

Characterizing Families of Cuts That Can Be Represented by Axis-Parallel Rectangles^{*}

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Abstract. A drawing of a family of cuts of a graph is an augmented drawing of the graph such that every cut is represented by a simple closed curve and vice versa.

We show that the families of cuts that admit a drawing in which every cut is represented by an axis-parallel rectangle are exactly those that have a cactus model that can be rooted such that edges of the graph that cross a cycle of the cactus point to the root. This includes the family of all minimum cuts of a graph. The proof also yields an efficient algorithm to construct a drawing with axis-parallel rectangles if it exists.

1 Introduction

A cut of a graph is a partition of its vertex set into two non-empty subsets. In a drawing of a graph, it is therefore natural to represent a cut by a closed curve partitioning the plane into two regions containing one subset each.

When a set of cuts is drawn in this way, the curves can intersect so that their union might contain closed curves that represent other cuts not contained in that set. We are interested in families of cuts that can be drawn without creating confusing non-members. In particular, we consider the problem of drawing families of cuts such that every member is represented by an axis-parallel rectangle and vice versa.

Our main result is a characterization of all families of cuts that can be represented by axis-parallel rectangles, namely those that can be modeled by a cactus containing a certain, rootable, node. These include the important family of all minimum cuts of a graph [4]. We give an algorithm to test this property. Provided a set of cuts of a graph with n vertices and m edges is represented by a cactus model, the test works in $\mathcal{O}(nm)$ time. We also show how to construct a drawing with axis-parallel rectangles if one exists. The construction works in two

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steps. First, we solve the problem for families of minimal cuts of a planar connected graph. Then, we use similar planarization techniques as in [3] to extend the result to general graphs.

The paper is organized as follows. In Sect 2, we define drawings of families of cuts. In Sect. 3, we introduce the cactus model and show that its existence is necessary for the existence of a drawing with axis-parallel rectangles. Using hierarchically clustered graphs (briefly introduced in Sect. 4), we give an additional necessary condition for families of cuts that admit a drawing with axis-parallel rectangles in Section 5 and show how to test it. To show that the conditions are also sufficient, we construct a drawing in Section 6.

2 Drawing Families of Cuts

Throughout this paper, let $G = (V, E)$ denote a simple, connected undirected graph with n vertices and m edges. A drawing \mathcal{D} of G maps vertices on distinct points in the plane and edges on simple curves connecting the drawings of their incident vertices. The interior of an edge must not contain the drawing of a vertex. A drawing is planar, if edges do not intersect but in common end points. Let $S, T \subseteq V$. With $G(S)$ we denote the subgraph of G that is induced by S and with $E(S, T)$ we denote the set of edges that are incident to a vertex in S and a vertex in T . A *cut* of G is a partition $C = \{S, \bar{S}\}$ of the vertex set V into two non-empty subsets S and $\bar{S} := V \setminus S$. We say that the cut $\{S, \bar{S}\}$ is induced by S . The edges in $E(C) = E(S, \bar{S})$ are the *cut-edges* of C . A *minimum cut* is a cut with the minimum number of cut-edges among all cuts of G . A *minimal cut* is a cut $\{S, \bar{S}\}$ that is inclusion-minimal, i.e. there is no cut $\{T, \bar{T}\}$ with $E(T, \bar{T}) \subsetneq E(S, \bar{S})$. Note that both, $G(S)$ and $G(\bar{S})$, are connected if $\{S, \bar{S}\}$ is a minimal cut of the connected graph G .

A *drawing of a cut* $C = \{S, \bar{S}\}$ of G in a drawing $\mathcal{D}(G)$ is a simple closed curve γ , such that

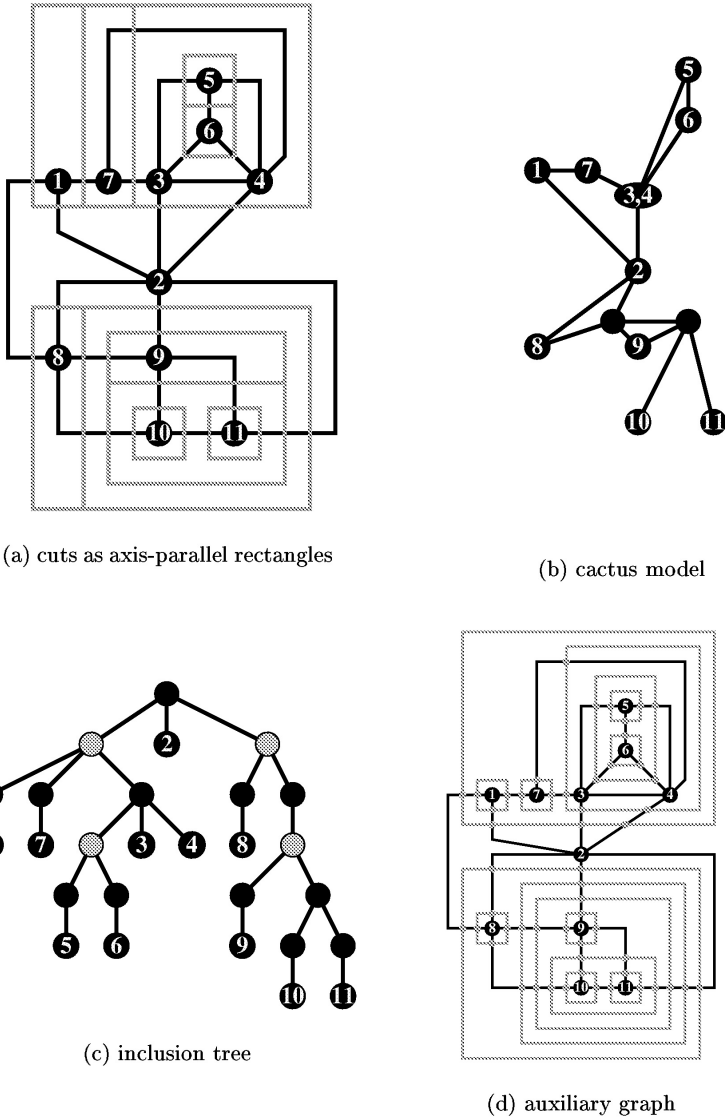
- γ separates S and \bar{S} , i.e. the drawings of edges and vertices in $G(S)$ and $G(\bar{S})$, respectively, are in different connected regions of $\mathbb{R}^2 \setminus \gamma$, and
- $|\mathcal{D}(e) \cap \gamma| = 1$ for $e \in E(C)$, i.e. the drawing of a cut-edge crosses the drawing of the cut exactly once.

Let \mathcal{C} be a set of cuts of a graph G . A mapping \mathcal{D} is a (*planar*) *drawing* of a graph G and a family of cuts \mathcal{C} of G , if

1. $\mathcal{D}(G)$ is a (planar) drawing of G and
2. $\mathcal{D}(C)$ is a drawing of C in $\mathcal{D}(G)$ for every cut $C \in \mathcal{C}$, and
3. every simple closed curve $\gamma \subseteq \bigcup_{C \in \mathcal{C}} \mathcal{D}(C)$ is a drawing of some cut in \mathcal{C} .

Note that the third condition eliminates any potential ambiguity regarding which cuts are in the family.

A drawing \mathcal{D} of a graph and a family \mathcal{C} of cuts is a *drawing with axis-parallel rectangles*, if every cut is drawn as an axis-parallel rectangle – more precisely, if every simple closed curve in $\bigcup_{C \in \mathcal{C}} \mathcal{D}(C)$ is an axis-parallel rectangle. See Fig. 1(a) for a planar drawing with axis-parallel rectangles of the set of all minimum and minimum+1 cuts of a graph.



(a) cuts as axis-parallel rectangles

(b) cactus model

(c) inclusion tree

(d) auxiliary graph

Fig. 1. A drawing of the family of all minimum and minimum+1 cuts of a graph with axis-parallel rectangles. In the cactus model of the cuts, φ is indicated by node labels, and cycle-replacement nodes in the inclusion tree constructed from the cactus are shown in grey. The auxiliary graph is shown in a c-planar drawing of the hierarchical clustering represented by the inclusion tree.

3 Necessity of a Cactus Model

A *cactus* is a connected graph in which every edge belongs to at most one cycle. A *cactus model* for a set \mathcal{C} of cuts of a graph $G = (V, E)$ is a pair (\mathcal{G}, φ) that

consists of a cactus $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a mapping $\varphi : V \rightarrow \mathcal{V}$ such that \mathcal{C} is modeled by the minimal cuts of \mathcal{G} , i.e.,

$$\mathcal{C} = \{\{\varphi^{-1}(S), \varphi^{-1}(\overline{S})\}; \{S, \overline{S}\} \text{ is a minimal cut of } \mathcal{G}\}.$$

To avoid confusion, the vertices of the cactus \mathcal{G} are called *nodes*. A node $\nu \in \mathcal{V}$ is called *empty* if $\varphi^{-1}(\nu) = \emptyset$.

An important family of cuts that has a cactus model is the set of all minimum cuts of a connected graph [4]. A cactus model for the minimum and minimum+1 cuts of the graph in Fig. 1(a) is given in Fig. 1(b).

While not every family of cuts has a cactus model, we show that only those that do can have a drawing with axis-parallel rectangles. Two cuts $\{S, \overline{S}\}$ and $\{T, \overline{T}\}$ cross, if and only if the four corners $S \cap T$, $S \setminus T$, $T \setminus S$, and $\overline{S} \cup \overline{T}$ are non-empty. The four cuts induced by the four corners of two crossing cuts, respectively, are called *corner cuts*. The cut induced by $(S \setminus T) \cup (T \setminus S)$ is called the *diagonal cut*.

Theorem 1 ([5]). *A set \mathcal{C} of cuts of the graph G can be modeled by a cactus if and only if, for any two crossing cuts $\{S, \overline{S}\}$ and $\{T, \overline{T}\}$ in \mathcal{C} ,*

- *the four corner cuts are in \mathcal{C} , and*
- *the diagonal cut is not in \mathcal{C} .*

If a cactus model exists, there is always one with $O(n)$ nodes.

The properties of crossing cuts in this characterization are implied by overlapping axis-parallel rectangles.

Lemma 1. *If a set of cuts has a drawing with axis-parallel rectangles, it has a cactus model.*

Proof. Let \mathcal{D} be a drawing of a set \mathcal{C} of cuts with axis-parallel rectangles and suppose that \mathcal{C} contains crossing cuts $\{S, \overline{S}\}$ and $\{T, \overline{T}\}$. There are essentially the seven cases indicated in Fig. 2 for the drawings of two crossing cuts by axis-parallel rectangles. Since the cases in Fig. 2(a) contain simple closed curves that are not axis-parallel rectangles, only the case in Fig. 2(b) needs to be considered.

Let $D_S, D_T \subseteq \mathbb{R}^2$ be the rectangular regions bounded by $\mathcal{D}(\{S, \overline{S}\})$ and $\mathcal{D}(\{T, \overline{T}\})$, respectively. Then regions $D_S \cap D_T$, $D_S \setminus D_T$, $D_T \setminus D_S$, and $D_S \cup D_T$ are bounded by axis-parallel rectangles in $\mathcal{D}(\{S, \overline{S}\}) \cup \mathcal{D}(\{T, \overline{T}\})$. These are drawings of the four corner cuts of $\{S, \overline{S}\}$ and $\{T, \overline{T}\}$. Hence, they are in \mathcal{C} .

On the other hand, suppose the diagonal cut C induced by $(S \setminus T) \cup (T \setminus S)$ is in \mathcal{C} and let D_C be the rectangular region bounded by its drawing $\mathcal{D}(C)$. Without loss of generality, suppose that $\mathcal{D}(S) \subset D_S$ and $\mathcal{D}(T) \subset D_T$. Either D_C contains $\mathcal{D}(G(S \setminus T))$ and $\mathcal{D}(G(T \setminus S))$, but not $\mathcal{D}(G(S \cap T))$, or it contains $\mathcal{D}(G((S \setminus T) \cup (T \setminus S)))$, but not $\mathcal{D}(G(S \setminus T \cup T \setminus S))$. In the first case, the drawing contains a simple closed curve bounding the region $D_C \cap D_S \cap D_T$ (thus inducing an empty cut), and in the second case the union of the three rectangles contains a simple closed curve that is not an axis-parallel rectangle. \square

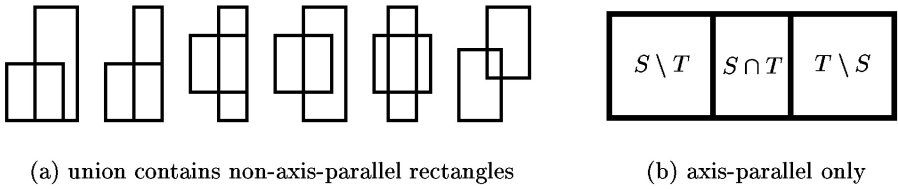


Fig. 2. Drawings of two crossing cuts $\{S, \bar{S}\}$ and $\{T, \bar{T}\}$ with axis-parallel rectangles.

4 Cactus-Induced Hierarchical Clusterings

Since a cactus model is necessary for a drawing to exist, we can make use of a transformation originally developed for drawing the particular family of all minimum cuts of a planar connected graph [1]. Given a cactus model (\mathcal{G}, φ) of a family \mathcal{C} of G , we use the tree $T = \mathcal{T}(\mathcal{G}, \varphi)$ that is constructed as follows.

1. Choose a node r of \mathcal{G} as root.
2. Replace each cycle c of \mathcal{G} by a star, i.e. delete every edge of c , add a new empty node ν_c – called *cycle-replacement node* of c – to \mathcal{G} , and for every node ν of c , add an edge $\{\nu_c, \nu\}$ to \mathcal{G} .
3. For each $v \in V$, add v to the vertex set of \mathcal{G} and add an edge $\{v, \varphi(v)\}$ to \mathcal{G} .

The construction is illustrated in Fig. 1(c) and yields a triple (G, T, r) of

- a graph $G = (V, E)$,
- a tree T , and
- an inner vertex r of T
- where the set of leaves of T is exactly V .

This is the hierarchically clustered graph model introduced by Feng et al. [6]. G is called the *underlying graph* and T the *inclusion tree* of (G, T, r) . Inner vertices of T are called *nodes*. We denote the tree T rooted at r by (T, r) . Each node ν of T represents the *cluster* $V_r(\nu)$ of leaves in the subtree of (T, r) rooted at ν . Note that $\{\{V_r(\nu), \overline{V_r(\nu)}\}; \nu \neq r \text{ node of } \mathcal{T}(\mathcal{G}, \varphi)\}$ equals the subset \mathcal{C}_{\parallel} of *non-crossing cuts* of \mathcal{C} , i.e. the set of cuts in \mathcal{C} that do not cross any other cut in \mathcal{C} . A *c-planar drawing* of a hierarchically clustered graph (G, T, r) consists of

- a planar drawing of the underlying graph G and
- an inclusion representation¹ of the rooted tree (T, r) such that
- each edge crosses the boundary of the drawing of a node of T at most once.

¹ In an inclusion representation of a rooted tree (T, r) , each node of T is represented by a simple closed region bounded by a simple closed curve. The drawing of a node or leaf ν of T is contained in the interior of the region representing a node μ of T if and only if μ is contained in the path from ν to r in T . The drawings of two nodes μ and ν are disjoint if neither μ is contained in the path from ν to r nor ν is contained in the path from μ to r in T .

Note that the vertices of G are the leaves of T and thus have the same drawing. Also note, that a c -planar drawing of $(G, \mathcal{T}(\mathcal{G}, \varphi), r)$ contains a planar drawing of the set of all non-crossing cuts \mathcal{C}_{\parallel} . A hierarchically clustered graph (G, T, r) is c -planar if it has a c -planar drawing. It is *completely connected*, if each cluster and the complement of each cluster induces a connected subgraph of G . A hierarchically clustered graph with planar underlying graph does not have to be c -planar. However, a completely connected hierarchically clustered graph is c -planar if only the underlying graph is planar [2].

According to the construction in [1,9], we associate an auxiliary graph $G_{\mathcal{D}}$ with a c -planar drawing \mathcal{D} of a hierarchically clustered graph. Let V' be the set of points, in which drawings of edges and boundaries of drawings of clusters intersect. Then the vertex set of $G_{\mathcal{D}}$ is $V \cup V'$. The edge set of $G_{\mathcal{D}}$ contains two types of edges. For an edge $e = \{v, w\}$ of G , let v_1, \dots, v_k be the points in $\mathcal{D}(e) \cap V'$ in the order they occur in the drawing of e from v to w . Then $G_{\mathcal{D}}$ contains the edges $\{v, v_1\}, \{v_1, v_2\}, \dots, \{v_k, w\}$. Let $\nu \neq r$ be a node of T . Let v_1, \dots, v_k be the points in $\partial\mathcal{D}(\nu) \cap V'$ in the order they occur in the boundary $\partial\mathcal{D}(\nu)$ of the drawing of ν . Then $G_{\mathcal{D}}$ contains the edges $\{v_1, v_2\}, \dots, \{v_{k-1}, v_k\}, \{v_k, v_1\}$. The cycle v_1, \dots, v_k of $G_{\mathcal{D}}$ is called the *boundary cycle* of ν . (To avoid loops and parallel edges, additional vertices of degree two may be inserted into boundary cycles). See Fig. 1(d).

5 Towards a Characterization

Another necessary condition for families of cuts that have a drawing with axis-parallel rectangles depends also on the edges in the graph. Let (\mathcal{G}, φ) be the cactus model of a set \mathcal{C} of cuts of a graph G . For a cycle $c : \nu_1, \dots, \nu_k$ in \mathcal{G} let $V_i := V_{\nu_c}(\nu_i), i = 1, \dots, k$. An edge $\{v, w\}$ of G *crosses cycle* c if there are $1 \leq i, j \leq k$ such that

$$v \in V_i, w \in V_j, \text{ and } i - j \not\equiv \pm 1 \pmod k.$$

If (\mathcal{G}, φ) is the cactus of all minimum cuts of G , then no edge of G crosses a cycle of \mathcal{G} . In general, it depends on the edges that cross a cycle of the cactus model, whether a set of cuts has a drawing with axis-parallel rectangles. More precisely, if \mathcal{C} has a drawing \mathcal{D} with axis-parallel rectangles and e crosses the cycle c then there exists an $i \in \{1, \dots, k\}$ such that e is incident to a vertex in V_i and the drawing of \bar{V}_i is contained in the simple region bounded by $\mathcal{D}(V_i, \bar{V}_i)$. This statement is further formalized in the next lemma.

Lemma 2. *A family of cuts that has a cactus model (\mathcal{G}, φ) has a drawing with axis-parallel rectangles only if the root r of $\mathcal{T}(\mathcal{G}, \varphi)$ can be chosen such that*

(R) *each edge of G that crosses a cycle c of \mathcal{G} is incident to a vertex in $V \setminus V_r(\nu_c)$.*

Proof. Suppose that the set of cuts modeled by (\mathcal{G}, φ) has a drawing \mathcal{D} with axis-parallel rectangles. Let r be a node of $T = \mathcal{T}(\mathcal{G}, \varphi)$ such that for every node $\nu \neq r$ of T the set $V_r(\nu)$ is contained in the simple region bounded by

$\mathcal{D}(\{V_r(\nu), \overline{V_r(\nu)}\})$. Using the fact that $\overline{V_r(\mu)} = V_\mu(r)$ for two adjacent nodes r and μ of T , it can be shown that such a node r exists.

Let c be a cycle of \mathcal{G} and let e be an edge of G that crosses c . Let ν_1, \dots, ν_k be the descendants of ν_c such that ν_1, \dots, ν_k is a path in c . Suppose that e is not incident to a vertex in $V \setminus V_r(\nu_c)$. Then $e \in E(V_r(\nu_i), V_r(\nu_j))$ for some $1 < i + 1 < j \leq k$. Let $S = \bigcup_{\ell=1}^{i+1} V_r(\nu_\ell)$ and $T = \bigcup_{\ell=i+1}^k V_r(\nu_\ell)$. Then, on one hand, $e \in E(S \setminus T, T \setminus S)$. But, on the other hand, $\{S, \overline{S}\}$ and $\{T, \overline{T}\}$ are two crossing cuts that are modeled by (\mathcal{G}, φ) such that S and T are contained in the simple closed region D_S and D_T bounded by $\mathcal{D}(\{S, \overline{S}\})$ and $\mathcal{D}(\{T, \overline{T}\})$, respectively. Hence, we have again the situation as indicated in Fig. 2(b). By the definition of drawings of cuts, the drawing of the edges and vertices of $G(S \cup T)$ has to be contained inside $D_S \cup D_T$ and the drawing of the edges and vertices of $\mathcal{D}(G(\overline{S} \cap \overline{T}))$ may not intersect $D_S \cap D_T$. This implies that $E(S \setminus T, T \setminus S) = \emptyset$. □

In the following, a node of the inclusion tree $T = \mathcal{T}(\mathcal{G}, \varphi)$ is called *rootable*, if it fulfills Condition (R) of the previous lemma. We give an algorithm for finding all rootable nodes of the inclusion tree $T = \mathcal{T}(\mathcal{G}, \varphi)$. We assume that the size of \mathcal{G} is in $\mathcal{O}(n)$.

We use a node and edge array `CROSSED` on T to store the information about edges of G that cross a cycle of the cactus. Let c be a cycle of \mathcal{G} . `CROSSED`(ν_c) is true if and only if there is an edge of G that crosses c . `CROSSED`($\{\nu_c, \nu\}$) is true for an edge $\{\nu_c, \nu\}$ of T if and only if `CROSSED`(ν_c) is true and each edge that crosses c is incident to a vertex in $V_{\nu_c}(\nu)$. For all other edges and nodes of T `CROSSED` is false. Clearly, it can be tested in linear time, which cycles are crossed by an edge. Hence, `CROSSED` can be computed in $\mathcal{O}(mn)$ time. The next lemma is a reformulation of Condition (R).

Lemma 3. *A node r of T is rootable if and only if for each node ν of T with `CROSSED`(ν) = TRUE there exists an adjacent node μ of ν with `CROSSED`{ ν, μ } = TRUE such that r is contained in the subtree of (T, ν) rooted at μ .*

Hence, we can obtain the possible roots for constructing a drawing with axis-parallel rectangles of \mathcal{C} – possibly including some cycle-replacement nodes – by deleting for all marked nodes ν the subtrees of (T, ν) rooted at those adjacent nodes μ of ν for which $\{\nu, \mu\}$ is not marked. Proceeding first from the leaves to an arbitrary root and then from this root to the leaves of T , this can be done in time $\mathcal{O}(n)$.

By constructing a drawing, we show in the next section that the conditions in Lemma 2 are also sufficient.

6 The Drawing

Let (\mathcal{G}, φ) continue to be a cactus model of a set \mathcal{C} of cuts of the graph G and let $T = \mathcal{T}(\mathcal{G}, \varphi)$ be the inclusion tree constructed in Sect. 4. We assume that the root r of T is a rootable node. We show how to construct a drawing with

axis-parallel rectangles for \mathcal{C} . In a first step, we consider the case that \mathcal{C} is a set of minimal cuts of a connected planar graph. Using planarization, we generalize the result to general sets of cuts of not necessarily planar graphs.

6.1 Planar Graphs

In this section, we assume that \mathcal{C} is a set of minimal cuts of a planar connected graph G . We show how the ideas for drawing the set of all minimum cuts of a planar graph [1] yield a drawing with axis-parallel rectangles of \mathcal{C} . We briefly sketch the general construction and explain in more details the parts that differ from the case of drawings for the set of minimum cuts. The construction starts with a c-planar drawing \mathcal{D} of the hierarchically clustered graph (G, T, r) . Such a c-planar drawing always exists: Since \mathcal{C} is a set of minimal cuts of a connected graph it follows that (G, T, r) is completely connected and hence c-planar.

Let $c : \nu_1, \dots, \nu_\ell$ be a cycle of \mathcal{G} . Suppose that ν_c is a descendent of ν_ℓ in the tree T rooted at r . Let $V_i = V_{\nu_c}(\nu_i), i = 1, \dots, \ell$ and let $V_0 = V_\ell$. Let $i \in \{1, \dots, \ell - 1\}$ and let e_1, \dots, e_k be the sequence of edges incident to a vertex in V_i in the cyclic order around V_i . Then $E(V_i, V_{i-1})$ and $E(V_i, V_{i+1})$ are non-empty subsequences of e_1, \dots, e_k , i.e. suppose $e_1 \in E(V_i, V_{i-1})$ and $e_k \notin E(V_i, V_{i-1})$, then there are indices $1 < k_1 \leq k_2 < k_3 \leq k$ such that

$$\underbrace{e_1, \dots, e_{k_1}}_{E(V_i, V_{i-1})}, \underbrace{e_{k_1+1}, \dots, e_{k_2}}_{E(V_i, V_\ell)}, \underbrace{e_{k_2+1}, \dots, e_{k_3}}_{E(V_i, V_{i+1})}, \underbrace{e_{k_3+1}, \dots, e_k}_{E(V_i, V_\ell)}. \tag{1}$$

Further, $E(V_\ell, V_1)$ and $E(V_\ell, V_{\ell-1})$ are non-empty subsequence of the edges around $V \setminus V_\ell$. This observation can be proved in the following two steps.

1. Since G is connected and \mathcal{C} is a set of minimal cuts, it follows that the sets $E(V_i, V_{i-1}), E(V_i, V_{i+1}), i = 1, \dots, \ell - 1$ are non-empty.
2. It follows from Step 1 and the c-planarity that $E(V_i, V_{i-1}), E(V_i, V_{i+1}), i = 1, \dots, \ell - 1$ are subsequences of the cyclic sequence of edges around V_i and that $E(V_\ell, V_1), E(V_\ell, V_{\ell-1})$ are subsequences of the cyclic sequence of edges around $V \setminus V_\ell$.

Besides, let $e \in E(V_i, V_j)$ for some $i, j \in \{1, \dots, \ell\}, i \neq j$. Then c-planarity implies that the two vertices in the auxiliary graph $G_{\mathcal{D}}$ that represent the intersection of e with the boundary of ν_i and ν_j (or the boundary of ν_c , if $i = \ell$ or $j = \ell$), respectively, are adjacent. Hence the situation in the auxiliary graph $G_{\mathcal{D}}$ is as indicated in Fig. 3a. For $i \in \{1, \dots, \ell - 1\}$, let p_i^\pm be the path in $G_{\mathcal{D}}$ that is induced by the intersection of the edges in $E(V_i, V_{i\pm 1})$ with the boundary of ν_i . Similarly, let $p_0^+ (p_\ell^-)$ be the path in $G_{\mathcal{D}}$ that is induced by the intersection of the edges in $E(V_\ell, V_1) (E(V_\ell, V_{\ell-1}))$ with the boundary of ν_c .

A planar drawing of \mathcal{C} with axis-parallel rectangles can now be obtained as follows. First, for each i , paths p_i^+ and p_{i+1}^- are united to one path. The parts of the boundary-cycle of ν_c that are not in p_0^+ or p_ℓ^- are removed. The result can be seen in Fig. 3b. Finally, at each end of each thus united path p_i^+ / p_{i+1}^- ,

$i = 1, \dots, \ell - 2$, an additional vertex is inserted as indicated in Fig. 3c. In the following, we refer to these vertices as *cycle-path end-vertices*. Let G' be the graph that results from $G_{\mathcal{D}}$ by doing this for every cycle of \mathcal{G} . Applying a min-cost flow approach for quasi-orthogonal drawings [10,8,7,11] with some restriction on the flow to G' yields a drawing of \mathcal{C} with axis-parallel rectangles. There are two restrictions on the flow necessary

1. the flow over a boundary edge from outside a boundary cycle into the inside of a boundary cycle is zero
2. the flow from a cycle-path end-vertex into the inside of a boundary cycle is minimum, i.e. there is a rectangle.

It can be shown that there is always a feasible flow for the thus restricted flow network.

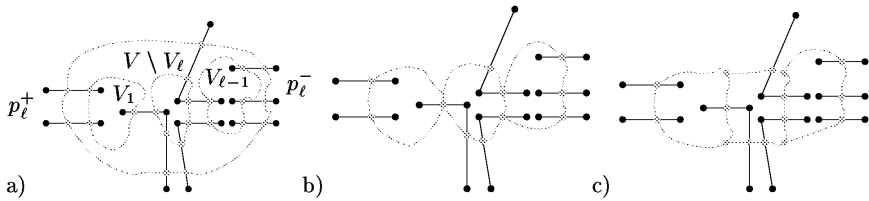


Fig. 3. Constructing a drawing for a set of cuts from a c-planar drawing.

6.2 General Graphs

In this section, we extend the results on planar drawings for families of minimal cuts to not necessarily planar drawings. Similar to the method described in [3] for hierarchically clustered graphs, the idea for the construction uses planarization techniques. Recall that (\mathcal{G}, φ) is a cactus model for a family of cuts of G . We assume again that the root r of $T = \mathcal{T}(\mathcal{G}, \varphi)$ is rootable.

1. Let $G_P = (V, E_P)$ be a planar connected graph on the same set of vertices as G , such that \mathcal{G} models a set of minimal cuts and each edge of G_P that crosses a cycle c of \mathcal{G} is incident to a vertex in $V \setminus V_r(\nu_c)$.
2. Construct the auxiliary graph G'_P for $(G_P, \mathcal{G}, \varphi)$ as it is described in Sect. 6.1.
3. Construct a planar graph G' , by inserting the remaining edges of G into G'_P such that each edge crosses the boundary cycle of a cut at most once.
4. Apply an algorithm for quasi-orthogonal drawings with the restrictions indicated in Sect. 6.1 to G' .
5. Delete the edges that do not correspond to original edges in G .

Remarks to Step 1. It is not always possible to choose the graph G_P with the properties required in Step 1 as a subgraph of G . The following lemma shows, however, that at least some graph with the properties required in Step 1 exists.

Lemma 4. *Let (\mathcal{G}, φ) be a cactus model for a set of cuts of a graph with vertex set V . Then there exists a planar connected graph H with vertex set V such that*

1. \mathcal{G} models a set of minimal cuts of H and
2. no edge of H crosses a cycle of \mathcal{G} .

Proof. Let μ be a node of T that is adjacent to a leaf v . In a first step, we start with an empty graph. Proceeding from bottom to top of the tree T rooted at μ , we can construct a graph $H_0 = (V, E_0)$ such that

1. for any node $\nu \neq \mu$ the subgraph of H_0 induced by $V_\mu(\nu)$ is a path,
2. no edge of H_0 crosses a cycle of \mathcal{G} , and
3. $E_0(V_\mu(\nu), \overline{V_\mu(\nu)}) = \emptyset$ if ν is adjacent to μ .

In a second step, for each node ν of T that is adjacent to μ let v_1 and v_2 be the two vertices of degree one or let $v_1 = v_2$ be the vertex of degree zero in the subgraph of H_0 induced by $V_\mu(\nu)$, respectively. Add edges $\{v, v_1\}$ and $\{v, v_2\}$ to H_0 . Finally, for all leaves $w \neq v$ of T that are adjacent to r , add an edge $\{v, w\}$ to H_0 .

It is now easy to see that no edge of the thus constructed graph H crosses a cycle of \mathcal{G} and that \mathcal{G} models a set of minimal cuts. H is a cactus and hence it is planar and connected. □

Let $H = (V, E_H)$ be the graph from the previous lemma and let G_P be a maximal planar subgraph of $(V, E \cup E_H)$ such that H is a subgraph of G_P . Then G_P is planar and connected, \mathcal{G} models a set of minimal cuts of G_P and r is rootable. Hence, G_P fulfills the properties required in Step 1.

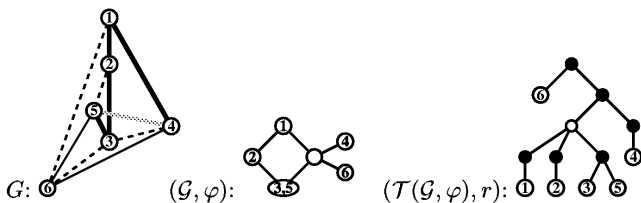


Fig. 4. A non-planar graph G and a cactus model. Deleting the grey edge yields a maximal planar subgraph. Solid thick black edges induce a spanning path in every cluster $S \neq V$ of $(G, \mathcal{T}(\mathcal{G}, \varphi), r)$. All solid black edges indicate the graph H constructed in the proof of Lemma 4.

Remarks to Step 3. We use an extension of planarization-techniques that is similar to the method introduced by Di Battista et al. [3] for drawing non-planar hierarchically clustered graphs. Edges in G that are not represented in G'_P are routed iteratively through the dual graph of G'_P , replacing every crossing of an edge and a dual in the rout by a vertex of degree four. Suppose, we want to insert an edge $\{v, w\}$. Let F be the set of faces of G'_P . Let $E^*_{\{v,w\}}$ be the following set of arcs. For every edge e of G'_P , let $f_1, f_2 \in F$ be the faces that are incident to e . Then $E^*_{\{v,w\}}$ contains the *dual arcs* $(f_1, f_2)_e$ and $(f_2, f_1)_e$. Further, $E^*_{\{v,w\}}$ contains the arcs $(f, x), (x, f), x \in \{v, w\}, f \in F, f$ incident to x . There are no other arcs contained in $E^*_{\{v,w\}}$. In general, any simple path in the *extended dual* $G^*_{\{v,w\}} = (\{v, w\} \cup F, E^*_{\{v,w\}})$ from v to w can be used as a rout for the edge $\{v, w\}$. To achieve that the drawing of $\{v, w\}$ does not cross a boundary cycle twice, we use a restricted version of the extended dual. Let $v = \nu_1, \nu_2, \dots, \nu_\ell = w$ be the path in T between v and w . Let $k \in \{2, \dots, \ell - 1\}$ be such that ν_{k-1} and ν_{k+1} are descendants of ν_k . For each boundary edge e of G'_P ,

- if e is contained in the drawing γ of $\{V_r(\nu_i), \overline{V_r(\nu_i)}\}, i = 2, \dots, k - 1$, delete the dual arc of e that is directed from the outside of γ to the inside of γ from $E^*_{\{v,w\}}$.
- if e is contained in the drawing γ of $\{V_r(\nu_i), \overline{V_r(\nu_i)}\}, i = k + 1, \dots, \ell - 1$, delete the dual arc of e that is directed from the inside of γ to the outside of γ from $E^*_{\{v,w\}}$.
- if e is contained in the drawing of a cut $\{V_r(\nu), \overline{V_r(\nu)}\}$ for a node $\nu \neq \nu_i, i = 2, \dots, k - 1, k + 1, \dots, \ell - 1$ of T , delete both dual arcs of e from $E^*_{\{v,w\}}$.

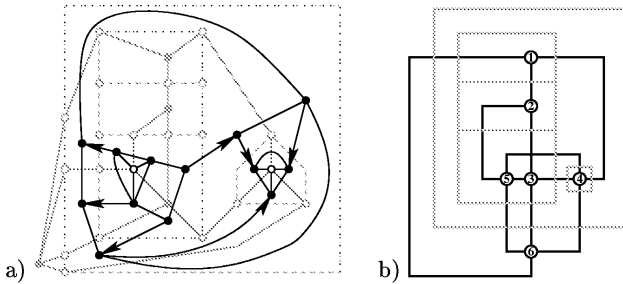


Fig. 5. a) Black edges show a connected component of the restricted extended dual for adding the edge $\{5, 4\}$ from 5 to 4 of the graph in Fig. 4. Bidirected edges are indicated by simple curves without arrows. b) A drawing with axis-parallel rectangles for the family of cuts modeled by (\mathcal{G}, φ) in Fig. 4.

The thus restricted dual (see Fig. 5 for an illustration) guarantees that any path from v to w crosses any boundary cycle at most once. On the other hand, there

is a path from v to w in the restricted dual if and only if either ν_k is not a cycle-replacement node or ν_k is a cycle-replacement node and ν_{k-1}, ν_{k+1} are adjacent in \mathcal{G} . Hence, by Condition (R), there is a path from v to w . Summarizing, we have shown the following characterization

Theorem 2. *A family \mathcal{C} of (minimal) cuts of a (planar) graph G has a (planar) drawing with axis-parallel rectangles if and only if*

1. \mathcal{C} has a cactus model (\mathcal{G}, φ) and
2. the root r of $\mathcal{T}(\mathcal{G}, \varphi)$ can be chosen such that each edge that crosses a cycle c is incident to a vertex in $V \setminus V_r(\nu_c)$.

Provided a cactus model is given, it can be tested in time $\mathcal{O}(nm)$ whether a drawing with axis-parallel rectangles exists.

References

1. U. Brandes, S. Cornelsen, and D. Wagner. How to draw the minimum cuts of a planar graph. In J. Marks, editor, *Proceedings of the 8th International Symposium on Graph Drawing (GD 2000)*, volume 1984 of *Lecture Notes in Computer Science*, pages 103–114. Springer, 2001.
2. S. Cornelsen and D. Wagner. Completely connected clustered graphs. To appear in *Proceedings of the 29th International Workshop on Graph Theoretic Concepts in Computer Science (WG 2003)*.
3. G. Di Battista, W. Didimo, and A. Marcandalli. Planarization of clustered graphs. In M. Jünger and P. Mutzel, editors, *Proceedings of the 9th International Symposium on Graph Drawing (GD 2001)*, volume 2265 of *Lecture Notes in Computer Science*, pages 60–74. Springer, 2002.
4. Y. Dinitz, A. V. Karzanov, and M. Lomonosov. On the structure of a family of minimal weighted cuts in a graph. In A. Fridman, editor, *Studies in Discrete Optimization*, pages 290–306. Nauka, 1976. (in Russian).
5. Y. Dinitz and Z. Nutov. A 2-level cactus model for the system of minimum and minimum+1 edge-cuts in a graph and its incremental maintenance. In *Proceedings of the 27th Annual ACM Symposium on the Theory of Computing (STOC '95)*, pages 509–518. ACM, The Association for Computing Machinery, 1995.
6. Q. Feng, R. F. Cohen, and P. Eades. Planarity for clustered graphs. In P. Spirakis, editor, *Proceedings of the 3rd European Symposium on Algorithms (ESA '95)*, volume 979 of *Lecture Notes in Computer Science*, pages 213–226. Springer, 1995.
7. U. Fößmeier and M. Kaufmann. Drawing high degree graphs with low bend numbers. In F. J. Brandenburg, editor, *Proceedings of the 3rd International Symposium on Graph Drawing (GD '95)*, volume 1027 of *Lecture Notes in Computer Science*, pages 254–266. Springer, 1996.
8. G. W. Klau and P. Mutzel. Quasi orthogonal drawing of planar graphs. Technical Report MPI-I-98-1-013, Max-Planck-Institut für Informatik, Saarbrücken, Germany, 1998. Available at <http://data.mpi-sb.mpg.de/internet/reports.nsf>.
9. D. Lütke-Hüttmann. Knickminimales Zeichnen 4-planarer Clustergraphen. Master's thesis, Universität des Saarlandes, 1999. (Diplomarbeit).
10. R. Tamassia. On embedding a graph in the grid with the minimum number of bends. *SIAM Journal on Computing*, 16:421–444, 1987.
11. R. Tamassia, G. Di Battista, and C. Batini. Automatic graph drawing and readability of diagrams. *IEEE Transactions on Systems, Man and Cybernetics*, 18(1):61–79, 1988.