Gram Spectrahedra of Quadratic Forms on Varieties of (Almost) Minimal Degree

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Introduction

A polynomial $f \in \mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \ldots, x_n]$ is a *sum of squares* (*sos*) if it can be written as $f = p_1^2 + \cdots + p_r^2$ for some $p_1, \ldots, p_r \in \mathbb{R}[\underline{x}]$. The tremendous importance of sums of squares in real algebraic geometry stems from the easy observation that a representation of f as above provides a simple algebraic certificate for the global nonnegativity of our polynomial.

The question concerning the relationship between nonnegativity and the existence of sos representations dates back at least to Hilbert [Hil]. In 1888, he determined the pairs (n, d) for which every nonnegative polynomial in n variables of degree at most $2d$ is a sum of squares. However, even in these cases it is not immediately clear how to find an explicit sos representation of f. In the other cases, the first question is whether there is such a representation at all. Fortunately, the so-called Gram matrix method, originally due to Choi, Lam and Reznick [CLR] can be used to tackle both problems: Let $f \in \mathbb{R}[\underline{x}]$ be a polynomial of degree $2d$ and write $\mathfrak{m}$ for the column vector containing all monomials of degree up to d in some fixed order. A *Gram matrix* of f is a real symmetric matrix G with $\mathfrak{m}^T G \mathfrak{m} = f$. Our f is a sum of squares if and only if there exists a positive semidefinite Gram matrix of f, and finding an explicit representation amounts to explicitly determining such a matrix.

If f actually is a sum of squares, there are usually many inequivalent ways of representing f as such. The set of all positive semidefinite (*psd*) Gram matrices of f parametrizes all representations of f as a sum of squares, up to orthogonal equivalence ([CLR]). As this set is the intersection of the cone of positive semidefinite matrices with an affine-linear space, it is a spectrahedron, the *Gram spectrahedron* $\mathrm{Gram}(f)$ of the polynomial f. Roughly speaking, the topic of this thesis is the analysis of the facial structure of Gram spectrahedra in various settings.

Spectrahedra are fundamental objects in semidefinite programming, a subfield of convex optimization which has applications in approximation theory, control theory, combinatorial optimization and engineering, see [BPT] and [WSV]. Studying $\mathrm{Gram}(f)$ from a convex algebro-geometric point of view is relevant for optimizing linear functions over all sums-of-squares representations of f. The term Gram spectrahedron was coined by Plaumann, Sturmfels and Vinzant in [PSV, Section 6]. The authors study the rank-minimal extreme points of $\mathrm{Gram}(f)$ and the line segments connecting those points in the case where f is a ternary quartic. Moreover, they present a numerical experiment related to semidefinite programming over this spectrahedron. Gram spectrahedra of binary sextics are discussed in [ORSV, Section 5], where they are related to Kummer surfaces in $\mathbb{P}^3$.

The emphasis on rank-minimal extreme points is no coincidence. Every nonnegative binary form $f \in \mathbb{R}[x, y]_{2d}$ of degree $2d$ can be written as a sum of two squares and there are 2^{d-1} inequivalent ways of doing so if f has distinct roots ([CLR, Example 2.13]). In his forecited work, Hilbert also showed that any nonnegative ternary quartic $f \in \mathbb{R}[x, y, z]_4$ is a sum of at most three squares. The fact that there are (up to orthogonal equivalence) precisely eight representations of f as a sum of three squares if the curve in $\mathbb{P}^2$ defined by f is smooth is a more recent result, see [PRSS]. For higher degree or larger number of variables, there are nonnegative polynomials which cannot be written as sums of squares, according to Hilbert's classification. But even if we know that a given polynomial f has an sos representation, it is in general not at all clear how many summands one needs at least or how many different representations of shortest length there are. Thus, there is often a particular interest in understanding the points of minimum rank in $\mathrm{Gram}(f)$ since these correspond to the sos representations of f of shortest length.

Questions of nonnegativity and sums of squares can be considered in a more general context, for example for forms in the homogeneous coordinate ring of certain real projective varieties. In Theorem 4.0.1 we quote a result of Blekherman, Smith and Velasco [BSV] separating the varieties where every nonnegative quadratic form is a sum of squares from those where this is not true. This gives a broad generalization of Hilbert's classification. For varieties of minimal degree – those for which every nonnegative quadratic form actually is a sum of squares – Blekherman, Plaumann, Sinn and Vinzant [BPSV] determined the length of a general nonnegative quadratic form (see Theorem 4.1.5). A similar result for the length of sums of squares of linear forms on varieties of almost minimal degree was obtained by Chua, Plaumann, Sinn and Vinzant [CPSV]. It comprises earlier ones of Scheiderer which give the shortest length of sos representations of ternary sextics and quaternary quartics ([Sch17, Theorems 4.1 and 4.2]). We quote it as Theorem 6.1.1 and take it as a motivation to study Gram spectrahedra in the context of varieties of almost minimal degree.

In addition to the original results, the article [CPSV] also presents a systematic study of Gram spectrahedra and collects known results and open questions. We will therefore often refer to it. However, its focus is still on extreme points and not much is said on the facial structure of Gram spectrahedra. In fact, already in the most basic case possible, namely binary forms, little was known about higher-dimensional faces when we started our work. Yet, the tools for analyzing the facial structure of spectrahedra were provided by Ramana and Goldman [RG] back in the 1990s. In particular, they gave a characterization of the faces of a spectrahedron by observing that all matrices in the relative interior of a face share the same kernel. Replacing symmetric matrices by symmetric tensors, Scheiderer gives a coordinate-free review of their results in [Sch22, Section 2]. Afterwards, he demonstrates how his new coordinate-free approach to Gram spectrahedra can be used to calculate dimensions of faces. We will mainly adopt this approach as it gives direct access to the polynomials involved in an sos representation. Nevertheless, Gram matrices will be used at some points where it is more convenient to do so.

In Chapter 1 we recall important notions from convex geometry and (semi-) algebraic geometry. We also include some introductory remarks on sums of squares, positive semidefinite matrices and spectrahedra. As suggested by the title, the aim of this thesis is to investigate Gram spectrahedra of quadratic forms on varieties of minimal or almost minimal degree. We focus on toric such varieties which are embedded into projective space using the lattice points $P \cap \mathbb{Z}^n$ of normal lattice polytopes $P \subseteq \mathbb{R}^n$. In these cases, we can interpret the results in terms of polynomials with given Newton polytopes. In addition to the general concepts widely used in algebraic geometry, Chapter 1 thus also gives a brief outline of toric varieties and lattice polytopes with an emphasis on properties important to this work.

Chapter 2 is a detailed introduction to Gram spectrahedra, where we recall the classical Gram matrix method as well as Scheiderer's coordinate-free approach. The general setting is as follows: Given an $\mathbb{R}$-algebra A, a finite-dimensional vector subspace $V \subseteq A$ and some $f \in A$, the *Gram spectrahedron* of f relative to V is defined to be

$$\mathrm{Gram}_V(f) := \{\vartheta \in \mathsf{S}_2 V : \vartheta \succeq 0,\ \mu(\vartheta) = f\}$$

where $\mathsf{S}_2 V$ denotes the space of symmetric tensors in $V \otimes V$ and $\mu \colon V \otimes V \to A$ is the multiplication map defined by $p \otimes q \mapsto pq$ (p, $q \in V$). For a start, the reader can think of the most important applications. These are firstly the case where $P \subseteq \mathbb{R}^n$ is a lattice polytope and $f \in \mathbb{R}[\underline{x}]_{2P}$ is a polynomial with Newton polytope contained in $2P$. We then take $V = \mathbb{R}[\underline{x}]_P$. And secondly, $f \in \mathbb{R}[X]_2$ being a quadratic form in the homogeneous coordinate ring of a real projective subvariety $X \subseteq \mathbb{P}^n$, for which we consider the space $V = \mathbb{R}[X]_1$ of linear forms on X. In these cases, V is omitted from the notation.

We are particularly interested in the facial structure of the Gram spectrahedron $\mathrm{Gram}_V(f)$. According to Scheiderer's work, a face $F \subseteq \mathrm{Gram}_V(f)$ is characterized by the *image* $U := \mathrm{span}(p_1, \dots, p_r) \subseteq V$ of any psd *Gram tensor* $\vartheta = \sum_{i=1}^r p_i \otimes p_i \in \mathrm{Gram}_V(f)$ in the relative interior of F (Proposition 2.3.4) and its dimension only depends on $\mathrm{rk}(F) := \dim(U)$ and $\dim(UU)$ (Proposition 2.3.9). Here, $UU = \mu(\mathsf{S}_2 U)$ is the linear subspace of A spanned by all products pq with p, $q \in U$. Therefore, a central task is to gain a better understanding of $\dim(UU)$ in terms of $\dim(U)$.

In Chapter 3 we analyze Gram spectrahedra of binary forms. Some results of this chapter are also contained in the author's previously published article [May21] and the following explanations partly overlap with the introduction written for that article. We start our analysis by showing which pairs $(\mathrm{rk}(F), \dim(F))$ of ranks and dimensions of faces $F \subseteq \mathrm{Gram}(f)$ can occur. In doing so, we observe substantial dimension gaps (see Example 3.1.11 for an illustration).

As Laurent and Poljak [LP96] write in their study of the elliptope, a polyhedral face is, in some sense, the most "nonsmooth part" of the boundary of a spectrahedron. In addition, polyhedra are the simplest examples of spectrahedra. For these reasons, we are interested in polyhedral faces of Gram spectrahedra. It turns out that Gram spectrahedra of sufficiently general binary forms have polyhedral faces of large dimension, but we can understand them better starting from a Hermitian point of view.

Writing a polynomial $f \in \mathbb{R}[\underline{x}]$ as a *Hermitian sum of squares*, that is $f = p_1\overline{p_1} + \dots + p_r\overline{p_r}$ with $p_1, \dots, p_r \in \mathbb{C}[\underline{x}]$, is another possibility to certify its nonnegativity. On the one hand, a Hermitian square $p\overline{p}$ can be written as the sum of two squares $\mathrm{Re}(p)^2 + \mathrm{Im}(p)^2$ where $\mathrm{Re}(p) = (p + \overline{p})/2$ and $\mathrm{Im}(p) = (p - \overline{p})/2i$ are polynomials with real coefficients. This means that f is a real sum of squares if and only if it is a Hermitian sum of squares. On the other hand, the *Hermitian Gram spectrahedron* $\mathcal{H}^+(f)$, which parametrizes the representations of f as a Hermitian sum of squares, is an interesting object of convex algebraic geometry on its own terms and was also considered in [CPSV, Section 5]. Chapter 2 is thus complemented by the introduction of a Hermitian analog to coordinate-free symmetric Gram spectrahedra. Moreover, we prove a correspondence between the faces of the symmetric Gram spectrahedron and special faces of the Hermitian one which we propose to call *real symmetric faces* due to their nature (Section 2.5).

Let us return to the discussion of binary forms. Using an idea presented in [LP96], we give bounds on the dimension of polyhedral faces in (Hermitian and symmetric)

Gram spectrahedra. Conversely, we show that the Hermitian Gram spectrahedron of a general positive binary form contains a simplex face of the largest possible dimension. We then use our findings from the Hermitian case to show an analogous result in the real symmetric setting. These are some of the main results in Chapter 3 which can be phrased as follows:

Theorem (Theorems 3.3.13 and 3.5.12)**.** *Let k, $d \in \mathbb{N}$. If $d \geq \binom{k+1}{2}$, there is an open dense set of nonnegative binary forms f of degree $2d$ such that the Hermitian Gram spectrahedron $\mathcal{H}^+(f)$ contains a k-dimensional face which is a simplex whose vertices correspond to certain representations of f as a Hermitian square.*

If $d \geq (k+1)^2$, there is an open dense set of nonnegative binary forms f of degree $2d$ such that the symmetric Gram spectrahedron $\mathrm{Gram}(f)$ contains a k-dimensional face which is a simplex whose vertices correspond to certain representations of f as a sum of two real squares.

We prove this by giving an explicit construction of these polyhedral faces using the combinatorics of the complex roots of a binary form (cf. Theorem 3.3.9). We also discuss how many faces of this kind one should expect to obtain by taking the supporting face of any combination of $k+1$ rank-one extreme points in $\mathcal{H}^+(f)$. In the case of binary sextics ($d = 3$) with distinct roots, this leads to a complete understanding of the supporting faces for all $\binom{8}{3} = 56$ possible choices of three such points (see Section 3.4).

Another important result is Theorem 3.7.15, where we show that the Gram spectrahedron of a general nonnegative binary form $f \in \mathbb{R}[x,y]_{2d}$ contains faces of each rank $r \in \{2, \ldots, d+1\}$ of the minimum possible dimension (with respect to the rank). This generalizes a result due to Scheiderer ([Sch22, Theorem 5.3]), which deals with the ranks in the so-called Pataki interval where the corresponding faces of minimum possible dimension are extreme points.

Note that a binary form of degree $2d$ corresponds to a quadratic form on the rational normal curve of degree d, which is a variety of minimal degree. In Chapter 4 we widen our perspective and study Gram spectrahedra of quadratic forms on other varieties of this kind. Varieties of minimal degree are well-understood and completely classified (see e.g. [EH]). Disregarding some quadric hypersurfaces, they can be realized as toric varieties embedded with respect to the lattice points of a polytope. This led to an explicit characterization of the Newton polytopes for which all nonnegative polynomials are sums of squares (cf. [BSV, Section 6] and [CPSV, Theorem 2.1]). We revisit this result in Sections 4.2 and 4.3.

Quadratic forms on the Veronese surface $v_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$ correspond to ternary quartics. The latter have traditionally always attracted a lot of attention and, as already said, Gram spectrahedra were first studied in this case. Thanks to recent work by Vill [Vill], Gram spectrahedra of ternary quartics are essentially fully understood in terms of their facial structure.

The contributions of this thesis therefore relate predominantly to the remaining class of varieties of minimal degree, namely the rational normal scrolls. Theorem 4.6.2 is a generalization of the bounds for dimensions of faces in Gram spectrahedra from the case of rational normal curves (binary forms) to higher-dimensional scrolls. In Section 4.7 we present new inequalities which depend not only on the dimension but also on the structure of the rational normal scroll. This shows that one cannot expect to transfer the aforementioned Theorem 3.7.15 to general varieties of minimal degree. This relates to the following: Given $d \in \mathbb{N}$ and a positive binary form $f \in \mathbb{R}[x,y]_{2d}$, we can consider quadratic forms on toric surfaces of minimal degree whose Gram spectrahedra have the same dimension as $\mathrm{Gram}(f)$. For small d, a comparison between the structure of these spectrahedra and that of $\mathrm{Gram}(f)$ is drawn in Section 4.5.

Let us note that Vill provides remarkable bounds for dimensions of faces in cases which go beyond that of ternary quartics. This is done by bounding the codimension of UU inside $\mathbb{R}[\underline{x}]_{2d}$ for subspaces $U \subseteq \mathbb{R}[\underline{x}]_d$ of fixed codimension. In terms of Newton polytopes this amounts to P being the scaled $(n-1)$-dimensional standard simplex $d\Delta_{n-1} = \{\alpha \in \mathbb{R}^n_{\geq 0} : \sum_i \alpha_i = d\}$. As he always works in this setting and obtains the best bounds for subspaces U of small codimension, this is not exactly what we are after when we think of all the possible shapes that polytopes defining varieties of (almost) minimal degree can take.

Chapter 5 is shorter than the others and deals with miscellaneous topics which at first glance might seem unrelated. We explain the composition of that chapter in its introduction. At this point we simply mention that Section 5.1 originates from a joint work with Julian Vill, where we used a result due to de Carli Silva and Tunçel [CST] to determine the dimension of normal cones of Gram spectrahedra (Theorem 5.1.18). This can be used to identify the vertices of the Gram spectrahedron (cf. Theorem 5.1.21), i.e., points with full-dimensional normal cone which are of certain significance in optimization.

Spectrahedra are the feasible regions in semidefinite programming problems. In Theorem 5.3.1 we show that every spectrahedron that does not contain an affine line is (linearly isomorphic to) the Gram spectrahedron of a quadratic form in some quotient ring of the type $\mathbb{R}[x_1,\dots,x_n]/I$ where $I \subseteq \mathbb{R}[\underline{x}]$ is a homogeneous ideal.

Chapter 6 then deals with Gram spectrahedra in the context of varieties of almost minimal degree. The forecited result of Chua et al. concerning the length of sos representations of quadratic forms on such varieties comprises toric varieties where one has lovely interpretations in terms of polynomials with certain Newton polytopes, just as the ternary sextics and quaternary quartics which we mentioned above. In [CPSV, Remark 3.7] the authors indicate how to find the lattice polytopes defining other toric varieties satisfying the hypotheses of their theorem. We elaborate on this in Sections 6.2 and 6.3. The main result concerning the facial structure of Gram spectrahedra in this setting is Theorem 6.4.8, where we present dimension bounds which take the same shape as for varieties of minimal degree.

As a first step, the proof uses the combinatorial structure of Gorenstein polytopes of degree 2, which have been classified by Batyrev and Juny [BJ]. On our way we will also employ results from the fascinating theory of (reflexive) lattice polytopes (see e.g. [BN08], [Nill]). When the combinatorial approach is no longer sufficient, we take a closer look from an algebraic point of view. As interesting special cases we obtain results for the facial structure of Gram spectrahedra of polynomials $f \in \mathbb{R}[x,y]$ with $\deg_x(f) \leq 4$, $\deg_y(f) \leq 4$ (Example 6.5.11) and for polynomials $f \in \mathbb{R}[x,y,z]$ which are sums of squares of multiaffine polynomials (Example 6.6.8).

We collect some still open problems at the end of this thesis.

CHAPTER 1

Preliminaries

This chapter is meant to introduce the reader to some terms and definitions that are used in this thesis. Of course, not all of them are equally important to our work and some of them will only appear at one point or another, whereas others are ubiquitous.

Many notions are quite common, for example those presented in Section 1.1 on algebraic geometry or in Section 1.2 dealing with basics in convex geometry. In Section 1.3 we consider lattice polytopes which are used in Section 1.4 for the construction of projective toric varieties. In preparation for Chapters 4 and 6, where we work with (toric) varieties of (almost) minimal degree, we also introduce special lattice polytopes like the reflexive ones or those that can be written as Cayley sums. In Section 1.5 we discuss sums of squares of polynomials with a focus on their Newton polytopes. At some points we include lemmata and propositions for later reference. This applies in particular to Section 1.6 where we debate generalities about the definition of spectrahedra and present some preparatory results on positive semidefinite matrices.

By $\mathbb{N} = \{1, 2, 3, \ldots\}$ we denote the set of positive integers, whereas $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of nonnegative integers. For m, $n \in \mathbb{N}$ and a field K, we write $\mathrm{M}_{m\times n}(K)$ for the set of $m \times n$-matrices over K. If $m = n$, we also write $\mathrm{M}_n(K) := \mathrm{M}_{n\times n}(K)$. The transpose of $A \in \mathrm{M}_{m\times n}(K)$ is denoted by A^T. For any rectangular matrix A over the field $\mathbb{C}$ of complex numbers, we define $A^* := \overline{A}^T$, where the overbar denotes the complex conjugation of all entries.

1.1. Algebraic and semialgebraic geometry

Let K be a field and let V be a finite-dimensional vector space. For $0 \neq v \in V$ we write $[v] := Kv = \{\lambda v : \lambda \in K\}$. The *projective space* $\mathbb{P}_V$ associated to V is the set of all one-dimensional subspaces of V, that is

$$\mathbb{P}_V = \{[v] : v \in V \setminus \{0\}\}.$$

The *(projective) dimension* of $\mathbb{P}_V$ is defined to be $\dim(\mathbb{P}_V) = \dim(V) - 1$. For $V = K^{n+1}$ we get the n-dimensional projective space over K which we denote by $\mathbb{P}_{K^{n+1}} = \mathbb{P}^n(K)$. We endow $\mathbb{P}^n(K)$ with homogeneous coordinates, that is to say, for $0 \neq \xi = (\xi_0, \ldots, \xi_n) \in K^{n+1}$ we write $(\xi_0 : \cdots : \xi_n) := [\xi] \in \mathbb{P}^n(K)$. Then $(\xi_0 : \cdots : \xi_n) = (\eta_0 : \cdots : \eta_n)$ if and only if there exists $c \in K^*$ such that $\eta_i = c\xi_i$ for all $i \in \{0, \ldots, n\}$. We simply write $\mathbb{P}^n$ for the n-dimensional projective space over the field $\mathbb{C}$ of complex numbers.

1.1.1. Let $K \in \{\mathbb{R}, \mathbb{C}\}$. An *affine K-variety* is the zero set $\mathcal{V}(f_1, \ldots, f_r)$ in $\mathbb{A}^n = \mathbb{C}^n$ of finitely many polynomials $f_1, \ldots, f_r \in K[x_1, \ldots, x_n]$, i.e.,

$$\mathcal{V}(f_1, \ldots, f_r) = \{v \in \mathbb{A}^n : f_1(v) = \cdots = f_r(v) = 0\}.$$

The *vanishing ideal* of a subset $X \subseteq \mathbb{A}^n$ is

$$\mathfrak{I}(X) = \{f \in K[x_1, \ldots, x_n] : f(v) = 0 \text{ for all } v \in X\}.$$

The *coordinate ring* of an affine K-variety $X \subseteq \mathbb{A}^n$ is $K[X] = K[x_1, \ldots, x_n]/\mathfrak{I}(X)$.

1.1.2. Let $f_1, \ldots, f_r \in K[x_0, \ldots, x_n]$ be homogeneous polynomials (forms). In order to distinguish between the affine and the projective setting, we write

$$\mathcal{V}_+(f_1, \ldots, f_r) := \{\xi \in \mathbb{P}^n : f_1(\xi) = \cdots = f_r(\xi) = 0\} \subseteq \mathbb{P}^n$$

for the *projective zero set* of $f_1, \ldots, f_r$ and $\mathfrak{I}_+(X)$ for the homogeneous ideal generated by all forms in $K[x_0, \ldots, x_n]$ vanishing identically on a subset $\varnothing \neq X \subseteq \mathbb{P}^n$. For $X = \varnothing \subseteq \mathbb{P}^n$ one defines $\mathfrak{I}_+(X) = \langle x_0, \ldots, x_n \rangle \subseteq K[x_0, \ldots, x_n]$.

A subset $X \subseteq \mathbb{P}^n$ is a *projective K-variety* if X is the zero set of finitely many homogeneous polynomials in $K[x_0, \ldots, x_n]$. For a projective K-variety $X \subseteq \mathbb{P}^n$, we call $\mathfrak{I}_+(X)$ the *homogeneous vanishing ideal* of X. The *homogeneous coordinate ring* of X is $K[X] = K[x_0, \ldots, x_n]/\mathfrak{I}_+(X)$, a finitely generated, reduced and $\mathbb{Z}$-graded K-algebra. Note that this is the same notation as in the affine case. However, this is unambiguous as X is always given as a subset of $\mathbb{A}^n$ or $\mathbb{P}^n$.

We always endow $\mathbb{P}^n$ with the *K-Zariski topology* whose closed sets are precisely the projective K-varieties. Those carry the induced subspace topology. For every homogeneous $f \in K[x_0, \ldots, x_n]$, we write $\mathrm{D}_+(f) := \mathbb{P}^n \setminus \mathcal{V}_+(f)$ for the Zariski-open set where f does not vanish.

1.1.3. Let X be a topological space. We say that X is *irreducible* if $X \neq \varnothing$ and the following holds: whenever $X = X_1 \cup X_2$ with closed subsets X_1, $X_2 \subseteq X$, then already $X_1 = X$ or $X_2 = X$. The *(Krull) dimension* of X, denoted as $\dim(X)$, is the maximum length m of chains

$$\varnothing \neq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_m$$

of closed irreducible subsets of X.

Consequently, a nonempty affine K-variety $V \subseteq \mathbb{A}^n$ is irreducible (with respect to the K-Zariski topology) if and only if $\mathfrak{I}(V)$ is a prime ideal of $K[x_1, \ldots, x_n]$. We have $\dim(V) = \dim K[V]$, where $\dim K[V]$ is the Krull dimension of $K[V]$. For a nonempty projective K-variety $X \subseteq \mathbb{P}^n$, the irreducibility of X is equivalent to $\mathfrak{I}_+(X)$ being a homogeneous prime ideal of $K[x_0, \ldots, x_n]$. The dimension of X is given by $\dim K[X] - 1$.

1.1.4. Let $X \subseteq \mathbb{P}^n$ be an irreducible projective K-variety of dimension m and let $K[X] = K[x_0, \ldots, x_n]/\mathfrak{I}_+(X)$ be its homogeneous coordinate ring. The map $d \mapsto \dim K[X]_d$ ($d \in \mathbb{N}_0$) is the *Hilbert function* of the graded ring $K[X]$. There exists a unique polynomial $p_X \in \mathbb{Q}[x]$ of degree $m = \dim(X)$ with $p_X(d) = \dim K[X]_d$ for $d \gg 0$, the *Hilbert polynomial* of $X \subseteq \mathbb{P}^n$. Note that the Hilbert polynomial of a projective variety depends on the projective embedding of the latter. Let c be the leading coefficient of p_X. The *degree* of X is defined as $\deg(X) := c \cdot m! \in \mathbb{N}$. If X is *nondegenerate*, that is not contained in any hyperplane, we have

$$\deg(X) \geq n - m + 1 = \operatorname{codim}(X) + 1$$

according to [Harr, Corollary 18.12].

We say that X is a *variety of minimal degree* if X is nondegenerate and $\deg(X) = \operatorname{codim}(X) + 1$. An irreducible nondegenerate projective variety $X \subseteq \mathbb{P}^n$ is a *variety of almost minimal degree* if $\deg(X) = \operatorname{codim}(X) + 2$.

1.1.5 Definition. Let $(R, \mathfrak{m})$ be a Noetherian local ring. We say that $f_1, \ldots, f_s \in \mathfrak{m}$ form a *regular sequence* if for any $i \in \{1, \ldots, s\}$ the element $\overline{f_i} \in R/\langle f_1, \ldots, f_{i-1}\rangle$ is not a zero divisor. The *depth* of R is defined to be the maximum length of any regular sequence in R. The local ring R is *Cohen-Macaulay* if its depth equals its (Krull) dimension. An arbitrary Noetherian ring R is said to be *Cohen-Macaulay* if every localization of R at a maximal ideal $\mathfrak{m} \subseteq R$ is a Cohen-Macaulay ring in the sense above.

A variety X is *Cohen-Macaulay* if its local rings $(\mathcal{O}_{X,\xi}, \mathfrak{m}_{X,\xi})$ are Cohen-Macaulay for all $\xi \in X$.

1.1.6. Now let $\varnothing \neq X \subseteq \mathbb{P}^n$ be a projective K-variety ($K \in \{\mathbb{R}, \mathbb{C}\}$). The *affine cone* over X is

$$\hat{X} := \{0\} \cup \{v \in \mathbb{A}^{n+1} : v \neq 0,\ [v] \in X\} \subseteq \mathbb{A}^{n+1}.$$

If $I := \mathfrak{I}_+(X) \subseteq K[x_0, \ldots, x_n]$ is the homogeneous vanishing ideal of X, then I (considered as an ordinary ideal in $K[x_0, \ldots, x_n]$) is also the vanishing ideal of $\hat{X}$. Therefore, the affine cone over X is an affine K-variety in $\mathbb{A}^{n+1}$ and we have $\dim \hat{X} = \dim X + 1$ (see also [Hart, Chapter I, Exercise 2.10]).

We say that X is *arithmetically Cohen-Macaulay* (*aCM*) if the homogeneous coordinate ring $R = K[x_0, \ldots, x_n]/I$ of X is a Cohen-Macaulay ring. Since the affine variety $\hat{X}$ has the same coordinate ring as X, this is equivalent to saying that $\hat{X}$ is a Cohen-Macaulay variety.

1.1.7. Let $V \subseteq \mathbb{A}^n$ be an affine $\mathbb{R}$-variety. We write $V(\mathbb{R}) = V \cap \mathbb{R}^n$ for the set of $\mathbb{R}$-rational points of V. If $X \subseteq \mathbb{P}^n$ is a projective $\mathbb{R}$-variety, a point $\xi \in X$ is called *real* if it has a real representative, i.e., if there is some $v \in \mathbb{A}^{n+1}(\mathbb{R}) = \mathbb{R}^{n+1}$ with $\xi = [v]$. The set of real points of X is denoted by $X(\mathbb{R})$.

Semialgebraic geometry. In contrast to the field of complex numbers, the reals admit a natural ordering that makes $\mathbb{R}$ an ordered field. It thus makes sense to study not only solutions to systems of polynomial equations but also to allow for inequalities if we work over $\mathbb{R}$. This is where real algebra and especially semialgebraic geometry comes in. For a comprehensive treatment of the rich theory we refer to the textbooks [BCR] or [KS]. One can engage in semialgebraic geometry over any real closed field, and some of the most important results in this context are only obtained in this way. However, for our purposes it is sufficient to stick with the familiar field $\mathbb{R}$. Unless stated otherwise, we always consider $\mathbb{R}^n$ with its Euclidean topology.

1.1.8 Definition. A subset $S \subseteq \mathbb{R}^n$ is *semialgebraic* if S is a finite boolean combination (obtained by taking finite intersections, finite unions and complements) of sets of the form $\{v \in \mathbb{R}^n : f(v) > 0\}$ with $f \in \mathbb{R}[x_1, \ldots, x_n]$.

1.1.9. Let $\pi\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, $\pi(v, w) = w$ be the projection. The Tarski-Seidenberg principle ensures that for any semialgebraic set $S \subseteq \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$, its projection $\pi(S)$ is a semialgebraic subset of $\mathbb{R}^n$ (see [BCR, Theorem 2.2.1]).

Given any semialgebraic set $S \subseteq \mathbb{R}^n$, we have the following pleasant consequences of the projection theorem: The closure $\overline{S}$, the interior $\operatorname{int}(S)$, the boundary ∂S and the convex hull $\operatorname{conv}(S)$ of S are again semialgebraic. A map $\phi\colon A \to B$ between semialgebraic sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ is said to be *semialgebraic* if its graph is a semialgebraic subset of $\mathbb{R}^{n+m}$. Images and preimages of semialgebraic subsets under a semialgebraic mapping are semialgebraic. The aforementioned results can be found in [BCR, Section 2.2].

1.1.10. Let $S \subseteq \mathbb{R}^n$ be a semialgebraic set and let $V \subseteq \mathbb{A}^n$ be the Zariski closure of S (with respect to the $\mathbb{R}$-Zariski topology). The *(semialgebraic) dimension* of S is defined to be the dimension of V in the $\mathbb{R}$-Zariski topology. In particular, we have $\dim(S) = \dim K[V]$.

We quote the following useful fact from [BCR, Propositions 2.8.2 and 2.8.13].

1.1.11 Proposition. *Let $S \subseteq \mathbb{R}^n$ be a semialgebraic set. We write $\overline{S}$ for the (Euclidean) closure of S. Then* $\dim(\overline{S} \setminus S) < \dim(S) = \dim(\overline{S})$.

1.2. Convex sets

In this section we introduce some standard terms and definitions from convex geometry. The main ones that constantly appear in this thesis are the notions of faces and supporting faces as well as their dimensions. We refer to the textbooks by Webster [Web] and Barvinok [Bar].

Let V be a finite-dimensional real vector space. As we want to make use of some basic topological notions, we first note the following: The space $\mathbb{R}^n$, endowed with the usual Euclidean topology, is a topological vector space, i.e., the vector addition $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and the scalar multiplication $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous. Moreover, any isomorphism $\mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism. On an n-dimensional $\mathbb{R}$-vector space V, there is exactly one topology for which an (and therefore every) isomorphism $V \cong \mathbb{R}^n$ is a homeomorphism, and we always endow V with this topology.

1.2.1. For any subsets A, $B \subseteq V$ and any $\lambda \in \mathbb{R}$, we define the *Minkowski sum* $A + B = \{v + w : v \in A,\, w \in B\} \subseteq V$ and the *dilation* $\lambda A = \{\lambda v : v \in A\} \subseteq V$.

1.2.2. The *line segment* between v, $w \in V$ is $[v, w] := \{(1-\lambda)v + \lambda w : \lambda \in [0, 1]\}$. A subset $C \subseteq V$ is called *convex* if for all v, $w \in C$ the line segment $[v, w]$ is contained in C. Obviously, the intersection of a family of convex sets in V is again convex. Thus, for any $S \subseteq V$ we can define the *convex hull* $\operatorname{conv}(S)$ of S to be the smallest convex set $C \subseteq V$ that contains S. The convex hull of S is equal to the set of all *convex combinations* of points from S (see [Bar, Theorem I.2.1]), that is to say

$$\operatorname{conv}(S) = \left\{ \sum_{i=1}^{m} \lambda_i v_i \in V : m \in \mathbb{N},\, \forall\, i = 1, \ldots, m : v_i \in S,\, \lambda_i \geq 0 \text{ and } \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

A nonempty convex set that is closed under multiplication by nonnegative scalars is called a *convex cone*. Equivalently, a set $\varnothing \neq C \subseteq V$ is a convex cone if $C + C \subseteq C$ and $\mathbb{R}_{\geq 0} C \subseteq C$. A convex cone $C \subseteq V$ is *pointed* if $C \cap (-C) = \{0\}$.

The *dimension* of a convex set $C \subseteq V$ is the dimension of its affine hull $\operatorname{aff}(C)$. By $\operatorname{relint}(C)$ we denote the *relative interior* of C, i.e., the interior of C relative to $\operatorname{aff}(C)$. As we work in a finite-dimensional vector space, the relative interior of any nonempty convex set is nonempty (cf. [Web, Theorem 2.3.1]). We say that C is *full-dimensional* if $\operatorname{aff}(C) = V$. It is easy to see that this is equivalent to $\operatorname{int}(C) \neq \varnothing$.

Linear maps preserve (relative) interiors of convex sets, a fact that will be important later on and whose proof we thus include here.

1.2.3 Lemma (see [CPSV, Lemma 1.5]). *Let m, $n \in \mathbb{N}$ and let $\phi\colon \mathbb{R}^m \to \mathbb{R}^n$ be a surjective linear map. For every convex set $C \subseteq \mathbb{R}^m$ with nonempty interior we have $\phi(\operatorname{int}(C)) = \operatorname{int}(\phi(C))$.*

Proof. As ϕ is a surjective linear map between finite-dimensional vector spaces, it is an open map. Therefore, $\phi(\operatorname{int}(C)) \subseteq \operatorname{int}(\phi(C))$.

Conversely, let $x \in \mathbb{R}^n \setminus \phi(\operatorname{int}(C))$. Since ϕ is surjective, the affine subspace $\phi^{-1}(x) \subseteq \mathbb{R}^m$ is nonempty. Consequently, $\operatorname{int}(C)$ and $\phi^{-1}(x)$ are disjoint nonempty convex sets in $\mathbb{R}^m$. By a separation theorem for convex sets ([Web, Theorem 2.4.10]), they can be properly separated. That is to say, there exists a linear functional $l \in (\mathbb{R}^m)^\vee$ and an $a \in \mathbb{R}$ such that $l|_{\phi^{-1}(x)} = a$ and $l|_{\operatorname{int}(C)} > a$. As C is convex and $\operatorname{int}(C) \neq \varnothing$, we have $C \subseteq \overline{\operatorname{int}(C)}$ and therefore $l|_C \geq a$.

The linear functional l is constant on the fibers of ϕ. Indeed, let $w \in \ker(\phi)$. Then $w = v-(v-w)$ for any $v \in \phi^{-1}(x)$. As $\phi(v-w) = \phi(v) = x$, we have $l(w) = 0$. Hence, there is a well-defined linear functional $0 \neq l' \in (\mathbb{R}^n)^\vee$ given by $l' \circ \phi = l$. We obtain $l'(x) = a$ and $l'(\phi(C)) = l(C) \subseteq [a, \infty)$. As $l' \neq 0$, this means that x cannot be an interior point of $\phi(C)$. □

We are mainly interested in the structure of the (relative) boundary of convex sets. For us, the most important concept from convex geometry is the notion of faces.

1.2.4 Definition ([Web, Section 2.6]). A convex subset F of a convex set C is called a *face* of C if whenever $(1-\lambda)v + \lambda w \in F$ for some v, $w \in C$ and $0 < \lambda < 1$, then already v, $w \in F$.

1.2.5. We state some important facts concerning the faces of a convex set C. They can also be found in Section 2.6 of Webster's book [Web], especially Theorems 2.6.5 and 2.6.10 should be mentioned.

According to the definition, $\varnothing$ and C itself are faces of C. Faces of C other than $\varnothing$ and C are called *proper faces* of C.

The intersection of any nonempty family of faces of C is again a face of C. Hence, for every $v \in C$ there is a unique face F of C with $v \in \operatorname{relint}(F)$, called the *supporting face* of v and denoted by $\operatorname{suppface}(v)$. It can be obtained as the intersection of all faces of C containing v. Moreover, the relative interiors of all faces of C form a partition of C. In particular, the relative boundary of C is the union of the proper faces of C.

Zero-dimensional faces of C are points and are called *extreme points* of C. By $\operatorname{Ex}(C)$ we denote the set of all extreme points of C. A one-dimensional face of a compact convex set is an *edge*. If our convex set is a *polytope* P, that is the convex hull of finitely many points, an extreme point of P is also called a *vertex* of P and a face of dimension $\dim(P) - 1$ is a *facet*. We write $F \prec P$ if F is a facet of the polytope P.

1.2.6 Lemma. *Let $C \subseteq V$ be convex. Let v, $w \in C$ and write $F = \operatorname{suppface}(w) \subseteq C$. Then $v \in F$ if and only if there exists $\varepsilon \in (0,1)$ such that $\frac{1}{1-\varepsilon}(w - \varepsilon v) \in C$.*

Proof. For $v = w$ there is nothing to show. So let $v \neq w$ and assume that $v \in F$. As $w \in \operatorname{relint}(F)$, we can prolong the line segment $[v, w] \subseteq F$ beyond w without leaving F. In other words, there are $u \in F$ and $\varepsilon \in (0,1)$ such that $w = (1-\varepsilon)u + \varepsilon v$. Then $\frac{1}{1-\varepsilon}(w - \varepsilon v) = u \in F \subseteq C$.

Conversely, we assume that there exists $\varepsilon \in (0,1)$ such that $u := \frac{1}{1-\varepsilon}(w - \varepsilon v) \in C$. Then $(1-\varepsilon)u + \varepsilon v = w \in F$. As F is a face of C, we obtain $u, v \in F$. □

1.2.7. Let $V^\vee = \operatorname{Hom}(V, \mathbb{R})$ be the dual space of V. For a linear functional $l \in V^\vee$ and $u \in V$ we write the evaluation as a bilinear pairing $\langle l, u \rangle := l(u)$. For $l \in V^\vee$ and $c \in \mathbb{R}$ we define

$$H_{l,c} = \{u \in V : \langle l, u \rangle = c\} \subseteq V \text{ and } H^+_{l,c} = \{u \in V : \langle l, u \rangle \geq c\} \subseteq V.$$

If $l \neq 0$, then $H_{l,c}$ is an affine hyperplane and $H^+_{l,c}$ is a closed half-space in V. A *polyhedron* is a finite intersection of closed half-spaces, i.e., a set P of the form

$$P = \bigcap_{i=1}^{r} H^+_{l_i,c_i}$$

with $r \in \mathbb{N}_0$, $0 \neq l_i \in V^\vee$ and $c_i \in \mathbb{R}$, $i = 1, \ldots, r$. A classical theorem tells us that polytopes are precisely the compact polyhedra (see [Zie, Theorem 1.1]).

Let $C \subseteq V$ be convex. A hyperplane $H_{l,c}$ as above *isolates* C if $C \subseteq H^+_{l,c}$. In this case, the hyperplane is called a *supporting hyperplane* of C. A face F of C is *exposed* if $F \in \{\varnothing, C\}$ or if there exists a supporting hyperplane H of C with $H \cap C = F$. It is well-known that a polyhedron has only finitely many faces and that they are all exposed and polyhedral sets themselves. Moreover, if a polyhedron P has a nonempty face of dimension k, then P has faces of all dimensions from k to $\dim(P)$. These statements are contained in [Web, Theorem 3.2.2], for example.

1.2.8. For $u \in V$ we let $u^\circ := \{l \in V^\vee : \langle l, u \rangle \geq -1\}$. For every set $A \subseteq V$ we define the *polar set* A° by

$$A^\circ := \bigcap_{u \in A} u^\circ = \{l \in V^\vee : \langle l, u \rangle \geq -1 \text{ for all } u \in A\}.$$

This is a closed convex set in $V^\vee$ that contains 0. We canonically identify V with the dual space of $V^\vee$ and thereby consider the *bipolar* $A^{\circ\circ} = (A^\circ)^\circ$ as a subset of V, that is to say,

$$A^{\circ\circ} = \{u \in V : \langle l, u \rangle \geq -1 \text{ for all } l \in A^\circ\}.$$

Obviously, we have $A \subseteq A^{\circ\circ}$. As a consequence of the Bipolar Theorem ([Bar, Theorem IV.1.2]) we see that $A = A^{\circ\circ}$ if and only if A is a closed convex set containing the origin. We will be particularly interested in polar sets of lattice polytopes.

1.3. Lattice polytopes

Our main reference for lattice polytopes and toric varieties is the textbook by Cox, Little and Schenck [CLS].

1.3.1. A *lattice* is a free abelian group of finite rank. To a lattice M we associate the real vector space $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$. A *lattice polytope* in $M_\mathbb{R}$ is the convex hull of a finite subset of the lattice M. Obviously, a polytope $P \subseteq M_\mathbb{R}$ is a lattice polytope if and only if all vertices of P lie in M, i.e., they are lattice points. So, we see immediately that both faces and Minkowski sums of lattice polytopes are lattice polytopes themselves. When we say that $P \subseteq \mathbb{R}^n$ is a lattice polytope, we mean that P is a lattice polytope in the sense of the above definition where the underlying lattice is $M = \mathbb{Z}^n$.

Normal and very ample polytopes. Fix a lattice M. In Section 1.4 we will define the projective toric variety X_P associated to a lattice polytope P. This variety and its interplay with P will become important when we study Gram spectrahedra of quadratic forms on X_P in Chapter 4 and Chapter 6. However, in order to get the properties of P reflected in X_P and vice versa, P must not have too few lattice points. One possibility to guarantee that P contains sufficiently many lattice points is the notion of normality.

1.3.2 Definition ([CLS, Def. 2.2.9])**.** A lattice polytope $P \subseteq M_{\mathbb{R}}$ is *normal* if

$$(kP) \cap M + (lP) \cap M = ((k+l)P) \cap M$$

for all k, $l \in \mathbb{N}$.

1.3.3 Remark. The inclusion "$\subseteq$" in Definition 1.3.2 is always satisfied. Indeed, for v, $w \in P$ and k, $l \in \mathbb{N}$ we have $kv + lw = (k+l)(\frac{k}{k+l}\,v + \frac{l}{k+l}\,w) \in (k+l)P$. The crucial point of the definition is that in a normal polytope P every lattice point of the dilation $(k+l)P$ can be written as a sum of a lattice point of kP and one of lP. Equivalently, we could require that

$$\underbrace{P \cap M + \cdots + P \cap M}_{k \text{ times}} = (kP) \cap M$$

holds for all $k \in \mathbb{N}$. So normality means that P has enough lattice points to generate the lattice points in all integer multiples of P.

Obviously, one-dimensional lattice polytopes are normal. The principal result on normality is the following theorem.

1.3.4 Theorem ([CLS, Thm. 2.2.12 and Cor. 2.2.13])**.** *Let $n = \dim M_{\mathbb{R}}$ and let $P \subseteq M_{\mathbb{R}}$ be a full-dimensional lattice polytope. Then kP is normal for all $k \geq n-1$. In particular, every lattice polygon $P \subseteq \mathbb{R}^2$ is normal.*

1.3.5 Remark. The normality of a lattice polytope $P \subseteq M_{\mathbb{R}}$ can be checked efficiently using the software package `Normaliz` [1]: Consider the so-called cone of P, defined by

$$C(P) := \operatorname{cone}(P \times \{1\}) \subseteq M_{\mathbb{R}} \times \mathbb{R},$$

whose slice at height k is exactly the dilation kP for all $k \in \mathbb{N}$. By Gordan's Lemma (cf. [BG09, Lemma 2.9]), $S := C(P) \cap (M \times \mathbb{Z})$ is an affine semigroup. As $C(P)$ is pointed, we have $S \cap (-S) = \{0\}$. Therefore, S has a unique minimal system of generators, its *Hilbert basis* (see Proposition 2.14 and Definition 2.15 in [BG09]). By [CLS, Lemma 2.2.14], P is normal if and only if $(P \cap M) \times \{1\}$ is the Hilbert basis of $S = C(P) \cap (M \times \mathbb{Z})$. The computation of the Hilbert basis is implemented in `Normaliz`.

In another way, the presence of a sufficient number of lattice points in a polytope can be captured as follows:

1.3.6 Definition ([CLS, Def. 2.2.17])**.** A lattice polytope $P \subseteq M_{\mathbb{R}}$ is *very ample* if for every vertex $m_0 \in P$, the semigroup

$$S_{P,m_0} = \left\{ \sum_{m \in P \cap M} a_m(m - m_0) : a_m \in \mathbb{N}_0 \right\}$$

generated by the set $(P \cap M) - m_0$ is saturated in M. (Recall that a semigroup $S \subseteq M$ is *saturated* if whenever $km \in S$ for some $k \in \mathbb{N}$ and $m \in M$, then already $m \in S$.)

We note that every normal lattice polytope is very ample (see [CLS, Proposition 2.2.18]).

Facet presentation and normal fan of a polytope. The structure of the toric variety associated to a full-dimensional lattice polytope $P \subseteq M_{\mathbb{R}}$ is determined by the facets of P and their normal vectors. We thus recall the construction of the facet presentation and the normal fan of P from [CLS, Sections 2.2 and 2.3].

1.3.7. Let M and N be dual lattices with associated vector spaces $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, respectively. As before we write $\langle m, u \rangle := m(u)$ for $m \in M_{\mathbb{R}}$ and $u \in N_{\mathbb{R}}$.

Let $P \subseteq M_{\mathbb{R}}$ be a full-dimensional polytope (i.e., $\operatorname{int}(P) \neq \varnothing$) and let $F \prec P$ be a facet. Then F is cut out by a *unique* affine hyperplane. Thus, up to multiplication by a positive real number, there is a unique pair $(u_F, a_F) \in N_{\mathbb{R}} \times \mathbb{R}$ such that $H_F = H_{u_F, -a_F}$ is a supporting hyperplane of P with $F = P \cap H_F$ and $P \subseteq H_F^+$. We call u_F an *inward-pointing facet normal* of F. It follows that

$$P = \bigcap_{\substack{F \prec P \\ \text{facet}}} H_F^+ = \{m \in M_{\mathbb{R}} : \langle m, u_F \rangle \geq -a_F \text{ for all facets } F \prec P\}.$$

The inward-pointing facet normals of a facet F of P lie on a unique ray in $N_{\mathbb{R}}$. If P is a lattice polytope, then this cone is rational and thus has a unique ray generator $u_F \in N$. The corresponding (real) number a_F is an integer since $\langle m, u_F \rangle = -a_F$ for every vertex $m \in M$ of F. By this choice of u_F we obtain the *unique* description

$$P = \{m \in M_{\mathbb{R}} : \langle m, u_F \rangle \geq -a_F \text{ for all facets } F \prec P\},$$

which we call the *facet presentation* of the lattice polytope P.

1.3.8. The facet normals u_F of a full-dimensional lattice polytope $P \subseteq M_{\mathbb{R}}$ can be used to construct its normal fan: For every face Q of P, the *normal cone* of Q is the rational polyhedral cone σ_Q generated by the facet normals u_F where F runs over all facets of P that contain Q, that is to say,

$$\sigma_Q = \operatorname{cone}(u_F : F \text{ facet of } P \text{ with } Q \subseteq F).$$

In particular, if F is a facet of P, then σ_F is the ray generated by u_F, and $\sigma_P = \{0\}$ is the cone generated by the empty set.

The set $\Sigma_P = \{\sigma_Q : Q \text{ face of } P\}$ is the *normal fan* of P. Theorem 2.3.2 in [CLS] shows that this collection of pointed rational polyhedral cones is indeed a fan in $N_{\mathbb{R}}$.

Reflexive polytopes. We will be interested in a particular class of lattice polytopes, namely the so-called reflexive polytopes. These are polytopes P such that both P and its polar polytope P° are lattice polytopes. The concept of reflexive polytopes can be traced back to a work of Batyrev published in 1994 ([Bat]). Therein, he shows how reflexivity relates to a duality for Calabi-Yau varieties which is called mirror symmetry and is important in theoretical physics.

1.3.9 Definition ([BJ, Def. 1.9])**.** A lattice polytope $P \subseteq M_{\mathbb{R}}$ with $0 \in \operatorname{int}(P)$ is *reflexive* if P° is again a lattice polytope.

1.3.10 Remark. Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope with $0 \in P$.

(i) Since P is convex and closed, we have $P = P^{\circ\circ}$ by bipolarity (see 1.2.8). Moreover, a polytope is bounded. Therefore, P° can only be a polytope if 0 is an *interior* point of $P^{\circ\circ} = P$.

(ii) If P is a reflexive polytope, then 0 is an interior point of the lattice polytope P° and $(P^{\circ})^{\circ} = P$ is a lattice polytope. In other words, P° is reflexive as well. If, conversely, we assume that P° is reflexive, then the same argument implies that P must be reflexive.

1.3.11 Remark. There are many characterizations of reflexive polytopes. The interested reader could for instance consult Proposition 3.1.4 in [Nill] for the twelve "most important equivalences of reflexivity". One that we will use in Chapter 6 is that there are no lattice points lying between the affine hyperplane spanned by a facet and its parallel through the origin. Note that this also implies that the origin is the only interior lattice point of a reflexive polytope. Aside from that, we want to highlight another characterization that is phrased as "all facets have integral lattice distance one from the origin". This amounts to saying that a full-dimensional lattice polytope $P \subseteq M_{\mathbb{R}}$ with $0 \in \operatorname{int}(P)$ is reflexive if and only if in the facet presentation of P we have $a_F = 1$ for all facets F of P. The equivalence to reflexivity is clear since one can easily see that the vertices of P° are precisely the points $\frac{1}{a_F} u_F$ where F runs over the facets of P.

Cayley sums. A well-established process for constructing a new polytope from given lattice polytopes $\Delta_0, \ldots, \Delta_r \subseteq M_{\mathbb{R}}$ is taking their Minkowski sum. In the following we describe a construction that is less known but of utmost importance for us. In contrast to the Minkowski sum that lives in the same space $M_{\mathbb{R}}$, when building Cayley sums we enlarge the ambient space. Intuitively, one can think of iteratively placing each summand Δ_i ($i = 1, \ldots, r$) at height 1 in a new dimension. A nice property of this construction is that the given polytopes can be retrieved as certain faces of their Cayley sum. We now give the formal definition and subsequently illustrate it by some examples.

1.3.12 Definition ([BN07, Def. 1.11])**.** Let M' be a lattice, let $r \in \mathbb{N}_0$ and consider $r+1$ lattice polytopes $\Delta_0, \ldots, \Delta_r$ in $M'_{\mathbb{R}} \cong M' \otimes_{\mathbb{Z}} \mathbb{R}$. Let M'' be the lattice $\mathbb{Z}^r$, so that $M''_{\mathbb{R}} = \mathbb{R}^r$. The *Cayley polytope* associated to $\Delta_0, \ldots, \Delta_r$ is the convex hull of $(\Delta_0 \times \{0\}) \cup (\Delta_1 \times \{e_1\}) \cup \ldots \cup (\Delta_r \times \{e_r\})$ in $M_{\mathbb{R}} := M'_{\mathbb{R}} \oplus M''_{\mathbb{R}}$, where $e_1, \ldots, e_r$ are the standard basis vectors in $\mathbb{R}^r$. We will denote this polytope by $\Delta_0 * \cdots * \Delta_r$ and call it the *Cayley sum* of $\Delta_0, \ldots, \Delta_r$.

If $\dim(\Delta_1) = 0$, then $\Pi(\Delta_0) := \Delta_0 * \Delta_1$ will be called a *pyramid* over Δ_0.

1.3.13 Example. Let M' be the customary integer lattice, i.e., $M' = \mathbb{Z}^q$ for some $q \in \mathbb{N}$. Then the Cayley sum P of lattice polytopes $\Delta_0, \ldots, \Delta_r \subseteq \mathbb{R}^q$ lives in $\mathbb{R}^{q+r}$. Consider the case $q = 1$ and let $\Delta_i = [0, h_i] \subseteq \mathbb{R}$ for some nonnegative integers h_i ($i = 0, \ldots, r$). Then

$$\begin{aligned} P &= [0, h_0] * [0, h_1] * \cdots * [0, h_r] \\ &= \operatorname{conv}\big(([0, h_0] \times \{0\}) \cup ([0, h_1] \times \{e_1\}) \cup \ldots \cup ([0, h_r] \times \{e_r\})\big) \\ &\subseteq \mathbb{R} \times \mathbb{R}^r = \mathbb{R}^{r+1}. \end{aligned}$$

A Cayley polytope of $r+1$ segments $[0, h_i]$ as above is also called an $(r+1)$-dimensional *Lawrence prism* with *heights* $h_0, \ldots, h_r$ (cf. [BN07, Section 2]). Such

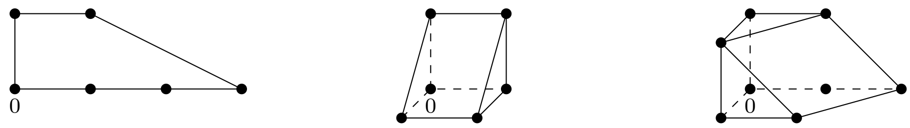
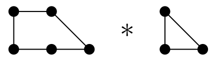


FIGURE 1.1. The polytopes $P_{(3,1)}$, $P_{(1,1,1)}$ and P_{11}.

Lawrence prisms will play a major role in Chapter 4 as they are linked to smooth rational normal scrolls. In Chapter 6 we will encounter Cayley polytopes of two or more lattice polygons. Figure 1.1 shows the Lawrence prisms $P_{(3,1)} := [0,3] * [0,1] \subseteq \mathbb{R}^2$ and $P_{(1,1,1)} := [0,1] * [0,1] * [0,1] \subseteq \mathbb{R}^3$ as well as the Cayley polytope



which we will see again under the name P_{11}.

1.4. Toric varieties

1.4.1. An *affine algebraic group* is a group G which is also an affine variety in such a manner that the multiplication $G \times G \to G$ and the inversion operation $G \to G$ are morphisms. An *n-dimensional* torus is an affine algebraic group isomorphic to $(\mathbb{C}^*)^n$. A torus T comes with two lattices: the group of its characters and the group of its one-parameter subgroups. A *character* of a torus T is a morphism $\chi\colon T \to \mathbb{C}^*$ of affine algebraic groups. A *one-parameter subgroup* of T is a morphism $\lambda\colon \mathbb{C}^* \to T$ of affine algebraic groups. The characters of T form a free abelian group $M := M_T$ of rank equal to the dimension of T. Likewise, the one-parameter subgroups of T constitute a lattice $N := N_T$ of the same rank.

1.4.2 Example. For every $m = (a_1, \dots, a_n) \in \mathbb{Z}^n$ the map

$$\chi^m\colon (\mathbb{C}^*)^n \to \mathbb{C}^*,\ (t_1, \dots, t_n) \mapsto t_1^{a_1} \cdots t_n^{a_n}$$

is a character of the torus $T := (\mathbb{C}^*)^n$. Similarly, every $u = (b_1, \dots, b_n) \in \mathbb{Z}^n$ gives a one-parameter subgroup of $(\mathbb{C}^*)^n$ through

$$\lambda^u\colon \mathbb{C}^* \to (\mathbb{C}^*)^n,\ t \mapsto (t^{b_1}, \dots, t^{b_n}).$$

In fact, the maps $\mathbb{Z}^n \to M_T$, $m \mapsto \chi^m$ and $\mathbb{Z}^n \to N_T$, $u \mapsto \lambda^u$ are group isomorphisms (see [Hum, §16]).

1.4.3. Let T be a torus with lattices M and N of characters and one-parameter subgroups, respectively. We will often write M as an additive group, but if we need a character in its capacity as a map, we will use the multiplicative notation and write χ^m for $m \in M$ even if $m \notin \mathbb{Z}^n$. The same applies for N where we sometimes write λ^u for $u \in N$.

There is a natural bilinear pairing $\langle -, - \rangle\colon M \times N \to \mathbb{Z}$ defined as follows: For a character χ^m and a one-parameter subgroup λ^u the map $\chi^m \circ \lambda^u\colon \mathbb{C}^* \to \mathbb{C}^*$ is a character of $\mathbb{C}^*$. According to Example 1.4.2, there is a unique $l \in \mathbb{Z}$ such that

$$(\chi^m \circ \lambda^u)(t) = t^l \quad \text{for all } t \in \mathbb{C}^*$$

and we can define $\langle m, u \rangle = l$.

If we fix an isomorphism $T \cong (\mathbb{C}^*)^n$ and identify M and N with $\mathbb{Z}^n$ by means of Example 1.4.2, then this pairing corresponds to the usual scalar product. In particular, it is nondegenerate and thus identifies N with $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ and M with

$\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Moreover, one obtains a canonical isomorphism $N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong T$, defined by $u \otimes t \mapsto \lambda^u(t)$, see [CLS, §1.1]. It is therefore customary to tie a torus to its lattice of one-parameter subgroups and write T_N.

1.4.4. An *affine toric variety* is an irreducible affine variety V containing a torus T_N as a Zariski-open subset such that the action of T_N on itself extends to an algebraic group action of T_N on V, i.e., an action $T_N \times V \to V$ given by a morphism. Analogously, a *projective toric variety* is an irreducible projective variety with the aforementioned properties. There are many equivalent ways of constructing affine toric varieties including the usage of lattice points, affine semigroups or so-called toric ideals. The most important construction for our purposes is the one using a finite set of lattice points. So let T_N be a torus with character lattice M and let $\mathcal{A} = \{m_1, \ldots, m_s\} \subseteq M$. We define

$$\begin{aligned} \Phi_{\mathcal{A}} \colon T_N &\to (\mathbb{C}^*)^s, \\ t &\mapsto (\chi^{m_1}(t), \ldots, \chi^{m_s}(t)). \end{aligned}$$

By $Y_{\mathcal{A}}$ we denote the Zariski closure of $\Phi_{\mathcal{A}}(T_N)$ in $\mathbb{C}^s$. Then $Y_{\mathcal{A}}$ is an affine toric variety. For a proof of this fact and details on the other constructions we refer the reader to Section 1.1 of the textbook [CLS].

At this point we want to highlight Proposition 1.1.11 in the said section. It implies that the vanishing ideal $\mathfrak{I}(Y) \subseteq \mathbb{C}[x_1, \ldots, x_s]$ of an affine toric variety $Y \subseteq \mathbb{C}^s$ is generated by binomials $\underline{x}^\alpha - \underline{x}^\beta$. Therefore, we can view Y as an $\mathbb{R}$-variety if necessary.

1.4.5. Let $\mathcal{A}$ be as above. We can compose $\Phi_{\mathcal{A}} \colon T_N \to (\mathbb{C}^*)^s \subseteq \mathbb{C}^s \setminus \{0\}$ with the quotient map $\pi \colon \mathbb{C}^s \setminus \{0\} \to \mathbb{P}^{s-1}$. Then the *projective toric variety* $X_{\mathcal{A}}$ is the Zariski closure of the image of $\pi \circ \Phi_{\mathcal{A}}$ in $\mathbb{P}^{s-1}$, i.e., $X_{\mathcal{A}}$ is the Zariski closure of

$$\{(\chi^{m_1}(t) : \cdots : \chi^{m_s}(t)) : t \in T_N\} \subseteq \mathbb{P}^{s-1}.$$

That $X_{\mathcal{A}}$ is indeed a toric variety and that its dimension is the dimension of the affine hull of $\mathcal{A}$ in $M_{\mathbb{R}}$ is proven in [CLS, Propositions 2.1.2 and 2.1.6]. We will suppress the quotient map π in the notation and simply write $\Phi_{\mathcal{A}}$ for the map $\pi \circ \Phi_{\mathcal{A}} \colon T_N \to \mathbb{P}^{s-1}$ like in the affine case. Whether we consider the map to affine or projective space will be understood from the context.

Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope. Since the set $P \cap M$ of lattice points is finite, this set defines a projective toric variety $X_{P \cap M}$. As already indicated in Section 1.3, this naive guess for the toric variety of P does not establish a satisfactory correspondence:

1.4.6 Example. Let $P = S_3 = \mathrm{conv}(0, e_1, e_2, e_3) \subseteq \mathbb{R}^3$ be the three-dimensional *unit simplex* and let $Q = \mathrm{conv}(0, e_1, e_2, e_1 + e_2 + 3e_3)$ be another 3-simplex. Both contain only four lattice points in $M = \mathbb{Z}^3$, namely their vertices. We have

$$\Phi_{P \cap \mathbb{Z}^3} \colon (\mathbb{C}^*)^3 \to \mathbb{P}^3,\ (x, y, z) \mapsto (1 : x : y : z)$$

and $X_{P \cap \mathbb{Z}^3} = \overline{\Phi_{P \cap \mathbb{Z}^3}((\mathbb{C}^*)^3)} = \overline{\mathrm{D}_+(x_0 \cdots x_3)} = \mathbb{P}^3$. Likewise, we consider the map

$$\Phi_{Q \cap \mathbb{Z}^3} \colon (\mathbb{C}^*)^3 \to \mathbb{P}^3,\ (x, y, z) \mapsto (1 : x : y : xyz^3).$$

For $(1 : a : b : c) \in \mathrm{D}_+(x_0 \cdots x_3)$ we let $x = a$, $y = b$, $z = \sqrt[3]{\frac{c}{xy}} \in \mathbb{C}^*$ and calculate $\Phi_{Q\cap\mathbb{Z}^3}(x, y, z) = (1 : a : b : ab\frac{c}{ab}) = (1 : a : b : c)$. Therefore, $X_{Q\cap\mathbb{Z}^3}$ contains $\mathrm{D}_+(x_0 \cdots x_3)$ as well and it follows $X_{Q\cap\mathbb{Z}^3} = \mathbb{P}^3$.

This is rather unsatisfactory because the lattice polytopes P and Q are quite different. For example, the nonzero vertices of P form a lattice basis of $\mathbb{Z}^3$, those of Q, however, do not.

This is where the property of being very ample becomes important. For the general definition of an abstract toric variety X_Σ associated to a fan Σ in $N_\mathbb{R}$ and obtained by gluing together affine varieties, we refer to [CLS, Section 3.1]. As a special case, using the normal fan Σ_P of a full-dimensional lattice polytope $P \subseteq M_\mathbb{R}$, one obtains the (abstract) toric variety X_{Σ_P} associated to P. By [CLS, Proposition 3.1.6], X_{Σ_P} is isomorphic to the projective toric variety

$$X_P := X_{(kP)\cap M}$$

where k is any positive integer such that kP is very ample. The latter is well-defined as an abstract variety. Indeed, an integer k as above exists by Theorem 1.3.4 and due to the fact that any normal polytope is very ample. If k and l are two such integers, then kP and lP clearly have the same normal fan, namely $\Sigma_{kP} = \Sigma_{lP} = \Sigma_P$. This means that although $X_{(kP)\cap M}$ and $X_{(lP)\cap M}$ live in different projective spaces, they are built from the same affine toric varieties glued together. When we want to use a specific embedding, we will say "X_P is embedded using kP" or "X_P is embedded with respect to the lattice points of kP".

We agree on the following convention:

1.4.7 Convention. Let $P \subseteq M_\mathbb{R}$ be a full-dimensional very ample lattice polytope. The projective toric variety X_P is embedded with respect to the lattice points of P, i.e., $X_P = X_{P\cap M} \subseteq \mathbb{P}^{s-1}$ with $s = |P \cap M|$.

X_P is not contained in any hyperplane in this embedding:

1.4.8 Proposition. *Let $M = \mathbb{Z}^n$ and let $P \subseteq M_\mathbb{R}$ be a full-dimensional very ample lattice polytope. Then the projective variety X_P is nondegenerate in the embedding with respect to the lattice points $P \cap M$.*

Proof. We write $P\cap M = \{m_1, \dots, m_s\}$ for the lattice points of P where $s = |P\cap M|$. Then $X_P \subseteq \mathbb{P}^{s-1}$ is the Zariski closure of

$$\{(t^{m_1} : \cdots : t^{m_s}) : t \in (\mathbb{C}^*)^n\} \subseteq \mathbb{P}^{s-1}.$$

For every $m \in M$ we obviously have

$$\{(t^{m_1} : \cdots : t^{m_s}) : t \in (\mathbb{C}^*)^n\} = \{(t^{m_1+m} : \cdots : t^{m_s+m}) : t \in (\mathbb{C}^*)^n\}.$$

So the points in $P\cap M$ give the same embedding of X_P as the points in $(P+m)\cap M = (P \cap M) + m$. We may therefore assume that $m_1, \dots, m_s \in \mathbb{N}_0^n$.

Let $f = a_1x_1 + \cdots + a_sx_s \in \mathbb{C}[x_1, \dots, x_s]_1$ be a linear polynomial vanishing on X_P. Then in particular $a_1t^{m_1} + \cdots + a_st^{m_s} = 0$ for all $t \in (\mathbb{C}^*)^n$. In other words, the polynomial $p := a_1y^{m_1} + \cdots + a_sy^{m_s} \in \mathbb{C}[\mathrm{y}]$, $\mathrm{y} = (y_1, \dots, y_n)$, vanishes on $(\mathbb{C}^*)^n$ and therefore on $\mathbb{C}^n$. Consequently, $p = 0$, so $a_1 = \cdots = a_s = 0$ and thus $f = 0$. □

Let $X \subseteq \mathbb{P}^{s-1}$ be a projective toric ($\mathbb{R}$)-variety. Then X is irreducible and contains a torus $T_N \cong (\mathbb{C}^*)^n$ for some n. Since $T_N \subseteq X$ and $(\mathbb{C}^*)^n \subseteq \mathbb{A}^n$ are each open and nonempty, and since $\mathbb{A}^n$ is irreducible as well, X and $\mathbb{A}^n$ are birationally equivalent. In other words, X is a *rational* variety. Therefore, the real points $X(\mathbb{R})$ are Zariski-dense in X. Although this fact seems to be well-known, we include a proof that is taken from the lecture notes for a lecture on algebraic geometry:

1.4.9 Proposition (cf. [Plau, Proposition 4.64]). *Let k be an infinite field and let $X \subseteq \mathbb{P}^n$ be a rational irreducible k-variety. Then the set $X(k)$ of k-rational points is Zariski-dense in X.*

Proof. Let $\bar{k}$ be an algebraic closure of k and write $\mathbb{A}^m = \bar{k}^m$ for any $m \in \mathbb{N}_0$. The assumption that X is rational says that there are $m \geq 0$ and nonempty open subsets $U \subseteq \mathbb{A}^m$, $W \subseteq X$, as well as an isomorphism $\varphi\colon U \to W$. Since φ is an isomorphism of k-varieties, we have $\varphi(U \cap k^m) \subseteq W(k)$. Since k^m is Zariski-dense in $\mathbb{A}^m$ (because k is infinite), $U \cap k^m$ is Zariski-dense in U. Consequently, $\varphi(U \cap k^m)$ is Zariski-dense in W and thus also in X. □

1.5. Sums of squares

1.5.1 Definition. Let A be an $\mathbb{R}$-algebra, let $f \in A$. We say that f is a *sum of squares* or *sos* for short if $f = p_1^2 + \cdots + p_r^2$ for some $p_1, \ldots, p_r \in A$. Any equality as above is said to be an *sos representation* of f.

When our algebra A is the polynomial ring $\mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \ldots, x_m]$, for example, we can evaluate f in points of $\mathbb{R}^m$. We say that f is *nonnegative* or *positive semidefinite* (*psd*) if $f(x) \geq 0$ for all $x \in \mathbb{R}^m$. Obviously, if a polynomial $f \in \mathbb{R}[\underline{x}]$ is sos then it is psd. Conversely, given a (nonnegative) polynomial $f \in \mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \ldots, x_m]$ one can ask for an sos representation $f = \sum_{i=1}^r p_i^2$ certifying the nonnegativity of f. In general, the search for such a representation would be computationally intractable if one could not restrict the search space to a reasonably small finite-dimensional subspace of $\mathbb{R}[\underline{x}]$. The main idea is that certain terms in $\sum_{i=1}^r p_i^2$ appear with same sign so that they cannot cancel. This can be made precise using the Newton polytope of f.

1.5.2 Definition. Let $f \in \mathbb{R}[\underline{x}]$. We write $f = \sum_{\alpha \in \mathbb{N}_0^m} c_\alpha x^\alpha$ with $c_\alpha \in \mathbb{R}$. The *support* of f is the finite set $\operatorname{supp}(f) := \{\alpha \in \mathbb{N}_0^m : c_\alpha \neq 0\}$. Its convex hull $\operatorname{Newt}(f) := \operatorname{conv}(\operatorname{supp}(f)) \subseteq \mathbb{R}^m$ is the *Newton polytope* of f.

The following well-known theorem is due to Reznick (cf. [Rez, Theorem 1]).

1.5.3 Theorem. *Let $r \in \mathbb{N}$, $p_1, \ldots, p_r \in \mathbb{R}[\underline{x}]$, and let $f := \sum_{i=1}^r p_i^2$. Then* $\operatorname{Newt}(f) = 2\operatorname{conv}(\operatorname{Newt}(p_1) \cup \cdots \cup \operatorname{Newt}(p_r))$.

For any lattice polytope $P \subseteq \mathbb{R}^m$ with vertices in $\mathbb{N}_0^m$, we write $\mathbb{R}[\underline{x}]_P$ for the vector space of all polynomials in $\mathbb{R}[\underline{x}]$ whose Newton polytope is contained in P. The above theorem says that in any sos representation $f = \sum_{i=1}^r p_i^2$ of a polynomial $f \in \mathbb{R}[\underline{x}]_{2P}$, we must have $p_i \in \mathbb{R}[\underline{x}]_P$ for all $i = 1, \ldots, r$. This observation was key to the study of sos representations via the so-called Gram matrix method introduced by Choi, Lam and Reznick in [CLR], which we recall in Section 2.1. The entirety of all expressions of f as a sum of squares is then parametrized by the polynomial's Gram spectrahedron. We explain this in detail in Chapter 2.

1.5.4. We can consider questions of nonnegativity and sums of squares also on certain projective varieties. Following the presentation in [BSV, Section 2], we consider a nondegenerate m-dimensional projective $\mathbb{R}$-variety $X \subseteq \mathbb{P}^n$ with Zariski-dense real points. Let $R := \mathbb{R}[X] = \mathbb{R}[x_0, \ldots, x_n]/\mathfrak{I}_+(X)$ be its homogeneous coordinate ring. Since X is nondegenerate, we have $R_1 = \mathbb{R}[x_0, \ldots, x_n]_1$. For $f \in R_2$ and $\xi \in X(\mathbb{R})$, we can define the *sign of f at ξ* by setting

$$\operatorname{sgn}_\xi(f) := \operatorname{sgn}\left(\tilde{f}(v)\right) \in \{-1, 0, 1\}$$

where $\tilde{f} \in \mathbb{R}[x_0, \ldots, x_n]_2$ is a representative of the residue class f and $0 \neq v \in \mathbb{A}^{n+1}(\mathbb{R})$ is an affine representative of ξ, i.e., a point with $[v] = \xi$. This is well-defined. Indeed, first of all, the real number $\tilde{f}(v)$ is independent of the choice of $\tilde{f}$ as for any other choice $\tilde{f}'$ we have $\tilde{f} - \tilde{f}' \in \mathfrak{I}_+(X)$. Finally, since the degree of f is even, choosing a different affine representative of ξ amounts to multiplying $\tilde{f}(v)$ by the square of a nonzero real number.

We say that f is *nonnegative* (or *positive*) on $X(\mathbb{R})$ if $\operatorname{sgn}_\xi(f) \geq 0$ (or $\operatorname{sgn}_\xi(f) > 0$) for all $\xi \in X(\mathbb{R})$. Slightly negligently, in this case we mostly speak of f being nonnegative (or positive) on X although the sign is only defined at real points of X. Clearly, if $f = p_1^2 + \cdots + p_r^2$ for some $p_1, \ldots, p_r \in R_1$ then f is nonnegative on X. As one can move to the d-th Veronese re-embedding of X in order to tackle forms in R_{2d}, it is often sufficient to consider quadratic forms on the adequate variety (see for example [BSV, Remark 4.6]). When it comes to comparing nonnegativity and sums of squares on varieties, one is thus interested in separating the varieties X where every nonnegative quadratic form on X is a sum of squares from those where this is not the case. This was done by Blekherman, Smith and Velasco in [BSV], see Theorem 4.0.1 for a quotation of their main result.

We study Gram spectrahedra of quadratic forms on X in both cases. In Chapter 4 we deal with varieties X of minimal degree where every nonnegative quadratic form actually is a sum of squares. We then proceed to the first case where this is no longer true, namely certain varieties of almost minimal degree (see Chapter 6). We are especially interested in toric varieties of this kind. Let us briefly mention how Newton polytopes from the beginning of this section are related to all this.

1.5.5. Fix a full-dimensional normal lattice polytope $P \subseteq \mathbb{R}^m$ with vertices in $\mathbb{N}_0^m$. Let $X_P \subseteq \mathbb{P}^n$ be the projective toric $\mathbb{R}$-variety that is embedded with respect to the lattice points $P \cap \mathbb{Z}^m$. Similarly to the Veronese embedding, polynomials in $\mathbb{R}[x_1, \ldots, x_m]_P$ correspond to linear forms on X_P. Since P is normal, every lattice point of $2P$ can be written as a sum of two lattice points in P. This ensures that polynomials f with $\operatorname{Newt}(f) \subseteq 2P$ correspond to quadratic forms in $\mathbb{R}[X_P]_2$. The relations between lattice points of P are reflected in the vanishing ideal of X_P. We can thus study sos representations of quadratic forms on X_P with the aid of the combinatorics of P, and use calculations in the polynomial ring to infer properties of the associated Gram spectrahedra.

1.6. Spectrahedra

The term *spectrahedron* was coined by Ramana and Goldman [RG]. They gave this name to the feasible regions of semidefinite programming problems since these sets involve the spectra of matrices and bear a resemblance to polyhedra. This section initiates the reader in the general notion of spectrahedra, whereas Gram spectrahedra and their facial structure are introduced in Chapter 2.

1.6.1. Let $n \in \mathbb{N}$. An $n \times n$ real symmetric matrix $A \in \operatorname{Sym}_n(\mathbb{R})$ is *positive semidefinite* (*psd*), written $A \succeq 0$, if $v^T A v \geq 0$ for all $v \in \mathbb{R}^n$. There are many equivalent characterizations of positive semidefinite matrices. The most prominent one is probably that $A \succeq 0$ if and only if all eigenvalues of A are nonnegative. Another, which is of great significance to us, is that A can be decomposed as a product $A = C^T C$ with a matrix $C \in \mathrm{M}_{r\times n}(\mathbb{R})$ where $r = \operatorname{rk}(A) = \operatorname{rk}(C)$. The set of all (real symmetric) psd matrices is denoted by $\operatorname{Sym}_n^+(\mathbb{R})$.

1.6.2 Remark. Let $(V, \langle -,-\rangle)$ be a Euclidean vector space. The *Gram matrix* of vectors $v_1, \ldots, v_n \in V$ with respect to the inner product $\langle -,-\rangle$ is the matrix $A = (a_{ij}) \in \operatorname{Sym}_n(\mathbb{R})$ with $a_{ij} = \langle v_i, v_j\rangle$ for $i, j = 1, \ldots, n$. The matrix A is positive semidefinite and $\operatorname{rk}(A) = \dim \operatorname{span}(v_1, \ldots, v_n)$, see [HJ, Theorem 7.2.10]. Gram matrices are named after the Danish mathematician Jørgen Pedersen Gram who is best-known for the Gram–Schmidt process in linear algebra. It is easy to see that every $A \in \operatorname{Sym}_n^+(\mathbb{R})$ is the Gram matrix of certain $v_1, \ldots, v_n \in \mathbb{R}^n$ where $\langle -,-\rangle$ is the standard inner product.

1.6.3 Proposition. $\operatorname{Sym}_n^+(\mathbb{R})$ *is a full-dimensional closed convex cone in* $\operatorname{Sym}_n(\mathbb{R})$ *that does not contain an affine line.*

Proof. We only show the last part of the statement since everything else is immediate. Assume there was an affine line contained in $\operatorname{Sym}_n^+(\mathbb{R})$. Then we find $A, B \in \operatorname{Sym}_n(\mathbb{R})$ with $B \neq 0$ such that $A + \lambda B \in \operatorname{Sym}_n^+(\mathbb{R})$ for all $\lambda \in \mathbb{R}$. Since $B \neq 0$, there is $v \in \mathbb{R}^n$ with $v^T B v \neq 0$. Let

$$\lambda_\pm = -\frac{v^T A v}{v^T B v} \pm 1.$$

Then $v^T(A + \lambda_\pm B)v = \pm v^T B v$. But this means that $A + \lambda_+ B$ is not positive semidefinite (if $v^T B v < 0$) or $A + \lambda_- B$ is not positive semidefinite (if $v^T B v > 0$), a contradiction. □

1.6.4 Definition. Let V be a finite-dimensional $\mathbb{R}$-vector space. A set $S \subseteq V$ is called a *spectrahedron* if there is a linear map $\varphi\colon V \to \operatorname{Sym}_n(\mathbb{R})$ (for some n) and a symmetric matrix $A_0 \in \operatorname{Sym}_n(\mathbb{R})$ such that

$$S = \{v \in V : A_0 + \varphi(v) \succeq 0\}.$$

1.6.5 Remark. (a) Definition 1.6.4 says that S is the preimage of $\operatorname{Sym}_n^+(\mathbb{R})$, the cone of positive semidefinite real symmetric matrices, under the affine-linear map $v \mapsto A_0 + \varphi(v)$.

(b) Let $S \subseteq \mathbb{R}^m$ be a spectrahedron and let $(e_1, \ldots, e_m)$ be the standard basis of $\mathbb{R}^m$. Writing $\varphi(e_i) = A_i$ for some $A_i \in \operatorname{Sym}_n(\mathbb{R})$ $(i = 1, \ldots, m)$, we obtain

$$S = \{x \in \mathbb{R}^m : A_0 + x_1 A_1 + \cdots + x_m A_m \succeq 0\}.$$

An expression of the form $A_0 + x_1 A_1 + \cdots + x_m A_m \succeq 0$ is called a *linear matrix inequality* (*LMI*). Therefore, spectrahedra (in $\mathbb{R}^m$) are precisely the solution sets of linear matrix inequalities. Of course, by choosing coordinates on V we can also identify a spectrahedron in V with a set of the above form.

1.6.6 Proposition. *Let $S \subseteq V$ be a nonempty spectrahedron, say $S = \{v \in V : A_0 + \varphi(v) \succeq 0\}$ for a linear map $\varphi\colon V \to \mathrm{Sym}_n(\mathbb{R})$ and $A_0 \in \mathrm{Sym}_n(\mathbb{R})$. Let $v_0 \in S$. For every subspace $U \subseteq V$, the following are equivalent:*

(i) S contains the affine subspace $v_0 + U$,
(ii) $U \subseteq \ker(\varphi)$.

Proof. Let $U \subseteq V$ be a linear subspace. If $U \subseteq \ker(\varphi)$, then

$$A_0 + \varphi(v_0 + u) = A_0 + \varphi(v_0) + \varphi(u) = A_0 + \varphi(v_0) \succeq 0$$

for all $u \in U$. Therefore, S contains the affine subspace $v_0 + U$.

For the converse let $v_0+U \subseteq S$. Then $(A_0+\varphi(v_0))+\varphi(U) \subseteq \mathrm{Sym}_n^+(\mathbb{R})$. However, by Proposition 1.6.3 the cone $\mathrm{Sym}_n^+(\mathbb{R})$ does not contain an affine line. Hence, we must have $\varphi(U) = 0$, so $U \subseteq \ker(\varphi)$. □

1.6.7 Corollary. *Let $S = \{v \in V : A_0 + \varphi(v) \succeq 0\}$ be a nonempty spectrahedron as in Proposition 1.6.6. Then S contains an affine line if and only if $\ker(\varphi) \neq \{0\}$.* □

1.6.8 Remark. By Proposition 1.6.6, a spectrahedron $S \subseteq V$ is linearly isomorphic to $S' \times \ker(\varphi)$ where S' is a spectrahedron in $V/\ker(\varphi)$ that does not contain any affine line. Aside from that, $\overline{\varphi}\colon V/\ker(\varphi) \to \mathrm{im}(\varphi)$ is an isomorphism. Therefore, a spectrahedron is (linearly isomorphic to) the Cartesian product of an affine slice of $\mathrm{Sym}_n^+(\mathbb{R})$ and $\mathbb{R}^k$ for some $n,\, k \in \mathbb{N}_0$.

1.6.9. In particular, for any affine-linear subspace L of $\mathrm{Sym}_n(\mathbb{R})$, we consider $S = L \cap \mathrm{Sym}_n^+(\mathbb{R})$ a spectrahedron. This is justified by the following observation: By choosing $A_0 \in \mathrm{Sym}_n(\mathbb{R})$ and linearly independent $A_1, \ldots, A_m \in \mathrm{Sym}_n(\mathbb{R})$ with $L = A_0 + \mathrm{span}(A_1, \ldots, A_m)$, we could identify S with $\{x \in \mathbb{R}^m : A_0 + \sum_{i=1}^m x_i A_i \succeq 0\}$. However, we prefer to think of S as a set of matrices since valuable properties and the geometrical structure are ignored or hidden when transitioning from $\mathrm{Sym}_n(\mathbb{R})$ to $\mathbb{R}^m$. In this thesis all spectrahedra will be of the form $S = L \cap \mathrm{Sym}_n^+(\mathbb{R})$ for an affine-linear subspace $L \subseteq \mathrm{Sym}_n(\mathbb{R})$, or coordinate-free incarnations of these creatures. The coordinate-free approach will be introduced in Chapter 2. In consideration of the fact that in the literature often no distinction is made between $L \cap \mathrm{Sym}_n^+(\mathbb{R})$ and the solution set of an LMI, we consider that sufficiently general.

We are going to see that every spectrahedron $L \cap \mathrm{Sym}_n^+(\mathbb{R})$ is linearly isomorphic to the Gram spectrahedron of a quadratic form in a finitely generated graded $\mathbb{R}$-algebra (Theorem 5.3.1).

The following well-known statements about positive semidefinite matrices will be useful when we study the facial structure of spectrahedra. As we will also deal with a Hermitian version of spectrahedra, we formulate them for both real symmetric and complex Hermitian matrices. To this end, we denote by $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ the field of real or complex numbers.

1.6.10 Proposition. *Let $A \in \mathrm{M}_n(\mathbb{K})$ be symmetric ($\mathbb{K} = \mathbb{R}$) or Hermitian ($\mathbb{K} = \mathbb{C}$). Then $\mathrm{im}(A) = \ker(A)^\perp$ in $\mathbb{K}^n$.*

Proof. Let $u \in \ker(A)$ and $v \in \mathbb{K}^n$. Then

$$\langle Av, u\rangle = (Av)^* u = v^* A u = 0.$$

This shows $\operatorname{im}(A) \subseteq \ker(A)^\perp$. Since both spaces have the same dimension, they have to be equal. □

1.6.11 Lemma. *Let* A, $B \in \mathrm{M}_n(\mathbb{K})$ *be (real symmetric or complex Hermitian) psd matrices. Then* $\ker(A+B) = \ker(A) \cap \ker(B)$ *and* $\operatorname{im}(A+B) = \operatorname{im}(A) + \operatorname{im}(B)$.

Proof. The inclusion $\ker(A) \cap \ker(B) \subseteq \ker(A+B)$ is clear. For the other inclusion we let $u \in \mathbb{K}^n$ with $(A+B)u = 0$. Then also

$$0 = u^*(A+B)u = u^*Au + u^*Bu.$$

Since both summands are (real and) nonnegative, we must have $u^*Au = u^*Bu = 0$. As A is positive semidefinite, we can write $A = C^*C$ for some $C \in \mathrm{M}_n(\mathbb{K})$. This gives

$$0 = u^*C^*Cu = (Cu)^*(Cu),$$

so that $Cu = 0$ and therefore $Au = 0$. That $Bu = 0$ follows analogously.

Using Proposition 1.6.10, we get

$$\begin{aligned}\operatorname{im}(A+B) = \ker(A+B)^\perp = (\ker(A) \cap \ker(B))^\perp \\ \supseteq \ker(A)^\perp + \ker(B)^\perp = \operatorname{im}(A) + \operatorname{im}(B),\end{aligned}$$

and the opposite inclusion is obvious. □

1.6.12 Lemma. *Let* A, $B \in \mathrm{M}_n(\mathbb{K})$ *be (real symmetric or complex Hermitian) matrices with* $\operatorname{im}(B) \subseteq \operatorname{im}(A)$. *Assume that* $A \succeq 0$. *Then there is* $\varepsilon > 0$ *with* $A - \varepsilon B \succeq 0$.

Proof. Let $U := \ker(A)$. Then $U = \operatorname{im}(A)^\perp \subseteq \operatorname{im}(B)^\perp = \ker(B)$ by Proposition 1.6.10. Since A is positive definite on $U^\perp$, there is $\varepsilon > 0$ with $(A - \varepsilon B)|_{U^\perp} \succ 0$. Moreover, $U \subseteq \ker(A - \varepsilon B)$. Using $\mathbb{K}^n = U \oplus U^\perp$, we see that $A - \varepsilon B \succeq 0$. □

CHAPTER 2

An introduction to Gram spectrahedra

As explained in the introduction to this thesis, the Gram matrix method introduced by Choi, Lam and Reznick [CLR] provides a computational method to decide whether a given polynomial f is a sum of squares (see also [PW]). In the language of convex optimization, one wants to decide the feasibility of a semidefinite programming problem and ideally find a feasible point. We recall the Gram matrix method in Section 2.1.

Yet, the main object of this thesis is the Gram spectrahedron of f, the entirety of all positive semidefinite Gram matrices of f. We are interested in its structure as a convex set. It turns out that the coordinate-free approach introduced by Scheiderer in [Sch22] is better suited for analyzing the faces of the spectrahedron. We present this approach in Section 2.2.

Ramana and Goldman [RG] provide useful tools for the analysis of the facial structure of spectrahedra. Following Scheiderer ([Sch22, Sections 2 and 3]), we give a coordinate-free review of their results with a focus on Gram spectrahedra. This is done in Section 2.3. Of particular importance is the dimension formula for faces of Gram spectrahedra (Proposition 2.3.9).

Chua, Plaumann, Sinn and Vinzant consider a Hermitian analogue of the Gram spectrahedron ([CPSV, Section 5]) and note that "Hermitian sums of squares enjoy some technical advantages for certain questions". In the author's opinion, they are also interesting in themselves, especially for binary forms which are the matter of Chapter 3. We introduce a coordinate-free version of Hermitian Gram spectrahedra in Section 2.4. Afterwards, we study the relationship between real symmetric and Hermitian Gram spectrahedra in general and on the level of their faces.

2.1. The Gram matrix method

2.1.1. Let A be an $\mathbb{R}$-algebra and let $V \subseteq A$ be a linear subspace of finite dimension n. Fix a basis $\mathcal{B}$ of V and let $\mathfrak{m}$ be the column vector that contains the elements of $\mathcal{B}$ in some fixed order. Assume that $f \in A$ has a representation $f = \sum_{i=1}^r p_i^2$ for some linearly independent $p_1, \dots, p_r \in V$. For each $i \in \{1, \dots, r\}$, let c_i be the vector of coefficients of p_i with respect to $\mathcal{B}$. In other words, we require that $p_i = c_i^T \mathfrak{m}$. Let $C \in \mathrm{M}_{r\times n}(\mathbb{R})$ be the matrix whose i-th row is c_i^T. Then

$$f = p_1^2 + \cdots + p_r^2 = (p_1, \dots, p_r)(p_1, \dots, p_r)^T = (C\mathfrak{m})^T (C\mathfrak{m}) = \mathfrak{m}^T (C^T C)\mathfrak{m},$$

and the matrix $C^T C = \sum_{i=1}^r c_i c_i^T$ is a positive semidefinite matrix of rank r.

Conversely, every positive semidefinite matrix $G \in \mathrm{Sym}_n(\mathbb{R})$ with $\mathfrak{m}^T G \mathfrak{m} = f$ leads to a representation of f as a sum of squares of elements from V. Indeed, we can find a factorization $G = C^T C$ with an $r \times n$-matrix C where $r = \mathrm{rk}(G)$. Using the i-th row of C as coefficient vector for $p_i \in V$, we get $f = \sum_{i=1}^r p_i^2$.

Any matrix $G \in \mathrm{Sym}_n(\mathbb{R})$ with $\mathfrak{m}^T G \mathfrak{m} = f$ is called a *Gram matrix* of f *relative to* V. (Note that we do not require G to be positive semidefinite.) The *Gram spectrahedron* of f (*relative to* V) is the set of all positive semidefinite such matrices.

2.1.2 Remark. In general, there is no one-to-one correspondence between representations of f as a sum of squares of r (linearly independent) elements from V and positive semidefinite rank-r Gram matrices of f relative to V. Indeed, let $U = (u_{ij}) \in \mathrm{O}(r)$ be an orthogonal $r \times r$-matrix and let $(q_1, \dots, q_r)^T = U(p_1, \dots, p_r)^T$, that is to say $q_i = \sum_{j=1}^r u_{ij} p_j$ for $i = 1, \dots, r$. If $G = C^T C$ is the Gram matrix associated to the representation $f = p_1^2 + \cdots + p_r^2$, we have $f = q_1^2 + \cdots + q_r^2$ and $G = C^T U^T U C = C^T C$ is also the Gram matrix associated to the latter representation since $U^T U = I_r$.

Two representations $f = p_1^2 + \cdots + p_r^2 = q_1^2 + \cdots + q_r^2$ such that both $p_1, \dots, p_r$ and $q_1, \dots, q_r$ are linearly independent are *orthogonally equivalent* if there is $U \in \mathrm{O}(r)$ such that $(q_1, \dots, q_r)^T = U(p_1, \dots, p_r)^T$. It is well-known that two sum-of-squares representations of f of length r give the same Gram matrix if and only if they are orthogonally equivalent (see [HJ, Theorem 7.3.11] or [CPSV, Lemma 1.4] which capture the main aspect of [CLR, Proposition 2.10]). Thus, up to orthogonal equivalence, the Gram spectrahedron of f relative to V parametrizes all representations of f as a sum of squares of elements from V.

2.1.3 Example/Remark. Typically, one is interested in the following case: Let $A = \mathbb{R}[x_1, \dots, x_m]$ and let $f \in A$. Choose a (preferably small) lattice polytope $P \subseteq \mathbb{R}^m$ such that $\mathrm{Newt}(f)$ is contained in $2P$. If $f = \sum_{i=1}^r p_i^2$ for some $p_i \in A$, we must have $\mathrm{Newt}(p_i) \subseteq P$ for all $i \in \{1, \dots, r\}$, according to Theorem 1.5.3. Thus, it makes sense to consider the Gram spectrahedron of f relative to $V = \mathbb{R}[x_1, \dots, x_m]_P$. Then f is a sum of squares if and only if this spectrahedron is nonempty.

In this setting, there is a canonical choice for a basis of V, namely to take a monomial basis. The most basic case possible is $m = 1$, where Newton polytopes are line segments. Consider, for example,

$$f = x^8 + 2x^7 - 6x^6 - 26x^5 + 101x^4 + 620x^3 + 248x^2 - 1872x + 3380 \in \mathbb{R}[x].$$

Then, with respect to the basis $(1, x, x^2, x^3, x^4)$ of $V = \mathbb{R}[x]_{\leq 4}$, the Gram matrices of f are of the form

$$G = \begin{pmatrix} 3380 & -936 & \lambda_1 & \lambda_3 & \lambda_6 \\ -936 & 248 - 2\lambda_1 & 310 - \lambda_3 & \lambda_5 & \lambda_4 \\ \lambda_1 & 310 - \lambda_3 & 101 - 2(\lambda_5 + \lambda_6) & -13 - \lambda_4 & \lambda_2 \\ \lambda_3 & \lambda_5 & -13 - \lambda_4 & -6 - 2\lambda_2 & 1 \\ \lambda_6 & \lambda_4 & \lambda_2 & 1 & 1 \end{pmatrix}$$

where $\lambda_1, \dots, \lambda_6 \in \mathbb{R}$. Setting

$$(\lambda_1, \dots, \lambda_6) = (-575, -11, 63, -9, -124, 24),$$

we obtain a positive definite Gram matrix of f. This shows that f is positive on $\mathbb{R}$ and that its (full) Gram spectrahedron is indeed a six-dimensional convex set. Admittedly, we constructed this f as

$$f = \prod_{j=1}^{4} (x - \alpha_j)(x - \overline{\alpha_j})$$

with $\{\alpha_1, \dots, \alpha_4\} = \{1 + i, 3 + 2i, -2 + 3i, -3 + i\}$, in order to ease calculations in what follows.

Consider the four-dimensional subspace $U \subseteq V$ with basis $\mathcal{B} = (q_1, \dots, q_4)$ where

$$\begin{aligned} q_1 &= x^4 + x^3 - 4x^2 + 6x - 52, & q_3 &= x^2 - x, \\ q_2 &= x^3 - 15x^2 - 24x + 26, & q_4 &= x - 13. \end{aligned}$$

With respect to this basis, the affine space of Gram matrices of f relative to U can be parametrized as

$$H = \begin{pmatrix} 1 & 0 & -\mu_2 & -\frac{1}{2}(\mu_1+\mu_2) \\ 0 & 1+2\mu_2 & \frac{1}{2}(\mu_1+61\mu_2) & \mu_2 \\ -\mu_2 & \frac{1}{2}(\mu_1+61\mu_2) & 4\mu_1+598\mu_2 & 130\mu_2 \\ -\frac{1}{2}(\mu_1+\mu_2) & \mu_2 & 130\mu_2 & 4\mu_1 \end{pmatrix}$$

with μ_1, $\mu_2 \in \mathbb{R}$. The arrangement of μ_1 and μ_2 in H is due to the fact that

$$\begin{aligned} 0 &= -q_1q_4 + q_2q_3 + 4q_4^2 + 4q_3^2 \quad \text{and} \\ 0 &= -2q_1q_3 - q_1q_4 + 2q_2^2 + 61q_2q_3 + 2q_2q_4 + 598q_3^2 + 260q_3q_4. \end{aligned}$$

The Gram spectrahedron of f relative to U is actually a two-dimensional convex set whose boundary consists of rank-three points except for two points of rank two. Later, we will see that the number of quadratic relations between $q_1, \dots, q_4$ determines the dimension of the Gram spectrahedron (cf. Corollary 2.3.10 and Remark 2.3.6).

For now, let us remark the following: Gram matrices definitely have their raison d'être because they are useful for so many calculations. However, there is no canonical choice for a basis of U. In this case, choosing coordinates and building the matrix H is rather cumbersome and hides the relevant structure. Moreover, since $U \subseteq V$, the Gram spectrahedron of f relative to U should be a subset of that relative to V in a canonical way. For analyzing the facial structure of Gram spectrahedra we thus prefer a coordinate-free approach. In this setting, for example, it will be obvious that $\mathrm{Gram}_U(f)$ is a face of $\mathrm{Gram}_V(f)$, notwithstanding that we have yet to introduce the notation just used.

2.2. Gram tensors and the Gram spectrahedron

The example in the previous section supports us in our motivation to work in a coordinate-free setting. Therefore, we are going to replace symmetric matrices by symmetric tensors, mostly following the presentation in [Sch22], where this new approach to spectrahedra and their facial structure was first introduced.

2.2.1. As before, we let A be an $\mathbb{R}$-algebra. By $\mathsf{S}_2A \subseteq A \otimes_{\mathbb{R}} A$ we denote the space of symmetric tensors, i.e., tensors that are invariant under the involution $p \otimes q \mapsto q \otimes p$. The multiplication map $\mu\colon A \times A \to A$, $(p,q) \mapsto pq$ is $\mathbb{R}$-bilinear and induces an $\mathbb{R}$-linear map $\mathsf{S}_2A \to A$. Let $V \subseteq A$ be a subspace of finite dimension. Given $f \in A$, the symmetric tensors $\vartheta \in \mathsf{S}_2V$ with $\mu(\vartheta) = f$ are called the *Gram tensors* of f, relative to V.

For matrices we have notions like *image*, *rank* and *positive semidefinite*. We explain these notions for symmetric tensors.

2.2.2. Let $V^\vee = \mathrm{Hom}(V, \mathbb{R})$ denote the dual space of V. A symmetric tensor $\vartheta = \sum_{i=1}^r p_i \otimes q_i \in \mathsf{S}_2V$ induces a symmetric bilinear form $b_\vartheta\colon V^\vee \times V^\vee \to \mathbb{R}$ given by

$b_\vartheta(\lambda,\mu)=\sum_{i=1}^r \lambda(p_i)\mu(q_i)$, as well as a linear map $\varphi_\vartheta\colon V^\vee\to V$ with

$$\varphi_\vartheta(\lambda)=\sum_{i=1}^r \lambda(p_i)q_i=\sum_{i=1}^r \lambda(q_i)p_i$$

for $\lambda\in V^\vee$. We define the *image* of ϑ to be the image of the linear map φ_ϑ and denote it by $\mathrm{im}(\vartheta)$. Consequently, the *rank* of ϑ is defined as $\mathrm{rk}(\vartheta)=\dim(\mathrm{im}(\vartheta))$.

2.2.3 Lemma. *Let $\vartheta=\sum_{i=1}^r p_i\otimes q_i\in \mathsf{S}_2V$. If both $p_1,\dots,p_r$ and $q_1,\dots,q_r$ are linearly independent, then $\mathrm{im}(\vartheta)=\mathrm{span}(p_1,\dots,p_r)=\mathrm{span}(q_1,\dots,q_r)$ and $\mathrm{rk}(\vartheta)=r$.*

Proof. That $\mathrm{im}(\vartheta)$ is contained in $\mathrm{span}(p_1,\dots,p_r)$ and in $\mathrm{span}(q_1,\dots,q_r)$ is obvious. In order to prove the other inclusions, we choose $\lambda_1,\dots,\lambda_r\in V^\vee$ with $\lambda_i(p_j)=\delta_{ij}$ for all $i,\,j\in\{1,\dots,r\}$. (These linear forms can be obtained by extending $p_1,\dots,p_r$ to a basis of V and taking $\lambda_1,\dots,\lambda_r$ from the corresponding dual basis.) It follows

$$\mathrm{im}(\vartheta)\ni\varphi_\vartheta(\lambda_i)=\sum_{j=1}^r \lambda_i(p_j)q_j=q_i$$

for all $i\in\{1,\dots,r\}$. Hence, we have $\mathrm{im}(\vartheta)=\mathrm{span}(q_1,\dots,q_r)$. The equality $\mathrm{im}(\vartheta)=\mathrm{span}(p_1,\dots,p_r)$ is shown analogously by choosing linear forms $\lambda_1,\dots,\lambda_r\in V^\vee$ with $\lambda_i(q_j)=\delta_{ij}$. □

From this we obtain the following simple observation that is nonetheless crucial to the analysis of the facial structure of spectrahedra.

2.2.4 Corollary ([Sch22, Lemma 2.3]). *Let $\vartheta\in\mathsf{S}_2V$ and let $U\subseteq V$ be a linear subspace. Then $\mathrm{im}(\vartheta)\subseteq U$ if and only if $\vartheta\in\mathsf{S}_2U$.* □

2.2.5. The symmetric tensor $\vartheta\in\mathsf{S}_2V$ is *positive semidefinite* if b_ϑ is positive semidefinite, that is to say $b_\vartheta(\lambda,\lambda)\geq 0$ for all $\lambda\in V^\vee$. In this case we write $\vartheta\succeq 0$. Furthermore, ϑ is *positive definite*, $\vartheta\succ 0$, if $b_\vartheta(\lambda,\lambda)>0$ for all $0\neq\lambda\in V^\vee$.

After fixing a linear basis $p_1,\dots,p_n$ of V, every $\vartheta\in\mathsf{S}_2V$ can be written as $\vartheta=\sum_{i,j=1}^n a_{ij}(p_i\otimes p_j)$ with $a_{ij}=a_{ji}\in\mathbb{R}$. In this way we could identify ϑ with the real symmetric matrix $(a_{ij})\in\mathrm{Sym}_n(\mathbb{R})$. Then, ϑ is positive semidefinite if and only if this matrix is. Hence, the set

$$\mathsf{S}_2^+V:=\{\vartheta\in\mathsf{S}_2V:\vartheta\succeq 0\}$$

of positive semidefinite tensors corresponds to the cone of real symmetric positive semidefinite $n\times n$-matrices and is thus a full-dimensional closed convex cone in S_2V. The fact that every real symmetric matrix can be diagonalized implies that every $\vartheta\in\mathsf{S}_2V$ can be written as $\vartheta=\sum_{i=1}^r\varepsilon_i(p_i\otimes p_i)$ with $r\geq 0$, $\varepsilon_i=\pm1$ and $p_1,\dots,p_r\in V$ linearly independent. Of course, $\vartheta\succeq 0$ if and only if all $\varepsilon_i=1$.

We are now ready to present the object that has the leading part in this thesis.

2.2.6 Definition ([Sch22, 3.2]). Let A be an $\mathbb{R}$-algebra, let $V\subseteq A$ be a finite-dimensional linear subspace and let $f\in A$. The *(symmetric) Gram spectrahedron* of f, relative to V, is the set of all positive semidefinite Gram tensors of f in S_2V, that is to say

$$\mathrm{Gram}_V(f):=\mathsf{S}_2^+V\cap\mu^{-1}(f).$$

2.2.7 Remark. By definition, $\mathrm{Gram}_V(f)$ is an affine-linear slice of the cone of positive semidefinite tensors. Thus, it is indeed a spectrahedron. Its elements are symmetric tensors $\vartheta = \sum_{i=1}^r p_i \otimes p_i$ with $r \geq 0$ and $p_1, \dots, p_r \in V$ such that $\sum_{i=1}^r p_i^2 = f$. Given ϑ as above, we may assume that $p_1, \dots, p_r$ are linearly independent. As we have seen in Section 2.1, if $\vartheta' = \sum_{i=1}^r q_i \otimes q_i$ is another psd Gram tensor of f with linearly independent $q_1, \dots, q_r \in V$, then $\vartheta = \vartheta'$ if and only if there exists an orthogonal matrix (u_{ij}) of size $r \times r$ such that $q_i = \sum_{j=1}^r u_{ij} p_j$ for $i = 1, \dots, r$.

2.2.8 Notation. Let $V \subseteq A$ be any linear subspace. Then VV denotes the linear subspace of A spanned by the products pq $(p,\, q \in V)$. In other words, $VV = \mu(\mathsf{S}_2 V)$. Moreover, we let

$$\Sigma V^2 = \left\{ \sum_{i=1}^r p_i^2 : r \in \mathbb{N},\, p_1, \dots, p_r \in V \right\}$$

be the convex cone of all elements in VV that can be written as a sum of squares of elements from V. Of course, if $U \subseteq V$ is another subspace then $UU \subseteq VV$ and $\Sigma U^2 \subseteq \Sigma V^2$.

In their nature as spectrahedra, Gram spectrahedra are closed, convex, semialgebraic sets. We determine the cases in which they are compact.

2.2.9 Lemma. *Let $V \subseteq A$ be a finite-dimensional linear subspace. The following are equivalent:*

(a) For all $r \in \mathbb{N}$ and $p_1, \dots, p_r \in V$, we have $p_1^2 + \dots + p_r^2 = 0$ only if $p_i = 0$ for all $i = 1, \dots, r$.
(b) $\mathrm{Gram}_V(f)$ is bounded for every $f \in \Sigma V^2$.
(c) There exists $f \in \Sigma V^2$ such that $\mathrm{Gram}_V(f)$ is bounded.

Proof. In order to prove "(a) $\Rightarrow$ (b)", we let $f \in \Sigma V^2$ and assume that the spectrahedron $\mathrm{Gram}_V(f)$ is unbounded. Since it is a closed convex set, it must contain a half-line ([Web, Theorem 2.5.1]). That is to say, there is $\vartheta \in \mathrm{Gram}_V(f)$ and $0 \neq \rho \in \mathsf{S}_2 V$ with $\vartheta + \lambda\rho \in \mathrm{Gram}_V(f)$ for all $\lambda \geq 0$. But then $\rho \succeq 0$, so that $\rho = \sum_{i=1}^r p_i \otimes p_i$ for some $r \in \mathbb{N}$ and $p_1, \dots, p_r \in V \smallsetminus \{0\}$. Moreover, $\mu(\vartheta) = f = \mu(\vartheta + \lambda\rho) = \mu(\vartheta) + \lambda\mu(\rho)$ for all $\lambda \geq 0$. Therefore, $0 = \mu(\rho) = p_1^2 + \dots + p_r^2$.

The implication "(b) $\Rightarrow$ (c)" is trivial. For "(c) $\Rightarrow$ (a)" we also give a proof by contrapositive. So let $r \in \mathbb{N}$ and $p_1, \dots, p_r \in V \smallsetminus \{0\}$ with $p_1^2 + \dots + p_r^2 = 0$. Then $\rho := \sum_{i=1}^r p_i \otimes p_i \in \mathsf{S}_2^+ V \smallsetminus \{0\}$. Let $f \in \Sigma V^2$ be arbitrary. Then $\mathrm{Gram}_V(f) \neq \varnothing$ and for every $\vartheta \in \mathrm{Gram}_V(f)$ we have $\vartheta + \lambda\rho \in \mathrm{Gram}_V(f)$ for all $\lambda \geq 0$. Thus, $\mathrm{Gram}_V(f)$ is unbounded. □

2.2.10 Remark. Condition (a) from Lemma 2.2.9 is satisfied in the cases we are primarily interested in. First of all, if $A = \mathbb{R}[x_1, \dots, x_m]$ is the polynomial ring then (a) is true for any subspace $V \subseteq A$. Indeed, let $r \in \mathbb{N}$ and $p_1, \dots, p_r \in A$ with $p_1^2 + \dots + p_r^2 = 0$. Then $\mathrm{Newt}(p_i) \subseteq \mathrm{Newt}(0) = \{0\}$ due to Theorem 1.5.3. Therefore, $p_i \in \mathbb{R}$ for all $i = 1, \dots, r$, and consequently $p_1 = \dots = p_r = 0$.

Now let $X \subseteq \mathbb{P}^n$ be an irreducible projective $\mathbb{R}$-variety such that the set $X(\mathbb{R})$ of real points is Zariski-dense. Such a variety is sometimes called *real*. This term seems not to be prevalent in the literature. Rather, some authors refer to projective $\mathbb{R}$-varieties as "real projective varieties" and we shall adopt this when quoting from their work. Be that as it may, the Artin-Lang Theorem tells us that the function field $\mathbb{R}(X)$ of X is a *real field*, i.e., it has an ordering. Hence, $-1 \notin \Sigma\mathbb{R}(X)^2$, or equivalently, if $a_1^2 + \cdots + a_r^2 = 0$ for some $a_1, \ldots, a_r \in \mathbb{R}(X)$, then already $a_1 = \cdots = a_r = 0$. For more details on real algebra we refer the interested reader to the textbook [KS]. Here, we consider the graded ring $A = \mathbb{R}[X] = \mathbb{R}[x_0, \ldots, x_n]/\mathfrak{I}_+(X)$, the homogeneous coordinate ring of X. Let $V = A_d$ be the vector space of degree-d forms for some $d \in \mathbb{N}_0$. If $0 \neq p_1, \ldots, p_r \in A_d$, then $p_1^2 + \cdots + p_r^2 \neq 0$ in A_{2d}. Indeed, A is a domain since X is irreducible, and the function field

$$\mathbb{R}(X) = \left\{ \frac{p}{q} : p,\, q \in \mathbb{R}[X] \text{ homogeneous, } q \neq 0,\, \deg(p) = \deg(q) \right\}$$

is the "homogeneous quotient field" of $\mathbb{R}[X]$. Thus, the claim follows from the aforementioned fact that $\mathbb{R}(X)$ is real.

The previous remark indicates that we will predominantly be working with compact Gram spectrahedra. Therefore, here seems to be a good point to also present a simple non-compact example.

2.2.11 Example (see also Example 5.3.6)**.** Let $A = \mathbb{R}[x,y]/\langle x^2+y^2 \rangle$ and let $V = A_1$. Then $\rho = x \otimes x + y \otimes y \in \mathsf{S}_2^+V$ and $\mu(\rho) = 0$. Thus, $\mathrm{Gram}_V(f)$ is unbounded for every $f \in A_2$. For example, let $f = \overline{x^2} \in A_2$, then $\mathrm{Gram}_V(f) = \{x \otimes x + \lambda\rho : \lambda \geq 0\}$.

2.3. Facial structure of (Gram) spectrahedra

The foundations for the study of the facial structure of spectrahedra were laid by Ramana and Goldman. They showed that all matrices in the relative interior of any face share the same kernel, thereby charaterizing the faces of a spectrahedron by these subspaces ([RG, Section 2.1]). For the analysis of Gram spectrahedra, however, it is advantageous to consider Scheiderer's equivalent characterization using the images of tensors instead (see Proposition 2.3.4 below). We thus continue as in Section 2.2.

Afterwards, we take a glimpse on zero-dimensional faces (extreme points) and on faces of potentially large dimension, which are those of rank $\dim(V) - 1$. For the former we present Pataki's result on possible ranks of extreme points, for the latter we observe how simple methods from linear algebra can be used to get them under better control.

Note that formally our setting is a little more restrictive at first glance, as we always assume that the vector space V is contained inside the algebra A. In retrospect, though, the focus on Gram spectrahedra is no loss of generality since every spectrahedron that does not contain an affine line is a Gram spectrahedron, see Section 5.3.

We begin this section by translating the facts proven at the end of Section 1.6 to our coordinate-free setting.

2.3.1 Lemma (cf. Lemmata 1.6.11 and 1.6.12)**.** *Let* $\vartheta,\, \vartheta' \in \mathsf{S}_2^+V$. *Then* $\mathrm{im}(\vartheta + \vartheta') = \mathrm{im}(\vartheta) + \mathrm{im}(\vartheta')$. *If* $\gamma \in \mathsf{S}_2V$ *is another symmetric tensor with* $\mathrm{im}(\gamma) \subseteq \mathrm{im}(\vartheta)$, *then there is* $\varepsilon > 0$ *with* $\vartheta - \varepsilon\gamma \succeq 0$. □

2.3.2. We fix a spectrahedron $S = L \cap \mathsf{S}_2^+V$ where L is an affine-linear subspace of S_2V. For any linear subspace $U \subseteq V$ we consider the set

$$\mathcal{F}(U) := \{\vartheta \in S : \mathrm{im}(\vartheta) \subseteq U\} \subseteq S.$$

By Corollary 2.2.4, we have $\mathcal{F}(U) = L \cap \mathsf{S}_2^+U$. Moreover, the fact that $\mathrm{im}(\vartheta + \vartheta') = \mathrm{im}(\vartheta) + \mathrm{im}(\vartheta')$ for all $\vartheta,\, \vartheta' \in \mathsf{S}_2^+V$ implies that $\mathcal{F}(U)$ is a face of S.

Conversely, to any nonempty face F of S we associate the subspace

$$\mathcal{U}(F) := \sum_{\vartheta \in F} \operatorname{im}(\vartheta).$$

Trivially, $F \subseteq \mathcal{F}(\mathcal{U}(F))$. We will see below that the opposite inclusion holds true as well, so that we obtain a bijection between nonempty faces of S and certain subspaces of V.

2.3.3 Definition. A linear subspace $U \subseteq V$ is *S-facial*, or a *face subspace* (for the given spectrahedron $S = L \cap \mathsf{S}_2^+ V$), if there exists $\vartheta \in S$ with $U = \operatorname{im}(\vartheta)$.

2.3.4 Proposition ([Sch22, Prop. 2.10])**.** *There is a natural inclusion-preserving bijection between the nonempty faces F of S and the face subspaces $U \subseteq V$ for S, given by $F \mapsto \mathcal{U}(F)$. The inverse is $U \mapsto \mathcal{F}(U)$.*

Proof. Let F be a nonempty face of S and let $U := \mathcal{U}(F)$. Since $\dim(U) < \infty$, there are $k \in \mathbb{N}$ and $\vartheta_1, \dots, \vartheta_k \in F$ such that $U = \sum_{i=1}^k \operatorname{im}(\vartheta_k)$. By Lemma 2.3.1, we have $U = \operatorname{im}(\vartheta)$ for any positive combination $\vartheta = \sum_{i=1}^k a_i \vartheta_i$ ($a_i \in \mathbb{R}_{>0}$). Choosing, for example, $a_1 = \dots = a_k = \frac{1}{k}$ ensures that $\vartheta \in F$. Thus, $U = \operatorname{im}(\vartheta)$ is a face subspace for S, meaning that the suggested map is well-defined.

We have already seen that $F \subseteq \mathcal{F}(\mathcal{U}(F))$. In order to prove the opposite inclusion, we let $\gamma \in \mathcal{F}(U)$. Then γ is psd and $\operatorname{im}(\gamma) \subseteq U = \operatorname{im}(\vartheta)$. According to Lemma 2.3.1, there exists $\varepsilon > 0$ such that $\vartheta - \varepsilon\gamma \succeq 0$. Consequently, also $\vartheta' := (1+\varepsilon)\vartheta - \varepsilon\gamma \succeq 0$. As ϑ' is an affine combination of ϑ and γ, it follows $\vartheta' \in L \cap \mathsf{S}_2^+ V = S$. Moreover,

$$\vartheta = \frac{1}{1+\varepsilon}\,\vartheta' + \frac{\varepsilon}{1+\varepsilon}\,\gamma,$$

so that $\vartheta \in F$ is a nontrivial convex combination of ϑ', $\gamma \in S$. Since F is a face of S, we must have $\gamma \in F$.

It remains to show that $U = \mathcal{U}(\mathcal{F}(U))$ for every S-facial subspace $U \subseteq V$. Choose $\vartheta \in S$ with $U = \operatorname{im}(\vartheta)$. Then $\vartheta \in \mathcal{F}(U)$ and hence $U \subseteq \mathcal{U}(\mathcal{F}(U)) = \sum_{\gamma \in \mathcal{F}(U)} \operatorname{im}(\gamma)$. The opposite inclusion is tautologically true. □

In particular we see that given an S-facial subspace $U \subseteq V$, the relative interior of $\mathcal{F}(U)$ is $\{\vartheta \in S : \operatorname{im}(\vartheta) = U\}$. The equivalent formulation which is obtained starting from a face of S leads to the important notion of ranks of faces:

2.3.5 Definition. Let $F \subseteq S$ be a face. Then, for every $\vartheta \in \operatorname{relint}(F)$, we have $\operatorname{im}(\vartheta) = \mathcal{U}(F)$ and $\operatorname{rk}(\vartheta) = \dim \mathcal{U}(F)$. This number is called the *rank* of F, denoted by $\operatorname{rk}(F)$.

As the bijection from Proposition 2.3.4 is inclusion-preserving, we have $\operatorname{rk}(F') < \operatorname{rk}(F)$ for any proper subface F' of F. We will make extensive use of this fact in the following chapters.

2.3.6 Remark (cf. 2.1.3)**.** Let V again be a finite-dimensional linear subspace of an $\mathbb{R}$-algebra A. Given $f \in A$, we consider the affine-linear subspace $L := \mu^{-1}(f) \cap \mathsf{S}_2 V$ of $\mathsf{S}_2 V$. Then $S = L \cap \mathsf{S}_2^+ V = \operatorname{Gram}_V(f)$. We have seen at the beginning of this section that for any linear subspace $U \subseteq V$,

$$\operatorname{Gram}_U(f) = L \cap \mathsf{S}_2^+ U = \mathcal{F}(U)$$

is a face of $\mathrm{Gram}_V(f)$. Conversely, every (nonempty) face $F \subseteq \mathrm{Gram}_V(f)$ is of the form $\mathrm{Gram}_U(f)$ for some subspace $U \subseteq V$. However, only by restricting ourselves to $\mathrm{Gram}_V(f)$-facial subspaces we get the nice bijection from Proposition 2.3.4 that enables us to study ranks (and also dimensions, as we will see) of faces by means of these subspaces.

We revisit Example 2.1.3. Let $(p_1, p_2, p_3, p_4) := (q_1, q_2, q_1 - 8q_4, q_2 + 8q_3)$. Then $f = p_1^2 + p_2^2 = p_3^2 + p_4^2$. Thus, the symmetric tensor

$$\vartheta := \frac{1}{2} \sum_{i=1}^{4} p_i \otimes p_i \in \mathsf{S}_2^+ U$$

satisfies $\mu(\vartheta) = f$. Moreover, we have $\mathrm{im}(\vartheta) = U$ since $p_1, \ldots, p_4$ constitute a basis of U. This shows that U is actually a face subspace for $\mathrm{Gram}_V(f)$. The face $F = \mathcal{F}(U)$ has rank 4 and contains the two rank-2 extreme points $p_1 \otimes p_1 + p_2 \otimes p_2$ and $p_3 \otimes p_3 + p_4 \otimes p_4$. According to Corollary 2.3.10 below, we have $\dim(F) = 2$.

For convenience, we say that a subspace $U \subseteq V$ is a face subspace for f rather than for $\mathrm{Gram}_V(f)$ if we work in a setting where V is fixed. We give another characterization of face subspaces that can be seen as a generalization of [CLR, Proposition 5.5].

2.3.7 Proposition. *Let $f \in \Sigma V^2$ and let $U \subseteq V$ be a linear subspace. Then U is a face subspace for the spectrahedron $\mathrm{Gram}_V(f)$ if and only if $f \in \mathrm{int}(\Sigma U^2) \subseteq UU$.*

In particular ($U := V$), the Gram spectrahedron $\mathrm{Gram}_V(f)$ contains a positive definite tensor if and only if $f \in \mathrm{int}(\Sigma V^2)$.

Proof. Consider the surjective linear map $\mu \colon \mathsf{S}_2 U \to UU$. It maps the cone $\mathsf{S}_2^+ U$ onto the sums-of-squares cone ΣU^2 in UU. By Lemma 1.2.3, we have $f \in \mathrm{int}(\Sigma U^2)$ if and only if there is a tensor $\vartheta \in \mathsf{S}_2^+ U$ whose rank equals the dimension of U, and such that $\mu(\vartheta) = f$. The condition on the rank is equivalent to $\mathrm{im}(\vartheta) = U$. Thus, such ϑ exists if and only if U is a face subspace for f.

For the last statement, note that V itself is a face subspace for $\mathrm{Gram}_V(f)$ if and only if f has a psd Gram tensor of (full) rank $\dim(V)$. □

We now work towards a dimension formula for faces of Gram spectrahedra. Like the other statements about general spectrahedra, the following lemma is originally due to Ramana and Goldman [RG].

2.3.8 Lemma. *Let $S = L \cap \mathsf{S}_2^+ V$, where L is an affine-linear subspace of $\mathsf{S}_2 V$. If F is a nonempty face of S and $U = \mathcal{U}(F)$ the associated face subspace, then $\mathrm{aff}(F) = L \cap \mathsf{S}_2 U$.*

Proof. We have seen before that $F = L \cap \mathsf{S}_2^+ U$. Therefore, $\mathrm{aff}(F) \subseteq L \cap \mathsf{S}_2 U$ is clear. For the other inclusion let $\gamma \in L \cap \mathsf{S}_2 U$. We fix $\vartheta \in \mathrm{relint}(F)$ so that $U = \mathrm{im}(\vartheta)$. Since $\mathrm{im}(\vartheta - \gamma) \subseteq U$, we can use Lemma 2.3.1 in order to find $\varepsilon > 0$ such that $\vartheta' := \vartheta - \varepsilon(\vartheta - \gamma) \succeq 0$. Now $\vartheta' = (1 - \varepsilon)\vartheta + \varepsilon\gamma \in L \cap \mathsf{S}_2^+ U = F$. Consequently, $\gamma = \frac{1}{\varepsilon}\vartheta' + \frac{\varepsilon - 1}{\varepsilon}\vartheta \in \mathrm{aff}(F)$. □

The preceding lemma allows us to compute dimensions of faces in Gram spectrahedra by means of linear algebra.

2.3.9 Proposition ([Sch22, Proposition 3.7]). *Let $f \in \Sigma V^2$. For $U \subseteq V$ a face subspace for f, the face $\mathcal{F}(U)$ of* $\mathrm{Gram}_V(f)$ *has dimension*

$$\dim \mathcal{F}(U) = \binom{\dim(U)+1}{2} - \dim(UU).$$

In particular, if $f \in \mathrm{int}(\Sigma V^2)$, then

$$\dim \mathrm{Gram}_V(f) = \binom{\dim(V)+1}{2} - \dim(VV).$$

Proof. According to Lemma 2.3.8, $\dim \mathcal{F}(U)$ is the dimension of the affine space $\mu^{-1}(f) \cap \mathsf{S}_2 U$. Thus, $\dim \mathcal{F}(U) = \dim(W)$ where W is the kernel of the multiplication map $\mu|_{\mathsf{S}_2 U} \colon \mathsf{S}_2 U \to UU$. Since this map is surjective, we obtain $\dim(W) = \dim(\mathsf{S}_2 U) - \dim(UU)$. Finally, we note that $\dim(\mathsf{S}_2 U) = \binom{\dim(U)+1}{2}$.

The additional statement follows from the fact that V is a face subspace for f if and only if $f \in \mathrm{int}(\Sigma V^2)$ (see Proposition 2.3.7). □

For $\Theta \subseteq \mathrm{Gram}_V(f)$ we denote by $\mathrm{suppface}(\Theta)$ the smallest face of $\mathrm{Gram}_V(f)$ that contains Θ. This face is called the *supporting face* of Θ. If $\Theta = \{\vartheta\}$, then $F := \mathrm{suppface}(\vartheta)$ is the unique face of $\mathrm{Gram}_V(f)$ that contains ϑ in its relative interior. Recall that we have $F = \mathcal{F}(\mathrm{im}(\vartheta))$.

We have seen in the proof of Proposition 2.3.9 that $\dim(F)$ is the dimension of the kernel of the surjective map $\mathsf{S}_2 U \to UU$ where $U = \mathrm{im}(\vartheta)$. We get:

2.3.10 Corollary. *Let $f = \sum_{i=1}^r p_i^2$ with $p_1, \ldots, p_r \in V$ linearly independent, let $\vartheta = \sum_{i=1}^r p_i \otimes p_i$ be the corresponding Gram tensor of f. The dimension of* $\mathrm{suppface}(\vartheta) \subseteq \mathrm{Gram}_V(f)$ *equals the number of independent linear relations between the $\binom{r+1}{2}$ products $p_i p_j$ $(1 \le i \le j \le r)$.*

Especially, ϑ is an extreme point of $\mathrm{Gram}_V(f)$ *if and only if these products are linearly independent. In the latter case, we say that the sequence $p_1, \ldots, p_r$ is* quadratically independent. □

2.3.11 Remark. Note that the dimension of the supporting face in $\mathrm{Gram}_V(f)$ of a tensor $\vartheta = \sum_{i=1}^r p_i \otimes p_i$ depends only on the linear subspace $U := \mathrm{span}(p_1, \ldots, p_r)$, but not on $f = \sum_{i=1}^r p_i^2$. It is therefore a main objective to develop a better understanding of $\dim(UU)$ in terms of $\dim(U)$, where U ranges over the linear subspaces of V.

2.3.12 Remark. Assume that $p_1, \ldots, p_r \in V$ are linearly independent. The subspace $U = \mathrm{span}(p_1, \ldots, p_r)$ can also be recovered from a Gram matrix G associated to the representation $f = p_1^2 + \cdots + p_r^2$ without computing a factorization of G. Indeed, using the notation from 2.1.1, we have $G = C^T C$ for a matrix $C \in \mathrm{M}_{r \times n}(\mathbb{R})$ such that $\mathrm{rk}(C) = r$ and $(p_1, \ldots, p_r) = \mathfrak{m}^T C^T$. As C has (full) rank $r = \dim(U)$, the entries of $(\mathfrak{m}^T C^T) C$ generate U. Consequently,

$$U = \mathrm{span}(\mathfrak{m}^T G e_j : j = 1, \ldots, n) = \mathrm{span}(e_j^T G \mathfrak{m} : j = 1, \ldots, n),$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. Thus, we will sometimes argue using matrices when it is more convenient.

Pataki inequalities. There is often a special interest in extreme points of spectrahedra. From an algebraic perspective, the Krein-Milman Theorem (or Minkowski's Theorem, cf. [Bar, Section 3.3 in Chapter II]) implies that a compact spectrahedron

S is the convex hull of its extreme points. Hence, when minimizing a linear form over S, there is always an extreme point among the set of optimal solutions. For more information on topics related to optimization we refer to Section 5.2 and to Section 5.1, where we study normal cones and vertices of Gram spectrahedra.

The following result is due to Pataki and can be found in [Pat, Corollary 3.3.4] (see also [NRS, Proposition 5]). It comes from the semidefinite programming literature and is thus originally formulated in terms of optimal solutions to SDPs and their duals. We give a formulation which better fits our context. It is taken from [CPSV, Proposition 3.1].

2.3.13 Proposition. *Let* $\dim(V) = n$ *and let* $L \subseteq \mathsf{S}_2V$ *be an affine-linear space of dimension* m. *The rank* r *of an extreme point of the spectrahedron* $L \cap \mathsf{S}_2^+V$ *satisfies*

$$\binom{r+1}{2} + m \leq \binom{n+1}{2}.$$

Furthermore, if the affine-linear space L *is chosen generically among all affine subspaces of dimension* m, *the smallest rank* r *of any point* $\vartheta \in L \cap \mathsf{S}_2^+V$ *also satisfies*

$$m \geq \binom{n-r+1}{2}.$$

The inequalities in Proposition 2.3.13 define an interval of possible ranks r for the extreme points of a general spectrahedron. The *Pataki interval* is the range of integers r satisfying these inequalities.

2.3.14 Remark. In the situation of Gram spectrahedra, we have $L = \mu^{-1}(f) \cap \mathsf{S}_2V$ for some $f \in A$. As $m = \dim(L) = \dim(\mathsf{S}_2V) - \dim(VV)$, the upper bound in Proposition 2.3.13 says

$$\binom{r+1}{2} \leq \dim(VV).$$

This can also be seen using the theory developed in this section: Indeed, if $\binom{r+1}{2} >$ $\dim(VV)$, then a sequence $p_1, \dots, p_r \in V$ cannot be quadratically independent.

That, for generic f, also the lower bound

$$\binom{n+1}{2} - \dim(VV) \geq \binom{n-r+1}{2}$$

for r has to be satisfied can be seen using a dimension count (see [CPSV, Proposition 3.2]).

A fact from linear algebra. Given $f \in \operatorname{int}(\Sigma V^2)$, any point ϑ in the (relative) interior of $\operatorname{Gram}_V(f)$ has $\operatorname{rk}(\vartheta) = \dim(V)$ (see Proposition 2.3.7). Thus, the maximum rank on the relative boundary of $\operatorname{Gram}_V(f)$ is $\dim(V) - 1$. When investigating faces F of this rank we can try to choose a particular linear basis of $U = \mathcal{U}(F)$ that simplifies the analysis of $\dim(UU)$. For this purpose we remark the following fact which is rather obvious but incredibly useful. For applications we think of $f \in \mathbb{R}[x_1, \dots, x_m]$ and $V = \mathbb{R}[x_1, \dots, x_m]_P$ with a lattice polytope $P \subseteq \mathbb{R}^m$ such that $\operatorname{Newt}(f) \subseteq 2P$.

2.3.15 Remark. Consider the polynomial ring $\mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \dots, x_m]$ together with a monomial order $\preceq$. Let $k \in \mathbb{N}$ and let $V \subseteq \mathbb{R}[\underline{x}]$ be a k-dimensional monomial subspace, that is a linear subspace generated by monomials $\mathfrak{m}_1 \succ \mathfrak{m}_2 \succ \cdots \succ \mathfrak{m}_k$ from $\mathbb{R}[\underline{x}]$. Assume that $U \subseteq V$ is a linear subspace with $\operatorname{codim}_V(U) = 1$. Then there exists an $i_0 \in \{1, \dots, k\}$ and constants $a_i \in \mathbb{R}$ $(i = 1, \dots, k,\ i \neq i_0)$ with $a_i = 0$ for $i > i_0$ and such that

$$\mathcal{B} := (\mathfrak{m}_i - a_i \mathfrak{m}_{i_0} : i \in \{1, \dots, k\} \smallsetminus \{i_0\})$$

is a basis of U. Indeed, start with any basis $\mathcal{C}$ of U and consider the matrix A of size $(k-1) \times k$ whose rows consist of the coefficients of the elements in $\mathcal{C}$ with respect to the basis $(\mathfrak{m}_1, \dots, \mathfrak{m}_k)$ of V. Then $\operatorname{rk}(A) = k - 1$ and transforming A to its reduced row echelon form gives a basis of the desired form.

2.4. Hermitian Gram spectrahedra

Another possibility to certify nonnegativity of a polynomial $f \in \mathbb{R}[x_1, \dots, x_m]$ is to write it as a Hermitian sum of squares, that is to say $f = p_1\overline{p_1} + \cdots + p_r\overline{p_r}$ where $p_1, \dots, p_r \in \mathbb{C}[x_1, \dots, x_m]$. Although f is a real sum of squares if and only if it is a Hermitian sum of squares, the Hermitian point of view allows for insights that might otherwise have remained hidden. Chua, Plaumann, Sinn and Vinzant also discuss the Hermitian analog to symmetric Gram spectrahedra and provide a first example showing that this theory can be used for a better understanding of ordinary (real symmetric) Gram spectrahedra in the case of binary forms (see [CPSV, Section 5]). We will present another remarkable application on polyhedral faces when we study Gram spectrahedra of binary forms in Chapter 3.

Hermitian Gram spectrahedra are interesting objects of convex algebraic geometry on their own terms. However, we are mainly interested in the real symmetric setting. Therefore, we decided not to present the Hermitian setting as a general framework comprising the real symmetric one, but rather as a useful extension to the latter. A drawback to this perception is that some statements in this section are only slightly adjusted repetitions of those in Section 2.3.

Similar to Scheiderer's method presented in Section 2.2, we pursue a coordinate-free approach. We begin by introducing the real space of Hermitian tensors. A little preparatory work is necessary. For instance, we need the complexification of a real vector space, a special case of extension of scalars which is described in many undergraduate textbooks, see for example [Hal, §77].

2.4.1 Definition. Let V and V' be complex vector spaces. A mapping $\phi\colon V \to V'$ is said to be *antilinear* if

$$\phi(v + w) = \phi(v) + \phi(w) \quad \text{and} \quad \phi(\lambda v) = \overline{\lambda}\phi(v)$$

for all v, $w \in V$ and all $\lambda \in \mathbb{C}$. An antilinear map $\phi\colon V \to V$ is called *antilinear involution* if $\phi \circ \phi = \mathrm{id}_V$.

2.4.2. Let V be a vector space over $\mathbb{R}$. The *complexification* $V_{\mathbb{C}}$ of V is the tensor product of V with the complex numbers, i.e., $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. Then $V_{\mathbb{C}}$ is a complex vector space and every $v \in V_{\mathbb{C}}$ can be written uniquely in the form

$$v = v_1 \otimes 1 + v_2 \otimes i$$

where v_1, $v_2 \in V$. We will drop the tensor product symbol and simply write $v = v_1 + iv_2$. We call v_1 the *real part* and v_2 the *imaginary part* of v. Multiplication by the complex number $a + ib$ $(a, b \in \mathbb{R})$ is given by the usual rule

$$(a + ib)(v_1 + iv_2) = (av_1 - bv_2) + i(bv_1 + av_2).$$

On $V_{\mathbb{C}}$ we have a natural antilinear involution $\phi\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$ given by $\phi(v_1 + iv_2) = v_1 - iv_2$. We also write $\overline{v} := \phi(v)$ for $v \in V_{\mathbb{C}}$.

For any linear subspace $U \subseteq V_{\mathbb{C}}$, we define the *complex conjugate* of U to be the subspace $\overline{U} := \phi(U) \subseteq V$. We consider the tensor product $U \otimes_{\mathbb{C}} \overline{U}$ with the antilinear involution

$$U \otimes_{\mathbb{C}} \overline{U} \to U \otimes_{\mathbb{C}} \overline{U}, \quad v \otimes \overline{w} \mapsto w \otimes \overline{v} \quad (v,\, w \in U).$$

The fixed locus of this map is the real subspace of *Hermitian tensors* in $U \otimes_{\mathbb{C}} \overline{U}$ which we will denote by $\mathsf{H}_2 U$. In order to avoid confusion, let us note that for $v \in U$, the Hermitian tensors $v \otimes \overline{v} \in \mathsf{H}_2 U$ and $\overline{v} \otimes v \in \mathsf{H}_2 \overline{U}$ are different objects in general.

2.4.3. A Hermitian tensor $\vartheta = \sum_{j=1}^{r} v_j \otimes \overline{w_j} \in \mathsf{H}_2 U$ $(v_j,\, w_j \in U)$ can be identified with a Hermitian sesquilinear form $b_\vartheta \colon U^\vee \times U^\vee \to \mathbb{C}$ given by $b_\vartheta(\lambda, \mu) = \sum_j \lambda(v_j)\overline{\mu(w_j)}$. Here, $U^\vee = \mathrm{Hom}(U, \mathbb{C})$ denotes the dual space of U. By the choice of a basis, ϑ resp. b_ϑ can be identified with a Hermitian matrix A. We can diagonalize ϑ, i.e., we can write $\vartheta = \sum_{j=1}^{r} \varepsilon_j (v_j \otimes \overline{v_j})$ with $r \geq 0$, $\varepsilon_j = \pm 1$ and $v_1, \ldots, v_r \in U$ linearly independent. In this case, the *rank* of ϑ is $\mathrm{rk}(\vartheta) := r$, and ϑ is *positive semidefinite* if all $\varepsilon_j = 1$, or equivalently, $A \succeq 0$. The *image* of ϑ is $\mathrm{im}(\vartheta) := \mathrm{span}(v_1, \ldots, v_r)$. Note that if $u_1, \ldots, u_n$ $(n = \dim_{\mathbb{C}}(U))$ is a basis of U and $A \in \mathrm{M}_n(\mathbb{C})$ is the Hermitian $n \times n$-matrix associated to ϑ with respect to this basis, then

$$\mathrm{im}(\vartheta) = \mathrm{span}_{\mathbb{C}} \left(\sum_{k=1}^{n} (Ae_j)_k u_k : j = 1, \ldots, n \right) \subseteq U.$$

2.4.4. Let A be an $\mathbb{R}$-algebra and let $V \subseteq A_{\mathbb{C}}$ be a complex subspace of finite dimension. By $V\overline{V}$ we denote the subspace of $A_{\mathbb{C}}$ which is $\mathbb{C}$-linearly generated by the products $p\overline{q}$ $(p,\, q \in V)$. The multiplication map $\mu \colon V \times \overline{V} \to V\overline{V}$, $(p, \overline{q}) \mapsto p\overline{q}$ is $\mathbb{R}$-bilinear and induces an $\mathbb{R}$-linear map $\mathsf{H}_2 V \to V\overline{V}$. Given $f \in A$, the Hermitian tensors $\vartheta \in \mathsf{H}_2 V$ with $\mu(\vartheta) = f$ are called the *Hermitian Gram tensors* of f, relative to V.

2.4.5 Definition. In the situation of 2.4.4, we define the *Hermitian Gram spectrahedron* of f, relative to V, to be the set of all positive semidefinite Hermitian Gram tensors of f in $\mathsf{H}_2 V$. That is to say,

$$\mathcal{H}_V^+(f) := \mathsf{H}_2^+ V \cap \mu^{-1}(f).$$

2.4.6 Remark (cf. Remark 2.2.7)**.** Let $B \in \mathrm{M}_n(\mathbb{C})$ be Hermitian. We can write $B = B_0 + iB_1$ where B_0 is a real symmetric matrix and B_1 is a real skew-symmetric matrix. Then B is positive semidefinite if and only if the matrix

$$\begin{pmatrix} B_0 & B_1 \\ -B_1 & B_0 \end{pmatrix} \in \mathrm{Sym}_{2n}(\mathbb{R})$$

is positive semidefinite (cf. [RG, Section 1.4]). We could thus consider Hermitian spectrahedra as ordinary real symmetric ones. This, however, is not the point of view we want to take.

Instead, let us mention that $\mathcal{H}_V^+(f)$ parametrizes the Hermitian sums-of-squares representations $f = \sum_{j=1}^{r} p_j \overline{p_j}$ with $p_j \in V$ $(j = 1, \ldots, r)$, up to the equivalence induced by the action of unitary matrices. Using [HJ, Theorem 7.3.11] again, this can be seen in the same way as in the real case.

Having defined the notion of Hermitian Gram spectrahedra, we turn to their facial structure. At this point it is useful that we have formulated some lemmata in Section 1.6 for both real symmetric and complex Hermitian matrices.

2.4.7 Lemma (cf. Lemma 2.3.1). *Given ϑ, $\vartheta' \in \mathsf{H}_2^+V$, it holds* $\operatorname{im}(\vartheta+\vartheta') = \operatorname{im}(\vartheta) + \operatorname{im}(\vartheta')$. *If $\gamma \in \mathsf{H}_2V$ is another symmetric tensor with* $\operatorname{im}(\gamma) \subseteq \operatorname{im}(\vartheta)$, *then there is $\varepsilon > 0$ with $\vartheta - \varepsilon\gamma \succeq 0$.* □

We can adopt the definition of face subspaces from the real symmetric case to obtain a bijection between the nonempty faces of $\mathcal{H}_V^+(f)$ and the face subspaces $U \subseteq V$ for the spectrahedron $\mathcal{H}_V^+(f)$.

2.4.8 Definition. A linear subspace U of V is $\mathcal{H}_V^+(f)$-*facial*, or a *face subspace* (for the given spectrahedron $\mathcal{H}_V^+(f)$), if there exists $\vartheta \in \mathcal{H}_V^+(f)$ with $U = \operatorname{im}(\vartheta)$.

2.4.9 Remark. Given $\vartheta \in \mathsf{H}_2V$ and a linear subspace $U \subseteq V$, we have $\operatorname{im}(\vartheta) \subseteq U$ if and only if $\vartheta \in \mathsf{H}_2U$ (cf. Corollary 2.2.4). Consider 2.3.2, Proposition 2.3.4 and Lemma 2.3.8, which contain statements about faces in symmetric spectrahedra, and their respective proofs. Replacing S_2 by H_2 (and consequently also S_2^+ by H_2^+) and using Lemma 2.4.7 for the argumentation, we can adopt the aforementioned statements and proofs from the real symmetric case almost word-for-word. We thus save ourselves the trouble of essentially copying them, and instead proceed to the dimension formula for faces.

2.4.10 Proposition. *For $U \subseteq V$ a face subspace for f, the face $\mathcal{F}(U)$ of $\mathcal{H}_V^+(f)$ has dimension*

$$\dim \mathcal{F}(U) = \dim_{\mathbb{C}}(U)^2 - \dim_{\mathbb{C}}(U\overline{U}).$$

Proof. The dimension of the convex set $\mathcal{F}(U)$ is the dimension of the (real) affine space $\mu^{-1}(f) \cap \mathsf{H}_2U$. Therefore, $\dim \mathcal{F}(U) = \dim_{\mathbb{R}}(W)$ where W is the kernel of the $\mathbb{R}$-linear map $\mu\colon \mathsf{H}_2U \to U\overline{U}$. The image of μ is $\operatorname{im}(\mu) = \operatorname{span}_{\mathbb{R}}(p\overline{q} : p,\, q \in U)$. Hence, $\dim_{\mathbb{R}}(\operatorname{im}(\mu)) = \dim_{\mathbb{C}}(\operatorname{im}(\mu)_{\mathbb{C}}) = \dim_{\mathbb{C}}(U\overline{U})$. So using that $\dim_{\mathbb{R}}(\mathsf{H}_2U) = \dim_{\mathbb{C}}(U)^2$, the claim follows from the rank-nullity theorem. □

2.4.11 Corollary. *Let $f = \sum_{j=1}^r p_j\overline{p_j}$ with $p_1, \dots, p_r \in V$ linearly independent, let $\vartheta = \sum_{j=1}^r p_j \otimes \overline{p_j}$ be the corresponding Hermitian Gram tensor of f. The dimension of the supporting face of ϑ in $\mathcal{H}_V^+(f)$ equals the number of independent linear relations between the r^2 products $p_j\overline{p_k}$ $(1 \le j, k \le r)$.* □

The concepts of Hermitian Gram tensors and facial subspaces can be used to give a straightforward proof of a fact presented in [CPSV].

2.4.12 Proposition (cf. [CPSV, Prop. 5.6]). *Let $P \subseteq \mathbb{R}^m$ be a polytope with vertices in $\mathbb{N}_0^m$. If $f \in \mathbb{R}[\underline{x}]_{2P}$ factors as $f = g\overline{g} \cdot h$, where $g \in \mathbb{C}[\underline{x}]$ and $h \in \mathbb{R}[\underline{x}]$, then the Hermitian Gram spectrahedron of h is linearly isomorphic to a face of $\mathcal{H}_{\mathbb{C}[\underline{x}]_P}^+(f)$.*

Proof. At first, we follow the argumentation in [CPSV]: Without loss of generality, we can assume that $2P$ equals the Newton polytope of f. Then $2P$ is the Minkowski sum of the polytopes $2\operatorname{Newt}(g)$ and $\operatorname{Newt}(h)$. Therefore, we can write $\operatorname{Newt}(h)$ as $2Q$ for some $Q \subseteq \mathbb{R}^m$ with integer vertices. We see that P is the Minkowski sum $\operatorname{Newt}(g) + Q$.

For the rest of the proof, we will argue using Hermitian Gram tensors instead of matrices. We choose some $\vartheta \in \operatorname{relint}(\mathcal{H}^+_{\mathbb{C}[\underline{x}]_Q}(h))$, for instance $\vartheta = \sum_{j=1}^r p_j \otimes \overline{p_j}$, with $p_1, \dots, p_r \in \mathbb{C}[\underline{x}]_Q$ linearly independent and $\mu(\vartheta) = h$. Let $U' = \operatorname{im}(\vartheta) = \operatorname{span}_{\mathbb{C}}(p_1, \dots, p_r) \subseteq \mathbb{C}[\underline{x}]_Q$ and consider $U := gU' \subseteq \mathbb{C}[\underline{x}]_P$. Then $(gp_1, \dots, gp_r)$ is a basis of $gU' = U$ and

$$\mu\left(\sum_{j=1}^r gp_j \otimes \overline{gp_j}\right) = g\overline{g} \cdot \sum_{j=1}^r p_j\overline{p_j} = g\overline{g} \cdot h = f.$$

Therefore, $U \subseteq \mathbb{C}[\underline{x}]_P$ is a facial subspace for $\mathcal{H}^+_{\mathbb{C}[\underline{x}]_P}(f)$ and the face $\mathcal{F}(U)$ of $\mathcal{H}^+_{\mathbb{C}[\underline{x}]_P}(f)$ is linearly isomorphic to $\mathcal{H}^+_{\mathbb{C}[\underline{x}]_Q}(h)$. □

The consequences in the case of binary forms are discussed in Section 3.2.

2.5. Real symmetric faces of Hermitian Gram spectrahedra

We have introduced (a coordinate-free version of) the customary notion of (real symmetric) Gram spectrahedra as well as an analogous concept in the complex Hermitian setting. Having studied their respective facial structures somewhat separately, this section is devoted to exploring first connections between the real symmetric and the Hermitian world.

Trivially, any real symmetric matrix can be considered as complex Hermitian. Vice versa, we can decompose a Hermitian matrix as $A_0 + iA_1$, where A_0 is a real symmetric and A_1 a real skew-symmetric matrix, and focus on its “real part” A_0. In this section, we study the analog for symmetric and Hermitian tensors. We first use it to re-prove the connection between the shortest length of Hermitian and real sums-of-squares representations. Afterwards, we establish a bijection between the faces of the symmetric Gram spectrahedron and certain faces of the Hermitian Gram spectrahedron which we propose to call *real symmetric* due to their nature.

2.5.1. Let A be an $\mathbb{R}$-algebra, let $V \subseteq A$ be a linear subspace of finite dimension and let $V_{\mathbb{C}} \subseteq A_{\mathbb{C}}$ be its complexification. Note that then $\overline{V_{\mathbb{C}}} = V_{\mathbb{C}}$ since complex conjugation is an antilinear involution on $V_{\mathbb{C}}$. We can consider $\mathsf{S}_2 V$ as a subspace of $\mathsf{H}_2 V_{\mathbb{C}}$. Clearly, every symmetric tensor $\vartheta = \sum_{j=1}^r p_j \otimes p_j$ with linearly independent $p_1, \dots, p_r \in V$ is also Hermitian since $\overline{p_j} = p_j$.

Here is a retraction of the embedding $\mathsf{S}_2 V \hookrightarrow \mathsf{H}_2 V_{\mathbb{C}}$.

2.5.2 Lemma. *There is a well-defined $\mathbb{R}$-linear surjective map $\psi \colon \mathsf{H}_2 V_{\mathbb{C}} \to \mathsf{S}_2 V$, $\vartheta \mapsto \frac{1}{2}(\vartheta + \overline{\vartheta})$. In particular, $\psi|_{\mathsf{S}_2 V} = \mathrm{id}_{\mathsf{S}_2 V}$.*

Proof. This is clear for matrices. Nevertheless, we give a proof in our setting. For $\vartheta \in \mathsf{H}_2 V_{\mathbb{C}}$, a priori $\frac{1}{2}(\vartheta + \overline{\vartheta}) \in V_{\mathbb{C}} \otimes V_{\mathbb{C}}$ so that ψ can be defined as a map whose codomain is the latter space. Obviously, ψ is $\mathbb{R}$-linear and $\mathsf{S}_2 V \subseteq \operatorname{im}(\psi)$. We have to show that actually $\frac{1}{2}(\vartheta + \overline{\vartheta}) \in \mathsf{S}_2 V$. Diagonalizing ϑ as in 2.4.3 and using the $\mathbb{R}$-linearity of ψ, it suffices to consider $\vartheta = p \otimes \overline{p}$ for $p \in V_{\mathbb{C}}$. Write $p = g + ih$ with $g, h \in V$, that is to say $g = \operatorname{Re}(p)$, $h = \operatorname{Im}(p)$. Then

$$\begin{aligned}\frac{1}{2}(\vartheta + \overline{\vartheta}) &= \frac{1}{2}\big((g + ih) \otimes (g - ih) + (g - ih) \otimes (g + ih)\big)\\ &= \frac{1}{2}\big(2(g \otimes g) - 2((ih) \otimes (ih))\big)\\ &= \operatorname{Re}(p) \otimes \operatorname{Re}(p) + \operatorname{Im}(p) \otimes \operatorname{Im}(p) \in \mathsf{S}_2 V.\end{aligned}$$ □

2.5.3 Corollary. *For every* $\vartheta \in \mathsf{H}_2^+ V_{\mathbb{C}}$*, we have* $\vartheta_0 := \frac{1}{2}(\vartheta + \overline{\vartheta}) \in \mathsf{S}_2^+ V$ *and*

$$\operatorname{rk}(\vartheta) \leq \operatorname{rk}(\vartheta_0) \leq 2\operatorname{rk}(\vartheta).$$

Proof. If ϑ is positive semidefinite, so is $\overline{\vartheta}$ and hence also their convex combination ϑ_0. Moreover, $\operatorname{im}(\vartheta_0) = \operatorname{im}(\vartheta + \overline{\vartheta}) = \operatorname{im}(\vartheta) + \operatorname{im}(\overline{\vartheta})$ (cf. Lemma 1.6.11) which implies the claim since $\operatorname{rk}(\vartheta) = \operatorname{rk}(\overline{\vartheta})$. □

2.5.4. Let $f \in A$ and let $\vartheta \in \mathsf{H}_2 V_{\mathbb{C}}$. Of course, ϑ is a Hermitian Gram tensor of f (relative to $V_{\mathbb{C}}$) if and only if $\overline{\vartheta}$ is such. Consequently, $\vartheta \in \mathcal{H}^+_{V_{\mathbb{C}}}(f)$ if and only if $\vartheta_0 = \frac{1}{2}(\vartheta + \overline{\vartheta}) \in \operatorname{Gram}_V(f)$.

2.5.5 Lemma. *Given* $\vartheta_0 \in \operatorname{Gram}_V(f)$ *of rank* r*, there exists* $\vartheta \in \psi^{-1}(\vartheta_0) \cap \mathcal{H}^+_{V_{\mathbb{C}}}(f)$ *with* $\operatorname{rk}(\vartheta) \leq \lceil \frac{r}{2} \rceil$.

Proof. Write $\vartheta_0 = \sum_{k=1}^r p_k \otimes p_k$ with $p_1, \dots, p_r \in V$ linearly independent. Consider

$$\vartheta := \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor} (p_{2k-1} + ip_{2k}) \otimes (p_{2k-1} - ip_{2k}) + \delta_{\text{odd}} \cdot p_r \otimes p_r,$$

where δ_{odd} is 1 if r is odd and 0 otherwise. This is a psd Hermitian Gram tensor of f, relative to $V_{\mathbb{C}}$, of rank at most $\lceil \frac{r}{2} \rceil$. By the proof of Lemma 2.5.2, $\psi(\vartheta) = \vartheta_0$. □

Particularly interesting are the tensors in $\operatorname{Gram}_V(f)$ and $\mathcal{H}^+_{V_{\mathbb{C}}}(f)$ of minimum rank because they give the shortest representations of f as a sum of squares or a Hermitian sum of squares, respectively. This is formalized using the notion of length.

2.5.6 Definition. Let A be an $\mathbb{R}$-algebra and let $f \in A$.

(a) The *length* of an sos representation $f = p_1^2 + \dots + p_r^2$ is r, the number of summands.

Let $V \subseteq A$ be a subspace of finite dimension.

(b) We say that $f = p_1^2 + \dots + p_r^2$ is an sos representation of f *over* V if $p_1, \dots, p_r \in V$.
(c) The *(real) length* (or *sum-of-squares length*) of f *relative to* V is the shortest length of any sos representation of f over V. If $f \notin \Sigma V^2$, its length relative to V is defined to be infinity.

2.5.7 Remark. (i) Every tensor in $\operatorname{Gram}_V(f)$ of rank r gives an sos representation of f (over V) of length r. Conversely, an sos representation $f = p_1^2 + \dots + p_r^2$ over V leads to the positive semidefinite Gram tensor $p_1 \otimes p_1 + \dots + p_r \otimes p_r \in \mathsf{S}_2^+ V$ of rank at most r. Therefore, the length of f relative to V actually equals the minimum rank in $\operatorname{Gram}_V(f)$.

(ii) If $f \in \Sigma V^2$ and $U \subseteq V$ is another subspace, then the length of f relative to U is finite if and only if U contains a face subspace for the spectrahedron $\operatorname{Gram}_V(f)$.

(iii) Given a polytope $P \subseteq \mathbb{R}^m$ with vertices in $\mathbb{N}_0^m$ and a polynomial $f \in \mathbb{R}[\underline{x}]_{2P}$ with Newton polytope contained in $2P$, it is customary to define the *length* of f to be its length relative to $V = \mathbb{R}[\underline{x}]_P$ (cf. Theorem 1.5.3).

2.5.8 Definition. Let A be an $\mathbb{R}$-algebra, let $f \in A$ and let $V \subseteq A_{\mathbb{C}}$ be any complex subspace of finite dimension. The *Hermitian (sum-of-squares) length* of f *relative to* V is the smallest r for which $f = p_1\overline{p_1} + \dots + p_r\overline{p_r}$ with $p_1, \dots, p_r \in V$, or equivalently, the minimum rank in $\mathcal{H}^+_V(f)$.

2.5.9 Proposition (see [CPSV, Proposition 5.3]). *Let $V \subseteq A$ be a subspace of finite dimension and let f in ΣV^2. If the real length of f relative to V is r and its Hermitian length relative to $V_{\mathbb{C}}$ is s, then $\lceil \frac{r}{2} \rceil = s$ (and $r \in \{2s-1, 2s\}$).*

Proof. Let r be the real length of f relative to V and let s be the Hermitian length of f relative to $V_{\mathbb{C}}$. Given $\vartheta \in \mathcal{H}^+_{V_{\mathbb{C}}}(f)$ of rank s, the point $\psi(\vartheta) \in \mathrm{Gram}_V(f)$ has rank at most $2s$ (Corollary 2.5.3). Thus, $r \leq 2s$, that is $\lceil \frac{r}{2} \rceil \leq s$.

For the reverse inequality, take $\vartheta_0 \in \mathrm{Gram}_V(f)$ of rank r. By Lemma 2.5.5, there is $\vartheta \in \psi^{-1}(\vartheta_0) \cap \mathcal{H}^+_{V_{\mathbb{C}}}(f)$ whose rank is at most $\lceil \frac{r}{2} \rceil$. Therefore, $s \leq \lceil \frac{r}{2} \rceil$. □

For the rest of this section we let $f \in A$ and consider the Gram spectrahedron of f relative to a finite-dimensional subspace $V \subseteq A$ as well as its Hermitian Gram spectrahedron relative to $V_{\mathbb{C}} \subseteq A_{\mathbb{C}}$. Via the map ψ, any face of $\mathrm{Gram}_V(f)$ induces a face of $\mathcal{H}^+_{V_{\mathbb{C}}}(f)$.

2.5.10 Proposition. *Let $F \subseteq \mathrm{Gram}_V(f)$ be a face. Consider*

$$F_{\mathcal{H}} := \psi^{-1}(F) \cap \mathcal{H}^+_{V_{\mathbb{C}}}(f).$$

Then $F_{\mathcal{H}}$ is a face of $\mathcal{H}^+_{V_{\mathbb{C}}}(f)$ with $F_{\mathcal{H}} \cap \mathrm{Gram}_V(f) = F$. Moreover, if $\vartheta \in \mathrm{relint}(F)$, then $F_{\mathcal{H}}$ is the supporting face of ϑ in the Hermitian Gram spectrahedron $\mathcal{H}^+_{V_{\mathbb{C}}}(f)$ and $\mathrm{rk}(F_{\mathcal{H}}) = \mathrm{rk}(\vartheta) = \mathrm{rk}(F)$.

Proof. Of course, $F_{\mathcal{H}} \subseteq \mathcal{H}^+_{V_{\mathbb{C}}}(f)$ is convex. Let ρ_1, $\rho_2 \in \mathcal{H}^+_{V_{\mathbb{C}}}(f)$ and $0 < \lambda < 1$ such that $(1-\lambda)\rho_1 + \lambda\rho_2 \in F_{\mathcal{H}}$. Then $\psi(\rho_1)$, $\psi(\rho_2) \in \mathrm{Gram}_V(f)$ and $(1-\lambda)\psi(\rho_1) + \lambda\psi(\rho_2) \in F$. The fact that F is a face implies $\psi(\rho_1)$, $\psi(\rho_2) \in F$. Thus, ρ_1, $\rho_2 \in F_{\mathcal{H}}$. This shows that $F_{\mathcal{H}}$ is indeed a face of $\mathcal{H}^+_{V_{\mathbb{C}}}(f)$.

Obviously, $F \subseteq F_{\mathcal{H}} \cap \mathrm{Gram}_V(f)$. For the opposite inclusion let $\vartheta \in F_{\mathcal{H}} \cap \mathrm{Gram}_V(f)$. Then $\vartheta = \overline{\vartheta}$ and $\vartheta = \psi(\vartheta) \in F$.

For $\vartheta \in \mathrm{relint}(F)$ we have $\mathrm{rk}(F) = \mathrm{rk}(\vartheta)$. Since $\vartheta \in F_{\mathcal{H}}$, the supporting face of ϑ in $\mathcal{H}^+_{V_{\mathbb{C}}}(f)$ is contained in $F_{\mathcal{H}}$. On the other hand, we have $\mathrm{rk}(\vartheta) \geq \mathrm{rk}(F_{\mathcal{H}})$ since the rank of ϑ is maximal among all elements of $\psi^{-1}(\vartheta) \cap \mathsf{H}^+_2 V_{\mathbb{C}}$ (see Corollary 2.5.3). Hence, $\mathrm{rk}(\vartheta) = \mathrm{rk}(F_{\mathcal{H}})$ and $F_{\mathcal{H}} = \mathrm{suppface}_{\mathcal{H}^+_{V_{\mathbb{C}}}(f)}(\vartheta)$. □

2.5.11 Corollary. *Let $F \subseteq \mathrm{Gram}_V(f)$ be a face of rank r. For the face $F_{\mathcal{H}} = \psi^{-1}(F) \cap \mathcal{H}^+_{V_{\mathbb{C}}}(f)$ of the Hermitian Gram spectrahedron $\mathcal{H}^+_{V_{\mathbb{C}}}(f)$ we have*

$$\dim(F_{\mathcal{H}}) = \dim(F) + \binom{r}{2}.$$

Proof. Fix a point $\vartheta \in \mathrm{relint}(F)$. We consider the face subspace $U = \mathcal{U}(F) = \mathrm{im}(\vartheta) \subseteq V$ and write $U = \mathrm{span}_{\mathbb{R}}(p_1, \ldots, p_r)$ with $p_1, \ldots, p_r \in V$. Then $U_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}(p_1, \ldots, p_r) \subseteq V_{\mathbb{C}}$ is a face subspace for $\mathcal{H}^+_{V_{\mathbb{C}}}(f)$ and the corresponding face is $F_{\mathcal{H}}$ by Proposition 2.5.10. As $\overline{U_{\mathbb{C}}} = U_{\mathbb{C}}$ and $\dim_{\mathbb{C}}(U_{\mathbb{C}}) = \dim_{\mathbb{R}}(U)$, Propositions 2.4.10 and 2.3.9 imply that

$$\begin{aligned}
\dim(F_{\mathcal{H}}) &= r^2 - \dim_{\mathbb{C}}(U_{\mathbb{C}}\overline{U_{\mathbb{C}}}) \\
&= r^2 - \dim_{\mathbb{R}}(UU) \\
&= r^2 - \left(\binom{r+1}{2} - \dim(F)\right) \\
&= \binom{r}{2} + \dim(F).
\end{aligned}$$

□

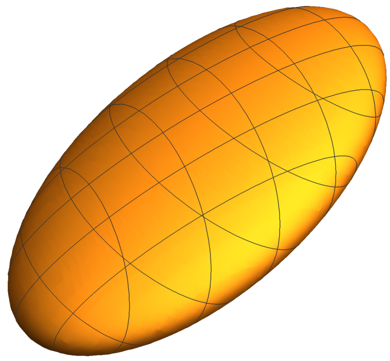


FIGURE 2.1. The supporting face in $\mathcal{H}^+_{V_\mathbb{C}}(f)$ of a rank-three extreme point of $\mathrm{Gram}_V(f)$.

2.5.12 Example. Let $f \in \mathbb{R}[x,y]_{2d}$ be a nonnegative binary form of degree $2d$ and let $V = \mathbb{R}[x,y]_d$. Assume that $f = g^2 + h^2$ with linearly independent $g, h \in V$. Consider the corresponding tensor $\vartheta = g \otimes g + h \otimes h \in \mathrm{Gram}_V(f)$ of rank two. Then ϑ is an extreme point of $\mathrm{Gram}_V(f)$ (see Chapter 3) and the supporting face of ϑ in $\mathcal{H}^+_{V_\mathbb{C}}(f)$ is an edge whose center is ϑ and whose endpoints are the Hermitian rank-one tensors $p \otimes \overline{p}$ and $\overline{p} \otimes p$ with $p = g + ih \in \mathbb{C}[x,y]_d = V_\mathbb{C}$.

2.5.13 Proposition. *Let $G \subseteq \mathcal{H}^+_{V_\mathbb{C}}(f)$ be a nonempty face. The following are equivalent:*

(i) $G = F_\mathcal{H}$ *for some nonempty face* $F \subseteq \mathrm{Gram}_V(f)$.
(ii) $\mathrm{relint}(G) \cap \mathrm{Gram}_V(f) \neq \varnothing$.
(iii) *The face subspace* $U = \mathfrak{U}(G) \subseteq V_\mathbb{C}$ *satisfies* $\overline{U} = U$.

Proof. That (i) implies (ii) is contained in Proposition 2.5.10. For "(ii) $\Rightarrow$ (iii)" we take $\vartheta \in \mathrm{relint}(G) \cap \mathrm{Gram}_V(f)$. Then $U = \mathfrak{U}(G) = \mathrm{im}(\vartheta)$ is $\mathbb{C}$-linearly spanned by elements from V. As these are invariant under complex conjugation, we see that $\overline{U} = U$.

Finally, we prove "(iii) $\Rightarrow$ (i)". Write $U = \mathrm{im}(\vartheta)$ for some $\vartheta \in \mathrm{relint}(G)$, say $\vartheta = \sum_{j=1}^r p_j \otimes \overline{p_j}$ with $p_1, \dots, p_r \in U$. As $\overline{U} = U$, we have $\overline{p_j} \in U$ for every $j \in \{1, \dots, r\}$. Therefore, $\mathrm{Re}(p_j) = \frac{1}{2}(p_j + \overline{p_j}) \in U$ and $\mathrm{Im}(p_j) = \frac{1}{2i}(p_j - \overline{p_j}) \in U$. Hence, U is $\mathbb{C}$-linearly generated by $\mathrm{Re}(p_j), \mathrm{Im}(p_j) \in V$. Consider $\vartheta_0 = \psi(\vartheta) \in \mathrm{Gram}_V(f)$ and $F = \mathrm{suppface}(\vartheta_0) \subseteq \mathrm{Gram}_V(f)$. Then $F_\mathcal{H} = G$ since $U = \mathrm{im}(\vartheta_0)_\mathbb{C}$. □

2.5.14 Remark. Let ϑ_0 be an extreme point of $\mathrm{Gram}_V(f)$ and consider the associated face $G = \{\vartheta_0\}_\mathcal{H} = \psi^{-1}(\vartheta_0) \cap \mathcal{H}^+_{V_\mathbb{C}}(f)$ of the Hermitian Gram spectrahedron $\mathcal{H}^+_{V_\mathbb{C}}(f)$. For every $\vartheta \in G$, we have $\overline{\vartheta} \in G$ and $\vartheta_0 = \psi(\vartheta) = \frac{1}{2}(\vartheta + \overline{\vartheta})$ is the midpoint of the line segment $[\vartheta, \overline{\vartheta}]$. This means that G is a centrally symmetric convex set whose center is the real point ϑ_0.

Motivated by this observation, we suggest to call a face $\varnothing \neq G \subseteq \mathcal{H}^+_{V_\mathbb{C}}(f)$ *real symmetric* if it satisfies the equivalent conditions (i)-(iii) from Proposition 2.5.13. This perception of a certain symmetry with respect to real points is also justified by the fact that for every $\vartheta \in G$ also $\overline{\vartheta} \in G$ and the midpoint of the segment $[\vartheta, \overline{\vartheta}]$ lies in $G \cap \mathrm{Gram}_V(f)$.

2.5.15 Corollary. *The map $F \mapsto F_{\mathcal{H}}$ is an inclusion- and rank-preserving bijection between the (nonempty) faces $F \subseteq \mathrm{Gram}_V(f)$ and the real symmetric faces $G \subseteq \mathcal{H}^+_{V_{\mathbb{C}}}(f)$.*

2.5.16 Remark. Let C be a convex set in some finite-dimensional real vector space. For faces $F_1, F_2 \subseteq C$ one defines their *join* $F_1 \vee F_2$ to be the smallest face of C containing both F_1 and F_2. The *meet* $F_1 \wedge F_2$ is the intersection $F_1 \cap F_2$. In this way the set of faces of C carries the structure of a *lattice* (in the sense of partially ordered sets).

We can extend our definition of real symmetric faces of $\mathcal{H}^+_{V_{\mathbb{C}}}(f)$ to include the empty face $\varnothing \subseteq \mathcal{H}^+_{V_{\mathbb{C}}}(f)$. As the map $F \mapsto F_{\mathcal{H}} = \psi^{-1}(F) \cap \mathcal{H}^+_{V_{\mathbb{C}}}(f)$ is compatible with taking joins and meets of faces, we see that it actually induces an isomorphism between the face lattice of $\mathrm{Gram}_V(f)$ and the lattice of real symmetric faces of $H^+_{V_{\mathbb{C}}}(f)$.

2.5.17 Example. Let $\vartheta_0 \in \mathrm{Gram}_V(f)$ be an extreme point of rank three. According to Corollary 2.5.11, the face $G := \{\vartheta_0\}_{\mathcal{H}}$ of $\mathcal{H}^+_{V_{\mathbb{C}}}(f)$ is three-dimensional. Points in its relative interior have rank three (Proposition 2.5.10) and every other point must be an extreme point of rank two. Indeed, for any $\vartheta \in G$, we have $2\,\mathrm{rk}(\vartheta) \geq \mathrm{rk}(\vartheta_0) = 3$ by Corollary 2.5.3, so that ϑ cannot have rank one. This also shows that G itself has no positive-dimensional proper faces. The general appearance of the face $G \subseteq \mathcal{H}^+_{V_{\mathbb{C}}}(f)$ is depicted in Figure 2.1.

CHAPTER 3

Gram spectrahedra of binary forms

In this chapter we investigate Gram spectrahedra in the most basic case possible, namely binary forms. By a binary form, we mean a homogeneous polynomial f in two variables x, y with real coefficients. Of course, the form $f \in \mathbb{R}[x, y]$ can only be nonnegative on $\mathbb{P}^1(\mathbb{R})$ if the degree $\deg(f) = 2d$ is even. Let us fix some notation before continuing the discussion.

3.0.1 Notation. Let $d \in \mathbb{N}$. We write $\Sigma_{2d} = \Sigma\mathbb{R}[x, y]_d^2$ for the sums-of-squares cone in $\mathbb{R}[x, y]_{2d}$. As is well-known, this semialgebraic set is a pointed closed convex cone with nonempty interior (in $\mathbb{R}[x, y]_{2d}$) and its elements are precisely the nonnegative binary forms of degree $2d$.

For $f \in \Sigma_{2d}$, let $\mathrm{Gram}(f)$ always be the (full) Gram spectrahedron of f, that is to say, $\mathrm{Gram}(f) := \mathrm{Gram}_V(f)$ with $V = \mathbb{R}[x, y]_d$. Similarly, we simply write $\mathcal{H}^+(f)$ for the Hermitian Gram spectrahedron of f relative to $V_\mathbb{C} = \mathbb{C}[x, y]_d$.

3.0.2 Convention. We sometimes say that "a *general* $f \in \Sigma_{2d}$ has property P" or "a *general* nonnegative binary form of degree $2d$ has property P". By this we shall mean that the subset $S \subseteq \Sigma_{2d}$ of all f that have property P contains an open dense subset of Σ_{2d}. To be very clear, "open dense" refers to the subspace topology on Σ_{2d} inherited from the Euclidean topology on $\mathbb{R}[x, y]_{2d} \cong \mathbb{R}^{2d+1}$.

We note the following: Assume that S is semialgebraic. Then it is sufficient to require that S is dense in Σ_{2d}. Indeed, if $\overline{S} = \Sigma_{2d}$, then

$$\dim(\Sigma_{2d} \setminus S) < \dim(\Sigma_{2d})$$

by Proposition 1.1.11. Therefore, $\Sigma_{2d} \setminus S$ is contained in some hypersurface of the affine space $\mathbb{C}[x, y]_{2d}$. As hypersurfaces are (Euclidean) closed and nowhere dense, S contains a subset that is open and dense in Σ_{2d}.

Any nonnegative binary form can be written as a sum of two real squares and every such representation gives us a rank-two point in the Gram spectrahedron. Below, we recall the construction of the finitely many rank-two points in $\mathrm{Gram}(f)$ which are intimately connected with the representations of f as a single Hermitian square. A point of rank one exists in $\mathrm{Gram}(f)$ if and only if $f = p^2$ is a perfect square, which is certainly not true for general (nonnegative) $f \in \mathbb{R}[x, y]_{2d}$, see Example 3.0.4.

Already in the case of binary forms, little was known about the facial structure of Gram spectrahedra before we started our work. Some former results are presented in this section, while others will be mentioned at a later point where they better fit into the context.

3.0.3 Remark. Essentially, via $f(x, y) \mapsto f(x, 1)$ and $g(x) \mapsto y^d g(\frac{x}{y})$, a binary form of degree d is nothing but a univariate polynomial of degree at most d. Therefore, we will often work with univariate polynomials to simplify notation.

We choose homogeneous coordinates $(x : y)$ on $\mathbb{P}^1$. The *zeros* or *roots* of a binary form $f \in \mathbb{R}[x,y]_d$ are the points $\xi \in \mathbb{P}^1$ with $f(\xi) = 0$. A point $\xi \in \mathbb{P}^1(\mathbb{R})$ with $f(\xi) = 0$ is said to be a *real* root of f. We can interpret the projective line $\mathbb{P}^1$ as the union of an affine line $\mathbb{A}^1$ and a point at infinity, explicitly

$$\mathbb{P}^1 = \{(x : 1) \in \mathbb{P}^1 : x \in \mathbb{C}\} \cup \{(1 : 0)\}.$$

We thus keep in mind that a binary form $f \in \mathbb{R}[x,y]_d$ vanishes at $(1 : 0)$ if and only if f is divisible by y, which means that the degree of the univariate polynomial $f(x,1) \in \mathbb{R}[x]$ is strictly smaller than d.

We give a simple example that is meant to illustrate both Remark 3.0.3 and Convention 3.0.2.

3.0.4 Example. A general $f \in \Sigma_{2d}$ has distinct roots and is therefore not a perfect square.

Proof. Any $f \in \mathbb{R}[x,y]_{2d}$ is of the form

$$f = a_{2d}x^{2d} + a_{2d-1}x^{2d-1}y + \cdots + a_1xy^{2d-1} + a_0y^{2d}.$$

If f has a multiple root in $\mathbb{P}^1$, then $a_{2d} = a_{2d-1} = 0$ or the discriminant of the univariate polynomial $f(x,1)$ is zero. As this discriminant is a polynomial expression in the coefficients of f, we see that the set of $f \in \Sigma_{2d}$ with multiple roots is contained in a proper subvariety of $\mathbb{C}[x,y]_{2d}$.

Obviously, if $f = p^2$ for some $p \in \mathbb{R}[x,y]_d$, then every root of f has even multiplicity. □

3.0.5. Let $f \in \mathbb{R}[x,y]_{2d}$ be a nonnegative binary form. If $f \in \operatorname{int}(\Sigma_{2d})$, we get $\dim \operatorname{Gram}(f) = \binom{d}{2}$ by Proposition 2.3.9. Otherwise, $\operatorname{Gram}(f)$ does not contain any positive definite tensor (i.e., any tensor of rank $\dim(V) = d+1$), see Proposition 2.3.7, and the dimension of $\operatorname{Gram}(f)$ is smaller. In fact, the aforementioned proposition shows that then $\operatorname{Gram}(f) = \operatorname{Gram}_U(f)$ for a proper subspace $U \subseteq V$ with $f \in \operatorname{int}(\Sigma U^2)$.

Let us elaborate on this. Clearly, $f \in \partial\Sigma_{2d}$ if and only if f has a root $\xi \in \mathbb{P}^1(\mathbb{R})$. We write $\xi = [(a,b)]$ with $0 \neq (a,b) \in \mathbb{R}^2$. Consider any $\vartheta = \sum_{i=1}^r p_i \otimes p_i \in \operatorname{Gram}_U(f)$. From $\sum_{i=1}^r p_i(\xi)^2 = f(\xi) = 0$ we see that, for each $i \in \{1,\dots,r\}$, we must have $p_i(\xi) = 0$ which is why p_i is divisible by $l := bx - ay \in \mathbb{R}[x,y]_1$. Consequently, $f = l^2 g$ and we have a natural isomorphism between $\operatorname{Gram}_U(f)$ and $\operatorname{Gram}_{U'}(g)$ where $U' \subseteq \mathbb{R}[x,y]_{d-1}$ is a subspace with $U = lU'$. Hence, we will mostly assume that f is strictly positive on $\mathbb{P}^1(\mathbb{R})$.

3.0.6. Let $f \in \mathbb{R}[x,y]_{2d}$ be a positive binary form with distinct roots. Let us recall how rank-one tensors in $\mathcal{H}^+(f)$ and rank-two tensors in $\operatorname{Gram}(f)$ look like. For what follows, we refer to [CLR, Example 2.13] and [CPSV, Proposition 4.1], see also [Sch22, 6.1].

We may assume that $f(x,1) \in \mathbb{R}[x]$ is monic. If we denote its roots by $a_1,\dots,a_d$, $\overline{a_1},\dots,\overline{a_d} \in \mathbb{C}$, then there are precisely 2^d points of rank one in $\mathcal{H}^+(f)$, namely the tensors $p \otimes \overline{p}$ with $p = \prod_{j=1}^d (x - b_j y) \in \mathbb{C}[x,y]_d$ where $b_j \in \{a_j, \overline{a_j}\}$.

Every $\vartheta \in \operatorname{Gram}(f)$ with $\operatorname{rk}(\vartheta) = 2$ can be written as $\vartheta = g \otimes g + h \otimes h$ where $g, h \in \mathbb{R}[x,y]_d$. Then ϑ gives a factorization $f = g^2 + h^2 = (g + ih)(g - ih)$ as a Hermitian square, corresponding to a rank-one tensor in $\mathcal{H}^+(f)$. Conversely, any $p \otimes \overline{p} \in \mathcal{H}^+(f)$ induces a rank-two point $\vartheta = \psi(p \otimes \overline{p}) = \operatorname{Re}(p) \otimes \operatorname{Re}(p) + \operatorname{Im}(p) \otimes \operatorname{Im}(p)$

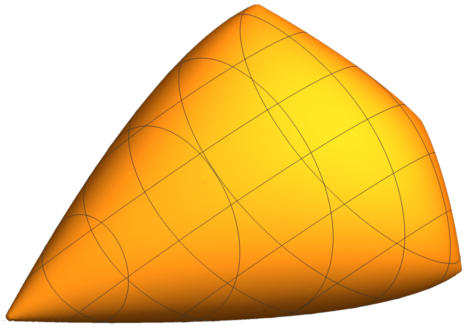


FIGURE 3.1. The Gram spectrahedron of a positive binary sextic with distinct roots.

in Gram(f) (cf. Lemma 2.5.2). Of course, $\overline{p} \otimes p$ gives the same point. However, as 2 is the smallest rank in Gram(f), ϑ is an extreme point and $\psi^{-1}(\vartheta) \cap \mathcal{H}^+(f)$ is an edge that contains only two rank-one tensors (see Example 2.5.12). Thus, Gram(f) contains precisely 2^{d-1} points of rank two.

3.0.7. As before, we let $f \in \mathbb{R}[x,y]_{2d}$ be a positive binary form with distinct roots. For $d \leq 3$, the structure of Gram(f) is well understood. If $d = 1$, then Gram(f) is a single point. For $d = 2$ we get a line segment whose endpoints have rank two. As $\dim \mathbb{R}[x,y]_2 = 3$, any positive combination of these points has rank three. The case $d = 3$ is more interesting. Gram(f) is then a three-dimensional spectrahedron. Except for the four points of rank two, its boundary consists of extreme points of rank three ([Sch22, Proposition 6.3]). In particular, there are no proper positive-dimensional faces. See Figure 3.1 for an illustration of the Gram spectrahedron. If f is sufficiently general, the algebraic boundary of Gram(f) is a Kummer surface (see [CPSV, Section 4.2] and [ORSV, Theorem 5.6]).

For larger d, the facts known about the facial structure of Gram(f) were essentially limited to the line segments connecting two extreme points of rank two, see the following result that we quote from [Sch22]. Following Scheiderer, we introduce the notations Ex(f) short hand for the set of all extreme points of Gram(f), and $\text{Ex}_r(f)$ for the set of rank-r extreme points.

3.0.8 Theorem ([Sch22, Theorem 6.4]). *Let $d \geq 4$. For all forms f in an open dense subset of Σ_{2d}, the following is true:*

(a) $d = 4$: For each of the $\binom{8}{2} = 28$ pairs $\vartheta \neq \vartheta'$ in $\text{Ex}_2(f)$, the interval $[\vartheta, \vartheta']$ is contained in the boundary of Gram(f). *For precisely 16 of these pairs, $[\vartheta, \vartheta']$ is a face of* Gram(f). *These 16 edges form a graph isomorphic to $K_{4,4}$, the complete bipartite graph on two sets of four points each.*

(b) $d \geq 5$: For any two $\vartheta \neq \vartheta'$ in $\text{Ex}_2(f)$, the line segment $[\vartheta, \vartheta']$ is a face of Gram(f).

For $d = 4$ we have $\dim \text{Gram}(f) = 6$. Note that part (a) of Theorem 3.0.8 does not tell us the dimension of the supporting face F of $[\vartheta, \vartheta']$ in the twelve cases where this line segment is not an edge. It follows from Corollary 3.1.8 that $\dim(F) = 2$ in these cases.

This chapter is organized as follows. Using the theory developed in Chapter 2, we examine the relationship between rank and dimension of faces $F \subseteq \mathrm{Gram}(f)$ and we show which pairs $(\mathrm{rk}(F), \dim(F))$ can occur. This is done in Section 3.1. The subsequent Section 3.2 deals with the same question in the Hermitian setting. We will obtain similar dimension bounds but also point out a different behavior (see Corollary 3.2.5).

In Section 3.3 we give bounds on the dimension of polyhedral faces in Hermitian Gram spectrahedra. Conversely, we show that $H^+(f)$ contains a simplex face of the largest possible dimension if f is a sufficiently general positive binary form (Theorem 3.3.13). Our proof of this fact is constructive and uses the combinatorics of rank-one extreme points in $H^+(f)$. Section 3.4 elaborates on the number of simplex faces of $H^+(f)$ obtained via this construction and on an estimate of the actual total number of such faces. The techniques developed in the Hermitian setting are then used to find large polyhedral faces in (real symmetric) Gram spectrahedra (Theorem 3.5.12). Most of the results contained in Sections 3.1–3.3 and 3.5 also appear in the same or a very similar way in the author's previously published article [May21].

In Section 3.6 we discuss various approaches to bounding the Carathéodory number of Gram spectrahedra. This is where chains of faces are of certain significance. These are also of independent interest and to be examined.

Let $d \in \mathbb{N}$ and let $f \in \mathbb{R}[x,y]_{2d}$ be a positive binary form. Any extreme point of $\mathrm{Gram}(f)$ is a face whose dimension is as small as possible regarding its rank. We are going to say that a face $F \subseteq \mathrm{Gram}(f)$ has expected dimension if the corresponding property for $\dim(F)$ is satisfied. See 3.7.1 for a precise definition of this notion. Generalizing a result of Scheiderer concerning the extreme points, which gave an affirmative answer to a question in [CPSV], we show that $\mathrm{Gram}(f)$ has faces of expected dimension for all ranks $r \in \{2, \dots, d+1\}$ as soon as f is sufficiently general (Theorem 3.7.15).

3.1. Dimension bounds for faces

In this section we use the formula for the dimension of faces in Gram spectrahedra (Proposition 2.3.9) together with an estimate for $\dim(UU)$ to give bounds for the dimensions of faces in terms of their ranks. In doing so, we observe substantial dimension gaps.

3.1.1. Let $f \in \Sigma_{2d}$ and let $F \subseteq \mathrm{Gram}(f)$ be a face of rank r. The lower bound for $\dim(F)$ is obvious: Let $U = \mathcal{U}(F) \subseteq \mathbb{R}[x,y]_d$ denote the face subspace corresponding to F. Since $UU \subseteq \mathbb{R}[x,y]_{2d}$, we have $\dim(UU) \leq 2d+1$, so that the dimension of $F = \mathcal{F}(U)$ is at least $\binom{r+1}{2} - (2d+1)$. Below, we prove an upper bound.

3.1.2 Proposition. *Let $U \subseteq \mathbb{R}[x]$ be a subspace of dimension r. Then the dimension of UU is at least $2r-1$.*

Proof. We can choose a basis $p_1, \dots, p_r$ of U such that $\deg(p_i) < \deg(p_{i+1})$ for all $i = 1, \dots, r-1$. Then

$$p_1p_1, p_1p_2, p_2p_2, p_2p_3, p_3p_3, \dots, p_{r-1}p_r, p_rp_r$$

are $2r-1$ polynomials in UU of pairwise different degree. Therefore, they are linearly independent. □

In other words, if $U \subseteq \mathbb{R}[x,y]_d$ is a subspace of codimension k, then the codimension of UU in $\mathbb{R}[x,y]_{2d}$ is at most $2k$.

3.1.3 Corollary. *Let $f \in \mathbb{R}[x,y]_{2d}$ be a nonnegative binary form and let $F \subseteq \mathrm{Gram}(f)$ be a face of rank r. Then $\dim(F) \le \binom{r-1}{2}$.*

Proof. Consider the subspace $U = \mathcal{U}(F)$ of $\mathbb{R}[x,y]_d$. By Proposition 3.1.2,

$$\dim(F) = \binom{r+1}{2} - \dim(UU) \le \binom{r+1}{2} - (2r-1) = \binom{r-1}{2}. \qquad \square$$

3.1.4 Remark. Let $\mathrm{rk}(F) = r$ and write $k = (d+1) - r$, that is $k = \dim\ker(\vartheta)$ for $\vartheta \in \mathrm{relint}(F)$. Thereby, we can represent both inequalities for the dimension of F in a uniform way:

$$\binom{d-k}{2} - 2k \le \dim(F) \le \binom{d-k}{2}.$$

3.1.5. Let $U \subseteq \mathbb{R}[x,y]_d$ be a subspace of codimension 1 and assume that there is a point $\xi \in \mathbb{P}^1(\mathbb{R})$ such that every $p \in U$ vanishes at ξ. Then the codimension of $UU \in \mathbb{R}[x,y]_{2d}$ is 2. On the one hand, it is at most 2, as we have seen before. On the other hand, every $f \in UU$ is divisible by $(v_2x - v_1y)^2$ where $v = (v_1, v_2) \in \mathbb{R}^2$ is a point with $\xi = [v]$.

We now work towards a converse which consequently improves the upper bound for the maximum dimension of a proper face in Gram spectrahedra of binary forms. Let $f \in \Sigma_6$ ($d = 3$). By Corollary 3.1.3, a proper face of $\mathrm{Gram}(f)$ has dimension at most $\binom{d-1}{2} = 1$. In fact, we already know that $\mathrm{Gram}(f)$ has no faces of dimension 1 if f is strictly positive on $\mathbb{P}^1(\mathbb{R})$ (see 3.0.7). In other words, if $\mathrm{Gram}(f)$ has a rank-d face F of dimension $\binom{d-1}{2}$, i.e., $U = \mathcal{U}(F)$ attains the lower bound $\dim(UU) = 2d-1$, then $f(\xi) = 0$ for some $\xi \in \mathbb{P}^1(\mathbb{R})$ and therefore $\mathrm{Gram}(f) = F$. We will show that this generalizes to arbitrary degree $d \ge 3$. The main part of the proof is the following lemma.

3.1.6 Lemma. *Let $d \ge 3$ and let $U \subseteq \mathbb{R}[x]_{\le d}$ be a d-dimensional subspace. If $1 \notin U$ and $\dim(UU) = 2d-1$, then the elements of U have a common real root.*

Proof. The case $d = 3$ is discussed in the previous remark. We assume $d \ge 4$ and proceed by induction on d. Due to $1 \notin U$, we can choose a basis $(p_1, \dots, p_d)$ of U where $p_i = x^i + \lambda_i$ for some $\lambda_i \in \mathbb{R}$ (cf. Remark 2.3.15). Consider $U' := \mathrm{span}(p_1, \dots, p_{d-1}) \subseteq \mathbb{R}[x]_{\le d-1}$. Since $\deg(p_d) = d$ is bigger than the degree of any element in U', the proof of Proposition 3.1.2 shows that $\dim(U'U') \le \dim(UU) - 2 = 2(d-1) - 1$. The right hand side coincides with the lower bound from 3.1.2, so we must have equality. By induction, $p_1, \dots, p_{d-1}$ have a common root $a \in \mathbb{R}$.

The choice of a simple basis for U makes it easy to read off bases for $U'U'$ and UU:

$$p_1p_1, p_1p_2, p_2p_2, \dots, p_{d-2}p_{d-1}, p_{d-1}p_{d-1}$$

are $2(d-1) - 1 = \dim(U'U')$ elements of $U'U'$ of pairwise different degrees, so they constitute a basis for $U'U'$ which is completed to a basis of UU by $p_{d-1}p_d$ and p_dp_d. We consider the representation of $p_1p_d \in UU$ with respect to this basis. Since $d \ge 3$ and for reasons of degree, the coefficients of $p_{d-1}p_d$ and p_dp_d are zero. Hence $p_1p_d \in U'U'$. But all elements of $U'U'$ have a double root in a, so $(x-a)^2$ divides $p_1p_d = (x-a)p_d$. Therefore, also $p_d(a) = 0$. $\square$

3.1.7 Proposition. *Let $d \ge 3$ and let $U \subseteq \mathbb{R}[x,y]_d$ be a d-dimensional subspace. If $\dim(UU) = 2d-1$, then there exists a $\xi \in \mathbb{P}^1(\mathbb{R})$ such that $p(\xi) = 0$ for all $p \in U$.*

Proof. We tackle the problem in its univariate formulation. For this purpose, we write $U = \operatorname{span}(p_1, \ldots, p_d) \subseteq \mathbb{R}[x]_{\leq d}$, $\dim(U) = d$, and assume that $\dim(UU) = 2d-1$. If $\deg(p_i) < d$ for all $i = 1, \ldots, d$, then $(1:0) \in \mathbb{P}^1(\mathbb{R})$ is a common root of the corresponding degree-d forms $y^d \cdot p_i(\frac{x}{y})$, see Remark 3.0.3. Therefore, we may assume that p_d is monic, $\deg(p_d) = d$ and $\deg(p_i) < d$ for all $i < d$.

Consider $U' := \operatorname{span}(p_1, \ldots, p_{d-1}) \subseteq \mathbb{R}[x]_{\leq d-1}$. The same argument as in the proof of Lemma 3.1.6 shows that $\dim(U'U') = 2(d-1)-1$. By induction, either $p_1, \ldots, p_{d-1}$ are of degree smaller than $d-1$ or they have a common real root. Suppose $\deg(p_i) < d-1$ for all $i = 1, \ldots, d-1$. Then U' is a subspace of $\mathbb{R}[x]_{\leq d-2}$ of dimension $d-1$, so $U' = \mathbb{R}[x]_{\leq d-2}$ and $U'U' = \mathbb{R}[x]_{\leq 2d-4}$. However, $p_d = x^d + q(x)$ for some $q \in \mathbb{R}[x]_{\leq d-1}$ and $x^{d-3}p_d$, $x^{d-2}p_d$, $p_d^2 \in UU$ are of degree $2d-3$, $2d-2$ and $2d$, respectively, which would imply that $\dim(UU) \geq 2d$, a contradiction. We conclude that $p_1, \ldots, p_{d-1}$ have a common real root. In particular, $1 \notin U'$ and since every polynomial in $U \smallsetminus U'$ has degree d, also $1 \notin U$. Finally, Lemma 3.1.6 gives the desired conclusion. □

3.1.8 Corollary. *Let $d \geq 3$. Let $f \in \mathbb{R}[x,y]_{2d}$ be positive on $\mathbb{P}^1(\mathbb{R})$. If $F \subseteq \operatorname{Gram}(f)$ is a face of rank d, then $\dim(F) < \binom{d-1}{2}$.*

In other words: If $f \in \mathbb{R}[x,y]_{2d}$ is a nonnegative binary form and if $\operatorname{Gram}(f)$ contains a face F of dimension $\binom{d-1}{2}$, then already $F = \operatorname{Gram}(f)$ (and f has a root in $\mathbb{P}^1(\mathbb{R})$).

Proof. We have already seen that $\dim(F) \leq \binom{d-1}{2}$ (cf. Corollary 3.1.3). Choose $\vartheta \in \operatorname{relint}(F)$, $\vartheta = \sum_{i=1}^d p_i \otimes p_i$ with linearly independent $p_1, \ldots, p_d \in \mathbb{R}[x,y]_d$. If we had $\dim(F) = \binom{d-1}{2}$, we would have $\dim(UU) = 2d-1$ for $U = \mathcal{U}(F) = \operatorname{im}(\vartheta)$. But then $p_i(\xi) = 0$ for some $\xi \in \mathbb{P}^1(\mathbb{R})$ $(i = 1, \ldots, d)$ by Proposition 3.1.7, and thus $f(\xi) = p_1(\xi)^2 + \cdots + p_d(\xi)^2 = 0$. □

3.1.9 Remark. Even for large d and a positive binary form $f \in \mathbb{R}[x,y]_{2d}$, we can have faces of rank $d-1$ and dimension $\binom{d-2}{2}$ in $\operatorname{Gram}(f)$. Consider, for instance,

$$U := \operatorname{span}\Big(x^i y^{d-2-i}(x^2+y^2) : i = 0, \ldots, d-2\Big) \subseteq \mathbb{R}[x,y]_d.$$

Then $\dim(U) = d-1$ and $\dim(UU) = \dim \mathbb{R}[x,y]_{2(d-2)} = 2d-3$. Consequently, the Gram spectrahedron of every $f \in \operatorname{int}(\Sigma U^2)$ contains a face as above. Thus, for $r = d-1$, we cannot further improve the bound from Corollary 3.1.3 by assuming that f is positive on $\mathbb{P}^1(\mathbb{R})$.

Having analyzed the proper faces of maximum rank, let us also drop some words on low ranks. Since two linearly independent forms in $\mathbb{R}[x,y]_d$ are always quadratically independent (Proposition 3.1.2), rank-two points are always extreme points of Gram spectrahedra by Corollary 2.3.10. This need not be true for points of rank three. However, for $d \geq 3$ and general $f \in \Sigma_{2d}$, any rank-three point of $\operatorname{Gram}(f)$ is an extreme point, as the following proposition shows.

3.1.10 Proposition. *Let $d \geq 3$. For general $f \in \Sigma_{2d}$, the Gram spectrahedron* $\operatorname{Gram}(f)$ *does not contain a positive-dimensional face of rank* 3.

Proof. Let $f \in \Sigma_{2d}$ such that f is not the square of a real polynomial. Then any extreme point of $\operatorname{Gram}(f)$ has rank at least 2. Suppose that F is a positive-dimensional face of $\operatorname{Gram}(f)$ with $\operatorname{rk}(F) = 3$. F is compact and convex, whence

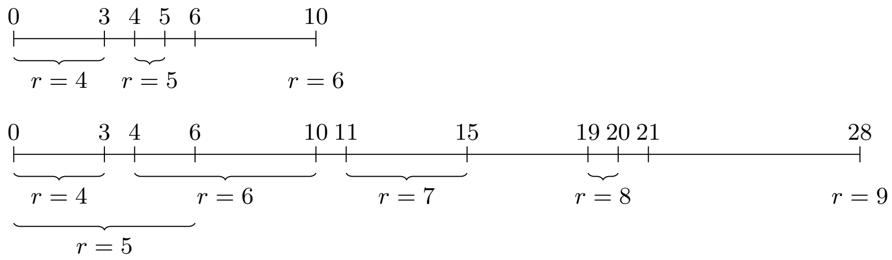


FIGURE 3.2. Intervals of possible dimensions of faces $F \subseteq \mathrm{Gram}(f)$ of rank r for $d = 5$ and $d = 8$.

$F = \mathrm{conv}(\mathrm{Ex}(F))$. For any extreme point $\vartheta \in \mathrm{Ex}(F) \subseteq \mathrm{Ex}(f)$, the set $\{\vartheta\}$ is a proper face of F, so $\mathrm{rk}(\vartheta) < \mathrm{rk}(F)$. It follows $\mathrm{rk}(\vartheta) = 2$. Due to $\dim(F) \geq 1$ we have at least two distinct extreme points ϑ, $\vartheta' \in \mathrm{Ex}(F)$. Consider the supporting face F' of $[\vartheta, \vartheta']$. For general f we have $\dim \mathcal{U}(F') = 4$, see [Sch22, 6.5]. But this implies $\mathrm{rk}(F') = 4 > 3 = \mathrm{rk}(F)$, which is impossible since $F' \subseteq F$. □

3.1.11 Example. Figure 3.2 illustrates the possible combinations of $\mathrm{rk}(F)$ and $\dim(F)$ in the Gram spectrahedron of a positive binary form $f \in \Sigma_{2d}$ for $d = 5$ and $d = 8$. For the sake of clarity, we do not plot faces of rank 2 or 3 in the charts. For general f, such faces are always extreme points of the spectrahedron (see Proposition 3.1.10).

We are going to show in Section 3.7 that the Gram spectrahedron of any sufficiently general positive binary form $f \in \mathbb{R}[x, y]_{2d}$ contains faces of each rank $r = 2, \ldots, d{+}1$, whose dimension is equal to the lower bound $\max\{0, \binom{r+1}{2} - (2d{+}1)\}$ from the beginning of this section.

3.2. Hermitian Gram spectrahedra of binary forms

We introduced a Hermitian version of Gram spectrahedra in Section 2.4. In this short section, we first establish bounds for the dimensions of faces in Hermitian Gram spectrahedra of binary forms. Afterwards, we revisit Proposition 2.4.12 in order to show that, in contrast to the real symmetric case, upper bounds are always attained.

The great benefit of the Hermitian approach comes from the fact that any nonnegative binary form $f \in \mathbb{R}[x, y]_{2d}$ factors completely into a Hermitian square. Moreover, if f is strictly positive, the different factorizations correspond to partitions of the roots of f into two subsets of size d (see 3.0.6). Thus, the rank-one extreme points of $\mathcal{H}^+(f)$ can be tackled well with combinatorial considerations. We will see this even more impressively in Section 3.3 when we deal with polyhedral faces.

The Hermitian Gram spectrahedron of a positive binary form $f \in \mathbb{R}[x, y]_{2d}$ has dimension d^2. For the symmetric Gram spectrahedron we have $\dim \mathrm{Gram}(f) = \binom{d}{2}$ and in Corollary 3.1.3 we have shown that a face of $\mathrm{Gram}(f)$ of rank r has dimension at most $\binom{r-1}{2}$. We get analogous bounds in the Hermitian case:

3.2.1 Proposition. *Let $f \in \mathbb{R}[x, y]_{2d}$ be a nonnegative binary form. Let $F \subseteq \mathcal{H}^+(f)$ be a face of rank r. Then* $\max\{0, r^2 - (2d + 1)\} \leq \dim(F) \leq (r - 1)^2$.

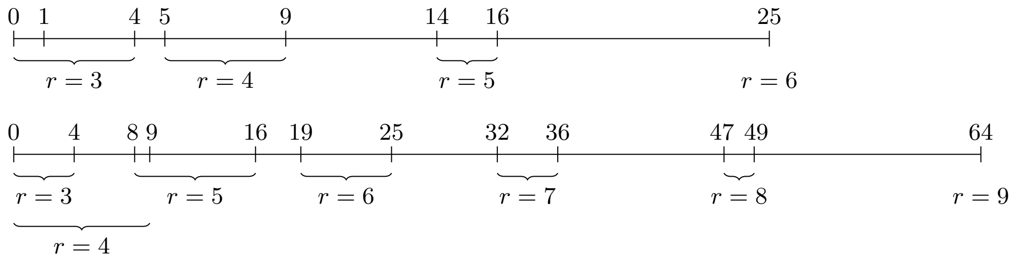


Figure 3.3. Intervals of possible dimensions of faces $F \subseteq \mathcal{H}^+(f)$ of rank r for $d = 5$ and $d = 8$.

Proof. Consider the face subspace $U = \mathcal{U}(F)$ of $\mathbb{C}[x,y]_d$. As $U\overline{U} \subseteq \mathbb{C}[x,y]_{2d}$, trivially $\dim_{\mathbb{C}}(U\overline{U}) \leq 2d+1$. Combining this with the formula from Proposition 2.4.10 gives the lower bound.

For the upper bound, the same argument as in the real case shows that $\dim_{\mathbb{C}}(U\overline{U})$ is at least $2r-1$ (cf. Proposition 3.1.2). Therefore,

$$\dim(F) = r^2 - \dim_{\mathbb{C}}(U\overline{U}) \leq r^2 - (2r-1) = (r-1)^2. \qquad \square$$

3.2.2 Remark (cf. Remark 3.1.4)**.** Let $\mathrm{rk}(F) = r$. Writing $k = (d+1) - r$, that is $k = \dim\ker(\vartheta)$ for $\vartheta \in \mathrm{relint}(F)$, we obtain $(d-k)^2 - 2k \leq \dim(F) \leq (d-k)^2$.

3.2.3 Example (cf. Example 3.1.11)**.** Figure 3.3 illustrates the possible combinations of $\mathrm{rk}(F)$ and $\dim(F)$ for faces of the Hermitian Gram spectrahedron of a positive binary form $f \in \Sigma_{2d}$ for $d = 5$ and $d = 8$. For the sake of clarity, we do not plot faces of rank one or two in the charts. A face of rank one is always an extreme point of $\mathcal{H}^+(f)$, while a face of rank two can be either an extreme point or the edge between two rank-one tensors.

The following proposition and corollary are in the spirit of Proposition 2.4.12. We are going to see that for positive $f \in \mathbb{R}[x,y]_{2d}$ the Hermitian Gram spectrahedron of f contains a face of rank d and dimension $(d-1)^2$. In contrast to that, the symmetric Gram spectrahedron does not contain a face of rank d and dimension $\binom{d-1}{2}$, as we have shown in Corollary 3.1.8.

3.2.4 Proposition (cf. [CPSV, Cor. 5.7])**.** *Let $f \in \mathbb{R}[x,y]_{2d}$ be a positive binary form with distinct roots. Then $\mathcal{H}^+(f)$ contains 2^d tensors of rank one. The sum of rank-one tensors $p_1 \otimes \overline{p_1}, \dots, p_s \otimes \overline{p_s}$ in $\mathcal{H}^+(f)$ satisfies*

$$\mathrm{rk}\left(\sum_{k=1}^{s} p_k \otimes \overline{p_k}\right) \leq d + 1 - \deg(\gcd(p_1, \dots, p_s)).$$

For each $2 \leq s \leq 2^d$, there are s rank-one tensors in $\mathcal{H}^+(f)$ whose sum has rank at most $\lceil \log_2(s) \rceil + 1$.

Proof. Let $g = \gcd(p_1, \dots, p_r)$ and $\deg(g) = e$, for instance $p_k = g q_k$ for some $q_k \in \mathbb{C}[x,y]_{d-e}$ $(k = 1, \dots, s)$. Then

$$\begin{aligned}\mathrm{rk}\left(\sum_{k=1}^{s} p_k \otimes \overline{p_k}\right) &= \dim\mathrm{span}(p_1, \dots, p_s) = \dim\mathrm{span}(q_1, \dots, q_s)\\ &\leq \dim \mathbb{C}[x,y]_{d-e} = (d-e) + 1 = d + 1 - \deg(g).\end{aligned}$$

For the rest of the proof we may assume that $f(x,1)$ is monic. We have seen before that $\mathcal{H}^+(f)$ contains 2^d tensors of rank one (cf. 3.0.6). Writing $a_1,\dots,a_d$, $\overline{a_1},\dots,\overline{a_d} \in \mathbb{C}$ for the roots of $f(x,1)$, these are precisely the tensors $p\otimes\overline{p}$ with $p=\prod_{j=1}^d(x-b_jy)\in\mathbb{C}[x,y]_d$ where $b_j\in\{a_j,\overline{a_j}\}$.

For the last part of the claim let $2\le s\le 2^d$ and $e\in\mathbb{N}_0$ such that $2^{d-e-1}<s\le 2^{d-e}$. Consider $g=\prod_{j=1}^e(x-a_jy)$, and choose s pairwise different elements $q_1,\dots,q_s$ of the set

$$\left\{\prod_{j=e+1}^{d}(x-b_jy) : b_j\in\{a_j,\overline{a_j}\}\right\}$$

(which has cardinality 2^{d-e}). Let $p_j=gq_j$. Then $p_1\otimes\overline{p_1},\dots,p_s\otimes\overline{p_s}$ are rank-one Hermitian Gram tensors of f, and g is the greatest common divisor of $p_1,\dots,p_s$. Hence,

$$\operatorname{rk}\left(\sum_{k=1}^{s}p_k\otimes\overline{p_k}\right)\le d+1-\deg(g)=(d-e)+1=\lceil\log_2(s)\rceil+1. \qquad \square$$

3.2.5 Corollary. *Let $f\in\mathbb{R}[x,y]_{2d}$ be a positive binary form. For each $1\le r\le d+1$ there is a face $F\subseteq\mathcal{H}^+(f)$ with $(\operatorname{rk}(F),\dim(F))=(r,(r-1)^2)$. In particular, the upper bound in Proposition 3.2.1 is sharp.*

Proof. Fix $r\in\{1,\dots,d+1\}$. Since a binary form factors completely into Hermitian squares, we can write $f=g\overline{g}\cdot h$, where $h\in\mathbb{R}[x,y]_{2(r-1)}$ is again a positive binary form and $g\in\mathbb{C}[x,y]_{d-(r-1)}$. By Proposition 2.4.12, $\mathcal{H}^+(h)$ is linearly isomorphic to a face F of $\mathcal{H}^+(f)$. Because h is in the interior of $\Sigma_{2(r-1)}$, any tensor in the relative interior of $\mathcal{H}^+(h)$ has rank r, and $\dim(\mathcal{H}^+(h))=(r-1)^2$. $\square$

3.3. Polyhedral faces of Hermitian Gram spectrahedra

Laurent and Poljak [LP96] analyze the facial structure of the *elliptope* $\mathcal{E}_{n\times n}$, which is the set of all correlation matrices of size $n\times n$, i.e., psd symmetric matrices whose diagonal entries are all equal to one. They are also interested in the polyhedral faces of those spectrahedra. Some techniques presented in the proof of Theorem 4.1 in [LP96] turn out to be helpful for understanding polyhedral faces in Hermitian Gram spectrahedra:

3.3.1 Theorem. *Let $f\in\mathbb{R}[x,y]_{2d}$ be a nonnegative binary form of degree $2d$. Let F be a polyhedral face of $\mathcal{H}^+(f)$ of dimension k. Then $\binom{k+1}{2}\le d$. Moreover, if all vertices of F are rank-one tensors, then F is a simplex.*

Proof. Let $F_0\subseteq F_1\subseteq\ldots\subseteq F_k:=F$ be a chain of faces of F, where $\dim(F_j)=j$ for all j. We must have $\operatorname{rk}(F_j)\ge j+1$, so $r:=\operatorname{rk}(F)\ge k+1$. Therefore, by Proposition 2.4.10,

$$k=\dim(F)\ge r^2-(2d+1)\ge(k+1)^2-(2d+1),$$

which is equivalent to $\binom{k+1}{2}\le d$.

Suppose now that all vertices of F are rank-one tensors resp. rank-one matrices, for instance $\operatorname{Ex}(F)=\{v_jv_j^*: j\in J\}$. Then $\mathcal{U}(F)=\operatorname{span}(v_j: j\in J)$ and $\dim\mathcal{U}(F)=\operatorname{rk}(F)\ge k+1$. Choose $k+1$ linearly independent vectors $v_0,\dots,v_k$ from $\{v_j: j\in J\}$. Then the vertices $v_jv_j^*$ $(j=0,1,\dots,k)$ affinely span the polyhedron F. We show that those are the only vertices of F. Assume X is another vertex of F. Then $X = \sum_{j=0}^{k} \alpha_j v_j v_j^*$ with $\sum_{j=0}^{k} \alpha_j = 1$. Let $l \in \{0, 1, \ldots, k\}$ and let

$$0 \neq u \in \operatorname{span}(v_j : j = 0, 1, \ldots, k, j \neq l)^{\perp} \cap \operatorname{span}(v_0, v_1, \ldots, v_k).$$

Since X is psd, we obtain

$$0 \leq u^* X u = \sum_{j=0}^{k} \alpha_j u^* v_j v_j^* u = \alpha_l u^* v_l v_l^* u = \alpha_l |u^* v_l|^2.$$

But $u^* v_l \neq 0$ and therefore $\alpha_l \geq 0$. This means that X is contained in the convex hull of $v_0 v_0^*, \ldots, v_k v_k^*$, a contradiction. We conclude that F is a simplex. □

3.3.2 Corollary. *If $F \subseteq \mathcal{H}^+(f)$ is a polyhedral face of dimension k and all vertices of F are rank-one tensors, then F is a simplex with vertices $\vartheta_j = p_j \otimes \overline{p_j}$ $(j = 0, \ldots, k)$ and the linear relations between the products $p_j \overline{p_l}$ are generated by the k obvious relations $p_0 \overline{p_0} = p_j \overline{p_j}$ for $j = 1, \ldots, k$.* □

Let $d \in \mathbb{N}$. We aim to construct a positive binary form $f \in \mathbb{R}[x, y]_{2d}$ with distinct roots such that the following holds: For all $k \in \mathbb{N}$ for which the inequality $\binom{k+1}{2} \leq d$ from Theorem 3.3.1 is satisfied, there is a polyhedral face of dimension k in the Hermitian Gram spectrahedron of f. We will need some preliminary work. Note that 3.3.3–3.3.5 do not require binary forms and could thus also be applied in more general settings.

3.3.3 Proposition. *Let F be a face of $\mathcal{H}^+(f)$, $\operatorname{rk}(F) = r$. If there is a basis $p_1, \ldots, p_r$ of $U = \mathcal{U}(F)$ such that $f = p_1 \overline{p_1} + \cdots + p_r \overline{p_r}$ and the linear relations among the products $p_j \overline{p_k}$ $(1 \leq j, k \leq r)$ only involve the Hermitian squares $p_1 \overline{p_1}, \ldots, p_r \overline{p_r}$, then F is polyhedral.*

Proof. We write $\mathfrak{p} = (p_1, \ldots, p_r)^T$. With respect to the basis $p_1, \ldots, p_r$ of U, the Gram matrices of f relative to U are of the form $I_r + A$ where A is a Hermitian $r \times r$-matrix with $\mathfrak{p}^T A \overline{\mathfrak{p}} = 0$. For $j \neq k$ there is no linear relation among the generators of $U\overline{U}$ which involves $p_j \overline{p_k}$. Therefore, any such A is diagonal. Thus, for this basis of U, the elements of F correspond to the solutions of a diagonal linear matrix inequality, so F is polyhedral. □

3.3.4 Remark. In the situation of the preceding proposition, any $\vartheta \in F$ has a representation $\vartheta = \sum_{j=1}^{r} a_j^2 (p_j \otimes \overline{p_j})$ with $a_j \in \mathbb{R}$. Indeed, if D is the (diagonal) Hermitian Gram matrix associated to ϑ with respect to the basis $p_1, \ldots, p_r$ of U, then we can choose a_j to be a square root of $D_{jj} \in \mathbb{R}_{\geq 0}$ $(j = 1, \ldots, r)$.

3.3.5 Corollary. *Let $\vartheta_0, \ldots, \vartheta_k \in \mathcal{H}^+(f)$. Let F be the smallest face of $\mathcal{H}^+(f)$ containing $\vartheta_0, \ldots, \vartheta_k$ (equivalently: F is the supporting face of $\frac{1}{k+1}(\vartheta_0 + \cdots + \vartheta_k)$ in the Hermitian Gram spectrahedron of f). If $\dim(F) = k$ and $\operatorname{rk}(F) = \sum_{i=0}^{k} \operatorname{rk}(\vartheta_i)$, then F is polyhedral.*

Proof. We write

$$\vartheta_i = \sum_{j=1}^{r_i} p_j^{(i)} \otimes \overline{p_j^{(i)}},$$

where $r_i = \operatorname{rk}(\vartheta_i)$. By the assumption on the rank of F, the family $(p_j^{(i)} : 0 \le i \le k,\ 1 \le j \le r_i)$ is linearly independent. Now we have at least the k relations

$$\sum_{j=1}^{r_0} p_j^{(0)} \overline{p_j^{(0)}} = \sum_{j=1}^{r_i} p_j^{(i)} \overline{p_j^{(i)}} \quad (i = 1, \ldots, k).$$

Because of $\dim(F) = k$, there are no further independent relations. Hence, F is polyhedral (see Proposition 3.3.3). □

3.3.6 Remark. In particular, if f is a binary form and if the supporting face F of $k+1$ rank-one extreme points of $\mathcal{H}^+(f)$ has rank $k+1$ (which is the maximal possible rank of F) and dimension k (which is the minimal possible dimension in this situation), then F is polyhedral.

These conditions also imply that all vertices of F are rank-one tensors. Indeed, if $F_0 \subseteq F_1 \subseteq \ldots \subseteq F_k := F$ was a chain of faces of F where $\operatorname{rk}(F_0) \ge 2$ and $\dim(F_j) = j$ for all j, we would have $\operatorname{rk}(F) \ge k+2$, a contradiction. So by Theorem 3.3.1, F is even a simplex.

In our construction of polyhedral faces, we will often be in the situation that we want to find a form $s \in \mathbb{C}[x,y]_k$ which does not divide any nonzero element of a given subspace $U \subseteq \mathbb{C}[x,y]_d$ of dimension $k = \deg(s)$.

3.3.7 Proposition. *Let $U \subseteq \mathbb{C}[x]_{\le d}$ be a linear subspace of dimension $\dim(U) = k \le d$. Then there are $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and a basis $p_1, \ldots, p_k$ of U such that $p_l(\lambda_l) \neq 0$ and $p_j(\lambda_l) = 0$ for all $j > l$. In particular, whenever $p \in U$ vanishes in $\lambda_1, \ldots, \lambda_k$ then $p = 0$.*

Proof. We use the following inductive procedure to find scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and to construct a basis of U with the desired properties. Start with any basis $q_1^{(1)}, \ldots, q_k^{(1)}$ of U. If $l \in \{1, \ldots, k-1\}$, choose $\lambda_l \in \mathbb{C} \setminus \mathcal{V}(q_l^{(l)})$. Then set

$$q_j^{(l+1)} := q_l^{(l)}(\lambda_l) \cdot q_j^{(l)} - q_j^{(l)}(\lambda_l) \cdot q_l^{(l)}, \quad j = l+1, \ldots, k.$$

This guarantees that $q_l^{(l)}(\lambda_l) \neq 0$ and $q_j^{(l+1)}(\lambda_l) = 0$ for all $j \ge l+1$. Furthermore, $q_1^{(1)}, q_2^{(2)}, \ldots, q_l^{(l)}, q_{l+1}^{(l+1)}, \ldots, q_k^{(l+1)}$ is again a basis of U. Also, if $\zeta \in \mathbb{C}$ was a common root of the polynomials $q_l^{(l)}, \ldots, q_k^{(l)}$ in the step before (as is the case for $\lambda_1, \ldots, \lambda_{l-1}$), it still is a common root of $q_l^{(l)}, q_{l+1}^{(l+1)}, \ldots, q_k^{(l+1)}$. By construction, setting $p_l := q_l^{(l)}$ $(l = 1, \ldots, k)$ yields the desired basis.

Suppose $p \in U$ with $p(\lambda_l) = 0$ for $l = 1, \ldots, k$. We have $p = \sum_{j=1}^k \mu_j p_j$ for some $\mu_j \in \mathbb{C}$ and

$$0 = p(\lambda_1) = \sum_{j=1}^k \mu_j p_j(\lambda_1) = \mu_1 p_1(\lambda_1).$$

Since $p_1(\lambda_1) \neq 0$, we deduce $\mu_1 = 0$. Iterating this argument, one successively shows that all μ_j's are zero. □

3.3.8 Remark. We see from the proof of Proposition 3.3.7 that (for a fixed subspace U) the scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ can be chosen from an open dense subset of $\mathbb{C}^k$. Indeed: We start with an arbitrary basis $p_1^{(1)}, \ldots, p_k^{(1)} \in \mathbb{C}[x]$ of U. If $l \in \{1, \ldots, k-1\}$, we set

$$\begin{aligned} p_j^{(l+1)}(x_1,\dots,x_l,x_{l+1}) := {} & p_l^{(l)}(x_1,\dots,x_{l-1},x_l)\cdot p_j^{(l)}(x_1,\dots,x_{l-1},x_{l+1}) \\ & - p_j^{(l)}(x_1,\dots,x_{l-1},x_l)\cdot p_l^{(l)}(x_1,\dots,x_{l-1},x_{l+1}) \\ & \in \mathbb{C}[x_1,\dots,x_l,x_{l+1}], \end{aligned}$$

for all $j = l+1,\dots,k$. For example,

$$p_j^{(2)}(x_1,x_2) = p_1^{(1)}(x_1)\cdot p_j^{(1)}(x_2) - p_j^{(1)}(x_1)\cdot p_1^{(1)}(x_2).$$

So if $p_1^{(1)}(\lambda_1) \neq 0$, then $p_j^{(2)}(\lambda_1,x) \in \mathbb{C}[x]$ coincides with $q_j^{(2)} \in \mathbb{C}[x]$ from the construction in Proposition 3.3.7. We set $q_j := p_j^{(j)} \in \mathbb{C}[x_1,\dots,x_j] \subseteq \mathbb{C}[x_1,\dots,x_k]$. Using induction, one can show that the set of suitable scalars contains

$$\left\{ (\lambda_1,\dots,\lambda_k) \in \mathbb{C}^k : \prod_{j=1}^{k} q_j(\lambda_1,\dots,\lambda_k) \neq 0 \right\}.$$

This set is nonempty and open in the Zariski topology on $\mathbb{C}^k$, and therefore also open and dense in the Euclidean topology.

The set of all $(\lambda_1,\dots,\lambda_k) \in \mathbb{C}^k$ with $\lambda_{j'} \neq \lambda_j, \overline{\lambda_j}$ for all $j' \neq j$ is open and dense as well. Therefore, we will assume that $|\{\lambda_j, \overline{\lambda_j} : j = 1,\dots,k\}| = 2k$ whenever needed.

We find polyhedral faces using the rank-one extreme points of $\mathcal{H}^+(f)$. The set of these points is denoted by $\mathrm{Ex}_1(\mathcal{H}^+(f))$.

3.3.9 Theorem. *Let $k \in \mathbb{N}$ and $d = \binom{k+1}{2}$. Then there exists a positive binary form $f \in \mathbb{R}[x,y]_{2d}$ with distinct roots such that $\mathcal{H}^+(f)$ contains a simplex face F with $(\mathrm{rk}(F), \dim(F)) = (k+1,k)$ and $\mathrm{Ex}(F) \subseteq \mathrm{Ex}_1(\mathcal{H}^+(f))$.*

Proof. We proceed by induction on k. Let $k = 1$, so $d = 1$. If $f \in \mathbb{R}[x,y]_2$ is positive, then $\mathcal{H}^+(f)$ is an interval of rank $d+1 = 2 = k+1$ whose extreme points have rank one. Now assume that $k \geq 2$, $d' = \binom{k}{2}$ and that we have a positive binary form $g \in \mathbb{R}[x,y]_{2d'}$ with distinct roots such that $\mathcal{H}^+(g)$ contains a polyhedral face F' with $(\mathrm{rk}(F'), \dim(F')) = (k, k-1)$ and $\mathrm{Ex}(F') \subseteq \mathrm{Ex}_1(\mathcal{H}^+(g))$. Then $U' := \mathfrak{U}(F')$ is spanned by some linearly independent $p_1,\dots,p_k \in \mathbb{C}[x,y]_{d'}$ with $g = p_j\overline{p_j}$ $(j = 1,\dots,k)$. Moreover, $\dim_{\mathbb{C}}(U') = k$ and $U'\overline{U'} = \mathbb{C}[x,y]_{2d'}$.

Let $\alpha_1,\dots,\alpha_{d'},\overline{\alpha_1}\dots,\overline{\alpha_{d'}} \in \mathbb{C}$ denote the (distinct) roots of $g(x,1)$. If $k \geq 3$, then $\dim_{\mathbb{C}}(U') = k \leq \binom{k}{2} = d'$ and we can use Proposition 3.3.7 to find $\beta_1,\dots,\beta_k \in \mathbb{C}$ such that

$$|\{\alpha_j, \overline{\alpha_j} : j = 1,\dots,d'\} \cup \{\beta_j, \overline{\beta_j} : j = 1,\dots,k\}| = 2d' + 2k = 2\binom{k+1}{2} = 2d,$$

and that whenever $p \in U'$ vanishes in $(\beta_1 : 1),\dots,(\beta_k : 1) \in \mathbb{P}^1$ then $p = 0$. (In the case $k = 2$ this is trivial since then $U' \subseteq \mathbb{C}[x,y]_1$, and a univariate linear polynomial has exactly one zero).

We define $s := \prod_{j=1}^{k}(x - \beta_j y) \in \mathbb{C}[x,y]_k$, and $t := p_l$ for some $l \in \{1,\dots,k\}$, e.g. $t = p_1$. Then $f := (st)\overline{(st)} = s\overline{s}\cdot g \in \mathbb{R}[x,y]_{2d}$ has distinct roots. Consider $F := \mathrm{suppface}(\vartheta_0,\dots,\vartheta_k) \subseteq \mathcal{H}^+(f)$ where $\vartheta_0 = st \otimes \overline{st}$ and $\vartheta_j = \overline{s}p_j \otimes s\overline{p_j}$ for $j = 1,\dots,k$. We have to show that $(\mathrm{rk}(F), \dim(F)) = (k+1,k)$ (that F is a simplex is then clear by Remark 3.3.6). The rank of F is given by the dimension of the subspace

$$U := \mathcal{U}(F) = \operatorname{span}(st, \overline{s}p_1, \dots, \overline{s}p_k) = \mathbb{C} \cdot st + \overline{s}U'.$$

If $\alpha st = \overline{s}p$ for some $\alpha \in \mathbb{C}$ and $p \in U'$, then s divides p since s and $\overline{s}$ are coprime. Therefore, $p(\beta_j, 1) = 0$ for all j, which implies $p = 0$. We conclude that $U = (\mathbb{C} \cdot st) \oplus \overline{s}U'$ and $\operatorname{rk}(F) = \dim_{\mathbb{C}}(U) = \dim_{\mathbb{C}}(U') + 1 = k + 1$. Furthermore,

$$\begin{aligned} U\overline{U} &= s\overline{s} \cdot U'\overline{U'} + s^2t\overline{U'} + \overline{s^2t}U' + \mathbb{C} \cdot \underbrace{(st)\overline{(st)}}_{\in s\overline{s} \cdot U'\overline{U'}} \\ &= s\overline{s} \cdot \mathbb{C}[x,y]_{2d'} + s^2t\overline{U'} + \overline{s^2t}U'. \end{aligned}$$

We first show that $s^2t\overline{U'} \cap \overline{s^2t}U' = \{0\}$. Let $u, v \in U'$ such that $s^2t\overline{u} = \overline{s^2t}v$. Since s^2t and $\overline{s^2t}$ have no roots in common, this implies that s^2t divides v. So if $v \neq 0$, we would get

$$\deg(v) \geq \deg(s^2t) = 2\deg(s) + \deg(t) > \deg(t) = \deg(v).$$

Therefore, the sum $s^2t\overline{U'} + \overline{s^2t}U'$ is direct. Now we show that also

$$(s\overline{s} \cdot \mathbb{C}[x,y]_{2d'}) \cap (s^2t\overline{U'} \oplus \overline{s^2t}U') = \{0\}. \tag{3.3.1}$$

Suppose that $s\overline{s}q = s^2t\overline{u} + \overline{s^2t}v$ where $q \in \mathbb{C}[x,y]_{2d'}$ and $u, v \in U'$. Then

$$s(\overline{s}q - st\overline{u}) = \overline{s^2t}v,$$

and hence s divides v. But since v is in U', $v = 0$ by the choice of s. Analogously, we see that $u = 0$. This proofs (3.3.1). To sum up,

$$\begin{aligned} \dim_{\mathbb{C}}(U\overline{U}) &= \dim_{\mathbb{C}}(\mathbb{C}[x,y]_{2d'}) + 2\dim_{\mathbb{C}}(U') \\ &= 2d' + 1 + 2k \\ &= 2(d' + k) + 1 \\ &= 2d + 1, \end{aligned}$$

and therefore $\dim(F) = (k+1)^2 - 2\binom{k+1}{2} - 1 = k$. □

3.3.10 Example. The proof of Theorem 3.3.9 was constructive. We illustrate the inductive argument and the combinatorial structure of the problem by giving explicit examples of polynomials $f^{(k)}$ for $k \leq 3$. Again, we identify $\mathbb{R}[x,y]_d$ with $\mathbb{R}[x]_{\leq d}$ to simplify notation. We can start with an arbitrary positive $f^{(1)} \in \mathbb{R}[x]$ of degree 2, say $f^{(1)} = x^2 - 2x + 2 = (x - \alpha)(x - \overline{\alpha})$ with $\alpha = 1 + i \in \mathbb{C} \setminus \mathbb{R}$. For $k = 2$ we first choose any $\beta_1, \beta_2 \in \mathbb{C}$ such that $\alpha, \overline{\alpha}, \beta_1, \overline{\beta_1}, \beta_2, \overline{\beta_2}$ are pairwise distinct, for example $\beta_1 = 3 + 2i$ and $\beta_2 = -1 + 4i$. Setting $s = (x - \beta_1)(x - \beta_2)$ and $t = x - \alpha$ leads to

$$f^{(2)} := st\overline{st} = x^6 - 6x^5 + 28x^4 - 120x^3 + 409x^2 - 594x + 442.$$

The 2-dimensional polyhedral face F of $\mathcal{H}^+(f^{(2)})$ we constructed in the proof is the triangle with vertices $\eta_j = q_j \otimes \overline{q_j}$ $(j = 0, 1, 2)$ where

$$\begin{aligned} q_0 &= (x - \beta_1)(x - \beta_2)(x - \alpha), \\ q_1 &= (x - \overline{\beta_1})(x - \overline{\beta_2})(x - \alpha), \\ q_2 &= (x - \overline{\beta_1})(x - \overline{\beta_2})(x - \overline{\alpha}). \end{aligned}$$

One can easily check that F is really polyhedral by verifying that for the subspace $U = \operatorname{span}_{\mathbb{C}}(q_0, q_1, q_2) \subseteq \mathbb{C}[x]_{\leq 3}$ we have $U\overline{U} = \mathbb{C}[x]_{\leq 6}$ (see Remark 3.3.6). The left

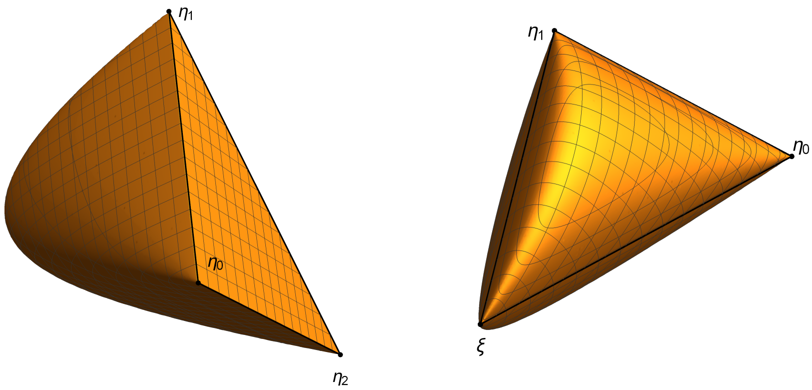


FIGURE 3.4. The polyhedral face F (on a 3-dimensional slice) and the non-polyhedral face F'.

picture in Figure 3.4 shows a 3-dimensional slice of $\mathcal{H}^+(f^{(2)})$ and the polyhedral face F. However, if we let

$$q = (x - \beta_1)(x - \overline{\beta_2})(x - \overline{\alpha}),$$

then the supporting face F' of the three rank-one extreme points η_0, η_1 and $\xi := q \otimes \overline{q}$ is three-dimensional. Indeed, for the subspace $W = \operatorname{span}_{\mathbb{C}}(q_0, q_1, q)$ we get $\dim_{\mathbb{C}}(W\overline{W}) = 6 = \dim_{\mathbb{C}}(W)^2 - 3$. The right picture in Figure 3.4 shows the face F' of $\mathcal{H}^+(f^{(2)})$. Again, we highlight the rank-one extreme points and the edges between them.

Now let $k = 3$. Starting with q_0, q_1, q_2 and following the algorithm presented in Proposition 3.3.7, we see that $\gamma_1 = -2+3i$, $\gamma_2 = -3+i$ and $\gamma_3 = 5+3i$ is one out of infinitely many possible choices for the roots of our next factor $(x-\gamma_1)(x-\gamma_2)(x-\gamma_3)$. With this special choice we get

$$f^{(3)} = x^{12} - 6x^{11} + 9x^{10} - 18x^9 + 497x^8 + 774x^7 - 2397x^6 - 9966x^5 + 21182x^4 + 247992x^3 + 439068x^2 - 1426776x + 1953640.$$

The 3-dimensional polyhedral face of $\mathcal{H}^+(f^{(3)})$ we constructed in the proof is the tetrahedron with vertices $\vartheta_j = p_j \otimes \overline{p_j}$ $(j = 0, \dots, 3)$ where

$$p_0 = (x - \gamma_1)(x - \gamma_2)(x - \gamma_3)q_0,$$
$$p_j = (x - \overline{\gamma_1})(x - \overline{\gamma_2})(x - \overline{\gamma_3})q_{j-1} \quad (j = 1, 2, 3).$$

As outlined above, to verify that this face is polyhedral one can check that the family $\{p_j\overline{p_l} : 0 \le j, l \le 3\}$ spans $\mathbb{C}[x]_{\le 12}$ (see Remark 3.3.6).

We have shown so far that we can construct polynomials which have polyhedral faces of the largest possible dimension in their Hermitian Gram spectrahedra. Now we want to show that the same holds true for "almost all" nonnegative binary forms. This is the content of Theorem 3.3.13.

3.3.11 Definition. Let $k \in \mathbb{N}$ and $d = \binom{k+1}{2}$. We define P_{2d} to be the set of all $f \in \mathbb{R}[x, y]_{2d}$ such that $f \in \operatorname{int}(\Sigma_{2d})$ has distinct roots and $\mathcal{H}^+(f)$ contains a simplex face F with $(\operatorname{rk}(F), \dim(F)) = (k + 1, k)$ and $\operatorname{Ex}(F) \subseteq \operatorname{Ex}_1(\mathcal{H}^+(f))$.

The reader is kindly advised not to confuse $\mathtt{P}_{2d}$ with the set of nonnegative forms of degree $2d$ which is sometimes denoted by P_{2d} or $\mathcal{P}_{2d}$. We chose our notation $\mathtt{P}_{2d}$ in order to refer to the particular polyhedral faces contained in the Hermitian Gram spectrahedra of the polynomials in this set.

3.3.12. Consider the set W of all tuples $(f, p_0, p_1, \dots, p_k) \in \mathbb{R}[x,y]_{2d} \times \mathbb{C}[x,y]_d^{k+1}$ such that the following holds: $f \in \operatorname{int}(\Sigma_{2d})$ has distinct roots (i.e., the discriminant $D(f(x,1)) \neq 0$), $f = p_0\overline{p_0} = \dots = p_k\overline{p_k}$, and for $U = \operatorname{span}_{\mathbb{C}}(p_0, \dots, p_k) \subseteq \mathbb{C}[x,y]_d$ we have $\dim_{\mathbb{C}}(U) = k+1$ and $U\overline{U} = \mathbb{C}[x,y]_{2d}$. Separating real and imaginary part of the coefficients, we can express all these conditions by polynomial equations and inequalities (over $\mathbb{R}$) in those coefficients. For instance, $U\overline{U} = \mathbb{C}[x,y]_{2d}$ if and only if not all $(2d+1)$-minors of the matrix containing the coefficients of $p_j\overline{p_{j'}}$ $(0 \leq j, j' \leq k)$ vanish. These minors are polynomial expressions in the real and imaginary parts of the coefficients of $p_0, \dots, p_k$. Therefore, W can be seen as an $\mathbb{R}$-semialgebraic set. By Remark 3.3.6, $\mathtt{P}_{2d}$ is the projection of W onto the first component and hence semialgebraic itself (cf. 1.1.9).

3.3.13 Theorem. *Let $k \in \mathbb{N}$ and $d \geq \binom{k+1}{2}$. The Hermitian Gram spectrahedron of a general nonnegative binary form $f \in \mathbb{R}[x,y]_{2d}$ contains a simplex face F with $(\operatorname{rk}(F), \dim(F)) = (k+1, k)$ and $\operatorname{Ex}(F) \subseteq \operatorname{Ex}_1(\mathcal{H}^+(f))$.*

Proof. It suffices to prove the theorem for $d = \binom{k+1}{2}$. According to 3.3.12, $\mathtt{P}_{2d}$ is semialgebraic. Thus, the assertion follows if we can show that $\mathtt{P}_{2d}$ is dense in Σ_{2d} (cf. 3.0.2).

We will prove this by induction on k. For $k = 1$ (i.e., $d = 1$) the Hermitian Gram spectrahedron of any positive $f \in \mathbb{R}[x,y]_2$ is an interval of rank 2 whose extreme points have rank one. Therefore, $\mathtt{P}_1 = \operatorname{int}(\Sigma_2)$. Now let $k \geq 2$, $d = \binom{k+1}{2}$ and define $d' = \binom{k}{2} = d - k$. Let $h \in \operatorname{int}(\Sigma_{2d})$ and let $U \subseteq \operatorname{int}(\Sigma_{2d})$ be an open neighborhood of h. We have to show that U contains some $\tilde{h} \in \mathtt{P}_{2d}$. Consider the multiplication map

$$\psi\colon \mathbb{R}[x,y]_{2d'} \times \mathbb{R}[x,y]_{2k} \to \mathbb{R}[x,y]_{2d}, \quad (f,g) \mapsto fg.$$

Without loss of generality, we can assume that the coefficient of x^{2d} in h is equal to 1. Since h is positive on $\mathbb{P}^1(\mathbb{R})$ and $\deg(h) = 2d = 2(d'+k)$, we can write

$$h = \underbrace{\prod_{j=1}^{d'}(x - \alpha_j y)(x - \overline{\alpha_j} y)}_{=:f \in \operatorname{int}(\Sigma_{2d'})} \cdot \underbrace{\prod_{j=1}^{k}(x - \beta_j y)(x - \overline{\beta_j} y)}_{=:g \in \operatorname{int}(\Sigma_{2k})},$$

so that $h = \psi(f,g)$. The fact that ψ is continuous implies that $\psi^{-1}(U)$ is open. In particular, there are open neighborhoods $V \subseteq \operatorname{int}(\Sigma_{2d'})$ of f and $W \subseteq \operatorname{int}(\Sigma_{2k})$ of g such that $\psi(V \times W) \subseteq U$. By induction, $\mathtt{P}_{2d'} \subseteq \Sigma_{2d'}$ is dense in the Euclidean topology. Therefore, we find a polynomial $\tilde{f} \in \mathtt{P}_{2d'} \cap V$. Now, the set of suitable factors for $\tilde{f}$ is dense in Σ_{2k} (see the proof of Theorem 3.3.9 and Remark 3.3.8). This means that there is $\tilde{g} \in W$ such that $\tilde{h} := \tilde{f}\tilde{g} \in \mathtt{P}_{2d}$. Moreover, $\tilde{h} = \psi(\tilde{f}, \tilde{g}) \in \psi(V \times W) \subseteq U$. We conclude that $\overline{\mathtt{P}_{2d}} = \Sigma_{2d}$, and this completes the proof. □

3.4. Counting polyhedral faces

The constructive proof of Theorem 3.3.9 points out how the combinatorics of the roots of a sufficiently general nonnegative binary form can be used in order to find a particular polyhedral face in its Hermitian Gram spectrahedron. Motivated by a question of Sinn at a SIAM conference where the article [May21] was presented, we now take a closer look on the number of polyhedral faces of this kind.

3.4.1 (The Hermitian Gram spectrahedron of a binary sextic.)**.** Let $f \in \mathbb{R}[x, y]_{2d}$ be a general positive binary form with distinct roots. We want to take a closer look on the case where f is a sextic, so let $d = 3$. There are $2^d = 8$ extreme points of rank one in the Hermitian Gram spectrahedron of f. We analyze the supporting face for any combination of three of these extreme points. Figure 3.4 shows two possible outcomes. We will see here that we get three fundamentally different types of faces and that they correspond to the types of triangles in a cube.

We have $\binom{8}{3} = 56$ possible combinations and we have already shown that at least one leads to a face which is a 2-simplex (triangle). But in our construction we made some choices and we have already mentioned that also different choices will give triangular faces. To simplify notation, we replace f by $f(x, 1) \in \mathbb{R}[x]$ and we let $\alpha_1, \overline{\alpha_1}, \alpha_2, \overline{\alpha_2}, \alpha_3, \overline{\alpha_3} \in \mathbb{C} \setminus \mathbb{R}$ denote the roots of this univariate polynomial. The eight rank-one tensors $p \otimes \overline{p}$ in $\mathcal{H}^+(f)$ can be encoded using the set $\{1, -1\}^3$ of vertices of the cube $C = [-1, 1]^3$. Having '1' at position j in a triple shall mean that we take $(x - \alpha_j)$ as a factor in p, while a '-1' at position j would suggest to take $(x - \overline{\alpha_j})$ instead. For example, the point $(1, -1, 1)$ stands for $p = (x - \alpha_1)(x - \overline{\alpha_2})(x - \alpha_3)$.

Now, let us take three (pairwise different) rank-one tensors $\eta_j = p_j \otimes \overline{p_j} \in \mathcal{H}^+(f)$ $(j = 0, 1, 2)$ and the corresponding vertices v_0, v_1, v_2 of the cube that encode for the polynomials p_0, p_1, p_2, respectively. As we have seen in the proof, $F := \text{suppface}(\eta_0, \eta_1, \eta_2)$ will be a triangle as soon as two of the three polynomials p_0, p_1, p_2 are complex conjugates (the third polynomial can be arbitrary). This means that two of the points v_j will be opposite vertices of C, joined by a space diagonal of the cube and completed to a triangle inside C by any of the other six vertices of C. There is a total of 24 right-angled triangles in C that "stand" on a face diagonal of the cube like the one in the left picture in Figure 3.5. Therefore, our construction gives 24 triangular faces in $\mathcal{H}^+(f)$ and every rank-one tensor is contained in exactly nine of those.

Furthermore, these are the only polyhedral faces in $\mathcal{H}^+(f)$ separate from edges: According to 3.3.1, if $F \subseteq \mathcal{H}^+(f)$ is a polyhedral face, then $\dim(F) \leq 2$ and F is a simplex as soon as all its vertices are rank-one tensors. We are going to see that the other 32 combinations of three rank-one tensors do not give polyhedral faces. Moreover, F cannot contain an extreme point of rank two since this would imply $\text{rk}(F) \geq 4$ and hence $\dim(F) \geq 9$, which is absurd.

In the language of (hyper)graphs – that we do not introduce here – this means:

3.4.2 Corollary. *Let $f \in \mathbb{R}[x, y]_{2d}$ be a general positive binary form with distinct roots. Consider the hypergraph H with vertices $V = \text{Ex}_1(\mathcal{H}^+(f))$ where a set $\Theta \subseteq V$ of vertices is connected by a hyperedge if and only if $\text{suppface}(\Theta) \subseteq \mathcal{H}^+(f)$ is a triangular face. Then H is a 3-uniform and 9-regular hypergraph.* □

3.4.3. We continue our analysis from 3.4.1. The next case is the one where our polynomials p_0, p_1, p_2 have a common root, say α_j. (Note that since $d = 3$, it is not possible that they have a common factor of degree 2.) Then, setting $U = \text{span}_{\mathbb{C}}(p_0, p_1, p_2)$ as usual, every element of $U\overline{U}$ is divisible by $(x - \alpha_j)(x - \overline{\alpha_j})$, meaning that $U\overline{U}$ has codimension 2 in $\mathbb{C}[x]_{\leq 6}$. Therefore, $F = \mathcal{F}(U)$ will be a

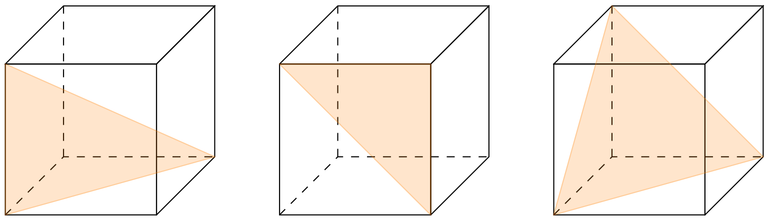


FIGURE 3.5. The three types of triangles in a cube.

face of dimension $3^2 - 5 = 4$ in the Hermitian Gram spectrahedron of f. In our geometrical encoding this situation corresponds to the fact that v_0, v_1, v_2 lie on a common facet of C like in the middle picture in Figure 3.5. On each of the six facets of the cube C we have four triangles with vertices in $\{1, -1\}^3$, resulting in a total of 24 faces of this type.

The right picture in Figure 3.5 shows one of the remaining eight triangles in C. They are obtained in the following way: Choose any one of the eight vertices v, then the three vertices that are "one edge away from v" form an equilateral triangle. For the corresponding polynomials, we always have the relations $p_0\overline{p_0} = p_1\overline{p_1} = p_2\overline{p_2}$. Using symbolic computation in SINGULAR [2] or `Mathematica` [3], one verifies that in this case, there is exactly one additional independent linear relation between the products $p_j\overline{p_k}$ $(0 \le j, k \le 2)$ (always under the assumption that the roots of f are sufficiently general). Therefore, we have eight combinations of three rank-one extreme points of $\mathcal{H}^+(f)$ that will result in a three-dimensional supporting face.

3.4.4 Proposition. *Let $k \in \mathbb{N}$ and $d_k = \binom{k+1}{2}$. Let $f \in \Sigma_{2d_k}$ be a general nonnegative binary form. Then the number of faces in $\mathcal{H}^+(f)$, which are k-simplices with rank-one vertices, is at least n_k, where n_k is given by the recurrence formula*

$$n_1 = 1 \quad \textit{and} \quad n_k = \binom{d_k}{k} \cdot 2^k \cdot k \cdot n_{k-1} \quad (k \ge 2).$$

Proof. We analyze our construction in the proof of Theorem 3.3.9 and the choices we made therein. The case $k = 1$ is clear since $\mathcal{H}^+(f)$ is an interval whose end points have rank one. So let $k \ge 2$. First note that $d_k = d_{k-1} + k$. Recall that we factored f as $f = (st)\overline{st} = s\overline{s} \cdot g$ with $g \in \Sigma_{2d_{k-1}}$, $s \in \mathbb{C}[x, y]_k$ and $t \in \mathbb{C}[x, y]_{d_{k-1}}$. The polynomial f has d_k pairs of complex conjugate roots. We can choose d_{k-1} of these pairs which shall be the roots of $g \in \Sigma_{2d_{k-1}}$ and the remaining k pairs are the roots of $s\overline{s}$.

By induction, $\mathcal{H}^+(g)$ contains at least n_{k-1} polyhedral faces of the desired form. We fix one of them and call it F'. Its vertices were denoted by $p_j \otimes \overline{p_j}$ $(j = 1, \ldots, k)$. Then we had k options for the choice of $t = p_j$.

There are 2^k possibilities to distribute k complex points and their conjugates between s and $\overline{s}$, and any of these choices leads to a new simplex face in $\mathcal{H}^+(f)$. This explains the factor 2^k. Finally, as mentioned above, we have $\binom{d_k}{d_{k-1}} = \binom{d_k}{k}$ options for the definition of g.

Of course, we need to make sure that there is no double counting. We show that if the set $M = \{st, \overline{s}p_1, \ldots, \overline{s}p_k\}$ was obtained using our construction, then st is uniquely determined by M. Unfortunately, at least for $k > 3$, this is not as obvious as it might look at first glance. As our proof of this fact is rather technical, we postpone it to Lemma 3.4.5. However, since $\overline{s}p_1, \ldots, \overline{s}p_k$ are divisible by $\overline{s}$ and we have $t = p_l$ for some $l \in \{1, \dots, k\}$, one easily gets s, as well as t and $\{p_1, \dots, p_k\}$, once st is identified in M. □

3.4.5 Lemma. *If the set $M = \{st, \overline{s}p_1, \dots, \overline{s}p_k\}$ was obtained using the construction from Theorem 3.3.9, then st is uniquely determined by M.*

Proof. Without loss of generality, we may assume that $t = p_1$. Note that $p_1, \dots, p_k$ have no common roots and that the element st is distinguished by the following facts: The elements of $M' = M \setminus \{st\}$ have precisely k roots in common, there is an element $h \in M'$ such that exactly d_{k-1} roots of h and st are the same (and k are conjugated), and for all $h \neq h' \in M'$ more than k roots of h' and st are conjugated.

We use these facts to identify st. Suppose for contradiction that also the set $\{s't', \overline{s'}p'_1, \dots, \overline{s'}p'_k\}$ was obtained using our construction, that it equals M and we have $s't' \neq st$. Then $\overline{s'} \mid st$ and $\overline{s'} \mid \overline{s}p_j$ for $k-1$ distinct values of $j \in \{1, \dots, k\}$. We may assume that

$$p_1 = t = \overline{s'}q, \quad p_2 = s'q_2, \quad p_j = \overline{s'}q_j \quad (j \geq 3)$$

for some q, $q_2, \dots, q_k \in \mathbb{C}[x, y]$ of degree $d_{k-1} - k = d_{k-2} - 1$. Thus, we can rewrite

$$\begin{aligned} M &= \{s\overline{s'}q, \overline{ss'}q, \overline{s}s'q_2, \overline{ss'}q_3, \dots, \overline{ss'}q_k\} \\ &= \{s'\overline{s}q_2\} \cup \overline{s'}\{sq, \overline{s}q, \overline{s}q_3, \dots, \overline{s}q_k\}. \end{aligned}$$

Now, the set $\widetilde{M} := \{sq, \overline{s}q, \overline{s}q_3, \dots, \overline{s}q_k\}$ must have originated from our construction. Consequently, there must be an element $h_0 \in \widetilde{M}$ that has the same properties with respect to $\widetilde{M}$ that st has with respect to M. In particular, there exists $h \in \widetilde{M} \setminus \{h_0\}$ such that precisely $k-1$ roots of h and h_0 are conjugated. Since $\deg(s) = k$, we have $h_0 \neq sq$. Furthermore, there exists a polynomial $s'' \in \mathbb{C}[x, y]_{k-1}$ such that $s'' \mid h_0$ and $\overline{s''} \mid h'$ for all $h_0 \neq h' \in \widetilde{M}$. Therefore, $h_0 \neq \overline{s}q$, and we can assume that $h_0 = \overline{s}q_3$. Rearranging leaves us with

$$\widetilde{M} = \{s''\overline{s}u_3\} \cup \overline{s''}\{su, \overline{s}u, \overline{s}u_4, \dots, \overline{s}u_k\},$$

where u, $u_3, \dots, u_k \in \mathbb{C}[x, y]$ are forms of degree $d_{k-2} - 1 - (k-1) = d_{k-3} - 2$. The same reasoning as before shows that u and $k-4$ of the u_j's must have a common factor of degree $k-2$. When iterating the argument for $l < k$, we get $k-l$ forms of degree $d_{k-l} + 1 - l$ that should have a common factor of degree $k - l + 1$. However, for sufficiently large values of l, this is absurd. □

3.4.6. Let $k \in \mathbb{N}$ and let $d = d_k = \binom{k+1}{2}$. If $f \in \mathbb{R}[x, y]_{2d}$ is a positive binary form with distinct roots, there are exactly 2^d rank-one extreme points in $\mathcal{H}^+(f)$. We know that a sophisticated choice of $k+1$ of these can give us a face of $\mathcal{H}^+(f)$ that is a k-simplex whose vertices are precisely the points we have chosen.

Our construction gives n_k such faces, where a formula for n_k is given in Proposition 3.4.4. For small values of k, we list the corresponding numbers n_k in the fourth column of Table 3.1. We have seen in 3.4.1 that $n_2 = 24$ is indeed the correct number in the case of binary sextics. To get a sense of the true number for larger k, let us approach the problem from the other direction. There is a situation where it is obvious that the supporting face of $k+1$ points $\eta_j = p_j \otimes \overline{p_j} \in \mathrm{Ex}_1(\mathcal{H}^+(f))$ $(j = 0, \dots, k)$ cannot be polyhedral. Indeed, if $p_0, \dots, p_k$ have a common root, then $U\overline{U}$ will fail to be all of $\mathbb{C}[x, y]_{2d}$. We want to count how many of the $\binom{2^d}{k+1}$ possible combinations of $k+1$ rank-one points are actually of this type. To this end, we have to solve the following combinatorial problem:

Table 3.1. The number of choices that obviously lead to non-polyhedral faces.

k	$d = \binom{k+1}{2}$	$\binom{2^d}{k+1}$	n_k	$\lvert\mathcal{S}(k,d)\rvert$	$\lvert\mathcal{S}(k,d)\rvert/\binom{2^d}{k+1}$
2	3	56	24	24	0.429
3	6	635376	11520	333280	0.525
4	10	$\approx 9.29 \cdot 10^{12}$	$\approx 1.55 \cdot 10^{8}$	$\approx 4.39 \cdot 10^{12}$	0.473
5	15	$\approx 1.72 \cdot 10^{24}$	$\approx 7.44 \cdot 10^{13}$	$\approx 6.51 \cdot 10^{23}$	0.379
6	21	$\approx 3.54 \cdot 10^{40}$	$\approx 1.55 \cdot 10^{21}$	$\approx 9.97 \cdot 10^{39}$	0.282

> Let $k, d \in \mathbb{N}$, let $C = [-1,1]^d$ be a d-dimensional hypercube and let $V = \{1,-1\}^d$ be the set of its vertices. Calculate the number $|\mathcal{S}(k,d)|$ where
> $$\mathcal{S}(k,d) = \{S \subseteq V : |S| = k+1 \text{ and } S \subseteq F \text{ for a proper face } F \subseteq C\}.$$

In other words, if $S \subseteq V$ has cardinality $k+1$, then $S \in \mathcal{S}(k,d)$ if and only if there is a coordinate in which all elements of S coincide. We note without proof that

$$|\mathcal{S}(k,d)| = \sum_{i=1}^{d} (-1)^{i-1} \cdot 2^i \cdot \binom{d}{i} \cdot \binom{2^{d-i}}{k+1}.$$

In the last column of Table 3.1 we compare this number to the number of all possible combinations of $k+1$ rank-one points. Although the fraction $|\mathcal{S}(k,d)|/\binom{2^d}{k+1}$ is not neglectable for small k and $d = \binom{k+1}{2}$, it is obvious that it tends to 0 as k goes to infinity.

3.4.7 Example ($d = 6$)**.** Let $f \in \mathbb{R}[x,y]$ be a general positive binary form of degree $2d = 12$. Since the roots of f are distinct, there are $2^6 = 64$ factorizations of our polynomial into a Hermitian square $f = p_j\overline{p_j}$ and they correspond to the rank-one extreme points $p_j \otimes \overline{p_j} \in \mathcal{H}^+(f)$, $j = 1,\dots,64$. Choosing four distinct numbers $j_1,\dots,j_4 \in \{1,\dots,64\}$ gives a subspace $U := \operatorname{span}(p_{j_1},\dots,p_{j_4}) \subseteq \mathbb{C}[x,y]_6$ and thereby a face $\mathcal{F}(U) \subseteq \mathcal{H}^+(f)$ which is the supporting face of the corresponding four rank-one extreme points and whose dimension is determined by $\dim(U)$ and $\dim(U\overline{U})$.

We have $\dim(U) = 3$ for exactly 240 choices. Indeed, $2^{d-2} \cdot \binom{d}{d-2} = 2^4 \cdot \binom{6}{2} = 240$ is precisely the number of choices where $p_{j_1},\dots,p_{j_4}$ are divisible by a form p of degree 4. Dividing by p leaves us with elements in the three-dimensional space $\mathbb{C}[x,y]_2$. Hence, $\dim(U) = 3$ for general f. We identify $\operatorname{Ex}_1(\mathcal{H}^+(f))$ with the set $\{1,-1\}^6$ of vertices of the six-dimensional hypercube as before. Then, in the situation above, there are four coordinates in which our four vertices coincide. Consequently, they are the vertices of a 2-face of the 6-cube (and there are 240 such faces). In the Hermitian Gram spectrahedron of f this means that $\mathcal{F}(U)$ is a face of rank $\dim(U) = 3$ and dimension $\dim \mathcal{F}(U) = \dim(U)^2 - \dim(U\overline{U}) = 9 - 5 = 4$.

Any other supporting face of four rank-one extreme points of $\mathcal{H}^+(f)$ has rank 4. As we have seen before, there are 333280 cases in which U has a base-point, and we have discussed 240 in detail. If $\dim(U) = 4$ and the maximum degree of any form dividing all elements of U is 1, 2 or 3, then the codimension of $U\overline{U}$ in $\mathbb{C}[x,y]_{12}$ will be at least 2, 4 or equal to 6, respectively. The resulting faces in $\mathcal{H}^+(f)$ have dimension ≥ 5.

TABLE 3.2. The supporting faces of four rank-one extreme points in $\mathcal{H}^+(f)$ for the form $f \in \mathbb{R}[x,y]_{12}$ from Example 3.4.7.

$\dim(U\overline{U})$	5	7	8	9	10	11	12	13
$\dim \mathcal{F}(U)$	4	9	8	7	6	5	4	3
#	240	10240	720	78256	0	249120	10482	286318

To illustrate this with a concrete example, let

$$f = x^{12} - 40\,x^{11}y + 875\,x^{10}y^2 - 11780\,x^9y^3 + 118883\,x^8y^4 - 933160\,x^7y^5 + 6457025\,x^6y^6 - 33994220\,x^5y^7 + 164813516\,x^4y^8 - 607654000\,x^3y^9 + 1439521700\,x^2y^{10} - 3874262000\,xy^{11} + 8923300000\,y^{12}.$$

The roots of f are the $(\alpha : 1) \in \mathbb{P}^1(\mathbb{C})$ where

$$\alpha \in \{-2 \pm 6i,\ -1 \pm 3i,\ 9 \pm 8i,\ 9 \pm 10i,\ 1 \pm 7i,\ 4 \pm i\}.$$

For any possible choice of four rank-one extreme points of $\mathcal{H}^+(f)$ we calculated the rank and the dimension of their supporting face using `Mathematica` [3] – a calculation that took about nine hours. The results are listed in Table 3.2. The faces $\mathcal{F}(U)$ of rank 4 and dimension 3 are tetrahedral. While our construction gives $n_3 = 11520$ faces of $\mathcal{H}^+(f)$ of this type, the number 286318 in this example shows that there are many more ways to obtain such faces.

Let $k \in \mathbb{N}$ and $d = \binom{k+1}{2}$. A general philosophy is that $U\overline{U}$ should be equal to $\mathbb{C}[x,y]_{2d}$ if the $(k+1)$-dimensional subspace $U \subseteq \mathbb{C}[x,y]_d$ is sufficiently general. Given $f \in \Sigma_{2d}$, this could mean that $\operatorname{suppface}(\vartheta_0, \ldots, \vartheta_k) \subseteq \mathcal{H}^+(f)$ will be a k-simplex for "most" choices of $k+1$ points $\vartheta_0, \ldots, \vartheta_k \in \operatorname{Ex}_1(\mathcal{H}^+(f))$. These considerations and some computational evidence from sampling leads to the following conjecture:

3.4.8 Conjecture. *Let $k \in \mathbb{N}$ and $d = \binom{k+1}{2}$. For $f \in \mathbb{R}[x,y]_{2d}$ we denote by $m_k(f)$ the number of faces in $\mathcal{H}^+(f)$ that are k-simplices with rank-one vertices. Let $m_k = \max\{m_k(f) : f \in \mathbb{R}[x,y]_{2d}\}$. Then $m_k/\binom{2^d}{k+1} \to 1$ for $k \to \infty$.*

3.5. Polyhedral faces of Gram spectrahedra

We will use the construction from the Hermitian case to construct polyhedral faces in the (symmetric) Gram spectrahedron of some binary forms. Many facts we proved in Section 3.3 translate nicely to the real symmetric case but others do not and some new phenomena can occur (see Remark 3.5.7 for an example). We start this section with an upper bound for the dimension of polyhedral faces.

3.5.1 Proposition. *Let $f \in \mathbb{R}[x,y]_{2d}$ be a general nonnegative binary form of degree $2d$. Let $F \subsetneq \operatorname{Gram}(f)$ be a polyhedral face of dimension $k \geq 1$. Then $\binom{k+3}{2} \leq 2d - 2$.*

Proof. There is a chain $F_0 \subseteq F_1 \subseteq \ldots \subseteq F_k := F$ of faces of F where $\dim(F_i) = i$ for all $i = 0, \ldots, k$. The corresponding chain of ranks of these faces has to be strictly increasing. Since f is general, $\operatorname{Gram}(f)$ does not contain neither points of rank 1 nor positive-dimensional faces of rank 3. Therefore, $\operatorname{rk}(F_1) \geq 4$ and $r = \operatorname{rk}(F) \geq k+3$. So the estimation we get in the real symmetric case is

$$k = \dim(F) \geq \binom{r+1}{2} - (2d+1) \geq \binom{k+4}{2} - (2d+1),$$

which is equivalent to the inequality in the claim. □

3.5.2 Proposition. *Let F be a face of* $\mathrm{Gram}(f)$, $\mathrm{rk}(F) = r$. *If there is a basis $p_1, \dots, p_r$ of $U = \mathcal{U}(F)$ such that $f = p_1^2 + \cdots + p_r^2$ and the quadratic relations among the p_i's only involve the squares $p_1^2, \dots, p_r^2$, then F is polyhedral.*

Proof. Note that $p_j = \overline{p_j}$ for all j, and that UU is generated by the products $p_j p_k$, where $1 \leq j \leq k \leq r$. Aside from these modifications, the proof is the same as for Proposition 3.3.3. □

3.5.3 Remark. In the situation of the preceding proposition, any $\vartheta \in F$ has a representation $\vartheta = \sum_{i=1}^{r} (a_i p_i) \otimes (a_i p_i)$ with $a_i \in \mathbb{R}$. Indeed, if D is the (diagonal) Gram matrix associated to ϑ with respect to the basis $p_1, \dots, p_r$ of U, then we can choose a_i to be a square root of $D_{ii} \geq 0$ $(i = 1, \dots, r)$.

3.5.4 Corollary (cf. Corollary 3.3.5). *Let $F \subseteq \mathrm{Gram}(f)$ be the supporting face of $k+1$ points $\vartheta_0, \dots, \vartheta_k \in \mathrm{Gram}(f)$. If $\dim(F) = k$ and $\mathrm{rk}(F) = \sum_{i=0}^{k} \mathrm{rk}(\vartheta_i)$, then F is polyhedral.* □

3.5.5 Remark. In particular, if $f \in \Sigma_{2d}$ is a binary form and if the supporting face F of $k+1$ rank-two extreme points of $\mathrm{Gram}(f)$ has rank $2(k+1)$ (which is the maximal possible rank of this face) and dimension k (which is the minimal possible dimension in this situation), then F is polyhedral.

In the situation of Remark 3.5.5, we can prove the following more detailed statement on the structure of the polyhedral face F. Recall that $\mathrm{Ex}_2(f)$ denotes the set of rank-two extreme points of $\mathrm{Gram}(f)$.

3.5.6 Proposition. *Let $f \in \Sigma_{2d}$ be a binary form and let $F \subseteq \mathrm{Gram}(f)$ be the supporting face of $k+1$ rank-two extreme points $\vartheta_0, \dots, \vartheta_k \in \mathrm{Ex}_2(f)$. If $\dim(F) = k$ and $\mathrm{rk}(F) = 2(k+1)$, then F is a simplex with vertices $\vartheta_0, \dots, \vartheta_k$, and the rank of any $\vartheta \in F$ is even.*

Proof. Using $\mathrm{rk}(\vartheta_i) = 2$, we write $\vartheta_i = q_{2i+1} \otimes q_{2i+1} + q_{2i+2} \otimes q_{2i+2}$ for every $i = 0, \dots, k$. By the assumption on the rank of F, we know that the family $\mathcal{B} = (q_j : j = 1, \dots, 2(k+1))$ is a basis of $U = \mathcal{U}(F) \subseteq \mathbb{R}[x,y]_d$. Furthermore, the quadratic relations between the elements of $\mathcal{B}$ are generated by

$$q_1^2 + q_2^2 = q_{2i+1}^2 + q_{2i+2}^2 \quad (i = 1, \dots, k).$$

So any quadratic relation is of the form

$$\sum_{i=0}^{k} \lambda_i \left(q_{2i+1}^2 + q_{2i+2}^2 \right) = 0, \tag{3.5.1}$$

where $\lambda_0, \lambda_1, \dots, \lambda_k \in \mathbb{R}$ with $\sum \lambda_i = 0$. Let $\vartheta \in F$. By Remark 3.5.3, there is a representation $\vartheta = \sum_{j=1}^{2(k+1)} (a_j q_j) \otimes (a_j q_j)$ with $a_j \in \mathbb{R}$. This leads to the quadratic relation

$$q_1^2 + q_2^2 = f = \mu(\vartheta) = \sum_{i=0}^{k} a_{2i+1}^2 q_{2i+1}^2 + a_{2i+2}^2 q_{2i+2}^2.$$

Comparing this to the general appearance (3.5.1) of a quadratic relation, we get $a_{2i+1}^2 = a_{2i+2}^2$ for all $i = 0, \dots, k$. Thus, $\mathrm{rk}(\vartheta) = 2 \cdot |\{i \in \{0, \dots, k\} : a_{2i+1} \neq 0\}|$ is

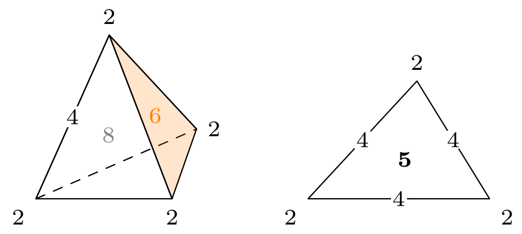


FIGURE 3.6. A schematic representation of the structure of a polyhedral face as in Proposition 3.5.6 and of the non-diagonalizable face in Example 3.5.16. The labels are meant to indicate the ranks of faces.

even. Besides, we can rewrite

$$\vartheta = \sum_{i=0}^{k} b_i^2 (q_{2i+1} \otimes q_{2i+1} + q_{2i+2} \otimes q_{2i+2})$$

where $b_i = a_{2i+1}$. Choose $l \in \{0, \dots, k\}$ with $b_l \neq 0$. Then

$$\mathcal{U}(\{\vartheta_l\}) = \operatorname{span}(q_{2l+1}, q_{2l+2}) \subseteq \operatorname{im}(\vartheta)$$

and thus $\vartheta_l \in \operatorname{suppface}(\vartheta)$. Hence, if $\vartheta \in \operatorname{Ex}(F)$ is an extreme point of $\operatorname{Gram}(f)$, then $\vartheta = \vartheta_l$. We conclude that $F = \operatorname{conv}(\operatorname{Ex}(F)) = \operatorname{conv}(\vartheta_0, \dots, \vartheta_k)$ is a simplex. □

3.5.7 Remark. For $d \leq 5$, Proposition 3.5.1 implies $k \leq 1$, i.e., there are no polyhedral faces bigger than edges. We are able to construct polynomials with two-dimensional polyhedral faces of rank 6 in their Gram spectrahedra as soon as $d \geq 9$ (see Theorem 3.5.11). Two-dimensional polyhedral faces can in principle also exist for $d \in \{6, 7, 8\}$. But the rank of such a face F has to be (at most) 5. This means that there has to be a linear dependency between the polynomials of which the different extreme points of F are made up. Furthermore, if $\operatorname{Ex}(F) \subseteq \operatorname{Ex}_2(f)$, then the face F is not *diagonalizable* as the following proposition shows. We give an example of a special polynomial $f \in \mathbb{R}[x, y]_{12}$ together with a two-dimensional non-diagonalizable polyhedral face $F \subseteq \operatorname{Gram}(f)$ in Example 3.5.16.

3.5.8 Proposition. *Let $f \in \mathbb{R}[x, y]_{2d}$ be a general nonnegative binary form of degree $2d$. (By general we mean at this point that $\operatorname{Gram}(f)$ has no positive-dimensional face of rank three, cf. Proposition 3.1.10). Let F be a (polyhedral) face of dimension k with $\operatorname{Ex}(F) \subseteq \operatorname{Ex}_2(f)$. If there is a basis $\mathcal{B}$ of $U = \mathcal{U}(F)$ such that all matrices associated to the Gram tensors in F are diagonal with respect to $\mathcal{B}$, then $r = \operatorname{rk}(F) \geq 2(k + 1)$ and $(k + 1)^2 \leq d$.*

Proof. Let $\mathcal{B} = (p_1, \dots, p_r)$ be such a basis of U. Since F is a k-dimensional polytope and $\operatorname{Ex}(F) \subseteq \operatorname{Ex}_2(f)$, we can choose $k+1$ distinct extreme points $\vartheta_0, \dots, \vartheta_k \in \operatorname{Ex}(F)$ of rank two. For $i = 0, \dots, k$ we consider the face subspaces $U_i = \mathcal{U}(\{\vartheta_i\}) \subseteq U$. By assumption, every $\vartheta \in F$ has a representation $\sum_{i=1}^{r} (a_i p_i) \otimes (a_i p_i)$ with $a_i \in \mathbb{R}$. This means that every facial subspace (for the given spectrahedron $F = \operatorname{Gram}_U(f)$) of U is generated by a subset of the p_i's. In particular, each U_i has a basis consisting of exactly two elements out of $p_1, \dots, p_r$. Assume that $r = \operatorname{rk}(F) < 2(k + 1)$. Then there are $j \in \{1, \dots, r\}$ and $i_1 \neq i_2$ such that $p_j \in U_{i_1} \cap U_{i_2} \neq \{0\}$. But then

$$2 < \operatorname{rk}(\vartheta_{i_1} + \vartheta_{i_2}) = \dim(U_{i_1} + U_{i_2}) \leq 3.$$

This would mean that ϑ_{i_1} and ϑ_{i_2} are contained in a positive-dimensional face of rank three. However, for general f, there is no such face in $\mathrm{Gram}(f)$ (Proposition 3.1.10).

Finally,

$$k = \dim(F) \geq \binom{r+1}{2} - (2d+1) \geq \binom{2(k+1)+1}{2} - (2d+1),$$

which can be simplified to $(k+1)^2 \leq d$. □

We will use the construction from the Hermitian case as a foundation for our construction in the real symmetric case. Since we aim for k-simplices with extreme points of rank two (instead of rank one), we have to allow for higher degree polynomials (cf. Proposition 3.5.8), meaning that we will have to introduce another factor to make up for the difference. Besides, we have to make sure that not only $U\overline{U}$ is big (as needed in the Hermitian case), but also UU. This means that we have to choose the factor slightly more carefully.

3.5.9 Lemma (cf. Proposition 3.3.7)**.** *Let $U \subseteq \mathbb{C}[x]_{\leq d}$ be a linear subspace of dimension $\dim(U) = k \leq d$. Then there are $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ such that the following holds: Whenever $p \in U$ and $p(\lambda_j) = 0$ for all j* or *$p(\overline{\lambda_j}) = 0$ for all j, then $p = 0$.*

Proof. We start with *two* arbitrary bases $q_1^{(1)}, \ldots, q_k^{(1)}$ and $\tilde{q}_1^{(1)}, \ldots, \tilde{q}_k^{(1)}$ of U. If $l \in \{1, \ldots, k-1\}$, we choose

$$\lambda_l \in \mathbb{C} \setminus \left(\mathcal{V}\left(q_l^{(l)}\right) \cup \mathcal{V}\left(\overline{\tilde{q}_l^{(l)}}\right)\right).$$

Then set

$$\begin{aligned} q_j^{(l+1)} &:= q_l^{(l)}(\lambda_l) \cdot q_j^{(l)} - q_j^{(l)}(\lambda_l) \cdot q_l^{(l)} \quad \text{and} \\ \tilde{q}_j^{(l+1)} &:= \tilde{q}_l^{(l)}(\overline{\lambda_l}) \cdot \tilde{q}_j^{(l)} - \tilde{q}_j^{(l)}(\overline{\lambda_l}) \cdot \tilde{q}_l^{(l)}, \quad j = l+1, \ldots, k. \end{aligned}$$

Among other things, this guarantees that $q_j^{(l+1)}(\lambda_l) = 0$ and $\tilde{q}_j^{(l+1)}(\overline{\lambda_l}) = 0$ for all $j \geq l+1$, whereas $q_l^{(l)}(\lambda_l) \neq 0$ and $\tilde{q}_l^{(l)}(\overline{\lambda_l}) \neq 0$. Everything else follows as in Proposition 3.3.7. Note that if $p \in U$ vanishes in $\lambda_1, \ldots, \lambda_k$, we can use the first basis to show that $p = 0$, while in the case of p vanishing in $\overline{\lambda_1}, \ldots, \overline{\lambda_k}$ we can use the second basis to come to the same conclusion. □

3.5.10 Lemma. *Let $k \in \mathbb{N}$ and $d = \binom{k+1}{2}$. The binary form $f \in \mathbb{R}[x,y]_{2d}$ in Theorem 3.3.9 can be constructed in such a way that for the subspace $U \subseteq \mathbb{C}[x,y]_d$ from the proof we have not only $U\overline{U} = \mathbb{C}[x,y]_{2d}$ but also $\dim_{\mathbb{C}}(UU) = \binom{k+2}{2}$.*

Proof. This is clear for $k = 1$. Let $k \geq 2$ and $d' = \binom{k}{2}$. By induction, we assume that we have a facial subspace $U' \subseteq \mathbb{C}[x,y]_{d'}$ for the Hermitian Gram spectrahedron of $g \in \mathbb{R}[x,y]_{2d'}$ such that $\dim_{\mathbb{C}}(U') = k$ and $U'\overline{U'} = \mathbb{C}[x,y]_{2d'}$ as in the proof of Theorem 3.3.9, and in addition $\dim_{\mathbb{C}}(U'U') = \binom{k+1}{2}$, i.e., U' is *quadratically independent.*

Given the roots $\alpha_j, \overline{\alpha_j} \in \mathbb{C}$ of $g(x,1)$, and using Lemma 3.5.9 if $k \geq 3$, we find $\beta_1, \ldots, \beta_k \in \mathbb{C}$ such that

$$|\{\alpha_j, \overline{\alpha_j} : j = 1, \ldots, d'\} \cup \{\beta_j, \overline{\beta_j} : j = 1, \ldots, k\}| = 2d,$$

and that whenever $p(\beta_1,1)=\cdots=p(\beta_k,1)=0$ or $p(\overline{\beta_1},1)=\cdots=p(\overline{\beta_k},1)=0$ for some $p\in U'$ then already $p=0$. We define s, t and f exactly as in the proof of 3.3.9, as well as

$$U=(\mathbb{C}\cdot st)\oplus\overline{s}U'.$$

It remains to show that the sum

$$UU=\mathbb{C}\cdot s^2t^2+s\overline{s}tU'+\overline{s}^2U'U'$$

is direct. Then the induction hypothesis implies

$$\dim_{\mathbb{C}}(UU)=1+\dim_{\mathbb{C}}(U')+\dim_{\mathbb{C}}(U'U')=1+k+\binom{k+1}{2}=\binom{k+2}{2}.$$

Suppose we have $s\overline{s}tu=\overline{s}^2w$ for some $u\in U'$ and $w\in U'U'$. Then $stu=\overline{s}w$, so $\overline{s}$ divides $u\in U'$. By construction, $u=0$. Hence, $s\overline{s}tU'\cap\overline{s}^2U'U'=\{0\}$. Finally, $s^2t^2\notin s\overline{s}tU'+\overline{s}^2U'U'$ is clear since $\overline{s}$ does not divide s^2t^2. ☐

After these minor technical preparations, we are ready to construct binary forms with nontrivial polyhedral faces in their (real symmetric) Gram spectrahedra. Note that the degree of the form constructed in Theorem 3.5.11 is larger than the lower bound on the degree given in Proposition 3.5.1. In the Hermitian case, however, the bound from Theorem 3.3.1 is realized by all sufficiently general forms. This new phenomenon in the real symmetric setting is also discussed in Remark 3.5.7 and Example 3.5.16. It is a relic of the fact that for a face $F\subseteq\mathrm{Gram}(f)$ and a point $\vartheta\in\mathrm{Ex}_2(f)$, there is a third option between $\mathrm{rk}(F)$ and $\mathrm{rk}(F)+\mathrm{rk}(\vartheta)$ for the rank of $\mathrm{suppface}(F\cup\{\vartheta\})\subseteq\mathrm{Gram}(f)$.

3.5.11 Theorem. *Let $k\in\mathbb{N}$ and $d=(k+1)^2$. Then there exists a positive binary form $f\in\mathbb{R}[x,y]_{2d}$ with distinct roots such that* $\mathrm{Gram}(f)$ *contains a simplex face F with* $(\mathrm{rk}(F),\dim(F))=(2(k+1),k)$ *and* $\mathrm{Ex}(F)\subseteq\mathrm{Ex}_2(f)$.

Proof. The proof heavily relies on the construction in the Hermitian case. Let $d_0=\binom{k+1}{2}$ and let $g\in\mathbb{R}[x,y]_{2d_0}$ be a positive binary form with distinct roots such that $\mathcal{H}^+(g)$ contains a simplex face F_0 with the following properties (see Theorem 3.3.9 and Lemma 3.5.10):

- $(\mathrm{rk}(F_0),\dim(F_0))=(k+1,k)$,
- $\mathrm{Ex}(F_0)\subseteq\mathrm{Ex}_1(\mathcal{H}^+(g))$,
- the subspace $U_0=\mathcal{U}(F_0)\subseteq\mathbb{C}[x,y]_{d_0}$ is quadratically independent, that is to say $\dim_{\mathbb{C}}(U_0U_0)=\binom{\dim(U_0)+1}{2}=\binom{k+2}{2}$.

Write $\mathrm{Ex}(F_0)=\{p_j\otimes\overline{p_j}:j=0,\dots,k\}$, so that $U_0=\mathrm{span}_{\mathbb{C}}(p_0,\dots,p_k)$. Due to Proposition 3.3.7, we can choose some $q\in\mathbb{C}[x,y]$ of degree $d-d_0=(k+1)^2-\binom{k+1}{2}=\binom{k+2}{2}=\dim_{\mathbb{C}}(U_0U_0)$ such that $f:=q\overline{q}\cdot g\in\mathbb{R}[x,y]_{2d}$ has distinct roots and q does not divide any nonzero element of $\overline{U_0U_0}$. Then $qp_j\otimes\overline{qp_j}$ are rank-one tensors in the Hermitian Gram spectrahedron of f. So

$$\vartheta_j:=\mathrm{Re}(qp_j)\otimes\mathrm{Re}(qp_j)+\mathrm{Im}(qp_j)\otimes\mathrm{Im}(qp_j)\in\mathrm{Ex}_2(f)$$

for all $j=0,\dots,k$. Consider $F=\mathrm{suppface}(\vartheta_j:j=0,\dots,k)\subseteq\mathrm{Gram}(f)$. For the facial subspace $U:=\mathcal{U}(F)\subseteq\mathbb{R}[x,y]_d$ of F we have

$$U_{\mathbb{C}}=\mathrm{span}_{\mathbb{C}}(qp_j,\overline{qp_j}:j=0,\dots,k)=qU_0+\overline{qU_0}$$

Since q and $\overline{q}$ are coprime and $U_0 \subseteq \mathbb{C}[x,y]_{d_0}$ with $d_0 = \binom{k+1}{2} < \binom{k+2}{2} = \deg(q)$, we see that

$$U_{\mathbb{C}} = qU_0 \oplus \overline{qU_0} \quad \text{and} \quad \operatorname{rk}(F) = \dim_{\mathbb{R}}(U) = \dim_{\mathbb{C}}(U_{\mathbb{C}}) = 2\dim_{\mathbb{C}}(U_0) = 2(k+1).$$

Next, we have to show that the sum

$$U_{\mathbb{C}}U_{\mathbb{C}} = q^2U_0U_0 + \overline{q^2U_0U_0} + q\overline{q}\mathbb{C}[x,y]_{2d_0}$$

is a direct sum. For $q^2U_0U_0 \cap \overline{q^2U_0U_0} = \{0\}$ we can use the same argument as above. Now let $q^2u + \overline{q}^2\overline{v} = q\overline{q}h$ for some $u, v \in U_0U_0$ and $h \in \mathbb{C}[x,y]_{2d_0}$. Then $q(\overline{q}h - qu) = \overline{q}^2\overline{v}$, and therefore q divides $\overline{v} \in \overline{U_0U_0}$. By the choice of q, $\overline{v} = 0$. Analogously, $u = 0$. Consequently,

$$\dim_{\mathbb{C}}(U_{\mathbb{C}}U_{\mathbb{C}}) = 2\dim_{\mathbb{C}}(U_0U_0) + 2d_0 + 1 = 2(d - d_0) + 2d_0 + 1 = 2d + 1.$$

This means that $\dim(F) = k$. Finally, Remark 3.5.5 and Proposition 3.5.6 imply that F is a simplex whose extreme points have rank two. □

3.5.12 Theorem. *Let $k \in \mathbb{N}$ and $d \geq (k+1)^2$. The Gram spectrahedron of a general nonnegative binary form $f \in \mathbb{R}[x,y]_{2d}$ contains a simplex face F with $(\operatorname{rk}(F), \dim(F)) = (2(k+1), k)$ and $\operatorname{Ex}(F) \subseteq \operatorname{Ex}_2(f)$.*

Proof. It is enough to prove the theorem for $d = (k+1)^2$. Let $\mathtt{Q}_{2d}$ denote the (semialgebraic) set of all $f \in \operatorname{int}(\Sigma_{2d})$ with a face of the desired form in $\operatorname{Gram}(f)$. With the same argumentation as in Theorem 3.3.13, it suffices to show that $\mathtt{Q}_{2d}$ is dense in Σ_{2d}. If $h \in \operatorname{int}(\Sigma_{2d})$, we have to find $\tilde{h} \in \mathtt{Q}_{2d}$ "close to" h.

Let $e = \binom{k+2}{2}$ and $d' = \binom{k+1}{2} = d - e$. Since the proof is conceptually the same as in the Hermitian case, we refrain from rigorously considering open neighborhoods and their images and preimages under a continuous map. We write $h = fg$ with $f \in \operatorname{int}(\Sigma_{2d'})$ and $g \in \operatorname{int}(\Sigma_{2e})$. From the Hermitian case we know that $\mathtt{P}_{2d'}$ is dense in $\Sigma_{2d'}$. Therefore, we choose $\tilde{f} \in \mathtt{P}_{2d'}$ "close enough to" f and a suitable factor $\tilde{g}$ which is "close enough to" g, and such that $\tilde{f}\tilde{g}$ is a positive binary form with distinct roots with a polyhedral face of the desired form in its (symmetric) Gram spectrahedron (cf. the construction in Theorem 3.5.11). Then $\tilde{f}\tilde{g} \in \mathtt{Q}_{2d}$ is "close to" h. □

3.5.13 Example. The first interesting case is $k = 2$ and $d = (k+1)^2 = 9$. The idea of the proof of Theorem 3.5.11 was to start with a polynomial $g \in \mathbb{R}[x]$ of degree $2\binom{k+1}{2} = 6$ such that the Hermitian Gram spectrahedron of g contains a two-dimensional polyhedral face whose corresponding face subspace fulfills some additional conditions (cf. Lemma 3.5.10). A simple calculation shows that we can use $g = f^{(2)} \in \mathbb{R}[x]_{\leq 6}$ and the subspace $U' = \operatorname{span}_{\mathbb{C}}(q_0, q_1, q_2) \subseteq \mathbb{C}[x]_{\leq 3}$, where q_0, q_1, q_2 and $f^{(2)}$ are as in Example 3.3.10. Next, we have to pick another set of six points $\gamma_1, \ldots, \gamma_6 \in \mathbb{C}$. We have shown that with probability one, a randomly chosen set will be good enough. But, for the sake of giving an explicit example, we use the algorithm of Lemma 3.5.9 and we thereby see that we can take γ_1, γ_2, γ_3 from Example 3.3.10 and $\gamma_4 = 2 + 7i$, $\gamma_5 = 4 + 6i$, $\gamma_6 = 7 + 5i$, for instance. Then $f := g \cdot \prod_{j=1}^{6}(x - \gamma_j)(x - \overline{\gamma_j})$ is a polynomial of degree $2d = 18$ with real coefficients. The three vertices of our two-dimensional face of $\operatorname{Gram}(f)$ are

$\vartheta_j = (h_j + \overline{h_j})/2 \otimes (h_j + \overline{h_j})/2 + (h_j - \overline{h_j})/2i \otimes (h_j - \overline{h_j})/2i$ for $j \in \{0,1,2\}$ where

$$\begin{aligned} h_0 &= (x-\gamma_1)\cdots(x-\gamma_6)(x-\beta_1)(x-\beta_2)(x-\alpha),\\ h_1 &= (x-\gamma_1)\cdots(x-\gamma_6)(x-\overline{\beta_1})(x-\overline{\beta_2})(x-\alpha),\\ h_2 &= (x-\gamma_1)\cdots(x-\gamma_6)(x-\overline{\beta_1})(x-\overline{\beta_2})(x-\overline{\alpha}). \end{aligned}$$

3.5.14 Remark. Let $k \in \mathbb{N}$ and $d = (k+1)^2$. Using the notation of Theorem 3.5.12, we fix some $f \in \mathbb{R}[x,y]_{2d}$ and a k-dimensional face $F \subseteq \mathrm{Gram}(f)$ as in the theorem. For the construction of F in the proof of Theorem 3.5.11, we used a certain polyhedral face in the Hermitian Gram spectrahedron of some form g of smaller degree. It might be tempting to think that F induces a polyhedral face of $\mathcal{H}^+(f)$ in some natural way, say by $F \mapsto F_{\mathcal{H}}$. While any rank-two point $\vartheta \in \mathrm{Gram}(f)$ induces an edge $\{\vartheta\}_{\mathcal{H}}$ of $\mathcal{H}^+(f)$ (see Example 2.5.12), the face $F_{\mathcal{H}}$, however, is never polyhedral. There is, of course, no reason for $F_{\mathcal{H}}$ to be polyhedral, as the intersection of a spectrahedron with the preimage of a polyhedral set under a linear map can take many shapes. That $F_{\mathcal{H}}$ is not polyhedral can be seen by a dimension count. We have $\mathrm{rk}(F) = 2(k+1)$ and $\dim(F) = k$. Consequently, $\dim(F_{\mathcal{H}}) = \dim(F) + \binom{\mathrm{rk}(F)}{2} = 2k^2 + 4k + 1$ (see Corollary 2.5.11). But then $\binom{\dim(F_{\mathcal{H}})+1}{2} = (k+1)^2(2k^2+4k+1) = d(2k^2+4k+1)$. Since $k \geq 1$, our claim follows from Theorem 3.3.1.

3.5.15 Remark. In Section 3.4 we presented a detailed analysis of the supporting faces for any combination of three rank-one extreme points in the Hermitian Gram spectrahedron of a binary sextic. Moreover, we counted the k-simplices emerging from our construction for any $k \in \mathbb{N}$ and $d = \binom{k+1}{2}$ which was already quite cumbersome. As we have to increase the degree to $d = (k+1)^2$ in order to get a (diagonalizable) k-simplex in the real symmetric setting, the problem gets even more unwieldy in this case. Thus, we only comment on the ratio of polyhedral faces.

Consider the polynomial f from Example 3.5.13. There are $2^{d-1} = 256$ rank-two extreme points in $\mathrm{Gram}(f)$, and thus almost 2.8 million possible combinations of three of them. We computed the dimension of the supporting face for 10000 randomly chosen such triples. More than 97.5% of these faces were two-dimensional and hence 2-simplices. In a similar numerical experiment for a form of degree $(3+1)^2 = 16$, the ratio of 3-simplices increased to more than 99.9%. These results are in line with the general philosophy that also led to Conjecture 3.4.8.

3.5.16 Example (cf. Remark 3.5.7)**.** Let $d = 6$. For special $f \in \mathbb{R}[x,y]_{12}$, we can get a two-dimensional polyhedral face in $\mathrm{Gram}(f)$. We identify $\mathbb{R}[x,y]_{12}$ with $\mathbb{R}[x]_{\leq 12}$ and consider the polynomial

$$\begin{aligned} f = x^{12} - 24x^{10} - 90x^9 + 1229x^8 + 3246x^7 - 7686x^6 - 44184x^5 \\ + 83780x^4 + 682104x^3 + 312360x^2 - 2198976x + 3677440. \end{aligned}$$

The roots of f are

$$1 \pm i,\ -3 \pm i,\ 3 \pm 2i,\ -2 \pm 3i,\ 5 \pm 3i,\ -4 \pm 4i.$$

Note that they sum to zero, which is why the monomial x^{11} does not occur in f. Let ϑ_1, ϑ_2, $\vartheta_3 \in \mathrm{Ex}_2(f)$ denote the positive semidefinite Gram tensors corresponding to the sos representations

$$\begin{aligned} f &= (x^6 - 14x^4 - 3x^3 + 350x^2 + 1134x - 1456)^2 \\ &\quad + (-2x^5 + 21x^4 + 27x^3 + 60x^2 - 442x - 1248)^2 \\ &= (x^6 - 20x^4 - 153x^3 + 86x^2 + 582x - 1888)^2 \\ &\quad + (4x^5 + 27x^4 - 9x^3 - 444x^2 + 2x - 336)^2 \\ &= (x^6 - 12x^4 - 45x^3 + 278x^2 + 1014x - 1248)^2 \\ &\quad + (-23x^4 - 3x^3 + 12x^2 - 114x - 1456)^2. \end{aligned}$$

We have $\mathrm{rk}(\vartheta_j + \vartheta_k) = 4$ for all $j \neq k \in \{1, 2, 3\}$, but the rank of the sum $\vartheta_1 + \vartheta_2 + \vartheta_3$ drops to 5. One can check that $F = \mathrm{suppface}(\{\vartheta_1, \vartheta_2, \vartheta_3\}) \subseteq \mathrm{Gram}(f)$ is a triangle. Consequently, F is an example of a non-diagonalizable polyhedral face.

3.6. Chains of faces and the Carathéodory number

Let $m \in \mathbb{N}$. Carathéodory's theorem states that every point $x \in \mathrm{conv}(S)$, where S is an arbitrary subset of $\mathbb{R}^m$, can be written as a convex combination of $m+1$ points from S, cf. [Bar, Section 2.3 in Chapter I]. As a compact convex subset $K \subseteq \mathbb{R}^m$ is the convex hull of its extreme points (Minkowski, Krein-Milman), every point in K is a convex combination of $m + 1$ extreme points of K. Of course, there are convex sets for which shorter convex combinations suffice. Take for example a closed ball $B = \{x \in \mathbb{R}^m : \|x\| \leq 1\}$ where any point is a convex combination of two extreme points. This leads to the notion of the Carathéodory number, which is equal to two for the ball B and which will be introduced for general compact convex sets below.

In this section we recall a well-known result that relates the Carathéodory number of K to the maximum length of chains of faces in K. We then present various approaches for estimating the Carathéodory number of Gram spectrahedra of binary forms.

3.6.1 Definition. Let $W \cong \mathbb{R}^m$ be a finite-dimensional vector space over the reals and let $S \subseteq W$. By $\mathrm{conv}_k(S)$ we denote the set of all points $p \in W$ which may be written as a convex combination of at most k points of S.

Let $K \subseteq W$ be compact and convex. The *Carathéodory number* $\mathrm{Car}(K)$ of K is the smallest integer k such that $K = \mathrm{conv}_k(\mathrm{Ex}(K))$.

Note that this is well-defined. Indeed, as explained above, $K = \mathrm{conv}_{m+1}(\mathrm{Ex}(X))$. In particular, $\mathrm{Car}(K) \leq m + 1$.

3.6.2 Remark. The Carathéodory number is often defined for pointed closed convex cones. Then, the definition is formulated in terms of the extreme rays of the cone. One can switch between those two settings and the definitions fit together properly.

Indeed, let $K \subseteq W$ be a compact convex set. Then the recession cone of K consists of the zero vector alone (cf. the discussion ahead of Theorem 2.5.6 in [Web]), so that

$$C := \{(\lambda, \lambda v) : \lambda \geq 0,\ v \in K\} \subseteq \mathbb{R} \times W$$

is the so-called *homogenization* K^h of K. It is well-known that C is a pointed closed convex cone and that we have a bijection between the nonempty faces F of K and the faces $G \not\subseteq \{0\} \times W$ of C which is given by homogenization and dehomogenization, that is to say

$$\begin{aligned} F \mapsto F^h &:= \{(\lambda, \lambda v) : \lambda \geq 0,\ v \in F\}, \\ \{v \in V : (1, v) \in G\} =: G^d &\leftarrow G, \end{aligned}$$

see for example [IL, Proposition 5].

On the other hand, a pointed closed convex cone $C \subseteq W$ has a compact base K. This is a convex subset $K \subseteq C$ with $C = \operatorname{cone}(K)$ such that there exists $\lambda \in W^\vee$ with $K = \{v \in C : \lambda(v) = 1\}$. Then $C = \{0\} \cup \{\lambda v : \lambda > 0, v \in K\}$, and C is affine-linear isomorphic to K^h.

3.6.3 Definition. We define the *Carathéodory number* $\operatorname{Car}(f)$ of a form $f \in \Sigma_{2d}$ to be the Carathéodory number of its Gram spectrahedron $\operatorname{Gram}(f)$ in the sense of Definition 3.6.1.

For $f \in \operatorname{int}(\Sigma_{2d})$ we have $\dim \operatorname{Gram}(f) = \binom{d}{2}$. Hence, Carathéodory's theorem gives us $\operatorname{Car}(f) \le \binom{d}{2} + 1$ as a first upper bound. We can use the facial structure of a compact convex set K to give a different upper bound. The following proposition and its corollary are well-known. They can be found in the thesis of Kunert ([Kun, Lemma 1.28 and Corollary 1.29]) in their corresponding conical formulations. These results are also contained in an article by Ito and Lourenço ([IL, Theorem 4 and Theorem 6]).

3.6.4 Proposition. *Let $K \neq \varnothing$ be a compact convex set. Then*

$$\operatorname{Car}(K) \le \max\{\operatorname{Car}(F) : F \subsetneq K \text{ face of } K\} + 1.$$

Proof. Let $x \in K$. If $x \in F$ for some proper face $F \subseteq K$ we are done, so we can assume that $x \in \operatorname{relint}(K)$. Let $y_0 \in K$ be an extreme point. Consider the affine line $x + \mathbb{R}(x - y_0) \subseteq \operatorname{aff}(K)$. On the one hand, we have $(1-\mu)x + \mu y_0 \in K$ for all $\mu \in [0,1]$. On the other hand, K is bounded, so K does not contain a ray. Therefore, and using that K is closed, there is some $\lambda_0 > 0$ such that $z := (1+\lambda_0)x - \lambda_0 y_0$ is contained in the relative boundary of K. Hence, $z \in F_0$ for a proper face $F_0 \subseteq K$. Consequently, we can find $k \le \operatorname{Car}(F_0)$ and $y_1, \dots, y_k \in \operatorname{Ex}(F_0)$ as well as $\lambda_1, \dots, \lambda_k \in \mathbb{R}_{\ge 0}$ with $\sum_{i=1}^k \lambda_i = 1$ such that $z = \sum_{i=1}^k \lambda_i y_i$. It follows that

$$x = \sum_{i=0}^{k} \frac{\lambda_i}{1+\lambda_0} y_i.$$

In particular, since $\operatorname{Ex}(F_0) \subseteq \operatorname{Ex}(K)$ we see that x is a convex combination of at most $\operatorname{Car}(F_0) + 1$ extreme points of K. □

3.6.5 Corollary. *Let $K \neq \varnothing$ be a compact convex set, let k be the maximum length of a chain*

$$\varnothing = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k = K$$

of faces of K. Then we have $\operatorname{Car}(K) \le k$. □

3.6.6 Remark. In general, we do not have equality in Corollary 3.6.5. For example, let K be the upper half of the unit disk in $\mathbb{R}^2$,

$$K = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1,\ y \ge 0\}.$$

Setting $F_1 := \{(1,0)\}$ and $F_2 := [-1,1] \times \{0\}$ gives a chain of faces $\varnothing = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq F_3 = K$ of length 3, but obviously $\operatorname{Car}(K) = 2$.

3.6.7 Proposition. *Let $d \ge 3$. For general $f \in \Sigma_{2d}$ we have $\operatorname{Car}(f) \le d - 1$.*

Proof. In any ascending chain of faces of $\mathrm{Gram}(f)$, ranks have to increase in every step. If

$$\varnothing = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k = \mathrm{Gram}(f)$$

is a chain of faces of maximum length, $F_1 = \{\vartheta\}$ is an extreme point of $\mathrm{Gram}(f)$. So $\mathrm{rk}(F_1) \geq 2$ since general f is not a square. Moreover, for general f, we cannot have $\mathrm{rk}(F_2) = 3$ since $\dim(F_2) > 0$ (see Proposition 3.1.10). Hence, $\mathrm{rk}(F_j) \geq j + 2$ for $j \geq 2$. In particular, $d + 1 = \mathrm{rk}(F_k) \geq k + 2$. This implies $\mathrm{Car}(f) \leq k \leq d - 1$, see Corollary 3.6.5. □

We now present various approaches for a lower bound on $\mathrm{Car}(f)$. The first one is a semialgebraic dimension count. It has been successfully applied for lower-bounding the Carathéodory number of the cone of nonnegative ternary quartics (cf. [Kun, Theorem 2.35]).

3.6.8. Let $S \subseteq \mathbb{R}^n$ be a semialgebraic set and let $\Delta \subseteq \mathbb{R}^k$ be the $(k-1)$-dimensional standard simplex, i.e., $\Delta = \{(a_1, \dots a_k) \in \mathbb{R}^k_{\geq 0} : \sum_{i=1}^k a_i = 1\}$. We consider the map

$$\phi\colon S^k \times \Delta \to \mathbb{R}^n, \quad \phi(s_1, \dots, s_k, a_1, \dots a_k) := \sum_{i=1}^k a_i s_i.$$

The image of ϕ is $\mathrm{conv}_k(S)$. Since ϕ is semialgebraic, we can bound the dimension of $\mathrm{conv}_k(S)$ by the dimension of the domain of ϕ ([BCR, Theorem 2.8.8]). This means

$$\dim \mathrm{conv}_k(S) \leq k \cdot \dim(S) + (k-1).$$

In particular situations, these considerations can be used to give a lower bound on Carathéodory numbers. Indeed, if $K \subseteq \mathbb{R}^n$ is a compact convex set for which $\mathrm{Ex}(K)$ is semialgebraic, then $\mathrm{Car}(K) > k$ for all k with $k \cdot \dim(\mathrm{Ex}(K)) + (k-1) < \dim(K)$.

3.6.9. We want to use the idea presented above in the case of Gram spectrahedra. From Scheiderer's work ([Sch22, Corollary 5.5]) we know that there is an open dense subset U of Σ_{2d} such that, for every $f \in U$ and every r in the Pataki range, we have

$$\dim \mathrm{Ex}_r(f) = \frac{1}{2}(r-2)(2d+1-r).$$

The set $\mathrm{Ex}(f)$ is the union of the sets $\mathrm{Ex}_r(f)$ where r runs through the Pataki range. Therefore, it is semialgebraic and its dimension is the maximum of the individual dimensions of the sets $\mathrm{Ex}_r(f)$ (see [BCR, Proposition 2.8.5]). The map

$$\rho\colon (0, d+1) \to \mathbb{R}, \quad r \mapsto \frac{1}{2}(r-2)(2d+1-r),$$

is monotonically increasing since it has derivative $\rho'(r) = d + \frac{3}{2} - r > 0$. This means that $\dim \mathrm{Ex}_r(f)$ is maximal when r is maximal in the Pataki range, i.e., $r = \lfloor \frac{1}{2}(\sqrt{16d+9} - 1) \rfloor$. We let

$$B(d) := \rho\left(\frac{1}{2}(\sqrt{16d+9} - 1)\right) = \frac{1}{2}d(\sqrt{16d+9} - 9) + \sqrt{16d+9} - 3.$$

We have $k < \mathrm{Car}(f)$ if $k \cdot B(d) + (k-1) < \dim \mathrm{Gram}(f) = \binom{d}{2}$. Rearranging, the latter is equivalent to

$$k < \frac{d^2 - d + 2}{d\left(\sqrt{16d+9} - 9\right) + 2\sqrt{16d+9} - 4}.$$

The expression on the right hand side is on the order of $\frac{1}{4}\sqrt{d}$, which is rather disappointing as a lower bound for $\mathrm{Car}(f)$.

3.6.10. We obtain a slightly better bound using the following observation. If r is maximal in the Pataki range, then $\mathrm{Car}(f) \geq \min\{k \in \mathbb{N} : kr \geq d+1\}$. Indeed, if $\vartheta \in \mathrm{relint}(\mathrm{Gram}(f))$ can be written as a convex combination of k extreme points $\vartheta_1, \ldots, \vartheta_k \in \mathrm{Ex}(f)$, then $d+1 = \mathrm{rk}(\vartheta) \leq \sum_{i=1}^{k} \mathrm{rk}(\vartheta_i) \leq kr$. Therefore,

$$\mathrm{Car}(f) \geq \frac{d+1}{\lfloor \frac{1}{2}\left(\sqrt{16d+9}-1\right)\rfloor},$$

where the expression on the right hand side grows as $\frac{1}{2}\sqrt{d}$ for $d \to \infty$.

Our best lower bound for the Carathéodory number comes from the construction of polyhedral faces in Gram spectrahedra.

3.6.11 Proposition. *For general $f \in \Sigma_{2d}$ we have* $\mathrm{Car}(f) \geq \lfloor\sqrt{d}\rfloor$.

Proof. By the definition of a face of a convex set,

$$\mathrm{Car}(f) \geq \max\{\mathrm{Car}(F) : F \text{ face of } \mathrm{Gram}(f)\}.$$

For general $f \in \Sigma_{2d}$ and any $k \in \mathbb{N}_0$ with $d \geq (k+1)^2$, $\mathrm{Gram}(f)$ contains a face F which is a k-dimensional simplex (Theorem 3.5.12). Therefore, $\mathrm{Car}(f) \geq \mathrm{Car}(F) = k+1$. In particular, this is true for $k+1 = \lfloor\sqrt{d}\rfloor$. □

3.6.12 Remark. We have discussed three different approaches to finding a lower bound for the Carathéodory number of Gram spectrahedra of binary forms: a semialgebraic dimension count, the subadditivity of ranks, and reducing to faces. Although the existence of large polyhedral faces is nontrivial, we find it a bit disappointing that our best bound for the Carathéodory number results from the most trivial approach, especially since our lower bound $\lfloor\sqrt{d}\rfloor$ and our upper bound $d-1$ differ by a factor of the order of magnitude of $\sqrt{d}$. Unfortunately, we were not able to demagnify this gap.

As remarked in 3.6.6, the existence of long chains of faces does not automatically lead to a lower bound on the Carathéodory number. Nevertheless, such chains are of independent interest and are thus studied for the rest of this section. As an aside, we will see that the upper bound $d-1$ for the length of a chain of faces in $\mathrm{Gram}(f)$ is not too bad in terms of its order of magnitude. Once more, we use the combinatorics of the roots.

3.6.13 Lemma. *Let $d \geq 3$ and let $f \in \mathbb{R}[x,y]_{2d}$ be a positive binary form with distinct roots. Then there are $d+1$ rank-two extreme points of* $\mathrm{Gram}(f)$ *which do not lie on a common proper face of* $\mathrm{Gram}(f)$.

Proof. Replace f by $f(x,1)$. We can assume that f is monic, and we denote the roots of f by $a_1, \ldots, a_d, \overline{a_1}, \ldots, \overline{a_d}$ so that $f = \prod_{j=1}^{d}(x-a_j)(x-\overline{a_j})$. We define

$$p_k = \prod_{\substack{j=1 \\ j\neq k}}^{d}(x-a_j)\cdot(x-\overline{a_k}) \quad \text{for } k = 1, \ldots, d, \text{ and } p_0 = \prod_{j=1}^{d}(x-a_j).$$

Then we have $f = p_k\overline{p_k} = \mathrm{Re}(p_k)^2 + \mathrm{Im}(p_k)^2$ for all $k \in \{0, 1, \dots, d\}$. Note that $\overline{p_k} \neq p_{k'}$ for any k, $k' \in \{0, \dots, d\}$ since $d \geq 3$. Hence, any pair $(p_k, \overline{p_k})$ gives a different point

$$\vartheta_k := \mathrm{Re}(p_k) \otimes \mathrm{Re}(p_k) + \mathrm{Im}(p_k) \otimes \mathrm{Im}(p_k) \in \mathrm{Ex}_2(f),$$

see also 3.0.6. By construction, for all j, $k \in \{1, \dots, d\}$ it holds

$$p_k(a_j) \begin{cases} = 0 & \text{if } j \neq k, \\ \neq 0 & \text{if } j = k. \end{cases}$$

From this we see that $\dim_{\mathbb{C}}(U') = d$ where $U' = \mathrm{span}_{\mathbb{C}}\{p_1, \dots, p_d\}$. Suppose that $p_0 = \sum_{k=1}^{d} \lambda_k p_k$ with $\lambda_1, \dots, \lambda_d \in \mathbb{C}$. Then $0 = p_0(a_j) = \lambda_j p_j(a_j)$ for all $j \in \{1, \dots, d\}$. Moreover, $p_j(a_j) \neq 0$, so that our supposition would imply $\lambda_j = 0$ and therefore $p_0 = 0$, a contradiction. Thus, the dimension of the space $U = \mathrm{span}_{\mathbb{C}}\{p_0, p_1, \dots, p_d\} = \mathbb{C}p_0 \oplus U'$ is $d + 1$. In particular, the supporting face F of the $d + 1$ extreme points $\vartheta_0, \dots, \vartheta_d$ of $\mathrm{Gram}(f)$ has rank $d + 1$. Indeed, its associated face subspace $\mathcal{U}(F) = \mathrm{span}_{\mathbb{R}}(\mathrm{Re}(p_k), \mathrm{Im}(p_k) : k = 0, \dots, d)$ has dimension $\dim_{\mathbb{R}}(\mathcal{U}(F)) = \dim_{\mathbb{C}}(U) = d + 1$. But this means that $F = \mathrm{Gram}(f)$. □

We have actually also shown that there is a collection of $d + 1$ rank-one extreme points in the Hermitian Gram spectrahedron of f whose supporting face equals $\mathcal{H}^+(f)$. This also provides another example where the upper bound in Proposition 3.2.4 is attained.

3.6.14 Remark. According to Lemma 3.6.13, there is a special choice of $d+1$ points of $\mathrm{Ex}_2(f)$ which do not lie on a common proper face of $\mathrm{Gram}(f)$. Note that this is not true for an arbitrary choice of $d+1$ points of $\mathrm{Ex}_2(f)$. Indeed, [CPSV, Corollary 5.8] implies that there is a proper face of $\mathrm{Gram}(f)$ that contains at least $2^{\lfloor \frac{d}{2} \rfloor - 1}$ extreme points of rank two.

If $d \geq 3$ and $f \in \mathrm{int}(\Sigma_{2d})$ has distinct roots, then $|\mathrm{Ex}_2(f)| = 2^{d-1} \geq d + 1$. In particular, Lemma 3.6.13 implies that there is no proper face of $\mathrm{Gram}(f)$ that contains all rank-two extreme points of $\mathrm{Gram}(f)$. We can use this fact to construct certain chains of faces.

3.6.15 Corollary. *Let $d \geq 3$ and let $f \in \mathbb{R}[x, y]_{2d}$ be a positive binary form with distinct roots. Let $F \subsetneq \mathrm{Gram}(f)$ be a face with $\mathrm{rk}(F) = r$. Then there is a face $F \subsetneq G \subseteq \mathrm{Gram}(f)$ with $\mathrm{rk}(G) \leq \mathrm{rk}(F) + 2$.*

Proof. As explained above, there is a $\vartheta \in \mathrm{Ex}_2(f)$ outside of F. We can take $G = \mathrm{suppface}(F \cup \{\vartheta\}) \supsetneq F$. Then, for any $\vartheta' \in \mathrm{relint}(F)$,

$$\mathrm{rk}(G) = \mathrm{rk}(\vartheta' + \vartheta) \leq \mathrm{rk}(\vartheta') + \mathrm{rk}(\vartheta) = r + 2.$$ □

3.6.16 Proposition. *Let $d \geq 3$. For any positive binary form $f \in \mathbb{R}[x, y]_{2d}$ with distinct roots there is a chain of faces of $\mathrm{Gram}(f)$ of length $k \geq \lceil \frac{d+1}{2} \rceil$.*

Proof. Let $F_0 = \varnothing$. If $i \geq 1$ and F_{i-1} is a proper face of $\mathrm{Gram}(f)$ with $\mathrm{rk}(F_{i-1}) \leq 2(i - 1)$, then there is a face $F_{i-1} \subsetneq F_i \subseteq \mathrm{Gram}(f)$ with $\mathrm{rk}(F_i) \leq 2i$ (as in the proof of Corollary 3.6.15 we could choose $F_i = \mathrm{suppface}(F_{i-1} \cup \{\vartheta_i\})$ for some $\vartheta_i \in \mathrm{Ex}_2(f) \setminus F_{i-1}$). This gives a chain

$$\varnothing = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \ldots \subseteq \mathrm{Gram}(f)$$

of faces with $\mathrm{rk}(F_i) \leq 2i$. The points in the relative interior of $\mathrm{Gram}(f)$ have rank $d+1$. Thus, the length of the chain is at least $\lceil \frac{d+1}{2} \rceil$. □

Now that we have constructed numerous faces with special properties, the following section is devoted to proving the existence of faces of each rank $2 \leq r \leq d+1$ whose dimension is the minimum possible dimension for the given rank r.

3.7. Faces of expected dimension

Pataki's theorem from the 1990s (see Proposition 2.3.13) gives an interval of possible ranks for the extreme points of a general spectrahedron. Already in 1980, Loewy [Loe] proved the following: Given any positive integer r such that $r^2 \leq n$, there exists a rank-r matrix A which is an extreme point of C_n, the set of all Hermitian correlation matrices of size $n \times n$. A *correlation matrix* is meant to be a positive semidefinite matrix whose diagonal entries are equal to 1. About ten years later, Grone, Pierce and Watkins [GPW] established the analogous fact for the set R_n of real correlation matrices: There exist extreme points of rank r in R_n if and only if $r^2 + r \leq 2n$. In our manner of speaking, this means that the elliptope $\mathcal{E}_{n\times n} = R_n$ has extreme points of all ranks in the Pataki interval.

To the best knowledge of the author, the elliptopes and their Hermitian relatives were the first classes of examples of spectrahedra of arbitrarily large dimension that have extreme points of all ranks in the Pataki interval. In their survey on Gram spectrahedra, Chua, Plaumann, Sinn and Vinzant ask whether there is a binary form (of large degree $2d$) whose Gram spectrahedron also has extreme points of all ranks in the Pataki interval ([CPSV, Question 4.2]). Subsequently, Scheiderer proved that for any d, this is true for all psd binary forms coming from an open dense set ([Sch22, Theorem 5.3]).

The philosophy behind is that a general sequence $p_1, \dots, p_r \in \mathbb{R}[x,y]_d$ of length r is quadratically independent if there is no obvious obstruction to it, that is to say, as long as $\binom{r+1}{2} \leq \dim \mathbb{R}[x,y]_{2d} = 2d+1$. The aim of this section is to generalize Scheiderer's result to longer sequences. Following the above conviction, any such sequence of binary forms should not satisfy more quadratic relations than absolutely necessary. In terms of spectrahedra, this means that for a general subspace $U \subseteq \mathbb{R}[x,y]_d$ of dimension r and any $f \in \mathrm{int}(\Sigma U^2)$, the face $\mathcal{F}(U) \subseteq \mathrm{Gram}(f)$ should be expected to have minimum possible dimension (given its rank r).

3.7.1 Definition. Let $f \in \mathbb{R}[x,y]_{2d}$ be a nonnegative binary form, and let $F \subseteq \mathrm{Gram}(f)$ be a face of rank $r \in \{1, \dots, d+1\}$. We say that F is a face of *expected dimension* if

$$\dim(F) = \max\left\{0, \binom{r+1}{2} - (2d+1)\right\}.$$

Note that this is the minimum possible dimension for any rank-r face (cf. Proposition 2.3.9).

3.7.2. Following the presentation in [Sch22, Section 4], we consider a field K and a graded K-algebra A. Let $d \in \mathbb{N}_0$. Recall that a sequence $p_1, \dots, p_r \in A_d$ is quadratically independent if the pairwise products $p_i p_j$ $(1 \leq i \leq j \leq r)$ are linearly independent. Equivalently, we can require that the p_i are a linear basis of a subspace $U \subseteq A_d$ for which the natural multiplication map $\mu \colon \mathsf{S}_2 U \to A_{2d}$ is injective.

The following definition suggests itself.

3.7.3 Definition. Given a finite-dimensional K-linear subspace $U \subseteq A_d$, we say that U is *quadratically generating* if the natural multiplication map $\mu\colon \mathsf{S}_2 U \to A_{2d}$ is surjective, i.e., if $UU = A_{2d}$.

A sequence $p_1, \ldots, p_r \in A_d$ is *quadratically generating* if the p_i are a linear basis of a quadratically generating subspace U of A_d.

For what follows, we fix the graded K-algebra $A := K[x,y] = \bigoplus_{d \geq 0} A_d$, where A_d is the $(d+1)$-dimensional vector space of binary forms of degree d. For easy reference, we include Scheiderer's result on quadratically independent binary forms.

3.7.4 Theorem ([Sch22, Theorem 4.2])**.** *Let K be an infinite field, and let d, $r \geq 1$ such that $\binom{r+1}{2} \leq 2d+1$. Then there exists a quadratically independent sequence $p_1, \ldots, p_r \in A_d$.*

The goal of this section is to prove the following analogous theorem:

3.7.5 Theorem. *Let K be an infinite field, let $d \geq 0$ and $1 \leq r \leq d+1$ such that $\binom{r+1}{2} \geq 2d+1$. Then there exists a quadratically generating sequence $p_1, \ldots, p_r \in A_d$.*

3.7.6 Remark. If one is interested in finding a quadratically independent sequence of length r in A_d, this is most difficult when d is minimal with $\binom{r+1}{2} \leq 2d+1$. In contrast, the more polynomials we are allowed to take, the easier it will be to find a quadratically generating sequence. So finding a quadratically generating sequence of length r tends to be hardest when $2d+1$ is smaller than but very close to $\binom{r+1}{2}$. A special situation occurs when these two values coincide: If $\binom{r+1}{2} = 2d+1$, any quadratically independent sequence in A_d is automatically quadratically generating, and the existence of such sequences is assured by Theorem 3.7.4. Therefore, we can assume that $\binom{r+1}{2} > 2d+1$.

Note that, as long as $r < d+1$, we can always extend a quadratically generating sequence $p_1, \ldots, p_r$ in A_d to a quadratically generating sequence $p_1, \ldots, p_r, p_{r+1}$ by taking any $p_{r+1} \in A_d \smallsetminus \operatorname{span}(p_1, \ldots, p_r)$. So the interesting cases are those where r is minimal with respect to $\binom{r+1}{2} > 2d+1$, as already suggested above.

The minimality of r implies $\binom{r}{2} \leq 2d+1$. Our proof then proceeds as follows: The first step only deals with the pairs r, d with $\binom{r+1}{2} > 2d+1$ and $\binom{r}{2} \leq 2d-1$. In this step we use Scheiderer's theorem on the existence of quadratically independent sequences of length $r-1$ in A_{d-1}, argue along the lines of his proof and add some modifications at the end, in order to find a quadratically generating sequence of length r in A_d. Beyond that, we have to fill the gaps we leave in the first step. This means that we have to prove the theorem for all pairs r, d with $\binom{r}{2} = 2d+1$ (these cases are trivial, see above) or $\binom{r}{2} = 2d$.

3.7.7. For the moment, we assume that the first step is already done and focus on $\binom{r}{2} = 2d$. The base case is $r = 1$ and $d = 0$ where we can take $p_1 = 1$. Let $r \geq 2$ and $\binom{r}{2} = 2d$. Using the first step and induction on r, we can assume that the theorem is proven for all smaller values $r' < r$ and $d' < d$ with $\binom{r'+1}{2} \geq 2d'+1$. In particular, since $\binom{(r-1)+1}{2} = 2d > 2(d-1)+1$, there is a quadratically generating sequence $q_1, \ldots, q_{r-1}$ in A_{d-1}. Consider the subspace $U' := \operatorname{span}(q_1 y, \ldots, q_{r-1} y) \subseteq A_d$. We have $U'U' = y^2 A_{2(d-1)}$. It follows that $U := U' \oplus K x^d \subseteq A_d$ is a quadratically generating subspace of dimension r. Indeed, we can assume that $\deg_x(q_1) = \max\{\deg_x(q_i) : 1 \le i \le r-1\}$. Then $\deg_x(x^{2d}) > \deg_x(x^d q_1 y) > \deg_x(pq)$ for all p, $q \in U'$ and thus $\dim(UU) \ge \dim(U'U') + 2 = 2d+1$.

For the rest of the discussion we let $\binom{r}{2} \le 2d-1$.

3.7.8 Remark. In Scheiderer's proof of Theorem 3.7.4, a marginal case distinction appears for small values of r where otherwise some dimensions of certain linear spaces under consideration would formally turn negative. We examine the cases with $r \le 5$ separately by hand. In this way we avoid another case distinction later on, but we can also briefly comment on constructive approaches to quadratically generating sequences.

The pairs (r, d) we have to consider are $(3, 2)$, $(4, 4)$ and $(5, 6)$. To simplify the notation, we switch to the univariate setting. In all three cases, we can give a quadratically generating sequence consisting of monomials: 1, x, x^2 as well as 1, x, x^3, x^4 and 1, x, x^3, x^5, x^6, respectively.

In general, there can be many quadratic relations among a list of monomials. Therefore, it might be interesting but not surprising that already for $(r, d) = (6, 9)$ there is no monomial subspace which is quadratically generating. [If we want $U \subseteq K[x]_{\le d}$ to be a monomial subspace with $UU = K[x]_{\le 2d}$, then U has to contain at least the monomials 1, x, x^{d-1}, x^d in order to have 1, x, x^{2d-1}, $x^{2d} \in UU$. When $d = 9$ and we want to have x^3 resp. x^{15} in UU, we also need to have (either) x^2 or x^3 resp. (either) x^6 or x^7 in U. But then we already reached $\dim(U) = 6$ and we end up with $x^5 \notin UU$.] For $d = 9$, a simple quadratically generating sequence of length $r = 6$ would be 1, x, x^3, x^7, $x^8 + x^5$, x^9.

We tried to come up with some simple method to explicitly construct sequences of quadratically generating fewnomials, but what we tried was either too simple and failed for larger values r, d or too complicated to really prove the conjectured generating property. So – without further ado – let us return to the proof which is nonconstructive.

We argue along the lines of Scheiderer's proof. As the existence of a single quadratically generating sequence of length r in A_d implies that a generic sequence in A_d of this length is quadratically generating, we can assume that the field K is algebraically closed. We adopt the following notation: For any $m \in \mathbb{N}$ and given $z_1, \dots, z_m \in \mathbb{P}^1 = \mathbb{P}^1(K)$, we let

$$W_d(z_1, \dots, z_m) := \{f \in A_d : f(z_1) = \cdots = f(z_m) = 0\}.$$

Moreover, we fix a point $\infty \in \mathbb{P}^1$ and a linear form $0 \ne l \in A_1$ with $l(\infty) = 0$.

There is a small technical subtlety. Recall that we are now assuming

$$r \ge 5, \ \binom{r+1}{2} > 2d+1 \text{ and } \binom{r}{2} \le 2d-1. \tag{3.7.1}$$

We need a statement similar to the one in [Sch22, Lemma 4.5]. However, this lemma was proven under different assumptions on r and d. For the sake of completeness, we give the argument with an adaptation to our situation. In particular, as $\binom{r}{2} \le 2d-1$, we can already use Scheiderer's final result 3.7.4 for $r-1$ and $d-1$.

3.7.9 Lemma (cf. [Sch22, Lemma 4.5]). *Under the assumptions (3.7.1) the following hold:*

(a) For any linear subspace $U \subseteq W_d(\infty)$ and any $0 \ne p \in U$,

$$\dim(pA_d \cap UU) \ge \max\{\dim(U), \dim(UU) - d + 1\}.$$

(b) There exists a subspace $U \subseteq W_d(\infty)$ with $\dim(U) = r-1$ and $\dim(UU) = \binom{r}{2}$, together with a form $p \in U$, such that equality holds in (a).

Proof. (a) For any $0 \neq p \in U$, the space $pA_d \cap UU$ contains pU. Therefore, its dimension is at least $\dim(U)$. Moreover, $pA_d + UU \subseteq W_{2d}(\infty)$ and consequently $\dim(pA_d \cap UU) = \dim(pA_d) + \dim(UU) - \dim(pA_d + UU) \geq (d+1) + \dim(UU) - 2d = \dim(UU) - d + 1$.

(b) According to Theorem 3.7.4, there exists a quadratically independent sequence $q_1, \ldots, q_{r-1} \in A_{d-1}$. We let $p_i := q_i l$ $(1 \leq i \leq r-1)$. Then $p_1, \ldots, p_{r-1}$ is a quadratically independent sequence in $W_d(\infty)$. For $V = \operatorname{span}(p_2, \ldots, p_{r-1})$ we have $\dim(VV) = \binom{r-1}{2}$. If $q \in W_d(\infty)$ has distinct zeros $z_1, \ldots, z_{d-1}, \infty$ in $\mathbb{P}^1$, we have $qA_d \cap VV = W_{2d}(z_1, \ldots, z_{d-1}) \cap VV$. This intersection has codimension $d-1$ in VV if q is sufficiently general. We can therefore modify $p_1 \in W_d(\infty)$ in such a way that $\dim(p_1 A_d \cap VV) = \binom{r-1}{2} - d + 1$ holds and the sequence $p_1, \ldots, p_{r-1}$ remains quadratically independent. Writing $U := \operatorname{span}(p_1, \ldots, p_{r-1}) = Kp_1 \oplus V$, we have $UU = p_1 U \oplus VV$ since U is quadratically independent. Therefore, $p_1 A_d \cap UU = p_1 U \oplus (p_1 A_d \cap VV)$, and this subspace has dimension $(r-1) + \binom{r-1}{2} - d + 1 = \binom{r}{2} - d + 1 = \dim(UU) - d + 1$. □

3.7.10. For the time being, we continue as in [Sch22]. Lemma 3.7.9 allows us to fix a quadratically independent subspace $U \subseteq W_d(\infty)$ with $\dim(U) = r-1$ and such that $\dim(pA_d \cap UU) \geq \binom{r}{2} - d + 1$ holds for all $0 \neq p \in U$, with equality holding for p sufficiently general. Recall that our aim is to extend U to a quadratically generating subspace of A_d of dimension r. We define $k \in \{1, \ldots, r-1\}$ by $\binom{r+1}{2} = 2d + 1 + k$.

Let $\mathbb{P}_U$ and $\mathbb{P}_{A_d}$ denote the projective spaces associated to the linear spaces U and A_d, respectively, and consider the closed subvariety

$$X := \{([p],[q]) \in \mathbb{P}_U \times \mathbb{P}_{A_d} : pq \in UU\}$$

of $\mathbb{P}_U \times \mathbb{P}_{A_d}$. Let $\pi_1 \colon X \to \mathbb{P}_U$ and $\pi_2 \colon X \to \mathbb{P}_{A_d}$ denote the projections onto the two components.

Using the projection π_1, we can calculate the dimension of X:

3.7.11 Lemma (cf. [Sch22, Lemma 4.7])**.** $\dim(X) = d - 1 + k$.

Proof. For every $0 \neq p \in U$ we have $([p],[p]) \in X$. Therefore, π_1 is surjective and the projective dimension of the fiber $\pi_1^{-1}([p])$ is $\dim(pA_d \cap UU) - 1$ for every $0 \neq p \in U$. This means that the generic fiber of π_1 has dimension $\binom{r}{2} - d$ (see 3.7.10). Using a fiber dimension theorem ([Harr, Corollary 11.13]), we see that the dimension of X is the sum of the dimension of $\pi_1(X)$ and the dimension of a generic fiber of π_1, so that $\dim(X) = (r-2) + \binom{r}{2} - d = \binom{r+1}{2} - d - 2 = 2d + 1 + k - d - 2 = d - 1 + k$. □

We will use this fact in a short while, but let us first take a closer look on the other projection π_2. For $0 \neq q \in A_d$ the fiber $\pi_2^{-1}([q])$ has projective dimension $\dim(qU \cap UU) - 1$. We relate this to the dimension of X by showing that π_2 is surjective as well.

3.7.12 Lemma. *The projection $\pi_2 \colon X \to \mathbb{P}_{A_d}$ is surjective.*

Proof. Let $0 \neq q \in A_d$. We have $\dim(qU) = \dim(U) = r - 1$ and therefore

$$\dim(UU) + \dim(qU) = \binom{r}{2} + r - 1 = \binom{r+1}{2} - 1 = 2d + k.$$

On the other hand, $qU + UU$ is contained in the space $W_{2d}(\infty)$ which has dimension $2d$. Thus, $\dim(qU \cap UU) \geq k$ and in particular $qU \cap UU \neq \{0\}$. This means that π_2 is surjective. □

3.7.13. Since the dimension of X is $d - 1 + k$ by Lemma 3.7.11, we see that the generic fiber of π_2 has (projective) dimension $k - 1$. Hence, for generically chosen $q \in A_d$ we have $\dim(qU \cap UU) = k$. We take such $q \in A_d$ with $q(\infty) \neq 0$. Then the r-dimensional subspace $U' := U \oplus Kq$ of A_d is quadratically generating. Indeed, since $q^2 \notin W_{2d}(\infty)$, we have

$$\begin{aligned}\dim(U'U') &= \dim(UU + qU) + 1\\ &= \dim(UU) + \dim(qU) - \dim(qU \cap UU) + 1\\ &= \binom{r}{2} + (r-1) - k + 1\\ &= \binom{r+1}{2} - k\\ &= 2d + 1.\end{aligned}$$

Hence, $U'U' = A_{2d}$ and this completes the proof of Theorem 3.7.5. □

Combining Scheiderer's result on quadratically independent sequences (Theorem 3.7.4) with the analogous statement on quadratically generating ones (Theorem 3.7.5), we get the following generalizations of Corollary 5.2 and Theorem 5.3 in [Sch22].

3.7.14 Corollary. *Let $d \in \mathbb{N}$ and let $1 \leq r \leq d+1$. Let $W_{d,r}$ be the set of r-tuples $(p_1, \ldots, p_r)$ in $(\mathbb{R}[x,y]_d)^r$ that satisfy precisely* $\max\{0, \binom{r+1}{2} - (2d+1)\}$ *independent quadratic relations. Then $W_{d,r}$ is open and dense in $(\mathbb{R}[x,y]_d)^r$.* □

3.7.15 Theorem. *Let $d \in \mathbb{N}$. There is an open dense set of nonnegative binary forms $f \in \mathbb{R}[x,y]_{2d}$ for which* $\mathrm{Gram}(f)$ *contains faces of expected dimension for all ranks $r \in \{2, \ldots, d+1\}$.*

Proof. Fix $r \in \{2, \ldots, d+1\}$ and consider the set

$$S_r := \{p_1^2 + \cdots + p_r^2 : (p_1, \ldots, p_r) \in W_{d,r}\}.$$

As $W_{d,r}$ is open and dense in $(\mathbb{R}[x,y]_d)^r$ (Corollary 3.7.14) and any nonnegative binary form is a sum of two squares, S_r is a dense semialgebraic subset of Σ_{2d}. Consequently, S_r contains a subset that is open and dense in Σ_{2d}, cf. 3.0.2.

Let $f \in S_r$. Then $f = p_1^2 + \cdots + p_r^2 = \mu(\vartheta)$ where $(p_1, \ldots, p_r) \in W_{d,r}$ and $\vartheta := \sum_{i=1}^r p_i \otimes p_i$. Let $F = \mathrm{suppface}(\vartheta) \subseteq \mathrm{Gram}(f)$. Then, according to Corollary 2.3.10 and Definition 3.7.1, F has expected dimension. Thus, we can take S to be $\bigcap_{r=2}^{d+1} S_r$ since this intersection in turn contains an open dense subset of Σ_{2d}. For every $f \in S$, the Gram spectrahedron of f has faces of expected dimension of all ranks $r \in \{2, \ldots, d+1\}$. □

Let $d \in \mathbb{N}$ and let $2 \leq r \leq d+1$. Given $f \in \Sigma_{2d}$, we write $\mathcal{S}_f(r)$ for the set of all psd Gram tensors of f of rank r whose supporting faces have (expected) dimension $\max\{0, \binom{r+1}{2} - (2d+1)\}$.

3.7.16 Corollary. *There is an open dense subset U of Σ_{2d} such that the following holds: For every $f \in U$ and every $2 \leq r \leq d+1$, the set $\mathcal{S}_f(r)$ is a semialgebraic set* *of dimension*

$$\dim \mathcal{S}_f(r) = r(d+1) - (2d+1) - \binom{r}{2} = \frac{1}{2}(r-2)(2d+1-r).$$

Proof. The key ingredients of the proof are Hardt's semialgebraic triviality theorem (see [BCR, Theorem 9.3.2]) that is used to prove a fiber dimension theorem for semialgebraic mappings, and a few observations on a certain action of the orthogonal groups $\mathrm{O}(r)$ (cf. Corollary 5.5 in [Sch22]). As this is elaborated in Vill's thesis, we refer to [Vill, Proposition 3.9.11], which implies the claim since $W_{d,r} \subseteq (\mathbb{R}[x,y]_d)^r$ is open and dense by Corollary 3.7.14. □

3.7.17 Corollary. *Let $d \in \mathbb{N}$. There is an open dense set of nonnegative binary forms $f \in \mathbb{R}[x,y]_{2d}$ for which $\mathcal{H}^+(f)$ contains faces of dimension $r^2-(2d+1)$ for all ranks r with $r \le d+1$ and $\binom{r+1}{2} \ge 2d+1$. For these ranks, this realizes the lower bound in Proposition 3.2.1.*

Proof. Applying Theorem 3.7.15 gives us an open dense subset $U \subseteq \Sigma_{2d}$ as desired. Indeed, let $f \in U$ and let $r \le d+1$ with $\binom{r+1}{2} \ge 2d+1$. Then $\mathrm{Gram}(f)$ contains a face F of rank r of expected dimension, that is $\dim(F) = \binom{r+1}{2} - (2d+1)$. Consequently, the face $F_{\mathcal{H}}$ of $\mathcal{H}^+(f)$ has dimension $r^2-(2d+1)$, according to Corollary 2.5.11. □

In Chapter 4 we will see that there is no obvious generalization of the results presented in this section to the case of quadratic forms on varieties of minimal degree. For an arbitrary nonnegative quadratic form f on a smooth rational normal surface in $\mathbb{P}^5$, we show that the Gram spectrahedron of f never contains an extreme point of highest rank in the Pataki interval (see 4.5.10). In Section 4.7 we will observe more general combinatorial obstructions for the existence of faces of expected dimension.

CHAPTER 4

Gram spectrahedra and varieties of minimal degree

As is generally known, a nonnegative binary form (of any degree) can be represented as a sum of two squares of real forms, and a nonnegative quadratic form (in any number n of variables) is a sum of n squares of real linear forms. Since 1888, when Hilbert's celebrated article [Hil] was published, we know that any nonnegative ternary quartic can be represented as a sum of three squares of real quadratic forms. Thus, in these cases nonnegativity is equivalent to being a sum of squares. In the same article Hilbert also showed that for any pair (n, d) with $n \geq 3$, $d \geq 2$ and $(n, d) \neq (3, 2)$, there exists a nonnegative n-variate form of degree $2d$ that is not a sum of squares.

Binary forms are treated in Chapter 3. In this chapter we place the case of binary forms, quadratic forms and ternary quartics in a broader context. Observe that a binary form of degree d corresponds to a linear form on the rational normal curve $v_d(\mathbb{P}^1) \subseteq \mathbb{P}^d$, where $v_d(\mathbb{P}^1)$ denotes the d-uple Veronese embedding of the projective line. Consequently, a binary form of degree $2d$ corresponds to a quadratic form on $v_d(\mathbb{P}^1)$. In the same way, one identifies an n-variate form of degree $2d$ with a quadratic form on $v_d(\mathbb{P}^{n-1})$. More generally, we can consider quadratic forms on an embedded projective $\mathbb{R}$-variety $X \subseteq \mathbb{P}^n$ with Zariski-dense real points $X(\mathbb{R})$ and ask for the relation between nonnegativity and the existence of sums-of-squares representations. Blekherman, Smith and Velasco established the following deep connection between this substantial question in real algebraic geometry and a classification of varieties originating from classical complex algebraic geometry.

4.0.1 Theorem ([BSV, Theorem 1.1])**.** *Let $X \subseteq \mathbb{P}^n$ be a real irreducible nondegenerate projective subvariety such that the set $X(\mathbb{R})$ of real points is Zariski-dense. Every nonnegative real quadratic form on X is a sum of squares of linear forms if and only if X is a variety of minimal degree.*

For any n, $d \in \mathbb{N}$, the varieties $v_d(\mathbb{P}^1)$ and $\mathbb{P}^n$ are varieties of minimal degree. The same holds true for $v_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$. Thus, Theorem 4.0.1 is indeed a generalization of Hilbert's result.

In this chapter we study Gram spectrahedra of quadratic forms on varieties of minimal degree. We recall the classification of these varieties in Section 4.1. At least the non-hypersurfaces among them can be realized as embedded projective toric varieties X_P for certain lattice polytopes $P \subseteq \mathbb{R}^m$ with vertices in $\mathbb{N}_0^m$. We then adopt a toric point of view and interpret quadratic forms on these varieties of minimal degree as polynomials with Newton polytope (contained in) $2P$. This is justified in Section 4.2 where we establish the fact that lattice equivalent polytopes give projectively equivalent varieties. We then make a detour to Ehrhart theory (Section 4.3), which is a powerful tool for counting lattice points in polytopes and has many beautiful applications beyond that. For us, Ehrhart theory primarily serves as preparation for Chapter 6. In addition, we use it to give an alternative proof for the characterization of lattice polytopes P for which every nonnegative polynomial $f \in \mathbb{R}[\underline{x}]_{2P}$ is a sum of squares. The remaining sections then deal with the individual cases of varieties of minimal degree.

4.1. Varieties of minimal degree

An important numerical invariant of a variety $X \subseteq \mathbb{P}^n$ is its quadratic deficiency. It was introduced in [Zak99], where it is also used to characterize the varieties of minimal degree. In the proof of Theorem 4.0.1, the quadratic deficiency plays an important role in distinguishing the varieties where the cones of nonnegative quadratic forms and sums of squares of linear forms, respectively, are equal, from those varieties where this is not the case. Furthermore, the quadratic deficiency helps us in calculating the dimension of Gram spectrahedra. Moreover, we will re-encounter this algebraic invariant in Chapter 6 when we deal with varieties of almost minimal degree. These should be enough reasons to put the definition at the beginning of this section!

4.1.1 Definition. Let $X \subseteq \mathbb{P}^n$ be an irreducible projective variety of dimension m and let $c := \operatorname{codim}(X) = n - m$. Let R be the homogeneous coordinate ring and $I = \mathfrak{I}_+(X)$ the homogeneous vanishing ideal of X. The *quadratic deficiency* $\varepsilon_2(X)$ of X is defined as

$$\varepsilon_2(X) := \binom{c+1}{2} - \dim(I_2).$$

By [L'v, Theorem 1.2], a nondegenerate irreducible projective variety $X \subseteq \mathbb{P}^n$ is contained in at most $\binom{c+1}{2}$ linearly independent quadrics. For nondegenerate varieties, the quadratic deficiency $\varepsilon_2(X)$ thus measures how far the number of linearly independent quadrics vanishing on X is apart from its theoretical maximum.

4.1.2 Lemma. *Using the notation of Definition 4.1.1, it holds*

$$\varepsilon_2(X) = \dim(R_2) - (n+1)(m+1) + \binom{m+1}{2}$$

Proof. We have $\dim(R_2) = \dim \mathbb{R}[x_0, \ldots, x_n]_2 - \dim I_2 = \binom{n+2}{2} - \dim I_2$. The claim thus follows from

$$\begin{aligned}\binom{n+2}{2} - (n+1)(m+1) + \binom{m+1}{2} &= \frac{1}{2}(n^2 + n - 2nm - m + m^2) \\ &= \frac{(n-m+1)(n-m)}{2} \\ &= \binom{c+1}{2}.\end{aligned}$$

□

4.1.3 Theorem ([Zak99, Corollary 5.8])**.** *For an irreducible projective variety $X \subseteq \mathbb{P}^n$ we have $\varepsilon_2(X) = 0$ if and only if X is a variety of minimal degree, that is* $\deg X = \operatorname{codim} X + 1$.

4.1.4 Remark. We can give a formula for the dimension of Gram spectrahedra in terms of the quadratic deficiency. Let $X \subseteq \mathbb{P}^n$ be a nondegenerate irreducible projective $\mathbb{R}$-variety of dimension m. Write $I = \mathfrak{I}_+(X)$ and let $R = \mathbb{R}[X] = \mathbb{R}[x_0, \ldots, x_n]/I$ be the homogeneous coordinate ring of X. If $f \in \operatorname{int}(\Sigma R_1^2)$, then $\operatorname{Gram}_{R_1}(f)$ contains a point of full rank $\dim(R_1) = n+1$ (see Proposition 2.3.7). We obtain

$$\begin{aligned}\dim \operatorname{Gram}_{R_1}(f) &= \binom{\dim(R_1)+1}{2} - \dim(R_2)\\ &= \binom{n-m+1}{2} - \varepsilon_2(X)\\ &= \binom{c+1}{2} - \varepsilon_2(X),\end{aligned}$$

in which we used Lemma 4.1.2 for the second equality. Note that $c = n - m$ again denotes the codimension of X. In particular, if X is a variety of minimal degree like in Theorem 4.0.1 and $f \in R_2$ is a quadratic form positive on $X(\mathbb{R})$, then $\dim \operatorname{Gram}_{R_1}(f) = \binom{c+1}{2}$ according to Theorem 4.1.3. Note that always $\dim \operatorname{Gram}_{R_1}(f) = \dim(I_2)$.

Here is a quantitative extension to Theorem 4.0.1, due to Blekherman, Plaumann, Sinn and Vinzant.

4.1.5 Theorem ([BPSV, Theorem 2.1]). *Let $X \subseteq \mathbb{P}^n$ be a nondegenerate irreducible real projective variety of minimal degree with dense real points. Then every quadratic form nonnegative on X is a sum of* $\dim(X)+1$ *squares in the homogeneous coordinate ring* $\mathbb{R}[X]$.

This means that the lowest rank of a positive semidefinite Gram tensor of a general positive quadratic form is $\dim(X)+1$. Note that this is also the smallest rank in the Pataki interval.

As a byproduct of the proof of Theorem 4.1.5 given in [BPSV], the authors deduce the fact that a generic $f \in \Sigma\mathbb{R}[X]_1^2$ has only finitely many Gram tensors of this rank. We sketch their line of reasoning.

4.1.6. Let $X \subseteq \mathbb{P}^n$ be a nondegenerate irreducible projective $\mathbb{R}$-variety with dense real points. For $k \in \mathbb{N}_0$, consider the map

$$\begin{aligned}\phi_{k+1} \colon (\mathbb{R}[X]_1)^{k+1} &\longrightarrow \mathbb{R}[X]_2,\\ (l_0,\ldots,l_k) &\longmapsto \sum_{i=0}^{k} l_i^2.\end{aligned} \tag{4.1.1}$$

Let $\dim(X) = m$ and let X be arithmetically Cohen-Macaulay. Then, for every $l_0,\ldots,l_m \in \mathbb{R}[X]_1$ with $X \cap \mathcal{V}_+(l_0,\ldots,l_m) = \varnothing$, the differential $\mathrm{d}\phi$ of $\phi := \phi_{m+1}$ at $(l_0,\ldots,l_m)$ satisfies

$$\operatorname{rk}(\mathrm{d}\phi(l_0,\ldots,l_m)) = (m+1)(n+1) - \binom{m+1}{2} \overset{4.1.2}{=} \dim(\mathbb{R}[X]_2) - \varepsilon_2(X), \tag{4.1.2}$$

see [BPSV, Lemma 2.2]. This follows from the fact that $l_0,\ldots,l_m$ is a homogeneous system of parameters in $\mathbb{R}[X]$ and therefore also a regular sequence since X is aCM. Consequently, the only syzygies among the l_i are the trivial ones (cf. [Mat, Theorem 16.5]), that is to say $l_i l_j = l_j l_i$ for $i \neq j$. The formula for the rank of $\mathrm{d}\phi(l_0,\ldots,l_m)$, i.e., of the map

$$\begin{aligned}\mathbb{R}[X]_1 \times \cdots \times \mathbb{R}[X]_1 &\longrightarrow \mathbb{R}[X]_2,\\ (h_0,\ldots,h_m) &\longmapsto 2\sum_{i=0}^{m} h_i l_i,\end{aligned}$$

then follows from the rank-nullity theorem.

Now let X be of minimal degree. By [EG, Theorem 4.2], X is arithmetically Cohen-Macaulay, and Theorem 4.1.3 gives $\varepsilon_2(X) = 0$. Thus, $\mathrm{d}\phi(l_0, \dots, l_m)$ is surjective for all $l_0, \dots, l_m \in \mathbb{R}[X]_1$ with $X \cap \mathcal{V}_+(l_0, \dots, l_m) = \varnothing$. Blekherman, Plaumann, Sinn and Vinzant then finish the proof of Theorem 4.1.5 by showing that the image of ϕ_{m+1} equals the cone of nonnegative quadratic forms in $\mathbb{R}[X]_2$. To this end they use a topological argument similar to the one given by Hilbert in his proof of the fact that every nonnegative ternary quartic is a sum of three squares.

4.1.7 Corollary ([BPSV, Corollary 2.4]). *(Let X be of minimal degree.) A generic $f \in \Sigma\mathbb{R}[X]_1^2$ has only finitely many Gram tensors of rank* $\dim(X) + 1$.

Proof. Consider $\phi := \phi_{m+1} \colon (\mathbb{R}[X]_1)^{m+1} \to \mathbb{R}[X]_2$. By Theorem 4.1.5, the image of ϕ equals $\Sigma\mathbb{R}[X]_1^2$. Sard's Theorem ([Mil, Section 2]) tells us that a generic $f \in \Sigma\mathbb{R}[X]_1^2$ is a regular value of ϕ. According to Lemma 1 in Section 2 of [Mil], for every such f the fiber $\phi^{-1}(f)$ has dimension

$$(m+1)\dim(\mathbb{R}[X]_1) - \dim(\mathbb{R}[X]_2) \overset{4.1.2}{=} \binom{m+1}{2}.$$

The orthogonal group $\mathrm{O}(m+1)$ acts faithfully on linearly independent linear forms. Since it has dimension $\binom{m+1}{2}$, we see that there are only finitely many orbits in the fiber of f. Finally, those orbits correspond to the psd Gram tensors of f of rank $m+1$. □

A more challenging task is to determine the number of Gram tensors of minimum rank. For the sake of completeness, we include a discussion of this problem in 4.1.9.

Classifying objects often makes them easier to handle. This applies also to the varieties for which we want to study Gram spectrahedra in this chapter.

Varieties of minimal degree have been classified by del Pezzo and Bertini. We refer to [EH] for a modern account. Of course, $\mathbb{P}^n$ itself is a variety of minimal degree for any $n \in \mathbb{N}$. In codimension 1, a variety of minimal degree is a quadric hypersurface. The classification for higher codimension is more interesting:

4.1.8 Theorem ([EH, Theorem 1]). *If $X \subseteq \mathbb{P}^n$ is a variety of minimal degree, then X is a cone over a smooth such variety. If X is smooth and* $\operatorname{codim}(X) > 1$, *then $X \subseteq \mathbb{P}^n$ is either a rational normal scroll or the Veronese surface $v_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$.*

Our aim is to study Gram spectrahedra of quadratic forms on varieties of minimal degree. So let $X \subseteq \mathbb{P}^n$ always be a (nondegenerate and irreducible) projective $\mathbb{R}$-variety of minimal degree with dense real points and let $f \in \mathbb{R}[X]_2$ be nonnegative on $X(\mathbb{R})$. As above we write $R = \mathbb{R}[X]$ and we consider the Gram spectrahedron of f relative to $V = R_1$.

Projective n-space. A real quadratic form on $X = \mathbb{P}^n$ is a homogeneous polynomial $f \in \mathbb{R}[x_0, \dots, x_n]_2$. Up to orthogonal equivalence, a positive definite quadratic form has exactly one representation as a sum of $n+1$ squares. The Gram spectrahedron of such an f is a single point.

Quadric hypersurfaces. In the case where our variety is a quadric hypersurface, there is not much to say either. The codimension of X equals 1 and its vanishing ideal is generated by a single form of degree 2. This means that $\dim \mathrm{Gram}_V(f) = 1$ if $f \in \mathrm{int}(\Sigma\mathbb{R}[X]_1^2)$. Hence, the Gram spectrahedron is a line segment whose interior points have rank $\dim(V) = n+1$. By Theorem 4.1.5, f is a sum of $\dim(X)+1 = n$ squares and this number coincides with the lower bound in the Pataki interval. For general f, both endpoints of $\mathrm{Gram}_V(f)$ thus have rank n.

Rational normal curves and scrolls. We start by recalling the construction of (smooth) rational normal scrolls (see also [Harr, Example 8.26]). Let natural numbers $m \geq 1$ and $d_0 \geq d_1 \geq \cdots \geq d_{m-1} \geq 1$ be given. We write $\bar{d} = (d_0, d_1, \ldots, d_{m-1}) \in \mathbb{N}^m$ and let

$$n+1 = \sum_{i=0}^{m-1} (d_i + 1) = \left(\sum_{i=0}^{m-1} d_i \right) + m = |\bar{d}| + m.$$

Fix m projectively independent linear subspaces $U_0, \ldots, U_{m-1}$ in $\mathbb{P}^n$ with $\dim(U_i) = d_i$. For every $i \in \{0, \ldots, m-1\}$ we choose a rational normal curve $\phi_i \colon \mathbb{P}^1 \to U_i$. Now, any $\eta \in \mathbb{P}^1$ gives an $(m-1)$-dimensional linear subspace $\phi_0(\eta) \vee \cdots \vee \phi_{m-1}(\eta)$ of $\mathbb{P}^n$. We let

$$X = X\left(\bar{d}\right) = \bigcup_{\eta \in \mathbb{P}^1} \left(\phi_0(\eta) \vee \cdots \vee \phi_{m-1}(\eta)\right).$$

Then X is an irreducible subvariety of $\mathbb{P}^n$, called a *rational normal scroll*, and we have $\dim(X) = m$. Choosing positive d_i's ensures that X is smooth, but one could also extend the construction to the case where one or more d_i's are zero in order to include cones over smooth scrolls. For the time being, we stick to the smooth case.

Choosing coordinates in $\mathbb{P}^n$ properly, we can realize X as the projective toric variety of a Lawrence prism like in Example 1.3.13. Let $P := P_{\bar{d}}$ be the Cayley sum of the m intervals $[0, d_i] \subseteq \mathbb{R}$ $(i = 0, 1, \ldots, m-1)$, so

$$\begin{aligned} P &= [0, d_0] * [0, d_1] * \cdots * [0, d_{m-1}] \\ &= \operatorname{conv}\big(([0, d_0] \times \{0\}) \cup ([0, d_1] \times \{e_1\}) \cup \ldots \cup ([0, d_{m-1}] \times \{e_{m-1}\})\big) \\ &\subseteq \mathbb{R} \times \mathbb{R}^{m-1} = \mathbb{R}^m. \end{aligned}$$

The number of lattice points of P in $\mathbb{Z}^m$ is $n+1 = |\bar{d}|+m$. For an illustration of the resulting Cayley polytopes we point the reader to Figure 1.1 that shows $P_{(3,1)} \subseteq \mathbb{R}^2$ and $P_{(1,1,1)} \subseteq \mathbb{R}^3$. In retrospect, P must be a smooth (and hence very ample) polytope given that we already know that X_P is a smooth rational normal scroll (cf. [CLS, Theorem 2.4.3 and Proposition 2.4.4]). But we can also argue differently in order to show that X_P can be embedded using the lattice points of P. Indeed, by Proposition 6.9 and Remark 6.8 in [BSV], P is normal. Hence, with the appropriate choice of coordinates, $X\left(\bar{d}\right) = X_P$ is the Zariski closure of the map

$$\begin{aligned} \Phi \colon \mathbb{C}^* \times (\mathbb{C}^*)^{m-1} &\to \mathbb{P}^n, \\ (s, x) = (s, x_1, \ldots, x_{m-1}) &\mapsto [u_0(s), u_1(s,x), \ldots, u_{m-1}(s,x)] \end{aligned}$$

with $u_0(s) = (1, s, \ldots, s^{d_0})$ and $u_i(s,x) = (x_i, x_i s, \ldots, x_i s^{d_i})$ for $i = 1, \ldots, m-1$.

If $m = 1$, then $X(d_0) \subseteq \mathbb{P}^{d_0}$ is the rational normal curve of degree d_0. For any $d \in \mathbb{N}$, a quadratic form on the rational normal curve $v_d(\mathbb{P}^1)$ corresponds to a binary form of degree $2d$. The Gram spectrahedra emerging from these cases are treated in detail in Chapter 3. In this chapter we thus advance to higher-dimensional rational normal scrolls. In Section 4.6 we will see that the bounds on the dimensions of faces in Gram spectrahedra generalize from $m = 1$ to arbitrary m.

The Veronese surface. The case of the Veronese surface is somewhat special since it corresponds to the exceptional case of ternary quartics in Hilbert's theorem on sums of squares. Ternary quartics have attracted much interest over the years and Gram spectrahedra of ternary quartics have already been studied in great detail, see for example [PSV] and [Vill]. We summarize the main results concerning their facial structure in Section 4.4.

Cones over varieties. We are going to explain the case of cones in Section 4.8. As an outlook, let us state the following: Let X be a variety of minimal degree and let Y be a cone over X. If $f \in \mathbb{R}[Y]_2$ is positive on $Y(\mathbb{R})$, then there exists a positive form $g \in \mathbb{R}[X]_2$ such that (up to a shift in ranks) the Gram spectrahedra of f and g are structurally identical.

Our analysis in Sections 4.5 to 4.7 will thus focus on the situation of quadratic forms on (smooth) rational normal scrolls.

4.1.9 (Gram tensors of minimum rank)**.** Let $X \subseteq \mathbb{P}^n$ as in Theorem 4.1.5 and let $m = \dim(X)$. In Corollary 4.1.7 we have seen that a generic quadratic form nonnegative on X has only finitely many inequivalent representations as a sum of $m+1$ squares. Generically, the precise number of these representations is $2^{\operatorname{codim}(X)} = 2^{n-m}$. As we have seen above, this is obviously true (for any value of m) when $\operatorname{codim}(X) \leq 1$. So let $\operatorname{codim}(X) > 1$. For $m = 1$ our X is a rational normal curve and the assertion follows from the corresponding fact about binary forms ([CLR, Example 2.13]). According to the classification of varieties of minimal degree, for $m = 2$ one has to deal with cones over rational normal curves, the Veronese surface $v_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$ and smooth rational normal surfaces. While cones are harmless, the count for ternary quartics is obtained in [PRSS]. Using some ideas presented in the aforementioned article, the case of rational normal surfaces is solved in [BPSV, Theorem 3.11]. For higher-dimensional smooth rational normal scrolls it was conjectured in [BPSV] that the number in question equals $2^{\operatorname{codim}(X)}$. This was finally proven by Hanselka and Sinn (see [HS, Corollary 5.7]) by means of the theory of quadratic forms.

4.2. Projective equivalence and lattice equivalent polytopes

We have seen that at least the non-hypersurfaces among the varieties of minimal degree are toric. As calculations in the polynomial ring can often be performed more easily than in the coordinate ring of a variety, we want to interpret quadratic forms on toric varieties of minimal degree as polynomials with prescribed Newton polytopes.

We devote this section to the following important fact that we will often use implicitly: Given two full-dimensional normal lattice polytopes P, $Q \subseteq M_{\mathbb{R}}$ with $n+1$ lattice points each, we consider the projective toric varieties X_P, $X_Q \subseteq \mathbb{P}^n$ embedded using the lattice points of P and Q, respectively. Then X_P and X_Q are projectively equivalent if and only if there is an affine-linear isomorphism of the lattice M that maps (the lattice points of) P to (those of) Q.

This statement is also mentioned in the proof of Theorem 2.1 in [CPSV] which characterizes the lattice polytopes P for which every nonnegative polynomial $f \in \mathbb{R}[\underline{x}]_{2P}$ is a sum of squares. As the authors note, it appears to be a folk theorem among the experts in toric geometry and there seems to be no suitable reference. Since we will frequently apply translations and lattice automorphisms to our polytopes, we should at least make sure that the homogeneous coordinate rings of the varieties remain in the same isomorphism class. This is done in Theorem 4.2.4. We prove the opposite implication in Theorem 4.2.6. For full disclosure let me note that this proof follows the answer of Knop on `Mathoverflow` to a question from Sinn. I thus want to join the authors of [CPSV] in thanking Friedrich Knop.

4.2.1 Definition. We say that two projective $\mathbb{C}$-varieties X, $Y \subseteq \mathbb{P}^n(\mathbb{C})$ are *projectively equivalent* if there exists an $a \in \mathrm{PGL}_{n+1}(\mathbb{C})$ such that $a(X) = Y$.

4.2.2 Remark. According to [Hart, Chapter II, Example 7.1.1], the projective linear group $\mathrm{PGL}_{n+1}(\mathbb{C}) = \mathrm{GL}_{n+1}(\mathbb{C})/(\mathbb{C}^* I_{n+1})$ is the full automorphism group of the projective space $\mathbb{P}^n$. In other words, every automorphism of projective n-space is a linear change of coordinates.

For projective varieties X, $Y \subseteq \mathbb{P}^n$, being projectively equivalent is a stronger property than being isomorphic as projective varieties. Indeed, if X and Y are projectively equivalent, then their homogeneous coordinate rings $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ are isomorphic as graded $\mathbb{C}$-algebras (cf. [Harr, Lecture 2]) which is not necessarily true if X and Y are isomorphic varieties.

However, two affine varieties are isomorphic if and only if their coordinate rings are. Since the affine cone $\hat{X} \subseteq \mathbb{A}^{n+1}$ over X has the same coordinate ring as X, we see that X and Y are projectively equivalent if and only if their affine cones $\hat{X}$ and $\hat{Y}$ are isomorphic affine varieties (see also [Harr, Exercise 20.10]).

4.2.3 Remark. We first note that translations are perfectly harmless. Indeed, let $\mathcal{A} = \{m_1, \dots, m_s\} \subseteq M$ be a set of s lattice points and let $m \in M$. For every torus element $t \in T_N$ we have $\chi^m(t) \in \mathbb{C}^*$ and thus

$$\begin{aligned}\Phi_{\mathcal{A}+m}(t) &= (\chi^{m_1+m}(t) : \dots : \chi^{m_s+m}(t))\\ &= (\chi^m(t)\chi^{m_1}(t) : \dots : \chi^m(t)\chi^{m_s}(t))\\ &= (\chi^{m_1}(t) : \dots : \chi^{m_s}(t))\\ &= \Phi_{\mathcal{A}}(t).\end{aligned}$$

Therefore, the projective varieties $X_{\mathcal{A}}$ and $X_{\mathcal{A}+m}$ are the same.

We now prove one direction of the result mentioned in the introduction of this section. This allows us to apply an affine-linear isomorphism to a polytope without significantly changing the homogeneous coordinate ring of the associated embedded projective variety.

4.2.4 Theorem. *Let P, $Q \subseteq M_{\mathbb{R}}$ be full-dimensional normal lattice polytopes with $n+1$ lattice points each. If Q is obtained from P by an affine-linear isomorphism of $M_{\mathbb{R}}$, that is a composition of a translation and a lattice automorphism, then X_P, $X_Q \subseteq \mathbb{P}^n$ (embedded with respect to the lattice points of P and Q, respectively) are projectively equivalent.*

Proof. As we have seen above, the projective embedding $X_P \subseteq \mathbb{P}^n$ is invariant under a translation of the lattice points. We can therefore assume that P is mapped to Q under a lattice automorphism ψ. It suffices to show that the affine cones $\hat{X}_P$ and $\hat{X}_Q$ have isomorphic coordinate rings. The discussion after Proposition 2.1.4 in [CLS] shows that $\hat{X}_P$ and $\hat{X}_Q$ are the affine toric varieties associated to the lattice points $(P \cap M) \times \{1\}$ and $(Q \cap M) \times \{1\}$, respectively, where the lattice is now $M \oplus \mathbb{Z}$. We may thus replace M by $M \oplus \mathbb{Z}$ and ψ by $\psi \times \mathrm{id}_{\mathbb{Z}}$, and are then reduced to showing the following: If there exists a lattice automorphism ψ with $\psi(\mathcal{A}) = \mathcal{A}'$ for some

$\mathcal{A}$, $\mathcal{A}' \subseteq M$ with s elements each, then the coordinate rings of the associated affine toric varieties $Y_{\mathcal{A}}$, $Y_{\mathcal{A}'} \subseteq \mathbb{C}^s$ are isomorphic.

We write $\mathcal{A} = \{m_1, \ldots, m_s\}$ and $\mathcal{A}' = \{m'_1, \ldots, m'_s\}$. Up to renumbering (which corresponds to a permutation of variables on the side of coordinate rings) we can assume that $\psi(m_i) = m'_i$ for $i = 1, \ldots, s$. We use the description of the vanishing ideal from [CLS, Proposition 1.1.9]: The map $\Phi_{\mathcal{A}} \colon T_N \to (\mathbb{C}^*)^s$ induces a map of character lattices $\mathbb{Z}^s \to M$ that sends the standard basis $e_1, \ldots, e_s$ to $m_1, \ldots, m_s$, and we consider its kernel

$$L_{\mathcal{A}} = \left\{ l \in \mathbb{Z}^s : \sum_{i=1}^{s} l_i m_i = 0 \right\}.$$

Then $\mathfrak{I}(Y_{\mathcal{A}}) = \langle \underline{x}^\alpha - \underline{x}^\beta : \alpha, \beta \in \mathbb{N}_0^s \text{ and } \alpha - \beta \in L_{\mathcal{A}} \rangle$. Since ψ is $\mathbb{Z}$-linear and injective, we immediately see that $L_{\mathcal{A}} = L_{\mathcal{A}'}$, where

$$L_{\mathcal{A}'} = \left\{ l \in \mathbb{Z}^s : \sum_{i=1}^{s} l_i \psi(m_i) = 0 \right\}.$$

Consequently, $\mathfrak{I}(Y_{\mathcal{A}}) = \mathfrak{I}(Y_{\mathcal{A}'})$, and this completes the proof. □

For the opposite direction we include the following preliminary remark linking torus automorphisms to lattice automorphisms.

4.2.5 Remark. We work with the torus $T = (\mathbb{C}^*)^n$ whose automorphism group $\operatorname{Aut}(T)$ is isomorphic to $\operatorname{GL}_n(\mathbb{Z})$. An automorphism $\gamma \in \operatorname{Aut}(T)$ thus corresponds to a matrix $C = (c_{ij}) \in \operatorname{GL}_n(\mathbb{Z})$ and for $t = (t_1, \ldots, t_n) \in T$ we have

$$\gamma(t) = (t_1^{c_{11}} \cdots t_n^{c_{1n}}, \ldots, t_1^{c_{n1}} \cdots t_n^{c_{nn}}) = \left(\prod_{j=1}^{n} t_j^{c_{ij}} \right)_{i=1,\ldots,n}.$$

Recall from Example 1.4.2 that every $m = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ gives a character χ^m of T by setting

$$\chi^m \colon (\mathbb{C}^*)^n \to \mathbb{C}^*, \ (t_1, \ldots, t_n) \mapsto t_1^{a_1} \cdots t_n^{a_n},$$

and that the map $\mathbb{Z}^n \to M_T$, $m \mapsto \chi^m$ is a group isomorphism. Using this identification we obtain the following:

$$\chi^m(\gamma(t)) = \prod_{i=1}^{n} \left(\prod_{j=1}^{n} t_j^{c_{ij}} \right)^{a_i} = \prod_{i=1}^{n} \prod_{j=1}^{n} t_j^{c_{ij} a_i} = \prod_{j=1}^{n} (t_j)^{\sum_{i=1}^{n} c_{ij} a_i} = \chi^{\gamma^\vee(m)}(t)$$

where by $\gamma^\vee(m)$ we denote the image of m under C^T, the transpose of C. Thus, γ induces an automorphism $\gamma^\vee$ of the (character) lattice M.

We are now ready to prove the converse of Theorem 4.2.4. The proof uses some deep results from the theory of algebraic groups.

4.2.6 Theorem. *Let X_P, $X_Q \subseteq \mathbb{P}^n$ be projective toric varieties, embedded using the lattice points of full-dimensional normal lattice polytopes P, $Q \subseteq M_{\mathbb{R}}$. If X_P and X_Q are projectively equivalent, then there exists an affine-linear isomorphism of $M_{\mathbb{R}}$ that maps P to Q.*

Proof. Let r be the dimension of the character lattice M associated to our torus $T := T_N \cong (\mathbb{C}^*)^r$. Let $m_0, \ldots, m_n$ denote the lattice points of P in some fixed order.

As the polytope P is normal, it is in particular very ample (see Section 1.3). Thus, the map

$$\begin{aligned}\Phi_{P\cap M}\colon T &\to \mathbb{P}^n,\\ t &\mapsto (\chi^{m_0}(t):\cdots:\chi^{m_n}(t))\end{aligned}$$

is an embedding of the torus T into $\mathbb{P}^n$ (cf. [BG09, Section 10.B, p. 368f.]). The torus T acts on $\mathbb{P}^n$ via $\Phi_{P\cap M}$:

$$t\cdot(\xi_0:\cdots:\xi_n) := (\chi^{m_0}(t)\xi_0:\cdots:\chi^{m_n}(t)\xi_n)$$

for any $t\in T$ and $\xi\in\mathbb{P}^n$. In other words, P gives a projective representation p of the group T:

$$\begin{aligned}p\colon T &\to \mathrm{PGL}_{n+1}(\mathbb{C}),\\ t &\mapsto p(t) := \mathrm{diag}(\chi^{m_0}(t),\dots,\chi^{m_n}(t))(\mathbb{C}^* I_{n+1}).\end{aligned}$$

In order to ease notation, we will avoid dragging along coset classes and simply remember that a "matrix" in $\mathrm{PGL}_{n+1}(\mathbb{C})$ is defined up to multiplication with some $\mu\in\mathbb{C}^*$. Since $\Phi_{P\cap M}$ is an embedding, the homomorphism p is injective.

Consider the group $G=\{g\in\mathrm{PGL}_{n+1}(\mathbb{C}) : g(X_P)=X_P\}$. The restriction homomorphism $G\to\mathrm{Aut}_{\mathbb{C}}(X_P)$ is injective since X_P is nondegenerate (cf. Proposition 1.4.8). Thus, the embedded r-dimensional torus $p(T)$ is a maximal torus of G. Indeed, according to [BG99, Theorem 5.4], the embedded torus $p(T)$ is even a maximal torus of $\mathrm{Aut}_{\mathbb{C}}(X_P)$. We comment on this fact in Remark 4.2.7.

Now assume that X_P and X_Q are projectively equivalent. Then there is an $a\in\mathrm{PGL}_{n+1}(\mathbb{C})$ with $a(X_Q)=X_P$. By q we denote the projective representation of T given by Q. Then

$$\begin{aligned}\bar q\colon T &\to \mathrm{PGL}_{n+1}(\mathbb{C}),\\ t &\mapsto \bar q(t) := aq(t)a^{-1}\end{aligned}$$

is another representation and we have $\bar q(t)(X_P)=X_P$ for every $t\in T$. Thus, also $\bar q(T)\subseteq G$ is a maximal torus of G. By [Hum, Corollary A in §21.3], maximal tori are conjugate in G, i.e., there exists $b\in G$ such that $p(T)=b\bar q(T)b^{-1}=(ba)q(T)(ba)^{-1}$. Hence, there is an automorphism $\gamma\in\mathrm{Aut}(T)$ such that

$$p(\gamma(t)) = (ba)q(t)(ba)^{-1}$$

for all $t\in T$. By Remark 4.2.5, γ induces an automorphism $\gamma^\vee$ of the lattice M. We replace P by $\gamma^\vee(P)$. Then we may assume that $\gamma=\mathrm{id}_T$. Now, p and q are conjugate by an element $c:=ba\in\mathrm{PGL}_{n+1}(\mathbb{C})$.

The $n+1$ eigenvalues of an element in $\mathrm{PGL}_{n+1}(\mathbb{C})$ are well-defined up to multiplication with some common factor $\mu\in\mathbb{C}^*$. For each $t\in T$, we have $p(t)=cq(t)c^{-1}$, so that – in this sense – the eigenvalues of $p(t)$ and $q(t)$ coincide. But $q(t)$ gives the diagonal matrix with entries $\chi^{m_0'}(t),\dots,\chi^{m_n'}(t)$ where $m_0',\dots,m_n'$ are the lattice points of Q. After renumbering, we can assume that

$$(\chi^{m_0}(t):\cdots:\chi^{m_n}(t)) = (\chi^{m_0'}(t):\cdots:\chi^{m_n'}(t))$$

for all $t\in T$. This means that for every $t\in T$ there is $\mu(t)\in\mathbb{C}^*$ such that $\chi^{m_i}(t)=\mu(t)\cdot\chi^{m_i'}(t)$ for all $i=0,\dots,n$. This defines a character $\mu: T\to\mathbb{C}^*$. In other words, μ is the character χ^m for some $m\in M$, so that the lattice points of P and Q, and therefore P and Q themselves, differ by a translation by m. □

4.2.7 Remark. The *Cremona group* $\mathrm{Cr}(\mathbb{P}^r(\mathbb{C}))$ is the group of birational automorphisms of projective r-space, sometimes also denoted as $\mathrm{Bir}(\mathbb{P}^r(\mathbb{C}))$. It is naturally identified with $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(x_1, \ldots, x_r))$, the group of $\mathbb{C}$-automorphisms of the rational function field in r variables. The Cremona $\mathrm{Cr}(\mathbb{P}^r(\mathbb{C}))$ contains the projective general linear group $\mathrm{PGL}_{r+1}(\mathbb{C})$. It is equal to this group for $r \in \{0, 1\}$ but it is strictly larger as soon as $r \geq 2$. There are some results on the structure of the Cremona for $r = 2$. However, because of the enormous size of the group there is little known about its structure for larger r.

Demazure studied linear algebraic subgroups of the Cremona groups in [Dem]. In connection with these studies he obtained results on automorphism groups of nonsingular complete toric varieties. These results may have motivated him to study such varieties. His article [Dem] published in 1970 is thus considered as the birth of toric geometry.

One of the many results therein can be phrased as follows: If X is a nonsingular complete toric variety, then $\mathrm{Aut}(X)$ is an affine algebraic group with T_N as maximal torus. For more information on the structure of $\mathrm{Aut}(X)$ as an algebraic group, we refer to Section 3.4 in Oda's textbook [Oda]. In the same section further results due to Demazure concerning the automorphism groups as well as the Cremona groups are discussed. In 1995, Cox proved the similar statement where nonsingular is replaced by simplicial ([Cox, Corollary 4.7]). One year later, Bühler [Büh] generalized the result in a diploma thesis to arbitrary complete toric varieties. In the above proof of Theorem 4.2.6 we cited the version of Bruns and Gubeladze for projective varieties. The remarks on the historical development of the said result are taken from the introduction of their article [BG99].

Let us finally note that $T_N \cong (\mathbb{C}^*)^r$ can even be considered a maximal torus in $\mathrm{Cr}(\mathbb{P}^r(\mathbb{C}))$. According to Demazure's fundamental work [Dem], the following is true: If $(\mathbb{C}^*)^k$ embeds as an algebraic subgroup in $\mathrm{Bir}(\mathbb{P}^r(\mathbb{C}))$, then $k \leq r$. If $k = r$, then the embedding is conjugate to an embedding into the group of diagonal matrices in $\mathrm{PGL}_{r+1}(\mathbb{C})$. This formulation is taken from [Dés, Theorem 8.4]. Déserti comments on this as follows:

> In other words the group of diagonal automorphisms plays the role of a maximal torus in $\mathrm{Bir}(\mathbb{P}^r(\mathbb{C}))$ and the Cremona group "looks like" a group of rank r.

Since X_P is an r-dimensional rational variety, we can also identify the Cremona $\mathrm{Cr}(\mathbb{P}^r(\mathbb{C}))$ with $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(X_P))$ or with the group of birational automorphisms of X_P in order to see that $p(T)$ is a maximal torus in G in the above proof.

We can use the results presented in this section in order to reduce the analysis of Gram spectrahedra of quadratic forms on varieties X_P of minimal degree to a manageable list of lattice polytopes.

4.2.8 Theorem ([BSV, Theorem 1.1 and §6], [CPSV, Theorem 2.1])**.** *Let $P \subseteq \mathbb{R}^k$ be a normal lattice polytope with vertices in $\mathbb{N}_0^k$. Suppose that every nonnegative polynomial $f \in \mathbb{R}[\underline{x}]_{2P}$ with Newton polytope $2P$ is a sum of squares. Then the lattice polytope P is, up to translation and an automorphism of the lattice, contained in one of the following lattice polytopes.*

(i) The m-dimensional unit simplex $S_m = \mathrm{conv}(0, e_1, \ldots, e_m) \subseteq \mathbb{R}^m$.
(ii) The Cayley polytope $[0, d_0] * [0, d_1] * \cdots * [0, d_{m-1}] \subseteq \mathbb{R}^m$ of m line segments, where $d_0 \geq d_1 \geq \cdots \geq d_{m-1} \geq 1$.
(iii) The two-dimensional simplex $2S_2 = \mathrm{conv}(0, 2e_1, 2e_2) \subseteq \mathbb{R}^2$.

(iv) *The free sum* conv $\big((Q \times \{0\}) \cup (\{0\} \times \Delta_{n-1})\big) \subseteq \mathbb{R}^m \times \mathbb{R}^n$, *where* $Q \subseteq \mathbb{R}^m$ *is one of the preceding polytopes (i)-(iii) and* $\Delta_{n-1} = \operatorname{conv}(e_1, \ldots, e_n) \subseteq \mathbb{R}^n$.

Note that the projective toric variety X_{S_m} associated to the unit simplex S_m from (i) is $\mathbb{P}^m$, the Cayley polytopes in (ii) give the smooth rational normal scrolls, the scaled simplex in (iii) corresponds to the Veronese surface $v_2(\mathbb{P}^2)$ in $\mathbb{P}^5$, and the polytopes in case (iv) give cones over these smooth varieties.

In the classification of varieties of minimal degree there was yet another class, namely quadric hypersurfaces. Before we prove Theorem 4.2.8 we discuss which of them are toric, and make sure that these are represented in the list above.

4.2.9 Remark. If $P \subseteq M_{\mathbb{R}} \cong \mathbb{R}^m$ is a full-dimensional normal lattice polytope that gives the projective embedding $X_P \subseteq \mathbb{P}^n$ with $n = |P \cap M| - 1$, then X_P is a hypersurface if and only if $n - m = 1$, that is to say $|P \cap M| = m + 2$.

For $m = 1$, the polytope P is a line segment so that (after a translation) $P = [0, 2]$. For $m = 2$, we have essentially two types of polytopes that lead to toric hypersurfaces in $\mathbb{P}^3$. The first type is represented by $\operatorname{conv}(0, 2e_1, e_2) \subseteq \mathbb{R}^2$, which is a pyramid over $[0, 2]$. Hence, its associated variety is a cone over $v_2(\mathbb{P}^1)$. The second type is represented by the square $P_{(1,1)} = [0, 1] * [0, 1] \subseteq \mathbb{R}^2$ corresponding to $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$. This shows that for $n \leq 3$, there are toric quadric hypersurfaces $X_P \subseteq \mathbb{P}^n$ and their corresponding polytopes are included in the list given in Theorem 4.2.8. In higher dimensions, of course, we always have quadric hypersurfaces that are cones over the preceding varieties.

Yet, there are no smooth quadric hypersurfaces in $\mathbb{P}^n$ that are toric as soon as $n \geq 4$. Indeed, let $\mathcal{Q} \subseteq \mathbb{P}^n$ be a smooth quadric hypersurface. If $\mathcal{Q}$ was toric, its automorphism group $\operatorname{Aut}(\mathcal{Q})$ would contain a torus of dimension $n-1$. However, it is well-known that $\operatorname{Aut}(\mathcal{Q})$ is isomorphic to the projective orthogonal group $\mathrm{PO}(n+1) = \mathrm{O}(n+1)/\{\pm I_{n+1}\}$. The rank of $\mathrm{O}(n+1)$ equals $\lfloor \frac{n+1}{2} \rfloor$. In other words, a maximal torus in $\mathrm{O}(n+1)$ has dimension $\lfloor \frac{n+1}{2} \rfloor$. In particular, a maximal torus in $\mathrm{PO}(n+1)$ has dimension at most $\lfloor \frac{n+1}{2} \rfloor$ (cf. [Hum, Corollary C in Section 21.3]). For $n \geq 4$, we have $\lfloor \frac{n+1}{2} \rfloor < n - 1$ so that $\mathcal{Q}$ cannot be toric.

Proof of Theorem 4.2.8. P has to be contained in a full-dimensional normal lattice polytope $Q \subseteq \mathbb{R}^m$ such that X_Q is a variety of minimal degree (Theorem 4.0.1). Using the classification of varieties of minimal degree and the discussion in Remark 4.2.9, we see that X_Q has to be projectively equivalent to one of the toric varieties given by the polytopes listed in (i)-(iv). By Theorem 4.2.6, there is an affine-linear isomorphism of $\mathbb{R}^m$ that maps Q to (and hence P into) one of these polytopes. □

Let us summarize the results of this section regarding their meaning for Gram spectrahedra of quadratic forms on varieties of minimal degree. As was pointed out in Section 4.1, Gram spectrahedra of dimension greater than one only arise for those varieties whose codimension is at least two and these varieties are toric. So let $X_Q \subseteq \mathbb{P}^n$ be a (not necessarily smooth) rational normal scroll or (a cone over) the Veronese surface and let $f \in \mathbb{R}[X_Q]_2$. In order to analyze the structure of $\mathrm{Gram}_{\mathbb{R}[X_Q]_1}(f)$, we can choose a lattice polytope $P \subseteq \mathbb{R}^m$ as in Theorem 4.2.8 that is obtained from $Q \subseteq \mathbb{R}^m$ via an affine-linear isomorphism of $\mathbb{R}^m$. By means of this isomorphism, we can identify "variables" (i.e., monomials of degree one) in $\mathbb{R}[X_Q] \cong \mathbb{R}[X_P]$ with monomials in $\mathbb{R}[y_1, \ldots, y_m]_P$, and quadratic forms in $\mathbb{R}[X_Q]_2$ with polynomials in $\mathbb{R}[y_1, \ldots, y_m]_{2P}$ since the relations between lattice points of P are reflected in the vanishing ideals of X_P and X_Q. Bounds for the dimensions of rank-r faces in Gram spectrahedra are usually obtained by estimating $\operatorname{codim}_{VV}(UU)$ where U ranges over the r-dimensional subspaces of $V = \mathbb{R}[X_Q]_1$. The discussion above shows that we might as well consider the r-dimensional subspaces U of $V = \mathbb{R}[y_1, \ldots, y_m]_P$. The latter approach is sometimes more convenient since one can work with explicitly given monomials and also make use of monomial orders.

4.3. A detour to Ehrhart theory

In the 1960s the French math teacher Eugène Ehrhart did research on the number of integer solutions to systems of linear equations and inequalities with integer coefficients (linear Diophantine systems). In other words, one of his interests was the number of lattice points in polytopes. The theory which originated from his work is used to count lattice points in polytopes, in their interiors or dilations, and is called *Ehrhart theory* in his honor.

We give a very brief introduction to Ehrhart theory ignoring many fundamental results like Ehrhart-Macdonald reciprocity, for example. For a comprehensive treatment of the topic we refer the interested reader to the textbook [BR]. The parts presented here will be used in Sections 6.2 and 6.3, where we identify the polytopes P that give special toric varieties of almost minimal degree. Thus, the following would also fit well there. However, we decided to include it at this point since it also leads to another nice characterization of toric varieties of minimal degree and can be used to give an alternative proof of Theorem 4.2.8 at the end of this section.

4.3.1 Remark. Let $P \subseteq M_{\mathbb{R}} \cong \mathbb{R}^m$ be a full-dimensional lattice polytope. A famous result by Ehrhart [Ehr] (see also [CLS, Theorem 9.4.2]) states that there is a polynomial $\operatorname{Ehr}_P(x) \in \mathbb{Q}[x]$ such that

$$\operatorname{Ehr}_P(k) = |(kP) \cap M| \tag{4.3.1}$$

for all $k \in \mathbb{N}$. So this polynomial, called the *Ehrhart polynomial* of P, counts the lattice points in dilations of P. We can also consider the generating function of the sequence in (4.3.1): The *Ehrhart series* of P is the formal power series

$$E_P(t) := \sum_{k=0}^{\infty} |(kP) \cap M| t^k.$$

It was proven by Stanley that

$$E_P(t) = \frac{h_0^* + h_1^* t + \cdots + h_m^* t^m}{(1-t)^{m+1}}$$

for some nonnegative integers h_i^* (cf. [Sta, Theorem 2.1]). The polynomial in the numerator is the *h^*-polynomial* of P. One defines the *degree* of P to be the greatest $i \in \{0, 1, \ldots, m\}$ such that $h_i^* \neq 0$. In other words, the degree of P is the degree of its h^*-polynomial.

Now let P be normal and let $X_P \subseteq \mathbb{P}^n$ be embedded with respect to $P \cap M$, the lattice points of P. The normality assumption ensures that $\mathbb{R}[X_P]$ is generated by $\mathbb{R}[X_P]_1$ and that

$$\dim(\mathbb{R}[X_P]_k) = |(kP) \cap M|$$

for all $k \in \mathbb{N}_0$. Consequently, in this case the Hilbert series $\sum_{k=0}^{\infty} \dim(\mathbb{R}[X_P]_k) t^k$ of X_P equals the Ehrhart series of P.

Let us calculate h_1^* and h_2^*. We write $R_k := \mathbb{R}[X_P]_k$ and compare the coefficients of the linear and of the quadratic terms in

$$(1-t)^{m+1}\sum_{k=0}^{\infty}\dim(R_k)t^k = h_0^* + h_1^* t + \cdots + h_m^* t^m.$$

By the binomial theorem we have

$$(1-t)^{m+1} = 1-(m+1)t + \binom{m+1}{2}t^2 \mp \cdots ,$$

so that $h_1^* = \dim(R_1) - (m+1)\dim(R_0)$ and

$$h_2^* = \dim(R_2) - (m+1)\dim(R_1) + \binom{m+1}{2}\dim(R_0).$$

We have $\dim(R_0) = 1$ and $\dim(R_1) = n+1$ since X_P is nondegenerate. Consequently, $h_1^* = n - m = \operatorname{codim}(X_P)$, and $h_2^* = \varepsilon_2(X_P)$ according to Lemma 4.1.2. Note that this is also proven in [BSV, Lemma 3.1].

4.3.2. A useful tool for calculating the degree of the projective toric variety $X_{P\cap M}$ of a very ample polytope $P \subseteq M_{\mathbb{R}}$ is the notion of the lattice normalized volume of P, which is also encoded in the Ehrhart polynomial of P.

We calculate volumes with respect to the lattice M. This is done as follows: Fix a lattice basis $v_1, \ldots, v_m$ of M and consider the simplex $\Delta_m := \operatorname{conv}(0, v_1, \ldots, v_m) \subseteq M_{\mathbb{R}} \cong \mathbb{R}^m$. Then the volume with respect to M is the usual m-dimensional Lebesgue measure on $\mathbb{R}^m$, scaled so that we have $\operatorname{vol}(\Delta_m) = \frac{1}{m!}$. The *lattice normalized volume* of a lattice polytope $P \subseteq M_{\mathbb{R}}$ is

$$\operatorname{Vol}(P) := m! \operatorname{vol}(P).$$

This means that the simplex Δ_m has normalized volume 1 and the fundamental parallelepiped $\{\sum_{i=1}^m \lambda_i v_i : 0 \le \lambda_i \le 1\}$ in $M_{\mathbb{R}}$ has normalized volume $m!$. When the underlying lattice M is $\mathbb{Z}^m$, the volume is the ordinary Lebesgue measure. Thus, for the m-dimensional unit simplex $S_m = \operatorname{conv}(0, e_1, \ldots, e_m) \subseteq \mathbb{R}^m$ we have $\operatorname{vol}(S_m) = \frac{1}{m!}$ as usual, and its lattice normalized volume is equal to 1. A fundamental parallelepiped for $\mathbb{Z}^m$ is the cube $[0,1]^m$.

As the volume of a lattice polytope P can be determined by counting lattice points in dilations of P, the connection between the volume of P and the degree of X_P is established by the Ehrhart polynomial of P.

4.3.3 Lemma. *Let $P \subseteq M_{\mathbb{R}} \cong \mathbb{R}^m$ be a very ample full-dimensional lattice polytope. Then $\deg X_P = \operatorname{Vol}(P)$ in the projective embedding of X_P given by the lattice points of P.*

Proof. The Ehrhart polynomial of P has degree m and leading coefficient $\operatorname{vol}(P) = \frac{\operatorname{Vol}(P)}{m!}$ (see for example [BR, Corollary 3.20]). Since P is very ample, we can use [CLS, Proposition 9.4.3] in order to see that Ehr_P equals the Hilbert polynomial of the toric variety X_P under the projective embedding given by the lattice points of P. By definition, the Hilbert polynomial has leading coefficient $\frac{\deg X_P}{m!}$. This completes the proof. $\square$

4.3.4 Corollary. *Let $P \subseteq M_{\mathbb{R}} \cong \mathbb{R}^m$ be a full-dimensional normal lattice polytope, and let $X_P \subseteq \mathbb{P}^n$ be embedded using $P \cap M$. For every $k \in \mathbb{N}$, the following are equivalent:*
(a) $\deg X_P = \operatorname{codim} X_P + k$,
(b) $\mathrm{Vol}(P) = |P \cap M| - m + (k-1)$.

Proof. Since P is very ample, we have $\deg X_P = \mathrm{Vol}(P)$ by Lemma 4.3.3. Hence, the claim follows from $\operatorname{codim} X_P = n - m = |P \cap M| - 1 - m$. □

Using the theory presented in this section, we can give an alternative proof of Theorem 4.2.8 that uses the classification of lattice polytopes of degree (at most) 1 developed by Batyrev and Nill [BN07].

Alternative proof of Theorem 4.2.8. P has to be contained in a full-dimensional normal lattice polytope $Q \subseteq \mathbb{R}^m$ such that X_Q is a variety of minimal degree (Theorem 4.0.1). As an aside, $\varepsilon_2(X_Q) = 0$ by Zak's result (Theorem 4.1.3), and therefore $h_2^* = 0$ where h_2^* is the coefficient in the quadratic term of the h^*-polynomial of Q, see Remark 4.3.1. One could then infer from [BSV, Proposition 6.7] that $h_3^* = \cdots = h_m^* = 0$.

Instead, we use Corollary 4.3.4 which says $\mathrm{Vol}(Q) = |Q \cap M| - m$. From the definition of the degree of Q and the equation $h_1^* = n - m = |Q \cap M| - m - 1$ one immediately gets $\deg(Q) \leq 1$ (cf. [BN07, Proposition 1.5]). According to Theorem 2.5 in [BN07], which classifies the lattice polytopes of degree at most 1, the polytopes in (i)-(iv) are indeed the only ones of this kind (up to translation and an automorphism of the lattice). Thus, there is an affine-linear isomorphism of $\mathbb{Z}^m$ that maps Q to one of these polytopes. □

Let P be one of the lattice polytopes in (i)-(iv) of Theorem 4.2.8. Recall that the Gram spectrahedron of a positive definite quadratic form on $\mathbb{P}^m$ is a single point, so that the case $P = S_m \subseteq \mathbb{R}^m$ is rather boring. We thus focus on the other cases. Quadratic forms on the Veronese surface $v_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$, the projective toric variety associated to the two-dimensional simplex $2S_2 = \mathrm{conv}(0, 2e_1, 2e_2) \subseteq \mathbb{R}^2$, correspond to ternary quartics. As mentioned before, their Gram spectrahedra have already been studied in great detail and we summarize these results in the following section. Afterwards, we present our contributions to the case of rational normal scrolls.

4.4. Ternary quartics

A (real) *ternary quartic* is a homogeneous polynomial $f \in \mathbb{R}[x, y, z]_4$. As early as 1888, Hilbert proved that every nonnegative ternary quartic can be written as a sum of three squares ([Hil]). More than a century later, in 2004, Powers, Reznick, Scheiderer and Sottile succeeded in showing that there are (up to orthogonal equivalence) exactly eight representations of f as a sum of three squares if the plane curve $\mathcal{V}_+(f) \subseteq \mathbb{P}^2$ is smooth ([PRSS]). In terms of Gram spectrahedra this means that $\mathrm{Gram}(f)$ has eight extreme points of minimum rank.

From a geometrical point of view, a ternary quartic describes a curve of degree four in the projective plane. The rich and enchanting theory of such curves dates back to the 19th century. Back then, Plücker computed the number of *bitangents* to the curve $\mathcal{V}_+(f) \subseteq \mathbb{P}^2$, and many other mathematicians including Cayley, Hasse, Salmon and Steiner contributed to the field. For a historical synopsis and references to original articles by these authors, we allude the interested reader to Dolgachev's book [Dol] which is also a useful source if one wants to learn about *Cayley octads* and *Steiner complexes*. Plaumann, Sturmfels and Vinzant use these objects in [PSV] to compute the 63 (complex) Gram matrices of rank 3 of a positive ternary quartic f describing a smooth curve $\mathcal{V}_+(f)$. They identify among them the eight positive semidefinite real symmetric Gram matrices leading to the representations of f as a sum of three real squares. The article [PSV] also makes the first steps toward a better understanding of Gram spectrahedra of ternary quartics as it analyzes the segments between the extreme points of rank 3 and when they are contained in the boundary of the spectrahedron.

This analysis was recently complemented by Vill. In particular, for general f and for every possible combination $\vartheta \neq \vartheta' \in \mathrm{Ex}_3(f)$, Vill gives the dimension of the supporting face of $[\vartheta, \vartheta']$ in $\mathrm{Gram}(f)$. The results of his work on Gram spectrahedra of ternary quartics are included in his thesis [Vill] that leads to a complete understanding of these spectrahedra in terms of their facial structure.

Let us summarize some results of [PSV] and [Vill]. In order to do so, we need the definition of the Steiner graph.

4.4.1 Definition ([Vill, Definition 4.3.1]). Let $f \in \mathbb{R}[x, y, z]_4$ be a ternary quartic describing a smooth curve $\mathcal{V}_+(f)$. The *Steiner graph* of f is the graph whose vertices correspond to the eight rank-three extreme points of $\mathrm{Gram}(f)$, and where two vertices are linked by an edge if and only if the line segment between the corresponding Gram tensors is contained in the boundary of $\mathrm{Gram}(f)$.

Let $f \in \mathbb{R}[x, y, z]_4$ be positive. In the case of ternary quartics, the cone of nonnegative forms coincides with the sums-of-squares cone so that we have $f \in \mathrm{int}(\Sigma\mathbb{R}[x, y, z]_2^2)$. Therefore, the spectrahedron $\mathrm{Gram}(f)$ is six-dimensional and any point in its relative interior has rank 6.

4.4.2. If $\mathcal{V}_+(f)$ is smooth, we have $|\mathrm{Ex}_3(f)| = 8$ according to [PRSS, Theorem 1.1]. The Steiner graph is a disjoint union of two complete graphs K_4 ([PSV, Theorem 6.2] and [Vill, Theorem 4.3.16]). Furthermore, under the smoothness assumption, there are always extreme points of rank 4 in $\mathrm{Gram}(f)$ ([Vill, Proposition 4.4.12]), the semialgebraic set of rank-5 extreme points is dense in the boundary of $\mathrm{Gram}(f)$ ([Vill, Theorem 4.4.23]) and we have two-dimensional faces of rank 5 ([Vill, Corollary 4.4.20]).

If f is generic, there are no other proper faces of positive dimension. In particular, there are no edges (of rank 4 or 5) in the Gram spectrahedron of a general ternary quartic. This means that the faces that we get from the edges in the Steiner graph are two-dimensional faces of rank 5 in this case. It is only possible to obtain an edge (then necessarily of rank 4) in the Gram spectrahedron of a (nongeneric) ternary quartic f describing a smooth curve if f is invariant under some automorphism of $\mathbb{C}^3$ of order 2. An upper bound for the number of edges in the Gram spectrahedron of such an f is six. To the best knowledge of the author, it is not known whether there actually exists a smooth ternary quartic for which the number of edges in $\mathrm{Gram}(f)$ exceeds three, the number reached by the Fermat quartic $f = x^4 + y^4 + z^4$. All statements in this passage are proven in [Vill, Section 4.4]. More precisely, they are taken from Remark 4.4.25, Corollary 4.4.8, Proposition 4.4.14 and Remark 4.4.15. We also refer to Vill's forthcoming article [Vill23] on Gram spectrahedra of ternary quartics.

4.5. Rational normal surfaces

Having ticked off the Veronese surface $v_2(\mathbb{P}^2)$, we turn our attention to an infinite subclass of varieties of minimal degree, namely the smooth rational normal scrolls. We study Gram spectrahedra of quadratic forms on these varieties and establish dimension bounds for their faces that are analogous to those in Corollary 3.1.3. In order to ease notation and to provide the reader with a better understanding of the general approach, we will first consider the case of surfaces ($m = 2$) before stating the results for arbitrary m in Section 4.6. For $|\bar{d}| \in \{3, 4\}$, we also compare Gram spectrahedra of quadratic forms on rational normal surfaces to those of binary forms of degree $2|\bar{d}|$.

4.5.1 Notation. Let $d \geq e \geq 1$ and let $P := P_{(d,e)}$ be the Cayley sum of $[0, d]$ and $[0, e]$, that is to say

$$\begin{aligned} P &= \operatorname{conv}\big(([0,d] \times \{0\}) \cup ([0,e] \times \{1\})\big) \\ &= \operatorname{conv}(\{(0,0),(d,0),(0,1),(e,1)\}) \subseteq \mathbb{R} \times \mathbb{R} = \mathbb{R}^2. \end{aligned}$$

In this setting, the surface $X := X_P$ is the Zariski closure of the image of the map

$$\begin{aligned} &(\mathbb{C}^*)^2 \to \mathbb{P}^{d+e+1} \\ &(s,x) \mapsto (1 : s : s^2 : \cdots : s^d : x : xs : \cdots : xs^e). \end{aligned}$$

A quadratic form $f \in \mathbb{R}[X]_2$ can be considered as a polynomial $f \in \mathbb{R}[s,x]_{2P}$, a polynomial whose Newton polytope is contained in $2P$. A different approach is taken in [BPSV, Section 3]. The authors bi-homogenize f in s and x with homogenizing variables t and y and consider f as a biform

$$f(s,t,x,y) = a(s,t)x^2 + 2b(s,t)xy + c(s,t)y^2,$$

bi-homogeneous of bidegree $(2d, 2)$, where a and b are divisible by $t^{2(d-e)}$ and t^{d-e}, respectively. In particular, $\mathcal{V}_+(f) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is singular as soon as $d \neq e$. We adopt the first point of view and work in the polynomial ring $\mathbb{R}[s,x]$ and its subspace $V := \mathbb{R}[s,x]_P = \operatorname{span}(1, s, \ldots s^d, x, xs, \ldots, xs^e)$.

In order to obtain an upper bound for rank-r faces in Gram spectrahedra, we need to provide a lower bound for $\dim(UU)$ where U ranges over the r-dimensional subspaces of V. This can be done as follows.

4.5.2. Fix some monomial order $\preceq$ on $\mathbb{R}[s,x]$. Given a subspace $U \subseteq V$ of dimension r, we choose a basis $\mathcal{B}$ of U with pairwise distinct leading monomials. Next, we subdivide the elements of our basis into two groups of polynomials: $p_1, \ldots, p_k$ have leading monomials not divisible by x and the leading monomials of $q_1, \ldots, q_l$ are divisible by x. Obviously, $k, l \geq 0$ and $k + l = r$. Moreover, we want to enumerate the elements such that leading monomials are ascending inside each group. This means $\operatorname{LM}(p_i) \prec \operatorname{LM}(p_{i+1})$ and $\operatorname{LM}(q_j) \prec \operatorname{LM}(q_{j+1})$ for all $i = 1, \ldots, k-1$ and $j = 1, \ldots, l-1$.

When we take all pairwise products of our basis elements and classify them according to their leading monomials, this will end in three categories: leading monomials which are not divisible by x; those divisible by x^2; and those divisible by x, but not by x^2. In the first category we have at least $2k - 1$ linearly independent elements (see Proposition 3.1.2). In the same way we find at least $2l - 1$ linearly independent elements in the second category. These two categories could be considered *pure* since they are obtained by multiplying two elements from the same group.

The third category is *mixed* because its elements arise by taking factors from distinct groups. For the count in the mixed category, we first assume that $k \geq l$ and consider the following list of polynomials in it:

$$p_1q_1,\ p_1q_2,\ p_2q_2,\ p_2q_3,\ p_3q_3,\ \ldots,\ p_lq_l,\ p_{l+1}q_l,\ \ldots,\ p_kq_l.$$

In each step in the above list the leading monomial becomes strictly larger. Therefore, if the third category is nonempty (that is to say $k, l \geq 1$), it contains at least $2l - 1 + (k - l) = (k + l) - 1$ linearly independent elements (the case $k < l$ is analogous). To sum up, if $k, l \geq 1$, then

$$\dim(UU) \geq (2k - 1) + (2l - 1) + (k + l - 1) = 3(k + l) - 3 = 3r - 3.$$

4.5.3 Example. Note that the condition $k, l \geq 1$ is important. Trivially, if $U \subseteq \mathbb{R}[s]$, we are thrown back to the case of univariate polynomials where we can only guarantee that $\dim(UU) \geq 2r - 1$. For a different example, consider $U = \operatorname{span}(q_1, q_2, q_3)$ where $q_i := s^{i-1}\big(b + s(a + x)\big)$ for some $a, b \in \mathbb{R}$. Then $\dim(UU) = 5 < 6 = 3 \cdot (3 - 1)$. The q_i's vanish on a common hyperbola $x = -a - \frac{b}{s}$ $(s \neq 0)$ if $b \neq 0$, or on the two lines $x = -a$ and $s = 0$, respectively, if $b = 0$.

Using a lexicographic order, we see that the condition $k, l \geq 1$ is characterized by the nontriviality of the intersection of U with $\mathbb{R}[s]$.

4.5.4 Lemma. *Let $\preceq = \preceq_{lex}$ with $x \succ s$. Consider an r-dimensional subspace $U \subseteq V$. Let $\mathcal{B}$ and k, l be as in 4.5.2. Then*

(a) $k \geq 1 \iff U \cap \mathbb{R}[s] \neq \{0\}$,

(b) $l \geq 1 \iff U \not\subseteq \mathbb{R}[s]$.

Proof. (a) Let $k \geq 1$. Then $x \nmid \operatorname{LM}(p_1)$. Since we are using the lexicographic order with $x \succ s$, the variable x does not occur in p_1. Therefore, $0 \neq p_1 \in U \cap \mathbb{R}[s]$.

Now let $k = 0$. Then $l = r$. Let $f = \lambda_1 q_1 + \cdots + \lambda_l q_l \in \mathbb{R}[s]$ for some $\lambda_j \in \mathbb{R}$. The q_j's have pairwise distinct leading monomials, each divisible by x. Since the leading monomial of $f \in \mathbb{R}[s]$ does not contain x, we must have $\lambda_j = 0$ for all $j = 1, \ldots, l$. Thus, $U \cap \mathbb{R}[s] = \{0\}$.

(b) If $l \geq 1$, then $x \mid \operatorname{LM}(q_1)$ and therefore $q_1 \in U \setminus \mathbb{R}[s]$. If $l = 0$, then $k = r$ and each element of a basis of U has a leading monomial that is not divisible by x. Since $x \succ s$, this means that $U \subseteq \mathbb{R}[s]$. □

4.5.5 Proposition. *Let $U \subseteq V$ be a subspace of dimension $r > e + 1$ and with $U \not\subseteq \mathbb{R}[s]$. Then* $\dim(UU) \geq 3r - 3$.

Proof. We use the lexicographic order with $x \succ s$. There are only $e + 1$ distinct monomials in V which are divisible by x. Any basis of U with pairwise different leading monomials thus contains at least one element of $U \cap \mathbb{R}[s]$, certifying that this intersection is nontrivial. Therefore, by the preceding lemma, the count from 4.5.2 applies. □

From this we obtain upper bounds for dimensions of faces in Gram spectrahedra, at least for faces of sufficiently large rank. Although we initially formulate and prove the following proposition only for the case of surfaces ($m = 2$), we express the dimension bounds in terms of m in order to stress the analogy to the case of binary forms (Corollary 3.1.3) and in anticipation of the general result for arbitrary m (see Theorem 4.6.2).

4.5.6 Proposition ($m = 2$). *Let $f \in \mathbb{R}[s,x]_{2P}$ be nonnegative on $\mathbb{R}^2$. Let $F \subseteq \mathrm{Gram}_V(f)$ be a face of rank r.*

(a) We have $\dim(F) \geq \binom{r-m}{2} - (m+1)(n+1-r)$.

(b) Assume that $f \notin \mathbb{R}[s]$. If $r > e+1$, then $\dim(F) \leq \binom{r-m}{2}$.

Proof. Recall that $n+1 = d+e+2$ is the number of lattice points in P.

(a) This follows by combining

$$\dim(F) \geq \binom{r+1}{2} - \dim(\mathbb{R}[s,x]_{2P})$$

with the formula for $\dim(\mathbb{R}[s,x]_{2P})$ from Lemma 4.1.2.

(b) Let $U = \mathfrak{U}(F)$. This is an r-dimensional subspace of V and $f \notin \mathbb{R}[s]$ implies $U \not\subseteq \mathbb{R}[s]$. According to Propositions 2.3.9 and 4.5.5, we thus have

$$\dim(F) \leq \binom{r+1}{2} - (3r-3) = \binom{r-2}{2} = \binom{r-m}{2}. \qquad \square$$

4.5.7. In the case of binary forms, we were able to improve the bound for the maximum dimension of a proper face by 1 if $d \geq 3$ and the form is strictly positive on $\mathbb{P}^1(\mathbb{R})$ (see Proposition 3.1.7). We can reinterpret the statement of the said proposition in our current setting: Let $X = v_d(\mathbb{P}^1) \subseteq \mathbb{P}^d$ be the rational normal curve of degree $d \geq 3$. We endow $\mathbb{P}^d$ with homogeneous coordinates $(z_0 : \cdots : z_d)$. The points on X are of the form $(1 : t : \cdots : t^d)$ for $t \in \mathbb{C}$ together with the point $(0 : \cdots : 0 : 1)$ at infinity. By $\mathbb{R}[X]$ we denote the homogeneous coordinate ring $\mathbb{R}[z_0, \ldots, z_d]/\mathfrak{I}_+(X)$. Let $U \subseteq V := \mathbb{R}[X]_1$ be a linear subspace of dimension d, that is to say $\mathrm{codim}_V(U) = 1$. If $\dim(UU) = 2d-1$ or, equivalently, $\mathrm{codim}_{VV}(UU) = m+1 = 2$, then there exists some $\xi \in X(\mathbb{R})$ such that $p(\xi) = 0$ for all $p \in U$. Indeed, in the language of binary forms, we have seen that the subspace of $\mathbb{R}[x,y]_d$ corresponding to U is either generated by $(x^j y^{d-j})_{j=0,\ldots,d-1}$ or by $(x^k y^{d-k} - a^k y^d)_{k=1,\ldots,d}$ for some $a \in \mathbb{R}$. In the first case, all elements of $U = \mathrm{span}(z_0, \ldots, z_{d-1})$ vanish in $(0 : \cdots : 0 : 1) \in X(\mathbb{R})$. The second case can be understood as $U = \mathrm{span}(z_k - a^k z_0 : k = 1, \ldots, d)$ with base-point $(1 : a : \cdots : a^d) \in X(\mathbb{R})$.

In a more cumbersome formulation, the condition $d \geq 3$ from the case of binary forms can equivalently be phrased as $\mathrm{codim}_{\mathbb{P}^d}(v_d(\mathbb{P}^1)) \geq 2$. We are going to generalize 4.5.7 to varieties $X \subseteq \mathbb{P}^n$ of minimal degree with $\mathrm{codim}(X) \geq 2$ in Section 4.9. Let us first consider an example that also serves as a base case for an inductive proof.

4.5.8 Example ($(d,e) = (2,1)$). We analyze the structure of Gram spectrahedra of quadratic forms on the rational normal surface $X \subseteq \mathbb{P}^4$ defined by the lattice polytope $P = P_{(2,1)}$. Let $f \in R_2$ be positive on $X(\mathbb{R})$. Then $\dim \mathrm{Gram}_{R_1}(f) = 3$, and points in the relative interior of the Gram spectrahedron have rank $n+1 = 5$. If f is generic, we have precisely four extreme points of rank $m+1 = 3$.

Since $\dim R_2 = 12$, the rank of an extreme point can be at most 4. On the other hand, according to Proposition 4.5.6, a face of rank $r = 4$ is at most one-dimensional. We are going to show that $\mathrm{Gram}_{R_1}(f)$ does not contain any edges for f positive on $X(\mathbb{R})$. Thus, for general positive $f \in R_2$, the structure of its Gram spectrahedron is the same as for a general positive binary sextic.

In order to exclude faces of dimension 1 we have to show the following: Let $U \subseteq V = R_1$ be a linear subspace of dimension $n = 4$, i.e., $\mathrm{codim}_V(U) = 1$. If $\mathrm{codim}_{VV}(UU) = 3 = m+1$, then $\mathcal{V}_+(U) \cap X(\mathbb{R}) \neq \varnothing$. For calculations it is more convenient to identify linear forms in R_1 with polynomials in $\mathbb{R}[s,x]_P = \operatorname{span}(1, s, s^2, x, xs)$ and consequently R_2 with $\mathbb{R}[s,x]_{2P}$. We are therefore working in this setting for the time being.

If $1 \notin U$, we can find a basis $\mathcal{B}$ of U of the form

$$s - \lambda_s,\ s^2 - \lambda_{s^2},\ x - \lambda_x,\ xs - \lambda_{xs}$$

with λ_s, λ_{s^2}, λ_x, $\lambda_{xs} \in \mathbb{R}$. Using a graded monomial order, one immediately sees that $\dim(UU) \geq 9$. We describe a simple procedure that can be used to prove that $\dim(UU) = 9$ if and only if there exists a point $(s,x) \in \mathbb{R}^2$ such that $p(s,x) = 0$ for all $p \in U$. Consider the 10×12-matrix A whose rows contain the coefficients of the pairwise products of elements in $\mathcal{B}$ with respect to a monomial basis of $\mathbb{R}[s,x]_{2P}$. There are six rows in A which contain a 1 that is the only non-zero element in its respective column. By A' we denote the matrix that is obtained from A by deleting these rows and the thereby arising zero columns. Then A' is a 4×6-matrix and the vanishing of (two of) the 4×4-minors implies that $\lambda_{xs} = \lambda_s \lambda_x$ and $\lambda_{s^2} = \lambda_s^2$. This means that

$$\mathcal{B} = (s - \lambda_s,\ s^2 - \lambda_s^2,\ x - \lambda_x,\ xs - \lambda_s\lambda_x).$$

Consequently, all elements of U vanish in (λ_s, λ_x). When we retranslate to the language of rational normal surfaces, this means that

$$U = \operatorname{span}(z_1 - \lambda_s z_0,\ z_2 - \lambda_s^2 z_0,\ z_3 - \lambda_x z_0,\ z_4 - \lambda_s\lambda_x z_0) \subseteq \mathbb{R}[X]_1,$$

and that a point of the form $(1 : \lambda_s : \lambda_s^2 : \lambda_x : \lambda_x\lambda_s) \in X(\mathbb{R})$ is contained in $\mathcal{V}_+(U) \cap X(\mathbb{R})$.

Let us return to the polyhedral setting and consider the case $1 \in U$. If $s \notin U$, the same procedure as above starting with a basis $\mathcal{B} = (1, s^2 - \lambda_{s^2} s, x - \lambda_x s, xs - \lambda_{xs} s)$ shows that we always have $\dim(UU) = 10$ in this case. We can thus move on to the case $s^2 \notin U$. A basis of U is here given by $(1, s, x + as^2, xs + bs^2)$, where a, $b \in \mathbb{R}$. The reduction step results in the matrix

$$A' = \begin{pmatrix} 0 & b & 1 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{pmatrix}.$$

Obviously, $\operatorname{rk}(A') < 3$ if and only if $a = 0$. In other words, $\dim(UU) = 9$ if and only if $a = 0$, independent of $b \in \mathbb{R}$. Using the correspondence between $\mathbb{R}[s,x]_P$ and R_1 this means the following: In case z_0, $z_1 \in U \subseteq R_1 \cong \mathbb{R}[z_0, \ldots, z_4]$ and $z_2 \notin U$, the assumption $\operatorname{codim}_{VV}(UU) = 3$ implies that U is generated by z_0, z_1, z_3 and $z_4 + bz_2$ for some $b \in \mathbb{R}$. A base-point of U is $(0 : 0 : 1 : 0 : -b) \in X(\mathbb{R})$.

The remaining cases – where x or xs is the single monomial (in $\mathbb{R}[s,x]_P$) that is not the leading monomial of an element in U – can be dealt with analogously. They result in a base-point of the form $(0 : 0 : 0 : 1 : -b) \in X(\mathbb{R})$ or $(0 : 0 : 0 : 0 : 1) \in X(\mathbb{R})$, respectively.

For later reference, we record the following fact that was proven in Example 4.5.8.

4.5.9 Lemma. *Let $m = 2$ and let $\bar{d} = (2,1)$. Consider the rational normal surface $X = X_{P_{\bar{d}}} \subseteq \mathbb{P}^4$ and its homogeneous coordinate ring $R = \mathbb{R}[X]$. Let $U \subseteq V = R_1$ be a linear subspace with* $\operatorname{codim}_V(U) = 1$. *If we have* $\operatorname{codim}_{VV}(UU) = m + 1$, *then* $\mathcal{V}_+(U) \cap X(\mathbb{R}) \neq \varnothing$. ☐

Binary octics and $|\bar{d}| = 4$. We consider the case $(d,e) = (2,2)$, that is to say $V = \operatorname{span}(1, s, s^2, x, xs, xs^2) \subseteq \mathbb{R}[s,x]$. Let f be a generic positive polynomial in V. Then $\dim \operatorname{Gram}_V(f) = \binom{d+e}{2} = 6$. By [BPSV, Theorem 3.11], the Gram spectrahedron of f has exactly $2^{d+e-1} = 2^{\operatorname{codim}(X_P)} = 8$ extreme points of (minimum) rank $\dim(X_P) + 1 = 3$. We want to compare properties of $\operatorname{Gram}_V(f)$ to those of $\operatorname{Gram}(g)$, where g is a generic positive binary form of degree $2(d+e) = 8$. First note that $\operatorname{Gram}(g)$ is also six-dimensional and contains eight extreme points of minimum rank.

4.5.10. Considering the Pataki inequalities from Section 2.3, it seems there might be a difference concerning extreme points of higher ranks: The Pataki interval for g is $\{2,3\}$, while that for f is $\{3,4,5\}$, containing one element more. However, a point of rank 5 can never be an extreme point of $\operatorname{Gram}_V(f)$. In order to prove this assertion, we show that, more generally, $\dim_{\mathbb{C}}(UU) \leq 14$ for any $\mathbb{C}$-linear subspace $U \subseteq V_{\mathbb{C}}$ of dimension 5.

For any $\lambda_1, \ldots, \lambda_5 \in \mathbb{C}$, we consider the polynomials

$$p_1 = s + \lambda_1,\ p_2 = s^2 + \lambda_2,\ p_3 = x + \lambda_3,\ p_4 = xs + \lambda_4,\ p_5 = xs^2 + \lambda_5$$

in $\mathbb{C}[s,x]_P$. Let $A(\lambda_1, \ldots, \lambda_5)$ be the 15×15-matrix that contains the coefficients of the products $p_i p_j$ $(1 \leq i < j \leq 5)$ with respect to a monomial basis of VV. Due to the simple appearance of the polynomials, this is a sparse matrix so that (at least with a computer) one easily checks that its determinant is identically zero. It suffices to consider subspaces $U = \operatorname{span}_{\mathbb{C}}(p_1, \ldots, p_5) \subseteq V_{\mathbb{C}}$ as above which do not contain 1. Indeed, this follows from a general observation that might also be useful elsewhere and is thus explained in Remark 4.5.11.

4.5.11 Remark. Let A be a $\mathbb{C}$-algebra and let $V \subseteq A$ be a linear subspace of dimension n. Let $r \leq n$ and $k \leq n - r$. We fix a basis $\mathcal{B}$ of V and thereby consider $V^r \cong (\mathbb{C}^n)^r$ as an affine space $\mathbb{A}^{nr}$ endowed with the Zariski topology. Choose linearly independent elements $q_1, \ldots, q_k \in V$ and write $q = (q_1, \ldots, q_k)$. Then

$$\mathcal{L}_q := \{(p_1, \ldots, p_r) \in V^r : p_1, \ldots, p_r, q_1, \ldots, q_k \text{ are linearly dependent}\}$$

is a closed algebraic set in V^r. Indeed, it is described by the vanishing of all minors of size $(r+k) \times (r+k)$ of the $(r+k) \times n$-matrix that contains variables for the coefficients of the p_i in its first r rows and the coefficients of $q_1, \ldots, q_k$ with respect to $\mathcal{B}$ in the last k rows. Consequently,

$$\mathcal{L}_q^c = \{(p_1, \ldots, p_r) \in V^r : \operatorname{span}(p_1, \ldots, p_r) =: U \text{ satisfies } \dim(U) = r \text{ and } U \cap \operatorname{span}(q_1, \ldots, q_k) = \{0\}\}$$

is open in the Zariski topology on V^r. As it is also nonempty, it is Zariski-dense since $\mathbb{A}^{nr}$ is an irreducible topological space. In particular, any Zariski-closed set $\mathcal{A} \subseteq V^r$ that contains $\mathcal{L}_q^c$ must already be equal to V^r.

Thus, if we want to show that a certain property holds true for all r-dimensional subspaces $U \subseteq V$, it suffices to show that it can be described by polynomial equations (where variables play the role of coefficients with respect to $\mathcal{B}$) and that it holds true for all r-dimensional subspaces U that intersect $\operatorname{span}(q_1, \ldots, q_k)$ trivially. As an example, we consider the set $\mathcal{A}$ consisting of all $(p_1, \ldots, p_r) \in V^r$ with at least l quadratic relations between $p_1, \ldots, p_r$. Then, $\mathcal{A}$ is given by the vanishing of all minors of a certain size in an $\binom{r+1}{2} \times n$-matrix.

To complete the argument in 4.5.10, we can take $r = 5$, $k = n - r = 1$, $q_1 = 1$ and $l = 1$. As was shown there, these settings ensure that $\mathcal{L}_q^c \subseteq \mathcal{A}$.

4.5.12 Remark. In the same manner one can show that there are no extreme points of rank 5 when $(d, e) = (3, 1)$. This shows that for $m = 2$ and $|\bar{d}| = 4$, there is no form $f \in \mathbb{R}[X_P]_2$ such that $\mathrm{Gram}_{\mathbb{R}[X_P]_1}(f)$ has extreme points of all ranks in the Pataki interval. The same statement applies when $|\bar{d}| = 6$, where we can rule out extreme points of rank 6 using the method introduced above.

In contrast, for $|\bar{d}| = 5, 7$, we generically get extreme points of rank 5 or of rank 6, respectively. Note that in these cases, the size of the Pataki interval is the same as for a binary form of the corresponding degree $2|\bar{d}|$, while its size was larger in the cases where we ruled out the maximum rank.

We are going to further analyze possible ranks of extreme points in Section 4.7, where we will see that also the size ratio between the entries of $\bar{d}$ becomes more important as $|\bar{d}|$ grows.

4.5.13. For now, let us return to the case $|\bar{d}| = 4$ and the comparison of $\mathrm{Gram}_V(f)$ and $\mathrm{Gram}(g)$, where $f \in \mathbb{R}[X_P]_2$ and $g \in \mathbb{R}[x, y]_8$ are generic positive forms. Recall that their Gram spectrahedra both contain eight extreme points of minimum rank. We turn our attention towards the line segments connecting these points. The interval $[\vartheta, \vartheta']$ connecting two extreme points $\vartheta \neq \vartheta' \in \mathrm{Ex}_2(g)$ is always contained in the boundary of $\mathrm{Gram}(g)$ since $\mathrm{rk}(\vartheta + \vartheta') \leq \mathrm{rk}(\vartheta) + \mathrm{rk}(\vartheta') = 4$. Consider the graph with vertex set $\mathrm{Ex}_2(g)$ where two vertices $\vartheta \neq \vartheta'$ are connected by an edge if and only if $[\vartheta, \vartheta']$ is an edge of $\mathrm{Gram}(g)$. By a result of Scheiderer, this graph is a $K_{4,4}$, a complete bipartite graph on two sets of four points each (see Theorem 3.0.8).

Interestingly, in our computer experiments we found the same structure on the set of rank-minimal extreme points of $\mathrm{Gram}(f)$ where f is a (not too special) quadratic form on a rational normal scroll with $|\bar{d}| = 4$. This led us to the following conjecture:

4.5.14 Conjecture. *Let $m \geq 2$ and let $d_0 \geq \cdots \geq d_{m-1} \geq 1$ be natural numbers with $|\bar{d}| = 4$, where $\bar{d} = (d_0, \ldots, d_{m-1})$. (This says $\bar{d} \in \{(3,1), (2,2), (2,1,1), (1,1,1,1)\}$.) Let X_P be the smooth rational normal scroll associated to the lattice polytope $P := P_{\bar{d}}$. If $f \in \mathbb{R}[X_P]$ is a general quadratic form, positive on $X_P(\mathbb{R})$, then we obtain a complete bipartite graph $K_{4,4}$ on two sets of four points each, whose vertices and (non)edges we interpret in the following way: The set of vertices is $\mathrm{Ex}_{m+1}(f)$. Two points $\vartheta \neq \vartheta' \in \mathrm{Ex}_{m+1}(f)$ are connected by an edge in our graph if and only if the interval $[\vartheta, \vartheta']$ is an edge of $\mathrm{Gram}(f)$, and if they belong to the same part (i.e., they are not connected by an edge) this interval intersects the relative interior of $\mathrm{Gram}(f)$.*

Unfortunately, we were not able to prove this conjecture. In the case of binary forms, the subdivision of the eight rank-two extreme points into two parts of four each arises from the combinatorics of roots in representations of f as Hermitian squares: Two different factorizations $f = p\overline{p} = q\overline{q}$ belong to the same subclass if and only if p and q have precisely two roots in common ([Sch22, 6.9]). For ternary quartics, one has a similar subdivision arising from the Cayley octad and bitangents, although the resulting graph is $K_4 \sqcup K_4$ (see Section 4.4). When dealing with rational normal surfaces ($m = 2$), $\mathcal{V}_+(f)$ can be regarded as the intersection of the surface defined by $P_{(d,e)}$ and a quadric given by f. According to [BPSV, Section 3], this curve C is an hyperelliptic curve. It has genus 3 and degree 8 in $\mathbb{P}^5$ (see [BPSV, Lemma 3.2]). As for ternary quartics, one has a bijection between the 2-torsion points of the Jacobian and so-called quadratic representations of f over the complex numbers. In the proof of [BPSV, Theorem 3.11], the authors also discuss how the real representations of f as a sum of three squares correspond to a coset in the group of real 2-torsion points of the Jacobian of C. However, we were not able to arrive at a meaningful split of those points that would explain the $K_{4,4}$-phenomenon we observed in experiments.

4.6. Smooth rational normal scrolls

In this short section we use the ideas presented in the previous one in order to obtain bounds similar to those in Propositions 4.5.5 and 4.5.6 for arbitrary m. We work in the polynomial ring $\mathbb{R}[s,\underline{x}] = \mathbb{R}[s, x_1, \ldots, x_{m-1}]$ and we fix the lexicographic monomial order with $s \prec x_1 \prec x_2 \prec \cdots \prec x_{m-1}$. To ease notation, we set $x_0 := 1$. Consider the vector space

$$V := \operatorname{span}\left(\{x_i s^j : i = 0, \ldots, m-1,\ j = 0, \ldots, d_i\}\right) \subseteq \mathbb{R}[s, \underline{x}]$$

generated by the monomials corresponding to the lattice points of $P = P_{\bar{d}}$ in $\mathbb{Z}^m$, where $\bar{d} = (d_0, d_1, \ldots, d_{m-1})$ is as in Section 4.1.

4.6.1. Let $U \subseteq V$ be a subspace of dimension r. We want to give a lower bound for the dimension of $UU \subseteq \mathbb{R}[s,\underline{x}]_{2P}$ provided that $\operatorname{codim}_V(U)$ is not too big. For the uniformity of the following argument, it is convenient to consider the elements of V as homogeneous (of degree 1) in the variables $x_0, x_1, \ldots, x_{m-1}$. In other words, let us multiply the generators $1, s, \ldots, s^{d_0}$ by x_0. We pick a basis

$$p_1^{(0)}, \ldots, p_{k_0}^{(0)}, p_1^{(1)}, \ldots, p_{k_1}^{(1)}, \ldots, p_1^{(m-1)}, \ldots, p_{k_{m-1}}^{(m-1)}$$

of U where $k_0, \ldots, k_{m-1}$ are nonnegative integers with $k_0 + \cdots + k_{m-1} = r$, and such that the leading monomials of the above polynomials form a strictly ascending chain with $x_i \mid \operatorname{LM}(p_l^{(i)})$ for all $i = 0, 1, \ldots, m-1$ and $l = 1, \ldots, k_i$.

As before, we obtain a lower bound for $\dim(UU)$ by taking the leading monomials of the pairwise products of our basis elements, categorizing them according to their divisibility by variables x_i and estimating the size of each category.

First of all, for each i, we have the category of monomials that are pure in the variable x_i, i.e., monomials divisible by x_i^2 (and therefore not divisible by any other variable x_j). Those leading monomials arise from the multiplication of a basis element $p_l^{(i)}$ with another one of the same type (i). In this category we have at least $2k_i - 1$ elements.

Let us assume that all $k_i \geq 1$ $(i = 0, 1, \ldots, m-1)$. Then, for every pair (i, j) with $0 \leq i < j \leq m-1$, we have the mixed category of monomials divisible by $x_i x_j$, which occur as leading monomials when multiplying a basis element of type (i) with one of type (j). As we argued before (see 4.5.2), the size of this category is at least $k_i + k_j - 1$.

It follows that

$$\begin{aligned}
\dim(UU) &\geq \sum_{i=0}^{m-1} (2k_i - 1) + \sum_{0 \leq i < j \leq m-1} (k_i + k_j - 1) \\
&= 2\left(\sum_{i=0}^{m-1} k_i\right) - m + (m-1)\left(\sum_{i=0}^{m-1} k_i\right) - \binom{m}{2} \\
&= (m+1)r - \binom{m+1}{2}.
\end{aligned}$$

4.6.2 Theorem. *Let $m \in \mathbb{N}$ and let $P = P_{\bar{d}} \subseteq \mathbb{R}^m$. Consider the smooth rational normal scroll $X_P \subseteq \mathbb{P}^n$ of dimension m. Let $f \in \mathbb{R}[X_P]_2$ be nonnegative and let $F \subseteq \mathrm{Gram}_{\mathbb{R}[X_P]_1}(f)$ be a face of rank r.*

(a) We have $\dim(F) \geq \binom{r-m}{2} - (m+1)(n+1-r)$.

(b) If $r \geq n+1-d_{m-1}$, then $\dim(F) \leq \binom{r-m}{2}$.

Proof. For (a) virtually the same proof as in Proposition 4.5.6 applies. For (b) we consider f as a polynomial in $\mathbb{R}[s, x_1, \dots, x_{m-1}]_{2P}$. Let $U = \mathcal{U}(F) \subseteq V = \mathbb{R}[s, \underline{x}]_P$ and choose a basis of U as in 4.6.1. The condition $r \geq n+1-d_{m-1}$ is equivalent to $r > \sum_{i=0}^{m-2}(d_i+1)$ and guarantees that $k_0, \dots, k_{m-1} \geq 1$ since d_{m-1} is the smallest among the d_i's. Hence,

$$\dim(F) = \binom{r+1}{2} - \dim(UU) \leq \binom{r+1}{2} - (m+1)r + \binom{m+1}{2} = \binom{r-m}{2}$$

according to our count in 4.6.1. □

4.6.3 Remark (cf. Remark 3.1.4)**.** Let $\mathrm{rk}(F) = r$. Using the corank, we get a different presentation of the inequalities from Theorem 4.6.2. Let $k = \dim\ker(\vartheta)$ for $\vartheta \in \mathrm{relint}(F)$. Substituting $k = (|\bar{d}| + m) - r = (n+1) - r$ into said inequalities results in

$$\binom{|\bar{d}| - k}{2} - (m+1)k \leq \dim(F) \leq \binom{|\bar{d}| - k}{2},$$

where for the upper bound we assume that $k \leq d_{m-1}$.

4.6.4 Remark. If d_{m-1} is comparatively small, part (b) of Theorem 4.6.2 applies to relatively few values of r since the rank of a boundary point of $\mathrm{Gram}_{\mathbb{R}[X_P]_1}(f)$ is at most n if f is positive on $X_P(\mathbb{R})$. Assuming that (the polynomial in $\mathbb{R}[s, \underline{x}]_{2P}$ corresponding to) f is not contained in $\mathbb{R}[s, x_1, \dots, x_{m-2}]$, we can relax the condition to $r \geq n+1-d_{m-2}$. The additional assumption is certainly fulfilled for every f that is positive on $X_P(\mathbb{R})$. Indeed, the absence of the variables corresponding to $x_{m-1}, x_{m-1}s, \dots, x_{m-1}s^{d_{m-1}}$ would imply that f vanishes in $(0 : \dots : 0 : 1) \in X_P(\mathbb{R})$, for example.

Note that the dimension bound in part (b) of Theorem 4.6.2 is also valid for singular rational normal scrolls, see 4.8.7. Finally, Section 4.9 is devoted to improving this bound in the case $r = n$.

4.7. More inequalities for the dimensions of faces

The lower bound for the dimension of a rank-r face of the Gram spectrahedron $\mathrm{Gram}_V(f)$ is always obtained by determining the smallest possible number of quadratic relations among the r generators of a corresponding face subspace. Up to now, we have followed an unsophisticated approach that leads to

$$\dim(F) \geq \max\left\{0, \binom{r+1}{2} - \dim(VV)\right\}$$

for any face $F \subseteq \mathrm{Gram}_V(f)$ of rank r. In particular, the given inequality suggests that a point of rank r, where $\binom{r+1}{2} \leq \dim(VV)$, could be an extreme point of the Gram spectrahedron. As mentioned before, the Gram spectrahedron of a general binary form actually contains extreme points of all ranks in the Pataki interval and even faces of "expected dimension" for any higher rank (cf. Theorem 3.7.15). However, if $m \geq 2$, we observe that the above inequality only depends on $\dim(VV)$

and therefore on the sum $|\bar{d}| = d_0 + \cdots + d_{m-1}$, but not on the individual values of the d_i or their size ratio. In the following, we will give additional inequalities (depending on the partial sums d_{m-1}, $d_{m-1}+d_{m-2}$, $d_{m-1}+d_{m-2}+d_{m-3}$ and so on) that further confine the possible dimensions of faces.

Quadratically independent subspaces. For the sake of simplicity, we start once more with the case $m = 2$. So let $d \geq e \geq 1$ and let $P = P_{(d,e)}$. Let $U = \mathrm{span}(p_1, \ldots, p_r) \subseteq V = \mathbb{R}[s,x]_P$ be a quadratically independent subspace of dimension r. We fix the lexicographic monomial order with $x \succ s$ and we assume that $\mathrm{LM}(p_j) \prec \mathrm{LM}(p_{j+1})$ for all $j \in \{1, \ldots, r-1\}$. Since V contains only $e+1$ monomials divisible by x, we conclude that $x \nmid \mathrm{LM}(p_i)$ for $i = 1, \ldots, r-(e+1)$ and therefore $p_i \in \mathbb{R}[s]$ for those i. As U is quadratically independent, especially the products $p_i p_j$ with $1 \leq i \leq j \leq r-(e+1)$ have to be linearly independent inside $\mathbb{R}[s]_{2d}$. Therefore, $\binom{r-e}{2} \leq 2d+1$.

4.7.1 Example. Let $m = 2$. Given $r \geq 3$, we choose the smallest possible $|\bar{d}| = d+e$ such that the inequality $\binom{r+1}{2} \leq 3(d+e)+3$ defining the Pataki interval is satisfied. This means that the Pataki interval corresponding to $|\bar{d}|$ is given by $[3, r]$. If, on the other hand, the inequality $\binom{r-e}{2} \leq 2d+1$ does not hold for our fixed r and a special choice of d and e with $d+e = |\bar{d}|$ then any r-tuple of elements in V fails to be quadratically independent. In this case, we cannot have an extreme point of rank r in the Gram spectrahedron of any $f \in \Sigma V^2$ although the Pataki bound would permit it.

The smallest r where this reasoning shows the non-exhaustion of the Pataki interval for $\bar{d} = (|\bar{d}|-1, 1)$ is $r = 9$ together with $\bar{d} = (13, 1)$. In other words: A general 8-tuple $(p_1, \ldots, p_8)$ of polynomials $p_i \in \mathbb{R}[s,x]_P$ is quadratically independent if $P = P_{(d,e)}$ with $d+e \geq 11$. Moreover, a general 9-tuple of polynomials in $\mathbb{R}[s,x]_P$ is quadratically independent if $P = P_{(d,e)}$ with $d+e = 14$ as soon as $e \geq 2$, but no such 9-tuple can be quadratically independent if $(d,e) = (13,1)$.

Next, we generalize the idea from above to the case $m \geq 2$. To this end, we return to the setting where $\bar{d} = (d_0, \ldots, d_{m-1})$ with $d_0 \geq \cdots \geq d_{m-1} \geq 1$ and $P = P_{\bar{d}}$.

4.7.2 Notation. Let $l \in \{0, \ldots, m-1\}$. In order to have a more concise notation for the sum of the first $l+1$ entries of $\bar{d}$, we write $|\bar{d}|_l := \sum_{i=0}^{l} d_i = d_0 + d_1 + \cdots + d_l$. For the rest of this section, we let $k_l := \sum_{i=l}^{m-1}(d_i + 1)$.

4.7.3. Let $U = \mathrm{span}(p_1, \ldots, p_r) \subseteq V = \mathbb{R}[s, \underline{x}]_P$ be a quadratically independent subspace of dimension r. As before, we use the lexicographic monomial order on $\mathbb{R}[s, \underline{x}]$ with $x_{m_1} \succ \cdots \succ x_1 \succ s$ and we assume that $\mathrm{LM}(p_j) \prec \mathrm{LM}(p_{j+1})$ for all $j \in \{1, \ldots, r-1\}$. For every $l \in \{1, \ldots, m-1\}$, the number of monomials in V that are divisible by one of the variables $x_{m-1}, x_{m-2}, \ldots, x_{l+1}, x_l$ is k_l. Therefore, for all $i = 1, \ldots, r-k_l$, the monomial $\mathrm{LM}(p_i)$ is not divisible by any of these variables. By the choice of our monomial order, this means that $p_i \in \mathbb{R}[s, x_1, \ldots, x_{l-1}]$. The fact that U is quadratically independent implies that the products $p_i p_j$ $(1 \leq i \leq j \leq r-k_l)$ are linearly independent, so

$$\binom{r-k_l+1}{2} \leq \dim(VV \cap \mathbb{R}[s, x_1, \ldots, x_{l-1}]) = (l+1)|\bar{d}|_{l-1} + \binom{l+1}{2}.$$

Formally, $l = m$ returns the upper bound in the Pataki interval. Indeed,

$$\dim(\mathbb{R}[s,\underline{x}]_{2P}) = (m+1)(n+1) - \binom{m+1}{2} = (m+1)|\bar{d}| + \binom{m+1}{2}.$$

We summarize the results from the preceding discussion in the next theorem.

4.7.4 Theorem. *Let $r \in \mathbb{N}$. If $U \subseteq V$ is a quadratically independent linear subspace of dimension r, then the inequalities*

$$\binom{r - \sum_{i=l}^{m-1}(d_i+1)+1}{2} \leq (l+1)\sum_{i=0}^{l-1} d_i + \binom{l+1}{2} \qquad (B_l)$$

hold for all $l \in \{1,\dots,m\}$. □

4.7.5 Example. Let $m = 3$. Then the inequalities (B_1), (B_2) and (B_3) read as follows:

$$\binom{r-(d_1+d_2)-1}{2} \leq 2d_0 + 1,$$
$$\binom{r-d_2}{2} \leq 3(d_0+d_1)+3,$$
$$\binom{r+1}{2} \leq 4(d_0+d_1+d_2)+6.$$

Consider $|\bar{d}| = 18$. Then the Pataki bound (B_3) tells us that $r \leq 12$ if we want to have r quadratically independent elements in V. For

$$\bar{d} \in \{(9,8,1),(10,7,1),(11,6,1),(12,5,1),(13,4,1),(14,3,1),(15,2,1)\},$$

that is $d_0 + d_1 = 17$ and $d_1 \geq 2$, the inequality (B_2) fails for $r = 12$ (the left hand side is 55 and the right hand side 54), whereas (B_1) would remain true for these values. Finally, if $\bar{d} = (16,1,1)$, then also (B_1) turns false.

We want to stress that there are also cases in which (B_1) is false but (B_2) is still true, take for example $\bar{d} = (13,1,1)$ and $r = 11$. In this sense, there are no trivial implications between the inequalities (B_l) for $l = 1,\dots,m-1$.

4.7.6 Remark. Furthermore, it should be mentioned that if r is in the Pataki interval but some inequality (B_l) does not hold, then the extent to which the inequality fails also gives a quantitative measure for the failure of an r-tuple to give an extreme point: If

$$\binom{r - \sum_{i=l}^{m-1}(d_i+1)+1}{2} = (l+1)\sum_{i=0}^{l-1} d_i + \binom{l+1}{2} + k$$

for some $k > 0$, then the dimension of a rank-r face in $\mathrm{Gram}_V(f)$ is at least k for every $f \in \Sigma V^2$.

So far, we derived additional inequalities that restrict the range of ranks that extreme points in Gram spectrahedra can have. The intuition behind is that if $d_0+\dots+d_{l-1}$ is relatively big (and $d_l+\dots+d_{m-1}$ is relatively small), we will get many products lying in $\mathbb{R}[s,x_1,\dots,x_{l-1}]$ and these can hardly be linearly independent. There is yet another effect, which we will explain in the following.

Quadratically generating subspaces. In analogy to Definition 3.7.3, we say that a subspace $U \subseteq V = \mathbb{R}[s,x]_P$ is *quadratically generating* if $UU = VV = \mathbb{R}[s,x]_{2P}$. We are going to show that the dimension r of a subspace $U \subseteq V$ has to be big enough if we want U to be quadratically generating and that "big enough" depends on the entries of $\bar{d}$. The intuition is that if d_{m-1} is relatively small compared to the other entries of $\bar{d}$, then there are very few monomials containing x_{m-1} in V but still a considerable amount of them in VV. This means that it is impossible to generate all these monomials in UU if the subspace $U \subseteq V$ is too small. An analogous reasoning applies for the sums $d_{m-1} + \cdots + d_l$.

4.7.7. As before, we choose a basis $p_1, \ldots, p_r$ of U with pairwise distinct leading monomials, but this time we want the polynomials to be in a descending order according to their leading monomials, i.e., $\mathrm{LM}(p_j) \succ \mathrm{LM}(p_{j+1})$ for all $j = 1, \ldots, r-1$. Then only the first $k_{m-1} = d_{m-1} + 1$ polynomials in this list can contain the variable x_{m-1}. More generally, for $l \in \{1, \ldots, m-1\}$, only the first k_l polynomials $p_1, \ldots, p_{k_l}$ could contain one of the variables $x_{m-1}, x_{m-2}, \ldots, x_{l+1}, x_l$. Therefore, in the pairwise products of the generators of U, these variables can only appear in $p_i p_j$ where

$$1 \le i \le j \le k_l \quad \text{or} \quad 1 \le i \le k_l,\ j > k_l.$$

The number of these products is

$$\binom{k_l + 1}{2} + k_l(r - k_l) = \frac{1}{2} k_l (2r + 1 - k_l).$$

On the other hand, we can count the number of monomials in VV which contain one of the variables $x_{m-1}, x_{m-2}, \ldots, x_{l+1}, x_l$. Let us begin with the monomials in VV that contain x_{m-1}. These are

$$x_{m-1}x_j, x_{m-1}x_j s, \ldots, x_{m-1}x_j s^{d_j + d_{m-1}} \quad \text{for } j = 0, 1, \ldots, m-1,$$

where we let $x_0 = 1$, and their number is $\sum_{j=0}^{m-1}(d_{m-1} + d_j + 1)$. If we want to count the monomials containing x_{m-1} or x_{m-2}, we have to add the number of monomials containing x_{m-2} but not x_{m-1}. This gives another $\sum_{j=0}^{m-2}(d_{m-2} + d_j + 1)$ elements, and so on. Adding everything up, the number of monomials in VV which contain one of the variables $x_{m-1}, x_{m-2}, \ldots, x_{l+1}, x_l$ is given by

$$\sum_{i=1}^{m-l} \sum_{j=0}^{m-i} (d_{m-i} + d_j + 1).$$

4.7.8 Lemma. *For $l \in \{1, \ldots, m-1\}$ it holds*

$$\sum_{i=1}^{m-l} \sum_{j=0}^{m-i} (d_{m-i} + d_j + 1) = (m-l)|\bar{d}| + (l+1)k_l + \binom{m-l}{2}. \tag{4.7.1}$$

Proof. We prove this using descending induction on l. The base case is $l = m-1$. We have

$$\sum_{j=0}^{m-1} (d_{m-1} + d_j + 1) = |\bar{d}| + m(d_{m-1} + 1) = |\bar{d}| + m \cdot k_{m-1},$$

so (4.7.1) holds for $l = m-1$. Now let $2 \le l \le m-1$ and assume that (4.7.1) holds for this l. Then

$$\begin{aligned}\sum_{i=1}^{m-(l-1)} \sum_{j=0}^{m-i} (d_{m-i} + d_j + 1) &= (m-l)|\bar{d}| + (l+1)k_l + \binom{m-l}{2} + \sum_{j=0}^{l-1}(d_{l-1} + d_j + 1) \\ &= (m-l)|\bar{d}| + (l+1)k_l + \binom{m-l}{2} \\ &\quad + l(d_{l-1}+1) + |\bar{d}| - \sum_{j=l}^{m-1} d_j \\ &= (m-l+1)|\bar{d}| + l \cdot k_{l-1} + \binom{m-l+1}{2}.\end{aligned}$$

This proves (4.7.1). □

Note that we can rewrite the right hand side of (4.7.1) as

$$(m-l)\left(|\bar{d}| + \frac{1}{2}(m+l+1)\right) + (l+1)\sum_{j=l}^{m-1} d_j.$$

To sum up, if $U \subseteq V$ is a linear subspace of dimension r and with $UU = VV$, then

$$\frac{1}{2}k_l(2r+1-k_l) \ge (m-l)\left(|\bar{d}| + \frac{1}{2}(m+l+1)\right) + (l+1)\sum_{j=l}^{m-1} d_j$$

for all $l \in \{1, \ldots, m-1\}$. Or, equivalently,

$$\frac{1}{2}\left(\sum_{j=l}^{m-1} d_j\right)\left(2(r-m) - 1 - \sum_{j=l}^{m-1} d_j\right) \ge (m-l)\big(|\bar{d}| + m - r\big). \qquad (C_l)$$

4.7.9 Theorem. *Let $r \in \mathbb{N}$. If $U \subseteq V$ is a quadratically generating linear subspace of dimension r, then $\binom{r+1}{2} \ge \dim(VV) = (m+1)|\bar{d}| + \binom{m+1}{2}$. Moreover, for every $l \in \{1, \ldots, m-1\}$, the inequality (C_l) holds.* □

4.7.10 Example. We revisit Example 4.7.1. For $m = 2$ and $|\bar{d}| = 14$ we have $\dim(VV) = 45 = \binom{9+1}{2}$, so $r = 9$ is the upper bound in the Pataki interval. Using these numbers, the condition (C_1) says $\frac{1}{2}d_1(13 - d_1) \ge 7$. We see that this condition fails if and only if $d_1 = 1$ which leads to the case of $\bar{d} = (13, 1)$. Therefore, a subspace $U \subseteq V$ of dimension 9 cannot be quadratically generating in this case. Of course, since $\binom{r+1}{2}$ just equals $\dim(VV)$ in this setting, this is equivalent to the fact that U cannot be quadratically independent, which we have already seen before.

The inequalities (C_l) give lower bounds for the dimension of a quadratically generating subspace in terms of the individual d_i's and their sums. In the language of Gram spectrahedra this means that only faces of sufficiently high rank can have what we called expected dimension before. We give another numerical example to illustrate this phenomenon.

4.7.11 Example. Let $m = 2$ and $|\bar{d}| = 209$. Pataki's bound for the rank r of an extreme point is $\binom{r+1}{2} \le 3 \cdot 209 + 3 = 630$. For $r = 35$ this holds with equality. On the other hand, for $r = 35$, condition (C_1) reads as $\frac{1}{2}d_1(65 - d_1) \ge 176$. As an

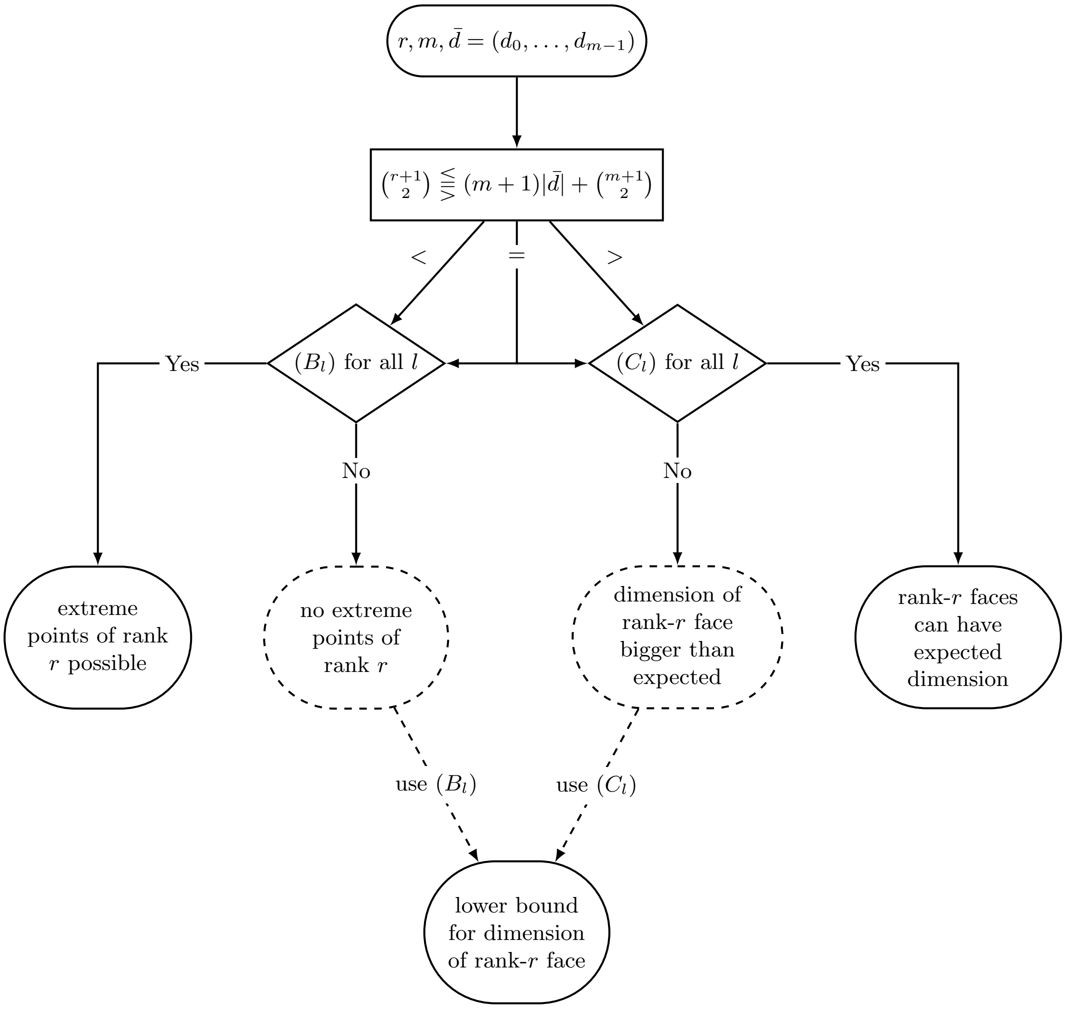


FIGURE 4.1. A flowchart to determine the minimum possible dimension of rank-r faces.

aside, we see that the Pataki interval cannot be exhausted if $d_1 \leq 5$. Let us take a closer look at the most extreme case $\bar{d} = (208, 1)$. In this case, Theorem 4.7.9 says that $2r - 5 \geq |\bar{d}| = 209$ if U is an r-dimensional quadratically generating subspace of V and thus leads to a face of expected dimension in the Gram spectrahedron of some $f \in \Sigma V^2$. This means that for all ranks $r \in \{35, 36, \dots, 106\}$, the faces of $\mathrm{Gram}_V(f)$ of rank r never achieve expected dimension. To be precise, if $r = 35 + k$ with $k \in \{0, 1, \dots, 71\}$, then a face of rank r has dimension at least $(144 - 2k)$ higher than expected dimension, which would be $\frac{1}{2}k(71 + k)$ in this case.

The flowchart in Figure 4.1 illustrates how the systems of inequalities obtained in this section can be used to determine the smallest possible dimension of a rank-r face in the Gram spectrahedron of a quadratic form $f \in \mathbb{R}[X_P]_2$ nonnegative on $X_P(\mathbb{R})$.

4.7.12 Example. For $m = 3$, the inequalities (C_1) and (C_2) read as follows:

$$\begin{aligned} \tfrac{1}{2}(d_1 + d_2)\big(2r - 7 - (d_1 + d_2)\big) + 2r &\geq 2(d_0 + d_1 + d_2) + 6, \\ \tfrac{1}{2}d_2(2r - 7 - d_2) + r &\geq (d_0 + d_1 + d_2) + 3. \end{aligned}$$

We give examples showing that they are not redundant. For $\bar{d} = (13, 1, 1)$ and $r = 11$ the first inequality holds true and the second does not. However, for $\bar{d} = (14, 6, 1)$ and $r = 13$ or for $\bar{d} = (11, 10, 1)$ and $r = 14$ the first inequality is false, while the second is true. In each of these cases the fact that $\binom{r+1}{2} \geq \dim(VV)$ suggests that we could have rank-r faces of expected dimension but our additional conditions exclude this possibility.

4.7.13 Remark. We have seen in Theorem 3.7.15 that the Gram spectrahedron of a general nonnegative binary form of degree $2d$ has rank-r faces of expected dimension for any $2 \leq r \leq d + 1$. Our analysis in the present section shows that one certainly cannot simply modify the proof of that case a bit in order to obtain a similar result for all rational normal scrolls. Rather, rank-r faces of expected dimension (in the sense of our definition) often do not occur at all and this depends essentially on the individual values of the d_i. Of course, we get a new lower bound for the dimension of a rank-r face for any $\bar{d}$ and one could conjecture that generically there are faces of this very dimension. We suspect that a proof could be pretty involved and would need many case distinctions. Besides, none of the inequalities (B_l) and (C_l) can explain the non-exhaustion of the Pataki interval that we observed for $\bar{d} = (2, 2)$, $(3, 1)$ and $\bar{d} = (3, 3)$, $(4, 2)$, $(5, 1)$ in Section 4.5, see Remark 4.5.12. One would therefore have to exclude further cases or find yet a different formula for the "new expected dimension".

4.8. Cones over varieties

Up to now, we were primarily interested in smooth varieties of minimal degree. In this section we deal with the case of (simple and iterated) cones over these varieties. More generally, we consider the projective cone Y over an arbitrary irreducible $\mathbb{R}$-variety $X \subseteq \mathbb{P}^n$ and relate the Gram spectrahedra of real quadratic forms on Y to those of forms in $\mathbb{R}[X]_2$. The take-home message from this section is that, up to a shift in ranks, the structure of Gram spectrahedra remains unchanged when going from X to Y. Corollary 4.8.6 gives a precise formulation of this statement.

4.8.1 Definition (see also [Harr, Example 3.1])**.** Let $\varnothing \neq X \subseteq \mathbb{P}^n$ be a projective variety. We consider $\mathbb{P}^n$ as a hyperplane in $\mathbb{P}^{n+1}$. Given any point $p \in \mathbb{P}^{n+1}$ not lying on the hyperplane $\mathbb{P}^n$, we define the *(projective) cone* $C_p(X)$ over X with vertex p to be the union

$$C_p(X) = \bigcup_{q \in X} (q \vee p)$$

of all lines joining p to points in X.

4.8.2 Remark. We can choose coordinates $(y : x_0 : \cdots : x_n)$ on $\mathbb{P}^{n+1}$ so that $X \subseteq \mathcal{V}_+(y) = \mathbb{P}^n$ and our point p is $(1 : 0 : \cdots : 0)$. If $I := \mathfrak{I}_+(X) \subseteq \mathbb{R}[x_0, \dots, x_n]$ is the homogeneous vanishing ideal of the projective $\mathbb{R}$-variety $X \subseteq \mathbb{P}^n$, then the vanishing ideal $\mathfrak{I}_+(Y)$ of $Y := C_p(X)$ is the ideal generated by I in $\mathbb{R}[y, x_0, \dots, x_n]$. Therefore, the cone over X is a projective variety in $\mathbb{P}^{n+1}$. By the choice of coordinates, Y is in fact the projective closure of the affine cone $\hat{X} \subseteq \mathbb{A}^{n+1}$ over X. Let X be irreducible. For the homogeneous coordinate rings we get $\mathbb{R}[Y] \cong \mathbb{R}[X][y]$ and we have $\dim Y = \dim X + 1$, see also [Harr, Lecture 11]. According to [Harr, Example 18.16], the degree $\deg C_p(X)$ of the projective cone over X equals $\deg X$, the degree of the variety X

itself. Therefore, if $\deg X = \operatorname{codim} X + k$ for some $k \geq 1$,

$$\deg Y = \operatorname{codim} X + k = n - \dim X + k = (n+1) - \dim Y + k = \operatorname{codim} Y + k.$$

In particular, if $X \subseteq \mathbb{P}^n$ is a variety of (almost) minimal degree, then $Y = C_p(X) \subseteq \mathbb{P}^{n+1}$ is also a variety of (almost) minimal degree.

For the rest of this section, we always assume that X is irreducible. The aim of this section is to make clear that the Gram spectrahedron of a general positive quadratic form on Y has the same structure as the Gram spectrahedron of a general positive quadratic form on X, where Y is a projective cone over X. This justifies that we have so far limited our analysis to smooth varieties of minimal degree. In [BPSV, Lemma 3.14], the authors use Schur complements to connect Gram matrices of a form $f \in \mathbb{R}[Y]_2$ to Gram matrices of a form $g \in \mathbb{R}[X]_2$ that is constructed from f by completing the square.

4.8.3 Definition. Let $m, n \geq 1$, let $A \in \mathrm{M}_n(\mathbb{R})$, $B \in \mathrm{M}_{n\times m}(\mathbb{R})$, $C \in \mathrm{M}_{m\times n}(\mathbb{R})$ and $D \in \mathrm{M}_m(\mathbb{R})$. Consider

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{m+n}(\mathbb{R}).$$

Let A be invertible. The *Schur complement* of (A in) M is $S := D - CA^{-1}B$.

It is an easy exercise in linear algebra to show that under the assumption that M is symmetric, the matrix M is positive (semi-)definite if and only if both A and S are positive (semi-)definite.

4.8.4. Let $X \subseteq \mathbb{P}^n$ be a nondegenerate (irreducible) projective variety and let $Y \subseteq \mathbb{P}^{n+1}$ be a cone over X. As above, we choose coordinates $(y : x_0 : \cdots : x_n)$ on $\mathbb{P}^n$ such that $X \subseteq \mathcal{V}_+(y)$ and Y is the cone over X with vertex $(1 : 0 : \cdots : 0)$. This means that

$$Y = \{(\lambda : \mu\xi_0 : \cdots : \mu\xi_n) \in \mathbb{P}^{n+1} : (\xi_0 : \cdots : \xi_n) \in X,\ (\lambda : \mu) \in \mathbb{P}^1\}.$$

A quadratic form $f \in \mathbb{R}[Y]_2$ can be written as $f = ay^2 + 2by + c$ with $a \in \mathbb{R}$, $b \in \mathbb{R}[X]_1$ and $c \in \mathbb{R}[X]_2$. If f is nonnegative and $a = 0$, then $b = 0$. In this case, $f = c$ is nonnegative on $X(\mathbb{R})$ and $\mathrm{Gram}_{\mathbb{R}[Y]_1}(f) \cong \mathrm{Gram}_{\mathbb{R}[X]_1}(c)$ naturally. We thus continue under the assumption $a > 0$.

The method of completing the square shows that f is positive on $Y(\mathbb{R})$ if and only if $c - \frac{b^2}{a}$ is positive on $X(\mathbb{R})$. Indeed, we can write

$$f = a\left(y + \frac{b}{a}\right)^2 + \left(c - \frac{b^2}{a}\right).$$

Let $\xi \in X(\mathbb{R})$. We choose some affine representative $v = (v_0, \ldots, v_n) \in \mathbb{R}^{n+1}$ with $[v] = \xi$. Then $b(v) \in \mathbb{R}$ and

$$\zeta := \left(-\frac{b(v)}{a} : v_0 : \cdots : v_n\right) \in Y(\mathbb{R})$$

for any choice of v. Since $f(\zeta) > 0$ and $\left(y + \frac{b}{a}\right)(\zeta) = 0$, we must have

$$\left(c - \frac{b^2}{a}\right)(\xi) > 0.$$

Thus, $c - \frac{b^2}{a} > 0$ on $X(\mathbb{R})$.

Any Gram matrix G of f is of the form

$$G = \begin{pmatrix} a & B^T \\ B & C \end{pmatrix} \in \mathrm{Sym}_{n+2}(\mathbb{R}),$$

where $B \in \mathbb{R}^{n+1}$ is the vector of coefficients of b and $C \in \mathrm{Sym}_{n+1}(\mathbb{R})$ is some Gram matrix of c. The Schur complement of (the 1×1-"block" $A = (a)$ in) G is $G' := C - a^{-1}BB^T$. This G' is a Gram matrix of $g := c - \frac{b^2}{a} \in \mathbb{R}[X]_2$. For fixed a and B, we thus consider the map

$$\Psi \colon \begin{pmatrix} a & B^T \\ B & C \end{pmatrix} = G \mapsto G' := C - a^{-1}BB^T,$$

which is a bijection between Gram matrices of f and Gram matrices of g. We show that this map respects the structure of the associated Gram spectrahedra $\mathrm{Gram}_V(f)$ and $\mathrm{Gram}_{V'}(g)$, where $V = \mathbb{R}[Y]_1$ and $V' = \mathbb{R}[X]_1$.

First of all, note that since $a > 0$, the matrix G is positive semidefinite if and only if the Schur complement G' is positive semidefinite. Furthermore, it holds

$$\begin{pmatrix} 1 & 0 \\ a^{-1}B & C - a^{-1}BB^T \end{pmatrix} \cdot \begin{pmatrix} a & B^T \\ 0 & I \end{pmatrix} = \begin{pmatrix} a & B^T \\ B & C \end{pmatrix} = G.$$

Since the matrix $M := \left(\begin{smallmatrix} a & B^T \\ 0 & I \end{smallmatrix}\right)$ is invertible, we have $\mathrm{rk}(G) = \mathrm{rk}(G') + 1$ and, more precisely,

$$\mathrm{im}(G) = \mathrm{im}\begin{pmatrix} 1 & 0 \\ a^{-1}B & C - a^{-1}BB^T \end{pmatrix} = \mathbb{R} \cdot \begin{pmatrix} 1 \\ a^{-1}B \end{pmatrix} \oplus \big(\{0\} \times \mathrm{im}(G')\big).$$

So if G_1 and G_2 are Gram matrices of f and G_1' and G_2' are their respective Schur complements, the above shows that $\mathrm{im}(G_1') \subseteq \mathrm{im}(G_2')$ implies $\mathrm{im}(G_1) \subseteq \mathrm{im}(G_2)$.

Now let G_1 and G_2 be positive semidefinite and let $\mathrm{im}(G_1) \subseteq \mathrm{im}(G_2)$. Let $F_i = \mathrm{suppface}(G_i) \subseteq \mathrm{Gram}_V(f)$ $(i = 1, 2)$. Then $F_1 \subseteq F_2$ and there exists $\varepsilon \in (0, 1)$ with $\frac{1}{1-\varepsilon}(G_2 - \varepsilon G_1) \in \mathrm{Gram}_V(f)$, see Lemma 1.2.6. This means that

$$G_2 - \varepsilon G_1 = \begin{pmatrix} (1-\varepsilon)a & (1-\varepsilon)B^T \\ (1-\varepsilon)B & C_2 - \varepsilon C_1 \end{pmatrix}$$

is positive semidefinite. Hence, also the Schur complement

$$\begin{aligned} &(C_2 - \varepsilon C_1) - \Big((1-\varepsilon)B\big((1-\varepsilon)a\big)^{-1}(1-\varepsilon)B^T\Big) \\ &\qquad = (C_2 - \varepsilon C_1) - (1-\varepsilon)a^{-1}BB^T = G_2' - \varepsilon G_1' \end{aligned}$$

is positive semidefinite. This shows that $\frac{1}{1-\varepsilon}(G_2' - \varepsilon G_1') \in \mathrm{Gram}_{V'}(g)$. Therefore, by Lemma 1.2.6, the supporting face of G_1' is contained in the supporting face of G_2'.

To summarize what has been shown up to here: The map Ψ provides a bijection between rank-r Gram matrices of f and rank-$(r-1)$ Gram matrices of g, it preserves positive semidefiniteness and respects the inclusion of faces in the corresponding Gram spectrahedra. Moreover, Ψ is an affine-linear map and the restriction of Ψ to an affine space maps this space to another affine space of same dimension, which is why also dimensions of faces will be preserved.

4.8.5 Remark. We can also establish the fact that dimensions of faces will be preserved by using the structure of the matrices and considering the associated face subspaces. Let G be a positive semidefinite Gram matrix of f as above and write $r = \mathrm{rk}(G)$. Let $\mathfrak{m} = (y, x_0, \ldots, x_n)^T$ and let e_i be the i-th standard basis vector in $\mathbb{R}^{n+2}$. According to Remark 2.3.12, the face subspace associated to the supporting face of G in $\mathrm{Gram}_V(f)$ is

$$U = \mathrm{span}(\mathfrak{m}^T G e_i : i = 1, \ldots, n+2) \subseteq \mathbb{R}[Y]_1.$$

Using column operations on a matrix means that we form linear combinations of the columns of the initial matrix. Therefore,

$$\tilde{U} := \mathrm{span}(\mathfrak{m}^T (GM) e_i : i = 1, \ldots, n+2) \subseteq U.$$

Since M is invertible, we have $\dim(\tilde{U}) = \dim(U)$ and hence $\tilde{U} = U$.

On the other hand, we have the face subspace

$$U' := \mathrm{span}\left((x_0, \ldots, x_n)\left(C - a^{-1}BB^T\right) e_i' : i = 1, \ldots, n+1\right) \subseteq \mathbb{R}[X]_1$$

associated to G', where e_i' denotes the i-th standard basis vector in $\mathbb{R}^{n+1}$. Since $\mathrm{rk}(G') = r - 1$, we can choose a basis $p_1, \ldots, p_{r-1} \in \mathbb{R}[X]_1$ of U'. The equality

$$\mathrm{im}(G) = \mathbb{R} \cdot \begin{pmatrix} 1 \\ a^{-1}B \end{pmatrix} \oplus \left(\{0\} \times \mathrm{im}\, G'\right)$$

tells us that $U = U' \oplus \mathbb{R} \cdot p_r$ with $p_r = y + a^{-1}b$. As the variable y does not occur in any element of $U'U'$, it is clear that the subspaces $U'U'$ and $p_r U$ of $\mathbb{R}[Y] = \mathbb{R}[X][y]$ intersect trivially. Therefore, $UU = U'U' \oplus p_r U$ and $\dim(UU) = \dim(U'U') + r$. In particular, the quadratic relations among $p_1, \ldots, p_{r-1}$ are the same as those among $p_1, \ldots, p_r$. This means that the supporting faces of G in $\mathrm{Gram}_V(f)$ and of G' in $\mathrm{Gram}_{V'}(g)$ have the same dimension.

4.8.6 Corollary. *Let $X \subseteq \mathbb{P}^n$ be a nondegenerate irreducible projective $\mathbb{R}$-variety such that the set $X(\mathbb{R})$ of real points is Zariski-dense. Let $Y \subseteq \mathbb{P}^{n+1}$ be a projective cone over X. Let $f \in \mathbb{R}[Y]_2$ be positive on $Y(\mathbb{R})$. Then there is a positive quadratic form $g \in \mathbb{R}[X]_2$ such that the Gram spectrahedra of f and g are isomorphic as convex sets.* □

4.8.7 (cf. Theorem 4.6.2)**.** Let us return to varieties of minimal degree, specifically to rational normal scrolls. For $m \geq 2$, we let $d_0 \geq d_1 \geq \cdots \geq d_{m-2} \geq 1$ and consider the (smooth) rational normal scroll $X_P \subseteq \mathbb{P}^{s-1}$ of dimension $m-1$ associated to the lattice polytope $P = P_{\bar{d}}$. Here, s denotes the number of lattice points in P. Let $Y \subseteq \mathbb{P}^s$ be a projective cone over X. Let $f \in \mathbb{R}[Y]_2$ be positive on $Y(\mathbb{R})$ and let $F \subseteq \mathrm{Gram}_{\mathbb{R}[Y]_1}(f)$ be a face of rank r. Writing $V = \mathbb{R}[Y]_1$ and $V' = \mathbb{R}[X]_1$, the Gram spectrahedron $\mathrm{Gram}_V(f)$ is isomorphic to $\mathrm{Gram}_{V'}(g)$ for some positive quadratic form $g \in \mathbb{R}[X]_2$. Let $F' \subseteq \mathrm{Gram}_{V'}(g)$ be the face corresponding to F under this isomorphism. Then $\mathrm{rk}(F') = r - 1$. If $r \geq s - d_{m-2}$, we get

$$\dim(F) = \dim(F') \leq \binom{(r-1)-(m-1)}{2} = \binom{r-m}{2} = \binom{\mathrm{rk}(F) - \dim(Y)}{2}.$$

4.8.8 Remark (Comments on the Pataki range)**.** Let $X \subseteq \mathbb{P}^{n-1}$ be an $(m-1)$-dimensional variety of minimal degree and let $Y \subseteq \mathbb{P}^n$ be a cone over X. We let $V = \mathbb{R}[Y]_1$ and $V' = \mathbb{R}[X]_1$ as before. Since X and Y are varieties of minimal degree, we have $\dim V'V' = mn - \binom{m}{2}$ and

$$\dim(VV) = (m+1)(n+1) - \binom{m+1}{2} = mn - \binom{m}{2} + (n+1).$$

We have seen that for a positive quadratic form $f \in \mathbb{R}[Y]_2$ there exists g, a positive quadratic form on X, such that we have an isomorphism between $\mathrm{Gram}_V(f)$ and $\mathrm{Gram}_{V'}(g)$ which maps points of rank r in $\mathrm{Gram}_V(f)$ to points of rank $r-1$ in $\mathrm{Gram}_{V'}(g)$.

In particular, if $\mathrm{Gram}_V(f)$ has extreme points of rank $m+1$ up to some r, then $\mathrm{Gram}_{V'}(g)$ has extreme points of rank m up to $r-1$ (and vice versa). However, the formula for $\dim(VV)$ already suggests that, for special choices of m and n, the Pataki interval for Y could be strictly bigger than for X. Indeed, if r, m, n are such that

$$\binom{r}{2} > mn - \binom{m}{2} \quad \text{and} \quad \binom{r+1}{2} \le mn - \binom{m}{2} + (n+1),$$

then $r-1$ is not in the Pataki interval for X, while r is in the Pataki interval for Y. But this means that the Pataki interval for Y cannot be exhausted. We therefore note that, in general, the Gram spectrahedron of a quadratic form on a projective cone does not contain extreme points of all ranks in the Pataki range.

4.8.9 Example. We can already observe the above mentioned phenomenon in the case where X is a curve. This means $m-1 = 1$ in our current setting. The first occurrence is $n = 5$ (and $r = 5$): Let $X = v_4(\mathbb{P}^1) \subseteq \mathbb{P}^4$ be the rational normal curve of degree 4. A quadratic form on X is a binary octic and the Pataki range is given by $\{2, 3\}$. Now let $Y \subseteq \mathbb{P}^5$ be a cone over X. The Pataki range for a general positive quadratic form on Y is $\{3, 4, 5\}$. Yet, we will never find an extreme point of rank 5 in the Gram spectrahedron of any $f \in \mathbb{R}[Y]_2$. Finally, we note that a quadratic form on Y corresponds to a polynomial in $\mathbb{R}[x, y]$ with Newton polytope contained in $2P$ where $P = \mathrm{conv}(\{(0,0), (4,0), (0,1)\})$.

4.9. Reducing the maximum dimension of proper faces

The goal of this section is to improve the upper bound for the dimension of a rank-maximal proper face of the Gram spectrahedron by 1 whenever this makes sense. For an m-dimensional variety $X \subseteq \mathbb{P}^n$ of minimal degree and $f \in \mathbb{R}[X]_2$, we have seen that $\mathrm{Gram}_{\mathbb{R}[X]_1}(f)$ is at most one-dimensional if $\mathrm{codim}(X) \le 1$. Consequently, what we want to show is the following: Under the assumption that $\mathrm{codim}(X) \ge 2$, the existence of a face $F \subseteq \mathrm{Gram}_{\mathbb{R}[X]_1}(f)$ with $\dim(F) = \binom{n-m}{2}$ means that $f(\xi) = 0$ for some $\xi \in X(\mathbb{R})$, and hence $f \notin \mathrm{int}(\Sigma \mathbb{R}[X]_1^2)$ so that $\mathrm{Gram}_{\mathbb{R}[X]_1}(f)$ is degenerated to its face F.

As indicated in 4.5.7 and the comments thereafter, we are going to give an inductive proof for this broad generalization of our result for binary forms. The following lemma can be seen as another explicit example that additionally completes the base cases.

4.9.1 Lemma. *Let $m = 3$ and $\bar{d} = (1,1,1)$. Consider the rational normal scroll $X = X_{P_{\bar{d}}} \subseteq \mathbb{P}^5$ and its homogeneous coordinate ring $R = \mathbb{R}[X]$. Let $U \subseteq V = R_1$ be a subspace with $\mathrm{codim}_V(U) = 1$. If $\mathrm{codim}_{VV}(UU) = m+1$, then $\mathcal{V}_+(U) \cap X(\mathbb{R}) \neq \varnothing$.*

Proof. We write $P := P_{\bar{d}}$. Due to the symmetry of $\bar{d}$, we can use linear coordinate changes on $\mathbb{P}^5$ that interchange the three rational normal curves involved in the construction of X_P. In terms of our identification of linear forms in R_1 with polynomials in $\mathbb{R}[s,x,y]_P = \operatorname{span}(1,s,x,xs,y,ys)$ this means that we can reduce to the analysis of two cases: either $1 \notin U$ or $1 \in U$, $s \notin U$.

If $1 \notin U$, we can find a basis $\mathcal{B}$ of U of the form

$$s - \lambda_s,\ x - \lambda_x,\ xs - \lambda_{xs},\ y - \lambda_y,\ ys - \lambda_{ys},$$

with λ_s, λ_x, λ_{xs}, λ_y, $\lambda_{ys} \in \mathbb{R}$. Using the same approach as in Example 4.5.8, we consider the 15×18-matrix A whose rows contain the coefficients of the pairwise products of elements in $\mathcal{B}$ with respect to a monomial basis of $\mathbb{R}[s,x]_{2P}$. The reduction step from A to A' where we successively delete rows that are obviously not contained in the span of the other rows (and finally the thereby resulting zero columns) leaves us with a 4×7-matrix A'. The vanishing of (some of) its 4×4-minors implies that $\lambda_{xs} = \lambda_s\lambda_x$ and $\lambda_{ys} = \lambda_s\lambda_y$. Consequently, all elements of U vanish in $(s,x,y) = (\lambda_s,\lambda_x,\lambda_y)$. In the setting where we consider U as a subspace of R_1 this means that a point of the form $(1 : \lambda_s : \lambda_x : \lambda_x\lambda_s : \lambda_y : \lambda_y\lambda_s) \in X(\mathbb{R})$ is contained in $\mathcal{V}_+(U) \cap X(\mathbb{R})$.

Now let $1 \in U$ and $s \notin U$. We start with a basis

$$(1, x - \lambda_x s, xs - \lambda_{xs} s, y - \lambda_y s, ys - \lambda_{ys} s).$$

The reduction step leads here to the matrix

$$A' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & -\lambda_{xs} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\lambda_{ys} \\ 0 & -\lambda_x & \lambda_x\lambda_{ys} & 1 & -\lambda_{ys} & 0 & 0 \\ -\lambda_y & 0 & \lambda_{xs}\lambda_y & 1 & 0 & -\lambda_{xs} & 0 \end{pmatrix}.$$

It is easy to see that $\operatorname{rk}(A') < 4$ if and only if $\lambda_x = \lambda_y = 0$. Therefore, the assumption $\operatorname{codim}_{VV}(UU) = m+1$ implies

$$U = \operatorname{span}(1, x, xs - as, y, ys - bs)$$

for some a, $b \in \mathbb{R}$. Using the correspondence between monomials 1, s, x, xs, y, ys in $\mathbb{R}[s,x,y]_P$ and variables $z_0,\dots,z_5$ in R_1, our subspace U is identified with

$$U = \operatorname{span}(z_0, z_2, z_3 - az_1, z_4, z_5 - bz_1) \subseteq R_1,$$

which has a base-point $(0 : 1 : 0 : a : 0 : b) \in X(\mathbb{R})$. □

Here is the main result of this section.

4.9.2 Theorem. *Let $m \in \mathbb{N}$, let $\bar{d} = (d_0, d_1, \dots, d_{m-1})$ with $|\bar{d}| \geq 3$. Let $P = P_{\bar{d}}$ and let $X_P \subseteq \mathbb{P}^n$ be the associated smooth rational normal scroll. Consider a liner subspace $U \subseteq V := \mathbb{R}[X_P]_1$ with $\operatorname{codim}_V(U) = 1$. Assume that $\operatorname{codim}_{VV}(UU) = m+1$. Then $\mathcal{V}_+(U) \cap X_P(\mathbb{R}) \neq \varnothing$, i.e., there exists $\xi \in X_P(\mathbb{R})$ such that $p(\xi) = 0$ for all $p \in U$.*

Note that the condition $|\bar{d}| \geq 3$ is equivalent to $\operatorname{codim}(X_P) = n - m = |\bar{d}| - 1 \geq 2$. A statement similar to that of Theorem 4.9.2 holds true for $X = v_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$ (cf. [Vill, Proposition 3.4.3 (ii)]). We show that we can also transfer it from a variety X of minimal degree to a projective cone over X.

4.9.3 Lemma. *Let $X \subseteq \mathbb{P}^{n-1}$ be an $(m-1)$-dimensional nondegenerate irreducible projective $\mathbb{R}$-variety of minimal degree with dense real points and assume that the following statement holds true: For every linear subspace $U' \subseteq V' := \mathbb{R}[X]_1$ of codimension 1 and with $\operatorname{codim}_{V'V'}(U'U') = m$, there exists $\xi' \in X(\mathbb{R})$ such that $p(\xi') = 0$ for all $p \in U'$.*

Let $Y \subseteq \mathbb{P}^n$ be a cone over X and let $U \subseteq V := \mathbb{R}[Y]_1$ be a linear subspace of codimension 1. Then $\operatorname{codim}_{VV}(UU) = m+1$ only if there is $\xi \in Y(\mathbb{R})$ such that $p(\xi) = 0$ for all $p \in U$.

Proof. We argue using the language of Gram spectrahedra. Take any f in the relative interior of ΣU^2. Then the Gram spectrahedron $\operatorname{Gram}_V(f)$ contains a face F such that $\mathcal{U}(F) = U$. Assume that f is positive. Then, according to Corollary 4.8.6, there is a quadratic form $g \in \mathbb{R}[X]_2$ and a face $F' \subseteq \operatorname{Gram}_{V'}(g)$ such that for the associated face subspace $U' := \mathcal{U}(F') \subseteq V'$ we have $\dim(U') = \dim(U) - 1$ and

$$\dim(F') = \dim(F) = \binom{n-m}{2} = \binom{(n-1)-(m-1)}{2}.$$

But this means precisely that $\operatorname{codim}_{V'}(U') = 1$ and $\operatorname{codim}_{V'V'}(U'U') = m$. Thus, $\mathcal{V}_+(U') \cap X(\mathbb{R}) \neq \varnothing$. In particular, g is not positive and neither is f. Consequently, there is a point $\xi \in Y(\mathbb{R})$ such that $p(\xi) = 0$ for all $p \in U$. $\square$

Combining Theorem 4.9.2 with the corresponding statement for the Veronese surface and with Lemma 4.9.3 that takes care of singular varieties, we obtain the desired result for Gram spectrahedra of quadratic forms on varieties of minimal degree.

4.9.4 Corollary. *Let $X \subseteq \mathbb{P}^n$ be an m-dimensional nondegenerate irreducible projective $\mathbb{R}$-variety of minimal degree with dense real points and let $\operatorname{codim}(X) = n-m \geq 2$. Let f be a quadratic form positive on $X(\mathbb{R})$. Then, for every proper face $F \subseteq \operatorname{Gram}_{\mathbb{R}[X]_1}(f)$, we have $\dim(F) \leq \binom{n-m}{2} - 1$.* $\square$

4.9.5. Before giving a proof of Theorem 4.9.2, let us first sketch our proof's structure. We proceed by a kind of induction where the base cases are those with $|\bar{d}| = 3$, i.e., $\bar{d} \in \{(3), (2,1), (1,1,1)\}$. We dealt with these cases in 4.5.7, Lemma 4.5.9 and Lemma 4.9.1, respectively. Given $\bar{d} = (d_0, d_1, \dots, d_{m-1})$ with $|\bar{d}| > 3$, we want to apply the induction hypothesis on $\bar{d}' = (d_0, d_1, \dots, d_{m-2}, d_{m-1} - 1)$ if $d_{m-1} > 1$ and on $\bar{d}' = (d_0, d_1, \dots, d_{m-2})$ if $d_{m-1} = 1$. To this end, we start with a suitably chosen basis $\mathcal{B}$ of U from which we delete one element q to obtain a basis $\mathcal{B}'$ of a smaller subspace $U' \subseteq \mathbb{R}[X_Q]_1$ where $Q = P_{\bar{d}'}$. The induction hypothesis will give us a point $\xi' \in X_Q(\mathbb{R})$ such that $p(\xi') = 0$ for all $p \in U'$. From ξ' we construct a point $\xi \in X_P(\mathbb{R})$ such that $p(\xi) = 0$ for all $p \in U$. Unfortunately, we have to distinguish several cases in this process.

Throughout, we will make extensive use of the combinatorics of the lattice points of P, which also give rise to the homogeneous vanishing ideal of X_P. We can interpret the subspace U as a space of linear forms on X_P or, alternatively, as a subspace of $\mathbb{R}[s, \underline{x}]_P$. As we did in the discussion of the base cases, we will switch back and forth between these two sides of the same coin. To keep track of things, it seems convenient to choose homogeneous coordinates

$$(z_{0,0} : \cdots : z_{0,d_0} : z_{1,0} : \cdots : z_{1,d_1} : \cdots : z_{m-1,0} : \cdots : z_{m-1,d_{m-1}})$$

on $\mathbb{P}^n$, so that the vanishing ideal of X_P is generated by all (2×2)-minors of the matrix

$$Z := \begin{pmatrix} z_{0,0} & \cdots & z_{0,d_0-1} & z_{1,0} & \cdots & z_{1,d_1-1} & \cdots\cdots & z_{m-1,0} & \cdots & z_{m-1,d_{m-1}-1} \\ z_{0,1} & \cdots & z_{0,d_0} & z_{1,1} & \cdots & z_{1,d_1} & \cdots\cdots & z_{m-1,1} & \cdots & z_{m-1,d_{m-1}} \end{pmatrix}$$

and the variable $z_{i,j} \in \mathbb{R}[X_P]_1$ corresponds to the monomial $x_i s^j \in \mathbb{R}[s, \underline{x}]_P$ for all $i = 0, \dots, m-1$ and $j = 0, \dots, d_i$. Using these coordinates, every point $\xi \in X_P$ has the form $\xi = [\xi_0, \dots, \xi_{m-2}, \xi_{m-1}]$, where

$$\xi_i = (y_i u^{d_i}, y_i u^{d_i-1} v, \dots, y_i v^{d_i}) \in \mathbb{C}^{d_i+1} \quad \text{for } i = 0, \dots, m-1$$

with $(0,0) \neq (u,v) \in \mathbb{C}^2$ and $(0, \dots, 0) \neq (y_0, \dots, y_{m-1}) \in \mathbb{C}^m$. When we say that a polynomial $p \in \mathbb{R}[s, \underline{x}]_P$ *vanishes* in a point $\xi \in X_P$, we mean that the linear form $\tilde{p} \in \mathbb{R}[X_P]_1$ corresponding to p vanishes in ξ, that is to say $\tilde{p}(\xi) = 0$. As indicated above, we identify p and $\tilde{p}$ and abusively denote both by p. In this sense, for example, the polynomial $1 \in \mathbb{R}[s, \underline{x}]_P$ "equals" $z_{0,0} \in \mathbb{R}[X_P]_1$ and vanishes in $(1 : 0 : \cdots : 0) \in X_P$.

On $\mathbb{R}[s, \underline{x}]$ we use the lexicographic monomial order with $s \prec x_1 \prec \cdots \prec x_{m-1}$. Restricting to $\mathbb{R}[s, \underline{x}]_P$, we can also compare *variables* $z_{i,j}$, $z_{i',j'} \in \mathbb{R}[X_P]_1$. We have $z_{i,j} \succ z_{i',j'}$ if and only if $i > i'$ or $i = i'$ and $j > j'$.

Proof of Theorem 4.9.2. We proceed by induction on $|\bar{d}|$. The base cases $\bar{d} \in \{(3), (2,1), (1,1,1)\}$ have already been proven (cf. 4.9.5). So let $|\bar{d}| \geq 4$.

We start with a basis of U as in 4.6.1. To restate this explicitly, in the interpretation of U as a subspace of $\mathbb{R}[s, \underline{x}]_P$ we work with a basis

$$\mathcal{B} := \left(p_1^{(0)}, \dots, p_{k_0}^{(0)}, p_1^{(1)}, \dots, p_{k_1}^{(1)}, \dots, p_1^{(m-1)}, \dots, p_{k_{m-1}}^{(m-1)}\right)$$

of U where $k_0, \dots, k_{m-1}$ are nonnegative integers with $k_0 + \cdots + k_{m-1} = \dim(U)$, and such that the leading monomials of the above polynomials form a strictly ascending chain with $x_i \mid \mathrm{LM}(p_l^{(i)})$ for all $i = 1, \dots, m-1$ and $l = 1, \dots, k_i$ and $x_i \nmid \mathrm{LM}(p_l^{(0)})$ for all $i = 1, \dots, m-1$ and $l = 1, \dots, k_0$. To ease notation and avoid unnecessary case distinctions, we sometimes write $x_0 = 1$. Since

$$\sum_{i=0}^{m-1} k_i = \dim(U) = n = -1 + \sum_{i=0}^{m-1} (d_i + 1),$$

we have $k_{i_0} = d_{i_0}$ for precisely one $i_0 \in \{0, \dots, m-1\}$ and $k_i = d_i + 1$ for all $i \neq i_0$. So we are guaranteed that $k_i \geq 1$ for all $i = 0, \dots, m-1$. Moreover, there is a single monomial $\mathfrak{m} = x_{i_0} s^{j_0} \in \mathbb{R}[s, \underline{x}]_P$ that is not a leading monomial of any of the above elements. We can thus achieve that every $p_l^{(i)}$ is of the form $\mathrm{LM}(p_l^{(i)}) - c_l^{(i)} \mathfrak{m}$ for some constant $c_l^{(i)} \in \mathbb{R}$ and $c_l^{(i)} = 0$ if $\mathfrak{m} \succ \mathrm{LM}(p_l^{(i)})$, cf. Remark 2.3.15.

We assign a special role to the polynomial $p_{k_{m-1}}^{(m-1)}$ with highest leading monomial and therefore denote it by q. The case $\mathrm{LM}(q) \neq x_{m-1} s^{d_{m-1}}$ is trivial. Indeed, in this case the elements of $\mathcal{B}$ are precisely all the other monomials and U has a base-point $(0 : \cdots : 0 : 1) \in X_P(\mathbb{R})$. So let $\mathrm{LM}(q) = x_{m-1} s^{d_{m-1}}$, i.e., $(i_0, j_0) \neq (m-1, d_{m-1})$.

Let $\mathcal{B}' = \mathcal{B} \setminus \{q\}$ and let $U' = \mathrm{span}(\mathcal{B}')$. Then $\dim(U') = \dim(U) - 1 = n - 1$. We distinguish two cases: $k_{m-1} \geq 2$ or $k_{m-1} = 1$. If $k_{m-1} \geq 2$, then we still have a polynomial in $\mathcal{B}'$ whose leading monomial is divisible by x_{m-1}. Thus, on the one hand, the count in 4.6.1 implies that $\dim(U'U') \geq (m+1)(n-1) - \binom{m+1}{2}$. On the other hand, we have $\dim(UU) \geq \dim(U'U') + (m+1)$ since the $m+1$ elements

$$p^{(0)}_{k_0}q,\ \ldots,\ p^{(m-2)}_{k_{m-2}}q,\ p^{(m-1)}_{k_{m-1}-1}q,\ q^2 \tag{4.9.1}$$

with pairwise distinct and ascending leading monomials generate a subspace that intersects $U'U'$ trivially. Consequently,

$$\dim(U'U') = (m+1)(n-1) - \binom{m+1}{2}. \tag{4.9.2}$$

We keep this fact in mind and turn to the case $k_{m-1} = 1$.

Recall that $\sum_{i=0}^{m-1} k_i = n = -1 + \sum_{i=0}^{m-1}(d_i+1)$ and therefore $k_{m-1} \geq 2$ as soon as $d_{m-1} \geq 2$. Hence, $k_{m-1} = 1$ implies $d_{m-1} = 1$ (and $m \geq 2$). Writing $\bar{d}' := (d_0, d_1, \ldots, d_{m-2})$ and $Q := P_{\bar{d}'}$, we have $U' \subseteq \mathbb{R}[s, x_1, \ldots, x_{m-2}]_Q$ in this situation. Both spaces have the same dimension $n-1$, so that they are equal. It follows that $\mathfrak{m} = x_{m-1}$ and consequently $q = x_{m-1}s - ax_{m-1}$ for some $a \in \mathbb{R}$. Using the language of varieties, the subspace $U \subseteq \mathbb{R}[X_P]_1$ has a base-point of the form $(0 : \cdots : 0 : 1 : a) \in X_P(\mathbb{R})$.

Now let $k_{m-1} = 2$ and $d_{m-1} = 1$. Then $U' \subseteq \mathbb{R}[C(X_Q)]_1 =: V'$ where Q is as above and $C(X_Q)$ denotes a projective cone over X_Q. Using equation (4.9.2), we see that $\operatorname{codim}_{V'V'}(U'U') = m+1$. As X_Q is an $(m-1)$-dimensional smooth rational normal scroll, we can use the induction hypothesis on X_Q together with Lemma 4.9.3 in order to obtain a point $\xi' \in C(X_Q)(\mathbb{R})$ with $p(\xi') = 0$ for all $p \in U'$.

We proceed by showing how to find ξ' for $k_{m-1}, d_{m-1} \geq 2$. In this case, we let $\bar{d}' := (d_0, d_1, \ldots, d_{m-2}, d_{m-1}-1)$ and $Q := P_{\bar{d}'}$. As discussed in 4.9.5, we can interpret $U' \subseteq \mathbb{R}[s, x_1, \ldots, x_{m-1}]_Q$ or $U' \subseteq \mathbb{R}[X_Q]_1$, where $X_Q \subseteq \mathbb{P}^{n-1}$ is an m-dimensional rational normal scroll embedded with respect to the lattice points of Q. According to equation (4.9.2), it holds $\operatorname{codim}_{\mathbb{R}[X_Q]_2}(U'U') = m+1$. By induction, there is a point $\xi' \in X_Q(\mathbb{R}) \subseteq \mathbb{P}^{n-1}$ such that $p(\xi') = 0$ for all $p \in U'$.

In any case, the point ξ' has the form $\xi' = [\xi_0, \ldots, \xi_{m-2}, \xi'_{m-1}]$, where

$$\begin{aligned} \xi_i &= (y_i u^{d_i}, y_i u^{d_i-1}v, \ldots, y_i v^{d_i}) \in \mathbb{R}^{d_i+1} \quad \text{for } i = 0, \ldots, m-2 \\ \text{and} \quad \xi'_{m-1} &= (y_{m-1}u^{d_{m-1}-1}, y_{m-1}u^{d_{m-1}-2}v, \ldots, y_{m-1}v^{d_{m-1}-1}) \in \mathbb{R}^{d_{m-1}} \end{aligned}$$

with $(0,0) \neq (u,v) \in \mathbb{R}^2$ and $(0, \ldots, 0) \neq (y_0, \ldots, y_{m-1}) \in \mathbb{R}^m$ (cf. 4.9.5).

For every $a \in \mathbb{R}$ we have $\xi^a := [\xi_0, \ldots, \xi_{m-2}, \xi'_{m-1}, a] \in \mathbb{P}^n(\mathbb{R})$. Considering U' as a subspace of $\mathbb{R}[X_P]_1$ again, we obtain $p(\xi^a) = 0$ for all $p \in U'$. Hence, we have to show that there exists an $a \in \mathbb{R}$ such that $\xi^a \in X_P(\mathbb{R})$ and $q(\xi^a) = 0$.

Recall that the elements of $\mathcal{B}'$ correspond to

$$z_{i,j} - \lambda_{i,j} z_{i_0,j_0} \in \mathbb{R}[X_P]_1$$

for $i = 0, \ldots, m-1$ and $j = 0, \ldots, d_i$ with $(i,j) \notin \{(i_0, j_0), (m-1, d_{m-1})\}$. Moreover, $\lambda_{i,j} \in \mathbb{R}$ and $\lambda_{i,j} = 0$ for $i < i_0$ or $i = i_0$ and $j < j_0$. The (i_0, j_0)-coordinate of ξ' is certainly nonzero. Indeed, if it was zero, evaluating the aforementioned elements in ξ' would imply that all entries of ξ' are zero, which is absurd for a point in projective space. In fact, the (i_0, j_0)-coordinate of ξ' must be the point's first nonzero coordinate.

Let us first consider the case $j_0 \neq 0$. Since $(y_0, \ldots, y_{m-1}) \neq (0, \ldots, 0)$, we must have $u = 0$. Then $\xi_i = (0, \ldots, 0, y_i v^{d_i})$ for $i = 0, \ldots, m-2$. If in addition $y_{m-1} = 0$, we can choose $a = c y_{i_0} v^{d_{i_0}} \in \mathbb{R}$. Then $q = z_{m-1,d_{m-1}} - c z_{i_0,d_{i_0}}$ vanishes in ξ^a. Besides, we have $\xi^a \in X_P(\mathbb{R})$ since the evaluation of the matrix Z from 4.9.5 in the point ξ^a contains only zeros in the first row.

We remark that (for $d_{m-1} > 1$) it is impossible to find an $a \in \mathbb{R}$ such that $\xi^a \in X_P$ if $u = 0$ and $y_{m-1} \neq 0$. However, that does not matter because we are going to show that the supposition $u = 0$ and $y_{m-1} \neq 0$ leads to $\dim(UU) > (m+1)n - \binom{m+1}{2}$, contradicting the hypothesis of the theorem, or back to the case $k_{m-1} = 1$. In order not to lose the thread, we outsource the proof of this fact to Lemma 4.9.6.

For the remainder of this proof we are therefore left with the case $j_0 = 0$. In this case, we have $u = 0$ only if $d_{m-1} = 1$ and $\xi' = [0, \dots, 0, \xi'_{m-1}]$ with $\xi'_{m-1} = (y_{m-1}) \in \mathbb{R} \setminus \{0\}$. But this means that $i_0 = m-1$ and $k_{m-1} = d_{m-1} = 1$. Thus, we may assume that $u \neq 0$. Dividing each coordinate in ξ'_{m-1} and in every ξ_i by $u^{d_{m-1}-1}$ gives the same point $\xi' \in \mathbb{P}^{n-1}$. Replacing v by $\frac{v}{u}$ and properly modifying the y_i's, we can thus achieve

$$\begin{aligned} \xi_i &= (y_i, y_i v, \dots, y_i v^{d_i}) \quad \text{for } i = 0, \dots, m-2 \\ \text{and} \quad \xi'_{m-1} &= (y_{m-1}, y_{m-1} v, \dots, y_{m-1} v^{d_{m-1}-1}) \end{aligned}$$

with $v \in \mathbb{R}$ and $(0, \dots, 0) \neq (y_0, \dots, y_{m-1}) \in \mathbb{R}^m$. Let $a := y_{m-1} v^{d_{m-1}}$. Then $\xi^a \in X_P(\mathbb{R})$ and we have to show that $q(\xi^a) = 0$.

We first assume that $i_0 < m-1$. Note that $y_{i_0} \neq 0$. Thus, we can achieve $y_{i_0} = 1$, so that $\xi_{i_0} = (1, v, \dots, v^{d_{i_0}})$. Consider the elements of $\mathcal{B}$ whose leading monomials are divisible by x_{i_0}. They were called $p_1^{(i_0)}, \dots, p_{d_{i_0}}^{(i_0)}$ at the beginning of the proof and are of the form

$$p_l^{(i_0)} = x_{i_0} s^l - \lambda_{i_0,l} x_{i_0} \quad \text{for all } l \in \{1, \dots, d_{i_0}\}.$$

Since they vanish in ξ, it follows that $\lambda_{i_0,l} = v^l$, i.e.,

$$p_l^{(i_0)} = x_{i_0} s^l - v^l x_{i_0} \qquad \text{for all } l \in \{1, \dots, d_{i_0}\}.$$

Recall from (4.9.1) that

$$UU = U'U' \oplus \operatorname{span}\left(p_{k_0}^{(0)} q, \dots, p_{k_{m-2}}^{(m-2)} q,\, p_{k_{m-1}-1}^{(m-1)} q,\, q^2\right).$$

If $k_{i_0} = d_{i_0} = 1$, then also $d_{m-1} = 1$ and a linear coordinate change on $\mathbb{P}^n$ that exchanges $(z_{i_0,0}, z_{i_0,1})$ and $(z_{m-1,1}, z_{m-1,0})$ transfers us back to the case $k_{m-1} = d_{m-1} = 1$. So let $k_{i_0} = d_{i_0} \geq 2$. Then $h := p_1^{(i_0)} q \in U'U'$ since every nonzero element in the second summand in the above decomposition has a leading monomial that is not that of h or any other element in $U'U'$. Now $h = x_{i_0}(s - v)(x_{m-1} s^{d_{m-1}} - c x_{i_0})$ must have a double root in $(s, x_{i_0}, x_{m-1}) = (v, 1, y_{m-1})$. In particular,

$$0 = \left.\frac{\partial h(s, x_{i_0}, x_{m-1})}{\partial s}\right|_{(v,1,y_{m-1})} = v^{d_{m-1}} y_{m-1} - c.$$

In other words, (the form in $\mathbb{R}[X_P]_1$ that corresponds to) q vanishes in $\xi^a \in X_P(\mathbb{R})$. Consequently, $\xi^a \in \mathcal{V}_+(U) \cap X_P(\mathbb{R}) \neq \varnothing$, as desired.

Finally, we consider the case $i_0 = m-1$. In this situation, $\xi' = [0, \dots, 0, \xi'_{m-1}]$ with $\xi'_{m-1} = (1, v, \dots, v^{d_{m-1}-1})$. Recall that $k_{m-1} = d_{m-1}$. We have

$$p_l^{(m-1)} = x_{m-1} s^l - v^l x_{m-1} \quad \text{for all } l \in \{1, \dots, d_{m-1} - 1\}.$$

The case $d_{m-1} = 1$ has been settled before. For $d_{m-1} \geq 3$, we obtain the desired conclusion $q = x_{m-1} s^{d_{m-1}} - v^{d_{m-1}} x_{m-1}$ in the same way as in the case of binary forms. So let $d_{m-1} = 2$. Consider the element $p := p_{d_{m-1}-1}^{(m-1)} = p_1^{(m-1)}$ with largest leading monomial in $\mathcal{B}'$. We have

$$q = x_{m-1} s^2 - c x_{m-1} \quad \text{and} \quad p = x_{m-1} s - v x_{m-1}.$$

As $d_0 \geq d_{m-1} = 2$, we have $1 \cdot q \in U'U'$ and consequently

$$U'U' \ni (1 \cdot q) - \big((s \cdot p) + v(1 \cdot p)\big) = (v^2 - c)x_{m-1}.$$

However, the smallest leading monomial in $U'U'$ divisible by x_{m-1} comes from the product $1 \cdot p$ and thus equals $x_{m-1}s$. Hence, $c = v^2$. We conclude that $\xi^a \in \mathcal{V}_+(U) \cap X_P(\mathbb{R}) \neq \varnothing$ and this completes the proof. □

4.9.6 Lemma. *Using the notation from the proof of Theorem 4.9.2, the following holds: If $u = 0$ and $y_{m-1} \neq 0$, then $k_{m-1} = 1$ or* $\operatorname{codim}_{VV}(UU) < m + 1$.

Proof. Let $u = 0$ and $y_{m-1} \neq 0$. We may assume that $\xi_i = (0, \ldots, 0, a_i)$ with $a_i \in \mathbb{R}$ ($i = 0, \ldots, m-2$) and $\xi'_{m-1} = (0, \ldots, 0, 1)$. Let us suppose that $k_{m-1} \geq 2$ and $\operatorname{codim}_{VV}(UU) = m + 1$. Recall from (4.9.1) that we then have

$$UU = U'U' \oplus \operatorname{span}\left(p^{(0)}_{k_0} q, \ldots, p^{(m-2)}_{k_{m-2}} q,\, p^{(m-1)}_{k_{m-1}-1} q,\, q^2\right).$$

Since the dimension of $U'U'$ is as small as it can possibly get, the leading monomials of elements in $U'U'$ are only those that we get as leading monomials of pairwise products of elements in $\mathcal{B}'$ (cf. the count in 4.6.1).

We first consider the case $a_i = 0$ for all $i \in \{0, \ldots, m-2\}$, i.e., $i_0 = m - 1$. Note that then $d_{m-1} = k_{m-1} \geq 2$. From $i_0 = m - 1$ it follows that $\mathcal{B}'$ consists of all monomials in $\mathbb{R}[s, \underline{x}]_P$ but the two largest. This means

$$\mathcal{B}' = (1, \ldots, s^{d_0}, x_1, \ldots, x_1 s^{d_1}, \ldots, x_{m-1}, \ldots, x_{m-1} s^{d_{m-1}-2})$$

(and $q = x_{m-1}s^{d_{m-1}} - c x_{m-1} s^{d_{m-1}-1}$). But then $s^{d_0-1}q$ has leading monomial $x_{m-1}s^{d_0+d_{m-1}-1}$, which is neither the product of two monomials in $\mathcal{B}'$ nor the leading monomial of any of the $m + 1$ elements in (4.9.1), a contradiction.

Now let $i_0 < m - 1$, that is $a_{i_0} \neq 0$. Consider the element $p := p^{(m-1)}_{k_{m-1}-1}$ with largest leading monomial in $\mathcal{B}'$. We have

$$q = x_{m-1}s^{d_{m-1}} - c x_{i_0} s^{d_{i_0}} \quad \text{and} \quad p = x_{m-1}s^{d_{m-1}-1} - l x_{i_0} s^{d_{i_0}}$$

for some $l \in \mathbb{R}$. Recall that $p(\xi') = 0$ and that the last coordinate of ξ' (corresponding to the index $(m-1, d_{m-1}-1)$) is nonzero, so that we must have $l \neq 0$. This fact will give us the desired contradiction in any of the following cases:

Let $i_0 = 0$. For $d_0 = 1$ we have $\bar{d} = (1, \ldots, 1)$, and $l = 0$ follows as in the case $1 \in U$, $s \notin U$ in Lemma 4.9.1. So let $d_0 \geq 2$. Arguing with leading monomials again, we see that we must have $s^{d_0-2}q \in U'U'$. Thus,

$$U'U' \ni s^{d_0-2}q - s^{d_0-1}p = l s^{2d_0-1} - c s^{2d_0-2}.$$

However, the largest leading monomial in $U'U'$ that is not divisible by any x_i ($i = 1, \ldots, m-1$) is $(s^{d_0-1})^2 = s^{2d_0-2}$. Consequently, $l = 0$, a contradiction.

Finally, if $0 < i_0 < m - 1$, then $\mathcal{B}'$ contains $1, s, \ldots, s^{d_0-1}, s^{d_0}$, while $x_{i_0}s^{d_{i_0}}$ is not the leading monomial of any element in $\mathcal{B}'$. In the same manner as before, we see that we now have $s^{d_0-1}q \in U'U'$. It follows

$$U'U' \ni s^{d_0-1}q - s^{d_0}p = l x_{i_0} s^{d_0+d_{i_0}} - c x_{i_0} s^{d_0+d_{i_0}-1}.$$

But $\operatorname{LM}(fg) \neq x_{i_0}s^{d_0+d_{i_0}}$ for all $f, g \in \mathcal{B}'$ and therefore $l = 0$, a contradiction. □

CHAPTER 5

Optimization related aspects of Gram spectrahedra

This chapter consists of various topics which are, in some sense, related to optimization. In Section 5.1 we study the dimension of normal cones of Gram spectrahedra at boundary points. This can be used to identify the spectrahedron's vertices. These are distinguished extreme points that are likely to occur as optimal solutions in minimization and maximization problems. Section 5.2 gives an interpretation of what optimization over Gram spectrahedra actually means in terms of sos representations of a polynomial. Afterwards, we show that any spectrahedron $S = L \cap \mathrm{Sym}_n^+(\mathbb{R})$ is the Gram spectrahedron of a quadratic form in a finitely generated graded $\mathbb{R}$-algebra (Theorem 5.3.1). As spectrahedra are the feasible regions of semidefinite programming problems (SDPs), this means that an SDP can be seen as an optimization problem over a Gram spectrahedron. The rest of Section 5.3 is then meant to present some familiar examples. For instance, we identify the Gram spectrahedron corresponding to the elliptope $\mathcal{E}_{n\times n}$.

Of course, this result could also have been presented in an earlier chapter of this work. However, now that the reader is already familiar with Gram spectrahedra of quadratic forms on varieties, it makes sense to include the present chapter at this point. The composition of this chapter can also be considered as a comparison of general spectrahedra and the special ones we studied in Chapter 3 and Chapter 4. Just as varieties of minimal degree stand out from others in the question of the relationship between nonnegativity and sums of squares, they also do so with regard to vertices in Gram spectrahedra (cf. Theorem 5.1.21 and Corollary 5.1.24). Furthermore, numerical experiments on optimization over Gram spectrahedra of binary forms re-emphasize the special structure of these spectrahedra (Section 5.2). In contrast, more general Gram spectrahedra do not have any characterizing features in the class of all spectrahedra not containing affine lines (Section 5.3).

5.1. Normal cones

Let K be a closed convex subset of a finite-dimensional real vector space. The normal cone of K at a point $x \in \partial K$ is the set of all linear forms that attain their maximum over K in x. A vertex of K is an extreme point whose normal cone is full-dimensional.

As de Carli Silva and Tunçel [CST] point out, vertices are very important objects from the point of view of optimization: Fix a full-dimensional closed convex set $K \subseteq \mathbb{R}^n$ and a point $x \in K$. If we choose a unit vector $c \in \mathbb{R}^n$ uniformly at random, the probability that x is an optimal solution for the problem of maximizing $l_c := \langle -, c\rangle$ over K is positive if and only if x is a vertex of K.

Vertices have been studied for spectrahedra arising from combinatorial optimization problems like the elliptope $\mathcal{E}_{n\times n}$, which can be viewed as a relaxation of the famous max-cut problem. Laurent and Poljak proved that $\mathcal{E}_{n\times n}$ has precisely 2^{n-1} vertices, namely the rank-one matrices in $\mathcal{E}_{n\times n}$ which are also called cut matrices, see [LP95, Theorem 2.5] and [LP96, Theorem 1.2]. Another example is the stable set problem for a graph G, whose relaxation leads to the so-called theta body of G. These examples and related ones are also analyzed in [CST] by means of the dimension of normal cones.

In this section we study normal cones and their dimensions for Gram spectrahedra. To this end, we transfer a result due to de Carli Silva and Tunçel to our situation. We therefore initially make use of their notation. Afterwards, we prove a dimension formula (Theorem 5.1.18) that is adapted to the context of sums of squares and can be applied to investigate vertices of Gram spectrahedra at the end of this section. For full disclosure let me note that the main results of this section were achieved in joint work with Julian Vill and are thus also included in his thesis [Vill].

Let $(\mathbb{E}, \langle -,- \rangle)$ be a finite-dimensional Euclidean vector space and let $\mathbb{E}^\vee$ be the dual space of $\mathbb{E}$.

5.1.1 Definition. Let $K \subseteq \mathbb{E}$ be a closed convex set and let $x \in \partial K$. The *normal cone* of K at x, denoted by $\mathcal{N}_K(x)$, is the set of all linear forms on $\mathbb{E}$ that attain their maximum over K in x, that is to say

$$\mathcal{N}_K(x) := \{l \in \mathbb{E}^\vee : \forall\, y \in K \; l(y) \leq l(x)\}\,.$$

The boundary point x is a *vertex* of K if its normal cone is full-dimensional, i.e., $\dim \mathcal{N}_K(x) = \dim(\mathbb{E}^\vee)$.

5.1.2 Remark. The inner product $\langle -,- \rangle$ on $\mathbb{E}$ defines an isomorphism $\Phi\colon \mathbb{E} \to \mathbb{E}^\vee$ by $(\Phi(v))(v') := \langle v, v' \rangle$ $(v, v' \in \mathbb{E})$. Under this identification, the normal cone of K at x is the set of all (outward) normal vectors of support hyperplanes of K at x.

Certain sources (e.g. [Sinn]) define the normal cone to be the set of all linear forms attaining their *minimum* over K at x. In this case one arrives at *inward* normal vectors. Of course, this makes no difference in terms of the cone's dimension.

5.1.3. Let K be a closed convex set and let $x \in \partial K$. Let $F = \operatorname{suppface}(x)$ and let $x' \in K$ be another point whose supporting face is F. Then the normal cones of K at x and at x' coincide. Indeed, let $l \in \mathcal{N}_K(x)$ then $l(y) \leq l(x) =: c$ for all $y \in K$. We show that $l(x') = l(x)$. The inequality $l(x') \leq l(x)$ is clear since $x' \in K$. Assume that $l(x') < l(x)$. Then $\{y \in K : l(y) = c\}$ is an exposed face of K that contains x but not x'. This is a contradiction because F was the smallest face of K containing x.

The discussion above justifies the following definition.

5.1.4 Definition. Let K be a closed convex set and let $F \subseteq K$ be a face. We define the *normal cone* of K at F, denoted by $\mathcal{N}_K(F)$, to be the normal cone of K at any point x in the relative interior of F.

As F itself is closed, this is equivalent to saying

$$\mathcal{N}_K(F) = \{l \in \mathbb{E}^\vee : \forall\, y \in K \; \forall\, x \in F \; l(y) \leq l(x)\}\,.$$

5.1.5 Remark. While exposed faces can be distinguished by their normal cones, this is no longer true for non-exposed faces. Consider Figure 5.1 for an example. We let $K \subseteq \mathbb{R}^2$ be the union of the upper half of the $\|\cdot\|_2$ unit ball (disk) and the lower half of the $\|\cdot\|_\infty$ unit ball (square),

$$K = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, y \geq 0\} \cup \{(x,y) \in \mathbb{R}^2 : \max\{|x|,|y|\} \leq 1, y \leq 0\}.$$

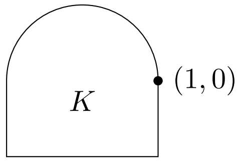


FIGURE 5.1. A compact convex set with a non-exposed extreme point.

The point $(1, 0)$ is a non-exposed extreme point of K, and the normal cones of K at all points $(1, y)$ with $-1 < y \leq 0$ coincide.

5.1.6 Remark. Let $K \subseteq \mathbb{E}$ be a compact convex set with nonempty interior. By a *perfect face* of K one understands a face F of K for which

$$\dim F + \dim \mathcal{N}_K(F) = \dim(\mathbb{E}).$$

It is well-known that all faces of a (full-dimensional) polytope $P \subseteq \mathbb{E}$ are perfect faces in this sense. In contrast, for example, all extreme points of a circle in $\mathbb{R}^2$ have one-dimensional normal cones so that these points are not perfect faces. In fact, any K as above has at most countably many perfect faces ([Schn, Theorem 2.2.5]).

In the case where a closed convex set $K \subseteq \mathbb{E}$ is described as the intersection of a polyhedron and a pointed closed convex cone with nonempty interior, de Carli Silva and Tunçel [CST] use a Strong Duality Theorem for conic optimization to obtain the following algebraic expression for the normal cones of K.

5.1.7 Proposition (see [CST, Proposition 2.1])**.** *Let $C \subseteq \mathbb{E}$ be a pointed closed convex cone with nonempty interior. Let $\mathcal{A}\colon \mathbb{E} \to \mathbb{R}^p$ and $\mathcal{B}\colon \mathbb{E} \to \mathbb{R}^q$ be linear functions. Let $a \in \mathbb{R}^p$ and $b \in \mathbb{R}^q$. Set $K := \{x \in C : \mathcal{A}(x) \leq a, \mathcal{B}(x) = b\}$. Suppose that $K \cap \operatorname{int}(C) \neq \varnothing$. If $x \in \partial K$, then*

$$\begin{aligned}\mathcal{N}_K(x) = &\{\mathcal{A}^*(y) : y \in \mathbb{R}^p_+,\ \operatorname{supp}(y) \cap \operatorname{supp}(\mathcal{A}(x) - a) = \varnothing\} \\ &+ \operatorname{im}(\mathcal{B}^*) - \left(C^* \cap \{x_0\}^\perp\right).\end{aligned}$$

In this context $\operatorname{supp}(v) := \{i \in \{1, \ldots, p\} : v_i \neq 0\}$ *is the* support *of $v \in \mathbb{R}^p$.*

We want to translate this result to the language of Gram spectrahedra.

5.1.8. If $(V, \langle -, - \rangle_V)$ and $(W, \langle -, - \rangle_W)$ are Euclidean vector spaces, we have an induced inner product $\langle -, - \rangle$ on $V \otimes W$ defined by

$$\langle v \otimes w, v' \otimes w' \rangle = \langle v, v' \rangle_V \cdot \langle w, w' \rangle_W \quad (v, v' \in V,\ w, w' \in W).$$

For $V = W$ we also obtain an inner product on $\mathsf{S}_2 V$ by restriction.

Now let A be an $\mathbb{R}$-algebra and $V \subseteq A$ a finite-dimensional linear subspace. Fix any inner product on V. We consider $\mathbb{E} := \mathsf{S}_2 V$ with the induced inner product. Abusively, we denote both the inner product on V and that on $\mathsf{S}_2 V$ by $\langle -, - \rangle$, but it should always be clear which one is meant.

5.1.9 Lemma. *The psd cone $C := \mathsf{S}_2^+ V \subseteq \mathbb{E}$ is self-dual.*

Proof. Let $0 \neq \vartheta \in \mathsf{S}_2 V$, say $\vartheta = \sum_{i=1}^r a_i(p_i \otimes p_i)$ with $p_i \in V$ and $a_i \in \mathbb{R} \smallsetminus \{0\}$. If $\vartheta \in \mathsf{S}_2^+ V$, then $a_i > 0$ for all i. Consequently, for every $\vartheta' = \sum_{j=1}^s q_j \otimes q_j \in \mathsf{S}_2^+ V$, we have

$$\langle \vartheta, \vartheta' \rangle = \sum_{i=1}^{r} \sum_{j=1}^{s} a_i \langle p_i, q_j \rangle^2 \geq 0.$$

Conversely, let $\langle \vartheta, \vartheta' \rangle \geq 0$ for all $\vartheta' \in \mathsf{S}_2^+ V$. We have to show that then $\vartheta \in \mathsf{S}_2^+ V$, that is to say $a_i > 0$ for all i. We can assume that $p_1, \ldots, p_r$ is an orthonormal basis of $\mathrm{im}(\vartheta)$. Fix $i \in \{1, \ldots, r\}$ and consider $\vartheta' := p_i \otimes p_i \in \mathsf{S}_2^+ V$. Then

$$a_i = \sum_{j=1}^{r} a_j \langle p_j, p_i \rangle^2 = \langle \vartheta, \vartheta' \rangle \geq 0.$$

Hence, $\mathsf{S}_2^+ V = (\mathsf{S}_2^+ V)^*$. □

5.1.10. For $f \in \Sigma V^2$, the Gram spectrahedron of f relative to V is given as the intersection of the cone $C = \mathsf{S}_2^+ V$ with the affine-linear subspace $\mu^{-1}(f)$, where $\mu \colon \mathsf{S}_2 V \to VV$, $p \otimes q \mapsto pq$ is the multiplication map. As we do not need further inequalities in the description of $K = \mathrm{Gram}_V(f)$, we can set $a = 0 \in \mathbb{R}$ and choose $\mathcal{A} \colon \mathbb{E} \to \mathbb{R}$ to be the zero map in Proposition 5.1.7, while $\mathcal{B} = \mu$ and $b = f \in VV \cong \mathbb{R}^q$ for some $q \in \mathbb{N}$. Summing up, Proposition 5.1.7 translates to the following:

5.1.11 Corollary. *Let $f \in \mathrm{int}(\Sigma V^2)$ and let $K = \{\vartheta \in \mathsf{S}_2^+ V : \mu(\vartheta) = f\}$ be the Gram spectrahedron of f relative to V. If $\vartheta \in \partial K$, then*

$$\mathcal{N}_K(\vartheta) = \mathrm{im}(\mu^*) - ((\mathsf{S}_2^+ V)^* \cap \{\vartheta\}^\perp) \subseteq (\mathsf{S}_2 V)^\vee.$$ □

Note that we used the fact from Proposition 2.3.7 saying $f \in \mathrm{int}(\Sigma V^2)$ if and only if $\mathrm{Gram}_V(f) \cap \mathrm{int}(\mathsf{S}_2^+ V) \neq \varnothing$. The condition $f \in \mathrm{int}(\Sigma V^2)$ is not severe. Indeed, if $f \notin \mathrm{int}(\Sigma V^2)$, then $\mathrm{Gram}_V(f) = \mathrm{Gram}_U(f)$ for a proper subspace $U \subseteq V$ with $f \in \mathrm{int}(\Sigma U^2)$. We could then develop the theory relative to U and would get the normal cone as a subset of $(\mathsf{S}_2 U)^\vee$.

In the following we want to determine the dimensions of the normal cones of $K = \mathrm{Gram}_V(f)$. We first give an easy upper bound from which we can infer an upper bound on the rank of vertices. Afterwards, we work towards a precise formula.

5.1.12 Corollary. *Let $\vartheta \in \partial K$. Then*

$$\dim \mathcal{N}_K(\vartheta) \leq \dim(VV) + \binom{\dim(\ker(\vartheta)) + 1}{2}$$

Proof. The multiplication map $\mu \colon \mathsf{S}_2 V \to VV$ is surjective. Therefore, the dual map μ^* is injective and we get $\dim(\mathrm{im}(\mu^*)) = \dim((VV)^\vee) = \dim(VV)$. If $A \in \mathbb{S}_+^m$, it is well-known that

$$\dim(\mathbb{S}_+^m \cap \{A\}^\perp) = \binom{\dim(\ker(A)) + 1}{2}.$$

Now the claim follows from Corollary 5.1.11. □

5.1.13 Example. Let $f \in \mathrm{int}(\Sigma_{2d})$ be a positive definite binary form of degree $2d$. We let $V = \mathbb{R}[x,y]_d$ and consider a Gram tensor $\vartheta \in K = \mathrm{Gram}(f)$ of rank r. By Corollary 5.1.12, we have

$$\dim \mathcal{N}_K(\vartheta) \leq 2d + 1 + \binom{d + 2 - r}{2}.$$

On the other hand, $\dim(\mathsf{S}_2 V) = \binom{d+2}{2} = \binom{d}{2} + (2d+1)$. This means that only (extreme points) of rank 1 or 2 can be vertices of Gram(f).

5.1.14 Lemma. *Let $f \in \operatorname{int}(\Sigma V^2)$, let $\vartheta \in \partial K$ and write $U = \operatorname{im}(\vartheta)$. Then*

$$\dim \mathcal{N}_K(\vartheta) = \dim(\mathsf{S}_2 V) - \dim\left(\ker(\mu) \cap \left(\mathsf{S}_2^+(U^\perp)\right)^\perp\right).$$

Proof. According to Corollary 5.1.11, we have

$$\begin{aligned}\left(\operatorname{span}\mathcal{N}_K(\vartheta)\right)^\perp &= \left(\operatorname{im}(\mu^*) - \operatorname{span}((\mathsf{S}_2^+ V)^* \cap \{\vartheta\}^\perp)\right)^\perp \\ &= \ker(\mu) \cap \left(\operatorname{span}((\mathsf{S}_2^+ V)^* \cap \{\vartheta\}^\perp)\right)^\perp \\ &= \ker(\mu) \cap \left((\mathsf{S}_2^+ V)^* \cap \{\vartheta\}^\perp\right)^\perp\end{aligned}$$

Using the isomorphism $\mathsf{S}_2 V \cong (\mathsf{S}_2 V)^\vee$ induced by the inner product, we consider $(\mathsf{S}_2^+ V)^* = \mathsf{S}_2^+ V$ (Lemma 5.1.9) and $\{\vartheta\}^\perp$ as subsets of $\mathsf{S}_2 V$. Write $\vartheta = \sum_{i=1}^r p_i \otimes p_i$ with a basis $p_1, \ldots, p_r$ of $U = \operatorname{im}(\vartheta)$. Let $\rho \in \mathsf{S}_2^+ V$, say $\rho = \sum_{j=1}^s q_i \otimes q_i$. We have $\rho \in \{\vartheta\}^\perp$ if and only if

$$0 = \langle \rho, \vartheta \rangle = \left\langle \sum_{i=1}^s q_i \otimes q_i, \sum_{j=1}^r p_j \otimes p_j \right\rangle = \sum_{i=1}^s \sum_{j=1}^r \langle q_i, p_j \rangle^2.$$

This means that $\rho \in \{\vartheta\}^\perp$ if and only if $q_j \in U^\perp$ for $j = 1, \ldots, s$. Consequently, $\mathsf{S}_2^+ V \cap \{\vartheta\}^\perp = \mathsf{S}_2^+(U^\perp)$. □

5.1.15 Lemma. *Let $U \subseteq V$ be a subspace, and let $U^\perp$ be the orthogonal complement of U in V. Then $(\mathsf{S}_2^+(U^\perp))^\perp = \operatorname{Sym}(U \otimes V)$, where $\operatorname{Sym}\colon V \otimes V \to \mathsf{S}_2 V$ is the symmetrization map $p \otimes q \mapsto \frac{1}{2}(p \otimes q + q \otimes p)$.*

Proof. $\operatorname{Sym}(U \otimes V)$ is generated by tensors of the form $p \otimes q + q \otimes p$, where $p \in U$, $q \in V$. Furthermore, $\mathsf{S}_2^+(U^\perp)$ is the conical hull of elementary tensors $g \otimes g$ with $g \in U^\perp$. Therefore, "$\supseteq$" follows from the fact that for all $p \in U$, $q \in V$ and $g \in U^\perp$, we have

$$\begin{aligned}\langle g \otimes g, p \otimes q + q \otimes p \rangle &= \langle g \otimes g, p \otimes q \rangle + \langle g \otimes g, q \otimes p \rangle \\ &= \underbrace{\langle g, p \rangle}_{=0} \langle g, q \rangle + \langle g, q \rangle \underbrace{\langle g, p \rangle}_{=0} = 0.\end{aligned}$$

For the opposite inclusion note that $\mathsf{S}_2 V = \mathsf{S}_2 U \oplus \mathsf{S}_2(U^\perp) \oplus \operatorname{Sym}(U \otimes U^\perp)$. Fix an orthonormal basis $q_1, \ldots, q_s$ of $U^\perp$. Given $\rho \in \mathsf{S}_2 V$, we write

$$\rho = \rho' + \sum_{1 \le i \le j \le s} \lambda_{ij} (q_i \otimes q_j + q_j \otimes q_i),$$

where $\rho' \in \mathsf{S}_2 U + \operatorname{Sym}(U \otimes U^\perp)$ and $\lambda_{ij} \in \mathbb{R}$. Note that $\langle \rho', g \otimes g \rangle = 0$ for all $g \in U^\perp$. Now assume that $\rho \in (\mathsf{S}_2^+(U^\perp))^\perp$. For any $g \in U^\perp$, it holds

$$0 = \langle \rho, g \otimes g \rangle = \sum_{1 \le i \le j \le s} 2\lambda_{ij} \langle q_i, g \rangle \langle q_j, g \rangle.$$

We specialize to $g = q_k$ for $k \in \{1, \ldots, s\}$ and get

$$0 = \sum_{1 \le i \le j \le s} 2\lambda_{ij} \delta_{ik} \delta_{jk} = 2\lambda_{kk},$$

and therefore $\lambda_{kk} = 0$. Next, we let $g = q_k + q_l$ with $1 \leq k < l \leq s$. Thereby, we obtain

$$0 = \sum_{1 \leq i < j \leq s} 2\lambda_{ij} \langle q_i, q_k + q_l \rangle \langle q_j, q_k + q_l \rangle = 2\lambda_{kl}.$$

Indeed, we have $(\delta_{ik} + \delta_{il})(\delta_{jk} + \delta_{jl}) = \delta_{ik}\delta_{jk} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{il}\delta_{jl}$. Using $i \neq j$, we see that the first and the last summand have to be 0. Since $i < j$ but $l > k$, also the third summand vanishes, leaving us with $\delta_{ik}\delta_{jl}$. To sum up, we have shown that $\lambda_{kl} = 0$ for all $1 \leq k \leq l \leq s$, that is to say $\rho = \rho' \in \mathrm{Sym}(U \otimes V)$. □

5.1.16 Remark. By Lemma 5.1.15, the orthogonal complement of $\mathrm{span}\,\mathcal{N}_K(\vartheta)$ is the vector space W of all Gram tensors of 0 in $\mathrm{Sym}(U \otimes V)$ where $U = \mathrm{im}(\vartheta)$. We have also seen that $\mathrm{Sym}(U \otimes V) \cong \mathsf{S}_2V/\mathsf{S}_2(U^\perp)$. Let $r = \dim(U) = \mathrm{rk}(\vartheta)$. Since $\mu|_{\mathrm{Sym}(U \otimes V)} \colon \mathrm{Sym}(U \otimes V) \to UV$ is surjective and W is the kernel of this map, we get

$$\begin{aligned}
\dim(W) &= \dim(\mathrm{Sym}(U \otimes V)) - \dim(UV) \\
&= \dim(\mathsf{S}_2V) - \dim(\mathsf{S}_2(U^\perp)) - \dim(UV) \\
&= \binom{\dim(V) + 1}{2} - \binom{\dim(V) - r + 1}{2} - \dim(UV) \\
&= r \cdot \dim(V) - \binom{r}{2} - \dim(UV).
\end{aligned}$$

The space UV can be understood by means of a familiar map.

5.1.17. For any $r \in \mathbb{N}$, we consider the sum-of-squares map

$$\phi_r \colon V^r \to VV, \quad (q_1, \ldots, q_r) \mapsto \sum_{i=1}^{r} q_i^2,$$

that we have already come across in 4.1.6 in a more specific setting. Let

$$\mathrm{d}\phi_r(\underline{p}) \colon V^r \to VV, \quad (q_1, \ldots, q_r) \mapsto 2\sum_{i=1}^{r} q_i p_i$$

be its differential at the point $\underline{p} = (p_1, \ldots, p_r) \in V^r$.

Let $p_1, \ldots, p_r \in V$ be linearly independent and let $U = \mathrm{span}(p_1, \ldots, p_r) \subseteq V$. The image of the map $\mathrm{d}\phi_r(\underline{p})$ is UV and depends only on the subspace U but not on the particular basis $\underline{p}$. Consequently, also its rank and the dimension of its kernel depend only on U and we can write $\dim\ker(\mathrm{d}\phi_r(U)) := \dim\ker(\mathrm{d}\phi_r(\underline{q}))$, where $\underline{q} = (q_1, \ldots, q_r)$ is any basis of U. Using this notation, we show:

5.1.18 Theorem. *Let $f \in \mathrm{int}(\Sigma V^2)$, let $K = \mathrm{Gram}_V(f)$ and let $\vartheta \in \partial K$. We write $U = \mathrm{im}(\vartheta)$ and $r = \dim(U)$. Then*

$$\dim \mathcal{N}_K(\vartheta) = \dim(\mathsf{S}_2V) - \dim\ker(\mathrm{d}\phi_r(U)) + \binom{r}{2}.$$

Proof. Let $W := \ker(\mu) \cap \mathrm{Sym}(U \otimes V)$. According to the Lemmata 5.1.14 and 5.1.15, we have to show that

$$\dim(W) = \dim\ker(\mathrm{d}\phi_r(U)) - \binom{r}{2}. \tag{5.1.1}$$

If $\underline{p} = (p_1, \ldots, p_r)$ is any basis of U, the image of $\mathrm{d}\phi_r(\underline{p})$ is UV. Therefore,

$$\dim\ker(\mathrm{d}\phi_r(U)) = r \cdot \dim(V) - \dim(UV).$$

Comparing this to the formula for $\dim(W)$ from Remark 5.1.16 shows the desired equality (5.1.1). □

5.1.19 Remark. Let $f \in \mathrm{int}(\Sigma V^2)$ and let $F \subseteq K = \mathrm{Gram}_V(f)$ be a nontrivial face. Consider the face subspace $U = \mathcal{U}(F)$ associated to F. We have seen in Proposition 2.3.9 that the dimension of F is essentially determined by $\dim(UU)$. Yet, the formulae for the dimension of $\mathcal{N}_K(F)$ obtained in this section show that the dimension of the normal cone of K at F is determined by $\dim(UV)$.

Let us first interpret this for comparatively large ranks. If U is quadratically generating, then $VV = UU \subseteq UV$ and hence $\dim(UU) = \dim(UV)$. Note that in this case, F is a face that has the smallest possible dimension among all faces of rank $r = \mathrm{rk}(F)$. Combining Proposition 2.3.9 with Theorem 5.1.18 (and Remark 5.1.16) leads to

$$\begin{aligned}\dim F + \dim \mathcal{N}_K(F) &= \dim(\mathsf{S}_2 V) + \dim(UV) - \dim(UU) - r(\dim(V) - r) \\ &= \dim(\mathsf{S}_2 V) - r(\dim(V) - r).\end{aligned}$$

We now turn our attention to vertices. As any vertex of K has to be an extreme point of K, we can of course only hope for vertices in small ranks since the associated face subspace has to be quadratically independent. Using Theorem 5.1.18, we first give a general criterion for a boundary point to be a vertex. Afterwards, we point out its meaning for varieties of minimal degree and beyond.

5.1.20 Corollary. *Let $f \in \mathrm{int}(\Sigma V^2)$ and let $\vartheta \in \partial\,\mathrm{Gram}_V(f)$ be a point of rank r. Let $p_1, \ldots, p_r$ be any basis of $\mathrm{im}(\vartheta)$. Then ϑ is a vertex of $\mathrm{Gram}_V(f)$ if and only if all relations $\sum_{i=1}^r q_i p_i = 0$ with $q_1, \ldots, q_r \in V$ are generated by the $\binom{r}{2}$ trivial relations $p_i p_j = p_j p_i$ for $i < j$.*

By this we mean that the subspace $\{(q_1, \ldots, q_r) \in V^r : \sum_{i=1}^r q_i p_i = 0\} \subseteq V^r$ is generated by $(0, \ldots, 0, p_j, 0, \ldots, 0, -p_i, 0, \ldots, 0)$ for $i < j$.

Proof. This follows directly from Theorem 5.1.18 since the normal cone at ϑ is full-dimensional if and only if $\dim\ker(\mathrm{d}\phi_r(\underline{p})) = \binom{r}{2}$. □

Let $X \subseteq \mathbb{P}^n$ be a nondegenerate irreducible projective $\mathbb{R}$-variety of dimension m such that $X(\mathbb{R})$ is Zariski-dense in X. Assume that X is of minimal degree. We consider the set

$$\begin{aligned}\mathcal{P} := \Big\{ f \in \mathbb{R}[X]_2 : &\ \mathrm{sgn}(f(\xi)) > 0 \text{ for all } \xi \in X(\mathbb{R}) \text{ and for all} \\ &\ \xi \in X \cap \mathcal{V}_+(f) \text{ there exists } i \in \{0, \ldots, n\} \text{ with } \frac{\partial f}{\partial x_i}(\xi) \neq 0 \Big\}\end{aligned}$$

of all strictly positive quadratic forms $f \in \mathbb{R}[X]_2$ such that $\mathcal{V}_+(f)$ has no singularities on X. Write $V := \mathbb{R}[X]_1$. The topological argument in the proof of Theorem 4.1.5 given in [BPSV] shows that $\mathcal{P}$ is open and dense in $\mathrm{im}(\phi_{m+1}) = \Sigma V^2$. We obtain the following theorem.

5.1.21 Theorem. *Let $f \in \mathcal{P}$. The vertices of $\mathrm{Gram}_V(f)$ are precisely the tensors $\vartheta \in \mathrm{Gram}_V(f)$ with $\mathrm{rk}(\vartheta) \leq m + 1$. For general f, the set of vertices of $\mathrm{Gram}_V(f)$ equals $\mathrm{Ex}_{m+1}(f)$.*

Proof. That points of rank bigger than $m+1$ cannot be vertices follows from a dimension count similar to that in Example 5.1.13, see also Remark 5.1.23. So let $\vartheta \in \mathrm{Gram}_V(f)$ be a point of rank $r \leq m+1$. Write $\vartheta = \sum_{i=1}^r p_i \otimes p_i$ with linearly independent $p_1, \ldots, p_r \in V$. Then $X \cap \mathcal{V}_+(p_1, \ldots, p_r) = \varnothing$ since $f = p_1^2 + \cdots + p_r^2$ would be singular in any point of this intersection. Now, we argue as in the proof of [BPSV, Lemma 2.2] (cf. 4.1.6). The sequence $p_1, \ldots, p_r \in \mathbb{R}[X]$ is part of a homogeneous system of parameters, and thus of a regular sequence, so that the first claim follows from Corollary 5.1.20. Furthermore, a general f is not a sum of fewer than $m+1$ squares. □

5.1.22 Remark. Note that by a result of Fawzi and Safey El Din ([FSED, Theorem 4]), the number of vertices of a spectrahedron (or more generally, a spectrahedral shadow) is finite. For $f \in \mathcal{P}$, this re-proves that $\mathrm{Gram}_V(f)$ contains only finitely many tensors of rank $r \leq m+1$ (cf. Corollary 4.1.7). However, the upper bound in [FSED] is much larger than the number $2^{\mathrm{codim}(X)}$ from 4.1.9.

The previous theorem solves the question regarding vertices in the case of quadratic forms on varieties of minimal degree, or equivalently $\varepsilon_2(X) = 0$. Now we want to use Corollary 5.1.12 for quadratic forms on varieties of given quadratic deficiency. This will also fill in the details in the first part of the proof of Theorem 5.1.21.

5.1.23 Remark. Let $X \subseteq \mathbb{P}^n$ be an irreducible nondegenerate projective $\mathbb{R}$-variety of dimension m. As before, we let $V = \mathbb{R}[X]_1$, $f \in \mathrm{int}(\Sigma V^2)$, and $\vartheta \in K = \mathrm{Gram}_V(f)$. As X is nondegenerate, we have $\dim(V) = n+1$ and $\dim(VV)$ is given by the deficiency formula from Lemma 4.1.2. Writing $r = \mathrm{rk}(\vartheta)$, Corollary 5.1.12 says

$$\begin{aligned}\dim(\mathsf{S}_2 V) - \dim \mathcal{N}_K(\vartheta) &\geq \dim(\mathsf{S}_2 V) - \dim(VV) - \binom{n+2-r}{2} \\ &= \frac{1}{2}\big(r-(m+1)\big)\big((2n+2)-(m+r)\big) - \varepsilon_2(X) \\ &=: \Delta(r).\end{aligned}$$

As an aside,

$$\begin{aligned}(2n+2)-(m+r) &= (n-m) + \big((n+1)-r\big) + 1 \\ &= \mathrm{codim}(X) + \dim(\ker(\vartheta)) + 1 \geq 1.\end{aligned}$$

The maximum possible rank is $\dim(V) = n+1$. Therefore,

$$\frac{\partial \Delta}{\partial r}(r) = n + \frac{3}{2} - r > 0,$$

so that $\Delta(r)$ is monotonically increasing in the rank r. We have

$$\Delta(m+2) = n - m - \varepsilon_2(X) = \mathrm{codim}(X) - \varepsilon_2(X).$$

For varieties of minimal degree, this means that points of rank $r \geq m+2$ cannot be vertices in Gram spectrahedra. (Note that $m+1 = n+1$ is the maximum possible rank in the case $\mathrm{codim}(X) = 0$.)

We consider (arithmetically Cohen-Macaulay) varieties with $\varepsilon_2(X) = 1$ in Chapter 6. We anticipate some facts so as to discuss the vertices of Gram spectrahedra at this point. If $X \subseteq \mathbb{P}^n$ is a hypersurface, then Gram spectrahedra of quadratic forms $f \in \Sigma\mathbb{R}[X]_1^2$ are single points (cf. Remark 4.1.4) of rank at most $\dim \mathbb{R}[X]_1 = n+1 = m+2$ and these points are trivially vertices of their respective spectrahedra. For $\mathrm{codim}(X) \geq 2$, we have $\Delta(r) \geq 1$ for all $r \geq m+2$ so that we cannot have vertices of these ranks. Since $m+2$ is the smallest rank in the Pataki interval in this case (see also Theorem 6.1.1), a generic sum of squares $f \in \mathbb{R}[X]_2$ has no vertices in its Gram spectrahedron. Of course, special forms with shorter sos representations can have vertices of smaller rank, see 6.4.3 for two-dimensional examples.

5.1.24 Corollary. *Let $X \subseteq \mathbb{P}^n$ be an irreducible nondegenerate projective $\mathbb{R}$-variety with Zariski-dense real points. Assume that* $\operatorname{codim}(X) \geq 2$. *Let X be aCM and let $\varepsilon_2(X) = 1$. If $f \in \Sigma\mathbb{R}[X]_1^2$ is generic, then* $\operatorname{Gram}_{\mathbb{R}[X]_1}(f)$ *has no vertices.*

In particular, there are no vertices in the Gram spectrahedron of a general ternary sextic or a general quaternary quartic.

Proof. The first statement was proven in Remark 5.1.23. That it includes the latter one is discussed after Corollary 6.1.2. □

5.1.25 Remark. Studying normal cones can also be motivated from a different point of view. For instance, they are useful for understanding the algebraic boundary of convex semialgebraic sets and their polar sets (cf. [Sinn]): Let $C \subseteq \mathbb{R}^n$ be a compact convex semialgebraic set with $0 \in \operatorname{int}(C)$ and consider the Zariski closure X of $\operatorname{Ex}(C)$ in $\mathbb{A}^n$. Via a condition on normal cones at certain extreme points, Sinn relates irreducible subvarieties of X to irreducible components of the projective closure of the algebraic boundary of C° ([Sinn, Corollary 3.9]).

In his thesis, Vill uses Sinn's result to show that, for a general nonnegative ternary quartic f, the boundary of the convex body dual to $\operatorname{Gram}(f)$ contains at least ten irreducible components ([Vill, Proposition 4.7.3]).

5.2. Optimization over Gram spectrahedra

We accentuated the importance of vertices in optimization at the beginning of Section 5.1. In this section we want to give a concise interpretation of what it means to optimize certain linear forms over Gram spectrahedra. This is also indicated in the introduction of Scheiderer's article [Sch22]. Subsequently, we present numerical experiments concerning the optimization over Gram spectrahedra which we relate to similar ones carried out by Nie, Ranestad and Sturmfels in [NRS].

5.2.1. Let A be a graded $\mathbb{R}$-algebra and let $V \subseteq A$ be a finite-dimensional subspace. Let $b\colon V \times V \to \mathbb{R}$ be any bilinear form. By the universal property of the tensor product, there is a unique linear map $l_b\colon V \otimes V \to \mathbb{R}$ with $l_b(v \otimes w) = b(v, w)$ for all v, $w \in V$.

Let us interpret this in terms of Gram spectrahedra. Let $f \in \Sigma V^2$. Fix a basis $\mathcal{B} = (q_1, \ldots, q_n)$ of V and let $p \in V$. Writing $p = \sum_{i=1}^n a_i q_i$ with $a_i \in \mathbb{R}$, we get

$$l_b(p \otimes p) = \sum_{i,j=1}^{n} a_i a_j b(q_i, q_j).$$

We see that minimizing a form which is homogeneous of degree 2 in the $\mathcal{B}$-coordinates of elements of V over all sos representations of f amounts to minimizing a linear form over $\operatorname{Gram}_V(f)$.

5.2.2 Example. Let $b\colon V \times V \to \mathbb{R}$ be the bilinear form defined by $b(q_i, q_j) = \delta_{ij}$ for all i, $j \in \{1, \ldots, n\}$. Then b is symmetric and positive definite (b is the scalar product

	rank 2	rank 3	rank 4
$d = 3$	61.70%	38.30%	
$d = 4$	40.11%	59.89%	
$d = 5$	16.28%	67.75%	15.97%

TABLE 5.1

	rank 2	rank 3	rank 4
$d = 3$	61.49%	38.51%	
$d = 4$	38.57%	61.43%	
$d = 5$	30.23%	67.87%	1.90%

TABLE 5.2

Distribution of the rank of the optimal matrix when minimizing the trace linear form or random linear forms, respectively.

on V that arises by declaring the given basis as orthonormal). For $p = \sum_{i=1}^n a_i q_i$ we get

$$l_b(p \otimes p) = \sum_{i=1}^n a_i^2 =: \|p\|_2^2.$$

Let $\vartheta = \sum_{i=1}^r p_i \otimes p_i \in \mathrm{Gram}_V(f)$ and write $p_i = \sum_{j=1}^n c_{ij} q_j$. The positive semidefinite Gram matrix corresponding to the psd Gram tensor ϑ of f is $G = \sum_{i=1}^r c_i c_i^T$, where $c_i = (c_{i1}, \dots, c_{in})^T$ (see 2.1.1). We obtain

$$l_b(\vartheta) = \sum_{i=1}^r \|p_i\|_2^2 = \sum_{i=1}^r \sum_{j=1}^n c_{ij}^2 = \sum_{i=1}^r \mathrm{tr}(c_i c_i^T) = \mathrm{tr}(G).$$

Thereby we see that minimizing the (squared) Euclidean norm of the vector of coefficients of the polynomials appearing in an sos representation of f amounts to minimizing the trace linear form over the Gram spectrahedron of f.

Numerical experiments. In a series of numerical experiments we optimized the trace linear form over Gram spectrahedra of binary forms and recorded the ranks of the optimal solutions. The programming was done with the `Julia` programming language [4]. Modeling was done with the `JuMP` [5] package and the SDPs were solved with `SCS` [6].

5.2.3. For a fixed degree $d \in \{3, 4, 5\}$, we generated 10000 tuples $(p_1, \dots, p_{d+1})$ of polynomials of degree d, where we drew all coefficients of the polynomials uniformly at random from $[-1, 1]$. For each such tuple we minimized the trace linear form over the Gram spectrahedron of $f := p_1^2 + \cdots + p_{d+1}^2$. The distribution of ranks of the optimal solutions found by the solver are recorded in Table 5.1.

In another experiment we also varied the linear form. To be specific, in addition to drawing the coefficients of $(p_1, \dots, p_{d+1})$ uniformly at random from $[-1, 1]$, we also drew the coefficients of the linear form that should be minimized over $\mathrm{Gram}(f)$ uniformly at random from $[-1, 1]$. This was also repeated 10000 times and the resulting distributions of ranks are recorded in Table 5.2.

5.2.4. Nie et al. describe a numerical experiment where they minimized random linear functions over random spectrahedra. The distribution of ranks of the optimal solutions are recorded in [NRS, Table 1]. In their notation, n is the size of the matrices and m is the number of variables. We can compare our findings to their results by looking at the cells with $n = d+1$ and $m = \binom{d}{2}$. For $d = 3$, the distribution of ranks we observed when minimizing the trace (or a random linear form) over Gram spectrahedra is very close to the distribution in the case of general spectrahedra with $(n, m) = (4, 3)$. However, for $d = 4$, we had significantly more optimal solutions of rank 2 than in the general case with $(n, m) = (5, 6)$ where the vast majority of optimal points seems to have rank 3. This reconfirms that Gram spectrahedra of binary forms form a special class of spectrahedra.

Table 1 in [NRS] does not contain a cell for $(n, m) = (6, 10)$ anymore, but here already the trace seems to be special compared to a general linear form. While the trace form contains $\frac{2}{3}$ $(= \frac{4}{6})$ of the variables in the cases $d = 3$ and $d = 4$, this number drops to six out of ten when $d = 5$.

5.2.5 Example. We give an example where the trace is minimized by a rank-three tensor. Consider

$$f = x^6 + 6x^5 + 15x^4 - 20x^3 - 101x^2 - 26x + 845.$$

This example was constructed such that the roots of f are $-3 \pm 2i$, $-2 \pm 3i$ and $2 \pm i$. The four points in $\mathrm{Ex}_2(f)$ correspond to the following representations of f as a sum of two squares:

$$\begin{aligned} f &= (x^3 + 3x^2 - 15x - 13)^2 + (-6x^2 - 8x + 26)^2 \\ &= (x^3 + 3x^2 + x - 29)^2 \quad + (-2x^2 - 8x - 2)^2 \\ &= (x^3 + 3x^2 - 5x + 13)^2 \quad + (-4x^2 + 2x + 26)^2 \\ &= (x^3 + 3x^2 + 3x - 19)^2 \quad + (2x + 22)^2. \end{aligned}$$

The trace (or squared norm of the vector of coefficients) is maximized by the Gram tensor corresponding to the first representation. This tensor has trace 1180. The traces of the other rank-two extreme points are 924, 900 and 868, respectively. However, the minimum value of the trace form on $\mathrm{Gram}(f)$ is 860 and it is attained in the rank-three extreme point corresponding to the representation

$$f = (x^3 + 3x^2 + x - 13)^2 + (2x^2 - 6)^2 + (8\sqrt{10})^2.$$

Let us round off this section with a philosophical thought. As was mentioned above, Gram spectrahedra of binary forms are certainly very special objects among the class of spectrahedra. Only the particular structure of a nonnegative binary form allows us to prove properties of this class of spectrahedra which are certainly not shared by every arbitrary spectrahedron. Just think of the many polyhedral faces of large dimension that we found in Section 3.5, for example. However, this perception changes when we extend our class of polynomials, as we will see in the next section.

5.3. Universality of Gram spectrahedra

We widen our class of polynomials to quadratic forms in finitely generated graded $\mathbb{R}$-algebras. The resulting Gram spectrahedra essentially range over the entirety of all spectrahedra. In fact, every spectrahedron that does not contain an affine line is (linearly isomorphic to) the Gram spectrahedron of a quadratic form in some quotient ring of the type $\mathbb{R}[x_1, \ldots, x_n]/I$ where $I \subseteq \mathbb{R}[\underline{x}]$ is a homogeneous ideal:

5.3.1 Theorem. *Let $n \in \mathbb{N}$ and let $L \subseteq \mathrm{Sym}_n(\mathbb{R})$ be an affine-linear space. Let $S = L \cap \mathrm{Sym}_n^+(\mathbb{R})$ be a spectrahedron. Then there is a finitely generated graded $\mathbb{R}$-algebra R and a quadratic form $f \in R_2$ such that S is the Gram spectrahedron of f (with respect to R_1).*

Proof. If L is m-dimensional, we can write

$$L = A_0 + \mathrm{span}(A_1, \ldots, A_m)$$

where A_0 is an arbitrary point in L and $(A_1, \dots, A_m)$ is a basis of the subspace associated to L. Using this notation, we have

$$S = \left\{ A \in \mathrm{Sym}_n^+(\mathbb{R}) \mid \exists\, \lambda \in \mathbb{R}^m : A = A_0 + \sum_{i=1}^{m} \lambda_i A_i \right\}.$$

We start with the polynomial ring $\mathbb{R}[x_1, \dots, x_n]$ and write $x = (x_1, \dots, x_n)^T$. For each $i = 1, \dots, m$, the symmetric real matrix A_i corresponds to a quadratic form $q_i \in \mathbb{R}[x_1, \dots, x_n]_2$ given by $q_i(x_1, \dots, x_n) = x^T A_i x$. Let $I := \langle q_1, \dots, q_m \rangle \subseteq \mathbb{R}[\underline{x}]$ be the (homogeneous) ideal generated by these quadratic forms. Consider the graded $\mathbb{R}$-algebra $R := \mathbb{R}[\underline{x}]/I$ and the quadratic form $f := \overline{x^T A_0 x} \in R_2$. (The bar here denotes the residue class modulo I.) Since $I_1 = \{0\}$, we have $R_1 \cong \mathbb{R}[\underline{x}]$ and $\dim R_1 = n$. The Gram spectrahedron of f with respect to R_1 is

$$\mathrm{Gram}_{R_1}(f) = \left\{ A \in \mathrm{Sym}_n^+(\mathbb{R}) : f = \overline{x^T A x} \right\}.$$

We claim that $S = \mathrm{Gram}_{R_1}(f)$. The inclusion $S \subseteq \mathrm{Gram}_{R_1}(f)$ is clear since by construction $\overline{x^T B x} = 0$ in R_2 for every $B \in \mathrm{span}(A_1, \dots, A_m)$. In order to show the other inclusion, we start with a matrix $A \in \mathrm{Sym}_n^+(\mathbb{R})$ that satisfies $f = \overline{x^T A x}$. This means $x^T A_0 x - x^T A x \in I_2$. Since I is generated by the quadratic forms $q_1, \dots, q_m$, we get

$$x^T (A_0 - A) x = \sum_{i=1}^{m} \mu_i (x^T A_i x)$$

for some $\mu \in \mathbb{R}^m$. Rearranging gives

$$x^T \left(\left(A_0 - \sum_{i=1}^{m} \mu_i A_i \right) - A \right) x = 0$$

and hence $A = A_0 - \sum_{i=1}^{m} \mu_i A_i \in S$. □

5.3.2 Remark. If S contains a positive definite matrix and we allow coordinate changes, one can achieve $A_0 = I_n$. Thus, one could reduce to $f = \overline{x_1^2 + \cdots + x_n^2} \in R_2$.

We consider some examples of familiar spectrahedra.

5.3.3 Example. Let $L = A_0 + \mathrm{span}(A_1, A_2, A_3) \subseteq \mathrm{Sym}_4(\mathbb{R})$ with

$$A_0 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 5 & -2 & 0 \\ 0 & -2 & 5 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The matrices in L are of the form

$$A = \begin{pmatrix} 1 & -1 & \lambda_1 & \lambda_2 \\ -1 & 5-2\lambda_1 & -2-\lambda_2 & \lambda_3 \\ \lambda_1 & -2-\lambda_2 & 5-2\lambda_3 & -1 \\ \lambda_2 & \lambda_3 & -1 & 1 \end{pmatrix}$$

with $\lambda \in \mathbb{R}^3$. We immediately recognize $S = L \cap \mathrm{Sym}_4^+(\mathbb{R})$ as the Gram spectrahedron of the binary form

$$g = x^6 - 2x^5y + 5x^4y^2 - 4x^3y^3 + 5x^2y^4 - 2xy^5 + y^6.$$

Nevertheless, we want to illustrate the construction in the proof of Theorem 5.3.1. The quadratic forms in $\mathbb{R}[x_0, \ldots, x_3]$ associated to the matrices A_1, A_2 and A_3 are

$$q_1 = 2x_0x_2 - 2x_1^2, \quad q_2 = 2x_0x_3 - 2x_1x_2, \quad q_3 = 2x_1x_3 - 2x_2^2.$$

Therefore, $I = \langle x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2 \rangle \subseteq \mathbb{R}[x_0, \ldots, x_3]$ and we recognize I as the homogeneous vanishing ideal of the rational normal curve $X = v_3(\mathbb{P}^1) \subseteq \mathbb{P}^3$. The quadratic form f we defined in the proof is the residue class of

$$x_0^2 - 2x_0x_1 + 5x_1^2 - 4x_1x_2 + 5x_2^2 - 2x_2x_3 + x_3^2$$

in $R := R[\underline{x}]/I = \mathbb{R}[X]$, the homogeneous coordinate ring of X. As was not to be expected otherwise, we have realized the given Gram spectrahedron of a binary sextic as the Gram spectrahedron of a quadratic form on the rational normal curve of degree 3.

5.3.4 Example. We consider the square $S = [-1,1]^2 \subseteq \mathbb{R}^2$, which is a spectrahedron in $\mathbb{R}^2$ given as the set of all $\lambda \in \mathbb{R}^2$ such that

$$\begin{pmatrix} 1-\lambda_1 & 0 & 0 & 0 \\ 0 & 1+\lambda_1 & 0 & 0 \\ 0 & 0 & 1-\lambda_2 & 0 \\ 0 & 0 & 0 & 1+\lambda_2 \end{pmatrix} \succeq 0.$$

We identify S with $L \cap \mathrm{Sym}_4^+(\mathbb{R})$ for $L = A_0 + \mathrm{span}(A_1, A_2)$, where A_0 is the identity matrix, $A_1 = \mathrm{diag}(-1,1,0,0)$ and $A_2 = \mathrm{diag}(0,0,-1,1)$. The resulting $\mathbb{R}$-algebra is $R = \mathbb{R}[x_1, \ldots, x_4]/\langle x_1^2 - x_2^2, x_3^2 - x_4^2 \rangle$ and we get $S = \mathrm{Gram}_{R_1}(f)$ for

$$f = \overline{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \overline{2(x_1^2 + x_3^2)}.$$

S has four extreme points of rank 2, namely

$$\mathrm{diag}(2,0,2,0),\ \mathrm{diag}(2,0,0,2),\ \mathrm{diag}(0,2,2,0),\ \mathrm{diag}(0,2,0,2),$$

and these four points correspond to the representations

$$\begin{aligned} &(\sqrt{2}x_1)^2 + (\sqrt{2}x_3)^2 \equiv (\sqrt{2}x_1)^2 + (\sqrt{2}x_4)^2 \\ &\qquad \equiv (\sqrt{2}x_2)^2 + (\sqrt{2}x_3)^2 \equiv (\sqrt{2}x_2)^2 + (\sqrt{2}x_4)^2 \pmod{\langle x_1^2 - x_2^2, x_3^2 - x_4^2 \rangle} \end{aligned}$$

of f as a sum of two squares.

5.3.5 Example. Let $n \in \mathbb{N}$. Let S be the elliptope $\mathcal{E}_{n \times n}$, that is

$$S = \{A \in \mathrm{Sym}_n(\mathbb{R}) : A \succeq 0,\ a_{ii} = 1 \text{ for all } i = 1, \ldots, n\}.$$

For i, $j \in \{1, \dots, n\}$, we denote by E_{ij} the matrix which has a 1 at position (i, j) and zeros everywhere else. Then $S = L \cap \mathrm{Sym}_n^+(\mathbb{R})$ for

$$L = I_n + \mathrm{span}(E_{ij} + E_{ji} : 1 \leq i < j \leq n).$$

We obtain $R = \mathbb{R}[x_1, \dots, x_n]/I$ where $I = \langle x_i x_j : 1 \leq i < j \leq n \rangle$. Note that

$$X := \mathcal{V}_+(I) = \{[e_i] : 1 \leq i \leq n\} \subseteq \mathbb{P}^{n-1},$$

i.e., the projective variety X defined by I is a set of n points in $\mathbb{P}^{n-1}$. Since I is generated by squarefree monomials, it is radical ([HH, Corollary 1.2.5]). Therefore, R is also the homogeneous coordinate ring of X. By the proof of Theorem 5.3.1, the elliptope is the Gram spectrahedron of the quadratic form

$$f = \sum_{i=1}^{n} \overline{x_i}^2 \in R_2.$$

It is well-known (see [LP95, Theorem 2.5]) that $\mathcal{E}_{n \times n}$ has precisely 2^{n-1} extreme points of rank one. Having identified the elliptope as the Gram spectrahedron of f, we also see that these points are precisely the rank-one tensors

$$(x_1 \pm \cdots \pm x_n) \otimes (x_1 \pm \cdots \pm x_n) \in \mathrm{Gram}_{R_1}(f).$$

5.3.6 Example (cf. Example 2.2.11)**.** In order to also give an explicit example of an unbounded (Gram) spectrahedron, we consider

$$S = \left\{ A \in \mathrm{Sym}_2^+(\mathbb{R}) \mid \exists\, \lambda \in \mathbb{R} : A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Obviously, S is linearly isomorphic to the half-line $\mathbb{R}_{\geq 0} = [0, \infty)$. Our construction from Theorem 5.3.1 shows that S is the Gram spectrahedron of $f := \overline{x^2} \in R_2$ relative to $V := R_1$, where $R = \mathbb{R}[x, y]/\langle x^2 + y^2 \rangle$. The unique extreme point of $\mathrm{Gram}_V(f)$ is the rank-one tensor $x \otimes x$ and the other points are the rank-two tensors $(1 + \lambda) x \otimes x + \lambda y \otimes y \in \mathsf{S}_2^+ V$ for $\lambda > 0$.

CHAPTER 6

Gram spectrahedra in the context of varieties of almost minimal degree

Having studied quadratic forms on varieties of minimal degree in Chapter 4, the next natural step is to proceed to varieties of almost minimal degree. For a quadratic form, being nonnegative is here no longer sufficient for being a sum of squares. However, if $f \in \mathbb{R}[X]_2$ is a sum of squares in the homogeneous coordinate ring of a real projective variety X of almost minimal degree, one can still ask for its length. This length is determined in [CPSV] and the result is pretty much in the spirit of the corresponding theorem for varieties of minimal degree (see Theorem 4.1.5 taken from [BPSV]). As it gives the minimum rank of any point in the Gram spectrahedron of f, we will quote the precise result below (Theorem 6.1.1). It includes two prominent special cases that have already been studied before by Scheiderer, namely ternary sextics as well as quartics in four variables. One can also consider these cases in a toric framework. In Sections 6.2 and 6.3 we identify the embedded projective toric varieties X_P that meet the prerequisites of Theorem 6.1.1. Afterwards, we study Gram spectrahedra of quadratic forms on these varieties and the dimensions of their faces.

The main result is Theorem 6.4.8 which generalizes the inequalities for the dimension of a maximal proper face from the case of varieties of minimal degree (whose quadratic deficiency is 0) to the case of our toric varieties X_P whose quadratic deficiency equals 1. In order to prove the said theorem, we analyze the combinatorics of the underlying polytope P and thus distribute the proof over Sections 6.5 to 6.7 according to the dimension of P.

6.1. Sums of squares on varieties of almost minimal degree

Studying Gram spectrahedra on varieties of almost minimal degree is mainly motivated by the following theorem due to Chua, Plaumann, Sinn and Vinzant. It can be considered the next step after Theorem 4.1.5 that dealt with varieties of minimal degree whose quadratic deficiency is zero. Their theorem is formulated for nondegenerate varieties with quadratic deficiency $\varepsilon_2(X) = 1$. By [Zak99, Proposition 5.10], any such variety is a hypersurface of degree at least 3 or a variety of almost minimal degree. Originally, Zak claimed that the converse was also true. In 2005, he revoked his proof in an annotated version of the forecited article which is available through his institutional website [Zak05]. Using results by Han and Kwak [HK] from 2015, in Section 6.2 we will show that a nondegenerate variety $X \subseteq \mathbb{P}^n$ with $\operatorname{codim}(X) \geq 2$ which is aCM and of almost minimal degree fulfills $\varepsilon_2(X) = 1$. In this sense, 6.1.1 is thus also a theorem on varieties of almost minimal degree.

6.1.1 Theorem ([CPSV, Thm. 3.5]). *Let $X \subseteq \mathbb{P}^n$ be an irreducible nondegenerate real projective variety with Zariski-dense real points. If X is arithmetically Cohen-Macaulay and the quadratic deficiency $\varepsilon_2(X)$ equals 1, then every sum of squares* *in* $\mathbb{R}[X]_2$ *is a sum of* $\dim(X) + 2$ *squares. This is the smallest rank in the Pataki interval.*

Using this Theorem one gets a different proof (cf. [CPSV, Corollary 3.6]) for the following results which are originally due to Scheiderer ([Sch17, Theorems 4.1 and 4.2]).

6.1.2 Corollary. *(i) Every ternary sextic that is a sum of squares is a sum of 4 squares.*
(ii) Every quartic in four variables that is a sum of squares is a sum of 5 squares.

A ternary sextic corresponds to a quadratic form in the homogeneous coordinate ring $\mathbb{R}[v_3(\mathbb{P}^2)]$ of the cubic Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^9$. A quartic in four variables corresponds to a quadratic form in $\mathbb{R}[v_2(\mathbb{P}^3)]$, the coordinate ring of the quadratic Veronese embedding of $\mathbb{P}^3$ in $\mathbb{P}^9$. One easily checks that these varieties have quadratic deficiency 1. That Veronese varieties are arithmetically Cohen-Macaulay is well-known and can for example be seen by writing the coordinate rings as rings of invariants under a suitable group action and then using the Theorem of Hochster and Eagon (see [HE, Proposition 13]) from the invariant theory of finite groups.

We refrain from giving a rigorous proof and emphasize the toric point of view instead. Fix $m \in \mathbb{N}$ and consider the m-dimensional unit simplex

$$S_m := \operatorname{conv}(0, e_1, \ldots, e_m) \subseteq \mathbb{R}^m.$$

The polytope S_m is normal and thus very ample. The lattice points of kS_m in $\mathbb{Z}^m$ correspond to the exponents of the $s_k = \binom{m+k}{k}$ monomials in $\mathbb{R}[x_1, \ldots, x_m]$ of total degree at most k. Consequently, for every $k \in \mathbb{N}$, we obtain an embedding $X_{kS_m} \subseteq \mathbb{P}^{s_k - 1}$, the k-th Veronese embedding of $\mathbb{P}^m$. Therefore,

$$v_3(\mathbb{P}^2) = X_{(3S_2)\cap\mathbb{Z}^2} \quad \text{and} \quad v_2(\mathbb{P}^3) = X_{(2S_3)\cap\mathbb{Z}^3}.$$

After a suitable translation, the lattice polytope $3S_2$ is a reflexive polygon (of type 9 in Figure 6.2) and $2S_3$ is a so-called Gorenstein polytope of degree 2 as well. (We will call it P_{15} later on.) We are going to see in Section 6.3 that the toric varieties corresponding to this kind of polytopes are aCM (Lemma 6.3.5) and have quadratic deficiency equal to 1 (Lemma 6.3.6). In particular, $v_3(\mathbb{P}^2)$ and $v_2(\mathbb{P}^3)$ meet the prerequisites of Theorem 6.1.1.

Toric varieties are intimately linked to certain lattice polytopes and quadratic forms on those varieties can be interpreted as polynomials with prescribed Newton polytope. Now that we have seen two such varieties that give instances of Theorem 6.1.1 with particularly appealing interpretations, it is natural to ask for all (embedded projective) toric varieties satisfying the same conditions. This question is also discussed in [CPSV, Remark 3.7], where the authors sketch how to obtain the polytopes defining such varieties. The following two sections are devoted to presenting this road in more detail. Afterwards, we will examine Gram spectrahedra of quadratic forms on these varieties.

6.2. From del Pezzo varieties to Gorenstein polytopes

Let X_P be the projective toric variety embedded with respect to the lattice points of a very ample lattice polytope $P \subseteq M_{\mathbb{R}}$. As indicated above, we want to understand when X_P satisfies the conditions of Theorem 6.1.1. To this end we assume that the variety X_P is arithmetically Cohen-Macaulay and of almost minimal degree, that is to say a (maximal) del Pezzo variety. We study its anticanonical divisor and a polyhedron associated to it. This will allow us to infer some properties of P and leads to the class of Gorenstein polytopes. Most of the contents of this and of the following section are also included in the author's unpublished master's thesis [May17].

Let $P \subseteq M_{\mathbb{R}}$ be a full-dimensional lattice polytope. In the theory of toric varieties one associates a divisor $\mathbf{D}_P$ on X_P to the polytope P. Conversely, if one starts with a T_N-invariant divisor D on a normal toric variety X_Σ, then one can construct a polyhedron $\mathbf{P}_D$ associated to D. Since these divisors and polyhedra play an important role in the study of del Pezzo varieties, we recall their constructions from [CLS, §4.2 and §4.3]. Beforehand, we also need the prime divisors discussed in [CLS, §4.1]. The following paragraphs are adopted almost verbatim from the aforementioned sections in the textbook.

6.2.1. Let X_Σ be the toric variety of a fan Σ in $N_{\mathbb{R}}$ and let $\dim N_{\mathbb{R}} = n$. By the Orbit-Cone Correspondence from [CLS, Theorem 3.2.6], k-dimensional cones $\sigma \in \Sigma$ correspond to $(n-k)$-dimensional T_N-orbits in X_Σ. We denote the set of one-dimensional cones (i.e., the rays) in Σ by $\Sigma(1)$. A ray $\rho \in \Sigma(1)$ thus gives the orbit $O(\rho)$ of codimension 1. Again by the Orbit-Cone Correspondence, the closure $\overline{O(\rho)}$ of this orbit is a union of T_N-orbits and therefore invariant under the T_N-action. Consequently, $\overline{O(\rho)}$ is a T_N-invariant prime divisor on X_Σ. We denote it by D_ρ to emphasize its nature as a divisor.

6.2.2. Let $P \subseteq M_{\mathbb{R}} \cong \mathbb{R}^n$ be a full-dimensional lattice polytope with facet presentation

$$P = \{m \in M_{\mathbb{R}} : \langle m, u_F \rangle \geq -a_F \text{ for all facets } F \prec P\},$$

where, as usual, $a_F \in \mathbb{Z}$ and $u_F \in N$ denotes the inward-pointing facet normal that is the minimal generator of the ray $\rho_F = \operatorname{cone}(u_F)$. The normal fan Σ_P consists of the cones σ_Q indexed by faces $Q \subseteq P$, where $\sigma_Q = \operatorname{cone}(u_F : Q \subseteq F)$, see 1.3.8. By [CLS, Proposition 2.3.8], the fan Σ_P is *complete*, i.e., $\bigcup_{\sigma\in\Sigma} \sigma = N_{\mathbb{R}}$. Furthermore, the vertices of P correspond to the maximal cones in $\Sigma_P(n)$ and the facets of P correspond to the rays in $\Sigma_P(1)$. The ray generators of the normal fan Σ_P are the facet normals u_F. The corresponding prime divisors in X_P from above will be denoted by D_F. Everything is now indexed by facets F of P and we can define

$$\mathbf{D}_P := \sum_F a_F D_F,$$

the *divisor associated to* P. As was mentioned before, the divisors D_F are invariant under the action of T_N so that also $\mathbf{D}_P$ is a T_N-invariant divisor.

If P is very ample, then the divisor $\mathbf{D}_P$ is very ample by [CLS, Proposition 6.1.10] and thus gives a projective embedding $i\colon X_P \hookrightarrow \mathbb{P}^{s-1}$ where $s = |P \cap M|$. Theorem II.7.1 in [Hart] shows that $\mathcal{O}_{X_P}(\mathbf{D}_P)$ is the restriction $\mathcal{O}_{X_P}(1) := i^*\mathcal{O}_{\mathbb{P}^{s-1}}(1)$ of $\mathcal{O}_{\mathbb{P}^{s-1}}(1)$ to X_P.

6.2.3. Conversely, we now start with a fan Σ and a T_N-invariant divisor $D = \sum_\rho a_\rho D_\rho$ on the normal toric variety X_Σ with the aim of defining a polyhedron $\mathbf{P}_D$. By [CLS, Proposition 4.1.2], for every $m \in M$, the character χ^m is a rational function on X_Σ and $\operatorname{div}(\chi^m) + D \geq 0$ is equivalent to

$$\langle m, u_\rho \rangle + a_\rho \geq 0 \text{ for all } \rho \in \Sigma(1),$$

where $u_\rho \in \rho \cap N$ is a minimal generator of the ray $\rho \in \Sigma(1)$. We use these inequalities to define the polyhedron

$$\mathbf{P}_D := \{m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\}.$$

Thus, $\mathbf{P}_D \cap M$ consists of all $m \in M$ for which the divisor $D + \operatorname{div}(\chi^m)$ is effective.

For the polyhedra associated to divisors the following calculation rules apply:

6.2.4 Lemma (cf. [CLS, Exercise 4.3.2])**.** *Let $D = \sum_\rho a_\rho D_\rho$ be a T_N-invariant Weil divisor in X_Σ. Let $k > 0$ and $m_0 \in M$. Then*

(i) $\mathbf{P}_{kD} = k\mathbf{P}_D$,
(ii) $\mathbf{P}_{D+\operatorname{div}(\chi^{m_0})} = \mathbf{P}_D - m_0$.

Proof. (i) Given $m \in M$, we have $\operatorname{div}(\chi^m) + kD \geq 0$ if and only if $\langle m, u_\rho \rangle + ka_\rho \geq 0$ for all $\rho \in \Sigma(1)$. Thus,

$$\begin{aligned} \mathbf{P}_{kD} &= \{m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle \geq -ka_\rho \text{ for all } \rho \in \Sigma(1)\} \\ &= \left\{m \in M_{\mathbb{R}} : \left\langle \frac{m}{k}, u_\rho \right\rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\right\} \\ &= \left\{m \in M_{\mathbb{R}} : \frac{m}{k} \in \mathbf{P}_D\right\} = k\mathbf{P}_D. \end{aligned}$$

(ii) Let $D' = D + \operatorname{div}(\chi^{m_0})$. Given $m \in M$, the divisor $D' + \operatorname{div}(\chi^m)$ is effective if and only if $a_\rho + \langle m_0, u_\rho \rangle + \langle m, u_\rho \rangle \geq 0$ for all $\rho \in \Sigma(1)$ (see [CLS, Proposition 4.1.2]). We obtain

$$\begin{aligned} \mathbf{P}_{D'} &= \{m \in M_{\mathbb{R}} : \langle m + m_0, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\} \\ &= \{m - m_0 \in M_{\mathbb{R}} : \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\} \\ &= \mathbf{P}_D - m_0. \end{aligned}$$

□

6.2.5 Remark. Given a full-dimensional lattice polytope $P \subseteq M_{\mathbb{R}}$ with facet presentation $P = \{m \in M_{\mathbb{R}} : \langle m, u_F \rangle \geq -a_F \text{ for all facets } F \prec P\}$, we defined the divisor $\mathbf{D}_P = \sum_F a_F D_F$. For every $m \in M$, the divisor $\mathbf{D}_P + \operatorname{div}(\chi^m)$ is effective if and only if $a_F + \langle m, u_F \rangle \geq 0$ for all facets F of P. According to the definitions we thus have $\mathbf{P}_{\mathbf{D}_P} = P$. In particular, the polyhedron $\mathbf{P}_{\mathbf{D}_P}$ is a polytope in this case.

Starting from a lattice polytope P we obtained a divisor $\mathbf{D}_P$ on the toric variety X_P. There is, however, another important divisor on a (normal) toric variety. Recall that on a normal variety X there is a Weil divisor D such that

$$\omega_X \cong \mathcal{O}_X(D),$$

where ω_X is the canonical sheaf of X. This is for example explained in the discussion preceding Definition 8.0.20 in the textbook [CLS] by Cox, Little and Schenck and in Theorem 8.0.4 therein. A Weil divisor D as above is called a *canonical divisor* of X and is often denoted by K_X. Note that the canonical divisor K_X is well-defined up to linear equivalence (see [CLS, Proposition 8.0.7]). The *anticanonical divisor* of X is $-K_X$.

An important aspect of the present section is to analyze the relationship between the anticanonical divisor $-K_{X_P}$ and the divisor $\mathbf{D}_P$ of a toric variety X_P. The following Theorem gives the canonical divisor of a general normal toric variety in terms of its prime divisors.

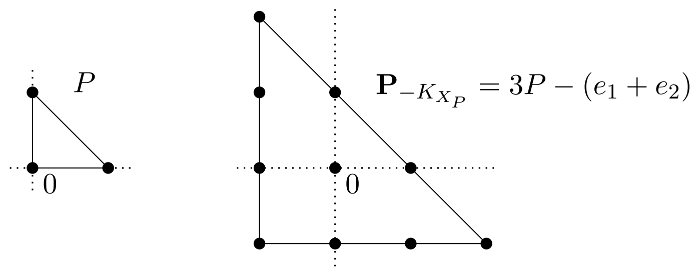


FIGURE 6.1. The polytopes from Example 6.2.7

6.2.6 Theorem ([CLS, Thm. 8.2.3]). *For a toric variety X_Σ, the canonical sheaf ω_{X_Σ} is given by*

$$\omega_{X_\Sigma} \cong \mathcal{O}_{X_\Sigma}\left(-\sum\nolimits_\rho D_\rho\right).$$

Thus $K_{X_\Sigma} = -\sum_\rho D_\rho$ is a torus-invariant canonical divisor on X_Σ. □

6.2.7 Example. Consider the 2-simplex $P = S_2 = \operatorname{conv}(0, e_1, e_2) \subseteq \mathbb{R}^2$. As P is a two-dimensional lattice polytope, it is normal and very ample. X_P is the Zariski closure of the image of

$$\Phi_{P\cap\mathbb{Z}^2} \colon (\mathbb{C}^*)^2 \to \mathbb{P}^2,\ (s,t) \mapsto (1 : s : t).$$

If we use homogeneous coordinates $(x : y : z)$ on $\mathbb{P}^2$, then $X_P = \overline{\mathrm{D}_+(xyz)} = \mathbb{P}^2$.

In its facet presentation, P is given by the inequalities $x_1 \geq 0$, $x_2 \geq 0$ and $-x_1 - x_2 \geq -1$. Thus, the facet normals are $u_{F_1} = e_1$, $u_{F_2} = e_2$ and $u_{F_3} = -e_1 - e_2$. By definition we have $\mathbf{D}_P = 0 \cdot D_{F_1} + 0 \cdot D_{F_2} + 1 \cdot D_{F_3} = D_{F_3}$. On the other hand, according to Theorem 6.2.6, the anticanonical divisor is given by

$$-K_{X_P} = D_{F_1} + D_{F_2} + D_{F_3}.$$

Then $\mathbf{P}_{-K_{X_P}} = \{m \in \mathbb{R}^2 : \langle m, u_{F_i}\rangle \geq -1,\ i = 1,2,3\}$ is a reflexive polytope which is described by the inequalities $x_1 \geq -1$, $x_2 \geq -1$ and $-x_1 - x_2 \geq -1$. We have

$$\mathbf{P}_{-K_{X_P}} = \operatorname{conv}(-e_1 - e_2, 2e_1 - e_2, -e_1 + 2e_2) = 3P - (e_1 + e_2).$$

According to [CLS, Proposition 5.24], for $m = e_1 + e_2$ we have

$$\operatorname{div}(\chi^m) = \sum_{i=1}^{3} \langle e_1 + e_2, u_{F_i}\rangle D_{F_i} = D_{F_1} + D_{F_2} - 2D_{F_3},$$

and therefore $-K_{X_P} = 3\mathbf{D}_P + \operatorname{div}(\chi^m) \sim 3\mathbf{D}_P$.

By definition, two divisors that are linearly equivalent differ only by a principal divisor. In the above example, where we considered the torus-invariant divisors $-K_{X_P}$ and $3\mathbf{D}_P$, it was even possible to take the divisor of a character. The following Lemma generalizes this observation.

6.2.8 Lemma. *Let D, $E \in \operatorname{Div}_{T_N}(X_\Sigma)$ with $D \sim E$. Then there is an $m \in M$ such that $D = E + \operatorname{div}(\chi^m)$.*

Proof. We have $D - E = \operatorname{div}(f) \in \operatorname{Div}_0(X_\Sigma)$ for some $f \in \mathbb{C}(X)^*$. Since $\operatorname{Div}_{T_N}(X_\Sigma)$ is a group, also $\operatorname{div}(f) \in \operatorname{Div}_{T_N}(X_\Sigma)$. According to [CLS, Theorem 4.1.3], there is an exact sequence

$$M \to \operatorname{Div}_{T_N}(X_\Sigma) \to \operatorname{Cl}(X_\Sigma) \to 0$$

where the first map is $m \mapsto \operatorname{div}(\chi^m)$ and the second sends a T_N-invariant divisor to its divisor class in $\operatorname{Cl}(X_\Sigma)$. Being a principal divisor, $\operatorname{div}(f)$ is contained in the kernel of the second and thus in the image of the first map. This gives an $m \in M$ as desired. □

We now turn our attention to the study of nondegenerate projective varieties which are arithmetically Cohen-Macaulay and whose quadratic deficiency equals 1 just like in Theorem 6.1.1. We will see that any such variety whose codimension is at least 2 is a variety of almost minimal degree. Varieties of almost minimal degree have been studied extensively by Fujita in the 1980s (e.g. [Fuj, Theorem I]) and in a series of papers by Brodmann and Schenzel. We refer to their article [BS] from 2007 on arithmetic properties of such varieties.

6.2.9 Definition. Let $X \subseteq \mathbb{P}^n$ be a nondegenerate projective variety. Following [BS, Definition 6.3], we call X a *maximal del Pezzo variety* if X is arithmetically Cohen-Macaulay and of almost minimal degree.

6.2.10 Proposition. *Let $X \subseteq \mathbb{P}^n$ be a nondegenerate projective variety with $c :=$ $\operatorname{codim} X \geq 2$. Then $\varepsilon_2(X) = 1$ if and only if X is a maximal del Pezzo variety.*

Proof. By [HK, Theorem 4.3], X is a maximal del Pezzo variety if and only if

$$h^0(\mathbb{P}^n, \mathcal{I}_X(2)) = \binom{c+1}{2} - 1.$$

Write $I = \mathfrak{I}_+(X)$ for the homogeneous vanishing ideal of X. As $H^0(\mathbb{P}^n, \mathcal{I}_X(k)) = I_k$, we have

$$\dim I_2 = \dim H^0(\mathbb{P}^n, \mathcal{I}_X(2)) = h^0(\mathbb{P}^n, \mathcal{I}_X(2)).$$

The assertion now follows directly from

$$\varepsilon_2(X) = \binom{c+1}{2} - \dim I_2.$$ □

6.2.11 Remark/Example. The proposition above only deals with varieties of codimension at least 2. The reason is that the quadratic deficiency of a hypersurface equals 1 as soon as its degree is at least 3. Indeed, let $X \subseteq \mathbb{P}^n$ be a hypersurface of degree ≥ 3. Then X has codimension $c = 1$ and for $I = \mathfrak{I}_+(X)$ we have $I_2 = \{0\}$. Therefore,

$$\varepsilon_2(X) = \binom{c+1}{2} - \dim I_2 = \binom{2}{2} - 0 = 1.$$

The case of hypersurfaces is also not particularly interesting in terms of Gram spectrahedra. If $P \subseteq \mathrm{M}_\mathbb{R} \cong \mathbb{R}^m$ is a full-dimensional very ample lattice polytope with s lattice points and $X_P \subseteq \mathbb{P}^{s-1}$ is a hypersurface, then

$$m = \dim(X_P) = (s-1) - 1.$$

But any Gram tensor of a sum of squares $f \in \mathbb{R}[X_P]_2$ can have rank at most $\dim \mathbb{R}[X_P]_1 = s = m+2$ anyway. This means that Gram spectrahedra are reduced to single points. Based on this observation, we will focus on varieties of almost minimal degree from now on.

6.2.12 Theorem (cf. [BS, Thm. 6.2])**.** *Let $X \subseteq \mathbb{P}^n$ be a normal projective variety of dimension $m > 0$. Let ω_X be the canonical sheaf of X. The following are equivalent:*

(i) X is arithmetically Cohen-Macaulay and of almost minimal degree.
(ii) $\omega_X \cong \mathcal{O}_X(1-m)$ and X is of almost minimal degree.
(iii) $\omega_X \cong \mathcal{O}_X(1-m)$ and X is arithmetically Cohen-Macaulay.

Proof. Since X is a normal projective variety, the canonical sheaf of X coincides with the dualizing sheaf and the claim follows from [BS, Theorem 6.2]. ☐

Using the above theorem in the case of a toric maximal del Pezzo variety X_P is the key ingredient for linking the divisor $\mathbf{D}_P$ to the anticanonical divisor $-K_{X_P}$. This will eventually lead us to the class of Gorenstein polytopes of degree 2 whose definition we give right now.

6.2.13 Definition ([BN08, Def. 1.2 and 1.5])**.** Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope.

(i) Let $m \in \operatorname{int}(P) \cap M$ be an interior lattice point of P. Then P is *reflexive with respect to m* if $P - m$ is reflexive in the sense of Definition 1.3.9.
(ii) Let $r \in \mathbb{N}$. We say P is *Gorenstein of index r* if rP is reflexive with respect to some $m \in \operatorname{int}(rP) \cap M$.
(iii) Let P be an n-dimensional Gorenstein polytope of index r. The number $d := n + 1 - r$ is called the *degree* of P.

6.2.14 Remark. The notion of degree has already been defined for arbitrary lattice polytopes using the Ehrhart series and the h^*-polynomial (see Remark 4.3.1). Proposition 1.14 in [BJ] shows that an n-dimensional Gorenstein polytope of index r indeed has degree $n + 1 - r$ according to this definition.

We will only be interested in n-dimensional Gorenstein polytopes of index $n-1$. As their degree is always 2, we often simply refer to them as Gorenstein polytopes of degree 2 without specifying the dimension.

6.2.15 Theorem. *Let $P \subseteq \mathbb{R}^n$ be a full-dimensional very ample lattice polytope. Let X_P be the projective toric variety embedded with respect to the lattice points of P. If X_P is arithmetically Cohen-Macaulay and of almost minimal degree, then P is a Gorenstein polytope of index $n-1$, or, equivalently, of degree 2.*

Proof. According to [CLS, Theorem 2.4.1], the toric variety X_P is normal. We have $\dim(X_P) = \dim(P) = n$. By assumption, X_P is arithmetically Cohen-Macaulay and of almost minimal degree so that $\omega_X \cong \mathcal{O}_X(1-n)$ by Theorem 6.2.12. This means that the anticanonical divisor satisfies $[-K_{X_P}] = [(n-1)\mathbf{D}_P]$. Proposition 4.2.10 in [CLS] shows that the divisor $\mathbf{D}_P$ is Cartier. The fact that the Cartier divisors form a subgroup of $\operatorname{Div}(X_P)$ implies that also $(n-1)\mathbf{D}_P$ is Cartier. Furthermore, this divisor is (very) ample by [CLS, Proposition 6.1.10]. To sum up, X_P is a complete (since projective) normal variety whose anticanonical divisor is Cartier and ample. Thus, X_P is a so-called *Gorenstein Fano variety* and the polyhedron $\mathbf{P}_{-K_{X_P}}$ associated to its anticanonical divisor is a reflexive lattice polytope ([CLS, Theorem 8.3.4]).

The two divisors $-K_{X_P}$ and $(n-1)\mathbf{D}_P$ are torus-invariant and linearly equivalent. Therefore, according to Lemma 6.2.8, there is an $m \in M$ such that

$$-K_{X_P} = (n-1)\mathbf{D}_P + \operatorname{div}(\chi^m).$$

Using the calculation rules from Lemma 6.2.4 and Remark 6.2.5, we obtain

$$\mathbf{P}_{-K_{X_P}} = \mathbf{P}_{(n-1)\mathbf{D}_P + \operatorname{div}(\chi^m)} = \mathbf{P}_{(n-1)\mathbf{D}_P} - m = (n-1)\mathbf{P}_{\mathbf{D}_P} - m = (n-1)P - m.$$

Hence, $(n-1)P$ is reflexive with respect to m. In other words, P is Gorenstein of index $n-1$. □

6.3. From Gorenstein polytopes to del Pezzo varieties

Due to Theorem 6.2.15, we are interested in Gorenstein polytopes of degree 2. These polytopes have been classified by Batyrev and Juny in [BJ]. Disregarding pyramids (which correspond to algebraic cones), there are – up to lattice equivalence – only 37 Gorenstein polytopes of degree 2 and their dimension is at most 5. A complete list can be found in [BJ] subsequent to Theorem 4.13. In the present section we show that for precisely 36 of these 37 polytopes, the associated toric variety X_P is indeed aCM in the embedding given by the lattice points of P, has quadratic deficiency 1 and consequently satisfies the hypotheses of Theorem 6.1.1.

6.3.1 Notation. By $\mathcal{GP}$ we denote a complete set of representatives for the Gorenstein polytopes of degree 2 which are not pyramids. The set $\mathcal{GP}$ thus consists of one representative for each of the 37 equivalence classes of polytopes from [BJ, Theorem 4.13]. Following the notation of Batyrev and Juny, we refer to the three-dimensional representatives as $P_1, \ldots, P_{15}$, the four-dimensional ones are called $Q_1, \ldots, Q_5$ and the only five-dimensional one is denoted by R_1.

Obviously, the two-dimensional Gorenstein polytopes of degree 2 are Gorenstein of index 1 and thus correspond to the reflexive lattice polygons depicted in Figure 6.2. Their nomenclature is adopted from [CLS, §8.3].

6.3.2 Lemma. *Every $P \in \mathcal{GP} \setminus \{P_1\}$ is normal.*

Proof. For $\dim P = 2$ the assertion follows from Theorem 1.3.4. For the remaining polytopes in $\mathcal{GP} \setminus \{P_1\}$ we checked the normality using `Normaliz` [1] (cf. Remark 1.3.5). □

In fact, P_1 is not normal as the following example shows.

6.3.3 Example. We consider the 3-simplex $P := \operatorname{conv}(0, v_1, v_2, v_3) \subseteq \mathbb{R}^3$ with $v_1 = e_1 + e_2$, $v_2 = e_1 + e_3$ and $v_3 = e_2 + e_3$. The lattice polytope P is a regular tetrahedron. We have

$$v := e_1 + e_2 + e_3 = \frac{1}{4}(2 \cdot 0 + 2 \cdot v_1 + 2 \cdot v_2 + 2 \cdot v_3) \in 2P,$$

but v is not a sum of two lattice points of P. Thus, P is not normal.

Actually, P is not very ample either. The semigroup S associated to the vertex 0 of P is generated by $P \cap \mathbb{Z}^3$. This semigroup is not saturated in $\mathbb{Z}^3$. Indeed, we have

$$2v = 2(e_1 + e_2 + e_3) = v_1 + v_2 + v_3 \in S,$$

but v is not contained in S.

On the other hand, v is an interior point of $2P$ and the polytope

$$2P - v = \operatorname{conv}(-e_1 - e_2 - e_3, e_1 + e_2 - e_3, e_1 - e_2 + e_3, -e_1 + e_2 + e_3)$$

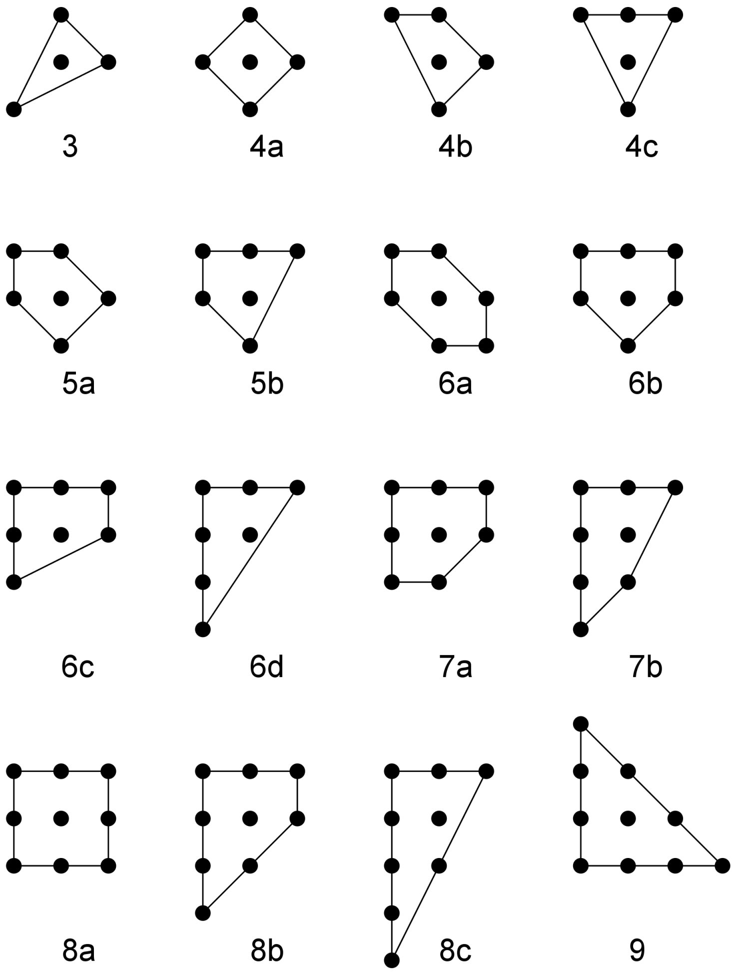


FIGURE 6.2. The 16 equivalence classes of reflexive lattice polygons in $\mathbb{R}^2$.

is given by the inequalities

$$\begin{aligned} -x_1 - x_2 - x_3 &\geq -1, \\ -x_1 + x_2 + x_3 &\geq -1, \\ x_1 - x_2 + x_3 &\geq -1, \\ x_1 + x_2 - x_3 &\geq -1. \end{aligned}$$

Consequently, it is reflexive (see Remark 1.3.11). This means that P is a possible realization of the degree-2 Gorenstein polytope P_1.

6.3.4 Proposition. *Let $P \in \mathcal{GP} \setminus \{P_1\}$ and let X_P be embedded using the lattice points of P. In this embedding X_P is a variety of almost minimal degree.*

Proof. Let $m = \dim(P)$. Batyrev and Juny give a formula ([BJ, Proposition 3.1]) for the volume of an m-dimensional Gorenstein polytope of degree 2:

$$\mathrm{Vol}(P) = |P \cap M| - m + 1.$$

By Lemma 6.3.2, P is normal. According to Corollary 4.3.4, we thus have $\deg X_P = \operatorname{codim} X_P + 2$. In other words, X_P is a variety of almost minimal degree. □

Our next goal is to show that the variety X_P satisfies $\varepsilon_2(X_P) = 1$.

6.3.5 Lemma (cf. [CLS, Exercise 9.2.8])**.** *Let $P \subseteq M_{\mathbb{R}}$ be a full-dimensional normal lattice polytope. The lattice points of P give a projective embedding of the toric variety X_P. In this embedding X_P is arithmetically Cohen-Macaulay.*

Proof. According to [CLS, Theorem 2.4.1], X_P is projectively normal in this embedding. This means that the affine cone Y of X_P is normal. Now Y is a normal (affine) toric variety (defined by $(P \cap M) \times \{1\}$, see the discussion following (2.1.4) in [CLS]) and is thus Cohen-Macaulay by [CLS, Theorem 9.2.9]. But this means precisely that X_P is arithmetically Cohen-Macaulay. □

6.3.6 Lemma. *Let $P \in \mathcal{GP} \setminus \{P_1\}$ and let X_P be embedded using the lattice points of P. Then $\varepsilon_2(X_P) = 1$.*

Proof. By Proposition 6.3.4, X_P is a variety of almost minimal degree. Using the normality of P, we see that X_P is arithmetically Cohen-Macaulay (Lemma 6.3.5) and thus a maximal del Pezzo variety. If $\operatorname{codim} X_P \geq 2$, then $\varepsilon_2(X_P) = 1$ follows from Proposition 6.2.10. We write $m = \dim P$ and $n = |P \cap M| - 1$. Then $\operatorname{codim} X_P = n - m$ and therefore $\operatorname{codim} X_P \geq 2$ if and only if $|P \cap M| \geq m + 3$. So we only have to check the polytopes that have fewer lattice points. These are P_2, Q_1 and the first polytope from our list of two-dimensional reflexive polytopes. If P is one of these three polytopes, then $|P \cap M| = m + 2$. This means $\operatorname{codim} X_P = 1$ and $\deg X_P = \operatorname{codim} X_P + 2 = 3$, so that X_P is a hypersurface of degree 3 and thus satisfies $\varepsilon_2(X_P) = 1$, see Example 6.2.11. □

Now that we have collected all the premises of Theorem 6.1.1 for the embedded toric varieties X_P, we can formulate it for the special case of these varieties.

6.3.7 Theorem. *Let $P \in \mathcal{GP} \setminus \{P_1\}$ and let X_P be embedded using the lattice points of P. Then every sum of squares in $\mathbb{R}[X_P]_2$ is a sum of $\dim(P) + 2$ squares.*

Proof. The variety $X := X_P$ is irreducible since it is toric. The discussion ahead of Proposition 1.4.9 shows that $X(\mathbb{R}) \subseteq X$ is Zariski-dense. Furthermore, X is nondegenerate by Proposition 1.4.8. By Lemma 6.3.2, P is normal so that X is aCM according to Lemma 6.3.5. Besides, we have $\varepsilon_2(X_P) = 1$ by Lemma 6.3.6. Since $\dim(X_P) = \dim(P)$, the claim now follows from Theorem 6.1.1. □

6.4. Gram spectrahedra on toric varieties of almost minimal degree

Let $P \subseteq \mathbb{R}^m$ be a normal Gorenstein lattice polytope of degree 2 which is not a pyramid. Up to an affine-linear isomorphism of the lattice, P is either one of the 16 reflexive polygons or one of the three-dimensional polytopes $P_2, \ldots, P_{15}$ or one of the four-dimensional polytopes $Q_1, \ldots, Q_5$ or the five-dimensional polytope R_1. Let $s := |P \cap \mathbb{Z}^m|$ be the number of lattice points. The associated projective toric variety is $X_P \subseteq \mathbb{P}^n$ $(n := s - 1)$. Let $R := \mathbb{R}[X_P]$ be its homogeneous coordinate ring. If $f \in R_2$ is sos, then f is a sum of $m + 2$ squares (Theorem 6.3.7). We want to analyze the Gram spectrahedron of f, relative to $V := R_1$.

We can also always interpret the results in terms of polynomials with certain Newton polytopes. Assume that $Q \subseteq \mathbb{R}^m$ is any lattice polytope with vertices in $\mathbb{N}_0^m$

that can be obtained from P via an affine-linear isomorphism of the lattice $\mathbb{Z}^m$. Then linear forms in R correspond to polynomials in $\mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \ldots, x_m]$ whose Newton polytope is contained in Q and $f \in R_2$ corresponds to a polynomial in $\mathbb{R}[\underline{x}]_{2Q}$ (cf. Section 4.2).

Let us start by discussing the rank-minimal points in $\mathrm{Gram}_V(f)$. Since $m+2$ is the smallest rank in the Pataki interval, this will also be the smallest rank of any tensor in $\mathrm{Gram}_V(f)$ if f is sufficiently general.

6.4.1 Remark. Recall that in the case of varieties of minimal degree, the Gram spectrahedron of a general nonnegative quadratic form contains only finitely many points of minimum rank $\dim(X)+1$, see Corollary 4.1.7. This is no longer true when we consider varieties of almost minimal degree or rather arithmetically Cohen-Macaulay varieties with quadratic deficiency $\varepsilon_2(X) = 1$. In general, we have infinitely many points of minimum rank $\dim(X)+2$ in this case. Indeed, Theorem 6.1.1 shows that only the image of $\phi := \phi_{m+2}$ equals $\Sigma\mathbb{R}[X]_1^2$, while that of ϕ_{m+1} is strictly smaller (cf. (4.1.1) for the definition of these maps). For generic $f \in \Sigma\mathbb{R}[X]_1^2$, the fiber $\phi^{-1}(f)$ thus has dimension

$$(m+2)(n+1) - \dim(\mathbb{R}[X]_2) \stackrel{4.1.2}{=} (n+1) + \binom{m+1}{2} - \varepsilon_2(X).$$

Subtracting $\binom{m+2}{2}$ for the action of the orthogonal group $\mathrm{O}(m+2)$, we arrive at

$$n - m - \varepsilon_2(X) = \mathrm{codim}(X) - 1,$$

so that $\mathrm{Gram}_{\mathbb{R}[X]_1}(f)$ contains infinitely many points of minimum rank as soon as $\mathrm{codim}(X) > 1$.

6.4.2. For easy reference, we restate Lemma 4.1.2 and Remark 4.1.4 in our toric setup. We have $\dim(V) = s = n+1$ and

$$\dim(VV) = (m+1)(n+1) - \binom{m+1}{2} + \varepsilon_2(X_P), \tag{6.4.1}$$

where $\varepsilon_2(X_P) = 1$ is the quadratic deficiency of X_P. When $c := \mathrm{codim}(X_P) = n-m$ denotes the codimension of X_P in $\mathbb{P}^n$ then

$$\dim \mathrm{Gram}_V(f) = \binom{c+1}{2} - \varepsilon_2(X_P) = \binom{n-m+1}{2} - \varepsilon_2(X_P)$$

if f lies in the interior of the sums-of-squares cone ΣV^2 in VV.

As the codimension of X_P determines the (maximum) dimension of Gram spectrahedra of quadratic forms on this variety, we first examine the varieties of small codimension.

Trivial Gram spectrahedra. We first consider the cases where the variety X_P is a hypersurface. In these cases, we have $c = (s-1) - m = 1$ and the defining polytope $P \subseteq \mathbb{R}^m$ has only $s = m+2$ lattice points. Thus, P is either a reflexive polygon of type 3, the three-dimensional polytope P_2 or the four-dimensional polytope Q_1. Note that X_P is a hypersurface of degree 3, as we have seen in the proof of Lemma 6.3.6. The Gram spectrahedron of a form $f \in \Sigma\mathbb{R}[X_P]_1^2$ is a single point which has rank $m+2$ for general f.

Two-dimensional Gram spectrahedra. Now let $\operatorname{codim}(X_P) = 2$. This means that P has $s = m+3$ lattice points. The respective m-dimensional Gorenstein polytopes of degree 2 are the reflexive polygons of type 4a, 4b and 4c as well as the higher-dimensional polytopes $P_3, \dots, P_7$, Q_2, Q_3 and R_1. If f is in the interior of the sums-of-squares cone $\Sigma\mathbb{R}[X_P]_1^2$, then $\dim \operatorname{Gram}_V(f) = \binom{c+1}{2} - \varepsilon_2(X_P) = 2$, where $V = \mathbb{R}[X_P]_1$.

So we are studying two-dimensional Gram spectrahedra here. Substituting $n = s-1 = m+2$ into equation (6.4.1) gives

$$\dim \mathbb{R}[X_P]_2 = \frac{1}{2}(m^2 + 7m + 8) = \binom{m+4}{2} - 2.$$

The rank r of an extreme point of $\operatorname{Gram}_V(f)$ thus has to fulfill $\binom{r+1}{2} \le \binom{m+4}{2} - 2$, that is to say $r \le m+2$. On the other hand, if f is generic, then $r \ge \dim(X_P)+2 = m+2$ by Theorem 6.1.1. Therefore, the Pataki interval consists of only one point, namely $\dim(X_P)+2$. To sum up, in the generic case, $\operatorname{Gram}_V(f)$ has a smooth boundary consisting of (infinitely many) points of rank $\dim(X_P)+2 = \dim(V)-1$, points in the relative interior of $\operatorname{Gram}_V(f)$ have rank $\dim(V)$, and the Carathéodory number of $\operatorname{Gram}_V(f)$ is 2.

It is natural to ask if we can have an edge on the boundary of $\operatorname{Gram}_V(f)$ for some $f \in \Sigma\mathbb{R}[X_P]_1^2$ or if the existence of an edge implies that the spectrahedron is degenerated into this edge.

6.4.3 Example. We consider the two-dimensional lattice polytope P of type 4a. A translation by $(1,1)$ moves P to the nonnegative orthant such that P defines the monomial space $V = \operatorname{span}(x,\, y,\, xy,\, x^2y,\, xy^2) \subseteq \mathbb{R}[x,y]$. Since we are aiming for an edge of rank 4 in a Gram spectrahedron, we have to find a hyperplane U in V which is not quadratically independent. Starting with

$$U = \operatorname{span}(y + \alpha x,\, xy + \beta x,\, x^2y + \gamma x,\, xy^2 + \delta x)$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, a quick Gröbner basis calculation shows that $\alpha = \beta = \gamma = 0$ if U is not quadratically independent. In order to give an explicit example, we let $\delta = 1$. Then $U = \operatorname{span}(y,\, xy,\, x^2y,\, xy^2 + x)$ is a face subspace corresponding to an edge in the Gram spectrahedron of

$$f := y^2 + (xy)^2 + (x^2y)^2 + (xy^2+x)^2 = x^2 + y^2 + 3x^2y^2 + x^4y^2 + x^2y^4.$$

With respect to the basis $(x,\, y,\, xy,\, x^2y,\, xy^2)$ of V, any Gram matrix of f is of the form

$$A(\lambda_1,\lambda_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1-\lambda_1 \\ 0 & 1 & 0 & -\lambda_2 & 0 \\ 0 & 0 & 1+2(\lambda_1+\lambda_2) & 0 & 0 \\ 0 & -\lambda_2 & 0 & 1 & 0 \\ 1-\lambda_1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The determinant of $A(\lambda_1,\lambda_2)$ decomposes into $\lambda_1(\lambda_1-2)(\lambda_2-1)(\lambda_2+1)(2\lambda_1+2\lambda_2+1)$. The Gram spectrahedron $\operatorname{Gram}_V(f)$ is a pentagon and its algebraic boundary is a union of five lines. More precisely, the five extreme points have rank three. With our choice of a basis, their coordinates are $(0,-\frac{1}{2})$, $(\frac{1}{2},-1)$, $(2,-1)$, $(0,1)$ and $(2,1)$. Obviously, $\operatorname{Car}(f) = 3$.

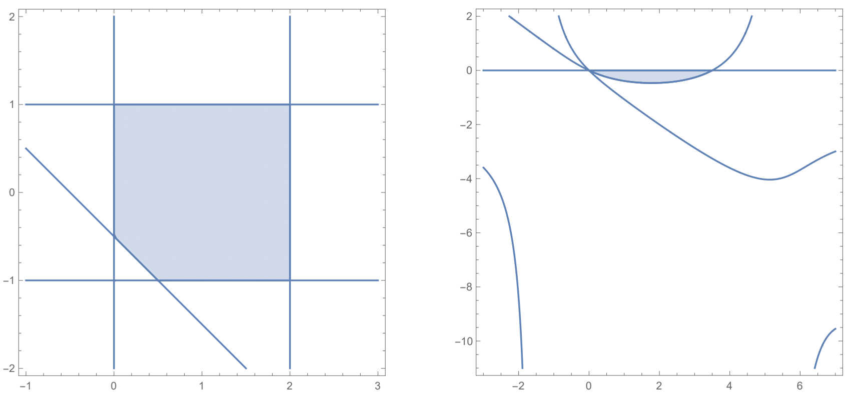


FIGURE 6.3. The spectrahedra $\mathrm{Gram}_V(f)$ and $\mathrm{Gram}_V(g)$ discussed in Example 6.4.3 and their algebraic boundaries.

There are more shapes that Gram spectrahedra of (nongeneric) quadratic forms on X_P can take. For example, the polynomial

$$g = x^2 - 2x^2y + y^2 + 3x^2y^2 + x^4y^2 + 6xy^3 - 4x^2y^3 - 6x^3y^3 + 13x^2y^4$$

can even be written as a sum of two squares, namely

$$g = (x - xy + 2xy^2)^2 + (y - x^2y + 3xy^2)^2.$$

The Gram spectrahedron of g is bounded by an edge connecting the rank-2 extreme point corresponding to the sos representation given above to a rank-3 extreme point. All other extreme points of $\mathrm{Gram}_V(g)$ have rank 4 and lie on an arc which is given by the quartic factor of the determinant. See Figure 6.3 for a visualization of the Gram spectrahedra discussed in this example.

Varieties of higher codimension. Let us finally turn to the Gorenstein polytopes $P \subseteq \mathbb{R}^m$ of degree 2 that have at least $m+4$ lattice points. Gram spectrahedra can then have a more interesting structure due to their larger dimensions.

Let $f \in \mathbb{R}[X_P]_2$ be a sum of squares. If $F \subseteq \mathrm{Gram}(f)$ is a face of rank r, then always

$$\begin{aligned}\dim(F) &\geq \binom{r+1}{2} - \dim(VV) \\ &= \binom{r-m}{2} - (m+1)(n+1-r) - \varepsilon_2(X_P),\end{aligned}$$

just like in the case of varieties of minimal degree.

For every $f \in \mathrm{int}(\Sigma R_1^2)$ and for proper faces of $\mathrm{Gram}_V(f)$ of maximum rank, that is to say $r = \mathrm{rk}(F) = \dim(V) - 1 = n$, we want to show that also the other inequality from Theorem 4.6.2 generalizes to

$$\dim(F) \leq \binom{r-m}{2} - \varepsilon_2(X_P).$$

Put differently, given any hyperplane $U \subseteq V$ with $\Sigma U^2 \cap \mathrm{int}(\Sigma V^2) \neq \varnothing$, we need to show that the space UU is big enough. Let us rephrase this in terms of the codimension of UU inside VV. By the formula for the dimension of faces, we have

$$\dim(F) = \binom{n+1}{2} - \dim(UU) = \binom{n+1}{2} + \mathrm{codim}_{VV}(UU) - \dim(VV).$$

Rearranging terms, we thus see that $\dim(F) \leq \binom{n-m}{2} - \varepsilon_2(X_P)$ if and only if

$$\operatorname{codim}_{VV}(UU) \leq \binom{n-m}{2} - \varepsilon_2(X_P) - \binom{n+1}{2} + \dim(VV).$$

Plugging in equation (6.4.1) for the dimension of VV, the latter condition simplifies to

$$\operatorname{codim}_{VV}(UU) \leq m+1.$$

We analyze the combinatorics of the underlying polytopes in order to find bounds for $\operatorname{codim}_{VV}(UU)$

6.4.4 Strategy. Let $U \subseteq V$ be a subspace of dimension r. After applying an affine-linear isomorphism of the lattice $\mathbb{Z}^m$, we can assume that P has vertices in $\mathbb{N}_0^m$. Identify V with $\mathbb{R}[\underline{x}]_P$ and fix a monomial order on $\mathbb{R}[\underline{x}]$. We consider the set $\mathrm{LM}(U)$ consisting of leading monomials of elements in U. We have $|\mathrm{LM}(U)| = r$ since we can choose a basis $p_1, \ldots, p_r$ of U with pairwise distinct leading monomials and with this choice $\mathrm{LM}(U) = \{\mathrm{LM}(p_i) : i = 1, \ldots, r\}$. Via the relation between monomials in V and lattice points in P, the set $\mathrm{LM}(U)$ corresponds to a subset L' of $L := P \cap \mathbb{Z}^m$ and the leading monomials of the pairwise products $p_i p_j$ are in bijection with the elements of $L' + L'$. Therefore, $\dim(UU) \geq |L' + L'|$. Put differently, we have

$$\operatorname{codim}_{VV}(UU) \leq |(2P) \cap \mathbb{Z}^m| - |L' + L'|.$$

As we are mainly interested in subspaces of codimension 1, we introduce the following notation:

6.4.5 Notation. Let L be a subset of the lattice $\mathbb{Z}^m$ and let $v \in L$. We write $L_v := L \smallsetminus \{v\}$.

6.4.6 Definition. Let $P \subseteq \mathbb{R}^m$ be a normal lattice polytope, let $L = P \cap \mathbb{Z}^m$ be the set of its lattice points and let $v \in L$. We define

$$\mathfrak{c}_v := |(2P) \cap \mathbb{Z}^m| - |L_v + L_v|$$

and call this value the *complemental strength* of v in $2P$.

6.4.7 Remark. Since P is normal, we have

$$(2P) \cap \mathbb{Z}^m = P \cap \mathbb{Z}^m + P \cap \mathbb{Z}^m = L + L.$$

Therefore, the complemental strength of v in $2P$ is the number of additional lattice points that can be reached by addition when v complements L_v to L again.

Another reason for the notation $\mathfrak{c}_v$ is that we can consider this number as a codimension when v is a vertex of a lattice polytope $P \subseteq \mathbb{R}^m_{\geq 0}$ and $Q := \operatorname{conv}(L_v)$ is normal. Indeed, let $U = \mathbb{R}[x_1, \ldots, x_m]_Q$. This is a linear hyperplane in the vector space $V = \mathbb{R}[x_1, \ldots, x_m]_P$. Then $\mathfrak{c}_v$ is the codimension of $UU = \mathbb{R}[x_1, \ldots, x_m]_{2Q}$ inside $VV = \mathbb{R}[x_1, \ldots, x_m]_{2P}$.

As discussed above (cf. Strategy 6.4.4), for an arbitrary subspace $U \subseteq V = \mathbb{R}[x_1, \ldots, x_m]_P$ of codimension 1, the complemental strength $\mathfrak{c}_v$ gives an upper bound for $\operatorname{codim}_{VV}(UU)$ where we let v be the lattice point of P corresponding to the (unique) monomial of V that is not contained in $\mathrm{LM}(U)$.

Recall that our aim is to prove an upper bound for the dimensions of maximal proper faces in Gram spectrahedra.

6.4.8 Theorem. *Let $P \subseteq M_\mathbb{R} \cong \mathbb{R}^m$ be a full-dimensional Gorenstein polytope of degree 2 with at least $m+4$ lattice points. Let $X_P \subseteq \mathbb{P}^n$ be the projective toric variety embedded with respect to the lattice points of P. For every $f \in \operatorname{int}(\Sigma\mathbb{R}[X_P]_1^2)$, we have $\dim \operatorname{Gram}_{\mathbb{R}[X_P]_1}(f) = \binom{n-m+1}{2} - \varepsilon_2(X_P)$. If $F \subseteq \operatorname{Gram}_{\mathbb{R}[X_P]_1}(f)$ is a proper face, then*

$$\dim(F) \leq \binom{n-m}{2} - \varepsilon_2(X_P).$$

The proof of Theorem 6.4.8 will keep us busy for quite a while. The approach introduced above, which initially uses only the combinatorics of lattice polytopes, works to some extent in this regard. Along the way we will encounter lovely results from the theory of reflexive polygons. However, when the approach fails, we will still need to take a closer look at the spaces U and UU. What we particularly like is that the analysis always leads back to polytopes we are already familiar with and thus establishes beautiful connections between Gram spectrahedra of quadratic forms on varieties of minimal and almost minimal degree.

We first note that it is enough to prove the theorem for those Gorenstein polytopes which are not pyramids over low dimensional ones. Indeed, if Q is a pyramid over P, then X_Q is a projective cone over X_P, which is why we will not observe new structural phenomena for Gram spectrahedra of quadratic forms in $\mathbb{R}[X_Q]$ (cf. Section 4.8).

We distribute the proof of Theorem 6.4.8 over the following sections. Section 6.5 deals with the case $m = 2$ where we need to consider the reflexive lattice polygons. In Section 6.6 we work through the three-dimensional polytopes $P_8, \ldots, P_{15}$ and Section 6.7 settles the remaining cases $P = Q_4, Q_5$.

6.4.9 Remark. From Theorem 6.4.8 we can infer much about the structure of Gram spectrahedra when $P \subseteq M_\mathbb{R} \cong \mathbb{R}^m$ has exactly $m + 4$ lattice points. In these cases, the Gram spectrahedron of a general quadratic form f on X_P that is a sum of squares (of linear forms) has dimension 5 and the points in its relative interior have rank $m+4$. Points of rank $m+2$ are extreme points, while points of rank $m+3$ can either be extreme or lie in the relative interior of an edge or a two-dimensional face whose boundary consists of rank-$(m + 2)$ points.

So for P having exactly $m + 4$ lattice points and general f, the inequality

$$\dim(F) \leq \binom{r-m}{2} - \varepsilon_2(X_P) \tag{6.4.2}$$

is not only valid for $r = n = m + 3$ (proper faces of maximum rank), but also for $r = n - 1 = m + 2$ simply because f does not have any shorter sos representations. Of course, for special f which has a representation with only $m + 1$ or even fewer summands, we can also have edges of rank $m + 2$ just like in Example 6.4.3.

In general, we therefore cannot expect to get inequality (6.4.2) for all $f \in \operatorname{int}(\Sigma\mathbb{R}[X_P]_1^2)$ if r is smaller than $n = \dim(\mathbb{R}[X_P]_1) - 1$.

6.5. Reflexive lattice polygons

In this section we will prove Theorem 6.4.8 in the case $m = 2$. This means that the projective variety X_P is a toric surface of almost minimal degree. The corresponding polytope P is a reflexive lattice polygon. Following our approach, we determine the complemental strength of every lattice point v of P. Ideally, we get $\mathfrak{c}_v \leq m + 1 = 3$ since then we are done. The richness, however, comes from those points with higher complemental strength. We first give examples where $\mathfrak{c}_v = m + 2 = 4$. Then we show that nothing worse can happen.

Beyond that, we prove that if v is a vertex of P such that the origin is contained in the interior of $Q := \operatorname{conv}(L_v)$, then Q is again reflexive (Proposition 6.5.4). For most combinations of a reflexive lattice polygon P (with at least six lattice points) and a vertex v, the above hypothesis is met and we then automatically get $\mathfrak{c}_v = 3$. At this point I want to thank Mateusz Michałek who suggested the subdivision of a reflexive polygon into triangles and pointed me to the partial addition on reflexive polytopes.

Apart from that, there will be a total of eleven special vertices in various polygons that we will consider separately. Finally, we would like to say a few words on the complemental strength of the interior lattice point of a reflexive polygon.

6.5.1 Example. Let P be the polytope of type 7b from Figure 6.2 and let $v = (1, 1)$ be the point in the right upper corner. The set $L_v + L_v$ misses four points of $(2P) \cap \mathbb{Z}^2$, so $\mathfrak{c}_v = 4$. This is illustrated in Figure 6.4 where the filled circles are the points of $L_v + L_v$ and the four blank squares represent the complemental strength of v in $2P$.

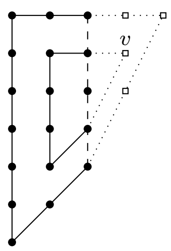


FIGURE 6.4. The situation in Example 6.5.1.

6.5.2 Example. Let P be the polytope of type 4a and let $v = 0 \in \operatorname{int}(P)$. Then $v \in L_v + L_v$ since P has two pairs of centrally symmetric vertices, but none of the four vertices of P is contained in $L_v + L_v$.

6.5.3 Lemma. *Let $P \subseteq \mathbb{R}^2$ be a reflexive lattice polygon and let $L := P \cap \mathbb{Z}^2$ be the set of its lattice points. If $v \in P$ is a vertex of P, then*

$$\mathfrak{c}_v = |(2P) \cap \mathbb{Z}^2| - |L_v + L_v| \leq 4.$$

Proof. Connecting every vertex of P to 0 by a line segment gives a subdivision of P into triangles which all have 0 as a common vertex. This subdivision is shown in Figure 6.5.

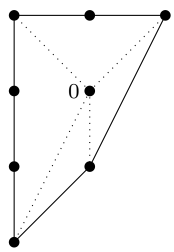


FIGURE 6.5. Subdivision of a reflexive polygon into triangles.

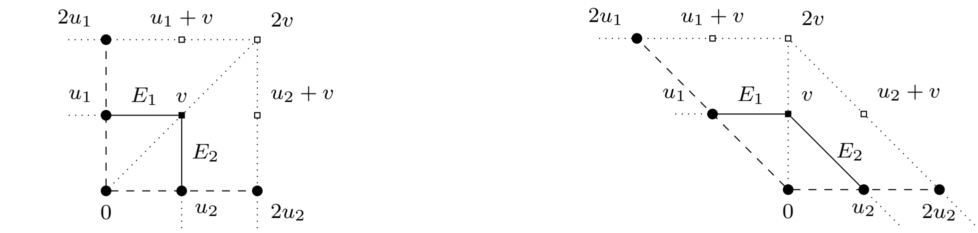


FIGURE 6.6. Some configurations that lead to $v \in L_v + L_v$.

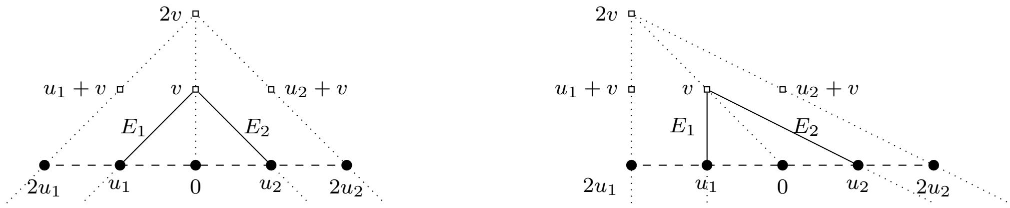


FIGURE 6.7. If u_1 and u_2 are centrally symmetric, then $v \notin L_v + L_v$.

We only need to analyze what happens in the two triangles that share the edge $[0, v]$. The point v lies on two edges E_1 and E_2 of P. Let u_1 and u_2 be the first lattice points of P we reach when walking from v along E_1 and E_2, respectively. Every two-dimensional lattice polygon is normal. Therefore, the lattice points in the two segments $[0, 2u_1]$ and $[0, 2u_2]$ as well as all lattice points of $2P$ that do not lie in $\operatorname{conv}(0, 2u_1, 2u_2)$ are contained in $L_v + L_v$. Note that since P is reflexive the edges E_1 and E_2 are distance one from 0 (see Remark 1.3.11). This means that the convex hull of 0, u_1, u_2, v does not contain any lattice points other than these four mentioned. Hence, there are only four points of $2L$ that $L_v + L_v$ might miss, namely v, $v + v$, $u_1 + v$, $u_2 + v$. $\square$

Lemma 6.5.3 and its proof raise the question of when we have $v \in L_v + L_v$. We can rephrase this and ask when v is the sum of two lattice points of P different from v. To answer this question, we make use of a powerful tool in the theory of reflexive polytopes developed by Nill in his thesis ([Nill, Proposition 3.3.1]). This is the existence of so-called generalized primitive relations for pairs of lattice points on the boundary. They give a partial addition on the set of lattice points of a reflexive polytope $P \subseteq M_{\mathbb{R}}$: If u, $w \in \partial P \cap M$ do not lie on a common facet of P, then either $u + w = 0$ or $u + w \in \partial P \cap M$.

The Figures 6.6 and 6.7 illustrate some geometric configurations where we actually get $v \in L_v + L_v$ or $v \notin L_v + L_v$, respectively.

6.5.4 Proposition. *Let $P \subseteq \mathbb{R}^2$ be a reflexive lattice polygon and let $L := P \cap \mathbb{Z}^2$ be the set of its lattice points. Let $v \in P$ be a vertex of P and let $Q = \operatorname{conv}(L_v)$. If $0 \in \operatorname{int}(Q)$, then Q is reflexive, $L_v + L_v = (2Q) \cap \mathbb{Z}^2$ and*

$$\mathfrak{c}_v = |L + L| - |L_v + L_v| \leq 3.$$

Proof. Let u_1, $u_2 \in L$ be as in the proof of Lemma 6.5.3. The assumption $0 \in \operatorname{int}(Q)$ implies that $E := [u_1, u_2]$ is an edge of Q and that $0 \notin \operatorname{conv}(v, u_1, u_2) =: \Delta$. We want to use [Nill, Proposition 3.3.1] in order to show that $u_1 + u_2 = v$. Since $E_1 \neq E_2$ and $P \neq \Delta$, the points u_1 and u_2 do not lie on a common facet (edge) of P. Furthermore, $u_1 + u_2 \neq 0$ by the assumption that $0 \in \operatorname{int}(Q)$. Therefore,

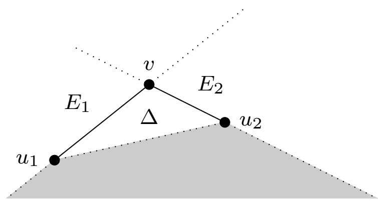


Figure 6.8. In the proof of Proposition 6.5.4 the location of the origin is restricted to the shaded area.

$u_1 + u_2 \in \partial P$ by [Nill, Proposition 3.3.1]. The same proposition tells us that u_1, u_2 is a $\mathbb{Z}$-basis of $\operatorname{span}(u_1, u_2) \cap \mathbb{Z}^2$ and that the following statement holds:

> There exists exactly one pair $(a_1, a_2) \in \mathbb{N}^2$ such that for the point $z := a_1u_1 + a_2u_2$, we have $z \in \partial P$, there is an edge of P which contains u_1 and z, and another edge of P contains u_2 and z. The values of a_1 and a_2 are then given by $a_i = |[u_i, z] \cap \mathbb{Z}^2| - 1$. If F is a facet (edge) of P that contains u_1 and z, then $\langle u_2, \eta_F\rangle = \frac{a_1-1}{a_2}$ where η_F is the unique inner normal of the facet F. (∗)

Our goal is to show that $z = v$. Since P is convex and $0 \in \operatorname{int}(P)$, we have

$$0 \in v + \mathbb{R}_{>0}(u_1 - v) + \mathbb{R}_{>0}(u_2 - v),$$

say $0 = v + \lambda(u_1 - v) + \mu(u_2 - v)$ with λ, $\mu \in \mathbb{R}_{>0}$. The fact $0 \notin \Delta$ implies $\lambda + \mu > 1$. See Figure 6.8 for an illustration. Therefore, the equality

$$\big((\lambda + \mu) - 1\big)v = \lambda u_1 + \mu u_2$$

shows that $v \in \mathbb{Z}^2$ is a linear combination of u_1 and u_2 with positive coefficients. On the other hand, u_1, u_2 is a $\mathbb{Z}$-basis of $\mathbb{Z}^2$. Thus, there are b_1, $b_2 \in \mathbb{N}$ such that $v = b_1u_1 + b_2u_2$. But we also have $v \in \partial P$ and u_i, $v \in E_i$ $(i = 1, 2)$. The uniqueness statement in (∗) implies that $(b_1, b_2) = (a_1, a_2)$ and $v = z$. For $i \in \{1, 2\}$ we defined u_i to be the first lattice point of P we reach when walking from v along the edge E_i, so $a_i = |[u_i, v] \cap \mathbb{Z}^2| - 1 = 1$. Consequently, $v = u_1 + u_2 \in L_v + L_v$. This shows $\mathfrak{c}_v \leq 3$ (cf. the proof of Lemma 6.5.3).

It remains to show that Q is reflexive. Let η_i be the inner normal of the facet E_i of P $(i = 1, 2)$. Then $\langle m, \eta_i\rangle = -1$ for all $m \in E_i$. Let $\nu := \eta_1 + \eta_2$ and let $m \in E = [u_1, u_2]$, say $m = (1 - \lambda)u_1 + \lambda u_2$ for some $\lambda \in [0, 1]$. Then

$$\begin{aligned}\langle m, \nu\rangle &= (1 - \lambda)\left(\langle u_1, \eta_1\rangle + \langle u_1, \eta_2\rangle\right) + \lambda\left(\langle u_2, \eta_1\rangle + \langle u_2, \eta_2\rangle\right)\\ &= -1 + (1 - \lambda)\langle u_1, \eta_2\rangle + \lambda\langle u_2, \eta_1\rangle\\ &= -1\end{aligned}$$

since $\langle u_2, \eta_1\rangle = \frac{a_1-1}{a_2} = 0$ and $\langle u_1, \eta_2\rangle = \frac{a_2-1}{a_1} = 0$. This shows that E has integral lattice distance one from the origin. Every other edge of Q is either an edge of P or a shortened version of E_1 or E_2 and thus also has integral lattice distance one from the origin. We conclude that Q is reflexive (cf. Remark 1.3.11). □

We continue in the vein of elementary geometric arguments. Bounding the complemental strength for non-extreme lattice points on the boundary of a reflexive polygon is thankfully less sophisticated.

6.5.5 Proposition. *Let $P \subseteq \mathbb{R}^2$ be a reflexive lattice polygon and let $L := P \cap \mathbb{Z}^2$ be the set of its lattice points. If $v \in \partial P$ is a lattice point in the relative interior of*

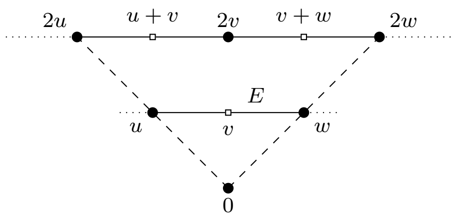


FIGURE 6.9. The proof of Proposition 6.5.5.

an edge, then

$$\mathfrak{c}_v = |(2P) \cap \mathbb{Z}^2| - |L_v + L_v| \leq 3.$$

Proof. Since v is not a vertex of P, there is a unique edge $E \subseteq \partial P$ containing v in its interior. There are two lattice points $u, w \in E \cap \mathbb{Z}^2$ that we reach first when walking from v along E in either direction and we have $v = \frac{u+w}{2}$.

Now, the relevant part of P is $\Delta = \operatorname{conv}(0, u, w)$. The lattice points of 2Δ are those of Δ and additionally the five points

$$u + u,\ u + v,\ v + v = u + w,\ v + w,\ w + w.$$

Therefore, the only points in $2L$ we possibly miss when we have to write them as a sum of two points in L_v are v, $u + v$ and $v + w$. □

Before returning to the vertices, we discuss the complemental strength of the interior lattice point.

6.5.6. Let $P \subseteq \mathbb{R}^2$ be a reflexive lattice polygon. In Proposition 6.5.5 we proved that the complemental strength of a point in the relative interior of an edge of P is at most 3, while vertices can have complemental strength up to 4. This intuitively makes sense since the loss of an extreme point cannot be compensated so easily. In the same vein, one should think that leaving out the origin does not lead to such a big loss in terms of points in $L_v + L_v$, as the origin is an interior lattice point of P. Already the example of a reflexive polygon of type 4a (Example 6.5.2) shows that this is not true in general. However, in the cases where P is a reflexive polygon with at least six lattice points we even get $\mathfrak{c}_0 \leq 2$. The precise values for every type are recorded in Table 6.1. It would be nice to have a geometric or combinatorial explanation for these numbers. We remark that they count the number of points in $\partial P \cap M$ that are not in the image of the partial addition introduced in [Nill, Proposition 3.3.1]. In fact, one can check that this partial addition reaches all points but those vertices we colored gray in Figure 6.11.

TABLE 6.1. The complemental strength of the interior point.

Type	5a	5b	6a	6b	6c	6d	7a	7b	8a	8b	8c	9
$\mathfrak{c}_0$	2	2	0	1	2	2	0	1	0	0	1	0

6.5.7 Remark/Example. Let P be a reflexive lattice polygon, let $L = P \cap \mathbb{Z}^2$ be the set of its lattice points and let $v \in L$ be a vertex of P. Proposition 6.5.4 deals with the case where $Q := \operatorname{conv}(L_v)$ contains an interior lattice point. We have shown that Q is then reflexive itself and thus belongs to one of the 16 equivalence classes of reflexive lattice polygons. Determining this specific equivalence class is easy since they can be distinguished by various criteria, e.g. by the number of edges or by the number of lattice points on an edge. An explicit lattice isomorphism that transforms Q to the standard representative of its equivalence class (like in Figure 6.2) is often obvious, for example when simple rotations or reflections suffice. We give some examples.

Let P be a polytope of type 6a and let v be any vertex of P. Then $0 \in \operatorname{int}(Q)$ and Q has to be of type 5a or 5b. But, in contrast to Q, a polytope of type 5b contains an edge with three lattice points. Therefore, Q must be of type 5a.

Now let $P = \operatorname{conv}((1,1),(-1,1),(-1,-2),(0,-1))$ be a polytope of type 7b and let $v = (-1,1)$. Then Q has precisely five edges and thus must be of type 6b since polytopes of type 6a, 6c and 6d have six, four and three edges, respectively. A lattice isomorphism that maps Q to the “standard form”

$$Q' = \operatorname{conv}((1,0),(1,1),(-1,1),(-1,0),(0,-1))$$

of this equivalence class is given by $\rho\colon e_1 \mapsto -e_1 - e_2,\ e_2 \mapsto e_1$. The polytopes Q and Q' are depicted in Figure 6.10.

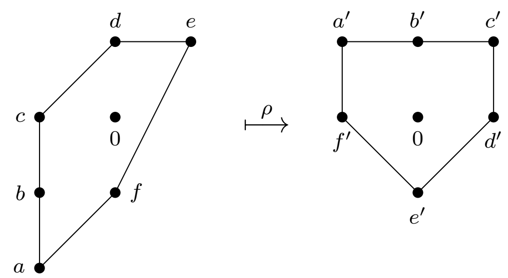


FIGURE 6.10. The lattice isomorphism ρ maps the reflexive polygon Q of type 6b to the standard form Q'.

For every combination of a reflexive lattice polygon $P \subseteq \mathbb{R}^2$ (with at least six lattice points) and a vertex v, we record the equivalence class of $Q = \operatorname{conv}(L_v)$ in Table 6.2. The vertices are listed counterclockwise starting at $(1,0)$. Recall that $S_2 = \operatorname{conv}(0, e_1, e_2)$ is the two-dimensional unit simplex and $P_{(d,e)}$ is the Cayley sum of intervals $[0,d]$ and $[0,e]$. The latter polytopes and its associated toric varieties of minimal degree – the rational normal scrolls – were discussed in Chapter 4. The eleven cases where Q is not reflexive again are discussed subsequently.

6.5.8 Remark. When studying the toric variety X_P associated to a reflexive lattice polygon $P \subseteq \mathbb{R}^2$, we prefer to use a representative of P's equivalence class that gives an embedding that is as simple as possible. As an example, we consider a polytope P of type 8c which is given as

$$P = \operatorname{conv}(e_1 + e_2, -e_1 + e_2, -e_1 - 3e_2) \subseteq \mathbb{R}^2$$

in the usual classification of reflexive lattice polygons like in Figure 6.2. Using the lattice points of this polytope, we would get $X_P = \overline{\operatorname{im}(\phi)} \subseteq \mathbb{P}^8$ for

$$\begin{aligned}\phi\colon (\mathbb{C}^*)^2 &\to \mathbb{P}^8,\\ (x,y) &\mapsto \left(\frac{1}{xy^3} : \frac{1}{xy^2} : \frac{1}{xy} : \frac{1}{x} : \frac{y}{x} : \frac{1}{y} : 1 : y : xy\right)\\ &= (1 : y : y^2 : y^3 : y^4 : xy^2 : xy^3 : xy^4 : x^2y^4).\end{aligned}$$

TABLE 6.2. Types of polytopes we get when deleting a certain vertex from a reflexive polygon.

5a	$(1,0)$	$(0,1)$	$(-1,1)$	$(-1,0)$	$(0,-1)$	
	$P_{(2,1)}$	4b	4a	4b	$P_{(2,1)}$	
5b	$(1,1)$	$(-1,1)$	$(-1,0)$	$(0,-1)$		
	$P_{(2,1)}$	4b	4c	$P_{(2,1)}$		
6a	$(1,0)$	$(0,1)$	$(-1,1)$	$(-1,0)$	$(0,-1)$	$(1,-1)$
	5a	5a	5a	5a	5a	5a
6b	$(1,0)$	$(1,1)$	$(-1,1)$	$(-1,0)$	$(0,-1)$	
	5b	5a	5a	5b	$P_{(2,2)}$	
6c	$(1,0)$	$(1,1)$	$(-1,1)$	$(-1,-1)$		
	$2S_2$	5b	5a	$P_{(2,2)}$		
6d	$(1,1)$	$(-1,1)$	$(-1,-2)$			
	$P_{(3,1)}$	5b	$2S_2$			
7a	$(1,0)$	$(1,1)$	$(-1,1)$	$(-1,-1)$	$(0,-1)$	
	6c	6b	6a	6b	6c	
7b	$(1,1)$	$(-1,1)$	$(-1,-2)$	$(0,-1)$		
	$P_{(3,2)}$	6b	6c	6d		
8a	$(1,1)$	$(-1,1)$	$(-1,-1)$	$(1,-1)$		
	7a	7a	7a	7a		
8b	$(1,0)$	$(1,1)$	$(-1,1)$	$(-1,-2)$		
	7b	7b	7a	7a		
8c	$(1,1)$	$(-1,1)$	$(-1,-3)$			
	$P_{(4,2)}$	7b	7b			
9	$(-1,2)$	$(-1,-1)$	$(2,-1)$			
	8b	8b	8b			

However, rotating by 90° counterclockwise and translating by $e_1 + e_2$, we can represent our polytope as $P = \operatorname{conv}(0, 4e_1, 2e_2) \subseteq \mathbb{R}^2$ and the resulting embedded toric variety X_P is the Zariski closure of the image of

$$\begin{aligned} \phi\colon (\mathbb{C}^*)^2 &\to \mathbb{P}^8, \\ (x,y) &\mapsto \left(1 : x : x^2 : x^3 : x^4 : y : xy : x^2y : y^2\right). \end{aligned}$$

The eleven cases from Table 6.2 where we cannot use Proposition 6.5.4 to exclude proper faces of dimension $\binom{n-m}{2}$ in Gram spectrahedra and thereby prove Theorem 6.4.8 are depicted in Figure 6.11. We first discuss a polytope with two special vertices in detail and then summarize the other special cases.

6.5.9 (The polytope 6d)**.** Consider the polytope P of type 6d as depicted in Figure 6.11. We choose a graded monomial order with $x \prec y$. The seven monomials

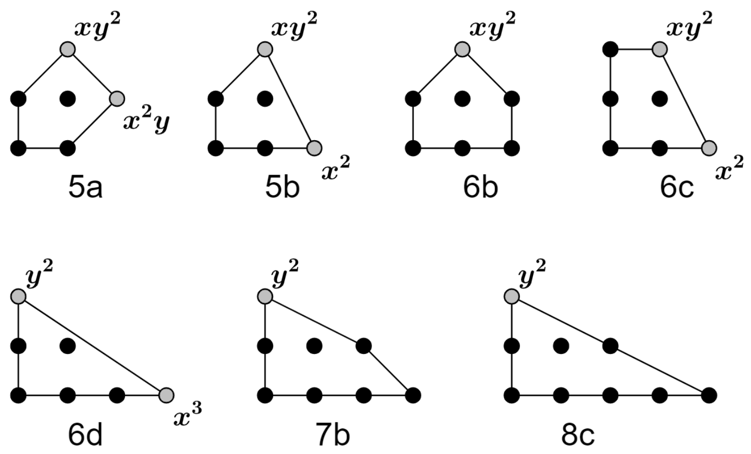


Figure 6.11. The reflexive polygons that require a more detailed analysis for the gray vertices.

corresponding to the lattice points of P are then ordered as follows:

$$1 \prec x \prec y \prec x^2 \prec xy \prec y^2 \prec x^3.$$

Let $U \subseteq V := \mathbb{R}[x,y]_P$ be a subspace of dimension 6. Note that $\dim(VV) = 19$. We have to show $\operatorname{codim}_{VV}(UU) \leq m+1 = 3$ if $\Sigma U^2 \cap \operatorname{int}(\Sigma V^2) \neq \varnothing$.

The two special cases we need to consider are $x^3 \notin \operatorname{LM}(U)$ or $y^2 \notin \operatorname{LM}(U)$. If $x^3 \notin \operatorname{LM}(U)$, then x^3 cannot appear in any element of U by the choice of the monomial order. But then $U = \mathbb{R}[x,y]_Q$ for $Q = 2S_2$. In the language of polynomials and Gram spectrahedra this would mean the following: Suppose we have a polynomial $f \in \mathbb{R}[x,y]_{2P}$ whose Gram spectrahedron $\operatorname{Gram}_V(f)$ has a (six-dimensional) face F with associated face subspace $\mathcal{U}(F) = U$. Then $f \in \mathbb{R}[x,y]_{2Q}$ is actually a ternary quartic (or a bivariate polynomial of degree 4 in our inhomogeneous setting) and its Gram spectrahedron does not have dimension 9 as for points in $\operatorname{int}(\Sigma V^2)$ but is degenerated to the face F. This means $\operatorname{Gram}_V(f) = F = \operatorname{Gram}_U(f)$ is the Gram spectrahedron of a ternary quartic.

Now, let us consider the case $y^2 \notin \operatorname{LM}(U)$. Start with any basis of U. By transforming the matrix of coefficients to its reduced row echelon form, we see that there is a basis of U of the form $(1, x, y, x^2, xy, q)$, where $q = x^3 + ay^2$ for some $a \in \mathbb{R}$. Using a computer algebra system to calculate minors, it is easy to see that $\dim(UU) \in \{15, 16\}$, and $\dim(UU) = 15$ if and only if $a = 0$. But we can also do it by hand: Let $U' = \operatorname{span}(1, x, y, x^2, xy)$, then $\dim(U'U') = 12$. First of all, we have $1 \cdot q = q \in U'U'$ and $x \cdot q = x^4 + axy^2 \in U'U'$. Next, for reasons of degree, the subspace $\operatorname{span}(x^2q, xy \cdot q, q^2)$ is 3-dimensional and its intersection with $U'U'$ is trivial. Therefore, we have $\dim(UU) = 15$ if $y \cdot q = x^3y + ay^3$ is contained in

$$W := U'U' \oplus \operatorname{span}(x^2q, xy \cdot q, q^2)$$

and $\dim(UU) = 16$ if $y \cdot q \notin W$. Obviously, $y \cdot q \in W$ if and only if $a = 0$ since $x^3y \in U'U' \subseteq W$ and $y^3 \notin W$. To sum up, $\operatorname{codim}_{VV}(UU) = 4$ if and only if the monomial y^2 does not appear in any element of U. But then $U = \mathbb{R}[x,y]_{Q'}$ for the polytope $Q' = P_{(3,1)}$. In this case, any $f \in \Sigma V^2$ whose Gram spectrahedron $\operatorname{Gram}_V(f)$ has a (six-dimensional) face F with associated face subspace $\mathcal{U}(F) = U$ essentially is a quadratic form on the variety $X_{Q'}$, a variety of minimal degree. Then the full Gram spectrahedron of f (relative to V or U) is F and the structure of this spectrahedron was studied in Chapter 4.

6.5.10. Consider the other polytopes from Figure 6.11. If only one vertex of the polytope P is colored gray, we can choose a monomial order that makes the monomial corresponding to this vertex the largest. For example, take the lexicographic order with $y \succ x$. Then xy^2 resp. y^2 resp. y^2 is the largest monomial among the monomials corresponding to lattice points of the polytope 6b resp. 7b resp. 8c. But whenever the largest monomial does not belong to $\mathrm{LM}(U)$ for a subspace $U \subseteq V = \mathbb{R}[x,y]_P$, it cannot appear in any element of U.

In the cases where we have two gray vertices with corresponding monomials $\mathfrak{m}_1$ and $\mathfrak{m}_2$, we choose a monomial order with $\mathfrak{m}_1 \succ \mathfrak{m}_2 \succ \mathfrak{m}$ for any monomial $\mathfrak{m}$ corresponding to a different lattice point of P. To be explicit, for the polytopes of type 5a, 5b and 6c we can take a graded order with $x \succ y$. Let P be one of these polytopes and let $U \subseteq V = \mathbb{R}[x,y]_P$ be a subspace of codimension 1. Again, if $\mathfrak{m}_1 \notin \mathrm{LM}(U)$, it is immediately clear that $\mathfrak{m}_1$ does not appear in any element of U. The case $\mathfrak{m}_2 \notin \mathrm{LM}(U)$ can be handled in a completely analogous manner as in 6.5.9 by either calculating minors of matrices (after reducing them to reasonable sizes by deleting rows or columns that are obviously linearly independent) or counting linearly independent elements in a set of generators of UU by hand. As this would not provide any new insights, we refrain from explicitly repeating these calculations once again.

The key point is that the result is always the same: If $\mathrm{codim}_{VV}(UU) = m+2 = 4$, then $U = \mathbb{R}[x,y]_Q$ for a lattice polytope Q corresponding to a variety X_Q of minimal degree. The exact type $2S_2$ or $P_{(d,e)}$ of the polytope Q we get by deleting a gray vertex from P can be read from Table 6.2. In particular, if $f \in \Sigma V^2$ and $U = \mathbb{R}[x,y]_Q$ is a facial subspace for $\mathrm{Gram}_V(f)$, then f essentially is a quadratic form on X_Q and its Gram spectrahedron $\mathrm{Gram}_V(f) = \mathrm{Gram}_U(f)$ has the respective structure that was studied in Chapter 4.

This concludes our analysis of Gram spectrahedra of quadratic forms on toric surfaces of almost minimal degree. Note that since $3S_2$ is Gorenstein of degree 2 and corresponds to a reflexive polygon of type 9, our results also include Gram spectrahedra of ternary sextics. Also the square (type 8a) has an interesting interpretation in terms of polynomials:

6.5.11 Example. Let $P = [0,2]^2$. Then P is a translation of the reflexive polygon $[-1,1]^2$ of type 8a. The polynomials in $\mathbb{R}[x,y]_{2P}$ are precisely those $f \in \mathbb{R}[x,y]$ with $\deg_x(f) \leq 4$ and $\deg_y(f) \leq 4$. Assume that $f \in \mathbb{R}[x,y]_{2P}$ is a sum of squares. We see from Theorem 6.3.7 that f is a sum of at most 4 squares of polynomials (whose degree in each variable is at most two). The Pataki interval is $\{4,5,6\}$. If f lies in the interior of the sos cone in $\mathbb{R}[x,y]_{2P}$, the Gram spectrahedron of f has dimension 20 and the dimension of a proper face is at most 14, according to Theorem 6.4.8 which we have proven so far for $m = 2$.

6.6. Three-dimensional Gorenstein polytopes of degree 2

We turn to dimension $m = 3$ and consider the m-dimensional Gorenstein polytopes of degree 2 that have at least $m+4 = 7$ lattice points. So let $P \in \{P_8, \ldots, P_{15}\}$, let $L = P \cap \mathbb{Z}^3$ be the set of lattice points of P and let $v \in L$ be a vertex of P. As before, we are interested in the type of the lattice polytope $Q = \mathrm{conv}(L_v)$. For all possible combinations of P and v, these types are recorded in Table 6.3. Note that we identify lattice points with their corresponding monomials. Whenever Q is again a Gorenstein polytope of degree 2, we automatically get $\mathfrak{c}_v = m + 1 = 4$ (cf. the dimension formula (6.4.1)). Elsewise, we have $\mathfrak{c}_v = 5$ and the strategy from 6.4.4 is not sufficient. In these cases, we have to take a closer look on UU.

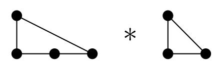
At three places in the table, a monomial is marked with a dagger. If this monomial is the one not contained in $\mathrm{LM}(U)$, then $\mathrm{codim}_{VV}(UU)$ can reach $m + 2 = 5$ although the monomial might still appear in U. We explain this and the consequences in more detail when we analyze P_9, the first case where the phenomenon occurs.

Lattice points $v \in P \cap \mathbb{Z}^3$ that are not extreme points of P are marked by a circle in Table 6.3 and the corresponding value $\mathfrak{c}_v$ is written down in a footnote. As one would expect, these values are smaller than the corresponding values for extreme points, but we do not have a clever proof of this fact.

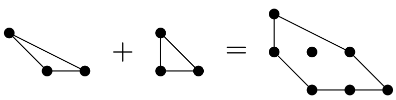
Due to space limitations we use the following abbreviations in Table 6.3: We write $\Pi\, 2S_2$ and $\Pi\, P_{(d,e)}$ instead of $\Pi\,(2S_2)$ and $\Pi\left(P_{(d,e)}\right)$ for a pyramid over the two-dimensional polytope $2S_2 = \mathrm{conv}(0, 2e_1, 2e_2)$ or the Cayley sum $P_{(d,e)} = [0, d] * [0, e]$, respectively.

6.6.1 Remark. It is usually easy to see the type of the polytope $Q = \mathrm{conv}(L_v)$. Nevertheless, we want to explain how one can determine this type by using the Cayley sum decompositions of $P \in \{P_8, \ldots, P_{14}\}$. Obviously, $Q = P$ if v is not an extreme point of P, so let v be a vertex. By [BJ, Proposition 4.12], $P = \Delta_1 * \Delta_2$ for plane lattice polytopes Δ_1 and Δ_2 whose Minkowski sum is a reflexive polygon. Removing v from P amounts to removing a vertex from Δ_1 or Δ_2. Let Δ_1' and Δ_2' be the resulting polytopes. To be clear, $\Delta_i' = \Delta_i$ for exactly one $i \in \{1, 2\}$. Now, Q is again one of the Gorenstein polytopes of degree 2 if and only if the Minkowski sum $\Delta_1' + \Delta_2'$ is a reflexive polygon ([BN08, Theorem 2.6]). Assume that the latter is the case. Then from the list of polytopes $P_1, \ldots, P_{14}$ only these remain in the running (that have the correct number of lattice points, of course, and) whose Cayley summands have a Minkowski sum lying in the equivalence class of $\Delta_1' + \Delta_2'$. After this step, there are at most two types that still come into consideration and they can be distinguished for example by their number of vertices. The next example illustrates this procedure with the polytope P_9.

6.6.2 Example. P_9 is the Cayley sum



of two lattice polytopes Δ_1 and Δ_2 whose Minkowski sum is a reflexive polygon of type 7b. If we delete the bottom left vertex in the first summand Δ_1, we are left with lattice polytopes Δ_1' and Δ_2' whose Minkowski sum

$\Delta_1' + \Delta_2' =$ 

is a reflexive polygon of type 6b (see 6.5.7). In [BJ, Table 2 on page 296] there are two Minkowski sum decompositions of a standard form of 6b. Replacing Minkowski sums by Cayley sums, one of these decompositions leads to P_5, a polytope that contains an edge with three lattice points and is therefore out of the question. (Moreover, P_5 has only five vertices.) Since the other decomposition leads to P_6, this must also be

TABLE 6.3. Types of polytopes we get when deleting a vertex from a three-dimensional Gorenstein polytope of degree 2, and complemental strength of non-extreme points.

$\boldsymbol{P_8}$	1	x	y	z	xy	xz	yz
	P_7	P_6	P_6	P_6	$P_{(1,1,1)}$	$P_{(1,1,1)}$	$P_{(1,1,1)}$

$\boldsymbol{P_9}$	1	x	$y^\dagger$	z	x^2	xz	yz
	P_6	$\circ^{\mathrm{a}}$	$\Pi\, P_{(2,1)}$	P_5	$P_{(1,1,1)}$	P_4	$\Pi\, P_{(2,1)}$

$\boldsymbol{P_{10}}$	1	x	y	$z^\dagger$	x^2	xy	yz	x^2y
	P_9	$\circ^{\mathrm{b}}$	P_9	$\Pi\, P_{(2,2)}$	P_9	$\circ^{\mathrm{b}}$	$\Pi\, P_{(2,2)}$	P_9

$\boldsymbol{P_{11}}$	1	x	y	z	x^2	xy	xz	yz
	P_8	$\circ^{\mathrm{c}}$	P_9	P_9	P_8	P_9	P_9	$P_{(2,1,1)}$

$\boldsymbol{P_{12}}$	1	x	y	z	xy	xz	yz	xyz
	P_8							

$\boldsymbol{P_{13}}$	1	x	y	$z^\dagger$	x^2	xy	xz	y^2
	P_9	$\circ^{\mathrm{d}}$	$\circ^{\mathrm{e}}$	$\Pi\, 2S_2$	P_9	$\circ^{\mathrm{e}}$	$\Pi\, 2S_2$	$P_{(2,1,1)}$

$\boldsymbol{P_{14}}$	1	x	y	z	x^2	xy	xz	y^2	yz
	P_{11}	$\circ^{\mathrm{f}}$	$\circ^{\mathrm{f}}$	P_{13}	P_{11}	$\circ^{\mathrm{f}}$	P_{13}	P_{11}	P_{13}

$\boldsymbol{P_{15}}$	1	x	y	z	x^2	xy	xz	y^2	yz	z^2
	P_{14}	$\circ^{\mathrm{g}}$	$\circ^{\mathrm{g}}$	$\circ^{\mathrm{g}}$	P_{14}	$\circ^{\mathrm{g}}$	$\circ^{\mathrm{g}}$	P_{14}	$\circ^{\mathrm{g}}$	P_{14}

(a) P_9: $\mathfrak{c}_x = 3$
(b) P_{10}: $\mathfrak{c}_x = \mathfrak{c}_{xy} = 3$
(c) P_{11}: $\mathfrak{c}_x = 2$
(d) P_{13}: $\mathfrak{c}_x = 2$
(e) P_{13}: $\mathfrak{c}_y = \mathfrak{c}_{xy} = 3$
(f) P_{14}: $\mathfrak{c}_x = \mathfrak{c}_y = \mathfrak{c}_{xy} = 2$
(g) P_{15}: $\mathfrak{c}_v = 2$ for all $v \in \{x, y, z, xy, xz, yz\}$

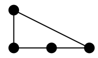
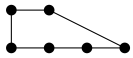
the type of $Q := \Delta_1' * \Delta_2'$. The linear map $\mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

induces a lattice isomorphism $\mathbb{Z}^3 \to \mathbb{Z}^3$ and maps Q to $P_6 + (-1, 1, 0)$.

If instead we start by deleting the bottom left vertex of Δ_2 (this point corresponds to the monomial z), we indeed get the same Minkowski sum, but now we are in the other case mentioned above that leads to the Cayley sum P_5.

In order to also have an example where $\Delta_1' + \Delta_2'$ is not reflexive anymore, let us remove the top left vertex from Δ_2. This point corresponds to yz. One immediately sees that in this case $Q = \Delta_1' * \Delta_2'$ is a pyramid over $P_{(2,1)}$. But also the fact that

$\Delta_1' + \Delta_2' =$  $+$ $=$  $= P_{(3,1)}$

is not reflexive proves that Q is not a Gorenstein polytope of degree 2 anymore.

Now that we have explained how to find the entries of Table 6.3, we address the (twelve) cases where $Q = \operatorname{conv}(L_v)$ fails to be Gorenstein of degree 2.

6.6.3 (The polytope P_8). We start with the polytope $P = P_8$ whose lattice points define the vector space $V = \mathbb{R}[x,y,z]_P = \operatorname{span}(1,x,y,z,xy,xz,yz)$. Let $U \subseteq V$ be a subspace of codimension 1. Note that $\dim(VV) = 23$ and that we would like to have $\operatorname{codim}_{VV}(UU) \leq 4$, that is to say $\dim(UU) \geq 19$, if U is facial for $\operatorname{Gram}_V(f)$ for any $f \in \operatorname{int}(\Sigma V^2)$. We use a graded monomial order with $x \prec y \prec z$ because then

$$1 \prec x \prec y \prec z \prec xy \prec xz \prec yz.$$

Looking at the entries of Table 6.3, we see that we only have to consider the three cases where one of the monomials xy, xz or yz is not contained in $\operatorname{LM}(U)$. The most delicate case is $xy \notin \operatorname{LM}(U)$. A basis of U is here given by $(1,x,y,z,q_1,q_2)$, where $q_1 = xz + axy$ and $q_2 = yz + bxy$ for some $a,\, b \in \mathbb{R}$. The pairwise products of the basis elements of U are homogeneous. Let $U' = \operatorname{span}(1,x,y,z)$. Then $U'U' = \mathbb{R}[x,y,z]_{\leq 2}$, this space has dimension 10 and it also contains q_1 and q_2. Moreover, the three products q_1^2, q_1q_2, q_2^2 of degree 4 are obviously linearly independent as they have distinct leading monomials. Therefore,

$$\dim(UU) = 10 + 3 + \dim\operatorname{span}(q_1x,\, q_1y,\, q_1z,\, q_2x,\, q_2y,\, q_2z) \leq 19.$$

The monomials occurring in $\mathbb{R}[x,y,z]_1 \cdot q_1 + \mathbb{R}[x,y,z]_1 \cdot q_2$ are

$$x^2y,\ x^2z,\ xy^2,\ xyz,\ xz^2,\ y^2z,\ yz^2.$$

Putting the coefficients of our six degree-3 generators in a matrix A, we obtain

$$A = \begin{pmatrix} a & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 1 & 0 & 0 \\ b & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 1 \end{pmatrix}.$$

Now, it is obvious that $\operatorname{rk}(A) = 4 + \operatorname{rk}(A')$ where

$$A' = \begin{pmatrix} 0 & a & 1 \\ b & 0 & 1 \end{pmatrix}$$

is the matrix we get from A by first deleting the first, third, fifth and sixth row of A and then all thereby arising zero columns. Hence, $\dim(UU) \in \{18, 19\}$, and $\dim(UU) = 18$ if and only if $a = b = 0$. If we actually have $\dim(UU) = 18$, then consequently $U = \mathbb{R}[x,y,z]_Q$ with a lattice polytope Q of type $P_{(1,1,1)}$.

The case where $xz \notin \operatorname{LM}(U)$ is similar but easier since we can start with a basis of the form $(1,x,y,z,xy,yz+axz)$ for some $a \in \mathbb{R}$. In case $yz \notin \operatorname{LM}(U)$, we have nothing to do since yz is the largest monomial and we immediately see that U is degenerated to $\mathbb{R}[x,y,z]_{Q'}$ for a lattice polytope Q' of type $P_{(1,1,1)}$.

6.6.4 (The polytope P_9). This polytope is particularly interesting since a new phenomenon occurs that we could not observe neither for the reflexive polygons nor for the first three-dimensional polytope we analyzed above. Up to now, $\operatorname{codim}_{VV}(UU) = m+2$ for a hyperplane $U \subseteq V$ always meant that there was a monomial that did not appear in U at all. As a consequence, for any $f \in \Sigma V^2$ that had U as a facial subspace for its Gram spectrahedron, the Newton polytope of f was actually smaller than the polytope we started with and the Gram spectrahedron was degenerated to a smaller dimension and had a structure we had already seen before. In the following discussion we will see that this shrinkage of the Newton polytope does not need to happen.

For our analysis of $P = P_9$ we use the lexicographic order with $z \prec x \prec y$. Then the monomials in $V = \mathbb{R}[x,y,z]_P$ are ordered as follows:

$$1 \prec z \prec x \prec xz \prec x^2 \prec y \prec yz.$$

The cases we need to comment on are those where one of the monomials x^2, y or yz is not contained in $\operatorname{LM}(U)$. If $yz \notin \operatorname{LM}(U)$, then yz does not appear in U at all, meaning that $U = \operatorname{span}(1,x,y,z,x^2,xz)$. The isomorphism of $\mathbb{R}[y]$-algebras $\phi\colon \mathbb{R}[x,y,z] \to \mathbb{R}[s,x,y]$, $x \mapsto s$, $z \mapsto x$, sends U to the subspace spanned by 1, s, s^2, x, xs and y. We recognize this subspace as $\mathbb{R}[s,x,y]_Q$ where $Q = \Pi\left(P_{(2,1)}\right)$ is a pyramid over $P_{(2,1)}$. As discussed in Section 4.8, the Gram spectrahedron of $f \in \Sigma U^2$ has the same structure as the Gram spectrahedron of a quadratic form on the variety $X_{P_{(2,1)}}$ of minimal degree.

If $x^2 \notin \operatorname{LM}(U)$, then a basis of U is given by $(1,z,x,xz,q_1,q_2)$, where $q_1 = y + ax^2$ and $q_2 = yz + bx^2$ for some a, $b \in \mathbb{R}$. As in 6.6.3, one can check that $\dim(UU) \in \{18,19\}$, and $\dim(UU) = 18$ if and only if $a = b = 0$, i.e., $U = \mathbb{R}[x,y,z]_{Q'}$ for the lattice polytope $Q' = P_{(1,1,1)}$ defining a variety $X_{Q'}$ of minimal degree.

Finally, we discuss the case $y \notin \operatorname{LM}(U)$. This amounts to $U = \operatorname{span}(p_1,\dots,p_6) \subseteq \mathbb{R}[x,y,z]_P$, where

$$(p_1,\dots,p_6) = (1,z,x,xz,x^2,yz+ay)$$

for some $a \in \mathbb{R}$. Interestingly enough, we have $\dim(UU) = 18$ for any $a \in \mathbb{R}$. So the new phenomenon that arises is that the codimension of UU inside VV can go up to 5 without any shrinkage of the Newton polytope. Therefore, we need a different argument in order to prove that $\operatorname{Gram}_V(f)$ cannot contain a face F with associated face subspace $\mathcal{U}(F) = U$ if $f \in \operatorname{int}(\Sigma V^2)$. To this end, let $f \in \Sigma V^2$ and let U be facial for the Gram spectrahedron $\operatorname{Gram}_V(f)$. Then $f \in \Sigma U^2$, say

$$f = \sum_{i=1}^{r} \left(\sum_{j=1}^{r} \lambda_{ij} p_j \right)^2$$

with $r = 6$ and $\lambda_{ij} \in \mathbb{R}$ for all i, $j \in \{1,\dots,r\}$. Let $\varepsilon > 0$. For

$$g(x,y,z) := f(x,y,z) - \varepsilon y^2 \in \mathbb{R}[x,y,z]_{2P}$$

it holds

$$g(0,y,-a) = \sum_{i=1}^{r} (\lambda_{i1} - a\lambda_{i2})^2 - \varepsilon y^2 < 0$$

for $y \gg 0$. Therefore, g is not positive semidefinite and in particular not a sum of squares, meaning that f is not in the interior of the sos cone ΣV^2. This is what was to be shown. Also note that for any f as above the Gram spectrahedron $\operatorname{Gram}_V(f)$ collapses to the face $F = \mathcal{F}(U)$.

For the remaining polytopes, the arguments are similar. For the convenience of the reader, we nevertheless give them in detail.

6.6.5 (The polytope P_{10})**.** For the polytope $P = P_{10}$ we have to analyze two vertices separately, namely those corresponding to z and yz. Take the lexicographic order with $z \succ y \succ x$. This gives

$$1 \prec x \prec x^2 \prec y \prec xy \prec x^2y \prec z \prec yz.$$

If the hyperplane $U \subseteq V = \mathbb{R}[x,y,z]_P$ is facial for $\mathrm{Gram}_V(f)$ for some $f \in \Sigma V^2$ and $yz \notin \mathrm{LM}(U)$, then no element of U contains the monomial yz and we obtain $U = \mathrm{span}(1,x,x^2,y,xy,x^2y,z) = \mathbb{R}[x,y,z]_Q$, where $Q = \Pi\left(P_{(2,2)}\right)$ is a pyramid over $P_{(2,2)}$. Similar as before, this means that the Newton polytope of f is $2Q$, the Gram spectrahedron of f is only six-dimensional and its structure is that of the Gram spectrahedron of a quadratic form on the variety $X_{P_{(2,2)}}$ of minimal degree.

For the case $z \notin \mathrm{LM}(U)$, we proceed along the lines of the last part of the analysis for P_9 (see 6.6.4). At this point we have $U = \mathrm{span}(p_1,\dots,p_7) \subseteq \mathbb{R}[x,y,z]_P$ where

$$(p_1,\dots,p_7) = (1,x,x^2,y,xy,x^2y,yz+az)$$

for some $a \in \mathbb{R}$. For any $a \in \mathbb{R}$ we get $\dim(UU) = 22$ and thereby $\mathrm{codim}_{VV}(UU) = 5$. But every $f \in \Sigma U^2$ is of the form $f = \sum_{i=1}^r (\sum_{j=1}^r \lambda_{ij}p_j)^2$ with $r = 7$ and some $\lambda_{ij} \in \mathbb{R}$ $(i, j \in \{1,\dots,r\})$. This implies that, for every $\varepsilon > 0$, the polynomial $g(x,y,z) := f(x,y,z) - \varepsilon z^2 \in \mathbb{R}[x,y,z]_{2P}$ is not positive semidefinite since

$$g(0,-a,z) = \sum_{i=1}^{r} (\lambda_{i1} - a\lambda_{i4})^2 - \varepsilon z^2 < 0$$

for $z \gg 0$. Therefore, $f \notin \mathrm{int}(\Sigma V^2)$.

6.6.6 (The polytope P_{11})**.** The only vertex of $P = P_{11}$ that needs our special attention is the one corresponding to the monomial yz. By taking the lexicographic order with $z \succ y \succ x$, this monomial becomes the largest in $\mathbb{R}[x,y,z]_P$ and we are done.

6.6.7 (The polytope P_{13})**.** Let us again take the lexicographic order with $z \succ y \succ x$ so that

$$1 \prec x \prec x^2 \prec y \prec xy \prec y^2 \prec z \prec xz.$$

Then the special vertices of $P = P_{13}$ correspond to the three largest monomials in $V = \mathbb{R}[x,y,z]_P$. Let $U \subseteq V$ with $\mathrm{codim}_V(U) = 1$ and let one of those three monomials be the one not contained in $\mathrm{LM}(U)$. The case $xz \notin \mathrm{LM}(U)$ is clear and leads to $U = \mathbb{R}[x,y,z]_Q$ with Q being a pyramid over $2S_2$. Thus, the Gram spectrahedron of any $f \in \Sigma V^2$ that has a (six-dimensional) face F with $\mathcal{U}(F) = U$ is reduced to this particular face F and the spectrahedron's structure is that of the Gram spectrahedron of a ternary quartic.

If $z \notin \mathrm{LM}(U)$, we can argue as for P_{10} (cf. 6.6.5): We get

$$(p_1,\dots,p_7) = (1,x,x^2,y,xy,y^2,xz+az)$$

with $a \in \mathbb{R}$. Once again, we have $\dim(UU) = 22$ irrespective of a. For f and g as in 6.6.5 we obtain

$$g(-a,0,z) = \sum_{i=1}^{r} (\lambda_{i1} - a\lambda_{i2} + a^2\lambda_{i3})^2 - \varepsilon z^2 < 0$$

for $z \gg 0$, leading to the conclusion that f lies on the boundary of ΣV^2.

The last case is $y^2 \notin \mathrm{LM}(U)$. There are $a, b \in \mathbb{R}$ such that

$$(p_1, \dots, p_5, q_1, q_2) = (1,\, x,\, x^2,\, y,\, xy, z + ay^2,\, xz + by^2)$$

is a basis of U. We determine the dimension of UU as a function of a and b. To this end, we subdivide the $\binom{7+1}{2} = 28$ pairwise products of basis elements into three groups according to their z-degree. The 15 elements $p_i p_j$ $(1 \leq i < j \leq 5)$ that have degree 0 in z generate a space W_0 of dimension 12. Furthermore, q_1^2, $q_1 q_2$, q_2^2 are the only products of z-degree 2 and their leading monomials are pairwise distinct. Let W_1 be the space spanned by the ten polynomials $p_i q_j$ $(i \in \{1, \dots, 5\},\, j \in \{1, 2\})$. By the isomorphism theorem $W_1/(W_0 \cap W_1) \cong (W_0 + W_1)/W_0$. Therefore,

$$\dim(UU) = 3 + \dim(W_0 + W_1) = 3 + \dim(W_0) + \dim\left((W_0 + W_1)/W_0\right).$$

In order to obtain a basis of $(W_0 + W_1)/W_0$, we can delete the monomials contained in W_0 from the generators of W_1 and discard duplicates that arise. The remaining eight polynomials are

$$z,\, xz,\, x^2z,\, x^3z,\, yz + ay^3,\, xyz + by^3,\, xyz + axy^3,\, x^2yz + bxy^3.$$

From this list we immediately see that $\dim(UU) \in \{22, 23\}$, and that $\dim(UU) = 22$ if and only if $xyz + by^3 = xyz + axy^3$, i.e., $a = b = 0$. To sum up, if $\dim(UU) = 22$, then $U = \mathbb{R}[x, y, z]_{Q'}$ where $Q' = P_{(2,1,1)}$ gives a toric variety of minimal degree.

This completes the proof of Theorem 6.4.8 for the case $m = 3$. Due to the simple and symmetric structure of $P_{15} = 2S_3$, the case of quartics in four variables has not required any extra effort.

6.6.8 Example. Another polytope with a particularly nice structure is the cube $P_{12} = [0, 1]^3$. Theorem 6.4.8 thus also deals with Gram spectrahedra of sums of squares of multiaffine polynomials in $\mathbb{R}[x, y, z]$. To sum up, a general such polynomial is a sum of squares of $m+2 = 5$ multiaffine polynomials, its Gram spectrahedron has dimension 9. The points in the relative interior of this spectrahedron have rank 8 while the Pataki range for the rank of extreme points is $\{5, 6\}$, and proper faces of maximum rank (i.e., of rank 7) have dimension at least 1 and at most 5.

In fact, using symbolic computation, one can check that $\dim_{\mathbb{C}}(UU) \leq 26 = \binom{7+1}{2} - 2$ for any $\mathbb{C}$-linear subspace $U \subseteq \mathbb{C}[x, y, z]_P$ that is spanned by polynomials of the form

$$x + \lambda_1,\, y + \lambda_2,\, z + \lambda_3,\, xy + \lambda_4,\, xz + \lambda_5,\, yz + \lambda_6,\, xyz + \lambda_7$$

with $\lambda_1, \dots, \lambda_7 \in \mathbb{C}$. It follows that a face of rank 7 is actually at least two-dimensional (cf. Remark 4.5.11).

6.7. Four- and five-dimensional Gorenstein polytopes of degree 2

In this section we discuss the four- and five-dimensional Gorenstein polytopes of degree 2 that are not pyramids over low dimensional ones. Recall that Q_1 leads to Gram spectrahedra that are reduced to a single point, while for Q_2, Q_3 and R_1 our Gram spectrahedra are generally two-dimensional. These spectrahedra were studied in Section 6.4. Therefore, there are only two more polytopes remaining, namely Q_4 and Q_5.

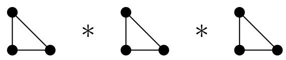
Every lattice point of Q_4 and Q_5 is an extreme point of the respective polytope. Table 6.4 lists the types of polytopes we get when cutting off one of these extreme points. We use the same notational conventions for pyramids as in Table 6.3. The dagger is also used in the same manner as before.

TABLE 6.4. Types of polytopes we get when deleting a certain vertex from Q_4 or Q_5.

$\boldsymbol{Q_4}$	1	x	y	z	w	xz	xw	yz	yw
	Q_5								

$\boldsymbol{Q_5}$	1	$x^\dagger$	y	z	$w^\dagger$	xz	yz	yw
	Q_2	$\Pi\, P_{(1,1,1)}$	Q_2	Q_2	$\Pi\, P_{(1,1,1)}$	$\Pi\, P_{(1,1,1)}$	Q_2	$\Pi\, P_{(1,1,1)}$

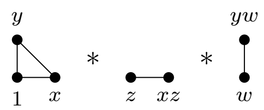
Due to the inherent symmetries of

$Q_4 =$ 

deleting any vertex of Q_4 always gives a polytope of type Q_5. Therefore, $\mathfrak{c}_v = 5 = m+1$ for every lattice point $v \in Q_4 \cap \mathbb{Z}^4$. The usage of complemental strength thus suffices to prove Theorem 6.4.8 in the case of $P = Q_4$.

Since half of the vertices of Q_5 lead to a polytope of type Q_2, it is worth taking a look at the latter.

6.7.1 Remark. The description of Q_2 as a Cayley polytope given in [BJ] is not correct. In fact it seems to be an accidental duplication of the description of Q_3 resulting in a four-dimensional polytope with six vertices. We want to find an actual representation of Q_2, a four-dimensional Gorenstein polytope of degree 2 with seven vertices. By [BJ, Proposition 4.12], Q_2 is a Cayley polytope whose three "summands" are plane lattice polytopes Δ_1, Δ_2, Δ_3 such that the Minkowski sum $\Delta_1 + \Delta_2 + \Delta_3$ is a reflexive polygon. In view of the number of vertices, only the decomposition at the very bottom of Table 1 on page 295 in [BJ] is eligible:



We want to show that Q_2 can indeed be realized as a polytope with vertices

$$(0,0,0,0),\ (1,0,0,0),\ (0,1,0,0),\ (0,0,1,0),\ (1,0,1,0),\ (0,0,0,1),\ (0,1,0,1).$$

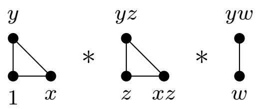
In anticipation of the result, we refer to the convex hull of these points as Q_2. Then $3Q_2$ is reflexive with respect to its interior point $(1,1,1,1)$ since the facet presentation of the polytope $3Q_2 - (1,1,1,1)$ is given by the (eight) inequalities

$$\begin{array}{rcl} x_i & \geq & -1 \quad (i = 1, \ldots, 4), \\ -x_1 - x_2 & \geq & -1, \\ -x_1 - x_4 & \geq & -1, \\ -x_2 - x_3 & \geq & -1, \\ -x_3 - x_4 & \geq & -1. \end{array}$$

The monomials corresponding to the lattice points of Q_2 are 1, x, y, z, xz, w, yw.

6.7.2 (The polytope Q_5)**.** In order to ease notation, we rearrange the summands in the description of Q_5 as a Cayley polytope given in [BJ] and consider $P := Q_5$ as

Cayley polytope



In the picture above, the vertices of P are labeled by the corresponding monomials. We use the lexicographic order with $w \succ x \succ y \succ z$. Thereby, the monomials in $V = \mathbb{R}[x, y, z, w]_P$ are ordered as follows:

$$1 \prec z \prec y \prec yz \prec x \prec xz \prec w \prec yw.$$

Let $L = P \cap \mathbb{Z}^4$ be the set of lattice points of P. Let $v \in \{1, z, y, yz\}$ and let $l_v \in L$ be the corresponding lattice point. Write $L_v := L \setminus \{l_v\}$. Then the polytope $Q_v := \operatorname{conv}(L_v)$ can (directly or after using a suitable lattice isomorphism) be recognized as a lattice polytope of type Q_2. Therefore,

$$|L_v + L_v| = |(2Q_2) \cap \mathbb{Z}^4| = |(2Q_5) \cap \mathbb{Z}^4| - (m + 1)$$

as desired. If, in contrast, we have $v \in \{x, xz, w, yw\}$, then Q_v is a pyramid over $P_{(1,1,1)}$ and therefore

$$|L_v + L_v| = |(2Q_5) \cap \mathbb{Z}^4| - (m + 2).$$

Hence, if $U \subseteq V$ is a linear subspace of codimension 1 and $v \notin \operatorname{LM}(U)$ for a $v \in \{x, xz, w, yw\}$, then we have to provide an argument that goes beyond counting lattice points in order to show that U is not facial for the Gram spectrahedron of any $f \in \operatorname{int}(\Sigma V^2)$ if $\operatorname{codim}_{VV}(UU) = 6$. We go through the four cases in descending order.

(i) Since yw is the largest monomial in V, the case $yw \notin \operatorname{LM}(U)$ is clear and results in $U = \mathbb{R}[x, y, z, w]_{Q_v}$.

(ii) If $w \notin \operatorname{LM}(U)$, then U has a basis of the form

$$(p_1, \ldots, p_7) = (1,\ z,\ y,\ yz,\ x,\ xz,\ yw + aw)$$

for some $a \in \mathbb{R}$ and we get $\operatorname{codim}_{VV}(UU) = m + 2 = 6$ irrespective of a. Any $f \in \Sigma U^2$ has a representation as in 6.6.5. For $\varepsilon > 0$ let $g(x, y, z, w) := f(x, y, z, w) - \varepsilon w^2 \in \mathbb{R}[x, y, z, w]_{2P}$. Then

$$g(0, -a, 0, w) = \sum_{i=1}^{r} (\lambda_{i1} - a\lambda_{i3})^2 - \varepsilon w^2 < 0$$

for $w \gg 0$. Thus, g is not a sum of squares and f lies on the boundary of ΣV^2.

(iii) In the case $xz \notin \operatorname{LM}(U)$, we work with the basis $(p_1, \ldots, p_5, q_1, q_2)$ of U where $(p_1, \ldots, p_5) = (1, z, y, yz, x)$ and

$$q_1 = w + axz, \quad q_2 = yw + bxz \quad (a,\, b \in \mathbb{R}).$$

We proceed as for P_{13} (see 6.6.7). The 15 elements $p_i p_j$ $(1 \le i < j \le 5)$ that have degree 0 in w generate a monomial space W_0 of dimension 14. Furthermore, q_1^2, $q_1 q_2$, q_2^2 are the only products of w-degree 2 and their leading monomials are pairwise distinct. Therefore,

$$\dim(UU) = 3 + 14 + \dim\left((W_0 + W_1)/W_0\right)$$

where W_1 is spanned by the ten polynomials $p_i q_j$ ($i \in \{1, \ldots, 5\}$, $j \in \{1, 2\}$). Deleting all monomials contained in W_0 from these ten polynomials and discarding the duplicate that arises, we are left with the nine elements

$$w,\ wz + axz^2,\ wy,\ wyz + axyz^2,\ wx + ax^2z,$$
$$wyz + bxz^2,\ wy^2,\ wy^2z + bxyz^2,\ wxy + bx^2z.$$

The only monomial of w-degree 1 that occurs more than once in this list is wyz. Hence, $\dim(UU) \in \{25, 26\}$, and $\dim(UU) = 25$ if and only if $wyz + axyz^2 = wyz + bxz^2$, that is to say $a = b = 0$. Since we have $\dim(VV) = 31$, this especially means that $U = \mathbb{R}[x, y, z, w]_{Q_{xz}}$ if $\operatorname{codim}_{VV}(UU) = 6$.

(iv) Finally, if $x \notin \mathrm{LM}(U)$, then a basis $(p_1, \ldots, p_7)$ of U is given by $(p_1, \ldots, p_4) = (1, z, y, yz)$,

$$p_5 = xz + ax, \quad p_6 = w + bx, \quad \text{and} \quad p_7 = yw + cx \quad (a,\, b,\, c \in \mathbb{R}).$$

Note that we have the quadratic relations $p_1p_4 - p_2p_3 = 0$ and

$$c \cdot p_1p_5 - a \cdot p_1p_7 - p_2p_7 - b \cdot p_3p_5 + a \cdot p_3p_6 + p_4p_6 = 0.$$

Therefore, $25 \le \dim(UU) \le \binom{7+1}{2} - 2 = 26$. We can use similar strategies as before to express $\dim(UU)$ as a function of a, b and c. The dimension of

$$W_0 = \operatorname{span}(p_ip_j : 1 \le i < j \le 5)$$

does not depend on a. Nevertheless, W_0 is not a monomial subspace anymore if $a \neq 0$. For this reason we have to be more prudent when calculating $\dim(W_0 + W_1)$ in this case, but what we ultimately get is

$$\dim(UU) = 25 \text{ if and only if } b = c = 0.$$

Consequently, if $\operatorname{codim}_{VV}(UU) = 6$, then

$$U = \operatorname{span}(1, z, y, yz, xz + ax, w, wx)$$

and $\Sigma U^2 \cap \operatorname{int}(\Sigma V^2) = \varnothing$. Indeed, $xz + ax$ vanishes in all points of $\mathbb{R}^4$ with z-coordinate $-a$. Therefore, if $f \in \Sigma U^2$ and $\varepsilon > 0$, then $g := f - \varepsilon x^2 \in VV$ and substituting $(0, -a, 0)$ for (y, z, w) gives a univariate quadratic polynomial with leading term $-\varepsilon x^2$. Thus, g is not psd. This proves that $f \notin \operatorname{int}(\Sigma V^2)$.

In each case, we established that a hyperplane $U \subseteq V$ with $\operatorname{codim}_{VV}(UU) = m + 2$ cannot be facial for $\operatorname{Gram}_V(f)$ if $f \in \operatorname{int}(\Sigma V^2)$. Thus, Theorem 6.4.8 is indeed true for $P = Q_5$.

This completes the proof of Theorem 6.4.8.

6.7.3 Remark. Although we had to discuss a noticeable number of cases separately in order to prove Theorem 6.4.8, the approach using what we called the complemental strength has already taken us quite far. It might be interesting to study the complemental strength of lattice points in more generality. For example, is it true that the complemental strength is maximized in an extreme point? If so, what are reasonable conditions on P that guarantee that any non-extreme lattice point has strictly less complemental strength than the "strongest" vertex? A good starting point for further research could be the class of reflexive lattice polytopes since these polytopes enjoy many favorable properties.

Open problems

Finally, we collect a few still open problems that we would like to see solved or that leave room for further research.

(1) Let $f \in \mathbb{R}[x,y]_{2d}$ be a positive binary form with distinct roots. For $d \leq 3$, we know that the (relative) boundary of $\mathrm{Gram}(f)$ consists of extreme points. In contrast, it is a union of positive-dimensional faces as soon as $d \geq 6$. This can be seen using a simple argument also given in [Sch22, Remark 5.6]: Let $\vartheta \in \mathrm{Ex}(f)$ and let $r = \mathrm{rk}(\vartheta)$. For $d \geq 6$, the inequality $\binom{r+1}{2} \leq 2d+1$ implies $r+2 \leq d$. We then take any $\vartheta' \in \mathrm{Ex}_2(f)$ different from ϑ and consider the supporting face F of $\{\vartheta, \vartheta'\}$ in $\mathrm{Gram}(f)$. As $\mathrm{rk}(F) \leq r+2 < d+1$, this is a proper positive-dimensional face that contains ϑ.

This argument does not apply for $d \in \{4,5\}$, where the Pataki bounds allow for extreme points of rank $d-1$. For general $f \in \Sigma_{2d}$, we know that $\mathrm{Gram}(f)$ actually contains an s-dimensional semialgebraic set of such points, where $s = \frac{1}{2}(d-3)(d+2)$, see Corollary 3.7.16. Under this assumption, we pose the following questions which have frustrated the author time and again:

(i) *Let $d = 4$. Is every $\vartheta \in \mathrm{Ex}_3(f)$ contained in a proper positive-dimensional face of* $\mathrm{Gram}(f)$*?*

(ii) *Let $d = 5$. Is every $\vartheta \in \mathrm{Ex}_4(f)$ contained in a proper positive-dimensional face of* $\mathrm{Gram}(f)$*?*

These questions are related to the fact that extreme points of rank greater than 2 are not well understood.

(2) Let $d \geq 3$. For general $f \in \Sigma_{2d}$, we have shown in Section 3.6 that the Carathéodory number $\mathrm{Car}(f)$ of the Gram spectrahedron of f satisfies

$$\lfloor \sqrt{d} \rfloor \leq \mathrm{Car}(f) \leq d-1.$$

Of course, $\mathrm{Car}(f) = 2$ for $d = 3$. But already for $d = 4$ we do not know if $\mathrm{Car}(f)$ is 2 or 3. *What is the precise Carathéodory number of the Gram spectrahedron of a general binary octic? More generally, what are better bounds on the Carathéodory number in case of higher degrees?*

(3) We mention a conjecture discussed in the body of this thesis. Let $X \subseteq \mathbb{P}^n$ be a smooth m-dimensional rational normal scroll and write $c := \mathrm{codim}(X) = n-m$. Consider the case $c = 3$. Then the Gram spectrahedron of a general quadratic form $f \in \mathbb{R}[X]_2$ positive on $X(\mathbb{R})$ contains eight extreme points of minimum rank $m+1$. *What is the structure of a graph describing the supporting faces of line segments connecting two of these points? Is it a complete bipartite graph $K_{4,4}$?* (cf. Conjecture 4.5.14)

(4) Analyze the "complemental strength" of lattice points (Definition 6.4.6) for other classes of lattice polytopes (see also Remark 6.7.3).

Zusammenfassung auf Deutsch

Übersetzung der Einleitung

Ein Polynom $f \in \mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \ldots, x_n]$ ist eine *Quadratsumme* (sum of squares [*sos*]), wenn es als $f = p_1^2 + \cdots + p_r^2$ mit $p_1, \ldots, p_r \in \mathbb{R}[\underline{x}]$ geschrieben werden kann. Die enorme Bedeutung von Quadratsummen in der reellen algebraischen Geometrie entstammt der einfachen Beobachtung, dass eine Darstellung von f als Summe von Quadraten ein einfaches algebraisches Zertifikat für die globale Nichtnegativität unseres Polynoms liefert.

Die Frage nach dem Zusammenhang zwischen Nichtnegativität und der Existenz von sos-Darstellungen geht mindestens bis auf Hilbert [Hil] zurück. Im Jahr 1888 bestimmte er die Paare (n, d), für die jedes nichtnegative Polynom in n Variablen vom Grad höchstens $2d$ eine Summe von Quadraten ist. Aber selbst in diesen Fällen ist es nicht sofort klar, wie man eine explizite sos-Darstellung von f findet. In den anderen Fällen stellt sich zunächst die Frage, ob es überhaupt eine solche Darstellung gibt. Praktischerweise kann die sogenannte Grammatrix-Methode, die zuerst von Choi, Lam und Reznick [CLR] eingeführt wurde, verwendet werden, um beide Probleme zu lösen: Sei $f \in \mathbb{R}[\underline{x}]$ ein Polynom vom Grad $2d$ und schreibe $\mathfrak{m}$ für den Spaltenvektor, der alle Monome vom Grad höchstens d in einer fixierten Reihenfolge enthält. Eine *Grammatrix* von f ist eine reelle symmetrische Matrix G mit $\mathfrak{m}^T G \mathfrak{m} = f$. Unser f ist genau dann eine Summe von Quadraten, wenn es eine positiv semidefinite Grammatrix von f gibt. Eine explizite Darstellung zu finden bedeutet, eine solche Matrix explizit zu bestimmen.

Wenn f tatsächlich eine Quadratsumme ist, gibt es in der Regel viele verschiedene Weisen, f als solche darzustellen. Die Menge aller positiv semidefiniter (*psd*) Grammatrizen von f parametrisiert alle sos-Darstellungen von f bis auf orthogonale Äquivalenz. Als Schnitt des Kegels der psd Matrizen mit einem affin-linearen Unterraum ist diese Menge ein Spektraeder, das *Gramspektraeder* $\mathrm{Gram}(f)$ des Polynoms f. Grob gesagt ist das Thema dieser Arbeit die Untersuchung der Seitenstruktur von Gramspektraedern in verschiedenen Kontexten.

Spektraeder sind grundlegende Objekte in der semidefiniten Optimierung, einem Teilgebiet der konvexen Optimierung mit Anwendungen in der Approximationstheorie, Kontrolltheorie, kombinatorischen Optimierung und Ingenieurswissenschaften, siehe [BPT] und [WSV]. Die Untersuchung von $\mathrm{Gram}(f)$ vom Standpunkt der konvexen algebraischen Geometrie ist relevant für die Optimierung linearer Funktionen über alle Quadratsummendarstellungen von f. Der Begriff des Gramspektraeders wurde von Plaumann, Sturmfels und Vinzant in [PSV, Abschnitt 6] geprägt. Die Autoren untersuchen die Extremalpunkte von $\mathrm{Gram}(f)$ von minimalem Rang und die Verbindungsstrecken zwischen diesen im Fall einer ternären Quartik f. Außerdem stellen sie ein numerisches Experiment mit Bezügen zur semidefiniten Optimierung über diesem Spektraeder vor. Gramspektraeder von binären Sextiken werden in [ORSV, Abschnitt 5] betrachtet, wo sie mit Kummerflächen in $\mathbb{P}^3$ in Beziehung gesetzt werden.

Es ist kein Zufall, dass der Schwerpunkt der Untersuchungen auf Extremalpunkten von minimalem Rang liegt. Jede nichtnegative binäre Form $f \in \mathbb{R}[x,y]_{2d}$ vom Grad $2d$ kann als Summe von zwei Quadraten geschrieben werden und es gibt 2^{d-1} grundlegend verschiedene solche Darstellungen, wenn die Nullstellen von f paarweise verschieden sind ([CLR, Beispiel 2.13]). Im bereits erwähnten Artikel von Hilbert zeigte dieser auch, dass jede nichtnegative ternäre Quartik $f \in \mathbb{R}[x,y,z]_4$ eine Summe von höchstens drei Quadraten ist. Die Tatsache, dass es (bis auf orthogonale Äquivalenz) genau acht Darstellungen von f als Summe von drei Quadraten gibt, wenn die durch f definierte Kurve in $\mathbb{P}^2$ glatt ist, ist ein Resultat jüngeren Datums, siehe [PRSS]. In höheren Graden oder wenn die Anzahl der Variablen größer ist, gibt es gemäß Hilberts Klassifikation nichtnegative Polynome, die nicht als Quadratsummen geschrieben werden können. Selbst wenn wir wissen, dass ein gegebenes Polynom f eine sos-Darstellung erlaubt, ist es im Allgemeinen nicht klar, wie viele Summanden man mindestens braucht oder wie viele verschiedene Darstellungen kürzester Länge existieren. Daher besteht oft ein besonderes Interesse daran, die Punkte von minimalem Rang in $\mathrm{Gram}(f)$ zu verstehen, da diese gerade den sos-Darstellungen von kürzester Länge entsprechen.

Fragen nach Nichtnegativität und Quadratsummendarstellungen können in einem allgemeineren Kontext betrachtet werden, beispielsweise für Formen im projektiven Koordinatenring von gewissen reellen projektiven Varietäten. In Theorem 4.0.1 zitieren wir ein Resultat von Blekherman, Smith und Velasco [BSV], das die Varietäten, für die jede nichtnegative quadratische Form eine Summe von Quadraten ist, von jenen abgrenzt, bei welchen dies nicht der Fall ist. Daraus ergibt sich eine weitgehende Verallgemeinerung der Hilbertschen Klassifikation. Für Varietäten von minimalem Grad – wo jede nichtnegative quadratische Form tatsächlich eine Quadratsumme ist – haben Blekherman, Plaumann, Sinn und Vinzant [BPSV] die Länge allgemeiner nichtnegativer quadratischer Formen bestimmt (siehe Theorem 4.1.5). Ein ähnliches Ergebnis für die Länge von Summen von Quadraten linearer Formen auf Varietäten von fast minimal Grad haben Chua, Plaumann, Sinn and Vinzant [CPSV] erhalten. Es umfasst frühere Sätze von Scheiderer, die die kürzeste Länge von sos-Darstellungen ternärer Sextiken und quaternärer Quartiken angeben ([Sch17, Theorem 4.1 und Theorem 4.2]). Wir zitieren es als Theorem 6.1.1 und sehen es als Motivation, Gramspektraeder im Kontext von Varietäten von fast minimalem Grad zu untersuchen.

Neben den neu erzielten Resultaten enthält der Artikel [CPSV] auch eine systematische Untersuchung von Gramspektraedern und sammelt bekannte Ergebnisse und offene Fragen. Wir werden uns daher oft auf ihn beziehen. Der Schwerpunkt des Artikels liegt jedoch immer noch auf den Extremalpunkten und es wird nicht viel über die Seitenstruktur von Gramspektraedern gesagt. Tatsächlich war bereits im einfachsten Fall, nämlich dem Fall der binären Formen, wenig über höherdimensionale Seiten bekannt, als wir unsere Arbeit begannen. Das Handwerkszeug für die Untersuchung der Seitenstruktur von Spektraedern wurde von Ramana und Goldman [RG] allerdings bereits in den 1990er Jahren bereitgestellt. Insbesondere haben sie die Seiten von Spektraedern dadurch charakterisiert, dass sie feststellten, dass alle Matrizen im relativen Inneren einer Seite denselben Kern haben. Indem er symmetrische Matrizen durch symmetrische Tensoren ersetzt, gibt Scheiderer in [Sch22, Abschnitt 2] einen Überblick über ihre Resultate, der ohne die Wahl von Koordinaten auskommt. Anschließend zeigt er, wie sein neuer koordinatenfreier Ansatz genutzt werden kann, um die Dimension von Seiten von Gramspektraedern zu berechnen. Wir werden hauptsächlich diesen Ansatz verwenden, da er einen direkten Zugriff auf die Polynome ermöglicht, die in einer Quadratsummendarstellung vorkommen. Nichtsdestotrotz werden wir an einigen Stellen Grammatrizen verwenden, wenn es zweckdienlich erscheint.

In Kapitel 1 erinnern wir an wichtige Begriffe aus der Konvexgeometrie und der (semi)algebraischen Geometrie. Außerdem machen wir einführende Bemerkungen zu Quadratsummen, positiv semidefiniten Matrizen und Spektraedern. Wie die Wahl des Titels bereits andeutet, ist das Ziel der Arbeit die Untersuchung von Gramspektraedern quadratischer Formen auf Varietäten von minimalem und fast minimalem Grad. Wir konzentrieren uns dabei auf torische Varietäten dieser Art, die mittels der Gitterpunkte $P \cap \mathbb{Z}^n$ eines normalen Gitterpolytops $P \subseteq \mathbb{R}^n$ in den projektiven Raum eingebettet sind. In diesen Fällen können wir die Ergebnisse im Hinblick auf Polynome mit vorgegebenen Newtonpolytopen interpretieren. Zusätzlich zu den allgemeinen Konzepten, die in der algebraischen Geometrie weit verbreitet sind, gibt Kapitel 1 daher auch einen kurzen Überblick über torische Varietäten und Gitterpolytope, wobei wir den Schwerpunkt auf die für die vorliegende Arbeit wichtigen Eigenschaften legen.

Kapitel 2 ist eine detaillierte Einführung in Gramspektraeder, wo wir sowohl an die klassische Grammatrix-Methode als auch an Scheiderers neuen koordinatenfreien Ansatz erinnern. Die allgemeine Situation ist folgende: Gegeben sei eine $\mathbb{R}$-Algebra A, ein endlichdimensionaler Untervektorraum $V \subseteq A$ und ein beliebiges $f \in A$. Dann ist das *Gramspektraeder* von f relativ zu V definiert als

$$\mathrm{Gram}_V(f) := \{\vartheta \in \mathsf{S}_2 V : \vartheta \succeq 0,\, \mu(\vartheta) = f\},$$

wobei $\mathsf{S}_2 V$ den Raum der symmetrischen Tensoren in $V \otimes V$ und $\mu\colon V \otimes V \to A$ die durch $p \otimes q \mapsto pq$ ($p, q \in V$) definierte Multiplikationsabbildung bezeichnen. Fürs Erste kann der Leser an die wichtigsten Anwendungen denken. Diese sind zum einen der Fall, dass $P \subseteq \mathbb{R}^n$ ein Gitterpolytop und $f \in \mathbb{R}[\underline{x}]_{2P}$ ein Polynom ist, dessen Newtonpolytop in $2P$ enthalten ist. Wir nehmen dann $V = \mathbb{R}[\underline{x}]_P$. Zum anderen denken wir an den Fall, dass $f \in \mathbb{R}[X]_2$ eine quadratische Form im projektiven Koordinatenring einer reellen projektiven Untervarietät $X \subseteq \mathbb{P}^n$ ist, wo wir dann den Raum $V = \mathbb{R}[X]_1$ der linearen Formen auf X betrachten. In diesen Fällen wird V in der Notation zumeist weggelassen.

Wir sind besonders an der Seitenstruktur des Gramspektraeders $\mathrm{Gram}_V(f)$ interessiert. Gemäß Scheiderers Arbeit ist eine Seite $F \subseteq \mathrm{Gram}_V(f)$ durch das *Bild* $U := \mathrm{span}(p_1, \ldots, p_r) \subseteq V$ eines beliebigen psd *Gramtensors* $\vartheta = \sum_{i=1}^r p_i \otimes p_i \in \mathrm{Gram}_V(f)$ im relativen Inneren von F charakterisiert (Satz 2.3.4) und ihre Dimension hängt nur von $\mathrm{rk}(F) := \dim(U)$ und $\dim(UU)$ ab (Satz 2.3.9). Hierbei ist $UU = \mu(\mathsf{S}_2 U)$ der lineare Unterraum von A, der von allen Produkten pq mit $p, q \in U$ aufgespannt wird. Eine zentrale Aufgabe besteht daher darin, ein besseres Verständnis von $\dim(UU)$ in Abhängigkeit von $\dim(U)$ zu erlangen.

In Kapitel 3 untersuchen wir Gramspektraeder von binären Formen. Einige Resultate dieses Kapitels sind auch in einem zuvor erschienen Artikel [May21] des Autors enthalten und die folgenden Ausführungen überschneiden sich teilweise mit der Einleitung desselben. Wir starten unsere Untersuchungen, indem wir zeigen, welche

Paare $(\mathrm{rk}(F), \dim(F))$ von Rängen und Dimensionen von Seiten $F \subseteq \mathrm{Gram}(f)$ überhaupt vorkommen können. Dabei zeigen sich große Dimensionslücken (siehe Beispiel 3.1.11 für eine Illustration).

Wie Laurent und Poljak [LP96] in ihrer Abhandlung über das Elliptop schreiben, ist eine polyedrische Seite in gewissem Sinne der „singulärste Teil" des Randes eines Spektraeders. Außerdem sind Polyeder die einfachsten Beispiele von Spektraedern. Aus diesen Gründen ist man auch an polyedrischen Seiten von Gramspektraedern interessiert. Es zeigt sich, dass Gramspektraeder hinreichend allgemeiner binärer Formen polyedrische Seiten von hoher Dimension enthalten. Wir können diese jedoch besser verstehen, wenn wir von einem hermiteschen Standpunkt aus beginnen.

Ein Polynom $f \in \mathbb{R}[\underline{x}]$ als *hermitesche Quadratsumme* zu schreiben, also als $f = p_1\overline{p_1} + \cdots + p_r\overline{p_r}$ mit $p_1, \ldots, p_r \in \mathbb{C}[\underline{x}]$, ist eine weitere Möglichkeit, seine Nichtnegativität zu zertifizieren. Einerseits kann jedes hermitesche Quadrat $p\overline{p}$ als die Summe von zwei Quadraten $\mathrm{Re}(p)^2 + \mathrm{Im}(p)^2$ geschrieben werden, wobei $\mathrm{Re}(p) = (p+\overline{p})/2$ und $\mathrm{Im}(p) = (p-\overline{p})/2i$ Polynome mit reellen Koeffizienten sind. Folglich ist f genau dann eine reelle Quadratsumme, wenn es eine hermitesche Quadratsumme ist. Andererseits ist das *hermitesche Gramspektraeder* $\mathcal{H}^+(f)$, das die Darstellungen von f als hermitesche Quadratsumme parametrisiert, aus sich selbst heraus ein interessantes Objekt der konvexen algebraischen Geometrie, das auch in [CPSV, Abschnitt 5] betrachtet wurde. Wir ergänzen daher Kapitel 2 durch die Einführungen eines hermiteschen Analogons zu koordinatenfreien symmetrischen Gramspektraedern. Darüber hinaus beweisen wir eine Korrespondenz zwischen den Seiten des symmetrischen Gramspektraeders und speziellen Seiten des hermiteschens, die wir aufgrund ihrer Beschaffenheit *reell symmetrisch* nennen wollen (Abschnitt 2.5).

Kehren wir zur Diskussion binärer Formen zurück. Unter Verwendung einer Idee, die in [LP96] vorgestellt wurde, geben wir Schranken für die Dimension polyedrischer Seiten in (hermiteschen und symmetrischen) Gramspektraedern. Umgekehrt zeigen wir, dass das hermitesche Gramspektraeder einer allgemeinen positiven binären Form eine Seite enthält, die ein Simplex der größtmöglichen Dimension ist. Anschließend verwenden wir unsere Erkenntnisse aus dem hermiteschen Fall, um ein analoges Resultat für den reell symmetrischen Fall zu zeigen. Dies sind einige der Hauptresultate in Kapitel 3, die wie folgt gefasst werden können:

Theorem (Theoreme 3.3.13 und 3.5.12)**.** *Seien k, $d \in \mathbb{N}$. Ist $d \geq \binom{k+1}{2}$, so gibt es eine offen-dichte Teilmenge von nichtnegativen binären Formen f vom Grad $2d$ derart, dass das hermitesche Gramspektraeder $\mathcal{H}^+(f)$ eine k-dimensionale Seite enthält, die ein Simplex ist, dessen Ecken gewissen Darstellungen von f als hermitesches Quadrat entsprechen.*

Ist $d \geq (k+1)^2$, so gibt es eine offen-dichte Teilmenge von nichtnegativen binären Formen f vom Grad $2d$ derart, dass das symmetrische Gramspektraeder $\mathrm{Gram}(f)$ *eine k-dimensionale Seite enthält, die ein Simplex ist, dessen Ecken gewissen Darstellungen von f als Summe von zwei reellen Quadraten entsprechen.*

Wir beweisen das, indem wir unter Verwendung der Kombinatorik der komplexen Nullstellen einer binären Form eine explizite Konstruktion dieser polyedrischen Seiten angeben (vgl. Theorem 3.3.9). Wir diskutieren auch, wie viele Seiten dieser Art man erwarten sollte, wenn man die Trägerseiten von je $k+1$ verschiedenen Rang-1 Extremalpunkten in $\mathcal{H}^+(f)$ betrachtet. Im Fall von binären Sextiken ($d = 3$) mit paarweise verschiedenen Nullstellen führt das zu einem vollständigen Verständnis der Trägerseiten aller $\binom{8}{3} = 56$ möglicher Kombinationen von drei solchen Punkten (siehe Abschnitt 3.4).

Ein weiteres wichtiges Resultat ist Theorem 3.7.15, wo wir zeigen, dass das Gramspektraeder einer allgemeinen nichtnegativen binären Form $f \in \mathbb{R}[x,y]_{2d}$ Seiten von jedem Rang $r \in \{2,\dots,d+1\}$ der kleinstmöglichen Dimension enthält (kleinstmöglich in Bezug auf den Rang r). Das verallgemeinert einen Satz von Scheiderer ([Sch22, Theorem 5.3]), welcher sich auf die Ränge im sogenannten Pataki-Intervall bezieht, wo die Seiten der entsprechenden kleinstmöglichen Dimension einfach Extremalpunkte sind.

Eine binäre Form vom Grad $2d$ entspricht einer quadratischen Form auf der rationalen Normalenkurve vom Grad d, wobei Letztere eine Varietät von minimalem Grad ist. In Kapitel 4 weiten wir unseren Blick und betrachten Gramspektraeder von quadratischen Formen auf Varietäten dieser Art. Varietäten von minimalem Grad sind gut verstanden und vollständig klassifiziert (siehe z.B. [EH]). Abgesehen von einigen Quadriken können sie alle als torische Varietäten realisiert werden, die mittels der Gitterpunkte eines Polytops eingebettet werden. Das führte zu einer expliziten Charakterisierung der Newtonpolytope, für die jedes nichtnegative Polynom eine Summe von Quadraten ist (vgl. [BSV, Abschnitt 6] und [CPSV, Theorem 2.1]). Wir kommen auf dieses Resultat in den Abschnitten 4.2 und 4.3 zurück.

Quadratische Formen auf der Veronesefläche $v_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$ entsprechen ternären Quartiken. Letztere haben traditionell immer viel Aufmerksamkeit auf sich gezogen und Gramspektraeder wurden wie bereits bemerkt zuerst in diesem Fall betrachtet. Dank kürzlich vollendeter Arbeit von Vill [Vill] sind Gramspektraeder von ternären Quartiken im Hinblick auf ihre Seitenstruktur im Wesentlichen vollständig verstanden.

Die Beiträge dieser Arbeit betreffen daher vorwiegend die verbleibende Klasse von Varietäten von minimalem Grad, nämlich die rationalen Normalenrollen. Theorem 4.6.2 ist eine Verallgemeinerung der Dimensionsschranken für Seiten in Gramspektraedern vom Fall der rationalen Normalenkurve (binäre Formen) auf höherdimensionale Rollvarietäten. In Abschnitt 4.7 zeigen wir neue Ungleichungen auf, die nicht nur von der Dimension, sondern auch von der Struktur der rationalen Normalenrolle abhängen. Das zeigt, dass man nicht erwarten kann, dass sich das zuvor genannte Theorem 3.7.15 auf allgemeine Varietäten von minimalem Grad übertragen lässt. Das hängt auch mit Folgendem zusammen: Wenn ein $d \in \mathbb{N}$ und eine positive binäre Form $f \in \mathbb{R}[x,y]_{2d}$ gegeben sind, können wir quadratische Formen auf torischen Flächen von minimalem Grad betrachten, deren Gramspektraeder die gleiche Dimension wie $\mathrm{Gram}(f)$ haben. Für kleine Werte von d ziehen wir in Abschnitt 4.5 einen Vergleich zwischen der Struktur dieser Spektraeder und der von $\mathrm{Gram}(f)$.

Es ist zu erwähnen, dass Vill auch über den Fall der ternären Quartiken hinaus bemerkenswerte Schranken für die Dimension von Seiten angibt. Das tut er, indem er für Unterräume $U \subseteq \mathbb{R}[\underline{x}]_d$ von fester Kodimension die Kodimension von UU in $\mathbb{R}[\underline{x}]_{2d}$ abschätzt. In der Sprache der Newtonpolytope bedeutet das, dass P das skalierte $(n-1)$-dimensionale Standardsimplex $d\Delta_{n-1} = \{\alpha \in \mathbb{R}^n_{\geq 0} : \sum_i \alpha_i = d\}$ ist. Da er immer unter dieser Annahme arbeitet und die besten Schranken für Unterräume von kleiner Kodimension erzielt, ist das nicht direkt auf unseren Fall übertragbar, denkt man doch an die ganzen verschiedenen Formen, die Polytope haben können, welche Varietäten von (fast) minimalem Grad definieren.

Kapitel 5 ist kürzer als die anderen und behandelt verschiedene Themen, die auf den ersten Blick ohne Bezug zueinander erscheinen könnten. Wir erläutern die Zusammensetzung dieses Kapitels in dessen Einleitung. An dieser Stelle sei daher nur erwähnt, dass Abschnitt 5.1 aus einer gemeinsamen Arbeit mit Julian Vill hervorgeht, wo wir Resultate von de Carli Silva und Tunçel [CST] nutzen, um die Dimension von Normalenkegeln von Gramspektraedern zu bestimmen (Theorem 5.1.18). Damit kann man die Spitzen dieser Spektraeder identifizieren (vgl. Theorem 5.1.21), also solche Punkte mit volldimensionalem Normalenkegel, die in der Optimierung von besonderer Bedeutung sind.

Spektraeder sind Mengen zulässiger Lösungen in Problemen der semidefiniten Programmierung. In Theorem 5.3.1 zeigen wir, dass jedes Spektraeder, das keine affine Gerade enthält, linear isomorph zum Gramspektraeder einer quadratischen Form in einem Quotientenring vom Typ $\mathbb{R}[x_1, \dots, x_n]/I$ ist, wobei $I \subseteq \mathbb{R}[\underline{x}]$ ein homogenes Ideal ist.

Kapitel 6 behandelt dann Gramspektraeder im Kontext von Varietäten von fast minimalem Grad. Das bereits erwähnte Resultat von Chua et al., welches die Länge von Quadratsummendarstellungen von quadratischen Formen auf solchen Varietäten betrifft, umfasst torische Varietäten, für die man wunderbare Interpretationen auf der Seite der Polynome mit gewissen Newtonpolytopen hat. Als Beispiele haben wir zuvor die ternären Sextiken und quaternäre Quartiken benannt. In [CPSV, Bemerkung 3.7] deuten die Autoren an, wie man die Gitterpolytope findet, die andere torische Varietäten definieren, welche ebenfalls die Voraussetzungen ihres Theorems erfüllen. Wir arbeiten das in den Abschnitten 6.2 und 6.3 sorgfältig aus. Das Hauptresultat in Bezug auf die Seitenstruktur von Gramspektraedern in diesem Rahmen ist Theorem 6.4.8, wo wir Dimensionsschranken zeigen, die die gleiche Gestalt annehmen wie für Varietäten von minimalem Grad.

Im ersten Schritt benutzt der Beweis nur die kombinatorische Struktur von Gorensteinpolytopen vom Grad 2, wobei Letztere von Batyrev und Juny [BJ] klassifiziert wurden. Dabei werden wir auch Resultate aus der faszinierenden Theorie der (reflexiven) Gitterpolytope anwenden (siehe z.B. [BN08], [Nill]). Wenn der kombinatorische Ansatz nicht mehr ausreichend ist, schauen wir aus einer algebraischen Perspektive genauer hin. Als interessante Spezialfälle erhalten wir Ergebnisse zur Seitenstruktur von Gramspektraedern von Polynomen $f \in \mathbb{R}[x, y]$ mit $\deg_x(f) \leq 4$, $\deg_y(f) \leq 4$ (Beispiel 6.5.11) und für Polynome $f \in \mathbb{R}[x, y, z]$, die Summen von Quadraten multiaffiner Polynome sind (Beispiel 6.6.8).

Am Ende dieser Arbeit benennen wir einige noch offene Probleme.

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