{n}\big)^n f \quad \text{for all } t \geq 0 \text{ and } f \in \mathrm{C_b}$$

such that Assumption 4.4.5 is valid. It holds $\mathrm{C}_\mathrm{b}^2 \subset D(A)$ and

$$(Af)(x) = \sup_{\lambda \in \mathbb{R}^d} \left(\frac{1}{2}\operatorname{tr}(\lambda\lambda^T D^2 f(x)) - \varphi(|\lambda|)\right) + \frac{1}{2}\operatorname{tr}(\Sigma D^2 f(x))$$

for all $f \in \mathrm{C}_\mathrm{b}^2$ and $x \in \mathbb{R}^d$, where $\Sigma := \int_{\mathbb{R}^d} yy^T\,\mu(\mathrm{d}y) \in \mathbb{S}_d^+$. Moreover, for every convex expectation space $(\Omega, \mathcal{H}, \bar{\mathcal{E}})$ and iid sequence $(\xi_n)_{n\in\mathbb{N}} \subset \mathcal{H}^d$ with $\bar{\mathcal{E}}[f(\xi_n)] = \mathcal{E}[f]$ for all $f \in \mathrm{C_b}$,

$$(S(t)f)(x) = \lim_{n\to\infty} \frac{t}{n}\bar{\mathcal{E}}\left[\frac{t}{n} f\left(x + \sqrt{\frac{t}{n}}\sum_{i=1}^n \xi_i\right)\right]$$

for all $t > 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$.

It turns out that, instead of the martingale constraint, it is sufficient to require that the couplings used to define the transport cost satisfy the condition

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} x^T a(y-x)\,\pi(\mathrm{d}x, \mathrm{d}y) = 0 \text{ for all } a \in \mathbb{S}^d, \tag{7.15}$$

where $\mathbb{S}^d$ consists of all symmetric $d \times d$-matrices. The set of all coupling $\pi \in \Pi(\mu, \nu)$ satisfying equation (7.15) is denote by $\Pi_0(\mu, \nu)$ and the corresponding transport cost is defined by

$$C_0(\mu, \nu) := \left(\inf_{\pi \in \Pi_0(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p\,\pi(\mathrm{d}x, \mathrm{d}y)\right)^{\frac{1}{p}},$$

where $\inf \emptyset = \infty$. The corresponding convex expectation is then defined by

$$\mathcal{E}\colon \mathrm{C}_\kappa \to \mathbb{R},\ f \mapsto \sup_{\nu \in \mathcal{P}_p^0} \left(\int_{\mathbb{R}^d} f(x)\,\nu(\mathrm{d}x) - \varphi(C_0(\mu, \nu))\right).$$

Let $I(0) := \mathrm{id}_{\mathrm{C_b}}$ and, for every $t > 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$,

$$(I(t)f)(x) := t\mathcal{E}\left[\frac{1}{t}f(x+\sqrt{t}\xi)\right], \quad \text{where } \xi := \mathrm{id}_{\mathbb{R}^d}.$$

Again, one can show that

$$(I(t)f)(x) = \sup_{\nu\in\mathcal{P}_p^0}\left(\int_{\mathbb{R}^d} f(x+y)\,\nu(\mathrm{d}y) - t\varphi\left(\frac{C_0(\mu_t,\nu)}{\sqrt{t}}\right)\right),$$

where $\mu_t := \mu\circ(\sqrt{t}\xi)^{-1}$. This leads to the following result.

Theorem 7.4.4. *The functional $\mathcal{E}$ is a convex expectation which is continuous from above and satisfies $\mathcal{E}[a\xi] = 0$ for all $a \in \mathbb{R}^d$. Hence, there exists a strongly continuous convex monotone semigroup $(S(t))_{t\geq 0}$ on $\mathrm{C_b}$ given by*

$$S(t)f = \lim_{n\to\infty} I\left(\tfrac{t}{n}\right)^n f \quad \textit{for all } t\geq 0 \textit{ and } f\in\mathrm{C_b}$$

such that Assumption 4.4.5 is valid. It holds $\mathrm{C_b^2}\subset D(A)$ and

$$(Af)(x) = \sup_{\lambda\in\mathbb{R}^d}\left(\frac{1}{2}\operatorname{tr}(\lambda\lambda^T D^2 f(x)) - \varphi(|\lambda|)\right) + \frac{1}{2}\operatorname{tr}(\Sigma D^2 f(x))$$

for all $f\in\mathrm{C_b^2}$ and $x\in\mathbb{R}^d$, where $\Sigma := \int_{\mathbb{R}^d} yy^T\,\mu(\mathrm{d}y)\in\mathbb{S}_d^+$. Moreover, for every convex expectation space $(\Omega,\mathcal{H},\bar{\mathcal{E}})$ and iid sequence $(\xi_n)_{n\in\mathbb{N}}\subset\mathcal{H}^d$ with $\bar{\mathcal{E}}[f(\xi_n)] = \mathcal{E}[f]$ for all $f\in\mathrm{C_b}$,

$$(S(t)f)(x) = \lim_{n\to\infty}\frac{1}{n}\bar{\mathcal{E}}\left[\frac{t}{n}f\left(x+\sqrt{\frac{t}{n}}\sum_{i=1}^n\xi_i\right)\right]$$

for all $t>0$, $f\in\mathrm{C_b}$ and $x\in\mathbb{R}^d$.

Proof. First, we show that $\mathcal{E}$ is a convex expectation which is continuous from above and satisfies $\mathcal{E}[a\xi]=0$ for all $a\in\mathbb{R}^d$. Let $f\in\mathrm{C}_\kappa$ and choose $c\geq 0$ with $|f(x)|\leq c(1+|x|^2)$ for all $x\in\mathbb{R}^d$. For every $\nu\in\mathcal{P}_p^0$, it follows from equation (7.15) that

$$\begin{aligned}\int_{\mathbb{R}^d}|y|^2\,\nu(\mathrm{d}y) - \int_{\mathbb{R}^d}|x|^2\,\mu(\mathrm{d}x) &= \int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^2\,\pi(\mathrm{d}x,\mathrm{d}y) + 2\int_{\mathbb{R}^d\times\mathbb{R}^d}x(y-x)\,\pi(\mathrm{d}x,\mathrm{d}y)\\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^2\,\pi(\mathrm{d}x,\mathrm{d}y)\end{aligned}$$

for all $\pi\in\Pi_0(\mu,\nu)$ and therefore Hölder's inequality yields

$$\int_{\mathbb{R}^d}|y|^2\,\nu(\mathrm{d}y) - \int_{\mathbb{R}^d}|x|^2\,\mu(\mathrm{d}x)\leq C_0(\mu,\nu)^2.$$

Hence, we can use $\lim_{c\to\infty}\frac{\varphi(c)}{c^2}=\infty$ to obtain

$$\mathcal{E}[|f|]\leq\sup_{\nu\in\mathcal{P}_p^0}\left(\int_{\mathbb{R}^d}c(1+|x|^2)\,\nu(\mathrm{d}x) - \varphi(C_0(\mu,\nu))\right)$$

$$
\begin{aligned}
&= \sup_{\nu\in\mathcal{P}_p^0}\left(\int_{\mathbb{R}^d} c(1+|x|^2)\,\mu(\mathrm{d}x) + \int_{\mathbb{R}^d} c|x|^2\,\nu(\mathrm{d}x) - \int_{\mathbb{R}^d} c|x|^2\,\mu(\mathrm{d}x) - \varphi(C_0(\mu,\nu))\right)\\
&\le \int_{\mathbb{R}^d} c(1+|x|^2)\,\mu(\mathrm{d}x) + \sup_{\nu\in\mathcal{P}_p^0}\big(C_0(\mu,\nu)^2 - \varphi(C_0(\mu,\nu))\big)\\
&\le \int_{\mathbb{R}^d} c(1+|x|^2)\,\mu(\mathrm{d}x) + \sup_{\nu\in\mathcal{P}_p^0}\big(C_0(\mu,\nu)^2 - \varphi(C_0(\mu,\nu))\big) < \infty.
\end{aligned}
$$

It follows from $\varphi(0)=0$ that $\mathcal{E}[c]=c$ for all $c\in\mathbb{R}$ and $\mathcal{E}\colon \mathrm{C}_\kappa\to\mathbb{R}$ is clearly convex and monotone. For every $a\in\mathbb{R}^d$, we use the fact that $\int_{\mathbb{R}^d} x\,\nu(\mathrm{d}x)=0$ for all $\nu\in\mathcal{P}_p^0$ and the condition $\varphi(0)=0$ to obtain $\mathcal{E}[a\xi]=0$. Let $(f_n)_{n\in\mathbb{N}}\subset \mathrm{C}_\kappa$ be a sequence with $f_n\downarrow 0$ and choose $c\ge 0$ with $f_1(x)\le c(1+|x|^2)$ for all $x\in\mathbb{R}^d$. Since

$$
\int_{\mathbb{R}^d} f_n(x)\,\nu(\mathrm{d}x) - \varphi(C_0(\mu,\nu)) \le \int_{\mathbb{R}^d} c(1+|x|^2)\,\mu(\mathrm{d}x) + C_0(\mu,\nu)^2 - \varphi(C_0(\mu,\nu))
$$

for all $\nu\in\mathcal{P}_p^0$ and $\lim_{c\to\infty}\frac{\varphi(c)}{c^2}=\infty$, there exists $R\ge 0$ with

$$
\mathcal{E}[f_n] = \sup_{\nu\in M}\left(\int_{\mathbb{R}^d} f_n(x)\,\nu(\mathrm{d}x) - \varphi(C_0(\mu,\nu))\right) \le \sup_{\nu\in M}\int_{\mathbb{R}^d} f_n(x)\,\nu(\mathrm{d}x) \quad \text{for all } n\in\mathbb{N},
$$

where $M:=\{\nu\in\mathcal{P}_p^0\colon C_0(\mu,\nu)\le R\}$. For every $\varepsilon>0$, we use $\mathcal{W}_p(\mu,\nu)\le C_0(\mu,\nu)$ and Lemma 7.3.9 to choose $r\ge 0$ with

$$
\sup_{\nu\in M}\int_{B(r)^c} c(1+|x|^2)\,\nu(\mathrm{d}x) \le \frac{\varepsilon}{2}.
$$

Moreover, we can use Dini's theorem to choose $n_0\in\mathbb{N}$ with

$$
\int_{\mathbb{R}^d} f_n(x)\,\nu(\mathrm{d}x) \le \int_{B(r)} f_n(x)\,\nu(\mathrm{d}x) + \int_{B(r)^c} c(1+|x|^2)\,\nu(\mathrm{d}x) \le \varepsilon
$$

for all $n\ge n_0$ and $\nu\in M$. We obtain $\mathcal{E}[f_n]\downarrow 0$ as $n\to\infty$. Now, Theorem 7.4.1 yields the existence of the semigroup $(S(t))_{t\ge 0}$. Furthermore, for every $f\in\mathrm{C}_\mathrm{b}^2$ and $x\in\mathbb{R}^d$,

$$
(Af)(x) = \mathcal{E}\left[\frac{1}{2}\xi^T D^2 f(x)\xi\right].
$$

Second, for every $f\in\mathrm{C}_\mathrm{b}$ and $x\in\mathbb{R}^d$, we show that

$$
(Af)(x) = \sup_{\lambda\in\mathbb{R}^d}\left(\frac{1}{2}\operatorname{tr}(\lambda\lambda^T D^2 f(x)) - \varphi(|\lambda|)\right) + \frac{1}{2}\operatorname{tr}(\Sigma D^2 f(x)).
$$

For every $\nu\in\mathcal{P}_p^0$ with $c:=C_0(\mu,\nu)<\infty$ and $\pi\in\Pi_0(\mu,\nu)$, inequality (7.15) yields

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^d} z^T D^2 f(x) z\,\nu(\mathrm{d}z) - \varphi(C_0(\mu,\nu))\\
&= \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} z^T D^2 f(x) z - y^T D^2 f(x) y\,\pi(\mathrm{d}y,\mathrm{d}z) + \frac{1}{2}\int_{\mathbb{R}^d} y^T D^2 f(x) y\,\mu(\mathrm{d}y) - \varphi(c)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} (z-y)^T D^2 f(x)(z-y)\,\pi(\mathrm{d}y,\mathrm{d}z) + \int_{\mathbb{R}^d\times\mathbb{R}^d} y^T D^2 f(x)(z-y)\,\pi(\mathrm{d}y,\mathrm{d}z) \\
&\quad + \frac{1}{2}\operatorname{tr}(\Sigma D^2 f(x)) - \varphi(c) \\
&\leq \frac{1}{2}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |y-z|^2\,\pi(\mathrm{d}y,\mathrm{d}z)\right)\sup_{|\lambda|=1}\lambda^T D^2 f(x)\lambda + \frac{1}{2}\operatorname{tr}(\Sigma D^2 f(x)) - \varphi(c).
\end{aligned}
$$

We take the supremum over $\pi\in\Pi_0(\mu,\nu)$ to obtain

$$
\begin{aligned}
(Af)(x) &\leq \sup_{c\geq 0}\left(\frac{1}{2}c^2\sup_{|\lambda|=1}\operatorname{tr}(\lambda\lambda^T D^2 f(x)) - \varphi(c)\right) + \frac{1}{2}\operatorname{tr}(\Sigma D^2 f(x)) \\
&= \sup_{\lambda\in\mathbb{R}^d}\left(\frac{1}{2}\operatorname{tr}(\lambda\lambda^T D^2 f(x)) - \varphi(|\lambda|)\right) + \frac{1}{2}\operatorname{tr}(\Sigma D^2 f(x)).
\end{aligned}
$$

In order to show the reverse inequality, let $\lambda\in\mathbb{R}^d$ and $\nu := \mu * \left(\frac{1}{2}\delta_\lambda + \frac{1}{2}\delta_{-\lambda}\right)\in\mathcal{P}_p^0$. Furthermore, let X, Z be independent random variables on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with $\mathbb{P}\circ X^{-1} = \mu$ and $\mathbb{P}\circ Z^{-1} = \frac{1}{2}\delta_\lambda + \frac{1}{2}\delta_{-\lambda}$. Define $Y := X + Z$ and $\pi := \mathbb{P}\circ(X,Y)^{-1}$. Clearly, it holds $\pi\in\Pi_0(\mu,\nu)$ and

$$
C_0(\mu,\nu) \leq \mathbb{E}[|X-Y|^p]^{\frac{1}{p}} = \mathbb{E}[|Z|^p]^{\frac{1}{p}} = |\lambda|.
$$

Hence, we can use the independence of X and Z to estimate

$$
\begin{aligned}
(Af)(x) &\geq \frac{1}{2}\int_{\mathbb{R}^d} y^T D^2 f(x) y\,\nu(\mathrm{d}y) - \varphi(C_0(\mu,\nu)) \\
&\geq \frac{1}{2}\mathbb{E}[(X+Z)^T D^2 f(x)(X+Z)] - \varphi(|\lambda|) \\
&= \frac{1}{2}\operatorname{tr}(\lambda\lambda^T D^2 f(x)) - \varphi(|\lambda|) + \frac{1}{2}\operatorname{tr}(\Sigma D^2 f(x)).
\end{aligned}
$$

Taking the the supremum over $\lambda\in\mathbb{R}^d$ yields

$$
(Af)(x) \geq \sup_{\lambda\in\mathbb{R}^d}\left(\frac{1}{2}\operatorname{tr}(\lambda\lambda^T D^2 f(x)) - \varphi(|\lambda|)\right) + \frac{1}{2}\operatorname{tr}(\Sigma D^2 f(x)). \qquad \square
$$

We want to discuss how Theorem 7.4.3 can be obtained from the previous proof. It follows from $\Pi_M(\mu,\nu)\subset\Pi_0(\mu,\nu)$ that $C_M(\mu,\nu)\geq C_0(\mu,\nu)$. Furthermore, the measure ν and coupling π that we have chosen to show

$$
(Af)(x) \geq \sup_{\lambda\in\mathbb{R}^d}\left(\frac{1}{2}\operatorname{tr}(\lambda\lambda^T D^2 f(x)) - \varphi(|\lambda|)\right) + \frac{1}{2}\operatorname{tr}(\Sigma D^2 f(x))
$$

satisfy $\pi\in\Pi_M(\mu,\nu)$ and $C_M(\mu,\nu)\leq|\lambda|$. In addition, Theorem 4.4.6 implies that the semigroups from Theorem 7.4.3 and Theorem 7.4.4 coincide. We conclude this subsection by showing that the semigroup from Theorem 7.4.4 corresponding to a supremum over an infinite dimensional set of arbitrary distributions can be approximated by uncertain random walks. To do so, we define the convex expectation

$$
\tilde{\mathcal{E}}\colon \mathrm{C}_\kappa\to\mathbb{R},\; f\mapsto \sup_{\lambda\in\mathbb{R}^d}\left(\frac{1}{2}\int_{\mathbb{R}^d} f(x+\lambda) + f(x-\lambda)\,\mu(\mathrm{d}x) - \varphi(|\lambda|)\right).
$$

Moreover, we define $J(0) := \mathrm{id}_{\mathrm{C_b}}$ and, for every $t > 0$, $f \in \mathrm{C_b}$ and $x \in \mathbb{R}^d$,

$$(J(t)f)(x) := t\tilde{\mathcal{E}}\left[\frac{1}{t}f(x+\sqrt{t}\xi)\right].$$

Corollary 7.4.5. *Denoting by $(S(t))_{t\geq 0}$ the semigroup from Theorem 7.4.4, we have*

$$S(t)f = \lim_{n\to\infty} J\big(\tfrac{t}{n}\big)^n f \quad \textit{for all } t \geq 0 \textit{ and } f \in \mathrm{C_b}.$$

Proof. As seen during the proof of Theorem 7.4.4, it holds

$$\begin{aligned}\tilde{\mathcal{E}}[f] &= \sup_{\lambda\in\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(x)\,\nu_\lambda(\mathrm{d}x) - \varphi(|\lambda|)\right)\\ &\leq \sup_{\lambda\in\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(x)\,\nu_\lambda(\mathrm{d}x) - \varphi(C_0(\mu,\nu_\lambda))\right) \leq \mathcal{E}[f]\end{aligned}$$

for all $f \in \mathrm{C}_\kappa$, where $\nu_\lambda := \mu * \big(\frac{1}{2}\delta_{-\lambda} + \frac{1}{2}\delta_\lambda\big)$. Hence, the functional $\tilde{\mathcal{E}}\colon \mathrm{C}_\kappa \to \mathbb{R}$ is a well-defined convex expectation which is continuous from above and satisfies $\tilde{\mathcal{E}}[a\xi] = 0$ for all $a \in \mathbb{R}^d$. By Theorem 7.4.1, there exists a semigroup $(T(t))_{t\geq 0}$ on C_κ given by

$$T(t)f = \lim_{n\to\infty} J\big(\tfrac{t}{n}\big)^n f \quad \text{for all } t \geq 0 \text{ and } f \in \mathrm{C_b}$$

with generator $(Bf)(x) = \tilde{\mathcal{E}}[\frac{1}{2}\xi^T D^2 f(x)\xi]$ for all $f \in \mathrm{C}_\mathrm{b}^2$ and $x \in \mathbb{R}^d$. For every $f \in \mathrm{C}_\mathrm{b}^2$ and $x \in \mathbb{R}^d$, it follows from Theorem 7.4.4 and a straightforward computation that

$$(Af)(x) = \sup_{\lambda\in\mathbb{R}^d}\left(\frac{1}{2}\operatorname{tr}(\lambda\lambda^T D^2 f(x)) - \varphi(|\lambda|)\right) + \operatorname{tr}(\Sigma D^2 f(x)) = (Bf)(x).$$

Theorem 4.4.6 implies $S(t)f = T(t)f$ for all $t \geq 0$ and $f \in \mathrm{C_b}$. □

Chapter 8

Convergence of infinitesimal generators and stability of convex monotone semigroups

8.1 Introduction

In this chapter, we consider sequences $(S_n)_{n\in\mathbb{N}}$ of convex monotone semigroups and give explicit conditions under which convergence of their infinitesimal generators $(A_n)_{n\in\mathbb{N}}$ guarantees convergence of the semigroups $(S_n)_{n\in\mathbb{N}}$. In the linear case, such results are classical, cf. Kurtz [124] and Trotter [159], and can be applied, for instance, to obtain convergence results for Markov processes, see, e.g., Ethier and Kurtz [68], Kertz [108, 109] and Kurtz [126]. In addition, based on the Crandall–Liggett theorem, c.f. Crandall and Liggett [50], the results can be extended to nonlinear semigroups which are generated by m-accretive or dissipative operators, see Brézis and Pazy [30] and Kurtz [125]. While this approach closely resembles the theory of linear semigroups, the definition of the nonlinear resolvent typically requires the existence of a unique classical solution of a corresponding fully nonlinear elliptic PDE. As pointed out in Evans [69] and Feng and Kurtz [75], the necessary regularity of classical solutions is, in general, delicate. This observation was, among others, one of the motivations for the introduction of viscosity solutions, cf. Crandall et al. [49] and Crandall and Lions [51]. In contrast to classical solutions, the latter have the stability property that, under mild conditions, limits of viscosity solutions are again viscosity solutions. A prominent example in this context is the vanishing viscosity method, where smooth solutions of semilinear second order equations converge uniformly to the unique viscosity solution of a fully nonlinear first order equation. Another remarkable stability property of viscosity solutions is that every monotone approximation scheme, which is consistent and stable, converges to the exact solution provided that the latter is unique, cf. Barles and Souganidis [9]. Based on this result, it is possible to derive explicit convergence rates for several numerical schemes for HJB equations, see, e.g., Barles and Jakobsen [7, 8], Briani et al. [33], Caffarelli and Souganidis [38], Jakobsen et al. [103] and Krylov [116–118]. Returning to the convergence of nonlinear semigroups, we would like to discuss the results from Feng and Kurtz [75] which are motivated by the large deviations principle for Markov processes. The approach in [75] combines results on dissipative operators with key arguments from viscosity theory and has been applied and extended in Feng [73], Feng et al. [74], Kraaij [112, 114] and Popovic [148]. Here, the resolvent equation $(\mathrm{id} - \lambda A)u = f$ for the limit operator $Au := \lim_{n\to\infty} A_n u$ is solved in the viscosity sense and needs to satisfy a comparison principle which does not hold a priori but has to be verified on a case-by-case basis. Since the existence of a solution is always guaranteed, the limit semigroup can then be constructed via the Euler formula $S(t)f := \lim_{n\to\infty} R(t/n)^n f$, where $R(\lambda)f$ denotes the unique viscosity solution of the resolvent equation. By construction, the semigroup $(S(t))_{t\geq 0}$ is generated by the operator A and one can show that $S(t)f = \lim_{n\to\infty} S_n(t)f$.

In contrast to the previously mentioned results, the present approach neither relies on the existence of nonlinear resolvents nor on the theory of viscosity solutions. Instead, the key arguments are based on the comparison principle, see Theorem 4.4.6, which uniquely determines semigroups via their generators evaluated at smooth functions. These results allow us to proceed in the following way. First, the limit semigroup is defined as $S(t)f := \lim_{l\to\infty} S_{n_l}(t)f$ for all (t, f) in a countable dense set, where the convergence for a subsequence $(n_l)_{l\in\mathbb{N}}$ is guaranteed by a relative compactness argument. Since we only require convergence w.r.t. the mixed topology rather than convergence w.r.t. the supremum norm, the latter can be verified by means of Arzéla-Ascoli's theorem. After an extension to arbitrary (t, f), we then show that the generator of $(S(t))_{t\geq 0}$ is given by $Af = \lim_{n\to\infty} A_n f$ for smooth functions f. At this point we would like to emphasize that the semigroups $(S_n)_{n\in\mathbb{N}}$ satisfy the comparison principle which transfers to $(S(t))_{t\geq 0}$. In particular, the limit semigroup does not depend on the choice of the convergent subsequence $(n_l)_{l\in\mathbb{N}}$ and therefore satisfies $S(t)f = \lim_{n\to\infty} S_n(t)f$. This stability result for convex monotone semigroups is stated in Theorem 8.2.3 while Theorem 8.2.6 also allows for discretizations in time and space. The latter covers a variety of approximation schemes of the form $S(t)f = \lim_{n\to\infty} I_n^{k_n} f$, where $(I_n)_{n\in\mathbb{N}}$ is a family of one-step operators describing the dynamics on a discrete time scale of size $h_n > 0$ with $h_n \to 0$ and $k_n h_n \to t$, see Theorem 8.2.8. This extension of the classical Chernoff approximation, cf. Chernoff [43, 44], includes finite-difference methods for HJB equations, cf. Barles and Jakobsen [8], Bonnans and Zidani [28] and Krylov [117, 118], and Markov chain approximations for stochastic control problems, cf. Dupuis and Kushner [64] and Fleming and Soner [77]. Moreover, Theorem 8.2.8 allows to construct nonlinear semigroups without relying on the existence of nonlinear resolvents, e.g., in the context of upper envelopes for families of linear semigroups, cf. Denk et al. [60], Nendel and Röckner [136] and Nisio [138]. Finally, we remark that our notion of a strongly continuous convex monotone semigroup coincides with the one in Goldys et al. [92], where it is shown that the function $u(t) := S(t)f$ is a viscosity solution of the abstract Cauchy problem $\partial_t u = Au$ with initial condition $u(0) = f$. Hence, our approach is consistent with the theory of viscosity solutions and thus Chernoff-type approximations can be seen as monotone schemes. We refer to Fleming and Soner [77, Chapter II.3] for a broad discussion on the relation between semigroups and viscosity solutions and to Yong and Zhou [168, Chapter 4] for an illustration of the interplay between the dynamic programming principle and viscosity solutions in a stochastic optimal control setting. In addition, convex monotone semigroups appear in the context of stochastic processes under model uncertainty, cf. Coquet et al. [47], Criens and Niemann [52], Fadina et al. [71], Hu and Peng [99], Krak et al. [115], Kühn [120], Neufeld and Nutz [137] and Peng [146]. In many situations, these semigroups admit a stochastic representation via BSDEs and 2BSDEs, cf. Cheridito et al. [42], El Karoui et al. [66], Kazi-Tani et al. [106] and Soner et al. [154]. For stability and approximation results for BSDEs, we refer to Briand et al. [31], Hu and Peng [100], Geiss et al. [87] and Papapantoleon et al. [140].

The abstract results are illustrated in a variety of applications which we briefly discuss at this point. In Subsection 8.3.1, we derive an implicit Euler formula and a Yosida approximation for upper envelopes of families of linear semigroups. Here, we would like to emphasize that difficulties in defining the resolvent of the supremum of linear operators are avoided by considering the supremum over linear resolvents rather than the nonlinear resolvent. We refer to Budde and Farkas [34], Cerrai [40], Kühnemund [121] and Pazy [141] for the corresponding results on linear semigroups in different settings. In Subsection 8.3.2 and Subsection 8.3.3, we study finite-difference methods, cf. Barles and Jakobsen [8], Bonnans and Zidani [28], Dupuis and Kushner [64] and Krylov [117, 118], as well as the stability of convex HJB equations, cf. Barron and Jensen [10, 11], Crandall and Lions [51], Frankowska [81] and Kraaij [114]. In Subsection 8.3.4, we consider explicit Euler schemes for Lipschitz ODEs with an additive noise term, where the distribution of the noise can be uncertain. Depending on the scaling of the noise this can either be seen as a robustness result which states that, regardless of possible numerical errors, the Euler scheme still converges to the solution of the ODE or as an Euler–Maruyama scheme for SDEs driven by G-Brownian motions, cf. Geng et al. [88, Section 3], Hu et al. [95] and Peng [143, 146]. The analysis of randomized Euler schemes is continued in Subsection 8.3.5, where we focus on asymptotic convergence rates by means of a large deviations approach, cf. Dupuis and Ellis [63], Dembo and Zeitouni [58], Feng and Kurtz [75] and Varadhan [162] leading to Freidlin–Wentzell-type results, cf. Feng and Kurtz [75] and Freidlin and Wentzell [82]. Finally, in Subsection 8.3.6 and Subsection 8.3.7, we provide Markov chain approximations for a class of stochastic control problems, cf. Dupuis and Kushner [64], Fleming and Soner [77] and Krylov [116], and continuous-time Markov processes with uncertain transition probabilities, cf. Bartl et al. [14] and Fuhrmann et al. [85].

8.2 Stability of convex monotone semigroups

Let $\kappa\colon \mathbb{R}^d \to \mathbb{R}$ be a bounded continuous function with

$$c_\kappa := \sup_{x\in\mathbb{R}^d} \sup_{|y|\le 1} \frac{\kappa(x)}{\kappa(x-y)} < \infty \tag{8.1}$$

We recall the following definition of a strongly continuous convex monotone semigroup.

Definition 8.2.1. A strongly continuous convex monotone semigroup is a family $(S(t))_{t\ge 0}$ of operators $S(t)\colon \mathrm{C}_\kappa(\mathbb{R}^d) \to \mathrm{C}_\kappa(\mathbb{R}^d)$ satisfying the following conditions:

(i) $S(0) = \mathrm{id}_{\mathrm{C}_\kappa(\mathbb{R}^d)}$ and $S(s+t)f = S(s)S(t)f$ for all $s,t \ge 0$ and $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$.

(ii) $S(t)f \le S(t)g$ for all $t \ge 0$ and $f, g \in \mathrm{C}_\kappa(\mathbb{R}^d)$ with $f \le g$.

(iii) $S(t)(\lambda f + (1-\lambda g) \le \lambda S(t)f + (1-\lambda)S(t)g$ for all $t \ge 0$, $f, g \in \mathrm{C}_\kappa(\mathbb{R}^d)$ and $\lambda \in [0,1]$.

(iv) The mapping $\mathbb{R}_+ \to \mathrm{C}_\kappa(\mathbb{R}^d),\ t \mapsto S(t)f$ is continuous for all $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$.

Furthermore, the generator of the semigroup is defined by

$$A\colon D(A) \to \mathrm{C}_\kappa(\mathbb{R}^d),\ f \mapsto \lim_{h \downarrow 0} \frac{S(h)f - f}{h},$$

where the domain $D(A)$ consists of all $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$ such that the previous limit exists. The Lipschitz set $\mathcal{L}^S$ consists of all $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$ such that there exist $c \geq 0$ and $t_0 > 0$ with

$$\|S(t)f - f\|_\kappa \leq ct \quad \text{for all } t \in [0, t_0],$$

while the upper Lipschitz set $\mathcal{L}^S_+$ consists of all $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$ such that there exist $c \geq 0$ and $t_0 > 0$ with

$$\|(S(t)f - f)^+\|_\kappa \leq ct \quad \text{for all } t \in [0, t_0].$$

We say that a set $\mathcal{C} \subset \mathrm{C}_\kappa(\mathbb{R}^d)$ is invariant if $S(t)\colon \mathcal{C} \to \mathcal{C}$ for all $t \geq 0$.

By definition, we have $D(A) \subset \mathcal{L}^S \subset \mathcal{L}^S_+$. We also recall that, in contrast to the sets $\mathcal{L}^S$ and $\mathcal{L}^S_+$, the domain $D(A)$ is, in general, not invariant under the nonlinear semigroup.

Assumption 8.2.2. Let $(S_n)_{n \in \mathbb{N}}$ be a sequence of strongly continuous convex monotone semigroups $(S_n(t))_{t \geq 0}$ on $\mathrm{C}_\kappa(\mathbb{R}^d)$ with $S_n(t)0 = 0$ for all $n \in \mathbb{N}$ and $t \geq 0$. We denote by A_n the generator of S_n and impose the following conditions:

(i) For every $r, T \geq 0$, there exists $c \geq 0$ with

$$\|S_n(t)f - S_n(t)g\|_\kappa \leq c\|f - g\|_\kappa$$

for all $n \in \mathbb{N}$, $t \in [0, T]$ and $f, g \in B_{\mathrm{C}_\kappa(\mathbb{R}^d)}(r)$.

(ii) For every $\varepsilon > 0$, $r, T \geq 0$ and $K \Subset \mathbb{R}^d$, there exist $c \geq 0$ and $K' \Subset \mathbb{R}^d$ with

$$\|S_n(t)f - S_n(t)g\|_{\infty,K} \leq c\|f - g\|_{\infty,K'} + \varepsilon$$

for all $n \in \mathbb{N}$, $t \in [0, T]$ and $f, g \in B_{\mathrm{C}_\kappa(\mathbb{R}^d)}(r)$.

(iii) For every $f \in \mathrm{Lip_b}(\mathbb{R}^d)$ and $\varepsilon > 0$, there exist $\delta, t_0 > 0$ and $(f_n)_{n \in \mathbb{N}} \subset \mathrm{C}_\kappa(\mathbb{R}^d)$ with $f_n \to f$ and

$$\|S_n(t)(\tau_x f_n) - \tau_x S_n(t) f_n\|_\kappa \leq \varepsilon t$$

for all $n \in \mathbb{N}$, $t \in [0, t_0]$ and $x \in B_{\mathbb{R}^d}(\delta)$.

(iv) For every $f \in \mathrm{Lip_b}(\mathbb{R}^d)$ and $T \geq 0$, there exists $r \geq 0$ with $S_n(t)f \in \mathrm{Lip_b}(\mathbb{R}^d, r)$ for all $n \in \mathbb{N}$ and $t \in [0, T]$.

(v) For every $f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$, there exists a sequence $(f_n)_{n \in \mathbb{N}}$ with $f_n \in D(A_n)$ for all $n \in \mathbb{N}$ and $f_n \to f$ such that the sequence $(A_n f_n)_{n \in \mathbb{N}}$ converges in $\mathrm{C}_\kappa(\mathbb{R}^d)$.

The following theorem is a consequence of the more general result that will be proved in the next subsection. In view of Theorem 4.4.6, the semigroup $(S(t))_{t \geq 0}$ is uniquely determined by its infinitesimal generator Af for $f \in \mathrm{C}_\mathrm{b}^\infty$.

Theorem 8.2.3. *Let $(S_n)_{n\in\mathbb{N}}$ be a sequence satisfying Assumption 8.2.2. Then, there exists a strongly continuous convex monotone semigroup $(S(t))_{t\geq 0}$ on $\mathrm{C}_\kappa(\mathbb{R}^d)$ with*

$$S(t)f = \lim_{n\to\infty} S_n(t_n)f_n \quad \text{for all } (f,t) \in \mathrm{C}_\kappa(\mathbb{R}^d)\times\mathbb{R}_+,$$

where $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ and $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\kappa(\mathbb{R}^d)$ are arbitrary sequences with $t_n \to t$ and $f_n \to f$. Furthermore, the semigroup $(S(t))_{t\geq 0}$ has the following properties:

(i) Let $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence with $f_n \in D(A_n)$ for all $n\in\mathbb{N}$ and $f_n \to f$ such that the sequence $(A_nf_n)_{n\in\mathbb{N}}$ converges in $\mathrm{C}_\kappa(\mathbb{R}^d)$. Then, it holds $f\in D(A)$ and $Af = \lim_{n\to\infty} A_nf_n$. This is particularly valid for all $f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$.

(ii) For every $r,T\geq 0$, there exists $c\geq 0$ with

$$\|S(t)f - S(t)g\|_\kappa \leq c\|f-g\|_\kappa \quad \text{for all } t\in[0,T] \text{ and } f,g\in B_{\mathrm{C}_\kappa(\mathbb{R}^d)}(r).$$

Furthermore, it holds $S(t)0 = 0$ for all $t\geq 0$.

(iii) For every $\varepsilon > 0$, $r,T\geq 0$ and $K \Subset \mathbb{R}^d$, there exist $c\geq 0$ and $K' \Subset \mathbb{R}^d$ with

$$\|S(t)f - S(t)g\|_{\infty,K} \leq c\|f-g\|_{\infty,K'} + \varepsilon$$

for all $t\in[0,T]$ and $f,g\in B_{\mathrm{C}_\kappa(\mathbb{R}^d)}(r)$.

(iv) For every $f\in \mathrm{Lip}_\mathrm{b}(\mathbb{R}^d)$ and $\varepsilon>0$, there exist $\delta, t_0 > 0$ with

$$\|S(t)(\tau_x f) - \tau_x S(t)f\|_\kappa \leq \varepsilon t$$

for all $t\in[0,t_0]$ and $x\in B_{\mathbb{R}^d}(\delta)$.

(v) The sets $\mathcal{L}^S$, $\mathcal{L}^S_+$ and $\mathrm{Lip}_\mathrm{b}(\mathbb{R}^d)$ are invariant.

8.2.1 Discretization in time and space

Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ and $(X_n)_{n\in\mathbb{N}}$ be two sequences of subsets $\mathcal{T}_n \subset \mathbb{R}_+$ and $X_n \subset \mathbb{R}^d$ such that, for every $t\geq 0$ and $x\in\mathbb{R}^d$, there exist $t_n\in\mathcal{T}_n$ and $x_n \in X_n$ with $t_n\to t$ and $x_n \to x$, respectively. We assume that $0\in\mathcal{T}_n$ and $s\pm t\in\mathcal{T}_n$ for all $n\in\mathbb{N}$ and $s,t\in\mathcal{T}_n$ with $s\geq t$. If $h_n := \inf(\mathcal{T}_n\setminus\{0\}) > 0$, it follows that $\mathcal{T}_n = \{kh_n \colon k\in\mathbb{N}_0\}$ is an equidistant grid. Otherwise, the set $\mathcal{T}_n$ is already dense. In addition, the sets X_n are supposed to be closed and satisfy $x+y\in X_n$ for all $n\in\mathbb{N}$ and $x,y\in X_n$. The space $\mathrm{C}_\kappa(X_n)$ consists of all continuous functions $f\colon X_n\to\mathbb{R}$ with

$$\|f\|_{\kappa,X_n} := \sup_{x\in X_n} |f(x)|\kappa(x) < \infty.$$

Let $B_{\mathrm{C}_\kappa(X_n)}(r) := \{f\in\mathrm{C}_\kappa(X_n)\colon \|f\|_{\kappa,X_n}\leq r\}$ for all $r\geq 0$. Moreover, for a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n\in\mathrm{C}_\kappa(X_n)$ and $f\in\mathrm{C}_\kappa(\mathbb{R}^d)$, we define $f := \lim_{n\to\infty} f_n$ if and only if

$$\sup_{n\in\mathbb{N}} \|f_n\|_{\kappa,X_n} < \infty \quad \text{and} \quad \lim_{n\to\infty} \|f-f_n\|_{\infty,K_n} = 0$$

for all $K\Subset\mathbb{R}^d$ with $K\cap X_n \neq\emptyset$ for all $n\in\mathbb{N}$, where $K_n := K\cap X_n$. The continuity of f guarantees that the previous limit is uniquely determined. Indeed, assume by contradiction that there exists another function $g \in \mathrm{C}_\kappa(\mathbb{R}^d)$ with $g = \lim_{n\to\infty} f_n$ and $f \neq g$. Then, there exists $\varepsilon > 0$ with $A := \{x \in \mathbb{R}^d \colon |f(x) - g(x)| > \varepsilon\} \neq \emptyset$. Fix $x \in A$ and choose a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in X_n$ and $x_n \to x$. Since A is open, there exists $n_0 \in \mathbb{N}$ with $x_n \in A$ for all $n \geq n_0$. For $K := \{x_n \colon n \in \mathbb{N}\} \cup \{x\}$, we obtain

$$\varepsilon \leq \|f - g\|_{\infty,K_n} \leq \|f - f_n\|_{\infty,K_n} + \|f_n - g\|_{\infty,K_n} \to 0 \quad \text{as } n \to \infty.$$

In addition, we also fix a family $\{S_n(t) \colon n \in \mathbb{N}, t \in \mathcal{T}_n\}$ of operators

$$S_n(t) \colon \mathrm{C}_\kappa(X_n) \to \mathrm{C}_\kappa(X_n),$$

and define $S_n(t)f := S_n(t)(f|_{X_n})$ for all $n \in \mathbb{N}$, $t \in \mathcal{T}_n$ and $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$, i.e., we consider $\mathrm{C}_\kappa(\mathbb{R}^d)$ as a subset of $\mathrm{C}_\kappa(X_n)$ using the possibility to restrict functions from $\mathbb{R}^d$ to X_n. Similar to Definition 8.2.1, a family $(S_n(t))_{t\in\mathcal{T}_n}$ is called convex monotone semigroup on $\mathrm{C}_\kappa(X_n)$ if the operators $S_n(t) \colon \mathrm{C}_\kappa(X_n) \to \mathrm{C}_\kappa(X_n)$ are convex and monotone for all $n \in \mathbb{N}$ and $t \in \mathcal{T}_n$, where functions are ordered pointwise, and satisfy $S_n(0)f = f$ and $S_n(s+t)f = S_n(s)S(t)f$ for all $n \in \mathbb{N}$, $s, t \in \mathcal{T}_n$ and $f \in \mathrm{C}_\kappa(X_n)$. If $\inf(\mathcal{T}_n \setminus \{0\}) > 0$, the generator of $(S_n(t))_{t\in\mathcal{T}_n}$ can not be defined as the limit

$$A_n f = \lim_{h\downarrow 0} \frac{S_n(h)f - f}{h}.$$

This observation motivates the following rather technical definition and corresponding condition (v) in Assumption 8.2.5. However, if $\mathcal{T}_n = \mathbb{R}_+$ for all $n \in \mathbb{N}$, it is sufficient to show that, for every $f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$, there exist $f_n \in D(A_n)$ with $f_n \to f$ such that the sequence $(A_n f_n)_{n\in\mathbb{N}}$ converges in $\mathrm{C}_\kappa(\mathbb{R}^d)$. Moreover, for Chernoff-type approximations, we only have to verify that the approximation scheme satisfies a reasonable consistency condition. For details, we refer to Theorem 8.2.6 and Theorem 8.2.8 below.

Definition 8.2.4. The asymptotic Lipschitz set $\mathcal{L}_0$ consists of all $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$ such that there exist $c \geq 0$ and sequences $(t_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ with $t_n \in \mathcal{T}_n \setminus \{0\}$, $f_n \in \mathrm{C}_\kappa(X_n)$, $f_n \to f$ and

$$\|S_n(t)f_n - f_n\|_{\kappa,X_n} \leq ct \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [0, t_n] \cap \mathcal{T}_n. \tag{8.2}$$

Furthermore, the asymptotic domain $\mathcal{D}_0$ consists of all $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$ such that there exist $g \in \mathrm{C}_\kappa(\mathbb{R}^d)$ and a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \in \mathrm{C}_\kappa(X_n)$ and $f_n \to f$ which satisfy the following conditions:

(i) There exist $c \geq 0$ and $(t_n)_{n\in\mathbb{N}}$ with $t_n \in \mathcal{T}_n \setminus \{0\}$ such that inequality (8.2) is valid.

(ii) For every $K \Subset \mathbb{R}^d$, there exists $(h_n)_{n\in\mathbb{N}}$ with $h_n \in (0, t_n] \cap \mathcal{T}_n$, $h_n \to 0$ and

$$\lim_{n\to\infty} \left\| \frac{S_n(h_n)f_n - f_n}{h_n} - g \right\|_{\infty,K_n} = 0. \tag{8.3}$$

For every $n \in \mathbb{N}$ and $r \geq 0$, we denote by $\mathrm{Lip}_\mathrm{b}(X_n, r)$ the set of all r-Lipschitz functions $f \colon X_n \to \mathbb{R}$ with $\|f\|_{\infty,X_n} \leq r$.

Assumption 8.2.5. Suppose that $(S_n)_{n\in\mathbb{N}}$ is a sequence of convex monotone semigroups $(S_n(t))_{t\in\mathcal{T}_n}$ on $\mathrm{C}_\kappa(X_n)$ with $S_n(t)0 = 0$ for all $n \in \mathbb{N}$ and $t \in \mathcal{T}_n$. In addition, the following conditions are satisfied:

(i) For every $r, T \geq 0$, there exists $c \geq 0$ with

$$\|S_n(t)f - S_n(t)g\|_{\kappa,X_n} \leq c\|f - g\|_{\kappa,X_n}$$

for all $n \in \mathbb{N}$, $t \in [0, T] \cap \mathcal{T}_n$ and $f, g \in B_{\mathrm{C}_\kappa(X_n)}(r)$.

(ii) For every $\varepsilon > 0$, $r, T \geq 0$ and $K \Subset \mathbb{R}^d$, there exist $c \geq 0$ and $K' \Subset \mathbb{R}^d$ with

$$\|S_n(t)f - S_n(t)g\|_{\infty,K_n} \leq c\|f - g\|_{\infty,K'_n} + \varepsilon$$

for all $n \in \mathbb{N}$, $t \in [0, T] \cap \mathcal{T}_n$ and $f, g \in B_{\mathrm{C}_\kappa(X_n)}(r)$.

(iii) For every $f \in \mathrm{Lip}_\mathrm{b}(\mathbb{R}^d)$ and $\varepsilon > 0$, there exist $\delta, t_0 > 0$ and a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \in \mathrm{C}_\kappa(X_n)$ for all $n \in \mathbb{N}$, $f_n \to f$ and

$$\|S_n(t)(\tau_x f_n) - \tau_x S_n(t)f_n\|_{\kappa,X_n} \leq \varepsilon t$$

for all $n \in \mathbb{N}$, $t \in [0, t_0] \cap \mathcal{T}_n$, $x \in B_{X_n}(\delta)$ and $y \in X_n$.

(iv) For every $f \in \mathrm{Lip}_\mathrm{b}(\mathbb{R}^d)$ and $T \geq 0$, there exists $r \geq 0$ with $S_n(t)f \in \mathrm{Lip}_\mathrm{b}(X_n, r)$ for all $n \in \mathbb{N}$ and $t \in [0, T] \cap \mathcal{T}_n$.

(v) It holds $\mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d) \subset \mathcal{D}_0$.

The following theorem is the main result of this article.

Theorem 8.2.6. *Let $(S_n)_{n\in\mathbb{N}}$ be a sequence satisfying Assumption 8.2.5. Then, there exists a strongly continuous convex monotone semigroup $(S(t))_{t\geq 0}$ on $\mathrm{C}_\kappa(\mathbb{R}^d)$ with*

$$S(t)f = \lim_{n\to\infty} S_n(t_n)f_n \quad \textit{for all } (f, t) \in \mathrm{C}_\kappa(\mathbb{R}^d) \times \mathbb{R}_+, \tag{8.4}$$

where $(t_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ are arbitrary sequences with $t_n \in \mathcal{T}_n$ and $f_n \in \mathrm{C}_\kappa(X_n)$ for all $n \in \mathbb{N}$, $t_n \to t$ and $f_n \to f$. Furthermore, the semigroup $(S(t))_{t\geq 0}$ has the following properties:

(i) It holds $\mathcal{L}_0 \subset \mathcal{L}^S$ and $\mathcal{D}_0 \subset D(A)$. Moreover, for every $f \in \mathcal{D}_0$ and $K \Subset \mathbb{R}^d$,

$$\lim_{n\to\infty} \left\| \frac{S_n(h_n)f_n - f_n}{h_n} - Af \right\|_{\infty,K_n} = 0, \tag{8.5}$$

where $(h_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ are arbitrary sequences satisfying the conditions (i) and (ii) in Definition 8.2.4. In particular, this is valid for all $f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$.

(ii) For every $r, T \geq 0$, there exists $c \geq 0$ with

$$\|S(t)f - S(t)g\|_\kappa \leq c\|f - g\|_\kappa \quad \textit{for all } t \in [0, T] \textit{ and } f, g \in B_{\mathrm{C}_\kappa(\mathbb{R}^d)}(r).$$

Furthermore, it holds $S(t)0 = 0$ for all $t \geq 0$.

(iii) *For every $\varepsilon > 0$, $r, T \geq 0$ and $K \Subset \mathbb{R}^d$, there exist $c \geq 0$ and $K' \Subset \mathbb{R}^d$ with*

$$\|S(t)f - S(t)g\|_{\infty,K} \leq c\|f - g\|_{\infty,K'} + \varepsilon$$

for all $t \in [0,T]$ and $f, g \in B_{\mathrm{C}_\kappa(\mathbb{R}^d)}(r)$.

(iv) *For every $f \in \mathrm{Lip}_{\mathrm{b}}(\mathbb{R}^d)$ and $\varepsilon > 0$, there exist $\delta, t_0 > 0$ with*

$$\|S(t)(\tau_x f) - \tau_x S(t)f\|_\kappa \leq \varepsilon t$$

for all $t \in [0, t_0]$ and $x \in B_{\mathbb{R}^d}(\delta)$.

(v) *The sets $\mathcal{L}^S$, $\mathcal{L}^S_+$ and $\mathrm{Lip}_{\mathrm{b}}(\mathbb{R}^d)$ are invariant.*

Proof. First, we construct the family $(S(t))_{t\geq 0}$ of operators $S(t)\colon \mathrm{C}_\kappa(\mathbb{R}^d) \to \mathrm{C}_\kappa(\mathbb{R}^d)$. Let $\mathcal{T} \subset \mathbb{R}_+$ be countable dense including zero. Due to Lemma 5.2.1, there exists a countable set $\mathcal{D} \subset \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$ such that, for every $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ with $\|f_n\|_\kappa \leq \|f\|_\kappa$ for all $n \in \mathbb{N}$ and $f_n \to f$. In addition, for every $t \in \mathcal{T}$, let $(t_n)_{n\in\mathbb{N}}$ be a sequence with $t_n \in \mathcal{T}_n$ and $t_n \to t$. For $t = 0$, we choose $t_n := 0$ for all $n \in \mathbb{N}$. Since Assumption 8.2.5(i) and (iv) imply that the sequence $(S_n(t_n)f)_{n\in\mathbb{N}}$ is bounded and uniformly equicontinuous, we can use Lemma 3.5.1 and a diagonalization argument to choose a subsequence $(n_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ such that the limit

$$S(t)f := \lim_{l\to\infty} S_{n_l}(t_{n_l})f \in \mathrm{C}_\kappa(\mathbb{R}^d) \tag{8.6}$$

exists for all $(f,t) \in \mathcal{D} \times \mathcal{T}$. In the order to extend the family $(S(t))_{t\in\mathcal{T}}$ to arbitrary points in time, we fix $f \in \mathcal{D}$ and $T \geq 0$. Since $f \in \mathcal{L}_0$, there exist $c_1 \geq 0$ as well as sequences $(h_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ with $h_n \in \mathcal{T}_n\backslash\{0\}$ and $f_n \in \mathrm{C}_\kappa(X_n)$ for all $n \in \mathbb{N}$, $f_n \to f$ and

$$\|S_n(h)f_n - f_n\|_{\kappa,X_n} \leq c_1 h \quad \text{for all } n \in \mathbb{N} \text{ and } h \in [0,h_n] \cap \mathcal{T}_n.$$

Let $s, t \in [0,T] \cap \mathcal{T}$ with $s \leq t$. We choose $s_n, t_n \in \mathcal{T}_n$ with $s_n \to s$ and $t_n \to t$ and define $k_n := \max\{k \in \mathbb{N}_0 \colon kh_n \leq t_n - s_n\}$ for all $n \in \mathbb{N}$. By Assumption 8.2.5(i), there exists $c_2 \geq 0$ with

$$\begin{aligned}
\|S_n(s_n)f_n - S_n(t_n)f_n\|_{\kappa,X_n} &= \|S_n(s_n)S_n(t_n - s_n)f_n - S_n(s_n)f_n\|_{\kappa,X_n} \\
&\leq c_2\|S_n(t_n - s_n)f_n - f_n\|_{\kappa,X_n} \\
&\leq c_2\|S_n(k_nh_n)S_n(t_n - s_n - k_nh_n)f_n - S_n(k_nh_n)f_n\|_{\kappa,X_n} \\
&\quad + c_2\sum_{i=0}^{k_n-1}\|S_n(ih_n)S_n(h_n)f_n - S_n(ih_n)f_n\|_{\kappa,X_n} \\
&\leq c_1c_2^2(t_n - s_n - k_nh_n) + c_2^2k_n\|S_n(h_n)f_n - f_n\|_{\kappa,X_n} \\
&\leq c(t_n - s_n)
\end{aligned}$$

for all $n \in \mathbb{N}$ and $c := 2c_1c_2^2$. Equation (8.6) yields

$$\|S(s)f - S(t)f\|_\kappa \leq c|s - t| \quad \text{for all } s, t \in [0,T] \cap \mathcal{T}. \tag{8.7}$$

Now, let $t \in [0,T]$ be arbitrary and choose a sequence $(\tilde{t}_n)_{n\in\mathbb{N}} \subset [0,T] \cap \mathcal{T}$ with $\tilde{t}_n \to t$. Inequality (8.7) guarantees that the limit

$$S(t)f := \lim_{n\to\infty} S(\tilde{t}_n)f \in \mathrm{C}_\kappa(\mathbb{R}^d) \tag{8.8}$$

exists and does not dependent on the choice of the approximating sequence $(\tilde{t}_n)_{n\in\mathbb{N}}$. In order to extend $(S(t))_{t\geq 0}$ to the whole space $\mathrm{C}_\kappa(\mathbb{R}^d)$, we fix $t \geq 0$ and $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$. By Assumption 8.2.5(ii) and the previous construction, for every $\varepsilon > 0$, $r \geq 0$ and $K \Subset \mathbb{R}^d$, there exist $c \geq 0$ and $K' \Subset \mathbb{R}^d$ with

$$\|S(t)g_1 - S(t)g_2\|_{\infty,K} \leq c\|g_1 - g_2\|_{\infty,K'} + \varepsilon \tag{8.9}$$

for all $g_1, g_2 \in B_{\mathrm{C}_\kappa(\mathbb{R}^d)}(r) \cap \mathcal{D}$. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ be a sequence with $\|f_n\|_\kappa \leq \|f\|_\kappa$ for all $n \in \mathbb{N}$ and $f_n \to f$. Due to Assumption 8.2.5(i) and equation (8.8), the sequence $(S(t)f_n)_{n\in\mathbb{N}}$ is bounded. Hence, by inequality (8.9), the limit

$$S(t)f := \lim_{n\to\infty} S(t)f_n \in \mathrm{C}_\kappa(\mathbb{R}^d)$$

exists and does not dependent on the choice of the approximating sequence $(f_n)_{n\in\mathbb{N}}$. The family $(S(t))_{t\geq 0}$ consists of convex monotone operators $S(t)\colon \mathrm{C}_\kappa(\mathbb{R}^d) \to \mathrm{C}_\kappa(\mathbb{R}^d)$ which have the desired properties (ii) and (iii) such that $S(t)\colon \mathrm{Lip}_\mathrm{b}(\mathbb{R}^d) \to \mathrm{Lip}_\mathrm{b}(\mathbb{R}^d)$ for all $t \geq 0$. Indeed, these properties are an immediate consequence Assumption 8.2.5 and the previous construction.

Second, we show that the family $(S(t))_{t\geq 0}$ is strongly continuous and satisfies

$$S(t)f = \lim_{l\to\infty} S_{n_l}(t_{n_l})f \quad \text{for all } (f,t) \in \mathrm{C}_\kappa(\mathbb{R}^d) \times \mathcal{T}, \tag{8.10}$$

where $(t_{n_l})_{l\in\mathbb{N}}$ is the same sequence as in equation (8.6). To do so, let $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$, $t \geq 0$, $\varepsilon > 0$ and $K \Subset \mathbb{R}^d$. Furthermore, let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ be a sequence with $f_n \to f$ and $r := \sup_{n\in\mathbb{N}} \|f_n\|_\kappa$. Due to property (iii), there exist $c \geq 0$ and $K' \subset \mathbb{R}^d$ with

$$\|S(s)g_1 - S(s)g_2\|_{\infty,K} \leq c\|g_1 - g_2\|_{\infty,K'} + \tfrac{\varepsilon}{6}$$

for all $s \in [0, t+1]$ and $g_1, g_2 \in B_{\mathrm{C}_\kappa(\mathbb{R}^d)}(r)$. Subsequently, we fix $n \in \mathbb{N}$ with

$$6c\|f - f_n\|_{\infty,K'} \leq \varepsilon.$$

Due to inequality (8.7) and $\inf_{x\in K} \kappa(x) > 0$, there exists $\delta \in (0,1]$ with

$$\begin{aligned}
\|S(s)f - S(t)f\|_{\infty,K} &\leq \|S(s)f - S(s)f_n\|_{\infty,K} + \|S(s)f_n - S(t)f_n\|_{\infty,K} \\
&\quad + \|S(t)f_n - S(t)f\|_{\infty,K} \\
&\leq 2c\|f - f_n\|_{\infty,K'} + \|S(s)f_n - S(t)f_n\|_{\infty,K} < \varepsilon
\end{aligned}$$

for all $s \geq 0$ with $|s - t| < \delta$ which shows that the mapping $\mathbb{R}_+ \to \mathrm{C}_\kappa(\mathbb{R}^d)$, $t \mapsto S(t)f$ is continuous. In order to verify equation (8.10), we fix $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$, $t \in \mathcal{T}$, $\varepsilon > 0$ and $K \Subset \mathbb{R}^d$. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ be a sequence with $f_n \to f$ and define $r := \sup_{n\in\mathbb{N}} \|f_n\|_\kappa$. Due to Assumption 8.2.5(ii) and property (iii), there exist $c \geq 0$ and $K' \Subset \mathbb{R}^d$ with

$$\|S_n(t_n)g_1 - S_n(t_n)g_2\|_{\infty,K_n} \leq c\|g_1 - g_2\|_{\infty,K'} + \tfrac{\varepsilon}{6},$$

$$\|S(t)g_1 - S(t)g_2\|_{\infty,K} \leq c\|g_1 - g_2\|_{\infty,K'} + \tfrac{\varepsilon}{6} \tag{8.11}$$

for all $n \in \mathbb{N}$ and $g_1, g_2 \in B_{\mathrm{C}_\kappa(\mathbb{R}^d)}(r)$. Subsequently, we fix $k \in \mathbb{N}$ with

$$6c\|f - f_k\|_{\infty,K'} \leq \varepsilon.$$

Equation (8.6) yields $l_0 \in \mathbb{N}$ with

$$\|S(t)f_k - S_{n_l}(t_{n_l})f_k\|_{\infty,K_{n_l}} \leq \tfrac{\varepsilon}{3} \quad \text{for all } l \geq l_0. \tag{8.12}$$

We combine inequality (8.11) and inequality (8.12) to obtain

$$\begin{aligned}\|S(t)f - S_{n_l}(t_{n_l})f\|_{\infty,K_{n_l}} &\leq \|S(t)f - S(t)f_k\|_{\infty,K} + \|S(t)f_k - S_{n_l}(t_{n_l})f_k\|_{\infty,K_{n_l}} \\ &\quad + \|S_{n_l}(t_{n_l})f_k - S_{n_l}(t_{n_l})f\|_{\infty,K_{n_l}} \\ &\leq 2c\|f - f_k\|_{\infty,K'} + \|S(t)f_k - S_{n_l}(t_{n_l})f_k\|_{\infty,K_{n_l}} + \tfrac{\varepsilon}{3} < \varepsilon\end{aligned}$$

for all $l \geq l_0$ and therefore $S(t)f = \lim_{l\to\infty} S_{n_l}(t_{n_l})f$.

Third, for every $t \geq 0$ and $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$, we show that

$$S(t)f = \lim_{l\to\infty} S_{n_l}(t_{n_l})f_{n_l}, \tag{8.13}$$

where $(t_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ are arbitrary sequences with $t_n \in \mathcal{T}_n$ and $f_n \in \mathrm{C}_\kappa(X_n)$ for all $n \in \mathbb{N}$, $t_n \to t$ and $f_n \to f$. We define $T := \sup_{n\in\mathbb{N}} t_n$ and $r := \sup_{n\in\mathbb{N}} c_\kappa\|f_n\|_{\kappa,X_n}$, where c_κ is defined by equation (8.1). Let $\varepsilon > 0$ and $K \Subset \mathbb{R}^d$. Due to Assumption 8.2.5(ii), there exist $c \geq 0$ and $K' \Subset \mathbb{R}^d$ with

$$\begin{aligned}\|S_n(s)g_1 - S_n(s)g_2\|_{\infty,K_n} &\leq c\|g_1 - g_2\|_{\infty,K_n'} + \tfrac{\varepsilon}{7}, \\ \|S(t)g_1 - S(t)g_2\|_{\infty,K} &\leq c\|g_1 - g_2\|_{\infty,K'} + \tfrac{\varepsilon}{7}\end{aligned} \tag{8.14}$$

for all $n \in \mathbb{N}$, $s \in [0,T] \cap \mathcal{T}_n$ and $g_1, g_2 \in B_{\mathrm{C}_\kappa(X_n)}(r)$. Moreover, there exist $n_0 \in \mathbb{N}$ and $g \in B_{\mathrm{C}_\kappa(\mathbb{R}^d)}(r) \cap \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$ with

$$\max\{\|f - g\|_{\infty,K'}, \|f_n - g\|_{\infty,K_n'}\} \leq \tfrac{\varepsilon}{7c} \quad \text{for all } n \geq n_0. \tag{8.15}$$

Indeed, let $\eta\colon \mathbb{R}^d \to \mathbb{R}$ be an infinitely differentiable function with $\eta \geq 0$, $\operatorname{supp}(\eta) \subset B_{\mathbb{R}^d}(1)$ and $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$. For every $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$, we define

$$g_n(x) := \int_{\mathbb{R}^d} \big((f(x-y) \wedge n) \vee (-n)\big)\eta_n(y)\,\mathrm{d}y,$$

where $\eta_n(y) := n^d\eta(ny)$. It holds $g_n \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$, $\|f - g_n\|_{\infty,K'} \to 0$ and equation (8.1) implies $\|g_n\|_\kappa \leq c_\kappa\|f\|_\kappa \leq r$ for all $n \in \mathbb{N}$. In combination with $\|f - f_n\|_{\infty,K_n'} \to 0$ this proves equation (8.15). It follows from the proof of equation (8.7) and $\inf_{x\in K'} \kappa(x) > 0$ that there exists $\delta > 0$ with

$$\|S_n(s_1)g - S_n(s_2)g\|_{\infty,K_n} \leq \tfrac{\varepsilon}{7} \tag{8.16}$$

for all $n \in \mathbb{N}$ and $s_1, s_2 \in [0,T] \cap \mathcal{T}_n$ with $|s_1 - s_2| < \delta$. Furthermore, since $(S(t))_{t\geq 0}$ is strongly continuous, we can fix $s \in [0,T] \cap \mathcal{T}$ with $|s - t| < \delta$ and

$$\|S(s)g - S(t)g\|_{\infty,K} \leq \tfrac{\varepsilon}{7}. \tag{8.17}$$

By equation (8.10), there exist a sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n \in [0,T]\cap\mathcal{T}_n$ and $l_0 \in \mathbb{N}$ with $n_{l_0} \geq n_0$, $|s - t_{n_l}| < \delta$ and

$$\|S_{n_l}(s_{n_l})g - S(s)g\|_{\infty,K_{n_l}} \leq \tfrac{\varepsilon}{7} \quad \text{for all } l \geq l_0. \tag{8.18}$$

We combine the inequalities (8.14)-(8.18) to obtain

$$\begin{aligned} &\|S_{n_l}(t_{n_l})f_{n_l} - S(t)f\|_{\infty,K_{n_l}} \\ &\leq \|S_{n_l}(t_{n_l})f_{n_l} - S_{n_l}(t_{n_l})g\|_{\infty,K_{n_l}} + \|S_{n_l}(t_{n_l})g - S_{n_l}(s_{n_l})g\|_{\infty,K_{n_l}} \\ &\quad + \|S_{n_l}(s_{n_l})g - S(s)g\|_{\infty,K_{n_l}} + \|S(s)g - S(t)g\|_{\infty,K} + \|S(t)g - S(t)f\|_{\infty,K} \\ &\leq c\|f_{n_l} - g\|_{\infty,K'_{n_l}} + c\|f-g\|_{\infty,K} + \tfrac{5\varepsilon}{7} \leq \varepsilon \end{aligned}$$

for all $l \geq l_0$. This implies $S_{n_l}(t_{n_l})f_{n_l} \to S(t)f$.

Fourth, we show that $(S(t))_{t\geq 0}$ is a semigroup. Clearly, it holds $S(0) = \mathrm{id}_{\mathrm{C}_\kappa(\mathbb{R}^d)}$. In order to show the semigroup property, let $s,t \geq 0$ and $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$. Choose $s_n, t_n \in \mathcal{T}_n$ with $s_n \to s$ and $t_n \to t$. For every $l \in \mathbb{N}$,

$$\begin{aligned} &S(s+t)f - S(s)S(t)f \\ &= \big(S(s+t)f - S_{n_l}(s_{n_l}+t_{n_l})f\big) + \big(S_{n_l}(s_{n_l})S_{n_l}(t_{n_l})f - S_{n_l}(s_{n_l})S(t)f\big) \\ &\quad + \big(S_{n_l}(s_{n_l})S(t)f - S(s)S(t)f\big). \end{aligned}$$

Equation (8.13) implies that the first and third term on the right-hand side convergence to zero as $l \to \infty$. Moreover, for every $\varepsilon > 0$ and $K \Subset \mathbb{R}^d$, it follows from Assumption 8.2.5(i) and (ii) that there exist $c \geq 0$ and $K' \Subset \mathbb{R}^d$ with

$$\|S_{n_l}(s_{n_l})S_{n_l}(t_{n_l})f - S_{n_l}(s_{n_l})S(t)f\|_{\infty,K_{n_l}} \leq c\|S_{n_l}(t_{n_l})f - S(t)f\|_{\infty,K'_{n_l}} + \varepsilon$$

for all $l \in \mathbb{N}$. Hence, we can use equation (8.13) to obtain

$$\lim_{l\to\infty} \big(S_{n_l}(s_{n_l})S_{n_l}(t_{n_l})f - S_{n_l}(s_{n_l})S(t)f\big) = 0$$

and therefore $S(s+t)f = S(s)S(t)f$. In particular, for every $t \geq 0$ and $f \in \mathcal{L}^S$, there exist $c \geq 0$ and $h_0 > 0$ with

$$\|S(h)S(t)f - S(t)f\|_\kappa \leq c\|S(h)f - f\|_\kappa \leq ch \quad \text{for all } h \in (0,h_0]$$

which shows that $S(t)\colon \mathcal{L}^S \to \mathcal{L}^S$ for all $t \geq 0$. Furthermore, let $f \in \mathcal{L}^S_+$ and $t \geq 0$. By definition, there exists $h_0 > 0$ with

$$r := \sup_{h\in(0,h_0]} \left\| \frac{(S(h)f - f)^+}{h} \right\|_\kappa + \|f\|_\kappa < \infty$$

and the semigroup property yields $S(h)S(t)f = S(t)S(h)f$ for all $h \geq 0$. It follows from Lemma 3.6.1, the monotonicity and the uniform Lipschitz continuity, there exists $c \geq 0$ with

$$\left(\frac{S(h)S(t)f - S(t)f}{h}\right)\kappa \leq \left(S(t)\left(f + \frac{(S(h)f-f)^+}{h}\right) - S(t)f\right)\kappa$$

$$\leq c\left\|\frac{(S(h)f-f)^+}{h}\right\|_\kappa \leq cr \quad \text{for all } h\in(0,h_0].$$

This shows that $S(t)\colon \mathcal{L}_+^S \to \mathcal{L}_+^S$ for all $t\geq 0$.

Fifth, we show that, for every $f\in \mathrm{Lip_b}(\mathbb{R}^d)$ and $\varepsilon>0$, there exist $\delta, t_0>0$ with

$$\|S(t)(\tau_x f)-\tau_x S(t)f\|_\kappa \leq \varepsilon t$$

for all $t\in[0,t_0]$ and $x\in B_{\mathbb{R}^d}(\delta)$. Let $f\in \mathrm{Lip_b}(\mathbb{R}^d)$ and $\varepsilon>0$. By Assumption 8.2.5(iii), there exist $\delta, t_0>0$ and a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n\in \mathrm{C}_\kappa(X_n)$, $f_n\to f$ and

$$\|S_n(t)(\tau_x f_n)-\tau_x S_n(t)f_n\|_{\kappa,X_n}\leq \varepsilon t \tag{8.19}$$

for all $n\in\mathbb{N}$, $t\in[0,t_0]\cap\mathcal{T}_n$, $x\in B_{X_n}(\delta)$ and $y\in X_n$. Let $t\in[0,t_0]$, $x\in B_{\mathbb{R}^d}(\delta)$ and $y\in\mathbb{R}^d$. Choose $t_n\in[0,t_0]\cap\mathcal{T}_n$ and $x_n,y_n\in X_n$ with $t_n\to t$, $x_n\to x$ and $y_n\to y$. For every $l\in\mathbb{N}$,

$$\begin{aligned}
&|S(t)(\tau_x f)-\tau_x S(t)f|(y)\\
&\leq |(S(t)(\tau_x f))(y)-(S(t)(\tau_x f))(y_{n_l})| + |S(t)(\tau_x f)-S_{n_l}(t_{n_l})(\tau_{x_{n_l}}f)|(y_{n_l})\\
&\quad + |S_{n_l}(t_{n_l})(\tau_{x_{n_l}}f)-\tau_{x_{n_l}}S_{n_l}(t_{n_l})f|(y_{n_l}) + |\tau_{x_{n_l}}S_{n_l}(t_{n_l})f-\tau_{x_{n_l}}S(t)f|(y_{n_l})\\
&\quad + |(\tau_{x_{n_l}}S(t)f)(y_{n_l})-(\tau_x S(t)f)(y)|.
\end{aligned}$$

The first and the last term on the right-hand side converge to zero because $S(t)(\tau_x f)$ and $S(t)f$ are continuous functions while equation (8.13) guarantees that the second and fourth term vanish as well. Moreover, due to inequality (8.19), the third term can be bounded by $\frac{\varepsilon t_{n_l}}{\kappa(y_{n_l})}$ which converges to $\frac{\varepsilon t}{\kappa(y)}$. We obtain

$$|(S(t)(\tau_x f))(y)-(\tau_x S(t)f)(y)|\kappa(y)\leq \varepsilon t.$$

Sixth, we show that $\mathcal{L}_0\subset\mathcal{L}^S$ and $\mathcal{D}_0\subset D(A)$. In addition, for every $f\in\mathcal{D}_0$ and $K\Subset\mathbb{R}^d$, we prove that

$$\lim_{n\to\infty}\left\|\frac{S_n(h_n)f_n-f_n}{h_n}-Af\right\|_{\infty,K_n}=0, \tag{8.20}$$

where $(h_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ are arbitrary sequences satisfying the conditions (i) and (ii) in Definition 8.2.4. Inequality (8.2), Assumption 8.2.5(i) and equation (8.13) guarantee that $\mathcal{L}_0\subset\mathcal{L}^S$. In the sequel, we fix $f\in\mathcal{D}_0$ and $K\Subset\mathbb{R}^d$. Let $(f_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be sequences as in the definition of the asymptotic domain. Furthermore, we also fix $\varepsilon>0$ and define $T:=\sup_{n\in\mathbb{N}}t_n$. Assumption 8.2.5(i) and the fact that $\kappa>0$ is continuous yield $\lambda_0\in(0,1]$ with

$$\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]\cap\mathcal{T}_n}\lambda_0\|S_n(t)f_n\|_{\kappa,X_n}\leq \inf_{x\in K}\frac{\kappa(x)\varepsilon}{3}. \tag{8.21}$$

We fix $\lambda\in(0,\lambda_0]$ with $\lambda T\leq\lambda_0$ and use inequality (8.2) to define

$$r:=\sup_{n\in\mathbb{N}}\sup_{h\in(0,t_n]\cap\mathcal{T}_n}\left\|\frac{S_n(h)f_n-f_n}{\lambda h}+f_n\right\|_{\kappa,X_n}<\infty. \tag{8.22}$$

Assumption 8.2.5(ii) guarantees the existence of $c \geq 0$ and $K' \Subset \mathbb{R}^d$ with

$$\|S_n(t)g_1 - S_n(t)g_2\|_{\infty,K_n} \leq c\|g_1 - g_2\|_{\infty,K_n'} + \frac{\varepsilon}{3} \tag{8.23}$$

for all $n \in \mathbb{N}$, $t \in [0,T] \cap \mathcal{T}_n$ and $g_1, g_2 \in B_{\mathrm{C}_\kappa(X_n)}(r)$. Choose $g \in \mathrm{C}_\kappa(\mathbb{R}^d)$ as in the definition of the asymptotic domain and a sequence $(h_n)_{n\in\mathbb{N}}$ with $h_n \in (0, t_n] \cap \mathcal{T}_n$ and $h_n \to 0$ satisfying equation (8.3) with K'. In particular, there exists $n_0 \in \mathbb{N}$ with

$$\left\| \frac{S_n(h_n)f_n - f_n}{h_n} - g \right\|_{\infty,K_n'} \leq \frac{\varepsilon}{3c} \quad \text{for all } n \geq n_0. \tag{8.24}$$

In the sequel, all functions are evaluated at a fixed point $x \in K$. For every $n \geq n_0$ and $k \in \mathbb{N}$ with $kh_n \leq T$, we use the semigroup property of $(S_n(t))_{t\in\mathcal{T}_n}$, Lemma 3.6.1 and inequality (8.21) to estimate

$$\begin{aligned}
S_n(kh_n)f_n - f_n &= \sum_{i=1}^k \big(S_n((i-1)h_n)S_n(h_n)f_n - S_n((i-1)h_n)f_n\big) \\
&\leq \lambda h_n \sum_{i=1}^k \left(S_n((i-1)h_n) \left(\frac{S_n(h_n)f_n - f_n}{\lambda h_n} + f_n \right) - S_n((i-1)h_n)f_n \right) \\
&\leq \lambda h_n \sum_{i=1}^k S_n((i-1)h_n) \left(\frac{S_n(h_n)f_n - f_n}{\lambda h_n} + f_n \right) + \frac{kh_n\varepsilon}{3}.
\end{aligned}$$

Furthermore, it follows from equation (8.22) and inequality (8.23) that

$$\begin{aligned}
&\lambda h_n \sum_{i=1}^k S_n((i-1)h_n) \left(\frac{S_n(h_n)f_n - f_n}{\lambda h_n} + f_n \right) + \frac{kh_n\varepsilon}{3} \\
&= \lambda h_n \sum_{i=1}^k S_n((i-1)h_n) \left(\frac{1}{\lambda} g + f_n \right) + \frac{kh_n\varepsilon}{3} \\
&\quad + \lambda h_n \sum_{i=1}^k \left(S_n((i-1)h_n) \left(\frac{S_n(h_n)f_n - f_n}{\lambda h_n} + f_n \right) - S_n((i-1)h_n) \left(\frac{1}{\lambda} g + f_n \right) \right) \\
&\leq \lambda h_n \sum_{i=1}^k S_n((i-1)h_n) \left(\frac{1}{\lambda} g + f_n \right) + ckh_n \left\| \frac{S_n(h_n)f_n - f_n}{h_n} - g \right\|_{\infty,K_n'} + \frac{2kh_n\varepsilon}{3}.
\end{aligned}$$

Combining the previous estimates with inequality (8.24) yields

$$S_n(kh_n)f_n - f_n \leq \lambda h_n \sum_{i=1}^k S_n((i-1)h_n) \left(\frac{1}{\lambda} g + f_n \right) + kh_n\varepsilon. \tag{8.25}$$

Fix $t \in [0,T]$ and define $k_n := \max\{k \in \mathbb{N}_0 \colon kh_n \leq t\}$ for all $n \in \mathbb{N}$. In addition, we define $i_n^s := \max\{i \in \mathbb{N} \colon (i-1)h_n \leq s\}$ for all $s \in [0,t]$ and $n \in \mathbb{N}$. Since $k_n h_n \to t$ and $i_n^s \to s$ for all $s \in [0,t]$, we can use equation (8.13), inequality (8.25), Assumption 8.2.5(i) and the dominated convergence theorem to conclude

$$S(t)f - f = \lim_{l\to\infty} \big(S_{n_l}(k_{n_l}h_{n_l})f_{n_l} - f_{n_l}\big)$$

$$
\begin{aligned}
&\leq \lim_{l\to\infty} \lambda \int_0^t \sum_{i=1}^{k_{n_l}} S_{n_l}((i-1)h_{n_l}) \left(\frac{1}{\lambda} g + f_{n_l}\right) \mathbb{1}_{[(i-1)h_{n_l}, ih_{n_l})}(s)\,\mathrm{d}s + \varepsilon t \\
&= \lambda \int_0^t \lim_{l\to\infty} S_{n_l}((i_{n_l}^s - 1)h_{n_l}) \left(\frac{1}{\lambda} g + f_{n_l}\right) \mathrm{d}s + \varepsilon t \\
&= \lambda \int_0^t S(s) \left(\frac{1}{\lambda} g + f\right) \mathrm{d}s + \varepsilon t. \qquad (8.26)
\end{aligned}
$$

The previous estimate holds uniformly for all $\lambda \in (0, \lambda_0]$ with $\lambda T \leq \lambda_0$ and $t \in [0, T]$, where λ_0 depends on $\varepsilon > 0$ and $K \Subset \mathbb{R}^d$. Concerning the lower bound, we observe that the semigroup property and Lemma 3.6.1 imply

$$
\begin{aligned}
&S_n(kh_n)f_n - f_n \\
&= -\sum_{i=1}^{k} \big(S_n((i-1)h_n)f_n - S_n((i-1)h_n)S_n(h_n)f_n\big) \\
&\geq -\lambda h_n \sum_{i=1}^{k} \left(S_n((i-1)h_n) \left(S_n(h_n)f_n - \frac{S_n(h_n)f_n - f_n}{\lambda h_n} \right) - S_n(ih_n)f_n \right)
\end{aligned}
$$

for all $k, n \in \mathbb{N}$ and $\lambda > 0$ with $\lambda h_n \leq 1$. Hence, similar to the previous arguments, one can show that there exists $\lambda_1 \in (0, 1]$ with

$$
S(t)f - f \geq -\lambda \int_0^t S(s) \left(-\frac{1}{\lambda} g + f\right) \mathrm{d}s - \varepsilon t \qquad (8.27)
$$

for all $\lambda \in (0, \lambda_1]$ with $\lambda T \leq \lambda_1$ and $t \in [0, T]$, where λ_1 also depends on $\varepsilon > 0$ and $K \Subset \mathbb{R}^d$. Since inequality (8.26) and inequality (8.27) hold uniformly for all $x \in K$, it follows from the strong continuity of $(S(t))_{t\geq 0}$ that

$$
\lim_{h\downarrow 0} \left\| \frac{S(h)f - f}{h} - g \right\|_{\infty, K} = 0.
$$

Note that the function $g \in \mathrm{C}_\kappa(\mathbb{R}^d)$ does neither depend on the choice of $K' \Subset \mathbb{R}^d$ nor on $\varepsilon > 0$. Since $K \Subset \mathbb{R}^d$ was arbitrary, we obtain $f \in D(A)$ with $Af = g$.

Sixth, we verify equation (8.4) by showing that the semigroup $(S(t))_{t\geq 0}$ does not depend on the choice of the convergent subsequence in equation (8.6). For every subsequence $(\tilde{n}_k)_{k\in\mathbb{N}} \subset \mathbb{N}$, there exist a further subsequence $(\tilde{n}_{k_l})_{l\in\mathbb{N}}$ and a strongly continuous convex monotone semigroup $(\tilde{S}(t))_{t\geq 0}$ which has the properties (i)-(v) and satisfies

$$
\tilde{S}(t)f = \lim_{l\to\infty} S_{\tilde{n}_{k_l}}\big(t_{\tilde{n}_{k_l}}\big) f_{\tilde{n}_{k_l}} \quad \text{for all } (f, t) \in \mathrm{C}_\kappa(\mathbb{R}^d) \times \mathbb{R}_+,
$$

where $(t_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ are arbitrary sequences with $t_n \in \mathcal{T}_n$ and $f_n \in \mathrm{C}_\kappa(X_n)$ for all $n \in \mathbb{N}$, $t_n \to t$ and $f_n \to f$. Hence, we can apply Theorem 4.4.6 to obtain

$$
S(t)f = \tilde{S}(t)f \quad \text{for all } (f, t) \in \mathrm{C}_\kappa(\mathbb{R}^d) \times \mathbb{R}_+.
$$

Since every subsequence has a further subsequence which converges to a limit that is independent of the choice of the subsequence, we conclude

$$
S(t)f = \lim_{n\to\infty} S_n(t_n)f_n \quad \text{for all } (f, t) \in \mathrm{C}_\kappa(\mathbb{R}^d) \times \mathbb{R}_+,
$$

where $(t_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ are arbitrary sequences with $t_n \in \mathcal{T}_n$ and $f_n \in \mathrm{C}_\kappa(X_n)$ for all $n\in\mathbb{N}$, $t_n \to t$ and $f_n \to f$. □

Theorem 8.2.3 follows from Theorem 8.2.6 by verifying $\mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d) \subset \mathcal{D}_0$.

Proof of Theorem 8.2.3. Let $f\in \mathrm{C}_\kappa(\mathbb{R}^d)$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence with $f_n\in D(A_n)$ for all $n\in\mathbb{N}$ and $f_n\to f$ such that the sequence $(A_nf_n)_{n\in\mathbb{N}}$ converges in $\mathrm{C}_\kappa(\mathbb{R}^d)$. For every $n\in\mathbb{N}$ and $t\geq 0$, it follows from Lemma 4.4.2 that

$$S_n(t)f_n - f_n \leq \int_0^t \big(S_n(s)(f_n + A_nf_n) - S_n(s)f_n\big)\,\mathrm{d}s. \tag{8.28}$$

In the sequel, we argue similarly to obtain a lower bound. Due to $f_n\in D(A_n)\subset \mathcal{L}^{S_n}$ and Assumption 8.2.2(i), the mapping $[0,\infty)\to\mathbb{R}$, $t\mapsto (S_n(t)f_n)(x)$ is Lipschitz continuous and therefore satisfies

$$(S_n(t)f_n - f_n)(x) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(S_n(s)f_n)(x)\,\mathrm{d}s \quad \text{for all } t\geq 0 \text{ and } x\in\mathbb{R}^d.$$

Moreover, for every $n\in\mathbb{N}$, $t\geq 0$ and $h\in(0,1]$, we can use Lemma 3.6.1 to estimate

$$\begin{aligned}
&\frac{S_n(t)f_n - S_n(t)S_n(h)f_n}{h} - S_n(t)(f_n - A_nf_n) - S_n(t)f_n \\
&\leq S_n(t)\left(\frac{f_n - S_n(h)f_n}{h} + S_n(h)f_n\right) - S_n(t)S_n(h)f_n + S_n(t)f_n - S_n(t)(f_n - A_nf_n) \\
&\leq \frac{1}{2}S_n(t)\left(2\left(S_n(h)f_n - f_n + A_nf_n - \frac{S_n(h)f_n - f_n}{h}\right) + f_n - A_nf_n\right) \\
&\quad - \frac{1}{2}S_n(t)(f_n - A_nf_n) - S_n(t)S_n(h)f_n + S_n(t)f_n.
\end{aligned}$$

The strong continuity of $(S_n(t))_{t\geq 0}$, Assumption 8.2.2(ii) and $f_n\in D(A_n)$ imply that the right-hand side converges to zero as $h\to 0$. Hence,

$$S_n(t)f_n - f_n \geq -\int_0^t \big(S_n(s)(f_n - A_nf_n) - S_n(s)f_n\big)\,\mathrm{d}s \tag{8.29}$$

for all $n\in\mathbb{N}$ and $t\geq 0$. Since $(A_nf_n)_{n\in\mathbb{N}}$ is bounded, it follows from inequality (8.28), inequality (8.29) and Assumption 8.2.2(i) that there exists $c\geq 0$ with

$$\|S_n(t)f_n - f_n\|_\kappa \leq ct \quad \text{for all } n\in\mathbb{N} \text{ and } t\in[0,1].$$

This shows that $f\in\mathcal{L}_0$. Furthermore, the limit $g:=\lim_{n\to\infty}A_nf_n\in \mathrm{C}_\kappa(\mathbb{R}^d)$ exists by assumption. Hence, for every $n\in\mathbb{N}$ and $K\Subset\mathbb{R}^d$, we can choose $h_n\in(0,1]$ with

$$\begin{aligned}
\left\|\frac{S_n(h_n)f_n - f_n}{h_n} - g\right\|_{\infty,K} &\leq \left\|\frac{S_n(h_n)f_n - f_n}{h_n} - A_nf_n\right\|_{\infty,K} + \|A_nf_n - g\|_{\infty,K} \\
&\leq \|A_nf_n - g\|_{\infty,K} + \tfrac{1}{n} \to 0.
\end{aligned}$$

This shows that $f\in\mathcal{D}_0$. In particular, Assumption 8.2.2(v) implies $\mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)\subset\mathcal{D}_0$. Now, the claim follows from Theorem 8.2.6. □

8.2.2 Chernoff-type approximations

Subsequently, we focus on the case, where $\mathcal{T}_n$ is an equidistant grid. For this purpose, let $(h_n)_{n\in\mathbb{N}} \subset (0,\infty)$ be a sequence with $h_n \to 0$ and define $\mathcal{T}_n := \{kh_n \colon k \in \mathbb{N}_0\}$ for all $n \in \mathbb{N}$. Furthermore, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of closed sets $X_n \subset \mathbb{R}^d$ with $x+y \in X_n$ for all $x, y \in X_n$ such that, for every $x \in \mathbb{R}^d$, there exist $x_n \in X_n$ with $x_n \to x$. Then, every semigroup $(S(t))_{t\in\mathcal{T}_n}$ is of the form

$$S_n(kh_n) = I_n^k f = \underbrace{(I_n \circ \ldots \circ I_n)}_{k \text{ times}} f \quad \text{with} \quad I_n f := S(h_n) f,$$

where $I_n^0 f := f$. Hence, the corresponding limit semigroup $(S(t))_{t\geq 0}$ from Theorem 8.2.6 is obtained by iterating a sequence $(I_n)_{n\in\mathbb{N}}$ of one-step operators. In many applications, the latter have an explicit representation from which one can derive properties of the one-step operators that are transferred to the limit semigroup. In the case $X_n = \mathbb{R}^d$, this procedure has been used to construct and approximate nonlinear semigroups, see Chapter 1, Chapter 5, Chapter 6, Chapter 7 and the references therein. Now, due to the possibility of discretizing the space via a sequence of subsets $X_n \subset \mathbb{R}^d$, we can show that, under reasonable stability and consistency conditions, every approximation scheme consisting of convex monotone operators converges to a convex monotone semigroup. Moreover, the latter is uniquely determined by its infinitesimal generator which satisfies

$$Af = \lim_{n\to\infty} \frac{I_n f - f}{h_n} \quad \text{for all } f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d).$$

In the sequel, we consider a sequence $(I_n)_{n\in\mathbb{N}}$ of operators $I_n \colon \mathrm{C}_\kappa(X_n) \to \mathrm{C}_\kappa(X_n)$ and impose the following assumption. Note that, apart from condition (ii), all the required properties are preserved during the iteration. Moreover, sufficient conditions on the one-step operators I_n which guarantee that the iterated operators I_n^k are uniformly equicontinuous on bounded sets w.r.t. the mixed topology have been discussed in Section 3.4.

Assumption 8.2.7. The operators $(I_n)_{n\in\mathbb{N}}$ are convex and monotone with $I_n 0 = 0$ for all $n \in \mathbb{N}$. Furthermore, they satisfy the following conditions:

(i) There exists $\omega \geq 0$ with

$$\|I_n f - I_n g\|_{\kappa,X_n} \leq e^{\omega h_n} \|f - g\|_{\kappa,X_n} \quad \text{for all } n \in \mathbb{N} \text{ and } f, g \in \mathrm{C}_\kappa(X_n).$$

(ii) For every $\varepsilon > 0$, $r, T \geq 0$ and $K \Subset \mathbb{R}^d$, there exist $c \geq 0$ and $K' \Subset \mathbb{R}^d$ with

$$\|I_n^k f - I_n^k g\|_{\infty,K_n} \leq c\|f - g\|_{\infty,K_n'} + \varepsilon$$

for all $k, n \in \mathbb{N}$ with $kh_n \leq T$ and $f, g \in B_{\mathrm{C}_\kappa(X_n)}(r)$.

(iii) There exist $\delta_0 \in (0,1]$ and $L \geq 0$ with

$$\|I_n(\tau_x f) - \tau_x I_n f\|_{\kappa,X_n} \leq Lrh_n|x|$$

for all $n \in \mathbb{N}$, $x \in B_{X_n}(\delta_0)$, $r \geq 0$ and $f \in \mathrm{Lip}_\mathrm{b}(X_n, r)$.

(iv) It holds $I_n\colon \mathrm{Lip_b}(X_n, r) \to \mathrm{Lip_b}(X_n, e^{\omega h_n} r)$ for all $r \geq 0$ and $n \in \mathbb{N}$.

(v) For every $f \in \mathrm{C_b^\infty}(\mathbb{R}^d)$, there exists $g \in \mathrm{C}_\kappa(\mathbb{R}^d)$ with $g = \lim_{n\to\infty} \frac{I_n f - f}{h_n}$.

We recall from Subsection 8.2.1 that the convergence in condition (v) means that

$$\sup_{n\in\mathbb{N}} \left\| \frac{I_n f - f}{h_n} \right\|_{\kappa, X_n} < \infty \quad \text{and} \quad \lim_{n\to\infty} \left\| \frac{I_n f - f}{h_n} - g \right\|_{\infty, K_n} = 0 \quad \text{for all } K \Subset \mathbb{R}^d.$$

We obtain the following result that will be frequently used in the rest of chapter.

Theorem 8.2.8. *Let $(I_n)_{n\in\mathbb{N}}$ be a sequence satisfying Assumption 8.2.7. Then, there exists a strongly continuous convex monotone semigroup $(S(t))_{t\geq 0}$ on $\mathrm{C}_\kappa(\mathbb{R}^d)$ with $S(t)0 = 0$ such that*

$$S(t)f = \lim_{n\to\infty} I_n^{k_n^t} f_n \quad \textit{for all } (f,t) \in \mathrm{C}_\kappa(\mathbb{R}^d) \times \mathbb{R}_+,$$

where $(k_n^t)_{n\in\mathbb{N}} \subset \mathbb{N}$ and $(f_n)_{n\in\mathbb{N}}$ with $f_n \in \mathrm{C}_\kappa(X_n)$ are arbitrary sequences satisfying $k_n^t h_n \to t$ and $f_n \to f$. Moreover, the following statements are valid:

(i) It holds $\mathrm{C_b^\infty}(\mathbb{R}^d) \subset D(A)$ and

$$Af = \lim_{n\to\infty} \frac{I_n f - f}{h_n} \quad \textit{for all } f \in \mathrm{C_b^\infty}(\mathbb{R}^d).[1]$$

(ii) It holds $\|S(t)f - S(t)g\|_\kappa \leq e^{\omega t}\|f - g\|_\kappa$ for all $t \geq 0$ and $f, g \in \mathrm{C}_\kappa(\mathbb{R}^d)$.

(iii) For every $\varepsilon > 0$, $r, T \geq 0$ and $K \Subset \mathbb{R}^d$, there exist $c \geq 0$ and $K' \Subset \mathbb{R}^d$ with

$$\|S(t)f - S(t)g\|_{\infty,K} \leq c\|f - g\|_{\infty,K'} + \varepsilon$$

for all $t \in [0,T]$ and $f, g \in B_{\mathrm{C}_\kappa(\mathbb{R}^d)}(r)$.

(iv) For every $r, t \geq 0$, $f \in \mathrm{Lip_b}(r)$ and $x \in B_{\mathbb{R}^d}(\delta_0)$,

$$\|S(t)(\tau_x f) - \tau_x S(t)f\|_\kappa \leq Lrte^{\omega t}|x|.$$

Furthermore, it holds $S(t)\colon \mathrm{Lip_b}(r) \to \mathrm{Lip_b}(e^{\omega t} r)$ for all $r, t \geq 0$.

Proof. We define $S_n(t) := I_n^k f$ for all $n \in \mathbb{N}$, $t := kh_n \in \mathcal{T}_n$ and $f \in \mathrm{C}_\kappa(X_n)$. In order to apply Theorem 8.2.6, we have to verify Assumption 8.2.5. The sequence $(S_n)_{n\in\mathbb{N}}$ consists of convex monotone semigroups on $\mathrm{C}_\kappa(X_n)$ with $S_n(t)0 = 0$ and Assumption 8.2.5(ii) is valid due to Assumption 8.2.7(ii). For every $k, n \in \mathbb{N}$ and $f, g \in \mathrm{C}_\kappa(X_n)$, it follows by induction from Assumption 8.2.7(i) that

$$\|I_n^k f - I_n^k g\|_{\kappa,X_n} \leq e^{\omega k h_n}\|f - g\|_{\kappa,X_n} \tag{8.30}$$

showing that Assumption 8.2.5(i) is satisfied. Next, we show by induction that

$$\|I_n^k(\tau_x f) - \tau_x I_n^k f\|_{\kappa,X_n} \leq Lrkh_n e^{\omega k h_n}|x|$$

[1] The statement is valid for all $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$ such that the limit $g := \lim_{n\to\infty} \frac{I_n f - f}{h_n} \in \mathrm{C}_\kappa$ exists.

for all $k, n \in \mathbb{N}$, $x \in B_{X_n}(\delta_0)$, $r \geq 0$ and $f \in \mathrm{Lip_b}(X_n, r)$. For $k = 1$, the claim holds due to Assumption 8.2.7(iii). Moreover, for the induction step, we use inequality (8.30) and Assumption 8.2.7(iv) to obtain

$$\begin{aligned}
&\|I_n^{k+1}(\tau_x f) - \tau_x I_n^{k+1} f\|_{\kappa,X_n} \\
&\leq \|I_n^k I_n(\tau_x f) - I_n^k(\tau_x I_n f)\|_{\kappa,X_n} + \|I_n^k(\tau_x I_n f) - \tau_x I_n^k I_n f\|_{\kappa,X_n} \\
&\leq e^{\omega k h_n}\|I_n(\tau_x f) - \tau_x I_n f\|_{\kappa,X_n} + Le^{\omega h_n} r k h_n e^{\omega k h_n}|x| \\
&\leq e^{\omega k h_n} L r h_n |x| + L r k h_n e^{\omega(k+1)k_n}|x| \leq Lr(k+1)h_n e^{\omega(k+1)h_n}|x|.
\end{aligned}$$

Hence, Assumption 8.2.5(iii) is satisfied and Assumption 8.2.5(iv) and (v) are an immediate consequence of Assumption 8.2.7(iv) and (v). Now, Theorem 8.2.6 yields the claim. □

8.3 Examples

8.3.1 Euler formula and Yosida approximation for upper envelopes

In this subsection, we derive an Euler formula and a Yosida-type approximation for upper envelopes of families of linear semigroups. The construction uses the supremum of linear resolvents rather than the resolvent of the supremum of linear operators as it is common both in the classical theory of m-accretive operators and the theory of viscosity solutions. We refer to [34] and [121] for an Euler formula for bi-continuous linear semigroups as well as to [40] for a Hille–Yosida theorem for weakly continuous linear semigroups on the space of bounded uniformly continuous functions and to [141] for the classical Yosida approximation. To that end, we consider a non-empty family $(A^\theta)_{\theta\in\Theta}$ of linear operators $A^\theta\colon D(A^\theta) \to \mathrm{C}_\kappa(\mathbb{R}^d)$ with domain $D(A^\theta) \subset \mathrm{C}_\kappa(\mathbb{R}^d)$ and resolvent $\varrho(A^\theta) \subset \mathbb{C}$. In the sequel, we provide certain nonlinear versions of the classical Euler formula and the Yosida approximation in order to construct a strongly continuous convex monotone semigroup $(S(t))_{t\geq 0}$ on $\mathrm{C}_\kappa(\mathbb{R}^d)$ whose generator is given by

$$Af = \sup_{\theta\in\Theta} A^\theta f \quad \text{for all } f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d).$$

Assumption 8.3.1. Suppose that the conditions (i)–(iv) or the conditions (i), (ii'), (iii) and (iv) from the following list are satisfied:

(i) There exists $\omega \in \mathbb{R}$ with $(\omega, \infty) \subset \bigcap_{\theta\in\Theta} \varrho(A^\theta)$ such that, for every $\lambda \in (\omega, \infty)$ and $\theta \in \Theta$,

$$\begin{aligned}
\|(\lambda-\omega)(\lambda - A^\theta)^{-1} f\|_\infty &\leq \|f\|_\infty \quad \text{for all } f \in \mathrm{Lip_b}(\mathbb{R}^d), \\
\|(\lambda-\omega)(\lambda - A^\theta)^{-1} f\|_\kappa &\leq \|f\|_\kappa \quad \text{for all } f \in \mathrm{C}_\kappa(\mathbb{R}^d).
\end{aligned}$$

In addition, the operators $(\lambda - A^\theta)^{-1}\colon \mathrm{C}_\kappa(\mathbb{R}^d) \to \mathrm{C}_\kappa(\mathbb{R}^d)$ are monotone for all $\lambda \in (\omega, \infty)$ and $\theta \in \Theta$.

(ii) There exists a bounded continuous function $\tilde{\kappa}\colon \mathbb{R}^d \to (0, \infty)$ with

$$\lim_{|x|\to\infty} \frac{\tilde{\kappa}(x)}{\kappa(x)} = 0 \quad \text{and} \quad \|(\lambda-\omega)(\lambda - A^\theta)^{-1} f\|_{\tilde{\kappa}} \leq \|f\|_{\tilde{\kappa}}$$

for all $\lambda \in (\omega, \infty)$ and $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$.

(ii') For every $\varepsilon > 0$, $r \geq 0$ and $K \Subset \mathbb{R}^d$, there exist a family $(\zeta_x)_{x\in K}$ of continuous functions $\zeta_x\colon \mathbb{R}^d \to \mathbb{R}$ and $\tilde{K} \Subset \mathbb{R}^d$ such that

(a) $0 \leq \zeta_x \leq 1$ and $\varphi_x(x) = 1$ for all $x \in K$,

(b) $\sup_{y\in\tilde{K}^c} \zeta_x(y) \leq \varepsilon$ for all $x \in K$,

(c) $\frac{r}{\kappa}(1-\zeta_x) \in \bigcap_{\theta\in\Theta} D(A^\theta)$ and $\sup_{\theta\in\Theta} \|A^\theta(\frac{r}{\kappa}(1-\zeta_x))\|_\kappa \leq \varepsilon$ for all $x \in K$.

(iii) There exists $L \geq 0$ with

$$(\lambda-\omega)^2\|\tau_x(\lambda-A^\theta)^{-1}f - (\lambda-A^\theta)^{-1}(\tau_x f)\|_\infty \leq Lr|x|$$

for all $r \geq 0$, $f \in \mathrm{Lip_b}(\mathbb{R}^d, r)$, $x \in \mathbb{R}^d$, $\lambda \in (\omega,\infty)$ and $\theta \in \Theta$.

(iv) It holds $\mathrm{C_b^\infty}(\mathbb{R}^d) \subset \bigcap_{\theta\in\Theta} D(A^\theta)$. Moreover, for every $f \in \mathrm{C_b^\infty}(\mathbb{R}^d)$ and $K \Subset \mathbb{R}^d$,

$$\sup_{\theta\in\Theta} \|A^\theta f\|_\kappa < \infty \quad \text{and} \quad \lim_{\lambda\to\infty} \|(\lambda-\omega)(\lambda-A^\theta)^{-1}A^\theta f - A^\theta f\|_{\infty,K} = 0.$$

The previous conditions are, for instance, satisfied for suitably bounded families of generators of Lévy processes, Ornstein–Uhlenbeck processes and geometric Brownian motions. In these cases, it is straightforward to verify the respective conditions for the transition semigroups and use the Laplace transform in order to transfer them to the resolvents. For a detailed illustration how the conditions can be verified for a large class of transition semigroups, we refer to [136, Section 6.3 and Section 7]. At this point we would like to emphasise that the nonlinear semigroups in [136] are not constructed via the resolvents but as so-called Nisio semigroups. Furthermore, due to Lemma 3.4.5 and Corollary 3.4.7, the conditions (ii) and (ii') both guarantee that Assumption 8.2.7(ii) is valid. While condition (ii) is satisfied for Ornstein–Uhlenbeck processes and geometric Brownian motions, it does not cover Lévy processes with non-integrable jumps. On the other hand, condition (ii') applies to Lévy processes with possibly non-integrable jumps but neither to Ornstein–Uhlenbeck processes nor geometric Brownian motions. In the sequel, let $(\lambda_n)_{n\in\mathbb{N}} \subset (\omega,\infty)$ be a sequence with $\lambda_n \to \infty$ and

$$h_n := \frac{1}{\lambda_n - \omega} \quad \text{for all } n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$ and $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$, we define

$$I_n f := \sup_{\theta\in\Theta} \lambda_n(\lambda_n - A^\theta)^{-1} f.$$

Moreover, let $X_n := \mathbb{R}^d$ and $\mathcal{T}_n := \{kh_n \colon k \in \mathbb{N}_0\}$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ and $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$, the resolvent identity implies

$$I_n f = f + \sup_{\theta\in\Theta} A^\theta(\lambda_n - A^\theta)^{-1} f.$$

Moreover, for every $f \in \bigcap_{\theta\in\Theta} D(A^\theta)$, it follows that

$$I_n f = f + \sup_{\theta\in\Theta} (\lambda_n - A^\theta)^{-1} A^\theta f. \tag{8.31}$$

Theorem 8.3.2. *Suppose that Assumption 8.3.1 is satisfied. Then, there exists a strongly continuous convex monotone semigroup $(S(t))_{t\geq 0}$ on $\mathrm{C}_\kappa(\mathbb{R}^d)$ with $S(t)0=0$ given by*

$$S(t)f = \lim_{n\to\infty} I_n^{k_n^t} f \quad \textit{for all } (f,t)\in \mathrm{C}_\kappa(\mathbb{R}^d)\times\mathbb{R}_+,$$

where $(k_n^t)_{n\in\mathbb{N}}\subset\mathbb{N}$ is an arbitrary sequence satisfying $k_n^t h_n\to t$, such that

$$\mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)\subset D(A) \quad \textit{and} \quad Af=\sup_{\theta\in\Theta} A^\theta f \quad \textit{for all } f\in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d).$$

Moreover, Assumption 4.4.5 is satisfied.

Proof. In order to apply Theorem 8.2.8, we have to verify Assumption 8.2.7. First, we show that $I_n\colon \mathrm{Lip_b}(\mathbb{R}^d,r)\to\mathrm{Lip_b}(\mathbb{R}^d,e^{\beta h_n}r)$ for all $n\in\mathbb{N}$ and $r\geq 0$, where

$$\beta := \omega+\frac{L\lambda_0}{\lambda_0-\omega} \quad \text{and} \quad \lambda_0:=\min_{n\in\mathbb{N}}\lambda_n>\omega.$$

For every $n\in\mathbb{N}$ and $f,g\in\mathrm{Lip_b}(\mathbb{R}^d)$, it follows from Assumption 8.3.1(i) that

$$\|I_nf-I_ng\|_\infty\leq\frac{\lambda_n}{\lambda_n-\omega}\|f-g\|_\infty=(1+\omega h_n)\|f-g\|_\infty\leq e^{\omega h_n}\|f-g\|_\infty.$$

Moreover, it holds $I_n0=0$ and Assumption 8.3.1(iii) implies

$$\|I_n(\tau_xf)-\tau_xI_nf\|_\infty\leq\frac{L\lambda_n}{\lambda_n-\omega}rh_n|x|\leq\frac{L\lambda_0}{\lambda_0-\omega}rh_n|x| \tag{8.32}$$

for all $n\in\mathbb{N}$, $r\geq0$, $f\in\mathrm{Lip_b}(\mathbb{R}^d,r)$ and $x\in\mathbb{R}^d$. Hence, for every $x,y\in\mathbb{R}^d$,

$$\begin{aligned}|(I_nf)(x+y)-(I_nf)(y)| &\leq |\tau_xI_nf-I_n(\tau_xf)|(y)+|I_n(\tau_xf)-I_nf|(y)\\ &\leq\frac{\lambda_0}{\lambda_0-\omega}Lrh_n|x|+(1+\omega h_n)r|x|\\ &=\left(1+\left(\omega+\frac{L\lambda_0}{\lambda_0-\omega}\right)h_n\right)r|x|\leq e^{\beta h_n}r|x|.\end{aligned}$$

We obtain that $I_nf\in\mathrm{Lip_b}(\mathbb{R}^d,e^{\beta h_n}r)$ and therefore Assumption 8.2.7(iv) is satisfied. Furthermore, inequality (8.32) and the fact that $\|\kappa\|_\infty<\infty$ yield Assumption 8.2.7(iii).

Second, we show that $I_n\colon\mathrm{C}_\kappa(\mathbb{R}^d)\to\mathrm{C}_\kappa(\mathbb{R}^d)$ for all $n\in\mathbb{N}$. Assumption 8.3.1(i) yields

$$\|I_nf-I_ng\|_\kappa\leq\frac{\lambda_n}{\lambda_n-\omega}\|f-g\|_\kappa=(1+\omega h_n)\|f-g\|_\kappa\leq e^{\omega h_n}\|f-g\|_\kappa$$

and thus Assumption 8.2.7(i) is satisfied. Moreover, the operators $I_n\colon\mathrm{C}_\kappa(\mathbb{R}^d)\to\mathrm{F}_\kappa(\mathbb{R}^d)$ are convex and monotone with $I_n0=0$. Let $f\in\mathrm{C}_\kappa(\mathbb{R}^d)$ and $(f_k)_{k\in\mathbb{N}}\subset\mathrm{Lip_b}(\mathbb{R}^d)$ be a sequence with $f_k\to f$. In case that Assumption 8.3.1(ii) is valid, Lemma 3.4.1 implies $I_nf=\lim_{k\to\infty}I_nf_k\in\mathrm{C}_\kappa(\mathbb{R}^d)$ and thus $I_n\colon\mathrm{C}_\kappa(\mathbb{R}^d)\to\mathrm{C}_\kappa(\mathbb{R}^d)$. In addition, it follows by induction that $\|I_n^kf\|_{\tilde\kappa}\leq e^{\omega kh_n}\|f\|_{\tilde\kappa}$ for all $k,n\in\mathbb{N}$ and $f\in\mathrm{C}_\kappa(\mathbb{R}^d)$. Hence, we can apply Lemma 3.4.5 to obtain that Assumption 8.2.7(ii) is satisfied. If

Assumption 8.3.1(ii') is valid, we can use Corollary 3.4.7 and Remark 3.4.11 instead. Indeed, equation (8.31) and Assumption 8.3.1(i) imply

$$\begin{aligned}\big\|I_n\big(\tfrac{r}{\kappa}(1-\zeta_x)\big)-\tfrac{r}{\kappa}(1-\zeta_x)\big\|_\kappa &\le \sup_{\theta\in\Theta}\big\|(\lambda_n-A^\theta)^{-1}A^\theta\big(\tfrac{r}{\kappa}(1-\zeta_x)\big)\big\|_\kappa \\ &\le \sup_{\theta\in\Theta}\big\|A^\theta\big(\tfrac{r}{\kappa}(1-\zeta_x)\big)\big\|_\kappa h_n \le \varepsilon h_n.\end{aligned}$$

Third, we verify Assumption 8.2.7(v). Let $f\in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$ and define

$$A_n f := \frac{I_n f - f}{h_n} \quad \text{for all } n\in\mathbb{N}.$$

For every $n\in\mathbb{N}$, it follows from equation (8.31) that

$$A_n f = \sup_{\theta\in\Theta}\big((\lambda_n-\omega)\lambda_n(\lambda_n-A^\theta)^{-1}f\big)-(\lambda_n-\omega) = \sup_{\theta\in\Theta}(\lambda_n-\omega)(\lambda_n-A^\theta)^{-1}A^\theta f.$$

Hence, for every $n\in\mathbb{N}$ and $K\Subset\mathbb{R}^d$, Assumption 8.3.1(i) yields

$$\begin{aligned}\|A_n f\|_\kappa &\le \sup_{\theta\in\Theta}\|(\lambda_n-\omega)(\lambda_n-A^\theta)^{-1}A^\theta f\|_\kappa \le \sup_{\theta\in\Theta}\|A^\theta f\|_\kappa, \\ \|A_n f - Af\|_{\infty,K} &\le \sup_{\theta\in\Theta}\|(\lambda_n-\omega)(\lambda_n-A^\theta)^{-1}A^\theta f - A^\theta f\|_{\infty,K}.\end{aligned}$$

and Assumption 8.3.1(iv) guarantees that Assumption 8.2.7(v) is satisfied. Now, the claim follows from Theorem 8.2.8, where property (iv) holds with the constants L and $\omega+L$ rather than $\frac{L\lambda_0}{\lambda_0-\omega}$ and β because of the fact that $\lambda_n\to\infty$. □

For the Yosida approximation, we additionally have to require norm convergence of the generators in condition (iv).

Assumption 8.3.3. Suppose that the conditions (i)–(iii) or the conditions (i), (ii') and (iii) from Assumption 8.3.1 are satisfied. Furthermore, the following statement is valid:

(iv') It holds $\mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)\subset\bigcap_{\theta\in\Theta}D(A^\theta)$. Moreover, for every $f\in\mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$,

$$\sup_{\theta\in\Theta}\|A^\theta f\|_\kappa<\infty \quad\text{and}\quad \lim_{\lambda\to\infty}\sup_{\theta\in\Theta}\|(\lambda-\omega)(\lambda-A^\theta)^{-1}A^\theta f - A^\theta f\|_\kappa = 0.$$

For every $n\in\mathbb{N}$ and $\theta\in\Theta$, we define

$$A_n^\theta := (\lambda_n-\omega)A^\theta(\lambda_n-A^\theta)^{-1}.$$

Since $A_n^\theta\colon \mathrm{C}_\kappa(\mathbb{R}^d)\to\mathrm{C}_\kappa(\mathbb{R}^d)$ is a bounded linear operator, we can define

$$J_n f := \sup_{\theta\in\Theta} e^{h_n A_n^\theta}f \quad \text{for all } n\in\mathbb{N} \text{ and } f\in\mathrm{C}_\kappa(\mathbb{R}^d),$$

where the operator exponential is given as the power series

$$e^{h_n A_n^\theta} := \sum_{k=1}^\infty \frac{(h_n A_n^\theta)^k}{k!}.$$

Since the resolvent identity

$$h_n A_n^\theta f = \lambda_n(\lambda_n - A^\theta)^{-1} f - f \tag{8.33}$$

holds for all $n \in \mathbb{N}$, $f \in \mathrm{C}_\kappa(\mathbb{R}^d)$ and $\theta \in \Theta$, we obtain

$$J_n f = e^{-1} \sup_{\theta\in\Theta} e^{\lambda_n(\lambda_n - A^\theta)^{-1}} f \quad \text{for all } n \in \mathbb{N} \text{ and } f \in \mathrm{C}_\kappa(\mathbb{R}^d). \tag{8.34}$$

Theorem 8.3.4. *Suppose that Assumption 8.3.3 is satisfied. Then, there exists a strongly continuous convex monotone semigroup $(T(t))_{t\geq 0}$ on $\mathrm{C}_\kappa(\mathbb{R}^d)$ with $T(t)0 = 0$ given by*

$$T(t)f = \lim_{n\to\infty} J_n^{k_n^t} f \quad \textit{for all } (f,t) \in \mathrm{C}_\kappa(\mathbb{R}^d) \times \mathbb{R}_+, \tag{8.35}$$

where $(k_n^t)_{n\in\mathbb{N}} \subset \mathbb{N}$ is an arbitrary sequence satisfying $k_n^t h_n \to t$, such that

$$\mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d) \subset D(B) \quad \textit{and} \quad Bf = \sup_{\theta\in\Theta} A^\theta f \quad \textit{for all } f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d).$$ [2]

Moreover, Assumption 4.4.5 is satisfied and therefore

$$S(t)f = T(t)f \quad \textit{for all } t \geq 0 \textit{ and } f \in \mathrm{C}_\kappa(\mathbb{R}^d).$$

Proof. First, for every $n \in \mathbb{N}$, $r \geq 0$, $f \in \mathrm{Lip}_\mathrm{b}(\mathbb{R}^d, r)$ and $x \in \mathbb{R}^d$, we show that

$$\|\tau_x J_n f - J_n(\tau_x f)\|_\infty \leq \frac{L\lambda_0 e^{\beta h_0}}{\lambda_0 - \omega} r h_n |x|, \tag{8.36}$$

where

$$\beta := \omega + \frac{L\lambda_0}{\lambda_0 - \omega} \quad \text{and} \quad \lambda_0 := \min_{n\in\mathbb{N}} \lambda_n > \omega.$$

Subsequently, we write $B_{n,\theta} := \lambda_n(\lambda_n - A^\theta)^{-1}$ for all $n \in \mathbb{N}$ and $\theta \in \Theta$. For every $n \in \mathbb{N}$, $r \geq 0$, $f \in \mathrm{Lip}_\mathrm{b}(\mathbb{R}^d, r)$, $x \in \mathbb{R}^d$ and $\theta \in \Theta$, Assumption 8.3.1(iii) implies

$$\|\tau_x B_{n,\theta} f - B_{n,\theta}(\tau_x f)\|_\infty \leq \frac{L\lambda_0}{\lambda_0 - \omega} r h_n |x|. \tag{8.37}$$

Furthermore, by induction, it follows from Assumption 8.3.1(i) that

$$\|B_{n,\theta} f - B_{n,\theta} g\|_\infty \leq (1 + \omega h_n)^k \|f - g\|_\infty. \tag{8.38}$$

for all $k, n \in \mathbb{N}$, $f, g \in \mathrm{Lip}_\mathrm{b}(\mathbb{R}^d)$ and $\theta \in \Theta$. Hence, similar to the proof of Theorem 8.3.2, one can show that

$$B_{n,\theta} \colon \mathrm{Lip}_\mathrm{b}(\mathbb{R}^d, r) \to \mathrm{Lip}_\mathrm{b}(\mathbb{R}^d, (1 + \beta h_n)^k r). \tag{8.39}$$

for all $n \in \mathbb{N}$, $r \geq 0$ and $\theta \in \Theta$. Combining the inequalities (8.37)–(8.39) yields

$$\|\tau_x B_{n,\theta}^k f - B_{n,\theta}^k(\tau_x f)\|_\infty \leq \sum_{l=1}^k \|B_{n,\theta}^{k-l}(\tau_x B_{n,\theta}^l f) - B_{n,\theta}^{k-l} B_{n,\theta}(\tau_x B_{n,\theta}^{l-1} f)\|_\infty$$

[2] Here, we denote by B the generator of $(T(t))_{t\geq 0}$.

$$\begin{aligned}
&\le \sum_{l=1}^{k}(1+\omega h_n)^{k-l}\|\tau_x B_{n,\theta}B_{n,\theta}^{l-1}f - B_{n,\theta}(\tau_x B_{n,\theta}^{l-1}f)\|_\infty \\
&\le \sum_{l=1}^{k}(1+\omega h_n)^{k-l}\frac{L\lambda_0}{\lambda_0-\omega}(1+\beta h_n)^{l-1} r h_n|x| \\
&\le k(1+\beta h_n)^{k-1}\frac{L\lambda_0}{\lambda_0-\omega} r h_n|x|
\end{aligned}$$

for all $n\in\mathbb{N}$, $r\ge 0$, $f\in \mathrm{Lip_b}(\mathbb{R}^d,r)$, $x\in\mathbb{R}^d$ and $\theta\in\Theta$. Equation (8.34) implies

$$\begin{aligned}
\|\tau_x J_n f - J_n(\tau_x f)\|_\infty &\le e^{-1}\sup_{\theta\in\Theta}\|\tau_x e^{B_{n,\theta}} - e^{B_{n,\theta}}(\tau_x f)\|_\infty \\
&\le e^{-1}\sup_{\theta\in\Theta}\sum_{k=0}^{\infty}\frac{\|\tau_x B_{n,\theta}^k f - B_{n,\theta}^k(\tau_x f)\|_\infty}{k!} \\
&\le e^{-1}\sum_{k=1}^{\infty}\frac{(1+\beta h_n)^{k-1}}{(k-1)!}\frac{L\lambda_0}{\lambda_0-\omega} r h_n|x| \\
&= \frac{L\lambda_0}{\lambda_0-\omega}e^{\omega h_n} r h_n|x| = \frac{L\lambda_0 e^{\omega h_0}}{\lambda_0-\omega} r h_n|x|
\end{aligned}$$

for all $n\in\mathbb{N}$, $r\ge 0$, $f\in \mathrm{Lip_b}(\mathbb{R}^d,r)$ and $x\in\mathbb{R}^d$.

Second, we verify Assumption 8.2.7. We have already shown that Assumption 8.2.7(iii) is satisfied. Moreover, it follows from equation (8.34) and equation (8.39) that

$$J_n\colon \mathrm{Lip_b}(\mathbb{R}^d,r)\to \mathrm{Lip_b}(\mathbb{R}^d,e^{\beta h_n}r) \quad \text{for all } n\in\mathbb{N} \text{ and } r\ge 0.$$

Similarly, one can derive from Assumption 8.3.1(i) that

$$\|J_n f - J_n g\|_\kappa \le e^{\omega h_n}\|f-g\|_\kappa \quad \text{for all } n\in\mathbb{N} \text{ and } f,g\in \mathrm{C}_\kappa(\mathbb{R}^d).$$

Moreover, the operators $I_n\colon \mathrm{C}_\kappa(\mathbb{R}^d)\to \mathrm{F}_\kappa(\mathbb{R}^d)$ are convex and monotone with $I_n 0 = 0$. Let $f\in \mathrm{C}_\kappa(\mathbb{R}^d)$ and $(f_k)_{k\in\mathbb{N}}\subset \mathrm{Lip_b}(\mathbb{R}^d)$ be a sequence with $f_k\to f$. In case that Assumption 8.3.1(ii) is valid, Lemma 3.4.1 implies $I_n f = \lim_{k\to\infty} I_n f_k \in \mathrm{C}_\kappa(\mathbb{R}^d)$ and thus $I_n\colon \mathrm{C}_\kappa(\mathbb{R}^d)\to \mathrm{C}_\kappa(\mathbb{R}^d)$. In addition, it follows by induction that $\|I_n^k f\|_{\tilde\kappa}\le e^{\omega k h_n}\|f\|_{\tilde\kappa}$ for all $k,n\in\mathbb{N}$ and $f\in \mathrm{C}_\kappa(\mathbb{R}^d)$. Hence, we can apply Lemma 3.4.5 to obtain that Assumption 8.2.7(ii) is satisfied. If Assumption 8.3.1(ii') is valid, we can use Corollary 3.4.7 and Remark 3.4.11 instead. Indeed, Assumption 8.3.1(i) implies

$$\begin{aligned}
\Big\|J_n\big(\tfrac{r}{\kappa}(1-\zeta_x)\big) - \tfrac{r}{\kappa}(1-\zeta_x)\Big\|_\kappa &\le \sup_{\theta\in\Theta}\int_0^1 \Big\|e^{s h_n A_n^\theta} h_n A_n^\theta\big(\tfrac{r}{\kappa}(1-\zeta_x)\big)\Big\|_\kappa \,\mathrm{d}s \\
&\le \sup_{\theta\in\Theta} e^{\omega h_n} h_n \Big\|A^\theta\big(\tfrac{r}{\kappa}(1-\zeta_x)\big)\Big\|_\kappa \le e^{\omega h_0}\varepsilon h_n.
\end{aligned}$$

It remains to verify Assumption 8.2.7(v). For every $n\in\mathbb{N}$, $f\in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$ and $\theta\in\Theta$, we use Assumption 8.3.1(i) and equation (8.33) to estimate

$$\Big\|\frac{e^{h_n A_n^\theta}f - f}{h_n} - A_n^\theta f\Big\|_\kappa \le \int_0^1 \|e^{s h_n A_n^\theta} A_n^\theta f - A_n^\theta f\|_\kappa$$

$$\begin{aligned}
&\le \int_0^1 \|e^{sh_n A_n^\theta} A^\theta f - A^\theta f\|_\kappa \\
&\le e^{\omega h_n} h_n \|A_n^\theta A^\theta f\|_\kappa \\
&= e^{\omega h_n} \|\lambda_n (\lambda_n - A^\theta)^{-1} A^\theta f - A^\theta f\|_\kappa \\
&\le e^{\omega h_n} \|(\lambda - \omega)(\lambda_n - A^\theta)^{-1} A^\theta f - A^\theta f\|_\kappa + \omega h_n \|A^\theta\|_\kappa.
\end{aligned}$$

Assumption 8.3.3(iv') and equation (8.34) imply

$$\lim_{n\to\infty} \left\| \frac{J_n f - f}{h_n} - \sup_{\theta\in\Theta} A^\theta f \right\|_\kappa = 0.$$

Now, the claim follows from Theorem 8.2.8 and Theorem 4.4.6. □

8.3.2 Finite-difference schemes for convex HJB equations

In this subsection, we provide finite-difference approximations for convex HJB equations. For simplicity, we focus on the one-dimensional case and constant coefficients and refer to [8, 28, 64, 117, 118] for the multi-dimensional case with Lipschitz coefficients. We fix $\kappa \equiv 1$ and denote by $\mathrm{C_b}(\mathbb{R})$ the space of all bounded continuous functions $f\colon \mathbb{R} \to \mathbb{R}$. Let $(\delta_n)_{n\in\mathbb{N}}$, $(h_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ be sequences in $(0,\infty)$ with $\delta_n, h_n \to 0$ and $\sigma_n \to \infty$. Furthermore, let $\varphi\colon \mathbb{R}_+ \to [0,\infty]$ be a function. We show that the finite-difference scheme

$$(I_n f)(x) := f(x) + h_n \sup_{\sigma\in[0,\sigma_n]} \left(\frac{\sigma^2}{2} \frac{f(x+\delta_n) - 2f(x) + f(x-\delta_n)}{\delta_n^2} - \varphi(\sigma) \right),$$

which is defined for all $n \in \mathbb{N}$, $f \in \mathrm{C_b}(\mathbb{R})$ and $x \in \mathbb{R}$, converges to a strongly continuous convex monotone semigroup $(S(t))_{t\ge 0}$ on $\mathrm{C_b}(\mathbb{R})$ whose generator is given by

$$Af = \sup_{\sigma\ge 0} \left(\frac{1}{2}\sigma^2 f'' - \varphi(\sigma) \right) \quad \text{for all } f \in \mathrm{C_b^\infty}(\mathbb{R}).$$

Assumption 8.3.5. Suppose that the following conditions are satisfied:

(i) For every $n \in \mathbb{N}$, there exists $\sigma \in [0, \sigma_n]$ with $\varphi(\sigma) = 0$.

(ii) It holds $\lim_{\sigma\to\infty} \varphi(\sigma)/\sigma^2 = \infty$.

(iii) It holds $\frac{\sigma_n^2 h_n}{\delta_n^2} \le 1$ for all $n \in \mathbb{N}$.

Condition (iii) is classical and guarantees that the finite-difference scheme is stable w.r.t. the supremum norm on the grid. Due to Theorem 4.4.6, the semigroup $(S(t))_{t\ge 0}$ appearing in the following theorem is unique.

Theorem 8.3.6. *Suppose that Assumption 8.3.5 is satisfied. Then, there exists a strongly continuous convex monotone semigroup $(S(t))_{t\ge 0}$ on $\mathrm{C_b}(\mathbb{R})$ with $S(t)0 = 0$ given by*

$$S(t)f = \lim_{n\to\infty} I_n^{k_n^t} f \quad \text{for all } (f,t) \in \mathrm{C_b}(\mathbb{R}) \times \mathbb{R}_+,$$

where $(k_n^t)_{n\in\mathbb{N}}\subset\mathbb{N}$ *is an arbitrary sequence satisfying* $k_n^t h_n\to t$, *such that*

$$\mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)\subset D(A)\quad\textit{and}\quad Af=\sup_{\sigma\geq 0}\Big(\frac{1}{2}\sigma^2 f''-\varphi(\sigma)\Big)\quad\textit{for all } f\in\mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d).$$

Moreover, Assumption 4.4.5 is satisfied.

Proof. We verify Assumption 8.2.7. For every $n\in\mathbb{N}$, $f\in\mathrm{C}_\mathrm{b}(\mathbb{R})$ and $x\in\mathbb{R}$,

$$(I_n f)(x)=\sup_{\sigma\in[0,\sigma_n]}\left(\left(1-\frac{\sigma^2 h_n}{\delta_n^2}\right)f(x)+\frac{\sigma^2 h_n}{2\delta_n^2}\big(f(x+2\delta_n)+f(x-2\delta_n)\big)-\varphi(\sigma)\right).$$

Hence, Assumption 8.3.5(iii) yields $I_n\colon\mathrm{Lip}_\mathrm{b}(\mathbb{R}^d,r)\to\mathrm{Lip}_\mathrm{b}(\mathbb{R}^d,r)$ for all $n\in\mathbb{N}$ and $r\geq 0$ while the identity $I_n(\tau_x f)=\tau_x I_n f$ holds by definition for all $f\in\mathrm{C}_\mathrm{b}(\mathbb{R})$. Moreover, the convex monotone operators $I_n\colon\mathrm{C}_\mathrm{b}(\mathbb{R})\to\mathrm{F}_\mathrm{b}(\mathbb{R})$ satisfy $I_n 0=0$ and

$$\|I_n f-I_n g\|_\infty\leq\|f-g\|_\infty\quad\text{for all } n\in\mathbb{N}\text{ and } f,g\in\mathrm{C}_\mathrm{b}(\mathbb{R}).$$

Now, let $f\in\mathrm{C}_\mathrm{b}^\infty(\mathbb{R})$. It follows from Taylor's formula that

$$\frac{\sigma^2}{2}\frac{f(x+\delta_n)-2f(x)+f(x-\delta_n)}{\delta_n^2}=\frac{\sigma^2}{2\delta_n^2}\int_0^{\delta_n}(\delta_n-s)\big(f''(x+s)+f''(x-s)\big)\mathrm{d}s$$

for all $n\in\mathbb{N}$, $\sigma\geq 0$ and $x\in\mathbb{R}$. Hence, by Assumption 8.3.5(ii), there exists $r\geq 0$ with

$$\left(\frac{I_n f-f}{h_n}\right)(x)=\sup_{\sigma\in[0,r]}\left(\frac{\sigma^2}{2\delta_n^2}\int_0^{\delta_n}(\delta_n-s)\big(f''(x+s)+f''(x-s)\big)\mathrm{d}s-\varphi(\sigma)\right)$$

for all $n\in\mathbb{N}$. Moreover, by increasing $r\geq 0$, we can assume that

$$\sup_{\sigma\geq 0}\Big(\frac{1}{2}\sigma^2 f''-\varphi(\sigma)\Big)=\sup_{\sigma\in[0,r]}\Big(\frac{1}{2}\sigma^2 f''-\varphi(\sigma)\Big).$$

We obtain

$$\frac{I_n f-f}{h_n}\to\sup_{\sigma\geq 0}\Big(\frac{1}{2}\sigma^2 f''-\varphi(\sigma)\Big)\quad\text{as } n\to\infty.$$

Moreover, for every $n\in\mathbb{N}$,

$$\left\|\frac{I_n f-f}{h_n}\right\|_\infty\leq\sup_{\sigma\geq 0}\big(\sigma\|f''\|_\infty-\varphi(\sigma)\big).$$

and therefore we obtain from Assumption 8.3.5(ii), Corollary 3.4.8 and Remark 3.4.11 that $I_n\colon\mathrm{C}_\mathrm{b}(\mathbb{R})\to\mathrm{C}_\mathrm{b}(\mathbb{R})$ for all $n\in\mathbb{N}$ and that Assumption 8.2.7(ii) is satisfied. □

8.3.3 Vanishing viscosity and stability of convex HJB equations

In this subsection, we provide a stability result for convex HJB equations which covers, as a special case, the vanishing viscosity method. We fix $\kappa\equiv 1$ and denote by $\mathrm{C}_\mathrm{b}(\mathbb{R}^d)$ the space of all bounded continuous functions $f\colon\mathbb{R}^d\to\mathbb{R}$. Let $(\varphi_n)_{n\in\mathbb{N}}$

be a sequence of functions $\varphi_n\colon \mathbb{S}_+^d \times \mathbb{R}^d \to [0,\infty]$, where $\mathbb{S}_+^d$ consists of all symmetric positive semi-definite $d \times d$-matrices, and define

$$H_n(a,b) := \sup_{(a',b')\in\mathbb{S}_+^d\times\mathbb{R}^d} \left(\frac{1}{2}\operatorname{tr}(aa') + b^T b' - \varphi_n(a',b')\right) \quad \text{for all } (a,b)\in\mathbb{R}^{d\times d}\times\mathbb{R}^d.$$

Here, we denote by $\operatorname{tr}(aa')$ the matrix trace and by $b^T b'$ the euclidean inner product. Under the assumption that $H_n \to H$, we will show that a sequence of semigroups $(S_n(t))_{t\geq 0}$, whose generators are given by

$$(A_n f)(x) = H_n(D^2 f(x), Df(x)) \quad \text{for all } f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d),$$

converges to a semigroup $(S(t))_{t\geq 0}$ with generator $Af = H(D^2 f, Df)$.

Assumption 8.3.7. Suppose that the following conditions are satisfied:

(i) For every $n \in \mathbb{N}$, there exists $(a,b) \in \mathbb{S}_+^d \times \mathbb{R}^d$ with $\varphi_n(a,b) = 0$.

(ii) The penalization functions grow superlinearly, i.e.,

$$\lim_{|a|+|b|\to\infty} \inf_{n\in\mathbb{N}} \frac{\varphi_n(a,b)}{|a|+|b|} = \infty.$$

(iii) There exists a function $H\colon \mathbb{R}^{d\times d}\times\mathbb{R}^d \to \mathbb{R}$ with

$$\lim_{n\to\infty} \sup_{(a,b)\in K} |H_n(a,b) - H(a,b)| = 0 \quad \text{for all } K \Subset \mathbb{S}_+^d \times \mathbb{R}^d.$$

In particular, condition (ii) guarantees that $(H_n)_{n\in\mathbb{N}}$ is a sequence of real-valued functions. Note that all the semigroups appearing in the following theorem are unique due to Theorem 4.4.6.

Theorem 8.3.8. *Suppose that Assumption 8.3.7 is satisfied. Then, there exists a sequence $(S_n)_{n\in\mathbb{N}}$ of strongly continuous convex monotone semigroups $(S_n(t))_{t\geq 0}$ on $\mathrm{C}_\mathrm{b}(\mathbb{R}^d)$ with*

$$\mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d) \subset D(A_n) \quad \text{and} \quad A_n f = H_n(D^2 f, Df)$$

for all $n \in \mathbb{N}$ and $f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$ satisfying Assumption 8.2.5. Hence, there exists another strongly continuous convex monotone semigroup $(S(t))_{t\geq 0}$ on $\mathrm{C}_\mathrm{b}(\mathbb{R}^d)$ with

$$S(t)f = \lim_{n\to\infty} S_n(t)f \quad \text{for all } (f,t) \in \mathrm{C}_\mathrm{b}(\mathbb{R}^d)\times\mathbb{R}_+$$

satisfying $\mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d) \subset D(A)$ and $Af = H(D^2 f, Df)$ for all $f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$.

Proof. First, we construct the semigroups $(S_n(t))_{t\geq 0}$ by applying Theorem 8.2.8. To do so, for every $n \in \mathbb{N}$, $t \geq 0$, $f \in \mathrm{C}_\mathrm{b}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we define

$$(I_n(t)f)(x) := \sup_{(a,b)\in\mathbb{S}_+^d\times\mathbb{R}^d} \big(\mathbb{E}[f(x + \sqrt{a}W_t + bt)] - \varphi_n(a,b)t\big),$$

where $\sqrt{a} \in \mathbb{S}_+^d$ denotes the unique matrix with $\sqrt{a}\sqrt{a} = a$. It is straightforward to show that the operators $I_n(t)\colon \mathrm{C}_\mathrm{b}(\mathbb{R}^d) \to \mathrm{C}_\mathrm{b}(\mathbb{R}^d)$ are well-defined and satisfy

- $I_n(t)$ is convex and monotone with $I_n(t)0 = 0$ for all $n \in \mathbb{N}$ and $t \geq 0$,
- $\|I_n(t)f - I_n(t)g\|_\infty \leq \|f-g\|_\infty$ for all $n \in \mathbb{N}$, $t \geq 0$ and $f, g \in \mathrm{C_b}(\mathbb{R}^d)$,
- $I_n(t)(\tau_x f) = \tau_x I_n(t) f$ for all $n \in \mathbb{N}$, $t \geq 0$, $f \in \mathrm{C_b}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,
- $I_n(t)\colon \mathrm{Lip_b}(\mathbb{R}^d, r) \to \mathrm{Lip_b}(\mathbb{R}^d, r)$ for all $n \in \mathbb{N}$ and $r, t \geq 0$.

Let $n \in \mathbb{N}$ and $f \in \mathrm{C_b^\infty}(\mathbb{R}^d)$. Due to Assumption 8.3.7(ii), there exists $r \geq 0$ with

$$H_n(D^2 f(x), Df(x)) = \sup_{|a|+|b|\leq r} \Big(\frac{1}{2}\operatorname{tr}(aD^2 f(x)) + b^T Df(x) - \varphi_n(a,b)\Big),$$
$$(I_n(t)f)(x) = \sup_{|a|+|b|\leq r} \big(\mathbb{E}[f(x+\sqrt{a}W_t + bt)] - \varphi_n(a,b)t\big)$$

for all $t \geq 0$ and $x \in \mathbb{R}^d$. Hence, for every $h > 0$, Itô's formula implies

$$\frac{I_n(h)f - f}{h} = \sup_{|a|+|b|\leq r} \left(\mathbb{E}\left[\frac{1}{h}\int_0^h \Big(\frac{1}{2}\operatorname{tr}(aD^2 f + b^T Df\Big)(\,\cdot\, + \sqrt{a}W_s + bs)\,\mathrm{d}s\right] - \varphi_n(a,b)\right)$$

and the right-hand side converges to $H_n(D^2 f, Df)$ as $h \to 0$. The previous equation also yields the estimate

$$\left\|\frac{I_n(h)f - f}{h}\right\|_\infty \leq \sup_{(a,b)\in\mathbb{S}^d_+\times\mathbb{R}^d} \Big(\frac{1}{2}|a|\cdot\|D^2 f\|_\infty + |b|\cdot\|Df\|_\infty - \varphi_n(a,b)\Big).$$

In the sequel, we fix $n \in \mathbb{N}$ and choose a sequence $(h_k)_{k\in\mathbb{N}} \subset (0,\infty)$ with $h_k \to 0$. Define $I_{n,k}f := I_n(h_k)f$ for all $f \in \mathrm{C_b}(\mathbb{R}^d)$. Due to the previous arguments and Corollary 3.4.8, the sequence $(I_{n,k})_{k\in\mathbb{N}}$ satisfies Assumption 8.2.7. Hence, by Theorem 8.2.8, there exists a strongly continuous convex monotone semigroup $(S_n(t))_{t\geq 0}$ on $\mathrm{C_b}(\mathbb{R}^d)$ with $S_n(t)0 = 0$ such that

$$S_n(t)f = \lim_{k\to\infty} (I_{n,k})^{m_k^t} f \quad \text{for all } (f,t) \in \mathrm{C_b}(\mathbb{R}^d)\times\mathbb{R}_+,$$

where $(m_k^t)_{k\in\mathbb{N}} \subset \mathbb{N}$ is an arbitrary sequence satisfying $m_k^t h_k \to t$ as $k \to \infty$. Moreover, the following statements are valid:

(i) $\mathrm{C_b^\infty}(\mathbb{R}^d) \subset D(A_n)$ and $A_n f = H_n(D^2 f, Df)$ for all $f \in \mathrm{C_b^\infty}(\mathbb{R}^d)$.

(ii) $\|S_n(t)f - S_n(t)g\|_\infty \leq \|f-g\|_\infty$ for all $n \in \mathbb{N}$, $t \geq 0$ and $f, g \in \mathrm{C_b}(\mathbb{R}^d)$.

(iii) For every $\varepsilon > 0$, $r, T \geq 0$ and $K \Subset \mathbb{R}^d$, there exist $c \geq 0$ and $K' \Subset \mathbb{R}^d$ with
$$\|S_n(t)f - S_n(t)g\|_{\infty,K} \leq c\|f-g\|_{\infty,K'} + \varepsilon$$
for all $t \in [0,T]$ and $f, g \in B_{\mathrm{C_b}(\mathbb{R}^d)}(r)$.

(iv) $S_n(t)(\tau_x f) = \tau_x S_n(t) f$ for all $n \in \mathbb{N}$, $t \geq 0$, $f \in \mathrm{C_b}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

(v) $S_n(t)\colon \mathrm{Lip_b}(\mathbb{R}^d, r) \to \mathrm{Lip_b}(\mathbb{R}^d, r)$ for all $n \in \mathbb{N}$ and $r, t \geq 0$.

In particular, due to Theorem 4.4.6, the semigroup $(S_n(t))_{t\geq 0}$ does not depend on the choice of the sequence $(h_k)_{k\in\mathbb{N}}$.

Second, we note that the conditions (i), (iii) and (iv) of Assumption 8.2.2 are clearly satisfied and that Assumption 8.3.7(iii) yields condition (v). Moreover, Assumption 8.3.7(ii) and Corollary 3.4.10 guarantee that Assumption 8.2.2(ii) is also valid. Hence, the claim about the semigroup $(S(t))_{t\geq 0}$ follows from Theorem 8.2.3. Again, due to Theorem 4.4.6, the latter is unique. □

8.3.4 Randomized Euler schemes for Lipschitz ODEs

Let $\psi\colon \mathbb{R}^d \to \mathbb{R}^d$ be a bounded Lipschitz continuous function. The aim of this subsection is to show that the unique solution of the parameter dependent ordinary differential equation

$$\begin{cases} \partial_t u(t,x) = \psi(u(t,x)), & t \geq 0, \\ u(0,x) = x, & x \in \mathbb{R}^d, \end{cases} \tag{8.40}$$

can be approximated by a randomized Euler scheme under model uncertainty. For that purpose, let $(h_n)_{n\in\mathbb{N}} \subset (0,\infty)$ be a sequence with $h_n \to 0$ and let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables on a sublinear expectation space $(\Omega, \mathcal{H}, \mathcal{E})$ satisfying $\mathcal{E}[|\xi_1|^3] < \infty$ and $\mathcal{E}[a\xi_1] = 0$ for all $a \in \mathbb{R}^d$. If $\mathcal{E}[\,\cdot\,] = \mathbb{E}[\,\cdot\,]$ is a linear expectation, then there exists a unique probability measure μ on $\mathcal{B}(\mathbb{R}^d)$ with

$$\mathbb{E}[f(\xi_n)] = \int_{\mathbb{R}^d} f(\xi_n)\,\mathrm{d}\mu \quad \text{for all } n \in \mathbb{N} \text{ and } f \in \mathrm{C_b}(\mathbb{R}^d).$$

Furthermore, if the distribution μ of the random variables $(\xi_n)_{n\in\mathbb{N}}$ is uncertain, a worst case approach consists in taking the supremum over a set of measures, i.e.,

$$\mathcal{E}[f(\xi_n)] := \sup_{\mu\in M} \int_{\mathbb{R}^d} f(\xi_n)\,\mathrm{d}\mu.$$

For more details, we refer Section 7.2. For every $n \in \mathbb{N}$, $\delta > 0$, $t \geq 0$ and $x \in \mathbb{R}^d$, we recursively define a randomized explicit Euler scheme under model uncertainty by

$$X_0^{n,\delta,x} := x \quad \text{and} \quad X_{(k+1)h_n}^{n,\delta,x} := X_{kh_n}^{n,\delta,x} + h_n\psi\left(X_{kh_n}^{n,\delta,x}\right) + \delta\sqrt{h_n}\xi_{k+1}$$

and show that $X_{k_n^t h_n}^{n,\delta_n,x} \to u(t,x)$ weakly for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, where $(\delta_n)_{n\in\mathbb{N}} \subset (0,\infty)$ and $(k_n^t)_{n\in\mathbb{N}} \subset \mathbb{N}$ are arbitrary sequences with $\delta_n \to 0$ and $k_n^t h_n \to t$. To be precise, we show that

$$\mathcal{E}\left[f\left(X_{k_n^t h_n}^{n,\delta_n,x}\right)\right] \to f(u(t,x)) \quad \text{for all } f \in \mathrm{C_b}(\mathbb{R}^d).$$

This means that, regardless of possible numerical errors, the Euler scheme still converges to the solution of equation (8.40). Furthermore, in the case that $\delta_n \to \delta > 0$, the Euler scheme converges to the solution of a stochastic differential equation which might be driven by a G-Brownian motion. We define

$$(I_{n,\delta}f)(x) := \mathcal{E}\left[f\left(x + h_n\psi(x) + \delta\sqrt{h_n}\xi_1\right)\right]$$

for all $n \in \mathbb{N}$, $\delta > 0$, $f \in \mathrm{C_b}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Let L be the Lipschitz constant of ψ.

Theorem 8.3.9. *Let $(\delta_n)_{n\in\mathbb{N}} \subset (0,\infty)$ with $\delta_n \to \delta \in \mathbb{R}_+$. Then, there exists a strongly continuous convex monotone semigroup $(S_\delta(t))_{t\geq 0}$ on $\mathrm{C_b}(\mathbb{R}^d)$ with $S(t)0 = 0$ such that*

$$S_\delta(t)f = \lim_{n\to\infty} (I_{n,\delta_n})^{k_n^t} f_n = \lim_{n\to\infty} \mathcal{E}\left[f\left(X_{k_n^t h_n}^{n,\delta_n,\cdot}\right)\right] \quad \text{for all } (f,t) \in \mathrm{C_b}(\mathbb{R}^d) \times \mathbb{R}_+,$$

where $(k_n^t)_{n\in\mathbb{N}} \subset \mathbb{N}$ is an arbitrary sequence with $k_n^t h_n \to t$. Moreover, the following statements are valid:

(i) *It holds* $\mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d) \subset D(A_\delta)$ *and*

$$(A_\delta f)(x) = \frac{1}{2}\delta^2 \mathcal{E}\big[\xi_1^T D^2 f(x)\xi_1\big] + Df(x)^T\psi(x) \quad \textit{for all } f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d).$$

(ii) *It holds* $\|S_\delta(t)f - S_\delta(t)g\|_\infty \leq \|f-g\|_\infty$ *for all* $t \geq 0$ *and* $f, g \in \mathrm{C}_\mathrm{b}(\mathbb{R}^d)$.

(iii) *For every* $\varepsilon > 0$, $r, T \geq 0$ *and* $K \Subset \mathbb{R}^d$, *there exist* $c \geq 0$ *and* $K' \Subset \mathbb{R}^d$ *with*

$$\|S_\delta(t)f - S_\delta(t)g\|_{\infty,K} \leq c\|f-g\|_{\infty,K'} + \varepsilon$$

for all $t \in [0,T]$ *and* $f, g \in B_{\mathrm{C}_\mathrm{b}(\mathbb{R}^d)}(r)$.

(iv) *For every* $r, t \geq 0$, $f \in \mathrm{Lip}_\mathrm{b}(r)$ *and* $x \in \mathbb{R}^d$,

$$\|S_\delta(t)(\tau_x f) - \tau_x S_\delta(t) f\|_\kappa \leq Le^{Lt} rt|x|.$$

Furthermore, it holds $S_\delta(t)\colon \mathrm{Lip}_\mathrm{b}(r) \to \mathrm{Lip}_\mathrm{b}(e^{Lt}r)$ *for all* $r, t \geq 0$.

In particular, we obtain $(S_\delta(t)f)(x) = \mathcal{E}[f(X_t^{\delta,x})]$ *for all* $t \geq 0$, $f \in \mathrm{C}_\mathrm{b}(\mathbb{R}^d)$ *and* $x \in \mathbb{R}^d$, *where* $X_t^{\delta,x}$ *denotes the unique solution of the parameter dependent ordinary (stochastic) differential equation*

$$\begin{cases} \mathrm{d}X_t^{\delta,x} = \psi(X_t^{\delta,x})\mathrm{d}t + \delta \mathrm{d}W_t, & t \geq 0, \\ X_0^{\delta,x} = x, & x \in \mathbb{R}^d, \end{cases} \tag{8.41}$$

which is driven by a G-Brownian motion $(W_t)_{t\geq 0}$ *with* $\mathcal{E}[W_1 W_1^T] = \mathcal{E}[\xi_1\xi_1^T]$.

Proof. In order to apply Theorem 8.2.8, we have to verify Assumption 8.2.7. It is straightforward to show that the operators $I_{n,\delta_n}\colon \mathrm{C}_\mathrm{b}(\mathbb{R}^d) \to \mathrm{C}_\mathrm{b}(\mathbb{R}^d)$ are well-defined and satisfy

- I_{n,δ_n} is convex and monotone with $I_{n,\delta_n}(t)0 = 0$ for all $n \in \mathbb{N}$,
- $\|I_{n,\delta_n} f - I_{n,\delta_n} g\|_\infty \leq \|f-g\|_\infty$ for all $n \in \mathbb{N}$ and $f, g \in \mathrm{C}_\mathrm{b}(\mathbb{R}^d)$,
- $\|\tau_x I_{n,\delta_n} f - I_{n,\delta_n}(\tau_x f)\|_\infty \leq Lrh_n|x|$ for all $n \in \mathbb{N}$, $f \in \mathrm{C}_\mathrm{b}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,
- $I_{n,\delta_n}\colon \mathrm{Lip}_\mathrm{b}(\mathbb{R}^d, r) \to \mathrm{Lip}_\mathrm{b}(\mathbb{R}^d, (1+Lh_n)r)$ for all $n \in \mathbb{N}$ and $r \geq 0$.

Moreover, for every $n \in \mathbb{N}$, $f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, Taylor's formula yields

$$\begin{aligned} &f(x + h_n\psi(x) + \delta_n\sqrt{h_n}\xi_1) \\ &= f(x) + Df(x)^T(h_n\psi(x) + \delta_n\sqrt{h_n}\xi_1) \\ &\quad + \frac{1}{2}(h_n\psi(x) + \delta_n\sqrt{h_n}\xi_1)^T D^2 f(x)(h_n\psi(x) + \sqrt{h_n}\delta_n\xi_1) + (R_n(f,x,\xi_1), \end{aligned}$$

where the reminder term can be estimated by

$$|R_n(f,x,\xi_1)| \leq \frac{|h_n\psi(x) + \delta_n\sqrt{h_n}\xi_1|^3}{6}\|D^3 f\|_\infty.$$

Since $\mathcal{E}[a\xi_1] = 0$ for all $a \in \mathbb{R}^d$, we obtain

$$\frac{(I_{n,\delta_n} f - f)(x)}{h_n} = \mathcal{E}\left[\frac{f(x + h_n\psi(x) + \delta_n\sqrt{h_n}\xi_1) - f(x)}{h_n}\right]$$
$$= Df(x)^T\psi(x) + \frac{\delta_n^2}{2}\mathcal{E}\left[\xi^T D^2 f(x)\xi\right] + (\tilde{R}_n(f, x, \xi_1),$$

where the reminder term can be estimated by

$$|\tilde{R}_n(f, x, \xi_1)| \leq \frac{1}{2}h_n|\psi(x)^T D^2 f(x)\psi(x)| + \frac{\mathcal{E}[|R_n(f, x, \xi_1)|]}{h_n}.$$

Since $\mathcal{E}[|\xi_1|^3] < \infty$ and $\|\psi\|_\infty < \infty$, we obtain

$$\frac{I_{n,\delta_n} f - f}{h_n} \to Df(\cdot)^T\psi + \frac{\delta^2}{2}\mathcal{E}\left[\xi^T D^2 f(\cdot)\xi\right] \quad \text{for all } f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d).$$

Moreover, the previous estimates and Corollary 3.4.8 guarantee that Assumption 8.2.7(ii) is satisfied and, since the random variables $(\xi_n)_{n\in\mathbb{N}}$ are iid, we obtain

$$(I_{n,\delta_n})^k f_n = \mathcal{E}\left[f\left(X_{kh_n}^{n,\delta_n,x}\right)\right]$$

for all $k, n \in \mathbb{N}$, $f \in \mathrm{C}_\mathrm{b}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Finally, by relying on the strong Markov property for stochastic differential equations driven by G-Brownian motions [95], one can show that $(T_\delta(t)f)(x) := \mathcal{E}[f(X_t^{\delta,x})]$ defines a strongly continuous monotone semigroup on $\mathrm{C}_\mathrm{b}(\mathbb{R}^d)$ satisfying Assumption 4.4.5. Now, the claim follows from Theorem 8.2.8 and Theorem 4.4.6. □

Corollary 8.3.10. *Let $\delta \geq 0$ and $(\delta_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be a sequence with $\delta_n \to \delta$. Then,*

$$S_\delta(t)f = \lim_{n\to\infty} S_{\delta_n}(t)f \quad \textit{for all } t \geq 0 \textit{ and } f \in \mathrm{C}_\mathrm{b}(\mathbb{R}^d).$$

Proof. Theorem 8.3.9 and Corollary 3.4.10 guarantee that Assumption 8.2.2 is satisfied. Indeed, it follows from the proof of Theorem 8.3.9 that

$$\lim_{n\to\infty}\left\|\frac{I_{n,\delta} f - f}{h_n} - A_\delta f\right\|_\infty = 0 \quad \text{for all } f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$$

and therefore Theorem 1.4.2 implies

$$\lim_{h\downarrow 0}\left\|\frac{S_\delta(h)f - f}{h} - A_\delta f\right\|_\infty = 0 \quad \text{for all } f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d).$$

Hence, the claim follows from Theorem 8.2.3. □

8.3.5 Large deviations for randomized Euler schemes

We continue our analysis of randomized Euler schemes from the previous subsection and focus on the convergence rate of the scheme by means of a large deviations approach, see [58, 63, 75, 162]. While the choice of the parameters δ_n and h_n is still arbitrary, the choice of the additional parameter α_n depends on δ_n and h_n. Moreover, we obtain a large deviations result for stochastic differential equations in the spirit of Freidlin–Wentzell [82]. Let $(h_n)_{n\in\mathbb{N}}$, $(\delta_n)_{n\in\mathbb{N}}$ and $(\alpha_n)_{n\in\mathbb{N}}$ be sequences in $(0,\infty)$. Moreover, let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables on a sublinear expectation space $(\Omega,\mathcal{H},\mathcal{E})$ satisfying $\mathcal{E}[|\xi_1|^3]<\infty$ and $\mathcal{E}[a\xi_1]=0$ for all $a\in\mathbb{R}^d$. For every $n\in\mathbb{N}$ and $f\in \mathrm{C_b}(\mathbb{R}^d)$, we define

$$J_n f := \frac{1}{\alpha_n}\log\left(I_n e^{\alpha_n f}\right),$$

where $I_n\colon \mathrm{C_b}(\mathbb{R}^d)\to \mathrm{C_b}(\mathbb{R}^d)$ is given by

$$(I_n f)(x) := \mathcal{E}\left[f\left(x+h_n\psi(x)+\delta_n\sqrt{h_n}\xi_1\right)\right].$$

In addition, for every $n\in\mathbb{N}$, $\delta>0$, $t\geq 0$ and $x\in\mathbb{R}^d$, we recursively define

$$X_0^{n,\delta,x} := x \quad\text{and}\quad X_{(k+1)h_n}^{n,\delta,x} := X_{kh_n}^{n,\delta,x} + h_n\psi\left(X_{kh_n}^{n,\delta,x}\right)+\delta\sqrt{h_n}\xi_{k+1}.$$

Theorem 8.3.11. *Assume that there exists $\delta,\gamma\geq 0$ with $\delta_n\to\delta$ and $\alpha_n\delta_n^2\to\gamma$. Furthermore, let $h_n\to 0$ and $\alpha_n h_n\to 0$. Then, there exists a strongly continuous convex monotone semigroup $(T_{\gamma,\delta}(t))_{t\geq 0}$ on $\mathrm{C_b}(\mathbb{R}^d)$ with $T_{\gamma,\delta}(t)0=0$ such that*

$$T_{\gamma,\delta}(t)f = \lim_{n\to\infty} J_n^{k_n^t} f = \lim_{n\to\infty}\frac{1}{\alpha_n}\log\left(\mathcal{E}\left[\exp\left(\alpha_n f\left(X_{k_n^t h_n}^{n,\delta_n,\cdot}\right)\right)\right]\right)$$

for all $t\geq 0$, $f\in \mathrm{C_b}(\mathbb{R}^d)$ and sequences $(k_n^t)_{n\in\mathbb{N}}\subset\mathbb{N}$ with $k_n^t\to t$. Moreover, the following statements are valid:

(i) It holds $\mathrm{C_b^\infty}(\mathbb{R}^d)\subset D(B_{\gamma,\delta})$ and

$$(B_{\gamma,\delta}f)(x) = \frac{1}{2}\mathcal{E}\left[\delta^2\xi_1^T D^2 f(x)\xi_1 + \gamma|Df(x)^T\xi_1|^2\right] + Df(x)^T\psi(x)$$

for all $f\in \mathrm{C_b^\infty}(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$.

(ii) It holds $\|T_{\gamma,\delta}(t)f - T_{\gamma,\delta}(t)g\|_\infty \leq \|f-g\|_\infty$ for all $t\geq 0$ and $f,g\in \mathrm{C_b}(\mathbb{R}^d)$.

(iii) For every $\varepsilon>0$, $r,T\geq 0$ and $K\Subset\mathbb{R}^d$, there exist $c\geq 0$ and $K'\Subset\mathbb{R}^d$ with

$$\|T_{\gamma,\delta}(t)f - T_{\gamma,\delta}(t)g\|_{\infty,K} \leq c\|f-g\|_{\infty,K'}+\varepsilon$$

for all $t\in[0,T]$ and $f,g\in B_{\mathrm{C_b}(\mathbb{R}^d)}(r)$.

(iv) For every $r,t\geq 0$, $f\in \mathrm{Lip_b}(r)$ and $x\in\mathbb{R}^d$,

$$\|T_{\gamma,\delta}(t)(\tau_x f) - \tau_x T_{\gamma,\delta}(t)f\|_\kappa \leq Le^{Lt}rt|x|.$$

Furthermore, it holds $T_{\gamma,\delta}(t)\colon \mathrm{Lip_b}(r)\to \mathrm{Lip_b}(e^{Lt}r)$ for all $r,t\geq 0$.

Proof. By relying on the estimates in the proof of Theorem 8.3.9, it is straightforward to show that the operators $J_n\colon \mathrm{C_b}(\mathbb{R}^d)\to \mathrm{C_b}(\mathbb{R}^d)$ are well-defined and satisfy

- J_n is convex and monotone with $J_n(t)0=0$ for all $n\in\mathbb{N}$,

- $\|J_n f - J_n g\|_\infty \le \|f-g\|_\infty$ for all $n\in\mathbb{N}$ and $f,g\in \mathrm{C_b}(\mathbb{R}^d)$,
- $\|\tau_x J_n f - J_n(\tau_x f)\|_\infty \le Lrh_n|x|$ for all $n\in\mathbb{N}$, $f\in \mathrm{C_b}(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$,
- $J_n\colon \mathrm{Lip_b}(\mathbb{R}^d,r)\to \mathrm{Lip_b}(\mathbb{R}^d,(1+Lh_n)r)$ for all $n\in\mathbb{N}$ and $r\ge 0$.

Indeed, one can show that $J_n(f+c)=J_nf+c$ and $J_n(\lambda f+(1-\lambda)g)\le \max\{J_nf, J_ng\}$ for all $n\in\mathbb{N}$, $f,g\in \mathrm{C_b}(\mathbb{R}^d)$, $c\in\mathbb{R}$ and $\lambda\in[0,1]$. Hence,

$$\begin{aligned}&\big(J_n(\lambda f+(1-\lambda)g)\big)(x)-\lambda(J_nf)(x)-(1-\lambda)(J_ng)(x)\\&=\Big(J_n\big(\lambda(f-(J_nf)(x))+(1-\lambda)(g-(J_ng)(x))\big)\Big)(x)\\&\le \max\big\{\big(J_n(f-(J_nf)(x))\big)(x),\big(g-(J_ng)(x))\big)(x)\big\}=0\end{aligned}$$

for all $n\in\mathbb{N}$, $f,g\in \mathrm{C_b}(\mathbb{R}^d)$ and $\lambda\in[0,1]$ which shows that J_n is convex. Moreover,

$$J_nf\le J_n\big(g+\|f-g\|_\infty\big)=J_ng+\|f-g\|_\infty$$

and reversing the roles of f and g yields $\|J_nf-J_ng\|_\infty\le\|f-g\|_\infty$. The remaining properties can be obtained similarly.

For every $n\in\mathbb{N}$, $f\in \mathrm{C_b^\infty}(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$, Taylor's formula yields

$$\begin{aligned}&\exp\Big(\alpha_n\big(f\big(x+h_n\psi(x)+\delta_n\sqrt{h_n}\xi\big)-f(x)\big)\Big)\\&=1+\alpha_nDf(x)^T(h_n\psi(x)+\delta_n\sqrt{h_n}\xi_1)+\frac{\alpha_n^2}{2}|Df(x)^T(h_n\psi(x)+\delta_n\sqrt{h_n}\xi_1)|^2\\&\quad+\frac{\alpha_n}{2}(h_n\psi(x)+\delta_n\sqrt{h_n}\xi_1^TD^2f(x)(h_n\psi(x)+\delta_n\sqrt{h_n}\xi_1+R_n(f,x,\xi_1),\end{aligned}$$

where the reminder term can be estimated by

$$|R_n(f,x,\xi_1)|\le\frac{\alpha_n|h_n\psi(x)+\delta\sqrt{h_n}\xi_1|^3}{6}\rho\left(\sum_{i=1}^3\|D^if\|_\infty\right)$$

for a suitable function $\rho\colon\mathbb{R}_+\to\mathbb{R}_+$ which satisfies $\rho(\varepsilon)\to0$ as $\varepsilon\to0$. We use the fact that $\mathcal{E}[a\xi_1]=0$ for all $a\in\mathbb{R}^d$ to conclude

$$\begin{aligned}&\mathcal{E}\Big[\exp\Big(\alpha_n\big(f\big(x+h_n\psi(x)+\delta_n\sqrt{h_n}\xi_1\big)-f(x)\big)\Big)\Big]\\&=1+\alpha_nh_nDf(x)^T\psi(x)+\frac{\alpha_n^2h_n^2}{2}|Df(x)^T\psi(x)|^2+\frac{\alpha_nh_n^2}{2}\psi(x)^TD^2f(x)\psi(x)\\&\quad+\frac{\alpha_nh_n}{2}\mathcal{E}\big[\alpha_n\delta_n^2|Df(x)^T\psi(x)|+\delta_n^2\xi_1^TD^2f(x)\xi_1\big]+\tilde{R}_n(f,x,\xi_1),\end{aligned}$$

where the reminder term can be estimated by $|\tilde{R}_n(f,x,\xi_1)|\le\mathcal{E}[|R_n(f,x,\xi_1)|]$. Moreover, by assumption, it holds $h_n\to0$ and $\alpha_nh_n\to0$. Hence, Taylor's formula implies

$$\begin{aligned}\frac{J_nf-f}{h_n}&=\frac{1}{\alpha_nh_n}\log\Big(\mathcal{E}\Big[\exp\Big(\alpha_n\big(f\big(x+h_n\psi(x)+\delta_n\sqrt{h_n}\xi_1\big)-f(x)\big)\Big)\Big]\Big)\\&=Df(x)^T\psi(x)+\frac{\alpha_nh_n}{2}|Df(x)^T\psi(x)|^2+\frac{h_n}{2}\psi(x)^TD^2f(x)\psi(x)\end{aligned}$$

$$+\frac{1}{2}\mathcal{E}\big[\alpha_n\delta_n|Df(x)^T\xi_1|^2+\delta_n^2\xi_1^TD^2f(x)\xi_1\big]+\hat{R}_n(f,x,\xi_1),$$

where the reminder term can be estimated by

$$|\hat{R}_n(f,x,\xi_1)|\leq\hat{\rho}\left(\sum_{i=1}^3\|D^if\|_\infty\right)r_n$$

for a suitable function $\hat{\rho}\colon\mathbb{R}_+\to\mathbb{R}_+$ satisfying $\hat{\rho}(\varepsilon)\to 0$ as $\varepsilon\to 0$ and a sequence $(r_n)_{n\in\mathbb{N}}\subset\mathbb{R}_+$ with $r_n\to 0$. Since $\delta_n\to\delta$, $\alpha_n\delta_n^2\to\gamma$, $h_n\to 0$ and $\alpha_nh_n\to 0$, we obtain

$$\lim_{n\to\infty}\left\|\frac{J_nf-f}{h_n}-\frac{1}{2}\mathcal{E}\big[\delta^2\xi_1^TD^2f(x)\xi_1+\gamma|Df(x)^T\xi_1|^2\big]-Df(x)^T\psi(x)\right\|_\infty=0$$

for all $f\in\mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$. In addition, the previous estimates and Corollary 3.4.8 guarantee that Assumption 8.2.7(ii) is satisfied and, since the random variables $(\xi_n)_{n\in\mathbb{N}}$ are iid, we obtain

$$(J_n^kf)(x)=\frac{1}{\alpha_n}\log\left(\mathcal{E}\left[\exp\left(\alpha_nf\left(X_{k_n^th_n}^{n,\delta_n,x}\right)\right)\right]\right)$$

for all $k,n\in\mathbb{N}$, $f\in\mathrm{C}_\mathrm{b}(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$. Now, the claim follows from Theorem 8.2.8. □

If the limit is a Hamilton–Jacobi semigroup, we obtain the following Laplace principle.

Corollary 8.3.12. *Let $(\delta_n)_{n\in\mathbb{N}}\subset(0,\infty)$ be a sequence with $\delta_n\to 0$ and $\delta_n^2n\to\infty$. Then,*

$$\delta_n^2\log\mathcal{E}\left[\exp\left(\frac{1}{\delta_n^2}f\left(\frac{\delta_n}{\sqrt{n}}\sum_{i=1}^n\xi_i\right)\right)\right]\to\sup_{y\in\mathbb{R}^d}\big(f(y)-\varphi(y)\big)$$

for all $f\in\mathrm{C}_\mathrm{b}(\mathbb{R}^d)$, where the rate function $\varphi\colon\mathbb{R}^d\to[0,\infty]$ is given by

$$\varphi(y):=\sup_{z\in\mathbb{R}^d}\big(y^Tz-\tfrac{1}{2}\mathcal{E}\big[|z^T\xi_1|^2\big]\big).$$

Proof. Applying Theorem 8.3.11 with $h_n:={}^1\!/\!_n$, $\alpha_n:={}^1\!/\!_{\delta_n^2}$, and $\psi\equiv 0$ yields

$$T_{1,0}(t)f=\lim_{n\to\infty}\delta_n^2\log\mathcal{E}\left[\exp\left(\frac{1}{\delta_n^2}f\left(\,\cdot\,+\frac{\delta_n}{\sqrt{n}}\sum_{i=1}^n\xi_i\right)\right)\right]\quad\text{for all } f\in\mathrm{C}_\mathrm{b}(\mathbb{R}^d,$$

$$B_{1,0}f=\frac{1}{2}\mathcal{E}[|Df(x)^T\xi_1|^2]\quad\text{for all } f\in\mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d).$$

In addition, similar to the proof Theorem 7.3.4, one can show that $(T_{1,0}(t))_{t\geq 0}$ can be represented by the Hopf–Lax formula

$$(T_{1,0}(t)f)(x)=\sup_{x\in\mathbb{R}^d}\big(f(x+ty)-\varphi(y)t\big)\quad\text{for all } t\geq 0 \text{ and } f\in\mathrm{C}_\mathrm{b}(\mathbb{R}^d).$$ □

Moreover, we obtain a Laplace principle corresponding to the Freidlin–Wentzell large deviations principle [82] which characterizes the convergence rate of the solution $X_t^{\delta,x}$ of the SDE (8.41), driven by a standard Brownian motion, to the solution $X_t^{0,x}=u(t,x)$ of the ODE (8.40). In the sublinear case, where the solution $X_t^{\delta,x}$ of the SDE (8.41) is driven by a G-Brownian motion, we refer to [41, 86] for corresponding large deviations principles.

Corollary 8.3.13. *Let $(\delta_n)_{n\in\mathbb{N}} \subset (0,\infty)$ be a sequence with $\delta_n \to 0$. Then,*

$$\delta_n^2 \log\left(\mathcal{E}\left[\exp\left(\frac{1}{\delta_n^2} f\left(X_t^{\delta_n,x}\right)\right)\right]\right) \to \sup_{y\in\mathbb{R}^d} \big(f(y) - \varphi(y)\big)$$

for all $f \in \mathrm{C_b}(\mathbb{R}^d)$, where the rate function $\varphi\colon \mathbb{R}^d \to [0,\infty]$ is given by

$$\varphi(y) := \sup_{z\in\mathbb{R}^d} \big(y^T z - \tfrac{1}{2}\mathcal{E}\big[|z^T\xi_1|^2\big]\big).$$

Proof. For every $n \in \mathbb{N}$, be denote by $(S_{\delta_n}(t))_{t\geq 0}$ the corresponding semigroup from Theorem 8.3.9. Moreover, for a fixed $n \in \mathbb{N}$, we choose $\alpha_k := {}^1\!/\!{}_{\delta_n^2}$ for all $k \in \mathbb{N}$ and choose a sequence $(h_k)_{k\in\mathbb{N}} \subset (0,\infty)$ with $h_k \to \infty$. Define

$$(I_{k,\delta_n} f)(x) := \mathcal{E}\left[f\left(x + h_n\psi(x) + \delta_n\sqrt{h_n}\xi_1\right)\right] \quad \text{and} \quad J_{k,\delta_n} := \frac{1}{\alpha_k}\log\left(I_{k,\delta_n} e^{\alpha_k f}\right)$$

for all $k,n \in \mathbb{N}$, $f \in \mathrm{C_b}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Theorem 8.3.9 and Theorem 8.3.11 imply

$$\begin{aligned}
\delta_n^2 \log\left(\mathcal{E}\left[\exp\left(\frac{1}{\delta_n^2} f\left(X_t^{\delta,\cdot}\right)\right)\right]\right) &= \delta_n^2 \log\left(S_{\delta_n}(t) e^{f/\delta_n^2}\right) \\
&= \lim_{k\to\infty} \delta_n^2 \log\left((I_{k,\delta_n}^{m_k^t} e^{f/\delta_n^2}\right) \\
&= \lim_{k\to\infty} (J_{k,\delta_n})^{m_k^t} f = T_{1,\delta_n}(t) f
\end{aligned}$$

for all $t \geq 0$, $f \in \mathrm{C_b}(\mathbb{R}^d)$ and sequences $(m_k^t)_{k\in\mathbb{N}} \subset \mathbb{N}$ with $m_k^t h_k \to t$. Furthermore, Theorem 8.3.11 and Corollary 3.4.10 guarantee that Assumption 8.2.2 is satisfied. Indeed, it follows from the proof of Theorem 8.3.11 and Theorem 1.4.2 that

$$\lim_{h\downarrow 0}\left\|\frac{T_{1,0}(h)f - f}{h} - B_{1,0} f\right\|_\infty = 0 \quad \text{for all } f \in \mathrm{C_b^\infty}(\mathbb{R}^d).$$

Hence, the claim follows from Theorem 8.2.3 and the representation of $(T_{1,0}(t))_{t\geq 0}$ by the Hopf–Lax formula from Corollary 8.3.12. □

8.3.6 Discretization of stochastic optimal control problems

In this subsection, we fix a Borel measurable function $\varphi\colon \mathbb{S}_+^d \times \mathbb{R}^d \to [0,\infty]$ and consider the value function of a dynamic stochastic control problem with finite time horizon given by

$$(T(t)f)(x) := \sup_{(a,b)\in\mathcal{A}} \left(\mathbb{E}\Big[f\Big(x + \int_0^t \sqrt{a_s}\,\mathrm{d}W_s + \int_0^t b_s\,\mathrm{d}s\Big)\Big] - \mathbb{E}\Big[\int_0^t \varphi(a_s,b_s)\,\mathrm{d}s\Big]\right),$$

where $(W_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion on a complete filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ satisfying the usual conditions and $\mathcal{A}$ consists of all predictable processes $(a,b)\colon \Omega\times\mathbb{R}_+ \to \mathbb{S}_+^d \times \mathbb{R}^d$ such that

$$\mathbb{E}\left[\int_0^t |a_s| + |b_s|\,\mathrm{d}s\right] < \infty \quad \text{for all } t \geq 0.$$

The aim of this subsection is to approximate $T(t)f$ by iterating a sequence of discrete static control problems given by

$$(I_n f)(x) := \sup_{(a,b)\in\mathcal{A}_n} \big(\mathbb{E}\big[f\big(x+\sqrt{ah_n}\xi_n + bh_n\big)\big] - \varphi_n(a,b)h_n\big),$$

where $\mathcal{A}_n \subset \mathbb{S}_+^d \times \mathbb{R}^d$, $x \in X_n \subset \mathbb{R}^d$ and $\varphi_n\colon \mathcal{A}_n \to [0,\infty]$. To be precise, we show that

$$T(t)f = \lim_{n\to\infty} I_n^{k_n^t} f \quad \text{for all } (f,t) \in \mathrm{C_b}(\mathbb{R}^d) \times \mathbb{R}_+,$$

where $h_n \to 0$ and $k_n^t h_n \to t$.

Subsequently, we formalize the definition of the operators $(I_n)_{n\in\mathbb{N}}$ and impose sufficient conditions to guarantee the convergence. Let $(h_n)_{n\in\mathbb{N}} \subset (0,\infty)$ be a sequence with $h_n \to 0$ and define $\mathcal{T}_n := \{kh_n\colon k \in \mathbb{N}_0\}$ for all $n \in \mathbb{N}$. Moreover, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of closed sets $X_n \subset \mathbb{R}^d$ with $x+y \in X_n$ for all $x,y \in X_n$, such that, for every $x \in \mathbb{R}^d$, there exists $x_n \in X_n$ with $x_n \to x$. We choose random variables $\xi_n\colon \Omega \to \mathbb{R}^d$ with $\mathbb{E}[\xi_n] = 0$, $\mathbb{E}[\xi_n\xi_n^T] = \mathbb{1} \in \mathbb{R}^{d\times d}$ and $\sup_{n\in\mathbb{N}} \mathbb{E}[|\xi_n|^3|] < \infty$ and subsets $\mathcal{A}_n \subset \mathbb{S}_+^d \times \mathbb{R}^d$. Suppose that

$$x + \sqrt{ah_n}\xi_n(\omega) + bh_n \in X_n$$

for all $n \in \mathbb{N}$, $x,y \in X_n$, $(a,b) \in \mathcal{A}_n$ and $\omega \in \Omega$. Finally, let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of functions $\varphi_n\colon \mathcal{A}_n \to [0,\infty]$ and define

$$\varphi_n^*\colon \mathbb{R}^{d\times d} \times \mathbb{R}^d \to \mathbb{R},\ (a,b) \mapsto \sup_{(a',b')\in\mathcal{A}_n} \left(\frac{1}{2}\operatorname{tr}(aa') + bb' - \varphi_n(a',b')\right),$$
$$\varphi^*\colon \mathbb{R}^{d\times d} \times \mathbb{R}^d \to \mathbb{R},\ (a,b) \mapsto \sup_{(a',b')\in\mathbb{S}_+^d\times\mathbb{R}^d} \left(\frac{1}{2}\operatorname{tr}(aa') + bb' - \varphi(a',b')\right).$$

Assumption 8.3.14. Suppose that the following conditions are satisfied:

(i) There exists $(a^*,b^*) \in \mathbb{S}_+^d \times \mathbb{R}^d$ with $\varphi(a^*,b^*) = 0$.

(ii) It holds $\lim_{|a|+|b|\to\infty} \frac{\varphi(a,b)}{|a|+|b|} = \infty$.

(iii) For every $n \in \mathbb{N}$, there exists $(a_n^*,b_n^*) \in \mathcal{A}_n$ with $\varphi_n(a_n^*,b_n^*) = 0$. Furthermore, it holds $\sup_{n\in\mathbb{N}}(|a_n^*| + |b_n^*|) < \infty$.

(iv) It holds $\lim_{|a|+|b|\to\infty} \inf_{n\in\mathbb{N}} \frac{\varphi_n(a,b)}{|a|+|b|} = \infty$.

(v) It holds $\lim_{n\to\infty} \sup_{|a|+|b|\le r} |\varphi_n^*(a,b) - \varphi^*(a,b)| = 0$ for all $r \ge 0$.

Theorem 8.3.15. *Suppose that Assumption 8.3.14 is satisfied. Then, there exists a strongly continuous convex monotone semigroup $(S(t))_{t\ge0}$ on $\mathrm{C_b}(\mathbb{R}^d)$ with $S(t)0 = 0$ given by*

$$S(t)f = \lim_{n\to\infty} I_n^{k_n^t} f \quad \textit{for all } (f,t) \in \mathrm{C_b}(\mathbb{R}^d) \times \mathbb{R}_+,$$

where $(k_n^t)_{n\in\mathbb{N}} \subset \mathbb{N}$ is an arbitrary sequence with $k_n^t h_n \to t$, such that

$$\mathrm{C_b^\infty}(\mathbb{R}^d) \subset D(A) \quad \textit{and} \quad Af = \sup_{(a,b)\in\mathbb{S}_+^d\times\mathbb{R}^d} \left(\frac{1}{2}\operatorname{tr}\big(aD^2f(x)\big) + b^TDf(x) - \varphi(a,b)\right)$$

for all $f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$. *Moreover, the conditions of Theorem 4.4.6 are satisfied and therefore*

$$S(t)f = T(t)f \quad \text{for all } t \geq 0 \text{ and } f \in \mathrm{C}_\kappa(\mathbb{R}^d).$$

Proof. In order to apply Theorem 8.2.8, we have to verify Assumption 8.2.7. It is straightforward to show that the operators $I_n \colon \mathrm{C}_\mathrm{b}(X_n) \to \mathrm{C}_\mathrm{b}(X_n)$ are well-defined and satisfy

- I_n is convex and monotone with $I_n 0 = 0$ for all $n \in \mathbb{N}$,
- $\|I_n f - I_n g\|_{\infty,X_n} \leq \|f-g\|_{\infty,X_n}$ for all $n \in \mathbb{N}$ and $f, g \in \mathrm{C}_\mathrm{b}(X_n)$,
- $\tau_x I_n f = I_n(\tau_x f)$ for all $n \in \mathbb{N}$, $f \in \mathrm{C}_\mathrm{b}(X_n)$ and $x \in X_n$,
- $I_n \colon \mathrm{Lip}_\mathrm{b}(X_n, r) \to \mathrm{Lip}_\mathrm{b}(X_n, r)$ for all $n \in \mathbb{N}$ and $r \geq 0$.

Now, let $f \in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d)$. For every $n \in \mathbb{N}$ and $x \in X_n$,

$$\begin{aligned}
&f(x + \sqrt{ah_n}\xi_n + bh_n) - f(x) \\
&= \int_0^1 Df(x + t(\sqrt{ah_n}\xi_n + bh_n))^T(\sqrt{ah_n}\xi_n + bh_n)\,\mathrm{d}t \\
&= \int_0^1 Df(x + t(\sqrt{ah_n}\xi_n + bh_n))bh_n\,\mathrm{d}t + \int_0^1 Df(x+tbh_n))^T\sqrt{ah_n}\xi_n\,\mathrm{d}t \\
&\quad + \int_0^1\int_0^1 \sqrt{ah_n}\xi_n^T D^2 f(x + st\sqrt{ah_n}\xi_n + tbh_n)\sqrt{ah_n}\xi_n t\,\mathrm{d}s\,dt.
\end{aligned}$$

Since $\mathbb{E}[\xi_n] = 0$ and $\mathbb{E}[|\xi_n|^2] = d$, we obtain

$$(I_n^{a,b}f - f)(x) - \varphi_n(a,b)h_n \leq \left(\frac{1}{2}d|a| \cdot \|D^2 f\|_\infty + |b| \cdot \|Df\|_\infty - \varphi_n(a,b)\right)h_n$$

for all $n \in \mathbb{N}$, $(a,b) \in \mathcal{A}_n$ and $x \in X_n$, where $(I_n^{a,b}f)(x) := \mathbb{E}[f(x + \sqrt{ah_n}\xi_n + bh_n)]$. Hence, due to Assumption 8.3.14(iv), there exists $r \geq 0$ with

$$I_n f = \sup_{(a,b)\in\mathcal{A}_n^r} \left(I_n^{a,b}f - \varphi_n(a,b)h_n\right) \quad \text{for all } n \in \mathbb{N}, \tag{8.42}$$

where $\mathcal{A}_n^r := \{(a,b) \in \mathcal{A}_n \colon |a| + |b| \leq r\}$. By increasing $r \geq 0$, we can assume that

$$\varphi_n^*(D^2 f(x), Df(x)) = \sup_{(a,b)\in\mathcal{A}_n^r} \left(\frac{1}{2}\operatorname{tr}(aD^2 f(x)) + b^T Df(x) - \varphi_n(a,b)\right) \tag{8.43}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$. For every $n \in \mathbb{N}$, $(a,b) \in \mathcal{A}_n^r$ and $x \in X_n$, Taylor's formula yields

$$\begin{aligned}
f(x + \sqrt{ah_n}\xi_n + bh_n) &= f(x) + (\sqrt{ah_n}\xi_n + bh_n)^T Df(x) \\
&\quad + \frac{1}{2}(\sqrt{ah_n} + bh_n)^T D^2 f(x)(\sqrt{ah_n}\xi_n + bh_n) + R_n(f, x\xi_n),
\end{aligned}$$

where the remainder term can be estimated by

$$|R_n(f,x,\xi_n)| \leq \frac{|\sqrt{ah_n}\xi_n + bh_n|^3}{6}\|D^3 f\|_\infty.$$

It follows from $\mathbb{E}[\xi_n]=0$ and $\mathbb{E}[\xi_n\xi_n^T]=\mathbb{1}$ that

$$\begin{aligned}\frac{I_n^{a,b}f-f}{h_n} &= \mathbb{E}\left[\frac{f(x+\sqrt{ah_n}\xi_n+bh_n)-f(x)}{h_n}\right] \\ &= b^T Df(x) + \frac{1}{2}\operatorname{tr}\big(aD^2f(x)\big) + \tilde{R}_n(f,x,\xi_n),\end{aligned}$$

where the remainder term can be estimated by

$$|\tilde{R}_n(f,x,\xi_n)| \leq \frac{1}{2}|b|^2 h_n |D^2 f(x)| + \mathbb{E}[|R_n(f,x,\xi_n)|].$$

Hence, we can use equation (8.42), equation (8.43), the condition $\sup_{n\in\mathbb{N}}\mathbb{E}[|\xi_n|^3]<\infty$ and Assumption 8.3.14(v) to obtain

$$\lim_{n\to\infty}\left\|\frac{I_n f - f}{h_n} - \varphi^*(D^2 f, Df)\right\|_\infty = 0 \quad \text{for all } f\in \mathrm{C}_\mathrm{b}^\infty(\mathbb{R}^d).$$

Moreover, the previous estimates and Corollary 3.4.8 guarantee that Assumption 8.2.7(ii) is satisfied while the results Section 6.1 yield that the semigroup $(T(t))_{t\geq 0}$ satisfies Assumption 4.4.5. Now, claim follows from Theorem 8.2.8 and Theorem 4.4.6. □

8.3.7 Markov chain approximations for perturbed transition semigroups

Let $(\psi_t)_{t\geq 0}$ be a family of functions $\psi_t\colon \mathbb{R}^d\to\mathbb{R}^d$ and $(\mu_t)_{t\geq 0}\subset\mathcal{P}_p(\mathbb{R}^d)$, where $\mathcal{P}_p(\mathbb{R}^d)$ consists of all probability measures $\mu\colon\mathcal{B}(\mathbb{R}^d)\to[0,1]$ with $\int_{\mathbb{R}^d}|x|^p\,\mu(\mathrm{d}x)<\infty$. In Subsection 8.3.7, we followed the framework of [14, 85] and studied Wasserstein perturbations of the linear transition semigroup

$$(R(t)f)(x) := \int_{\mathbb{R}^d} f(\psi_t(x)+y)\mu_t(\mathrm{d}y)$$

which can be constructed as the monotone limit

$$(T(t)f)(x) = \inf_{n\in\mathbb{N}} J(2^{-n}t)^{2^n}f = \lim_{n\to\infty} J(2^{-n}t)^{2^n}f,$$

where the one step operators are defined by

$$(J(t)f)(x) := \sup_{\nu\in\mathcal{P}_p(\mathbb{R}^d}\left(\int_{\mathbb{R}^d}(f(\psi_t(x)+y)\nu(\mathrm{d}y) - t\varphi\left(\frac{\mathcal{W}_p(\mu_t,\nu)}{t}\right)\right)$$

for all $t\geq 0$, $f\in\mathrm{C}_\mathrm{b}(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$. Here, we denote by

$$\mathcal{W}_p(\mu,\nu) := \left(\inf_{\pi\in\Pi(\mu,\nu)}\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^p\pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{p}} \quad \text{for all } \mu,\nu\in\mathcal{P}_p(\mathbb{R}^d)$$

the p-Wasserstein distance, where the set $\Pi(\mu,\nu)$ consists of all probability measures on $\mathcal{B}(\mathbb{R}^d\times\mathbb{R}^d)$ with first marginal μ and second marginal ν. Furthermore,

$$t\varphi(c/t) := \begin{cases} 0, & c=t=0,\\ \infty, & c\neq 0, t=0, \end{cases}$$

for a non-decreasing function $\varphi\colon \mathbb{R}_+\to[0,\infty]$. In this context, the family $(R(t))_{t\geq 0}$ is called the reference model and $(T(t))_{t\geq 0}$ is referred to as the perturbed transition semigroup. For related distributionally robust optimization problems and the description of non-parametric uncertainty using Wasserstein distances, we refer to [13, 22, 134].

The aim of this subsection is to provide a Markov chain approximation for the perturbed transition semigroup. To that end, let $(h_n)_{n\in\mathbb{N}}\subset(0,\infty)$ be a sequence with $h_n\to 0$ and define $\mathcal{T}_n := \{kh_n\colon k\in\mathbb{N}_0\}$ for all $n\in\mathbb{N}$. Moreover, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of closed sets $X_n\subset\mathbb{R}^d$ with $x+y\in X_n$ for all $x,y\in X_n$, such that, for every $x\in\mathbb{R}^d$, there exists $x_n\in X_n$ with $x_n\to x$. For every $n\in\mathbb{N}$, the set $\mathcal{P}_p(X_n)$ consists of all probability measures $\mu\colon\mathcal{B}(X_n)\to[0,1]$ with $\int_{X_n}|x|^p\,\mu(\mathrm{d}x)<\infty$ and is endowed with the p-Wasserstein distance

$$\mathcal{W}_p^{X_n}(\mu,\nu) := \left(\inf_{\pi\in\Pi(\mu,\nu)}\int_{X_n\times X_n}|x-y|^p\,\pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{p}}.$$

Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of functions $\psi_n\colon X_n\to X_n$ and $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures $\mu_n\in\mathcal{P}_p(X_n)$. For every $n\in\mathbb{N}$, $f\in\mathrm{C_b}(X_n)$ and $x\in X_n$, we define

$$(I_nf)(x) := \sup_{\nu\in\mathcal{P}_p(X_n)}\left(\int_{X_n} f(\psi_n(x)+y)\,\nu(\mathrm{d}y) - \varphi\left(\frac{\mathcal{W}_p^{X_n}(\mu_n,\nu)}{h_n}\right)h_n\right).$$

Furthermore, we define

$$A_0\colon D(A_0)\to\mathrm{C_b}(\mathbb{R}^d),\ f\mapsto \lim_{n\to\infty}\frac{1}{h_n}\left(\int_{X_n} f(\psi_n(\,\cdot\,)+y)\,\mu_n(\mathrm{d}y) - f\right),$$

where $D(A_0)$ consists of all functions $f\in\mathrm{C_b}(\mathbb{R}^d)$ such that the previous limit exists.

Assumption 8.3.16. Suppose that the following conditions are satisfied:

(i) It holds $\varphi(0)=0$ and $\lim_{c\to\infty}\frac{\varphi(c)}{c}=\infty$.

(ii) It holds $\lim_{n\to\infty}\int_{X_n}|y|\,\mu_n(\mathrm{d}y)=0$.

(iii) There exists $L\geq 0$ such that, for every $n\in\mathbb{N}$ and $x,y\in X_n$,

$$|\psi_n(x)-\psi_n(y)|\leq e^{Lh_n}|x-y| \quad\text{and}\quad |\psi_n(x+y)-x-\psi_n(y)|\leq Lh_n|x|.$$

Moreover, we assume that $\sup_{x\in K_n}|\psi_n(x)-x|\to 0$ for all $K\Subset\mathbb{R}^d$.

(iv) For every $r\geq 0$, there exists a sequence $(\delta_n)_{n\in\mathbb{N}}\subset(0,1]$ with $\delta_n\downarrow 0$ such that, for every $\lambda\in B_{\mathbb{R}^d}(r)$, there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n\in X_n$ and

$$\left|\frac{\lambda_n}{h_n}-\lambda\right|\leq\delta_n \quad\text{and}\quad \varphi\left(\frac{|\lambda_n|}{h_n}\right)-\varphi(|\lambda|)\leq\delta_n \quad\text{for all } n\in\mathbb{N}.$$

(v) It holds $\mathrm{C}_{\mathrm{b}}^{\infty}(\mathbb{R}^d) \subset D(A_0)$. Moreover, there exist $N \in \mathbb{N}$ and a non-decreasing function $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ with

$$\left\| \int_{X_n} f(\psi_n(\,\cdot\,) + y)\, \mu_n(\mathrm{d}y) - f \right\|_\infty \leq \rho\left(\sum_{i=1}^N \|D^i f\|_\infty \right) h_n$$

for all $n \in \mathbb{N}$ and $f \in \mathrm{C}_{\mathrm{b}}^{\infty}(\mathbb{R}^d)$.

Assumption 8.3.16(i) guarantees that

$$\varphi^*(a) := \sup_{b \geq 0} \big(ab - \varphi(b) \big) < \infty \quad \text{for all } a \geq 0.$$

In particular, the real-valued convex function $\varphi^*\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. We recall the following conditions from Section 8.3.7 which guarantees that the semigroup $(T(t))_{t\geq 0}$ satisfies Assumption 4.4.5.

Assumption 8.3.17. Suppose that the following conditions are satisfied:

(i) It holds $\lim_{t\downarrow 0} \int_{\mathbb{R}^d} |y|^p \,\mathrm{d}\mu_t(y) = 0$.

(ii) There exist $r > 0$ and $c \geq 0$ with $\mu_t(B_{\mathbb{R}^d}(r)^c) \leq ct$ for all $t \in [0,1]$.

(iii) It holds $\psi_t(0) = 0$ for all $t \geq 0$ and there exists $c \geq 0$ with

$$|\psi_t(x) - \psi_t(y) - (x - y)| \leq ct|x - y|$$

for all $x, y \in \mathbb{R}^d$ and $t \in [0,1]$.

(iv) It holds $A_0 f = \lim_{h\downarrow 0} \frac{R(h)f - f}{h}$ for all $f \in \mathrm{C}_{\mathrm{b}}^{\infty}(\mathbb{R}^d)$.

Theorem 8.3.18. *Suppose that Assumption 8.3.16 is satisfied. Then, there exists a strongly continuous convex monotone semigroup $(S(t))_{t\geq 0}$ on $\mathrm{C}_{\mathrm{b}}(\mathbb{R}^d)$ with $S(t)0 = 0$ given by*

$$S(t)f = \lim_{n\to\infty} I_n^{k_n^t} f_n \quad \text{for all } (f,t) \in \mathrm{C}_{\mathrm{b}}(\mathbb{R}^d) \times \mathbb{R}_+,$$

where $(k_n^t)_{n\in\mathbb{N}} \subset \mathbb{N}$ is arbitrary sequence with $k_n^t h_n \to t$, such that

$$\mathrm{C}_{\mathrm{b}}^{\infty}(\mathbb{R}^d) \subset D(A) \quad \text{and} \quad Af = A_0 f + \varphi^*(|Df|) \quad \text{for all } f \in \mathrm{C}_{\mathrm{b}}^{\infty}(\mathbb{R}^d).$$

Moreover, the conditions of Theorem 4.4.6 are satisfied and therefore

$$S(t)f = T(t)f \quad \text{for all } t \geq 0 \text{ and } f \in \mathrm{C}_{\kappa}(\mathbb{R}^d).$$

Proof. In order to apply Theorem 8.2.8, we have to verify Assumption 8.2.7. For every $n \in \mathbb{N}$, $r \geq 0$, $f \in \mathrm{Lip}_{\mathrm{b}}(X_n, r)$ and $x, y \in X_n$, it follows from Assumption 8.3.16(iv) that

$$|(I_n f)(x) - (I_n f)(y)| \leq r|\psi_n(x) - \psi_n(y)| \leq e^{Lh_n} r|x - y|,$$
$$|\tau_x I_n f - I_n(\tau_x f)|(y) \leq r|\psi_n(x + y) - x - \psi_n(y)| \leq Lrh_n|x|.$$

The operators $I_n\colon \mathrm{C_b}(X_n)\to \mathrm{F_b}(X_n)$ are convex and monotone with $I_n 0=0$ such that

$$\|I_n f - I_n g\|_{\infty,X_n} \le \|f-g\|_{\infty,X_n} \quad \text{for all } n\in\mathbb{N} \text{ and } f,g\in \mathrm{C_b}(X_n).$$

Moreover, for every $r\ge 0$ and $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ with

$$(I_n f)(x) = \sup_{\mathcal{W}_p^{X_n}(\mu_n,\nu)\le\varepsilon} (I_n^\nu f)(x) \tag{8.44}$$

for all $n\ge n_0$, $f\in B_{\mathrm{C_b}(X_n)}(r)$ and $x\in X_n$, where

$$(I_n^\nu f)(x) := \int_{X_n} f(\psi_n(x)+y)\,\nu(\mathrm{d}y) - \varphi\left(\frac{\mathcal{W}_p^{X_n}(\mu_n,\nu)}{h_n}\right) h_n.$$

Indeed, by Assumption 8.3.16(ii), there exists $n_0\in\mathbb{N}$ with $\varphi(\varepsilon/h_n)h_n\ge 2r$ and thus

$$I_n^\nu f \le -r \le I_n^{\mu_n} f \quad \text{for all } n\ge n_0 \text{ and } \mathcal{W}_p(\mu_n,\nu)\ge\varepsilon.$$

Now, let $f\in \mathrm{C_b^\infty}(\mathbb{R}^d)$. For every $n\in\mathbb{N}$, $x\in X_n$ and $\nu\in\mathcal{P}_p(X_n)$,

$$\begin{aligned}
&\frac{1}{h_n}\left(\int_{X_n} f(\psi_n(x)+y)\,\nu(\mathrm{d}y) - f(x) - \varphi\left(\frac{\mathcal{W}_p^{X_n}(\mu_n,\nu)}{h_n}\right) h_n\right)\\
&= \frac{1}{h_n}\left(\int_{X_n} f(\psi_n(x)+y)\,\nu(\mathrm{d}y) - \int_{X_n} f(\psi_n(x)+y)\,\mu_n(\mathrm{d}y)\right) - \varphi\left(\frac{\mathcal{W}_p^{X_n}(\mu_n,\nu)}{h_n}\right)\\
&\quad + \frac{1}{h_n}\left(\int_{X_n} f(\psi_n(x)+y)\mu_n(\mathrm{d}y) - f(x)\right).
\end{aligned} \tag{8.45}$$

It follows from Assumption 8.3.16(v), the Kantorowitsch transport duality, the fact that f is $\|Df\|_\infty$-Lipschitz and Hölder's inequality that

$$\begin{aligned}
&\frac{1}{h_n}\left(\int_{X_n} f(\psi_n(x)+y)\,\nu(\mathrm{d}y) - f(x) - \varphi\left(\frac{\mathcal{W}_p^{X_n}(\mu_n,\nu)}{h_n}\right) h_n\right)\\
&\le \frac{\mathcal{W}_p^{X_n}(\mu_n,\nu)\|Df\|_\infty}{h_n} - \varphi\left(\frac{\mathcal{W}_p^{X_n}(\mu_n,\nu)}{h_n}\right) + \rho\left(\sum_{i=1}^N \|D^i f\|_\infty\right)\\
&\le \varphi^*(\|Df\|_\infty) + \rho\left(\sum_{i=1}^N \|D^i f\|_\infty\right)
\end{aligned}$$

for all $n\in\mathbb{N}$, $x\in X_n$ and $\nu\in\mathcal{P}_p(X_n)$ and therefore

$$\left\|\frac{I_n f - f}{h_n}\right\|_\infty \le \varphi^*(\|Df\|_\infty) + \rho\left(\sum_{i=1}^N \|D^i f\|_\infty\right) \quad \text{for all } n\in\mathbb{N}. \tag{8.46}$$

Next, we show that, for every $K\Subset\mathbb{R}^d$ and $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ with

$$\left(\frac{I_n f - f}{h_n}\right)(x) \le A_0 f(x) + \varphi^*\big(|Df(x)|+\varepsilon\big) + \varepsilon \tag{8.47}$$

for all $n \geq n_0$ and $x \in K_n$. Since Df is continuous, there exists $\delta \in (0, \varepsilon/4]$ with

$$|Df(x) - Df(x+y)| < \tfrac{\varepsilon}{4} \quad \text{for all } x \in K \text{ and } y \in B_{\mathbb{R}^d}(4\delta). \tag{8.48}$$

Furthermore, by Assumption 8.3.16(i) and equation (8.44), we can choose $n_0 \in \mathbb{N}$ with

$$\mu_n(B_{X_n}(\delta)^c) < \frac{\varepsilon}{4} \quad \text{and} \quad \frac{1}{\delta}\int_{X_n} |y|\,\mu_n(\mathrm{d}y) < \frac{\varepsilon}{4} \quad \text{for all } n \geq n_0 \tag{8.49}$$

such that

$$(I_n f)(x) = \sup_{\mathcal{W}_p^{X_n}(\mu_n,\nu)\leq\delta^2} (I_n^\nu f)(x) \quad \text{for all } n \geq n_0 \text{ and } x \in X_n. \tag{8.50}$$

Let $n \geq n_0$, $x \in K_n$ and $\nu \in \mathcal{P}_p(X_n)$ with $\mathcal{W}_p^{X_n}(\mu_n,\nu) \leq \delta^2$. For an optimal coupling $\pi \in \Pi(\nu,\mu_n)$ and $q := \frac{p}{p-1}$, Hölder's inequality implies

$$\begin{aligned}
&\frac{1}{h_n}\left(\int_{X_n} f(\psi_n(x)+y)\,\nu(\mathrm{d}y) - \int_{X_n} f(\psi_n(x)+z)\,\mu_n(\mathrm{d}z)\right) - \varphi\left(\frac{\mathcal{W}_p^{X_n}(\mu_n,\nu)}{h_n}\right)\\
&= \frac{1}{h_n}\int_{X_n\times X_n} f(\psi_n(x)+y) - f(\psi_n(x)+z)\,\pi(\mathrm{d}y,\mathrm{d}z) - \varphi\left(\frac{\mathcal{W}_p^{X_n}(\mu_n,\nu)}{h_n}\right)\\
&= \frac{1}{h_n}\int_0^1\int_{X_n\times X_n} Df(\psi_n(x)+y+t(y-z))(y-z)\,\pi(\mathrm{d}y,\mathrm{d}z)\,\mathrm{d}t - \varphi\left(\frac{\mathcal{W}_p^{X_n}(\mu_n,\nu)}{h_n}\right)\\
&\leq \frac{a_n^\nu(x)}{h_n}\left(\int_{X_n\times X_n} |y-z|^p\,\pi(\mathrm{d}y,\mathrm{d}z)\right)^{\frac{1}{p}} - \varphi\left(\frac{\mathcal{W}_p^{X_n}(\mu_n,\nu)}{h_n}\right)\\
&= a_n^\nu(x)\frac{\mathcal{W}_p^{X_n}(\mu_n,\nu)}{h_n} - \varphi_n\left(\frac{\mathcal{W}_p^{X_n}(\mu_n,\nu)}{h_n}\right) \leq \varphi^*\big(a_n^\nu(x)\big),
\end{aligned} \tag{8.51}$$

where

$$a_n^\nu(x) := \int_0^1 \left(\int_{X_n\times X_n} |Df(\psi_n(x)+y+t(y-z)|^q\,\pi(\mathrm{d}y,\mathrm{d}z)\right)^{\frac{1}{q}}\mathrm{d}t.$$

It remains to show that

$$a_n^\nu(x) \leq |Df(x)| + \varepsilon \quad \text{for all } n \geq n_0. \tag{8.52}$$

For every $n \geq 0$ and $t \in [0,1]$, Minkowski's inequality and inequality (8.48) yield

$$\begin{aligned}
&\left(\int_{X_n\times X_n} |Df(\psi_n(x)+y+t(y-z)|^q\,\pi(\mathrm{d}y,\mathrm{d}z)\right)^{\frac{1}{q}} - |Df(x)|\\
&\leq \left(\int_{X_n\times X_n} |Df(\psi_n(x)+y+t(y-z) - Df(x)|^q\,\pi(\mathrm{d}y,\mathrm{d}z)\right)^{\frac{1}{q}}\\
&\leq \left(\int_{B_{X_n\times X_n}(\delta)} |Df(\psi_n(x)+y+t(y-z) - Df(x)|^q\,\pi(\mathrm{d}y,\mathrm{d}z)\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\|Df\|_\infty \pi\big(B_{X_n\times X_n}(\delta)^c\big) \\
&\leq 2\|Df\|_\infty \pi\big(B_{X_n\times X_n}(\delta)^c\big) + \varepsilon/4.
\end{aligned}
$$

Moreover, it follows from inequality (8.49) and Hölder's inequality that

$$
\begin{aligned}
\pi\big(B_{X_n\times X_n}(\delta)^c\big) &\leq \pi\big(B_{X_n}(\delta)^c \times X_n\big) + \pi\big(X_n \times B_{X_n}(\delta)^c\big) \\
&= \nu\big(B_{X_n}(\delta)^c\big) + \mu_n\big(B_{X_n}(\delta)^c\big) \\
&\leq \frac{1}{\delta}\int_{X_n} |y|\,\nu(\mathrm{d}y) + \frac{\varepsilon}{4} \\
&\leq \frac{1}{\delta}\mathcal{W}_p^{X_n}(\mu_n,\nu) + \frac{1}{\delta}\int_{X_n} |y|\,\mu_n(\mathrm{d}y) + \frac{\varepsilon}{4} \leq \delta + \frac{\varepsilon}{2} \leq \frac{3\varepsilon}{4}.
\end{aligned}
$$

Moreover, due to Assumption 8.3.16(v), there exists $n_1 \geq n_0$ with

$$
\left\| \frac{1}{h_n}\left(\int_{X_n} f(\psi_n(\,\cdot\,)+y)\,\mu_n(\mathrm{d}y) - f\right) - A_0 f\right\|_{\infty,K_n} \leq \varepsilon \quad \text{for all } n \geq n_1. \tag{8.53}
$$

Combing the equations (8.45) and (8.50) with the inequalities (8.51)–(8.53) yields that inequality (8.47) is valid. In order to obtain a similar lower bound, we show that, for every $K \Subset \mathbb{R}^d$ and $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ with

$$
\left(\frac{I_n f - f}{h_n}\right)(x) \geq A_0 f(x) - \varphi^*(|Df(x)|) - 2\varepsilon \tag{8.54}
$$

for all $n \geq n_0$ and $x \in K_n$. By Assumption 8.3.16(i), there exists $r \geq 0$ with

$$
\varphi^*(|Df(x)|) = \sup_{\lambda \in B_{\mathbb{R}^d}(r)} \big(\lambda Df(x) - \varphi(|\lambda|)\big) \quad \text{for all } x \in \mathbb{R}^d. \tag{8.55}
$$

Moreover, by Assumption 8.3.16(iv), there exists a sequence $(\delta_n)_{n\in\mathbb{N}} \subset (0,1]$ with $\delta_n \downarrow 0$ such that, for every $\lambda \in B_{\mathbb{R}^d}(r)$, there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \in X_n$ and

$$
\left|\frac{\lambda_n}{h_n} - \lambda\right| \leq \delta_n \quad \text{and} \quad \varphi\left(\frac{|\lambda_n|}{h_n}\right) - \varphi(|\lambda|) \leq \delta_n \quad \text{for all } n \in \mathbb{N}. \tag{8.56}
$$

In the sequel, we fix $\lambda \in B_{\mathbb{R}^d}(r)$ and choose such a sequence $(\lambda_n)_{n\in\mathbb{N}}$. Since the measures $\nu_n := \mu_n * \delta_{\lambda_n} \in \mathcal{P}_p(X_n)$ satisfy satisfy $\mathcal{W}_p^{X_n}(\mu_n,\nu_n) = |\lambda_n|$, we obtain

$$
\begin{aligned}
&\frac{1}{h_n}\left(\int_{X_n} f(\psi_n(x)+y)\,\nu_n(\mathrm{d}y) - \int_{X_n} f(\psi_n(x)+y)\,\mu_n(\mathrm{d}y)\right) - \varphi\left(\frac{\mathcal{W}_p^{X_n}(\mu_n,\nu_n)}{h_n}\right) \\
&= \frac{1}{h_n}\int_{X_n} f(\psi_n(x)+y+\lambda_n) - f(\psi_n(x)+y)\,\mu_n(\mathrm{d}y) - \varphi\left(\frac{|\lambda_n|}{h_n}\right) \\
&= \frac{\lambda_n}{h_n}\int_0^1\int_{X_n} Df(\psi_n(x)+y+t\lambda_n)\,\mu_n(\mathrm{d}y)\,\mathrm{d}t - \varphi\left(\frac{|\lambda_n|}{h_n}\right)
\end{aligned} \tag{8.57}
$$

for all $n \in \mathbb{N}$ and $x \in X_n$. Moreover, there exists $\delta > 0$ with

$$
|Df(x) - Df(x+y)| \leq \frac{\varepsilon}{4(1+r)} \quad \text{for all } x \in K \text{ and } y \in B_{\mathbb{R}^d}(3\delta). \tag{8.58}
$$

Since inequality (8.56) guarantees $|\lambda_n| \leq (1+r)h_n \to 0$ and due Assumption 8.3.16(ii) and (iii), we can choose $n_0 \in \mathbb{N}$ such that $|\lambda_n| \leq \delta$, $\sup_{x\in K_n} |\psi_n(x) - x| \leq \delta$,

$$2(1+r)\|Df\|_\infty \mu_n\big(B_{X_n}(\delta)^c\big) \leq \tfrac{\varepsilon}{4} \quad \text{and} \quad \big(1+\|Df\|_\infty\big)\delta_n \leq \tfrac{\varepsilon}{4} \tag{8.59}$$

for all $n \geq n_0$. Inequality (8.56), inequality (8.58) and inequality (8.51) imply

$$\begin{aligned}
&\frac{\lambda_n}{h_n}\int_0^1\int_{X_n} Df(\psi_n(x)+y+t\lambda_n)\,\mu_n(\mathrm{d}y)\,\mathrm{d}t - \varphi\left(\frac{|\lambda_n|}{h_n}\right) - \big(\lambda Df(x) - \varphi(|\lambda|)\big)\\
&\geq -\frac{|\lambda_n|}{h_n}\int_0^1\int_{X_n} |Df(\psi_n(x)+y+t\lambda_n) - Df(x)|\,\mu_n(\mathrm{d}y)\,\mathrm{d}t - \left|\frac{\lambda_n}{h_n} - \lambda\right| \cdot |Df(x)|\\
&\quad + \varphi(|\lambda|) - \varphi\left(\frac{|\lambda_n|}{h_n}\right)\\
&\geq -(1+r)\int_0^1\int_{B_{X_n}(\delta)} |Df(\psi_n(x)+y+t\lambda_n) - Df(x)|\,\mu_n(\mathrm{d}y)\,\mathrm{d}t\\
&\quad - 2(1+r)\|Df\|_\infty\mu_n\big(B_{X_n}(\delta)^c\big) - \tfrac{\varepsilon}{2} \geq -\varepsilon
\end{aligned} \tag{8.60}$$

for all $n \geq n_0$ and $x \in K_n$. Combining equation (8.45), inequality (8.53), equation (8.55) and inequality (8.60) yields that inequality (8.54) is valid. Hence, from inequality (8.47), inequality (8.54) and the continuity of φ^*, we obtain

$$\lim_{n\to\infty}\left\|\frac{I_n f - f}{h_n} - A_0 f - \varphi^*(|Df|)\right\|_{\infty,K_n} = 0.$$

In addition, inequality (8.46), Corollary 3.4.7 and Remark 3.4.11 guarantee that the operators $I_n\colon \mathrm{C_b}(X_n) \to \mathrm{C_b}(X_n)$ are well-defined and satisfy Assumption 8.2.7(ii) while the results in Section 8.3.7 yield that $(T(t))_{t\geq 0}$ satisfies the Assumption 4.4.5. Now, claim follows from Theorem 8.2.8 and Theorem 4.4.6. □

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