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Why Frege Did Not Plagiarize the Stoics: More on the Relationship Between Fregean and Stoic Logic

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ABSTRACT
In this paper we give a detailed comparison of the key elements of Frege's formal language of thought and apparently similar views in Stoic formal logic. That is, we compare their views on the following topics: connectives, negation, simple sentences, propositional content, predicates and their incompleteness, and quantifications. We show that in most of these cases the similarities between Frege's views and the Stoic views are only superficial. Frege's views are far more systematic, better developed and can in no case directly or fully be traced back to Stoic ideas. Furthermore, we show that Prantl, *pace Bobzien 2021*, is in *most* of these cases not a reliable source with respect to Stoic logic and that it is very unlikely that Frege's views on logic were influenced by Prantl's interpretation of Stoic formal logic.

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1. Introduction

Without any doubt, Frege made very important contributions to the development of modern logic. Most works, however, that aim to evaluate Frege's achievements have an important hermeneutical deficit; especially those that follow Michael Dummett's idea that Frege's contributions in logic are not connected to nineteenth-century philosophy or traditional logic. In fact, the German philosophical tradition of his time had a significant impact on Frege's thinking. In particular, achievements of the Kantian logical tradition and the Brentano school were made accessible to him by his correspondence partners and the intellectual philosophical microcosm in Jena, and helped him to develop his own original views. The most relevant contributions in this context are those of Herbart, E. Reinhold, Trendelenburg, Lotze, Sigwart, Windelband, Liebmann and Wundt.[1] Frege refined some of their approaches, and finally arrived at entirely new systematic results. Consequently, he could take this context for granted and had no reason to refer to it explicitly in his works. If these were as original as is generally assumed today, then to accuse Frege of plagiarism,[2] or even

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[1] For details, see *Gabriel 2013*; *Gabriel and Schlotter 2017*.
[2] We mean by 'plagiarism' like *Bobzien 2021* the practice of taking someone else's work or ideas and passing them off as one's own. Moral connotations are relative to our use of this term also not desired.

to claim that he *merely* revived and systematized overlooked or underrated philosophical views of the past would be unfair.

This very accusation has recently been levelled at Frege by Susanne Bobzien in her paper 'Frege plagiarized the Stoics' (*2021*). In this context, she has also presented a critique of the attempt in *Gabriel, Hülser and Schlotter 2009* to identify the Jena classical philologist Rudolf Hirzel as a probable mediator of Stoic positions. Of course, this attempt did not concern Frege's foundation of formal logic, which in its logical essence[3] was already available in *Frege 1879*, *1880/81*, a decade before Hirzel and Frege were able to exchange any ideas. It was rather a matter of clarifying the relationship between Frege's concept of 'thought' ('Gedanke') and the Stoic concept '*lekton*' (' λεκτόν'), whose similarities have already been frequently noted in the literature.

In this respect, a passage in Eduard Zeller's standard work *Die Philosophie der Griechen in ihrer geschichtlichen Entwicklung* is very instructive:

> The general object of it [the doctrine of the signified] is the thought [das Gedachte], or as the Stoics call it, what is said [das Ausgesprochene] (λεκτόν). By this name, in fact, they designated the content of thinking as such, the thought [den Gedanken] in the objective sense, in its distinction from the things to which the thoughts refer, from the words by which they are expressed, and from the soul-activity by which they are produced; and for this reason they maintained that only what is said is something incorporeal [...].[4]

Not only is the term 'thought' ('Gedanke') introduced here in Frege's objectivist sense as a translation for '*lekton*'. The role of thoughts is also described as representing things in reality *objectively*. In an early essay on logic, Frege had already pointed out the objectivity of 'judgeable contents [beurteilbare Inhalte]'.[5] This was Frege's early term for contents of judgements. Judgeable contents, however, *contained* (existing) things, properties and relations as objective constituents and did not *represent* them, neither in an objective nor in a subjective sense. For the early Frege, subjective presentations (i.e. 'Vorstellungen' in the psychological sense) fulfilled this task in a subjective sense. In this respect, Frege's view changed significantly between 1879 and 1891. Hence, the later Frege objectivized the former subjective presentations of things, properties and relations and identified them with thoughts in the same way as Zeller describes the Stoic position. However, Frege cannot be assumed to have had direct knowledge of Zeller's book, whereas Hirzel knew Zeller's work very well. He might have made Zeller's interpretation of the Stoic conception of logical content accessible to Frege. This claim is supported by the fact that Frege introduced the term 'thought' for the objective sense of the propositional sentence in 'Über Sinn und Bedeutung' (*1892*), some years after Hirzel had moved to Frege's house in 1889 as his tenant.

Furthermore, Zeller takes up the Stoic division into complete and incomplete *lekta*, which appears to hint at Frege's categorial distinction between complete and incomplete expressions, references and senses: '[...] what is said is either complete [vollständig] or incomplete [unvollständig]; complete, if it contains a finished statement, incomplete, if it contains an unfinished statement' (*Zeller 1865*, 79). Therefore, it is likely that Hirzel helped Frege out with Stoic terminology. Before Frege used the complete/incomplete terminology

[3] More details on this issue can be found in footnote 11.
[4] *Zeller 1865*, 78. This quote is not contained in the first edition, but in all editions from the second onwards.
[5] *Frege 1879a*, 7. See also: *Frege 1884*, §27, 37, Fn. ∗.

to mark an important distinction with respect to his logical semantics and his logical theory of content, he used different metaphors for this purpose. The chemical metaphor of 'being unsaturated [ungesättigt]' for concepts and functions can already be found in the letter to Stumpf from August 29, 1882 (*Frege 1976*, 164). The similar metaphor of being 'in need of supplementation [ergänzungsbedürftig]' is used in *Die Grundlagen der Arithmetik* (*1884*, §70). But only in *Frege 1891*, *1892b* and *1914*, does he use the expression 'unvollständig' ('incomplete') for the same purpose.

Whereas *Gabriel, Hülser and Schlotter 2009* pointed at Hirzel as a mediator of Stoic ideas to Frege, *Bobzien 2021* argued that Prantl's *Geschichte der Logik im Abendlande* was Frege's main source for Stoic logic.[6] For Frege's objectivist understanding of the term 'thought', however, Prantl cannot be consulted, since the latter interprets 'lekton' nominalistically and subjectivistically (*Prantl 1855*, 416). As it is well known, Prantl also misinterprets the distinction between complete and incomplete contents. He regards it as a distinction between complete and defective subjectless statements (*Prantl 1855*, 438–440, 444f., 471–473). With respect to both topics, Zeller is the more plausible candidate as a source and Hirzel as the intermediary.

With respect to Frege's formal logic, the situation is different. It is not unlikely that Frege knew Prantl's work, or at least had second-hand knowledge of it.[7] Bobzien very quickly and without convincing arguments rejects Rudolf Eucken as a mediator of the logical core of Stoic logic. In fact, Eucken knew Prantl's work excellently[8] and was one of the few philosophical supporters of Frege's career in Jena. Hence, we consider Eucken a very good candidate for such a mediating role. Anyway, it should not be disregarded that in the nineteenth century some aspects of Stoic thinking were already common knowledge, for instance the distinction between the judgeable content and the affirmative act of judgement (see *Gabriel 2013*, 292).[9] Furthermore, there are many philosophers who paved the way for the revolution in logic in the nineteenth century, and who were either well-informed about Stoic logic and referred to relevant sources in their works (e.g. cf. van Calker 1822; Hamilton 1863; Sigwart 1871, *1873*) or made use of Stoic ideas without knowing their origin. We also know from Heinrich Scholz' work on Frege's posthumous writings that Frege not only read *Sigwart 1873*, but that there existed a nineteen pages long notebook written by Frege to summarize Sigwart's views on logic (*Veraart 1976*, 103). Unfortunately, we do not know when Frege read *Sigwart 1873* and made these notes, since they got lost in 1945 at the end of World War II in Münster. On this basis, it is also a *prima facie* plausible option that Sigwart paved Frege's way to Prantl, since he like Eucken praises and quotes Prantl's history of logic. Frege was not trained in logic as a student, hence, he needed to inform

[6] *Zeller 1865* contains a new chapter on Stoic formal logic to correct mistakes of the first edition of this book and those in *Prantl 1855*.

[7] Among the books borrowed by Frege from the Jena University Library until 1879, there are neither Prantl's volumes nor any other works on logic (*Kreiser 1984*, 21).

[8] Ample evidence of this can be found in *Eucken 1879*, where Prantl's work is quoted more than 20 times. See also the preface of this work, where Eucken applauds Prantl for his achievements. Frege knew this work of Eucken quoted in *Frege 1884*, §32, fn. 56.

[9] It should be noticed that in modern times, J. Lipsius (1547–1606) founded Neostoicism, a widespread movement, rather influential until the French revolution. It made a modern version of Stoic thought popular in Europe. Though its influence was mainly a political one, some knowledge of Stoic logic was included, too, the more so, as many people read Seneca's writings. Insofar, and for further reasons, also in Frege's time various pieces of Stoic logical thinking were, so to speak, 'in the air'.

himself about the state of the art in logic and *Sigwart 1873* was probably the main source for him for this purpose.[10]

Yet, no matter how Frege became acquainted with Stoic logic, he did not plagiarize the ancient logicians, let alone Prantl. By systematizing various suggestions handed down from antiquity in some way and by developing them with formal rigour and new insights, he himself accomplished original ground-breaking research. It is a key feature of Frege's working method to take important traditional views on logic or philosophy as a starting point, and then either criticize and correct these views or develop them in a new direction on the basis of new and more systematic background assumptions.

Frege's most important achievements can be briefly listed as follows: the conception of the first and second level concepts as functions, the analysis of all expressions that make contributions to the truth-conditional content of the *Begriffsschrift* on the basis of the argument/function distinction, the conception of all logically relevant propositional connectives as functions from truth-values to truth-values, the mutual definability of all truth-functional connectives with the help of the material implication or the conjunction in combination with the negation, the complete rigorous formulation of the formal language of first and second level predicate logic, the axiomatization of these two languages, and the development of a calculus for both languages. Frege's *Begriffsschrift* (*1879a*) and *Frege 1880/81* already contained essentially all of his most important logical achievements.[11] The later work *Grundgesetze der Arithmetik* (*1893, 1903*) brought only further refinements, whereby the main changes concerned the philosophical foundation of logic, i.e. logical semantics as well as the logical conception of judgement.

In this paper, we give a detailed comparison of the key elements of Frege's formal language of thought and apparently similar views in Stoic formal logic. That is, we compare their views on the following topics: connectives, negation, simple sentences, propositional content, predicates and their incompleteness, and quantification. We show that in most of these cases the similarities between Frege's views and the Stoic views are only superficial. Frege's views are far more systematic, better developed, and can in no case directly and fully be traced back to Stoic ideas. Furthermore, we show that Prantl is in most of these cases not a reliable source with respect to Stoic logic and that it is very unlikely that Frege's views on logic were influenced by Prantl's interpretation of Stoic formal logic.

2. Two Possible Phases for an Influence of Stoic Formal Logic on Frege's Formal Logic

If one intends to argue that Stoic logic influenced Frege's logical investigations to any considerable extent, these influences must have taken place either in the *foundational period*

[10] Since Frege started thinking about a more suitable and general formal language of thought between 1874 and 1879 it is quite likely that Frege considered Sigwart's main work published in 1873, as the best up to date reference book on traditional logic.

[11] We, thereby, mean: (a) the development of a more powerful and more general theory of quantification than in the traditional Aristotelian or Stoic logic, (b) the unification of propositional logic with quantification theory, (c) a (traditional) axiomatization of classical first- and second-order logic, (d) the highlighting of the central deductive role of *modus ponens* in such a system, (e) the reduction of all logical connectives to the negation- and the conditional-stroke, (f) a function-theoretic interpretation of logical constants more in the tradition of the early Russell than the later Frege, and (g) the substitution of the subject-predicate analysis of judgements by a function-argument analysis of formulas, to name only the most important. We are certainly aware that Frege made important and substantial changes with respect to the *semantics* of his *Begriffsschrift* especially between 1884 and 1891.

between 1874 and 1879 or the *refinement period*[12] between 1884 and 1903.[13] The logical main work of the first phase is Frege's ground-breaking *Begriffsschrift*, and that of the second phase are the two volumes of *Grundgesetze der Arithmetik*. Changes between these two periods were too insubstantial with respect to logic, changes after 1903 did not concern formal logic itself. Hence, in the following we will compare Frege's view on logic in the highlighted two periods with the main results of Stoic logic.[14]

3. Frege and the Stoics on *Connective Sentences*

The propositional core of Frege's formal language of thought contains two primitive logical constants in both versions: The negation stroke and the conditionality stroke. Since Frege proposed an analysis of logical connectives based on the logical composition of facts or truth-values, he can define other connectives on the basis of these two primitives.

3.1. The Conditional

According to the first version of Frege's formal language from *the foundational period*, the connectives presumably stand for functions from judgeable contents to meta-semantic judgeable contents.[15] The early Frege identifies judgeable contents with circumstances or presentations (*Frege 1879*, §2). It is clear from elucidations in later works that Frege meant objective presentations[16] and that these objective presentations are the counterpart of subjective presentations that really were representing something (*Frege 1880/81*, 18–19). The best terms to specify these contents are certainly *circumstances* or *states of affairs* as translations of 'Umstände'. These states of affairs have objects, properties and relations as

[12] In opposition to one of our reviewers, we still think that 'refinement period' is an apt term. Since we use the ambiguous English term 'refinement' in the sense of 'improvement' but consider the former as more elegant.

[13] In *Bobzien 2021*, these two different options are not differentiated. Certainly, the period between 1884 and 1891 is in this respect of special importance, since Frege did not publish anything that concerns philosophy of logic or logic during this period.

[14] In *Bobzien 2021*, the comparison mainly focuses on an apparently similar logical *terminology* of Frege and the Stoics in general. We think that most of the highlighted resemblances are superficial; they mostly or completely disappear if one compares the views in sufficient detail. Furthermore, several textbooks on logic in the nineteenth century are full of Stoic terminology, since (a) there is a tradition of incorporating Stoic ideas that goes back to the ancient commentators and (b) propositional logic made a big comeback.

[15] *Frege 1879*, §§2–7. In opposition to *Heck and Mey 2013* and *Badesa and Bertran-San Millán 2017*, we interpret Frege's earlier notion of a function in a more uniform way and not as a mere notion of logical syntax. Partially based on later remarks of Frege on his earlier work and heuristic considerations (see for more details: section [4]), we think that in *Frege 1879* full-blooded functions are implicitly used as denotations of (i) logical and (ii) non-logical signs, which, according to our interpretation, are either (i) functions from objects to contents, or (ii) functions from functions to contents, or (iii) functions from contents to contents. Frege already used the notion of a function in his habilitation in its full-blooded sense (*Frege 1874*). There he distinguishes between the domain [Größe], the argument and the value of a function. We agree that his introduction of this notion to logic in 1879 is cryptic and minimalist, but we think that the overall use of this notion in his works provides good reasons to resist such an idiosyncratic interpretation. There are also good reasons why functions from contents to contents are meta-semantic, if they exist in the early system. Firstly, when Frege informally paraphrases the contents of complex sentences he consistently uses meta-semantic vocabulary, especially in the case of the paraphrases of negations. Secondly, since for any arbitrary content C, there are infinitely many different structured contents that are the contradictory opposite of C, and for any complex content C, there are infinitely many different equivalent structured contents, a meta-semantic view seems to be the only option to provide a one-to-one mapping in the described sense.

[16] *Frege 1884*, §27, 37, Fn. ∗, *1976*, 280, Fn. ∗15. In a letter to Jourdain, Frege comments on his early view on contents of judgements in *Frege 1879*. For this purpose, he distinguishes between subjective and objective presentations. According to him, subjective presentations are mental presentations by individual subjects of partially objective states of affairs, while objective presentations are the contents of logical judgements, and, hence, mind-independent. The same distinction can be found in *Frege 1884*, but supplemented with a rough statement of a conception of the structure and constituents of objective contents according to the earlier Frege.

constituents (*Frege 1880/81*, 18–19, *1882*, 189, *1884*, §§26–27, especially the footnote p. 37; § 70). In the early system, a sentence connective can only have expressions that denote a state of affairs as inputs. The function denoted by the conditionality sign requires as input two states of affairs, say *A* and *B*; and this function maps the two states of affairs to the meta-semantic state of affairs that states: *it is not the case that A does not occur and B does occur* (*Frege 1879*, §3, 6–7). This shows that the early Frege interpreted the conditionality sign as a *material conditional* with a quite peculiar semantics. This semantics is *functional*, but not truth-functional. A semantics of a connective is *truth-functional* in the literal sense if the connectives denote functions from truth-values to truth-values.[17] Frege's earlier semantics is also not a *compositional* semantics in the contemporary sense, since it does not provide an account how the *truth* of a sentence is determined on the basis of the semantic values of the constituent expressions. It only shows how the semantic contents of complex sentences are *composed* of the contents of the constituent expressions. Hence, it is a *composition theory* of content and not a compositional semantic theory in the outlined sense.

The later Frege, however, is aware of this problem: he makes a strict distinction between a semantic theory of sentences[18] and a theory about the composition of thoughts. On this basis, Frege assigns a *semantic value* instead of a content-constituting function to the connectivity stroke. This semantic value is a function from pairs of classical truth-values to truth-values, and it maps a complex sentence connected by this stroke to *the False* only if the antecedent has *the True* as value and the consequent *the False*; in all other cases the value of the function is *the True* (*Frege 1893*, §12, 20). In print, this view first appears in *Frege 1891*, and most importantly in *Frege 1893*.

What was the view of the Stoics on the conditional? Hellenistic positions on the conditional in general agreed about two things: (i) If 'If A, then B' is true, then B follows *in some sense* from A. (ii) 'If A, then B' is only then true if 'A and it is not the case that B' is false (*Frede 1974*, 81–82). According to the Stoic logicians, an additional *connection* between the antecedent and the consequent is required for the truth of the conditional. In this sense, the Stoics are the first known proponents of relevance logic. The main representative of the classical period of Stoic logic, Chrysippus, added the following connecting-conditions to complete the truth-conditions. According to him, a conditional sentence of the grammatical form 'If A, then B' is true *only if* the negation of B is incompatible with A (see *FDS* fr. 914, fr. 952–965, also fr. 473 and fr. 948–951; *Hülser 1987/88*; *Frede 1974*, 80–93; *Bobzien 2003*, 94–95; *Prantl 1855*, 456). The best current interpretations of the notion of incompatibility hold that A and B are in a logical, semantic, or nomological sense contradictory (*Frede 1974*, 83–89; *Bobzien 2003*, 95). Hence, this notion can also be applied to sentences like 'It is day' and 'It is night' if these two alternatives are considered as semantically exhaustive. So Stoic logic assumes a very strong connection between the antecedent and the consequent of a conditional, which is therefore not truth-functional, not purely extensional, and not compositional in our contemporary sense (*Hülser 2023*, 58–61, 71–73).

[17] Since, Frege did not have the notion of a truth-value or some (necessarily) equivalent notion in 1879, one cannot ascribe to him a truth-functional semantics in 1879. The Stoic semantics of connectives is truth-*conditional* since it uses the ordinary notions of truth and falsity and provides on this basis truth- and falsity-conditions of connective sentences. Literally speaking, Frege does not have a proper semantics in 1879, but only a theory that accounts for the contents of formulas and judgements.

[18] A semantics of sentences in its strict sense is *compositional*, and, hence, the truth or falsity of a sentence depends in a certain type-theoretical hierarchical order on the semantic values of constituent expressions.

The Stoic logicians formulated *truth-conditions* for conditional propositions, but they did not assign a *semantic value*, like the late Frege, or a specific *contribution to content*, like the early Frege, to the conditional connective. Like most traditional views on logical constants, the Stoics hold that they are *syncategorematic*, and hence make no direct and independent contribution to truth-conditions. It is Frege's ground-breaking contribution to logical semantics to provide a meaningful way to assign to certain *logical constants* an independent and specific contribution to the truth-conditions that is essential for the determination of the truth-value of a complex sentence.[19]

All this shows that there are substantial differences between the Stoic and the Fregean view on logic and connectives. Certainly, it is true that the ancient logicians were aware of the basic idea behind the material conditional, and some interpreters think that later Stoics had sympathies for this view.[20] As far as we know, the Megarian logician Philo was the first who claimed that *ordinary* conditionals are only then false if the antecedent is true and the consequent false. There, however, are also important differences between Frege's defence of the material conditional and ancient conceptions. Firstly, Frege chose to formalize the conditional in a very simple, general and useful way for logical purposes. The material conditional is the logically simplest thinkable conditional, and it is all that is required for Frege's logical purpose. Secondly, the early Frege had the view that the *ordinary* indicative conditional and the material conditional have different truth-conditions. He contrasted these two conditionals as follows: 'The causal connection that is encoded by 'if' will not be expressed by our sign [the conditionality stroke], although a judgement of this form can only be made if it is grounded in such a connection.'[21] Hence, he has an original nuanced view that distinguishes between the ordinary indicative conditional and a formal logical conditional, which no known ancient logician did. Thirdly, Frege's formal conditional is part of a *system* of truth-functional or fact-dependent connectives that no ancient philosopher endorsed. Finally, also Philo was not able to assign to the conditional an individual self-standing meaning.

Therefore, in all the above-mentioned respects Frege also did not plagiarize Philo's view on the conditional. He transformed well-known ancient insights in such a way that they became more powerful and useful, and that they could play an important role in Frege's revolution of logic in the nineteenth century.

In what sense could Prantl, then, have played a role in Frege's transformation and refinement of ancient views on logic of the conditional? Bobzien claimed that Prantl was

[19] It is important to notice that we can distinguish a basically Fregean and a Tarskian approach to compositionality. According to the Fregean view, truth-values are determined on the basis of *functional* application. In the case of the Tarskian approach, we have a *relational* account of compositionality. The semantic type of the logical forms provides or determines a specific relation that relates the semantic values of the constituent terms, and thereby constitutes the assignments of truth-values. In the case of atomic sentences, this relation is the set-membership relation; in the case of connectives, specific relations between truth-values. However, in classical Tarskian semantics the truth-conditions of quantified sentences are not compositional. *Pickel and Rabern 2016* have solved this problem. For more on this issue see *Rami 2021*.

[20] *Bobzien 2021*, 55–57. For doubts that we share on this view see: *Frede 1974*, 89–90.

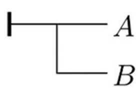
[21] *Frege 1879*, §5, 6 (our translation). The later Frege rejected this constraining condition of the material conditional. – A reviewer claimed that for Frege causal 'connections are captured by means of generality'. It is certainly true that it is claimed in *Frege 1879*, §12, 23, that causal connections can be *expressed* by universal generalisations of the form 'Every F is G' and according to the early logical analysis of such claims. However, we interpret Frege's claims in §5 and §12 as purporting a logical abstraction. Frege does not reduce causal relations to logical relations. The causal relation is the more fundamental ('grounded in'!) relation. However, for logical purposes, it is sufficient to use the material conditional to describe causal relations between individual events, and universal generalizations are logically sufficient expressions of law-like-generalizations according to Frege. This view is compatible with our distinction between Frege's early formal and informal view on the meaning of the conditional.

Frege's main source on Stoic logic. The first book of Prantl's *Geschichte der Logik im Abendlande*, published in four volumes, contains a long chapter on Stoic logic. In this chapter, Prantl discusses several views of the conditional, including Philo's und Chrysippus' view. Prantl presents the material conditional by following the presentation of this issue by Sextus Empiricus[22] in a surprisingly contemporary form, namely on the basis of the following forerunner of a truth-table (*Prantl 1855*, 454):

> (1) Antecedent *true*, consequent *true*, [...]
>
> (2) Antecedent *false*, consequent *false*, [...]
>
> (3) Antecedent *false*, consequent *true*, [...]
>
> (4) Antecedent *true*, consequent *false*. [...]
>
> Of these four combinations only the fourth was conceived as incorrect [...].

In *Bobzien 2021*, 52–57, it is noticed that we find a very similar apparent forerunner of a truth-table in *Frege 1879*, §5, 5. In opposition to Prantl, Frege does not use in these lists 'true' or 'false', but 'affirmed' and 'denied'. Didn't the early Frege believe that truth is the most important semantic property of sentences? What is the reason for his different terminology? In any case, there is an important difference concerning the *status* of Frege's list and Prantl's borrowed from Sextus. The latter represents the possible combinations of truth or falsity of the constituent propositions of a conditional. According to Philo's view as Prantl presents it following Sextus, specific combinations of the truth- or falsity-statuses of constituent propositions are sufficient for the truth of the whole proposition. Hence, Prantl uses the quoted list to determine the *truth-conditions* of a conditional. For Frege, his apparently similar list is used to constitute the *content* of a sentence of the *Begriffsschrift* that has the conditionality stroke as main connective. *Syntactically* for the early Frege there is only one connective sign, the conditionality stroke, and one modifying sign, the negation stroke. Semantically Frege reduces conditionals of the following general form:

(Con)



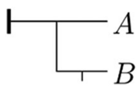
to meta-semantic contents that contain a conjunction and two negations. Hence, *Begriffsschrift*-sentences of the given form *mean* that *it is not the case that A does not obtain and B obtains*. In the end, the contents of Frege's early conditionals are equivalent with ordinary language conditionals conceived according to Philo. Nevertheless, Frege's apparently similar list of conjunctive combinations to those of Prantl or Sextus determines *contents* of formulas, not their *truth-values* or truth-statuses. In this sense, they are not comparable with contemporary truth-tables, which are a specific representation of *truth-functions*. Hence, if Frege has seen the above-mentioned list in Prantl, he did not just reproduce it, in his early work; he used it for a quite different and original purpose. The later Frege did not make use of truth-tables either, but he could have, since he had a genuine truth-functional conception of connectives.

[22] Prantl's list goes back to Sextus Empiricus, *Adv. Math.* VIII §449: 'For, as we showed earlier, they maintain that the conditional is sound when it begins with a true proposition and finishes with a true one, or when it begins with a false one and finishes with as false one, or when it begins with a false one and finishes with a true one, and is false in one way, when it begins with a true one and finishes with a false one. [...]' (transl. by R. Bett: *Sextus Empiricus 2005*).

Furthermore, Prantl's presentation of Philo's conception of the conditional is very critical and dismissive. Prantl himself seems to favour a view that assumes *some* non-causal connection between the antecedent and the consequent as necessary condition for the truth of the conditional. In any case, a possible reader, like, for example, Frege, would at least be forced by Prantl's critical comments and his review of different alternative conceptions to find good reasons to favour one of the outlined views over the others. Hence, Prantl did also not provide good reasons for Frege to make use of the material conditional, nor did Frege just plagiarize Philo's view if it was conveyed to him via Prantl, since his view is substantially different from this view in the four mentioned ways.

3.2. The Weak Disjunction

The early Frege extended the outlined strategy exemplified by the conditional to all other connectives in his formal language. That is, he represents connective sentences *syntactically* by means of combinations of the conditionality stroke and the negation stroke, but *semantically* these representations are reduced to combinations of conjunction and negation. Hence, the weak or inclusive disjunction is *syntactically* represented by Frege as follows:

(WD) 

Semantically it has the meaning that *it is not the case that A does not obtain and B does not obtain* (*Frege 1879*, §7, 10–11). The reason for this double-life is explained by Frege as follows: A *formulation* of his formal language by means of the conditionality stroke as the only primitive connective is founded in the choice of the logical calculus (*Frege 1879*, §7, 13). The Frege-Hilbert calculus[23] essentially requires formal representations by means of the conditional, since *modus ponens* is its only essential derivation rule.[24]

The later Frege got rid of such double-life of syntax and semantics. He again made use of the conditionality sign for proof-theoretic purposes; nevertheless, he was now able to assign to each possible connective its specific truth-functional denotation. On this basis, the formal representation of an inclusive disjunction given above denotes the True iff it is not that case that A and B denote the truth-value *the False* (*Frege 1893*, §12, 21).

Weak disjunction did not play an important role in classic Stoic logic.[25] Starting with the very strong exclusive disjunction of Chrysippus that was non-truth-conditional, and was stronger than our contemporary exclusive disjunction, there are different ways to weaken it, which were discussed in ancient logic (*Frede 1974*, 98–100). Frege's inclusive disjunction was only one of these options. The main reason for the classic Stoics to favour a

[23] It is an axiomatic proof system with a single essential rule of derivation: *modus ponens*. Certainly, there are important differences between Frege's and Hilbert's conception of such an axiomatic logical system, since Frege holds a more traditional foundationalist conception of axioms, and Hilbert a holistic modern alternative, but nevertheless they are close relatives.

[24] We are certainly aware that in *Frege 1879*, §24 two additional rules are introduced to simplify certain derivations. Even more of these additional rules are introduced in *Frege 1893*, but they are in both versions not essential derivation rules.

[25] A passage by the Roman lawyer Proculus (early 1st cent. AD) is our very oldest evidence for the weaker disjunctions in Stoicism. They were called *subdisjunctions*. See *Frede 1974*, 98–100, and *Hülser 2015*. These alternative disjunctions relax the incompatibility condition to more or less extent. Our contemporary exclusive and inclusive disjunctions are also among these variations. Since there are very few sources with respect to these variations, and since it was difficult to get access to them for a nineteenth century logician like (the early) Frege, it is quite unlikely that these views had any influence on Frege's conception of connectives.

stronger disjunction was probably the following: The Stoics consider five forms of inference of propositional logic as the bedrock of all logically valid arguments (*Frede 1974*; *Bobzien 1999*, *2003*); beside modus ponens, modus tollens and modus ponendo tollens also the following two versions of the disjunctive syllogism:

(DS1) Either P or Q; P, Not: Q (strong disjunctive syllogism)[26]

(DS2) Either P or Q; Not: Q, P (weak disjunctive syllogism)

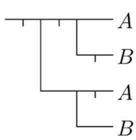
However, in opposition to (DS2), (DS1) is *invalid* if 'either … or … ' is interpreted as an expression of a weak disjunction in the Fregean sense. Hence, a stronger kind of disjunction is required to make both basal kinds of reasoning valid. Frege's choice for the weakest possible version of disjunction was again motivated by the norms of generality and simplicity. Hence, the choice of their main disjunction was motivated by classical Stoics and Frege in a different way.

Prantl also does not explicitly present *weak disjunction* as a sentential connective in his chapter on Stoic logic. Whereas weak disjunction does not play an important role in (classic) Stoic logic, in modern propositional logic it is the most important kind of disjunction because it is the *logical dual* of the truth-functional conjunction, which is manifested by the two de Morgan rules. In this respect, therefore, we see important conceptual and structural differences between the logic of Frege and the Stoic formal logic.

3.3. The Strong Disjunction

The early Frege was already aware that a stronger kind of disjunction could be introduced in his system. The familiar difference between Fregean weak and strong disjunction is that in the latter case the truth of both disjuncts leads to a false claim. One option for Frege to represent this strong truth-conditional disjunction syntactically is:

(DS)



Semantically this formula expresses a content that can easily be represented by adding a conjunct to the content of the weak disjunction. On this basis, (DS) means: 'it is not the case that A does not obtain and B does not obtain *and it is not that case that A obtains and B obtains*' (cf. *Frege 1879*, §7, 12). The later Frege made use of the very same logical structure, but interprets it in a different way, namely such that the structure (DS) expresses a truth-function that maps only pairs of different truth-values to *the True*.

Classical Stoic logic has as its essential component a disjunction that guarantees the validity of (DS1). It is not only stronger than Frege's weak disjunction, but also stronger than Frege's strong disjunction. The Stoic super-strong disjunction is a relevance connective and, therefore, not truth-functional. The truth of such a disjunction of the form 'Either A or B' requires that it is not the case that A and B are both false and that there is a strong incompatibility between A and B. We already know this notion of strong incompatibility:

[26] 'Not:' is typically a semi-technical device to provide a precise paraphrase of the Stoic view on sentential negation. Cf. *Bobzien 1999*, *2003*.

it says that A and B are contradictory in a logical, semantic or nomological sense (*Frede* 1974, 93–96; *Bobzien* 1999, 109–111, 2003, 96f.; *Hülser* 2015, 19–23). This feature provides a conceptual relation between the Stoic conditional and the Stoic strong disjunction. The early Stoic logic is one of the earliest versions of a *relevance* logic. Frege's logic is from the beginning a modern *classical* logic without any semantic relevance-constraints. For Frege, his notion of a strong disjunction is demanded by simplicity.[27] Furthermore, Frege's connectives form a system of connectives that are connected by their purely fact-dependent (early Frege) or truth-functional (later Frege) nature. For Frege, both kinds of disjunctions were part of a system of sixteen possible truth-functions. In this sense, there are systematic structural differences between Frege's conception of logic and the Stoics' conception of logic, differences that are important. Frege saw his conception of logic as competitive with and superior to the Stoics, and therefore cannot be seen as someone who reinvented or plagiarized Stoic logic. Only from a superficial perspective could such a claim appear credible. Prantl is himself aware of the strong and special nature of the central Stoic disjunction (cf. *Prantl* 1855, 460). Hence, there is no possible and interesting influence by Prantl with respect to Frege on this issue.

3.4. The Conjunction

With respect to conjunction, there is an exceptional agreement between Stoic logic and Frege regarding truth-conditions. The early Frege holds that a conjunctive state of affairs obtains iff both constituent states of affairs obtain (*Frege* 1879, §7, 12–13). The later Frege would interpret conjunction as a truth-function that maps only a pair of the truth-value *the True* to *the True*.[28] In a similar fashion, the Stoics held in general that a conjunction of the form 'Both A and B' is true iff both conjuncts are true (*Frede* 1974, 96–97; *Bobzien* 2003, 93–94). While for the early Frege the conjunction is the *semantically* most basic operation to create complex facts, for the Stoics the conjunction is of logical relevance mainly because of the inferential role of negated conjunction in arguments of this form:

(MPT) Not (P and Q); P; Not: Q (*Modus ponendo tollens*)

Against this background, one might expect that the Stoics in general distinguish between a strong and a weak negation of conjunctions, and that the strong version is required for arguments of the basal form (MPT). This strong version would add the condition that the two conjuncts of a negated conjunction must be strongly incompatible. A suggestion of this kind looks attractive and systematic, since all other four basal syllogisms of Chrysippus' logic have a major premise that contains a main connective that expresses a strong incompatibility relation. Furthermore, on this basis premises of the form 'Not (P and Q)' would in general come out as *a priori* truths. If one aims to provide a *demonstration*, an argument from evident and true premises to a true and evident conclusion, this special status is required for any premise of the argument. This strengthening, which was famously discussed

[27] Frege realized that the ordinary strong disjunction expresses some form of exclusiveness. However, unlike the Stoics, who semantically represented this additional exclusiveness, Frege conceived it merely as a pragmatic supplementation of the semantic core-meaning. That is, in our view, he opted for the claim that the ordinary language expression 'or' creates a conventional implicature. In this respect, Frege is not only a predecessor of Grice, but he also defends a distinctive view on the nature of the implicated content; cf. *Frege* 1879, §7, 12.

[28] However, since in *Frege* 1893 all truth-functions are reduced to negation and conditionality, Frege does not explicitly make use of a conjunction-stroke in his system.

and suggested by Galen[29] for the outlined reasons, was not acceptable for Chrysippus.[30] However, this choice makes the classical Stoic collection of basal arguments less systematic and arbitrary.[31] Frege's view, on the other hand, is much more systematic with respect to the semantics and the logical properties of the connectives.[32] Since his conjunction as all other connectives used by him is part of a system of related truth-functional dyadic connectives.

Yet, there are also good reasons to reject such a strengthening as proposed by Galen: Firstly: If 'Both A and B' would be treated as equivalent with the claim that A and B are not strongly incompatible and that A is true and B is true, the negation of such a claim would be weaker as desired. Secondly, it is also unclear how the postulation of a second kind of negation could provide the desired effect, since a sentential negation can be applied to every true or false sentence.

Prantl correctly points out that the Stoics had a standard conception of conjunction. However, in contrast to Frege's system, a connective of this kind seems out of place among the other connectives in the early Stoic system.

3.5. Further Connectives

Both Frege and the ancient Stoics made use of further logical connectives. Frege showed in both systems how our ordinary connective 'Neither … nor … ' can be represented in his formal language. As we already mentioned, his system of truth-functions has the potential to introduce sixteen truth-functional connectives in total. Later Stoic logicians added a few additional connectives to Chrysippus' logic, which are mainly *philosophically* motivated and do not belong to the logical core theory of the Stoics. Among these additional connectives are 'because' and 'since' (cf. *Diogenes Laertius VII,* §§71–74 (= *FDS* fr. 914) and *FDS* fr. 972ff. See also: *Bobzien 2003*, 93). This again shows that the Stoic view on this topic is more piecemeal and less systematic than Frege's conception.[33]

4. Frege and the Stoics on Negation

There are important differences between Frege's early and his later view on negation. *Syntactically,* in both periods Frege represents negation by means of the negation stroke that only can be applied to formulas that have a prefixed *content stroke* or *horizontal*. Hence, *syntactically* in the early system negation is an operation (i) on sentences that *express a judgeable content*, and in the later system, an operation (ii) on *any expression prefixed with the horizontal*. The horizontal guarantees semantically that the negation function has the

[29] Galen found it problematic that relative to a purely truth-dependent semantics of 'not' and 'and', 'Not (P and Q)' can contain two *completely unrelated* conjuncts: e.g., 'Peter is unmarried' and 'The sun is not shining today', and would be, hence, useless for demonstrations. Cf. *Galen 1964*, Chapter XIV.

[30] For more on this see *Frede 1974*, 96f.

[31] From our contemporary perspective, this collection is also not sufficient to establish a logically complete propositional logic. Cf. *Galen 1964*, Chapter XIV, 3 for a similar worry.

[32] As we all know: A sentence of the logical from 'Not: A and B' is reducible to 'Not: A or Not: B', if 'and', 'or' and 'not' are interpreted in a purely extensional or truth-functional way. Interestingly, this reduction also holds, if 'Not: A and B' is only then true if 'A' and 'B' are contradictory opposites and the same holds for the truth of 'A or B'. However, in all other cases, whether mixed or symmetric, such a reduction does not hold. Especially, if the compatibility condition is only conceived of as contrary opposition.

[33] In *Bobzien 2021*, Frege's view on 'because' is discussed. Unlike the later Stoics, 'because' is *not* considered as a logical connective by Frege. Furthermore, Frege holds in opposition to the Stoics 'A because B' iff 'A and B and if A, then B'. Such an analysis only makes sense if 'if, then' is not a material conditional, but this issue is not settled in *Frege 1892b*.

right and desired inputs, namely truth-values (*Frege 1893*, §5, 9–11). This syntactic characterization also foreshadows important semantic differences. A formula of the following form has completely different contents in Frege's two systems:

(NEG) $\qquad —\!\!\!\!\top\!\!\!\!— A$

In the early system, all adequate substitutions for '*A*' denote judgeable contents or states of affairs. The content-stroke in front of '*A*' syntactically marks an expression that denotes a full state of affairs, in opposition to expressions that only denote constituents of states of affairs (*Frege 1879*, §2, 2). *Semantically* sentences of the form (NEG) mean that *A does not obtain*. Hence, the early Frege reduces a negation that is syntactically represented as a *sentential* negation semantically to a *predicate* negation of the predicate 'obtain' that can only be applied to states of affairs (*Frege 1879*, §5, 10). Hence, the early Frege does not have a *genuine* and *fundamental* sentential negation in his formal system.

One motivation for the distinction between the propositional content and the designation of a Begriffsschrift sentence might have been a desire to eliminate the mismatch between syntax and semantics with respect to negation in the old system as well.[34] In the refinement period, Frege corrected this weakness, and distinguished between the *denotation* and the *content* of an expression of the form (NEG). According to the later Frege, the contents of such expressions are thoughts, and the denotations are truth-values. On this basis, the negation stroke denotes a function from truth-values to truth-values, which maps *the True* to *the False* and vice versa (*Frege 1891*, 22–23, *1893*, §5, 10–11). As a consequence, the negation stroke is now a full-fledged sentential negation from a syntactic as well as semantic perspective.

The Stoic conception of negation does in fact have some similarities with Frege's later conception. Since they regarded arguments of the form (MPT) as important and basal, the Stoics had to accept some kind of negation that operates either on sentential connectives or on whole sentences. They chose the second option, probably because it guarantees contradictory opposition in every case (See *FDS* fr. 914, fr. 920–946; *Hülser 1987/88*; *Frede 1974*, 69–73). The negation particle '*ou*' ('οὐ', 'οὐκ' or 'οὐχί') has a rather flexible syntax in ancient Greek and can be applied to different sorts of expressions. Semantically it has at least two different uses: (a) it can be applied to any meaningful expression to negate this expression, probably on the basis of a complement negation with respect to the extension of an expression; and (b) it can be applied to whole sentences to negate the content or state of affairs expressed by the sentence. The Stoics have a clear preference for the second use, which perfectly fits their goal of making sense of (MPT). Furthermore, they realized that if one puts the sentential negation '*ou*' in front of a whole sentence, the result is a new sentence that has the opposite truth-status of the original sentence. Hence, if 'A' is true, then '*ou* + A' is false, and the other way round. If one aims to simulate this kind of use in English, one has to extend ordinary English by a sentential negation particle 'Not', often marked by an additional colon. On this basis we can use 'Not: Socrates is walking' as a sentential negation of 'Socrates is walking' to simulate the intended and conventionalized use of negation by the Stoics.

[34] Certainly, this problem is of a more general nature, and it also affects the other main logical constant for the propositional part of Frege's formal language, the conditionality stroke. Furthermore, this problem has also a more substantial aspect, namely that the early Frege does not provide a proper compositional semantics for the *Begriffsschrift*.

There are obvious similarities between the Stoic conception of sentential negation and semantics of the negation stroke in the later version of Frege's *Begriffsschrift*. In the same sense as the one expression inverts the classical truth-status of an ordinary language assertoric sentence, the other expression inverts the classical truth-value of a *Begriffsschrift* sentence. Furthermore, the Stoics realized that one could iterate such a sentential negation, and that a simple iteration leads to a sentence with the same truth-status as the original one (See *FDS* fr. 914: §69; *Hülser 1987/88*).

Apart from these obvious similarities, there are also some subtle differences. Firstly, the later Frege provides a compositional semantics for negated sentences because his functional semantic theory allows him to assign a specific independent semantic value to the negation stroke, a truth-function. The Stoics had to treat their negation particle as a *syncategorematic* expression without an independent meaning.

Secondly, the Stoics also distinguished a third kind of privative term negation expressed in Greek by the prefixes '*a*-' and '*an*-'. Frege also realized that certain prefixes like 'in-', 'un-' or 'non-' can produce certain negative counterparts of adjectives and predicates (*Frege 1919*), but he considered privative negation as irrelevant for a language of thought.[35] The negation stroke is the only kind of negation that is available in the *Begriffsschrift* and it allows negation of not only atomic sentences, but sentences of any kind.

The conception of negation in early Frege leads to similar results as the Stoic negation with respect to the truth-status of negated sentences. However, it is completely different in nature, since it reduces sentential negation to the predicative negation of the predicate *to obtain*.

One might therefore think that Frege changed his mind here in response to his acquaintance with Hirzel. He knew the Stoic philosophy much better than Eucken did and he might have pointed out aspects of Stoic logic to Frege that even some other German commentators might have missed. That is one possible explanation. Another is that Frege made direct or indirect acquaintance with the work of Windelband, who had already used the notion of a truth-value to conceive judgements as assessments of the truth-value of a content (cf. *Gabriel 2013*, 293). Certainly, Frege's notion of a truth-value does not only play a role in judgements, but an equally important role in semantics, where it supplies assessments as arguments and values of specific functions. A reasonable hypothesis seems to be the following: Frege combined Windelband's ideas with inspirations from Boole's algebraic treatment of negation and other logical connectives (cf. *Boole 1854*) and fused them on the basis of his semantic framework that makes use of *functions* instead of algebraic notions or sets, which led to a new function-theoretic treatment of negation and other connectives. Hence, in this respect, at least two different influences are possible and probable that led to some of Frege's semantic innovations in the refinement period.

Prantl almost certainly played no important role in Frege's change of mind with respect to the content and semantics of the negation stroke. In *Prantl 1855*, one can find three passages on the Stoic view on negation. The first one (*Prantl 1855*, 444) only quotes and paraphrases Diogenes Laertius' distinction of negative propositions. In the second passage (*Prantl 1855*, 445), he distinguishes between the negation of the subject, the copula or the

[35] Privative negation is a negation that transforms a predicate A to a new predicate B such that B has the (relative or absolute) complement extension of the extension of A. In Frege's system it is certainly possible to distinguish between wide- and narrow-scope negations especially in the case of quantifications. However, semantically spoken both negations are sentential negations. Privative negation is a negation on the morpho-syntactic level.

predicate of a judgement. He thus interprets the Stoic view against an Aristotelian background.[36] Prantl claims that if the Stoics represent negation as negation of a whole sentence, they have to decide whether this sentential negation ultimately concerns the subject, copula or predicate of a judgement. According to him, the Stoics only considered negations of the subject or the copula (or verb) as logically relevant (cf. *Prantl 1855*, 445). Prantl thereby completely misunderstands the essence of the Stoic negation. Although he is aware of the fact that the Stoics favour a sentential negation, he represents this kind of negation carelessly in a third passage as a negation of the copula. According to Prantl, for the Stoics the negation of 'Socrates is walking' would be 'Socrates is not walking' (cf. *Prantl 1855*, 449).[37] Again, then, Prantl produces a very misleading and incorrect presentation of the Stoic view on negation. Therefore, even if Frege had known Prantl, the latter would have been of no use to Frege in understanding the core ideas behind Stoic negation.

5. Frege and the Stoics on Simple Sentences and Their Contents

The early Frege did not say much to elucidate his notion of simple sentences. They do not play a significant role in *Frege 1879*. However, if one takes into account *Frege 1880/81*, 18–19, *Frege 1882*, 189, *Frege 1884*, §§26–27, §70 and the following quote from 1879, things become importantly clearer (*Frege 1879*, §9, 15; our emphasis):

> If we think of the circumstance that hydrogen gas is lighter than carbon dioxide in our formal language, we can replace the sign for hydrogen gas with the sign for oxygen gas or that for nitrogen gas. This changes the sense in such a way that 'oxygen gas' or 'nitrogen gas' enters the *relationships* in which 'hydrogen gas' previously stood. By thinking of an expression as *variable* in this way, it is divided into an *invariant* part, which represents the totality of the *relations*, and into the sign, which is thought to be replaceable by others, and which signifies the *object* that is in these relations. I call the former component function, the latter its argument.

In our opinion, one can learn two different important things from this quote. Firstly, that Frege distinguishes between two constituent parts of sentences of his *Begriffsschrift*: *variable* expressions that are *functions*, and *invariant* expressions that are *arguments*. We interpret the invariant/variable distinction outlined in this quote as another forerunner of the 1891 distinction between *complete* and *incomplete* expressions. Secondly, one can learn from this quote that specific two-placed functional expressions denote *relations*, and their argument expressions denote *objects*. Based on *Frege 1880/81*, 18–19 and *Frege 1884*, §§26–27, §70, one can generalize this view to the claim that non-logical one-placed function-expressions denote *properties* and $n > 1$-placed function-expressions denote *relations*. Hence, it seems reasonable to hold that for the early Frege *simple sentences* consist of n primitive object-denoting constants of an n-ary functional expression that denotes a property or relation.

This view on the denotation of non-logical expressions can naturally be supplemented with a philosophical conception of functions that treats *properties* as unary functions from object to contents and *relations* as $n > 1$-placed functions from pairs of objects to contents.

[36] According to Aristotle's conception of negation, every form of negation is either a negation of the copula or of the predicate, and copula-negation is the more important kind of negation for logical purposes.

[37] To express his view Prantl uses the famous Stoic example 'It is day'. But due to its grammatical subject 'it', this example is inappropriate to illustrate the idea in English. Hence, we replaced it by a different example that is also known from the fragments on Stoic logic.

One can reasonably attribute such a view to the early Frege, if one is willing to accept three additional assumptions: (A1) Frege used the notion of a function (itself) throughout his career from 1874 on as an abstract mathematical device from analysis, with a range, values and arguments. The only difference between the original notion of a function in analysis is that it is restricted to numbers as values and arguments, while Frege generalizes the notion of a function for arguments and values of different sorts. This and the distinction between functions and function-symbols is interestingly also highlighted in *Frege 1879*, but probably overlooked by some philosophers:

> Since the symbol Φ occurs in the expression $\Phi(A)$ and can be thought of as replaced by other symbols [such as] Ψ, X, by means of which other functions of the argument A are then expressed, *$\Phi(A)$ can be regarded as a function of the argument Φ*. One sees here particularly clearly that the concept of function in Analysis, which in general I have followed, is far more restricted than the one developed here.[38]

We think that Frege is a bit sloppy in this passage concerning use and mentioning. Before 'are then expressed … ' he uses expressions like 'Φ' to refer to function-symbols. After 'are then expressed … ' he uses expressions like 'Φ' to designate functions themselves. On this basis, Frege clearly distinguishes in this passage from *Frege 1879* between function-expressions and functions as in earlier or later works.

(A2) In *Frege 1879* the focus of §9 is only on function-*expressions* and not functions as denotations of function-expressions themselves, because Frege's main aim in it is to justify his central thesis that the function/argument distinction is much more fruitful for a formal theory of judgements than the predicate/subject distinction. (A3) Function-expressions denote functions.

For Frege (A3) seems to be a conceptual truth, masked by the fact that he uses the term 'Vorstellung' when he speaks about contents and their constituents. We think that (A2) is clearly justified by the discursive structure and purpose of §9. (A1) seems to be justified by the cited quote and by the fact that in all works where Frege *explicitly* speaks about function-expressions and functions themselves, he considers them both in parallel ways as distinct kinds or expressions of entities that are variable, unsaturated and either incomplete or require completion by an argument (-expression). In the former case, the completion of an argument results in a value; in the latter case, the completion of an argument expression results in a complete expression. Therefore, we think the early *Begriffsschrift* if presented in a full and explicit way can be interpreted as making use of functions from objects to contents, and, as outlined above, functions from contents to contents.

The later Frege extends his early conception of simple sentences by adding two further types of singular terms: (a) terms that denote the value-ranges of functions and are formed by a specific abstraction-operator, (b) object-denoting formal counterparts of definite descriptions formulated by a second specific abstraction operator (*Frege 1893*, §9, §11).

The contents of simple sentences in the early period are states of affairs that contain n individual objects and if $n = 1$, a property, and if $n > 1$, an n-ary relation (*Frege 1880/81*, 18–19, *1882*, 189, *1884*, §70, 81–83). The later Frege assumed a different notion of content

[38] *Frege 1879*, §10, 18–19. This claim is similar in formulation and content with later related but more detailed claims on this issue made in *Frege 1891*, 1–3.

that contains abstract and objective presentations of individual objects, concepts and relations. A second difference is that he now conceives of the notion of an object in a semantic sense as the referent of a singular term, and conceives of concepts and relations in his technical and formal senses as functions from objects or n-tuples of objects to truth-values (*Frege 1892b*, *1893*, §§2–4).

The Stoic conceptions of these two issues are less clear because the traditional sources are not consistent. As far as we know, the Stoics mainly discussed simple propositions with a monadic predicate that expresses an abstract (incorporeal) concept (cf. *Frede 1974*, 51–73; *Bobzien 2003*, 88–91). Furthermore, they considered proper names, demonstratives, unspecific indefinite descriptions, and simple quantifier expressions as constituent expressions in subject-position of simple propositions (cf. *Frede 1974*, 53–66; *Bobzien 1999*, 97–101, *2003*, 88–90). How do the Stoics conceive of the contents (*lekta*) of simple propositions? Concerning this topic, we have agreement among scholars on one issue only: the abstract concepts that are expressed by monadic predicate terms are an essential component of the content of simple propositions (cf. *Long and Sedley 1987*, § 33 fr. G plus the final part of the commentary that follows; *Frede 1994*, 114f.). From our contemporary point of view, only sentences with two of the above four subject expressions count as simple propositions; those that contain proper names or demonstratives. On one kind of interpretation of simple propositions, both kinds of singular terms contribute the referent to the expressed content (cf. *Long 1971*, 75–82; *Long and Sedley 1987*, § 34 commentary in the middle). On another possible interpretation, that is only true for simple sentences with demonstratives. According to such a view, proper names contribute something else than the referent, namely an abstract presentation of an individual quality (essence) but not the individual itself. This quality necessarily coexists with the referent of the proper name (see on related views: *Frede 1994*, 120–127). A third kind of possible interpretation shares this view on names, but also assumes that demonstratives do not contribute demonstrated objects to propositions but an individual abstract representation of an act of demonstration. A fourth interpretation holds that proper names in apparently simple sentences (similar as in *Quine 1960*) are special kinds of predicates that at most apply to a single object (cf. *Alexander of Aphrodisias 2006*, 99). Hence, a sentence like 'Kallias is walking' means the same as 'There is a Kallias and he is walking' and is from a logical point of view a particular generalization.

If one compares these interpretations of the Stoic view with Frege's two views on this subject, one observes some similarities and some significant differences. Firstly, according to the early Frege, proper-name-like singular constants also contribute the denoted object to contents of simple sentences. Secondly, the early Frege considers concepts or properties as constituents of monadic simple sentences. However, he does this to conceive them of as objective entities of some sort, since he claims that states of affairs are in the same sense objective as the sun (*Frege 1879*, §2, 2–3). Thirdly, the later Frege also thinks that singular terms contribute abstract representation of the referents of these terms to the contents. However, unlike the Stoics he considers these modes of presentations as presentations of *individuals*.

But even Frege's earlier conception of content of simple sentences is more sophisticated than the view of the Stoics, since he distinguished between judgeable *contents* and *conceptual* contents of sentences (*Frege 1879*, §2, 2–4). Frege realizes that his early conception of content is too fine-grained for logical purposes, since according to it a simple sentence like 'Peter kissed Mary' does not express the same content as the simple sentence 'Mary received a kiss from Peter'.[39] However, according to Frege these two sentences have the same inferential potential, and, hence, have the same conceptual content. The later Frege was even more sophisticated in this respect. He rejected the view that propositional contents can be individuated by their constituents.[40] On this basis, different sentences with different logical forms can express the same content and these contents, therefore, have different form-relative decompositions (*Frege 1892a*, 199f., *1906a*, 203f.).

There is one essential difference between Long's interpretation of the Stoic view and Frege's early view. In opposition to the Stoics, Frege assumed that states of affairs obtain time-independently, and that thoughts are atemporally true (*Frege 1882*, 190, *1897*, 146–147, *1918*. See also: *Bobzien 2003*, 87). Consequently, Frege did not allow any context-sensitive expressions in the *Begriffsschrift*. Frege did not conceive of demonstratives or indexicals as *logically* respectable. He also wanted the *Begriffsschrift* to provide transparent representations of contents. That means there are also no ambiguities.

Apart from the outlined similarities, Frege's two conceptions of simple sentences and their contents are more or less substantially different from, and more sophisticated and complete than the Stoic counterparts. On this basis, there is no proper foundation for the claim that Frege just passed off work of the Stoics on these issues as his own or just systematized it.

In Prantl, one finds only the common syntactic distinctions between different kinds of simple sentences. With respect to the content of these sentences one finds nothing illuminating or correct (cf. *Prantl 1855*, 444).

6. Frege and the Stoics on *Predicates* and *Incomplete Expressions*

Many have noticed that Frege distinguished between *incomplete* and *complete* constituent expressions of well-formed sentences in a way that is apparently similar to the Stoics (cf. *Mates 1961*, 16f.; *Kneale 1962*). However, one has to be very careful here, since Frege actually uses three different but related distinctions concerning complete and incomplete expressions. Firstly, Frege distinguishes *syntactically* in general between expressions that *can* denote *arguments for a function* and those that denote the *functions* of the respective arguments. A first-order function can be the *argument* of a higher-order function, or the *function* of an object (*Frege 1891*, 26–27). Hence, the first distinction is a *relative* one. If a first-order function-expression *saturates* or *completes* a second-order function-expression, the result is an unsaturable, complete expression. Secondly, Frege distinguishes in an absolute, ontological sense between complete and incomplete items and their corresponding expressions. In this ontological sense, every object is complete, every function is incomplete (*Frege 1884*, §70, *1891*, 6–7, *1893*, §1). Objects are complete because they can only play the argument-role, while functions are incomplete, because they can play the function-role in at least some cases.[41] There is an infinite hierarchy of different functions with different arities and different kinds of argument places. Thirdly, every complete sentence requires for

[39] According to the later Frege, they express the same thought since they are cognitively equivalent.

[40] According to such a view, the following is the case: 'For every content A and B: $A = B$ iff A and B have the same structure and the same constituents relative to this structure'. The later Frege individuates or identifies contents on the basis of the relation of cognitive equivalence: 'For every content A and B: $A = B$ iff A and B have the same cognitive value (*Erkenntniswert*)'.

[41] This conception is structurally similar to an Aristotelian conception of the distinction between substances and universals. Substances cannot only be instances. Universals can be instances and be instantiated.

semantic reasons a functional expression that denotes a specific function and plays the role of a *predicate* in a sentence. For the later Frege, only functions to truth-values are concepts or relations and, hence, have a predicative nature (*Frege* 1892b, 129–130, 1906a, 103).

For the Stoics, neither expressions nor their denotations are in the outlined *syntactic*, *semantic* or *ontological* sense incomplete. They have a *pragmatic* notion of incompleteness. Only concepts can be *predicated* of individual things (cf. *Long* 1971, 77–78) or individual qualities (cf. *Frede* 1994, 121–124) in pragmatic sense. An expression that denotes a concept is incomplete *taken by itself*, according to the Stoics, since it *aims* to predicate the concept of an object. However, this only means that we cannot *merely* use a predicate to *state* something, more is required for this purpose (cf. *Frede* 1994, 113). Strictly speaking, the Stoic distinction between incomplete and complete expressions neither means the same as the Fregean nor is co-extensional with Frege's distinction. In Frege's system, the class of incomplete expressions is much larger than in Stoic logic. However, there is this interesting coincidence that Frege's notion of a *predicative* incomplete expression seems to have the same *extension* as the Stoic notions of an incomplete expression *simpliciter*. Nevertheless, these expressions play different roles in both systems. For Frege, predicates are not *predicated* of anything. The act of predication is not a constituent act of an act of a judgement. Predicates denote certain things that are required to build a complete content, but it is the judgement stroke that gives a sentence the force of an assertion or judgement. Hence, for Frege and the Stoics predicates play different explanatory roles, and are literally incomplete in different senses. Predicates are *semantically* or Frege-incomplete, since they express concepts (functions for truth-values) under which objects can fall. Predicates are *pragmatically* or Stoic-incomplete since they are predication-devices that are in need of a denoting subject to fulfil their pragmatic predicating function (cf. *FDS* fr. 696, §64). This shows that Frege's and the Stoics' definitions regarding predication and incompleteness are superficially similar and partly co-extensional, but different in nature.

7. Frege and the Stoics on *Quantified Sentences*

Frege's formal conception of quantified sentences is not motivated by any observations about quantification in natural language, but aims to systematize and generalize a familiar semantic practice in mathematics. This perspective on quantification is already quite explicitly stated in *Frege* 1879, 1:

> The symbols customarily used in the general theory of magnitude fall into two kinds. The first consists of the letters, each of which represents either a number or a function left undetermined. This indeterminacy makes it possible to use the letters to express the universal validity of sentences, as in:[42]

$$(a+b)c = ac + bc.$$

Frege uses two different sets of *letters* to transform any kind of simple statement to a generalization. For example, we could generalize the claim '$2+3=3+2$' by substituting the singular terms with different *Latin* letters (as Frege calls them), which leads to '$a+b=b+a$' (*Frege* 1891, 8–11). For him, the Latin letters allow us only to formulate *universal* quantifications. In more contemporary terms, one can interpret a sentence like

[42] See also: *Frege* 1879, IV, 1906b, 217.

'$a + b = b + a$' as an automatically universally closed sentence.[43] However, that is not what Frege *explicitly* says about such sentences. According to him, a Latin letter *indicates* certain objects *in an undetermined way* (*Frege 1893*, §8, 11, *1906b*, 217, *1976*, 117). We could say that Latin letters are indicators for satisfying objects or substitutable names of a specific type.

A simple universal generalization like 'Everything is F', can be formalized according to Frege as follows, where 'a' is interpreted as a *Latin letter* and not as a singular individual constant.[44]

(SU) $\vdash\!\!\!\text{———}\ \mathrm{F}(a)$

The remarks on the content of such a formula seem to be rather cryptic in *Frege (1879*, 19–20). There are *apparently* inconsistent hints that lead to the following two alternative views. According to the first hinted view, the formula (SU) means that:

> (EV1) The result of any supplementation of an object as argument of the first-order function F is an obtaining state of affairs (fact).

The second view claims that the formula means that:

> (EV2) The result of any insertion of any kind of name for an object as argument expression for the first-order function-expression 'F' is a complex expression that denotes an obtaining state of affairs (fact).

The first interpretation gives an *objectual* interpretation of the quantifiers, based on a quantification over arguments of specific functions. The second interpretation leads to a *substitutional* conception of quantification. We will not decide the issue here, which interpretation is the correct one.

The later Frege still made use of *Latin letters* for the purpose of expressing unrestricted universal generalizations, but he clearly committed himself to the objectual interpretation of quantification based on a different semantics. A formula of the form (SU) denotes the truth-value True iff the following holds:

> (EV1*) The first-level function $\mathrm{F}(\xi)$ has as value with respect to every argument the truth-value the True.

Frege was aware that such a view on generalization is useful for the formulation of *most logical laws*,[45] but too limited to express all relevant generalizations. On this basis, one cannot define or introduce a particular quantifier, since one cannot distinguish between the negation of a quantification and the quantification of a negation. That is, for example,

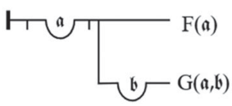
[43] That is, it is used with the implicit rule that every open sentence that contains unbound variables is universally closed by the implicit prefix of a universal quantifier. Cf. *Cumming 2008*, 540.

[44] The notion of a *Latin letter* should not be taken literally as one of our reviewers seemed to suggest. It is a technical term. A Latin letter is used in the position of a non-logical constant; it has a different non-denoting semantic function.

[45] One often overlooked feature of the *horizontal* interpreted as a function in truth-values is that it allows Frege to quantify with the same *type* of quantifier that quantifies over objects, also over truth-values, in particular, if a horizontal contains German letters, but no not-logical constants in its scope. That is, a horizontal can be combined with a Latin letter only, and lead to a meaningful sentence, or a German letter and the corresponding concavity. In contemporary type-theoretic semantics, truth-values are considered as expressions of type t, while individuals as entities of type e, which do not contain any truth-values in their domain. For Frege, truth-values are also individuals and can be denoted by non-logical constants or indicated by Latin or German letters. On this basis, a first-order quantification that *also* ranges over truth-values can be restricted by the combination of German letters and logical constants in such a way that truth-values are the only possible candidates to make the quantifications true. This allows Frege to formulate laws of logic in his system very easily without any sort of higher-order quantification.

we have only one opportunity to add a negation stroke to the formula (SU). It is unclear whether such a negation is the negation of the universal quantification or the universal quantification of the negation (*Frege 1893*, §8, 11–12). This is one of two motivations why Frege introduces the concavity of the horizontal to disambiguate negations with respect to universal generalizations (*Frege 1893*, §8, 11–14). A second motivation concerns scope-distinctions that one can make if at least two different Latin letters are used in a formula to quantify into two different argument positions of a function (*Frege 1880/81*, 21–23). For this purpose, Frege now inserts into the concavity the same *letter* that he uses to express a quantification. Furthermore, he distinguishes two types of letters with a quantifying purpose (*Frege 1879*, *1880/81*, *1893*). The *Latin letters* can only be used in generalizations as (SU), while those quantifications that allow for scope-distinction with respect to different quantification letters and negations, make use of so-called *German letters* inscribed into a concavity and used as quantificational letters *restricted* by the respective concavity. However, Frege does not have to change the truth-conditions of quantifications for this purpose; he just has to change the *formal representation* of quantifications. Here is an example of a quantification of the second type that exemplifies all the outlined special features:

(CU)



This formula contains the two German letters '𝔞' and '𝔟' that are part of two concavities. The concavity that contains '𝔞' has a wider scope than the one that contains '𝔟'. In both cases, the following formula contains '𝔞', or '𝔞' *and* '𝔟' as 'commanded' by the corresponding concavity. Furthermore, we have a negation stroke (i) before and (ii) behind the concavity that contains '𝔞', they represent (i) a narrow and (ii) wide scope negation. Any construction with at least one *German letter* inscribed in the concavity and another one corresponding *German letter* as part of some functional expression in the formula 'commanded' by the corresponding concavity allows us to express a *particular* quantification in Frege's system. This is all we have to know to understand Frege's conception of quantification that has its root in a mathematical practice that uses certain letters to express generalizations about numbers. Frege extends this practice from numbers to arbitrary objects, and he modifies it in the two ways mentioned above to enlarge the expressive power with regard to the original practice.

The Stoics develop their view on quantification on a significantly different basis. They develop their view (like *Quine 1974*) by focussing on ordinary *anaphoric* relations.[46] Let us elucidate this claim by focusing on an example from *Cicero De fato,* §12 (= *FDS* fr. 473):

(SE) If **someone** is born in the sign of the Dog Star, **he** will not die at sea.[47]

From a contemporary perspective, (SE) is a donkey-sentence. Such sentences contain a special sort of anaphoric construction. Firstly, 'someone' according to its literal meaning is

[46] The Stoic conception of quantification is more similar to Quine's views on this topic then Frege's. However, Frege is in one important respect closer to Aristotle than to the Stoics. He does not account for the semantics of quantifications on the basis of anaphoric relations or variable bindings, but on the basis of special subsumptions of lower-order concepts under higher-order concepts. The Aristotelian in a similar way thinks that the semantics of quantifications depend on specific relations between the extensions of two concepts. Frege's and Aristotle's view are both variations of a so-called view on generalized quantifiers. The Stoics and Quine are in spirit much closer to Tarski's semantics of quantifications.

[47] This example is quoted in *Prantl 1855*, 456, and used in *Bobzien 2021*, 63–68.

a particular quantifier. Secondly, the pronoun ‘he’ is used in the consequent of the conditional and the quantifier expression in the antecedent. Hence, syntactically the pronoun is outside the scope of the quantifier, and it should not be possible to bind it. However, as the Stoics already realized, the anaphoric pronoun ‘he’ can be bound by ‘someone’, which they call an indefinite phrase, and ‘someone’ strangely functions in (SE) as a universal quantifier with the widest possible scope.

As we can learn from Sextus Empiricus, who also discusses the semantics of generalizations of the form ‘If something is F, then it is G’, the Stoics also had a particular view on the truth-conditions of sentences like (SE). According to Sextus, ‘such a universal sentence is certainly false if at least one false instance is subsumed under it’ (*Sextus Empiricus, Adv. Math.* XI, §11 = *FDS* fr. 629). From this we can conclude that a sentence of the form ‘If something is F, then it is G’ is true iff all corresponding instances of the form ‘If a is F, then a is G’ are true. This shows that the Stoics had a substitutional view on quantification that is much more similar to our contemporary views than the more influential Aristotelian tradition.

However, Frege’s later view on this topic is superior to such a substitutional interpretation. Since the set of singular terms in ordinary languages is limited, the correctness of the outlined truth-conditions required an arbitrary technical extension of our ordinary vocabulary to make quantifications over infinite domains possible, whose members we cannot denote individually. If the early Frege in fact also championed a substitutional interpretation of quantifiers, which is unclear, he at least came to this conclusion on a completely *different path* than the Stoics. Furthermore, it is more likely that Frege was inspired in this respect by *Lotze* (*1974*, §68). In Lotze’s logic, universal judgements are interpreted as collections of singular judgements of a certain form. As Lotze, Frege does not specify the truth-conditions [Italics!] of universal generalizations as the Stoics in substitutional terms, but only the contents [Italics!] of these judgments.

In *Bobzien 2021*, 63–68, it is noticed that in *Frege 1923*, a late fragment on quantification, in which Frege aimed to present his views on quantification in informal terms, he, like the Stoics, uses donkey sentences to elucidate his view on universal quantifications. *Frege 1923*, 279, uses the example:

(FE) If someone is a human being, he is mortal.

At first sight, this appears to be a remarkable parallel between Frege and the Stoics. However, one should not overestimate this parallel and insinuate a direct influence of Stoic views on Frege. Firstly, sentences like (FE) were quite popular among nineteenth century German logicians. Kant famously held the view that categorial judgements like ‘All men are mortal’ are a different kind of judgements from hypothetical judgements (*Kant 1800*, §25, 106). Some logicians in the nineteenth century challenged this view and used sentences like (FE) to justify their views, most notably *Reinhold 1827*, §88, 188–189, *Lotze 1843*, 124, and *Sigwart 1873*, §27, 211–213. We know that Frege was at least personally acquainted with Sigwart’s views on logic.

Secondly, unlike these philosophers Frege does not argue that ordinary quantifications are in fact hypothetical judgements. He uses sentences like (FE) as the most natural ordinary-language paraphrases of his conditional treatment of universal generalizations that Frege favours for purely systematic reasons as he states in a famous letter to Husserl (*Frege 1983*, 106):

> I now use the expression 'all' in such a way that I neither indicate existence nor presuppose its fulfilment. The use of language cannot really be decisive here, because we are not concerned with the use in ordinary language. Rather, we can determine our use of language in logic according to logical needs. The reason for my stipulation is the simplicity.

This shows that Frege favoured his view on universal quantification for completely different reasons than the Stoics, who rely on certain canonical formulations of natural language.

Thirdly and most importantly, Frege even in his late works gives a significantly different analysis of sentences like (FE) than the Stoics. He also moves from (FE) to an instance of (FE), but not to give truth-conditions of (FE) on the basis of the truth of all instances. Instead, he starts with the instance 'If Napoleon is a human being, Napoleon is mortal', and substitutes 'Napoleon' with a *Latin letter*, to sketch on this basis his later *objectual* interpretation of universally closed generalizations.

It is true that there are *apparent* similarities between Frege's later elucidations of his conception of quantification and the remarks in ancient secondary sources on Stoic logic. Frege and the Stoics share some basic assumptions in opposition to the Aristotelian tradition, namely that quantifications can also contain (a) logical connectives, (b) relational predicates, and (c) more than one quantifier expression.[48] However, again as with respect to many other topics discussed in this paper, Frege's views are different in substantive respects. They are more systematic and have much more powerful expressive resources, based on his interesting generalization and modification of a common semantic practice in nineteenth century mathematics, and they are, if at all, only partially and indirectly influenced by Stoic insights.[49]

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[48] The latter was pointed out to us by Niko Strobach. Cf. *Cicero 2009 (De nat. deor.)*, III 22.

[49] There is another interesting topic that is neither fully covered in this paper nor in *Bobzien 2021*: Where there any direct or indirect influences of Stoic ideas on logic in the eighteenth and nineteenth centuries in general? Kant himself tried to incorporate certain 'new' aspects of logic into traditional Aristotelian logic that were already highlighted both by Theophrastus and Stoic logic. Many logicians in the Kantian tradition and in the Brentano-school intensified this influence. Maybe, the influence of Stoic logic on Frege's conception of logic can be exclusively traced back to the indirect influences of Stoic logic on Kantian logic. We will leave a detailed discussion of this topic for another occasion. – As to the present paper we would like to thank two anonymous reviewers for their careful, excellent, and very helpful comments; and Dustin Gooßens for corrections.

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