Universität Konstanz

A parabolic cross-diffusion system for granular materials

Gonzalo Galiano
Ansgar Jüngel
Julián Velasco

Konstanzer Schriften in Mathematik und Informatik
Nr. 177, Juni 2002

ISSN 1430–3558

A parabolic cross-diffusion system for granular materials

Gonzalo Galiano[*] Ansgar Jüngel[†] Julián Velasco[*]

Abstract

A cross-diffusion system of parabolic equations for the relative concentration and the dynamic repose angle of a mixture of two different granular materials in a long rotating drum is studied. The main feature of the system is the ability to describe the axial segregation of the two granular components. The existence of global-in-time weak solutions is shown by using entropy-type inequalities and approximation arguments. The uniqueness of solutions is proved if cross-diffusion is not too large. Furthermore, we show that in the non-segregating case, the transient solutions converge exponentially fast to the constant steady-state as time tends to infinity. Finally, numerical simulations show the long-time coarsening of the segregation bands in the drum.

Keywords. Strongly nonlinear parabolic system, cross-diffusion, segregation, existence of weak solutions, uniqueness of solutions, entropy-type estimates.

1991 Mathematics Subject Classification. 35K55, 76T25.

Acknowledgments. The authors acknowledge partial support from the Spanish-German Bilateral Project Acciones Integradas–DAAD. The first and the third author have been supported by the Spanish D.G.I. Project No. BFM2000-1324. The second author has been supported by the Deutsche Forschungsgemeinschaft, grants JU 359/3 (Gerhard-Hess Program) and JU 359/5 (Priority Program "Multiscale Problems"), the European TMR Project "Asymptotic methods in kinetic theory" (grant ERB-FM-BX-CT97-0157) and the AFF Project of the University of Konstanz.

1 Introduction

One important feature of granular materials, consisting of different components, is their ability to segregate under external agitation rather than to further mix. Mixtures of grains with different sizes in long rotating drums exhibit both radial and axial size segregation. Roughly speaking, radial segregation occurs during the first few revolutions of the drum and is often followed by slow axial

[*]Departamento de Matemáticas, Universidad de Oviedo, c/ Calvo Sotelo s/n, 33007 Oviedo, Spain. E-mail: galiano@orion.ciencias.uniovi.es, julian@orion.ciencias.uniovi.es.

[†]Fachbereich Mathematik und Statistik, Universität Konstanz, 78457 Konstanz, Germany. E-mail: ansgar.juengel@uni-konstanz.de.

segregation. Axial segregation leads to either a stable array of concentration bands or, after a very long time, to complete segregation [2, 3, 13].

Consider a mixture of two kind of particles with volume concentrations u_1, $u_2 \in [0,1]$, placed in a horizontal long narrow rotating cylinder of length $L > 0$. Let $u = u_1 - u_2 \in [-1,1]$ be the relative concentration of the mixture. Introduce the so-called dynamic angle of repose θ as the arctangent of the average slope of the free surface which is assumed to be flat (see Figure 1). The variables u and θ depend on the axial coordinate $z \in \Omega = (0,L)$ and on the time $t > 0$.

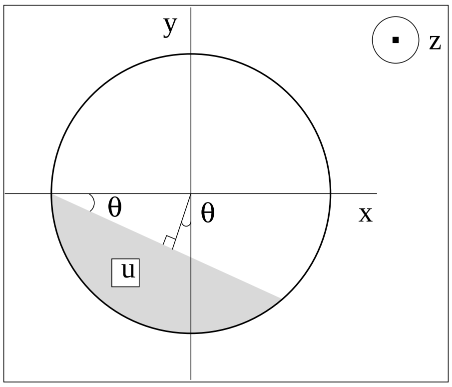


Figure 1: Relative concentration u and dynamical angle of repose θ in the geometry of the cross section of a rotating drum.

In [3] the following cross-diffusion system for the evolution of u and θ has been derived:

$$u_t - (\nu u_z - (1-u^2)\theta_z)_z = 0, \tag{1.1}$$
$$\theta_t - (\gamma u + \theta)_{zz} + \theta = \mu u \quad \text{in } Q_T := \Omega \times (0,T), \tag{1.2}$$

where the indices denote partial derivatives. The model (1.1)-(1.2) is obtained by averaging the mass conservation laws for the two components of the granular matter over the cross section of the cylinder, under the assumption that separation occurs only in a thin near-surface flow where the granular material is dilated and simply advected by the bulk flow.

The positive constant ν is related to the Fick diffusion constants arising in the surface fluxes of the two materials. The constant $\gamma > 0$ is proportional to the difference of the Fick diffusivities. Finally, μ is related to the difference of the static repose angles of the two kind of particles.

We impose periodic boundary conditions as in [3] and initial conditions for the variables:

$$\begin{array}{lll} u(0,\cdot) = u(L,\cdot), & u_z(0,\cdot) = u_z(L,\cdot) & \multirow{2}{*}{in $(0,T)$,} \\ \theta(0,\cdot) = \theta(L,\cdot), & \theta_z(0,\cdot) = \theta_z(L,\cdot) & \\ u(\cdot,0) = u_0, & \theta(\cdot,0) = \theta_0 & \text{in } \Omega. \end{array} \tag{1.3}$$

The terms $((1-u^2)\theta_z)_z$ and γu_{zz} in (1.1)-(1.2) are called *cross-diffusion* terms. It is well known that cross-diffusion seems to create pattern formation whereas diffusion tends to suppress pattern formation [10]. The final behavior of the solutions depends on the precise values of the parameters. We remark that segregation effects due to cross-diffusion are well known in population dynamics, and related cross-diffusion systems have been studied in mathematical biology (see, e.g., [11, 12]).

Mathematically, the parabolic system (1.1)-(1.2) has a full and non-symmetric diffusion matrix:

$$A := \begin{pmatrix} \nu & -(1-u^2) \\ \gamma & 1 \end{pmatrix}.$$

Problems with full diffusion matrix also arise, for instance, in semiconductor theory [5] and in non-equilibrium thermodynamics [7]. As a consequence, no classical maximum principle arguments and no regularity theory as for single equations are generally available for such kind of problems. Moreover, there are values for u and the parameters ν and γ for which A is not elliptic. The question arises if it is possible to prove the existence of global-in-time solutions.

The main aim of this paper is to prove that indeed the problem (1.1)-(1.3) admits a weak solution globally in time. The key of the proof is the observation that the system (1.1)-(1.2) possesses a functional whose time derivative is uniformly bounded in time if $|u| < 1$. Indeed, using the functions $\phi(u)$, where

$$\phi(s) := \frac{\gamma}{2} \log \frac{1+s}{1-s} \qquad \text{for } -1 < s < 1,$$

and θ in the weak formulation of (1.1) and (1.2), respectively, and adding the resulting equations leads to the inequality

$$\frac{d}{dt} \int_0^L \left(\Phi(u) + \frac{1}{2}\theta^2 \right) + \int_0^L (\gamma\nu u_z^2 + \theta_z^2) = \int_0^L (\mu u\theta - \theta^2) \le c, \tag{1.4}$$

where $c > 0$ only depends on μ and L. Here the function $\Phi(s) := \frac{\gamma}{2}(1-s)\log(1-s) + \frac{\gamma}{2}(1+s)\log(1+s) \ge 0$ is the primitive of ϕ such that $\Phi(0) = 0$. Observe that this estimate is purely formal since the values $|u| = 1$ are possible.

The estimate (1.4) has an important consequence. With the change of unknowns $u = g(v)$, where g is the inverse of ϕ, i.e. $g : \mathbb{R} \to (-1, 1)$ is given by

$$g(s) := \frac{e^{2s/\gamma} - 1}{e^{2s/\gamma} + 1}, \tag{1.5}$$

the system (1.1)-(1.2) becomes, for $|u| < 1$,

$$g(v)_t - (\nu g'(v)v_z - (1 - g(v)^2)\theta_z)_z = 0, \tag{1.6}$$

$$\theta_t - (\gamma g'(v)v_z + \theta_z)_z + \theta = \mu g(v). \tag{1.7}$$

Since $\gamma g' = 1 - g^2$, the diffusion matrix of the transformed problem

$$B := \begin{pmatrix} \nu g'(v) & -(1-g(v)^2) \\ \gamma g'(v) & 1 \end{pmatrix} \tag{1.8}$$

is elliptic:

$$(x, y)B(x, y)^T = \nu g'(v)|x|^2 + |y|^2 \ge 0 \quad \forall x, y \in \mathbb{R}.$$

This consequence is in some sense related to the equivalence between the existence of an entropy and the symmetrizability of hyperbolic conservation laws or parabolic systems [6, 9]. Indeed, using the definition of the (generalized) 'entropy'

$$\eta(s) := g(s)s - \chi(s) + \chi(0) \tag{1.9}$$

from [4] (first used in [1]), where $\chi' = g$, gives $\eta(v) = \Phi(g(v)) = \Phi(u)$, with Φ as above. In this sense, the functional $\Phi(u(t))+\theta(t)^2/2$ can be interpreted as an 'entropy' for the system (1.1)-(1.2) as long as $|u| < 1$. Instead of a symmetric positive definite matrix we only get an *elliptic* matrix B after the change of unknowns, which is sufficient for the existence analysis.

In order to make the above 'entropy' estimate rigorous, we have to overcome the difficulties near the points where $|u| = 1$. For the transformed problem (1.6)-(1.7) this difficulty translates into the fact that the matrix B is not uniformly elliptic. Therefore, we have to approximate (1.6)-(1.7) appropriately, see Section 2.

Our main existence result is as follows:

Theorem 1.1 *Let γ, $\nu > 0$, $\mu \geq 0$ and u_0, $\theta_0 \in L^2(\Omega)$ with $-1 \leq u_0 \leq 1$ in Ω. For any $T > 0$, there exists a weak solution (u, θ) of (1.1)-(1.2) such that*

$$\begin{aligned} u, \theta \in H^1(0, T; (H^1_{\text{per}}(\Omega))') \cap L^2(0, T; H^1_{\text{per}}(\Omega)), \\ -1 \leq u \leq 1 \quad \text{in } Q_T = \Omega \times (0, T). \end{aligned} \tag{1.10}$$

As explained above, the main difficulties of the proof of this theorem are that the system (1.1)-(1.2) is generally not elliptic and no maximum principle to show $|u| \leq 1$ is available.

The proof consists of three steps. First, instead of using the transformation g, we make a change of unknowns which takes into account the singular points $|u| = 1$ (Section 2.1). Then the parabolic problem is discretized in time by a recursive sequence of elliptic equations which can be solved each by Schauder's fixed point theorem (Section 2.2). Finally, a priori bounds independent of the time discretization parameter are obtained from an inequality similar to (1.4), and standard compactness results lead to the existence of a solution of the original problem (1.1)-(1.2) (Section 2.3). The bound on u can be proved by using Stampacchia's truncation method in the approximate problem.

We prove the uniqueness of solutions in a slightly smaller class of functions if the cross-diffusion is not too large (Section 3):

Theorem 1.2 *Let $\gamma < 4\nu$. Then, under the assumptions of Theorem 1.1 there exists at most one solution (u, θ) of (1.1)-(1.2) in the class of functions satisfying (1.10) and $\theta \in L^\infty(0, T; H^1_{\text{per}}(\Omega))$.*

Furthermore, we show that in the non-segregating case, the transient solutions converge to the constant steady-state solutions given by

$$\bar{u} = \frac{1}{L} \int_0^L u_0(z)dz, \quad \bar{\theta} = \frac{1}{L} \int_0^L \theta(z)dz,$$

and the rate of convergence is exponential (Section 4):

Theorem 1.3 *Let the assumptions of Theorem 1.1 hold and assume that* $|u_0| \leq c < 1$ *in* Ω *for some* $c < 1$, $\mu\bar{u} = \bar{\theta}$ *and*

$$\frac{\nu\gamma}{\mu^2} > \frac{L^4}{8(L^2+1)}. \tag{1.11}$$

Then there exist constants $c_0 > 0$*, depending on* u_0*,* θ_0*, and* $\delta_1, \delta_2 > 0$*, depending on the parameters, such that for all* $t > 0$*,*

$$\begin{aligned} \|u(t) - \bar{u}\|_{L^2(\Omega)} &\leq c_0 e^{-\delta_1 t}, \\ \|\theta(t) - \bar{\theta}\|_{L^2(\Omega)} &\leq c_0 e^{-\delta_2 t}. \end{aligned}$$

The constants c_0 and δ_1, δ_2 are defined in (4.1) and (4.4), respectively. The proof of the above result is based on careful estimates using the 'entropy' (1.9). Aranson et al. [3] have shown from linear stability theory that the condition $\mu > \nu$ is necessary to have size segregation. The assumption (1.11) shows that the condition $\mu > \nu$ needs *not* to be sufficient. In fact, there are parameter values for which *both* $\mu > \nu$ *and* (1.11) hold, i.e., the granular materials are not segregating.

Finally, we present in Section 5 some numerical examples showing the influence of the parameters on the segregation behavior of the system.

2 Proof of Theorem 1.1

2.1 Ideas of the proof

In this section we present and explain the approximations needed in the proof of Theorem 1.1. As already mentioned in the introduction, the function g provides an 'entropy' estimate only if $|u| < 1$. Since $u = \pm 1$ is possible, we use another change of unknowns which includes the points $u = \pm 1$. Let the assumptions of Theorem 1.1 hold and let $\alpha > 1$. Define the transformation $u = g_\alpha(v)$ with $g_\alpha : [-s_\alpha, s_\alpha] \to [-1, 1]$, given by

$$g_\alpha(s) := \alpha \frac{e^{2\alpha s/\gamma} - 1}{e^{2\alpha s/\gamma} + 1} \quad \text{and} \quad s_\alpha := \frac{\gamma}{2\alpha} \log \frac{\alpha+1}{\alpha-1}. \tag{2.1}$$

Observe that for $\alpha \to 1$, g_α equals g on $\mathbb{R}$, see (1.5). As the range of g_α is $[-1, 1]$, the critical points $u = \pm 1$ are included in that transformation. In the following we fix some $\alpha > 1$ and write again g for g_α.

With this change of unknowns we obtain the system (1.6)-(1.7), with periodic boundary conditions for v and θ and initial conditions

$$v(\cdot, 0) = v_0 := g^{-1}(u_0), \quad \theta(\cdot, 0) = \theta_0 \qquad \text{in } \Omega. \tag{2.2}$$

The new diffusion matrix B is given by (1.8). It holds for any $(x, y) \in \mathbb{R}^2$

$$\begin{aligned} (x, y)B(x, y)^T &= \nu g'(v)x^2 + y^2 + \left(\gamma g'(v) - (1 - g(v)^2)\right) xy \\ &= \nu g'(v)x^2 + y^2 + (\alpha^2 - 1)xy. \end{aligned}$$

Clearly, for $\alpha = 1$ the matrix becomes elliptic, and it seems reasonable that this will be also the case for $\alpha > 1$ sufficiently close to one. In fact, let (v, θ) be a weak solution to (1.1)-(1.2) and use v and θ as test functions in the weak formulation of (1.6)-(1.7), respectively, to obtain the identity

$$\begin{aligned}\int_\Omega \left(G(v(t)) + \frac{1}{2}\theta(t)^2 \right) + \int_0^t \int_\Omega \left(\nu g'(v)^2 v_z^2 + \theta_z^2 + \theta^2 \right) \\ = \int_\Omega \left(G(v_0) + \frac{1}{2}\theta_0^2 \right) - (\alpha^2 - 1) \int_0^t \int_\Omega v_z \theta_z + \int_0^t \int_\Omega \mu g(v)\theta,\end{aligned}$$

where G is defined by $G'(s) = sg'(s)$ and $G(0) = 0$, i.e.

$$G(s) = \frac{2\alpha s}{\gamma} \frac{e^{2\alpha s/\gamma}}{e^{2\alpha s/\gamma} + 1} + \log \frac{2}{e^{2\alpha s/\gamma} + 1}. \tag{2.3}$$

Since $|g|$ is bounded by one and $g' \geq (\alpha^2 - 1)/\gamma$ in $[-s_\alpha, s_\alpha]$, see Lemma 2.2, we can estimate

$$\begin{aligned}\int_\Omega \left(G(v(t)) + \frac{1}{2}\theta(t)^2 \right) + \int_0^t \int_\Omega \left(\frac{\nu}{\gamma}(\alpha^2 - 1) v_z^2 + \theta_z^2 \right) \\ \leq \int_\Omega \left(G(v_0) + \frac{1}{2}\theta_0^2 \right) - (\alpha^2 - 1) \int_0^t \int_\Omega v_z \theta_z + \int_0^t \int_\Omega (\mu|\theta| - \theta^2),\end{aligned} \tag{2.4}$$

as long as $-s_\alpha \leq v \leq s_\alpha$ in Q_t. Choosing $\alpha > 1$ small enough and applying Young's inequality, it is possible to control the second integral on the right-hand side by the integrals on the left-hand side. This gives the estimates $v_z \in L^2(0, T; L^2(\Omega))$ and $\theta \in L^2(0, T; H^1_{\rm per}(\Omega))$. The inequality (2.4) is made rigorous in Lemma 2.6 for a time-discretized version of (1.6)-(1.7).

Still there remain two difficulties: the elliptic operator corresponding to (1.6)-(1.7) is not uniformly elliptic (since g' is only positive, but not uniformly positive in $\mathbb{R}$), and we have to deal with time derivatives in $g(v)$ (instead of having time *and* space derivatives in v). The first difficulty can be overcome by adding a small number $\varepsilon > 0$ to the diffusion term containing $\nu g'(v)$ and to pass to the limit $\varepsilon \to 0$ after solving the approximate problem. To overcome the second difficulty we approximate the system by a semi-discrete problem in time (backward Euler method). This method is also interesting from a numerical point of view, see, e.g., [8].

The proof of Theorem 1.1 consists of the following steps:

1. Consider an approximate problem of (1.6)-(1.7) involving the additional diffusion parameter $\varepsilon > 0$ and the time discretization parameter $\tau > 0$.

2. Prove the existence of weak solutions of the approximate system by using Schauder's fixed-point theorem.

3. Deduce uniform estimates from an entropy-type estimate similar to (2.4).

4. Perform the limits $\varepsilon \to 0$ and $\tau \to 0$ ($\alpha > 1$ remains fixed).

2.2 A semi-discrete problem

The main objective of this section is to prove that for given $\tau > 0$ and $(\tilde{w}, \tilde{\theta}) \in (H^1_{\rm per}(\Omega))^2$, there exists a solution $(w, \xi) \in (H^1_{\rm per}(\Omega))^2$, satisfying $-s_\alpha \le w \le s_\alpha$ in Ω, of the problem

$$\begin{aligned} \frac{1}{\tau}(g(w) - g(\tilde{w})) - \big(\nu g'(w) w_z - (1 - g(w)^2)\xi_z\big)_z &= 0, & (2.5)\\ \frac{1}{\tau}(\xi - \tilde{\theta}) - (\gamma g'(w) w_z + \xi_z)_z + \xi &= \mu g(w) \quad \text{in } \Omega. & (2.6) \end{aligned}$$

This system is a time-discretized version of (1.6)-(1.7). The function $g(s)$ is defined as in (2.1) but we allow for arguments $s \in \mathbb{R}$. We shall use the following notion of weak solution.

Definition 2.1 *The pair (w, ξ) is called a* weak solution *of (2.5)-(2.6) if $(w, \xi) \in (H^1_{\rm per}(\Omega))^2$, $-s_\alpha \le w \le s_\alpha$ in Ω, the initial conditions in (1.3) are satisfied in the sense of $(H^1_{\rm per}(\Omega))'$, and for every $(\varphi, \psi) \in (H^1_{\rm per}(\Omega))^2$ we have*

$$\begin{aligned} \frac{1}{\tau}\int_\Omega (g(w) - g(\tilde{w}))\varphi + \int_\Omega \big(\nu g'(w) w_z - (1 - g(w)^2)\xi_z\big)\varphi_z &= 0, & (2.7)\\ \frac{1}{\tau}\int_\Omega (\xi - \tilde{\theta})\psi + \int_\Omega (\gamma g'(w) w_z + \xi_z)\psi_z + \int_\Omega \xi\psi &= \mu \int_\Omega g(w)\psi. & (2.8) \end{aligned}$$

As explained in Section 2.1, we approximate the system (2.5)-(2.6) by a system where an additional ellipticity constant $\varepsilon > 0$ is introduced: Find $(w, \xi) \in (H^1_{\rm per}(\Omega))^2$ such that in Ω

$$\begin{aligned} \frac{1}{\tau}(g(w) - g(\tilde{w})) - \big((\nu g'(w) + \varepsilon) w_z - (1 - g(w)^2)_+ \xi_z\big)_z + \varepsilon w &= 0, & (2.9)\\ \frac{1}{\tau}(\xi - \tilde{\theta}) - (\gamma g'(w) w_z + \xi_z)_z + \xi &= \mu g(w), & (2.10) \end{aligned}$$

where $s_+ = \max\{0, s\}$.

The function g possesses the following properties.

Lemma 2.2 *The function $g : \mathbb{R} \to (-\alpha, \alpha)$ defined by (2.1) satisfies $g \in C^\infty(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ and*

$$0 < g' \le \alpha^2/\gamma \quad \text{in } \mathbb{R}, \qquad g' \ge (\alpha^2 - 1)/\gamma \quad \text{in } [-s_\alpha, s_\alpha]. \tag{2.11}$$

Fix $\alpha > 1$ such that $2(\alpha^2 - 1) \le \nu/2\gamma$ and define $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ by

$$h_1 := \nu g' - \delta|\gamma g' - (1 - g^2)_+|, \quad h_2 := 1 - \frac{1}{\delta}|\gamma g' - (1 - g^2)_+|,$$

with $2(\alpha^2 - 1) \le \delta \le \nu/2\gamma$. Then

$$h_1 > 0, \quad h_2 \ge 1/2 \quad \text{in } \mathbb{R}, \quad \text{and} \quad h_1 \ge \frac{\nu}{2\gamma}(\alpha^2 - 1) \quad \text{in } [-s_\alpha, s_\alpha]. \tag{2.12}$$

Proof. We first observe that the function g is symmetric with respect to the origin, so we restrict our computations to the right semiplane. The C^∞ regularity of g is clear. It holds

$$g'(s) = 4\alpha^2 \frac{e^{2\alpha s/\gamma}}{\gamma(e^{2\alpha s/\gamma}+1)^2} > 0, \quad s \in \mathbb{R}. \tag{2.13}$$

This shows that g is increasing, and we deduce that $\|g\|_{L^\infty} \le \lim_{\sigma\to\infty} g(\sigma) = (1+\gamma)/\gamma$ and hence, $g \in L^\infty(\mathbb{R})$. The only critical point of g' is at $s = 0$, which is a local maximum point for this function. Inspecting the values of g' when $s \to \infty$ we deduce that g' has a global maximum at $s = 0$, with $g'(0) = \alpha^2/\gamma$. This proves $g' \le \alpha^2/\gamma$ in $\mathbb{R}$. On the other hand, the function g' attains its minimum in the set $[-s_\alpha, s_\alpha]$ at the boundary with value $(\alpha^2-1)/\gamma$. This shows (2.11).

To prove (2.12) we first observe that $1-g(s)^2 \ge 0$ if and only if $s \in [-s_\alpha, s_\alpha]$. Then, a straightforward computation shows that $\gamma g' - (1-g^2)_+ = \alpha^2 - 1$ in $[-s_\alpha, s_\alpha]$, and we conclude, using (2.11) and the bounds for δ, that in $[-s_\alpha, s_\alpha]$

$$h_1 = \nu g' - \delta(\alpha^2-1) \ge \frac{\nu(\alpha^2-1)}{\gamma} - \frac{\nu(\alpha^2-1)}{2\gamma} = \frac{\nu(\alpha^2-1)}{2\gamma} > 0$$

and

$$h_2 \ge 1 - \frac{\alpha^2-1}{\delta} \ge \frac{1}{2}.$$

In the set $\mathbb{R} \setminus [-s_\alpha, s_\alpha]$ it holds

$$h_1 = (\nu - \delta\gamma)g' \ge \frac{\nu}{2} g' > 0.$$

The maximum of g' in $\mathbb{R} \setminus [-s_\alpha, s_\alpha]$ is attained at $\pm s_\alpha$ which implies $g' \le (\alpha^2-1)/\gamma$ and therefore

$$h_2 \ge 1 - \frac{\alpha^2-1}{\delta} \ge 1 - \frac{\alpha^2-1}{2(\alpha^2-1)} = \frac{1}{2}$$

since $\delta \ge 2(\alpha^2-1)$. This proves (2.12). □

We prove the existence of a solution of (2.9)-(2.10) using Schauder's fixed point theorem. In order to define the fixed-point operator, we consider first the following linearized problem: Let $(\hat{w}, \hat{\xi}) \in (L^2(\Omega))^2$ be given and find $(w, \xi) \in (H^1_{\text{per}}(\Omega))^2$ such that

$$-\big((\nu g'(\hat{w}) + \varepsilon)w_z - (1-g(\hat{w})^2)_+ \xi_z\big)_z + \varepsilon w = \frac{1}{\tau}(g(\tilde{w}) - g(\hat{w})), \tag{2.14}$$

$$-(\gamma g'(\hat{w}) w_z + \xi_z)_z + \xi = \mu g(\hat{w}) + \frac{1}{\tau}(\tilde{\theta} - \hat{\xi}) \tag{2.15}$$

in Ω. The definition of a weak solution of problem (2.14)-(2.15) is similar to Definition 2.1.

Lemma 2.3 *Let $(\tilde{w}, \tilde{\theta}) \in (H^1_{\text{per}}(\Omega))^2$ and $(\hat{w}, \hat{\xi}) \in (L^2(\Omega))^2$ be given. Then there exists a unique weak solution of problem (2.14)-(2.15).*

Proof. We define the bilinear form $a : (H^1_{\rm per}(\Omega))^2 \times (H^1_{\rm per}(\Omega))^2 \to \mathbb{R}$,

$$a((w,\xi),(\varphi,\psi)) := \int_\Omega \left[\big((\nu g'(\hat w)+\varepsilon)w_z - (1-g(\hat w)^2)_+\xi_z\big)\varphi_z + \varepsilon w\varphi\right] + \int_\Omega \big((\gamma g'(\hat w)w_z + \xi_z)\psi_z + \xi\psi\big),$$

and the linear functional $f : (L^2(\Omega))^2 \to \mathbb{R}$,

$$f(\varphi,\psi) := \frac{1}{\tau}\int_\Omega \big((g(\tilde w) - g(\hat w))\varphi + (\tilde\theta - \hat\xi)\psi\big) + \mu \int_\Omega g(\hat w)\psi.$$

In order to apply the Lemma of Lax-Milgram, we have to check that a is continuous and coercive in $(H^1_{\rm per}(\Omega))^2 \times (H^1_{\rm per}(\Omega))^2$ and that f is continuous in $(L^2(\Omega))^2$. The continuity of a and f follows easily from the pointwise bounds of g and g' and the regularity of $\tilde w$, $\tilde\theta$, $\hat w$, and $\hat\xi$. For the coercivity of a we estimate

$$\begin{aligned} a((w,\xi),(w,\xi)) &= \int_\Omega \big((\nu g'(\hat w)+\varepsilon)|w_z|^2 + |\xi_z|^2 + \varepsilon|w|^2 + |\xi|^2\big) \\ &\quad + \int_\Omega \big((\gamma g'(\hat w) - (1-g(\hat w)^2)_+)w_z\xi_z\big) \\ &\geq \int_\Omega \big((\varepsilon + h_1(\hat w))|w_z|^2 + h_2(\hat w)|\xi_z|^2 + \varepsilon|w|^2 + |\xi|^2\big), \end{aligned}$$

using Young's inequality, where the functions h_1 and h_2 are defined in Lemma 2.2. The bounds (2.12) then imply that

$$a((w,\xi),(w,\xi)) \geq \min\{\varepsilon, 1/2\}\left(\|w\|^2_{H^1_{\rm per}(\Omega)} + \|\xi\|^2_{H^1_{\rm per}(\Omega)}\right),$$

and the coercivity of a is proved. □

In the following lemma we prove the existence of solutions of problem (2.9)-(2.10).

Lemma 2.4 *Let $(\tilde w, \tilde\theta) \in (H^1_{\rm per}(\Omega))^2$. Then there exists a unique weak solution of problem (2.9)-(2.10).*

Proof. We use the Schauder fixed point theorem. For this define the map $S : (L^2(\Omega))^2 \to (L^2(\Omega))^2$ by $S(\hat w, \hat\xi) = (w,\xi)$, where (w,ξ) is the weak solution of (2.14)-(2.15). We have to check that S is continuous and compact and that the set

$$\Lambda := \left\{u \in (L^2(\Omega))^2 : u = \lambda S(u)\right\},$$

for $\lambda \in [0,1]$, is bounded.

(i) *S is continuous.* Let $(\hat w_n, \hat\xi_n)_n$ be a sequence in $(L^2(\Omega))^2$ with $(\hat w_n, \hat\xi_n) \to (\hat w, \hat\xi)$ strongly in $(L^2(\Omega))^2$ as $n \to \infty$. Due to the coercivity of the bilinear form a, the corresponding sequence $(w_n, \xi_n) = S(\hat w_n, \hat\xi_n)$ is bounded in $(H^1_{\rm per}(\Omega))^2$, and therefore, there exists a pair $(w,\xi) \in (H^1_{\rm per}(\Omega))^2$ and a subsequence (w_{n_j}, ξ_{n_j}) such that $(w_{n_j}, \xi_{n_j}) \rightharpoonup (w,\xi)$ weakly in $(H^1_{\rm per}(\Omega))^2$.

On the other hand, since $g \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, we conclude, extracting a new subsequence if necessary, that $g(\hat{w}_{n_j}) \to g(\hat{w})$, $g'(\hat{w}_{n_j}) \to g'(\hat{w})$ and $(1-g(\hat{w}_{n_j})^2)_+ \to (1-g(\hat{w})^2)_+$ strongly in $L^\infty(\Omega)$. Passing to the limit $n_j \to \infty$ we obtain $S(\hat{w},\hat{\xi}) = (w,\xi)$.

(ii) *S is compact.* The compactness of S is just a consequence of the compactness of the embedding $H^1_{\rm per}(\Omega) \subset L^2(\Omega)$.

(iii) *Λ is bounded.* If $\lambda = 0$ then $\Lambda = \{(0,0)\}$ is trivially bounded. For $\lambda \in (0,1]$, the equation $S(\hat{w},\hat{\xi}) = \frac{1}{\lambda}(\hat{w},\hat{\xi})$ is equivalent to

$$\int_\Omega \Big(\big((\nu g'(\hat{w}) + \varepsilon)\hat{w}_z - (1-g(\hat{w})^2)_+\hat{\xi}_z\big)_z \varphi_z + \varepsilon \hat{w}\varphi \Big) = \frac{\lambda}{\tau}\int_\Omega (g(\tilde{w}) - g(\hat{w}))\varphi,$$

$$\int_\Omega \Big((\gamma g'(\hat{w})\hat{w}_z + \hat{\xi}_z)\psi_z + \hat{\xi}\psi \Big) = \lambda \int_\Omega \Big(\mu g(\hat{w}) + \frac{1}{\tau}(\tilde{\theta} - \hat{\xi}) \Big)\psi$$

Using $(\varphi,\psi) = (\hat{w},\hat{\xi})$ as a test function, adding the resulting integral identities and applying Young's inequality as in (2.12), we obtain

$$\int_\Omega \big((\varepsilon + h_1(\hat{w}))|\hat{w}_z|^2 + h_2(\hat{w})|\hat{\xi}_z|^2 + \varepsilon|\hat{w}|^2 + |\hat{\xi}|^2) \big) = \frac{\lambda}{\tau}\int_\Omega (g(\tilde{w}) - g(\hat{w}))\hat{w} + \lambda \int_\Omega \big(\mu g(\hat{w}) + \frac{1}{\tau}(\tilde{\theta} - \hat{\xi}) \big)\hat{\xi}.$$

Using again Young's inequality on the right-hand side of this equation and employing the estimate (2.12), we deduce

$$\int_\Omega \big(\varepsilon(|\hat{w}_z|^2 + |\hat{w}|^2) + |\hat{\xi}_z|^2 + |\hat{\xi}|^2 \big) \le \frac{\lambda^2}{\tau^2\varepsilon}\int_\Omega (g(\tilde{w}) - g(\hat{w}))^2 + \frac{2\lambda^2}{\tau^2}\int_\Omega \tilde{\theta}^2 + 2(\lambda\mu)^2 \int_\Omega |g(\hat{w})|^2,$$

and since $g \in L^\infty(\mathbb{R})$, the assertion follows. □

In the following we will derive uniform bounds for the solution of (2.9)-(2.10) which allow to pass to the limit $\varepsilon \to 0$. This will prove the existence of a solution of (2.5)-(2.6). First we prove the following auxiliary result.

Lemma 2.5 *Let $\varphi \in C(\mathbb{R})$ be non-decreasing with $\varphi(0) = 0$ and define $\Phi \in C^1(\mathbb{R})$ by $\Phi(s) := \int_0^s g'(\sigma)\varphi(\sigma)d\sigma$. Then it holds for all $s,t \in \mathbb{R}$*

$$\Phi(s) - \Phi(t) \le (g(s) - g(t))\varphi(s). \tag{2.16}$$

Proof. Let $s \ge t$. Then, on one hand,

$$\Phi(s) - \Phi(t) = \int_t^s g'(\sigma)\varphi(\sigma)d\sigma \le \int_t^s g'(\sigma)\varphi(s)d\sigma = (g(s) - g(t))\varphi(s),$$

and, on the other hand,

$$\Phi(s) - \Phi(t) \ge \int_t^s g'(\sigma)\varphi(t)d\sigma = (g(s) - g(t))\varphi(t),$$

and the result follows. □

Lemma 2.6 *Let $(\tilde w, \tilde\xi) \in (H^1_{\rm per}(\Omega))^2$ be such that $-s_\alpha \le \tilde w \le s_\alpha$ in Ω and let $(w_\varepsilon, \xi_\varepsilon) \in (H^1_{\rm per}(\Omega))^2$ be a solution of (2.9)-(2.10). Then the following estimates hold:*

$$
\begin{aligned}
&-s_\alpha \le w_\varepsilon \le s_\alpha \quad \text{in } \Omega, &(2.17)\\
&\int_\Omega \Big(G(w_\varepsilon) + \frac{1}{2}\xi_\varepsilon^2\Big) + C\tau \int_\Omega (|w_{\varepsilon z}|^2 + |\xi_{\varepsilon z}|^2 + |\xi_\varepsilon|^2)\\
&\le \int_\Omega \Big(G(\tilde w) + \frac{1}{2}\tilde\xi^2\Big) + C'\tau, &(2.18)
\end{aligned}
$$

for some positive constants C, C' independent of ε and τ, and G is defined in (2.3).

In addition, there exists a subsequence of $(w_\varepsilon, \xi_\varepsilon)$ (not relabeled) such that $(w_\varepsilon, \xi_\varepsilon) \to (w, \xi)$ weakly in $(H^1_{\rm per}(\Omega))^2$ and strongly in $(L^2(\Omega))^2$ as $\varepsilon \to 0$, and (w, ξ) is a weak solution of problem (2.5)-(2.6).

Proof. We use $\varphi(w_\varepsilon) := \max(w_\varepsilon - s_\alpha, 0)$ as a test function in the weak formulation of (2.9). Since φ is increasing and $\varphi(0) = 0$ we can employ Lemma 2.5. Let Φ be defined as in Lemma 2.5. Then, together with the identities $(1 - g(s)^2)_+ \varphi'(s) = 0$ for all $s \in \mathbb{R}$ and $\Phi(\tilde w) = 0$, we obtain

$$
0 \ge \frac{1}{\tau}\int_\Omega (g(w_\varepsilon) - g(\tilde w))\varphi(w_\varepsilon) \ge \int_\Omega (\Phi(w_\varepsilon) - \Phi(\tilde w)) = \int_\Omega \Phi(w_\varepsilon).
$$

This implies $\Phi(w_\varepsilon) = 0$ and therefore $w_\varepsilon \le s_\alpha$ in Ω. In a similar way we deduce $w_\varepsilon \ge -s_\alpha$ in Ω. Observe that these bounds imply that $(1-g(w_\varepsilon)^2)_+ = 1-g(w_\varepsilon)^2$ in Ω.

Now we use $(w_\varepsilon, \xi_\varepsilon)$ as a test function in the weak formulation of problem (2.9)-(2.10). Adding the corresponding integral identities and using property (2.16) we get, after multiplication by τ,

$$
\begin{aligned}
\int_\Omega \Big(G(w_\varepsilon) + \frac{1}{2}\xi_\varepsilon^2\Big) + \tau \int_\Omega \big(h_1(w_\varepsilon)|w_{\varepsilon z}|^2 + h_2(w_\varepsilon)|\xi_{\varepsilon z}|^2 + |\xi_\varepsilon|^2\big)\\
\le \mu\tau \int_\Omega g(w_\varepsilon)\xi_\varepsilon + \int_\Omega \Big(G(\tilde w) + \frac{1}{2}\tilde\xi^2\Big).
\end{aligned}
$$

Applying Young's inequality and the bounds (2.11) and (2.12) for g', h_1 and h_2, we deduce (2.18).

Finally, the uniform estimates (2.17) and (2.18) imply the existence of a subsequence (not relabeled) of $(w_\varepsilon, \xi_\varepsilon)$ and of a pair $(w, \xi) \in (H^1_{\rm per}(\Omega))^2$ such that, as $\varepsilon \to 0$,

$$
\begin{aligned}
w_\varepsilon &\overset{*}{\rightharpoonup} w && \text{weakly* in } L^\infty(\Omega), &(2.19)\\
w_{\varepsilon z} &\rightharpoonup w_z && \text{weakly in } L^2(\Omega), &(2.20)\\
\xi_\varepsilon &\rightharpoonup \xi && \text{weakly in } H^1_{\rm per}(\Omega).
\end{aligned}
$$

In fact, the convergences (2.19) and (2.20) imply $w_\varepsilon \rightharpoonup w$ weakly in $H^1_{\rm per}(\Omega)$ and thus, by the compactness of the embedding $H^1_{\rm per}(\Omega) \subset L^2(\Omega)$, we deduce for a subsequence, as $\varepsilon \to 0$, $w_\varepsilon \to w$ and $\xi_\varepsilon \to \xi$ strongly in $L^2(\Omega)$ and a.e. in Ω. These convergence results and the continuity of g and g' allow us to pass to the limit $\varepsilon \to 0$ in the weak formulation of problem (2.9)-(2.10) and to identify (w, ξ) as a weak solution of (2.5)-(2.6). □

2.3 End of the proof of Theorem 1.1

Let $T > 0$ and $N \in \mathbb{N}$ be given and let $\tau = T/N$ be the time step. We define recursively pairs $(v^k, \theta^k) \in (H^1_{\rm per}(\Omega))^2$, $k = 1, \ldots, N$, as the weak solution of the problem (2.5)-(2.6) corresponding to the data $(\tilde{w}, \tilde{\theta}) = (v^{k-1}, \theta^{k-1})$, and with $(v^0, \theta^0) = (v_0, \theta_0)$. Then we define the piecewise constant functions

$$v^\tau(x,t) := v^k(x) \quad \text{and} \quad \theta^\tau(x,t) := \theta^k(x) \quad \text{if } (x,t) \in \Omega \times ((k-1)\tau, k\tau],$$

for $k = 1, \ldots, N$, and introduce the discrete entropies

$$\eta^k := \int_\Omega \Big(G(v^k) + \frac{1}{2}|\theta^k|^2 \Big), \quad \eta^\tau(t) := \int_\Omega \Big(G(v^\tau(\cdot,t)) + \frac{1}{2}|\theta^\tau(\cdot,t)|^2 \Big). \quad (2.21)$$

We have the following consequence of Lemma 2.6.

Corollary 2.7 *There exist uniform bounds with respect to τ for the norms*

$$\|\eta^\tau\|_{L^\infty(0,T)}, \ \|v^\tau\|_{L^2(0,T;H^1_{\rm per}(\Omega))}, \ \|g(v^\tau)\|_{L^2(0,T;H^1_{\rm per}(\Omega))} \text{ and } \|\theta^\tau\|_{L^2(0,T;H^1_{\rm per}(\Omega))}.$$

In addition,

$$-s_\alpha \le v^\tau \le s_\alpha \quad \text{in } Q_T = \Omega \times (0,T). \quad (2.22)$$

Proof. From the 'entropy' inequality (2.18) we obtain

$$\eta^m - \eta^0 = \sum_{k=1}^m (\eta^k - \eta^{k-1}) \le C' m\tau - C\tau \sum_{k=1}^m \int_\Omega (|v^k_z|^2 + |\theta^k_z|^2 + |\theta^k|^2),$$

for $m = 1, \ldots, N$. Taking the maximum over m yields

$$\|\eta^\tau\|_{L^\infty(0,T)} + C \int_{Q_T} \big(|v^\tau_z|^2 + |\theta^\tau_z|^2 + |\theta^\tau|^2 \big) \le \eta^0 + C'T.$$

Since both g and g' are smooth and bounded we also deduce the estimate for $\|g(v^\tau)\|_{L^2(0,T;H^1_{\rm per}(\Omega))}$. Finally, (2.22) follows directly from (2.17). □

We need uniform estimates of the time derivatives. For this, we introduce the shift operator and linear interpolations in time. For $t \in ((k-1)\tau, k\tau]$, $k = 1, \ldots, N$, we define $\sigma_\tau v^\tau(\cdot,t) := v^{k-1}$ and $\sigma_\tau \theta^\tau(\cdot,t) := \theta^{k-1}$ in Ω. Setting $\delta t := (t/\tau - (k-1)) \in [0,1]$, we introduce

$$\tilde{g}^\tau := g(\sigma_\tau v^\tau) + \delta t\big(g(v^\tau) - g(\sigma_\tau v^\tau)\big), \quad \tilde{\theta}^\tau := \sigma_\tau \theta^\tau + \delta t\big(\theta^\tau - \sigma_\tau \theta^\tau\big) \quad (2.23)$$

in Q_T.

Lemma 2.8 *There exist uniform bounds with respect to τ for the norms*

$$\|\tilde{g}^\tau_t\|_{L^2(0,T;(H^1_{\rm per}(\Omega))')}, \quad \|\tilde{g}^\tau\|_{L^2(0,T;H^1_{\rm per}(\Omega))\cap L^\infty(Q_T)},$$
$$\|\tilde{\theta}^\tau_t\|_{L^2(0,T;(H^1_{\rm per}(\Omega))')} \quad \textit{and} \quad \|\tilde{\theta}^\tau\|_{L^2(0,T;H^1_{\rm per}(\Omega))}.$$

Proof. From the definition (2.23) of $\tilde{g}^\tau$ and equation (2.5) we compute

$$\tilde{g}^\tau_t = \frac{1}{\tau}\big(g(v^\tau) - g(\sigma_\tau v^\tau)\big) = \big(\nu g'(v^\tau)v^\tau_z - (1-g(v^\tau)^2)\theta^\tau_z\big)_z.$$

Using the boundedness of g' in $\mathbb{R}$ and Corollary 2.7 we obtain a uniform bound for $\|\tilde{g}^\tau_t\|_{L^2((0,T;H^1_{\rm per})')}$. Moreover, since g is bounded, it is clear that $\tilde{g}^\tau \in L^\infty(Q_T)$ for any $\tau \geq 0$. We also have

$$\tilde{g}^\tau_z = \delta t g'(v^\tau)v^\tau_z + (1-\delta t)g'(\sigma_\tau v^\tau)(\sigma_\tau v^\tau)_z. \tag{2.24}$$

Since $(\sigma_\tau v^\tau)_z = \sigma_\tau v^\tau_z$, the $L^\infty(Q_T)$ bound for $\tilde{g}^\tau$ together with (2.24) and Corollary 2.7 implies a uniform bound for $\|\tilde{g}^\tau\|_{L^2(0,T;H^1_{\rm per}(\Omega))}$. In a similar way we obtain uniform estimates for $\tilde{\theta}^\tau$. □

Proof of Theorem 1.1. The functions v^τ, θ^τ, $\tilde{g}^\tau$, $\tilde{\theta}^\tau$ satisfy the weak formulation

$$\int_0^T \langle \tilde{g}^\tau_t, \varphi\rangle + \int_{Q_T} \big(\nu g'(v^\tau)v^\tau_z - (1-g(v^\tau)^2)\theta^\tau_z\big)\varphi_z = 0, \tag{2.25}$$

$$\int_0^T \langle \tilde{\theta}_t, \psi\rangle + \int_{Q_T} \big(\gamma g'(v^\tau)v^\tau_z + \theta^\tau_z\big)\psi_z + \int_{Q_T} \theta^\tau\psi = \mu\int_{Q_T} g(v^\tau)\psi, \tag{2.26}$$

for any $\varphi, \psi \in L^2(0,T;H^1_{\rm per}(\Omega))$. The estimates of Lemma 2.8 allow us to extract a subsequence (not relabeled) such that, as $\tau \to 0$,

$$\tilde{g}^\tau_t \rightharpoonup u_t \qquad \text{weakly in } L^2(0,T;(H^1_{\rm per}(\Omega))'), \tag{2.27}$$
$$\tilde{g}^\tau \rightharpoonup u \qquad \text{weakly in } L^2(0,T;H^1_{\rm per}(\Omega)), \tag{2.28}$$
$$\tilde{g}^\tau \overset{*}{\rightharpoonup} u \qquad \text{weakly* in } L^\infty(Q_T),$$
$$\tilde{\theta}^\tau_t \rightharpoonup \theta_t \qquad \text{weakly in } L^2(0,T;(H^1_{\rm per}(\Omega))'), \tag{2.29}$$
$$\tilde{\theta}^\tau \rightharpoonup \theta \qquad \text{weakly in } L^2(0,T;H^1_{\rm per}(\Omega)). \tag{2.30}$$

The compact embedding $H^1_{\rm per}(\Omega) \subset L^\infty$, the convergence results (2.27)-(2.30) and Aubin's Lemma imply, up to a subsequence,

$$\tilde{g}^\tau \to u \qquad \text{strongly in } L^2(0,T;L^\infty(\Omega)), \tag{2.31}$$
$$\tilde{\theta}^\tau \to \theta \qquad \text{strongly in } L^2(0,T;L^\infty(\Omega)).$$

Moreover, Corollary 2.7 yields the existence of a subsequence such that

$$v^\tau \rightharpoonup v \qquad \text{weakly in } L^2(0,T;H^1_{\rm per}(\Omega)),$$
$$v^\tau \overset{*}{\rightharpoonup} v \qquad \text{weakly* in } L^\infty(Q_T),$$
$$g(v^\tau) \rightharpoonup \hat{u} \qquad \text{weakly in } L^2(0,T;H^1_{\rm per}(\Omega)), \tag{2.32}$$
$$\theta^\tau \rightharpoonup \hat{\theta} \qquad \text{weakly in } L^2(0,T;H^1_{\rm per}(\Omega)).$$

It holds $\tilde{g}^\tau - g(v^\tau) = \tau(\delta t - 1)\tilde{g}^\tau_t$, and therefore, by Lemma 2.8,

$$\|\tilde{g}^\tau - g(v^\tau)\|_{L^2(0,T;(H^1_{\rm per})')} \to 0 \quad \text{as } \tau \to 0. \tag{2.33}$$

Hence, $u = \hat{u}$. In a similar way we obtain $\theta = \hat{\theta}$. Finally,

$$\begin{aligned}
&\|g(v^\tau) - u\|_{L^1(0,T;L^2(\Omega))} \\
&\quad \le \|g(v^\tau) - \tilde{g}^\tau\|_{L^1(0,T;L^2(\Omega))} + \|\tilde{g}^\tau - u\|_{L^1(0,T;L^2(\Omega))} \\
&\quad \le \|g(v^\tau) - \tilde{g}^\tau\|^{1/2}_{L^1(0,T;(H^1_{\rm per}(\Omega))')}\|g(v^\tau) - \tilde{g}^\tau\|^{1/2}_{L^1(0,T;H^1_{\rm per}(\Omega))} \\
&\qquad + \|\tilde{g}^\tau - u\|_{L^1(0,T;L^2(\Omega))} \\
&\quad \le C\|g(v^\tau) - \tilde{g}^\tau\|^{1/2}_{L^2(0,T;(H^1_{\rm per}(\Omega))')} + \|\tilde{g}^\tau - u\|_{L^1(0,T;L^2(\Omega))} \\
&\quad \to 0,
\end{aligned} \tag{2.34}$$

as $\tau \to 0$. Therefore, $g(v^\tau) \to u$ strongly in $L^1(0,T;L^2(\Omega))$ and a.e. in Q_T. Now, letting $\tau \to 0$ in (2.25)-(2.26), we obtain, for $\varphi, \psi \in L^2(0,T;H^1_{\rm per}(\Omega))$,

$$\int_0^T \langle u_t, \varphi\rangle + \int_{Q_T} \big((\nu u_z - (1-u^2)\theta_z\big)\varphi_z = 0, \tag{2.35}$$

$$\int_0^T \langle \theta_t, \psi\rangle + \int_{Q_T} \big(\gamma u_z + \theta_z\big)\psi_z + \int_{Q_T} \theta\psi = \mu \int_{Q_T} u\psi. \tag{2.36}$$

This proves Theorem 1.1. □

3 Proof of Theorem 1.2

Let (u_1, θ_1) and (u_2, θ_2) be two weak solutions of (1.1)-(1.3) with the same initial data, satisfying (1.10) and $\theta_1 \in L^\infty(0,T;H^1_{\rm per}(\Omega))$. Set $Q_t = \Omega \times (0,t)$. The equations satisfied by $u = u_1 - u_2$ and $\theta = \theta_1 - \theta_2$ read

$$\begin{aligned}
u_t - \nu u_{zz} + \theta_{zz} &= \big((u_1+u_2)u\theta_{1z} + u_2^2\theta_z\big)_z, && (3.1)\\
\theta_t - \theta_{zz} + \theta &= \gamma u_{zz} + \mu u. && (3.2)
\end{aligned}$$

Take u and θ as test functions in the weak formulations of (3.1) and (3.2), respectively, and add (3.2), multiplied by some number $a > 0$, and (3.1) to obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (u(t)^2 + a\theta(t)^2) + \int_{Q_t} (\nu u_z^2 + a\theta_z^2 + a\theta^2) \\
&\quad = \int_{Q_t} (1 - a\gamma - u_2^2)u_z\theta_z + a\mu\int_{Q_t} u\theta - \int_{Q_t} (u_1+u_2)u\theta_{1z}u_z.
\end{aligned} \tag{3.3}$$

We apply Young's inequality to the second integral on the right-hand side:

$$a\mu\int_{Q_t} u\theta \le \frac{a\mu^2}{2}\int_{Q_t} u^2 + \frac{a}{2}\int_{Q_t}\theta^2.$$

For the third integral on the right-hand side of (3.3) we use the Gagliardo-Nirenberg inequality

$$\|u\|_{L^\infty(\Omega)} \le C_0\|u\|^{1/2}_{H^1(\Omega)}\|u\|^{1/2}_{L^2(\Omega)} \quad \forall u \in H^1(0,L)$$

and the Young inequality

$$x^{1/2}y^{3/2} \le \frac{\varepsilon}{2}x^2 + C(\varepsilon)y^2 \quad \forall x, y \ge 0,\ \varepsilon > 0.$$

Then, with the abreviation $C_1 = 2C_0\|\theta_{1z}\|_{L^\infty(0,T;L^2(\Omega))} < \infty$ and $|u_1|$, $|u_2| \le 1$,

$$\begin{aligned}
&\int_{Q_t}(u_1+u_2)u\theta_{1z}u_z \\
&\le 2\|u\|_{L^2(0,t;L^\infty(\Omega))}\|\theta_{1z}\|_{L^\infty(0,t;L^2(\Omega))}\|u_z\|_{L^2(0,t;L^2(\Omega))} \\
&\le C_1\|u\|^{1/2}_{L^2(0,t;L^2(\Omega))}\left(\|u\|^2_{L^2(Q_t)} + \|u_z\|^2_{L^2(Q_t)}\right)^{1/4}\|u_z\|_{L^2(0,t;L^2(\Omega))} \\
&\le C_1\left(\|u\|_{L^2(Q_t)}\|u_z\|_{L^2(Q_t)} + \|u\|^{1/2}_{L^2(Q_t)}\|u_z\|^{3/2}_{L^2(Q_t)}\right) \\
&\le \frac{\varepsilon}{2}\|u_z\|^2_{L^2(Q_t)} + \frac{C_1^2}{2\varepsilon}\|u\|^2_{L^2(Q_t)} + \frac{\varepsilon}{2}\|u_z\|^2_{L^2(Q_t)} + C(\varepsilon)C_1^4\|u\|^2_{L^2(Q_t)}.
\end{aligned}$$

With these inequalities we can estimate (3.3) as

$$\begin{aligned}
&\frac{1}{2}\left(\|u(t)\|^2_{L^2(\Omega)} + a\|\theta(t)\|^2_{L^2(\Omega)}\right) + \frac{a}{2}\|\theta\|^2_{L^2(Q_t)} \\
&\quad \le -\int_{Q_t}\left(-(|1-a\gamma|+1)|u_z||\theta_z| + (\nu-\varepsilon)u_z^2 + a\theta_z^2\right) \\
&\qquad + \left(\frac{a\mu^2}{2} + \frac{C_1^2}{2\varepsilon} + C(\varepsilon)C_1^4\right)\|u\|^2_{L^2(Q_t)}.
\end{aligned} \tag{3.4}$$

It remains to show that the quadratic form

$$A(x,y) = -(|1-a\gamma|+1)xy + (\nu-\varepsilon)x^2 + ay^2, \quad x, y \ge 0,$$

is non-negative. This is the case if and only if $\nu - \varepsilon \ge 0$ and

$$a(\nu-\varepsilon) - \frac{1}{4}(|1-a\gamma|+1)^2 \ge 0.$$

Now we choose $a = 1/\gamma$ and $\varepsilon = \nu - \gamma/4 > 0$ (since $\gamma < 4\nu$ by assumption). Then

$$a(\nu-\varepsilon) - \frac{1}{4}(|1-a\gamma|+1)^2 = \frac{\nu-\varepsilon}{\gamma} - \frac{1}{4} = 0.$$

Thus (3.4) implies that $u(t) = \theta(t) = 0$ in Ω for any $t > 0$. This proves Theorem 1.2. □

4 Proof of Theorem 1.3

Let (u,θ) be a weak solution of (1.1)-(1.3) given by Theorem 1.1. Let $\alpha > 1$ and set

$$c_0 = \frac{1}{2}\int_0^L \left(\gamma(u_0+1)\ln\frac{1+u_0}{1+\bar{u}} + \gamma(1-u_0)\ln\frac{1-u_0}{1-\bar{u}} + (\theta_0 - \bar{\theta})\right) dz. \quad (4.1)$$

Notice that c_0 is well defined even if $u_0(z) = \pm 1$. For the proof of Theorem 1.3 we need two simple lemmas:

Lemma 4.1 *Define the function* $\psi : [-1,1] \to \mathbb{R}$ *by*

$$\psi(u) = \frac{\gamma}{2\alpha}\ln\left(\frac{\alpha+u}{\alpha+\bar{u}}\,\frac{\alpha-\bar{u}}{\alpha-u}\right).$$

Then the function $\Psi : [-1,1] \to \mathbb{R}$*, defined by*

$$\Psi(u) = \frac{\gamma}{2\alpha}(\alpha+u)\ln\frac{\alpha+u}{\alpha+\bar{u}} + \frac{\gamma}{2\alpha}(\alpha-u)\ln\frac{\alpha-u}{\alpha-\bar{u}},$$

satisfies for all $u \in [-1,1]$*,*

$$\Psi'(u) = \psi(u), \quad \Psi''(u) = \frac{\gamma}{\alpha^2-u^2}, \quad \Psi(u) \geq \frac{\gamma}{2\alpha^2}(u-\bar{u})^2.$$

The lemma follows from Taylor expansion around $\bar{u}$:

$$\Psi(u) = \Psi(\bar{u}) + \Psi'(\bar{u})(u-\bar{u}) + \frac{1}{2}\Psi''(\xi)(u-\bar{u})^2 \geq \frac{\gamma}{2\alpha^2}(u-\bar{u})^2.$$

The second lemma is a Poincaré inequality:

Lemma 4.2 *For all* $v \in H^1_{\text{per}}(\Omega)$ *with* $\bar{v} = \int_0^L v(z)dz$ *it holds*

$$\|v-\bar{v}\|_{L^2(\Omega)} \leq \frac{L}{\sqrt{2}}\|v_z\|_{L^2(\Omega)}.$$

Proof. There exists $z_0 \in \Omega$ such that $v(z_0) = \bar{v}$. Then, integration of

$$|v(z)-\bar{v}|^2 = \left|\int_{z_0}^z v_z(s)ds\right|^2 \leq |z-z_0|\int_{z_0}^z v_z^2 ds,$$

for $z \in \Omega$, yields

$$\|v-\bar{v}\|_{L^2}^2 \leq \int_0^L v_z(s)^2 ds \int_0^L |z-z_0|dz \leq \frac{L^2}{2}\|v_z\|_{L^2}^2.$$

□

Proof of Theorem 1.3. We use $\psi(u) \in L^\infty(Q_T) \cap L^2(0,T;H^1_{\rm per}(\Omega))$ and $\theta - \bar\theta \in L^2(0,T;H^1_{\rm per}(\Omega))$ as test functions in the weak formulation of (1.1)-(1.2), respectively, and add the resulting equations:

$$\int_\Omega \left(\Psi(u(t)) + \frac{1}{2}(\theta(t)-\bar\theta)^2\right) + \int_{Q_t} (\nu\psi'(u)u_z^2 + \theta_z^2) \tag{4.2}$$
$$= \int_\Omega \left(\Psi(u_0) + \frac{1}{2}(\theta_0-\bar\theta)^2\right) + \int_{Q_t} ((1-u^2)\psi'(u)-\gamma)u_z\theta_z$$
$$+ \int_{Q_t} (\mu u - \theta)(\theta-\bar\theta).$$

For the second integral on the right-hand side we use Young's inequality:

$$\begin{aligned}\int_{Q_t} ((1-u^2)\psi'(u)-\gamma)u_z\theta_z &= \gamma\int_{Q_t} \frac{1-\alpha^2}{\alpha^2-u^2}u_z\theta_z \\ &\le \frac{\nu\gamma}{2}(\alpha^2-1)^{1/2}\int_{Q_t}\frac{u_z^2}{\alpha^2-u^2} + \frac{\gamma}{2\nu}(\alpha^2-1)^{3/2}\int_{Q_t}\frac{\theta_z^2}{\alpha^2-u^2} \\ &\le \frac{\nu\gamma}{2}(\alpha^2-1)^{1/2}\int_{Q_t}\frac{u_z^2}{\alpha^2-u^2} + \frac{\gamma}{2\nu}(\alpha^2-1)^{1/2}\int_{Q_t}\theta_z^2.\end{aligned}$$

Since $\mu\bar u = \bar\theta$, the last integral on the right-hand side of (4.2) becomes

$$\begin{aligned}\int_{Q_t}(\mu u-\theta)(\theta-\bar\theta) &= \mu\int_{Q_t}(u-\bar u)(\theta-\bar\theta) - \int_{Q_t}(\theta-\bar\theta)^2 \\ &\le \frac{\mu^2\delta}{2}\int_{Q_t}(u-\bar u)^2 + \left(\frac{1}{2\delta}-1\right)\int_{Q_t}(\theta-\bar\theta)^2,\end{aligned}$$

where we choose

$$\frac{L^2}{2(L^2+2)} < \delta < \frac{4\nu\gamma}{\mu^2L^2}.$$

This is possible by assumption (1.11). We employ Lemma 4.1 to estimate the first integral on the left-hand side of (4.2):

$$\int_\Omega \left(\Psi(u(t)) + \frac{1}{2}(\theta(t)-\bar\theta)^2\right) \ge \int_\Omega \left(\frac{\gamma}{2\alpha^2}(u(t)-\bar u)^2 + \frac{1}{2}(\theta(t)-\bar\theta)^2\right).$$

Finally, the second term on the left-hand side of (4.2) can be estimated by using Lemma 4.2:

$$\int_{Q_t}(\nu\psi'(u)u_z^2+\theta_z^2) \ge \int_{Q_t}\left(\frac{2\nu\gamma}{L^2}\frac{(u-\bar u)^2}{\alpha^2-u^2} + \frac{2}{L^2}(\theta-\bar\theta)^2\right).$$

Putting the above estimates together, we infer from (4.2)

$$\int_\Omega\left(\frac{\gamma}{2\alpha^2}(u(t)-\bar u)^2 + \frac{1}{2}(\theta(t)-\bar\theta)^2\right) \tag{4.3}$$
$$\le c_0^2 + \int_{Q_t}\left(\frac{\mu^2\delta}{2} - \frac{2\nu\gamma}{L^2} + \frac{\nu\gamma}{L^2}(\alpha^2-1)^{1/2}\right)\frac{(u-\bar u)^2}{\alpha^2-u^2}$$
$$+ \left(\frac{1}{2\delta} - \frac{L^2+2}{L^2} + \frac{\gamma}{\nu L^2}(\alpha^2-1)^{1/2}\right)\int_{Q_t}(\theta-\bar\theta)^2.$$

Observing that

$$\frac{(u-\bar{u})^2}{\alpha^2-u^2} \geq \frac{(u-\bar{u})^2}{\alpha^2},$$

we can let $\alpha \to 1$ in (4.3) to obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega \left(\gamma(u(t)-\bar{u})^2 + (\theta(t)-\bar{\theta})^2\right) \;&\leq\; c_0^2 - \int_{Q_t}\left(\frac{2\nu\gamma}{L^2} - \frac{\mu^2\delta}{2}\right)(u-\bar{u})^2 \\ &\quad - \left(\frac{L^2+2}{L^2} - \frac{1}{2\delta}\right)\int_{Q_t}(\theta-\bar{\theta})^2.\end{aligned}$$

Defining

$$\delta_1 = \frac{4\nu}{L^2} - \frac{\mu^2\delta}{\gamma} > 0, \quad \delta_2 = \frac{2(L^2+2)}{L^2} - \frac{1}{\delta} > 0, \tag{4.4}$$

the theorem follows from Gronwall's lemma. □

Remark 4.3 From the limits $\delta \to L^2/2(L^2+2)$ and $\delta \to 4\nu\gamma/\mu^2L^2$ we obtain upper bounds for the decay rates δ_1 and δ_2. Indeed, the first limit implies that

$$\delta_1 \nearrow \frac{\mu^2L^4 - 4(L^2+2)}{2L^2(L^2+2)}, \qquad \delta_2 \searrow 0,$$

and the second limit gives

$$\delta_2 \nearrow \frac{\mu^2L^4 - 8\nu\gamma(L^2+2)}{4\nu\gamma L^2}, \qquad \delta_1 \searrow 0.$$

5 Numerical examples

In this section we illustrate the long-time coarsening of the segregation bands in the drum by numerical experiments. For the numerical discretization, we use a time-discretized version of (1.6)-(1.7) (backward Euler method), as motivated by the existence analysis of Section 2, instead of discretizing directly (1.1)-(1.2). The space discretization is performed by using finite differences.

In the domain $Q_T = (0,L)\times(0,T)$ we define the grid

$$\left\{(x_i, t^{(n)}) \in Q_T : x_i = ih,\ i=0,\ldots,M,\ t^{(n)} = n\tau,\ n=0,\ldots,N\right\}$$

were $h = L/M$ is the space discretization parameter, $\tau = T/N$ the time discretization parameter, and $M, N \in \mathbb{N}$, $T > 0$.

For each time step, we have to solve the fully discretized nonlinear problem, consisting of the equations

$$\begin{aligned} &g'(v_i^{(n)})\frac{v_i^{(n)} - v_i^{(n-1)}}{\tau} \\ &= \frac{\nu}{h^2}\left[g'(v)_{i+1/2}^{(n)}\left(v_{i+1}^{(n)} - v_i^{(n)}\right) - g'(v)_{i-1/2}^{(n)}\left(v_i^{(n)} - v_{i-1}^{(n)}\right)\right] \\ &\quad - \frac{1}{h^2}\left[(1-g(v)^2)_{i+1/2}^{(n)}\left(\theta_{i+1}^{(n)} - \theta_i^{(n)}\right) - (1-g(v)^2)_{i-1/2}^{(n)}\left(\theta_i^{(n)} - \theta_{i-1}^{(n)}\right)\right]\end{aligned}$$

and

$$\begin{aligned}
\frac{\theta_i^{(n)}-\theta_i^{(n-1)}}{\tau} &= \frac{\gamma}{h^2}\left[g'(v)^{(n)}_{i+1/2}\left(v^{(n)}_{i+1}-v^{(n)}_i\right)-g'(v)^{(n)}_{i-1/2}\left(v^{(n)}_i-v^{(n)}_{i-1}\right)\right]\\
&\quad+\frac{1}{h^2}\left(\theta^{(n)}_{i-1}-2\theta^{(n)}_i+\theta^{(n)}_{i+1}\right)+\mu g(v^{(n)}_i)-\theta^{(n)}_i
\end{aligned}$$

in the variables $v_i^{(n)}$, $\theta_i^{(n)}$, for $i=1,\dots,M-1$, $n=1,\dots,N$. Here, we have used the notation

$$\begin{aligned}
g'(v)^{(n)}_{i\pm1/2} &= \frac{g'(v^{(n)}_{i\pm1})+g'(v^{(n)}_i)}{2},\\
(1-g(v)^2)^{(n)}_{i\pm1/2} &= 1-\frac{1}{2}\left(g(v^{(n)}_{i\pm1})^2+g(v^{(n)}_i)^2\right).
\end{aligned}$$

Discrete periodic boundary conditions are implemented in a standard way.

We solve the above nonlinear problem by a simple fixed-point strategy. The solution of the above nonlinear problem at time $t^{(n)}$ is the fixed point of the operator $S_n:\mathbb{R}^{2(M+1)}\to\mathbb{R}^{2(M+1)}$ defined by

$$S_n(\tilde v_0,\dots,\tilde v_M,\tilde\theta_0,\dots,\tilde\theta_M)=(v_0,\dots,v_M,\theta_0,\dots,\theta_M),$$

where $(v_0,\dots,v_M,\theta_0,\dots,\theta_M)$ is the solution of the linear problem

$$\begin{aligned}
&g'(\tilde v_i)\frac{v_i-v_i^{(n-1)}}{\tau}\\
&\quad=\frac{\nu}{h^2}\left[g'(\tilde v)_{i+1/2}(v_{i+1}-v_i)-g'(\tilde v)_{i-1/2}(v_i-v_{i-1})\right]\\
&\qquad-\frac{1}{h^2}\left[(1-g(\tilde v)^2)_{i+1/2}(\theta_{i+1}-\theta_i)-(1-g(\tilde v)^2)_{i-1/2}(\theta_i-\theta_{i-1})\right]
\end{aligned}$$

and

$$\begin{aligned}
\frac{\theta_i-\theta_i^{(n-1)}}{\tau} &= \frac{\gamma}{h^2}\left[g'(\tilde v)_{i+1/2}(v_{i+1}-v_i)-g'(\tilde v)_{i-1/2}(v_i-v_{i-1})\right]\\
&\quad+\frac{1}{h^2}(\theta_{i-1}-2\theta_i+\theta_{i+1})+\mu g(\tilde v_i)-\theta_i.
\end{aligned}$$

The dynamics of an initially preseparated initial state $u_0(z)=0.75\cos(80\pi z/L)$ with $L=30$ is shown in Figures 2 and 3. The initial short-wave perturbations produce decaying standing waves (Fig. 2a). The segregated bands emerge, and we observe metastable long-wave bands (Fig. 2b). Finally, after larger time $t>6$, the system segregates again (Fig. 3). Choosing a larger initial wave number (and a smaller value for γ), the system behavior is similar as in Figures 2 and 3, but the system starts to segregate after the metastable period at shorter time $t>2.5$ (Fig. 4).

We wish to understand the influence of the parameters γ and ν on the system behavior. In Fig. 5a, we have chosen a smaller value for γ compared to the situation of Fig. 4. Here, the short-wave perturbations persist on a longer time scale than in Fig. 4. For a larger value of ν (Fig. 5b), the period of quasistationary long-wave bands is longer than in Fig. 4.

An example of a non-segregating solution is presented in Fig. 6. We expect non-segregation since the parameters satisfy the condition (1.11). The granular materials further mix and do not segregate.

Finally, we present an example in which the initial perturbations are of long-wave type (Fig. 7). We observe again the influence of the parameter γ on the temporal evolution. In fact, we either obtain stable band arrays or a very slow coarsening of the band structure.

References

[1] H. W. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Math. Z.* 183 (1983), 311-341.

[2] I. Aranson and L. Tsimring. Dynamics of axial separation in long rotating drums. *Phys. Rev. Lett.* 82 (1992), 4643-4646.

[3] I. Aranson, L. Tsimring, and V. Vinokur. Continuum theory of axial segregation in a long rotating drum. *Phys. Rev. E* 60 (1999), 1975-1987.

[4] J.A. Carrillo, A. Jüngel, P. Markowich, G. Toscani, and A. Unterreiter. Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities. *Monatsh. Math.* 133 (2001), 1-82.

[5] P. Degond, S. Génieys, and A. Jüngel. An existence and uniqueness result for the stationary energy-transport model in semiconductor theory. *C. R. Acad. Sci. Paris* 324 (1997), 29-34.

[6] P. Degond, S. Génieys, and A. Jüngel. Symmetrization and entropy inequality for general diffusion systems. *C. R. Acad. Sci. Paris* 325 (1997), 963-968.

[7] P. Degond, S. Génieys, and A. Jüngel. A system of parabolic equations in nonequilibrium thermodynamics including thermal and electric effects. *J. Math. Pure Appl.* 76 (1997), 991-1015.

[8] G. Galiano, A. Jüngel, and M. Garzón. Semi-discretization in time and numerical convergence of a nonlinear cross-diffusion population model. To appear in *Numer. Math.*, 2002.

[9] S. Kawashima and Y. Shizuta. On the normal form of the symmetric hyperbolic-parabolic systems associated with the conservation laws. *Tohuku Math. J., II. Ser.* 40 (1988), 449-464.

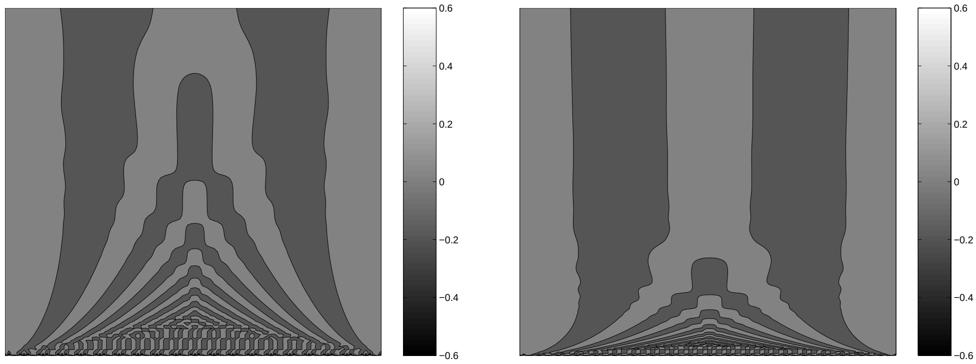
[10] Y. Lou and W.-M. Ni. Diffusion, self-diffusion and cross-diffusion. *J. Diff. Eqs.* 131 (1996), 79-131.

[11] M. Mimura and K. Kawasaki. Spatial segregation in competetive interaction-diffusion equations. *J. Math. Biol.* 9 (1980), 49-64.

[12] N. Shigesada, K. Kawasaki, and E. Teramoto. Spatial segregation of interacting species. *J. Theor. Biol.* 79 (1979), 83-99.

[13] O. Zik, S. Lipson, S. Shtrikman, and J. Stavans. Rotationally induced segregation of granular materials. *Phys. Rev. Lett.* 73 (1994), 644-647.

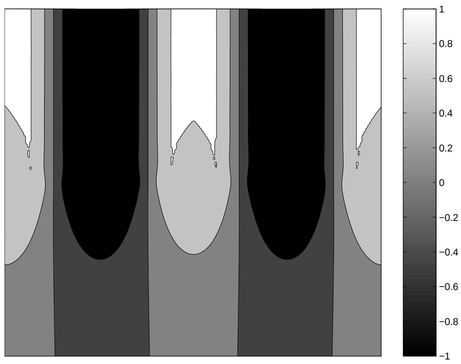
Figure 2: $\gamma = 100$, $\mu = 40$, $\nu = 0.5$, $L = 30$, $u_0(z) = 0.75\cos(80\pi z/L)$, $N = 1000$.



(a) $0 \le t \le 1.06$.

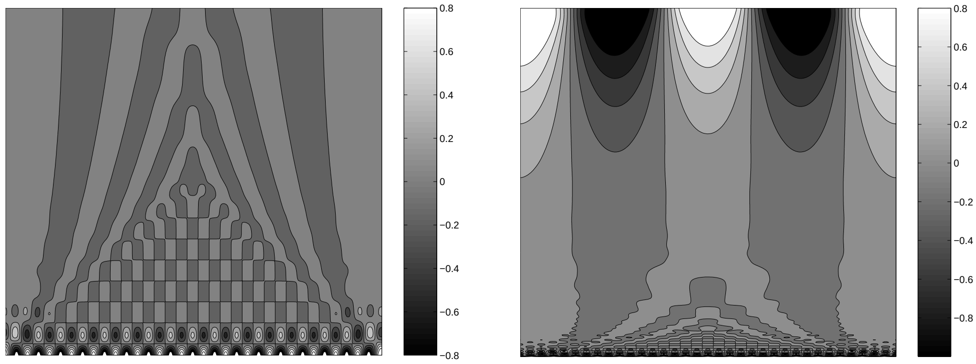
(b) $0 \le t \le 3.05$.

Figure 3: Continuation of Figure 2.



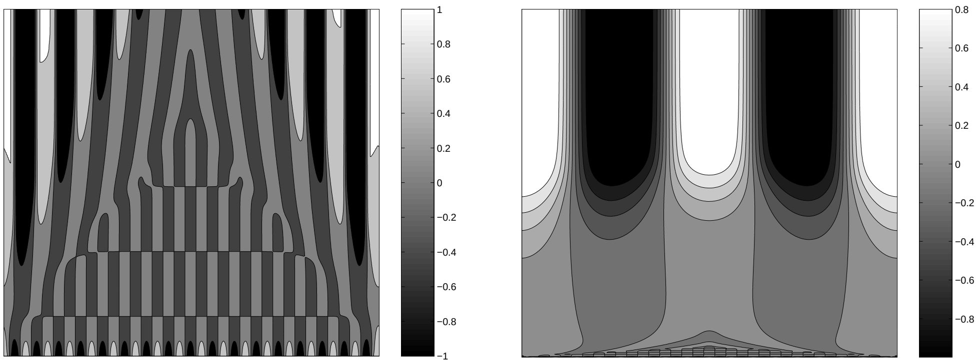
(a) $4.80 \le t \le 8.92$.

Figure 4: $\gamma = 40$, $\mu = 40$, $\nu = 0.5$, $L = 30$, $u_0(z) = 0.95\cos(32\pi z/L)$, $N = 1000$.



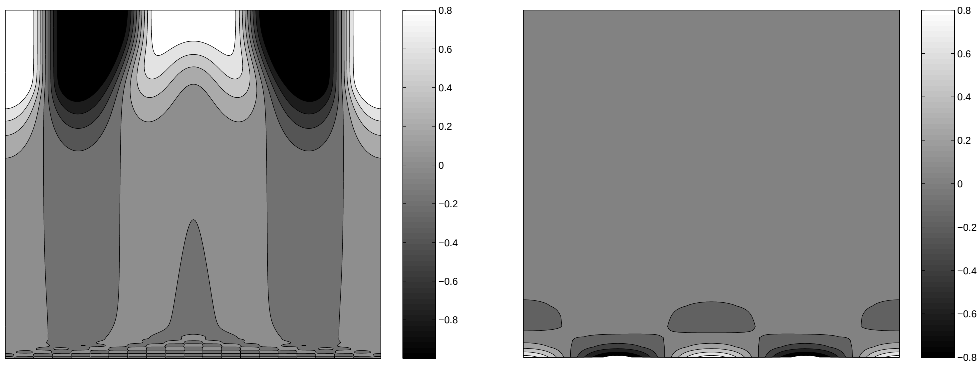
(a) $0 \le t \le 0.45$. (b) $0 \le t \le 3.78$.

Figure 5: Same data as Figure 4 except γ and ν.



(a) $\gamma = 4$, $\nu = 0.5$, $0 \le t \le 1.48$. (b) $\gamma = 40$, $\nu = 10$, $0 \le t \le 3.78$.

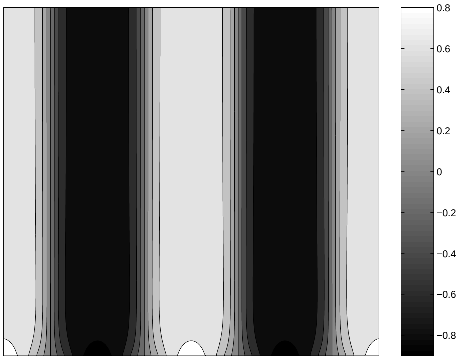
Figure 6:



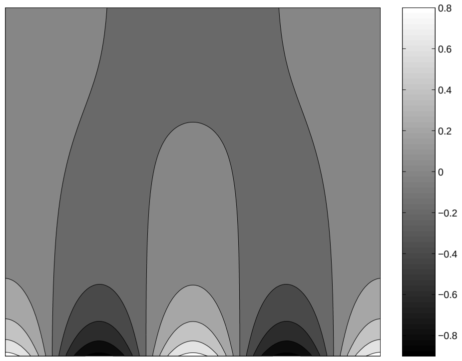
(a) $\gamma = 5$, $\mu = 4$, $\nu = 0.5$, $L = 30$, $u_0(z) = 0.05\cos(20\pi z/L)$, $N = 500$, $0 \le t \le 39.15$.

(b) $\gamma = 4$, $\mu = 1$, $\nu = 2$, $L = 1$, $u_0(z) = 0.9\cos(4\pi z/L)$, $N = 200$, $0 \le t \le 0.35$.

Figure 7: $\mu = 2$, $\nu = 1.6$, $L = 140$, $u_0(z) = 0.9\cos(4\pi z/L)$, $N = 200$.



(a) $\gamma = 5$, $0 \leq t \leq 500$.



(b) $\gamma = 100$, $0 \leq t \leq 1880$.