Cones of Locally Non-Negative Polynomials

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Introduction

The property of non-negativity of polynomials has been studied for a long time and the mathematicians looked especially for certificates of non-negativity. The most important certificate is the representation as a sum of squares of polynomials and there is a well-known theorem of Hilbert [Hil] from 1888 that solves the question if every non-negative homogeneous polynomial in n variables in degree $2d$ is a sum of squares of forms of degree d. However, the first explicit example of a non-negative polynomial that is not a sum of squares was found much later by Motzkin in [Mot] in 1967. Then, mathematicians were interested in more such examples and starting with Choi and Lam in [ChLa] and [ChLa2] one also studied the convex geometry by posing the question if these examples may be decomposed non-trivially or if they belong to extreme rays. Reznick and Knebusch also entered this topic (see [Rez] and [CKLR]). Again, some time passed until Blekherman, Iliman and Kubitzke regarded faces of higher dimensions and compared them with the sums of squares cone in [BlIlKu]. The named results just concern some faces of the cone. However, recent results from Blekherman et al. [BHCRS] and Kunert and Scheiderer [KuSch] consider all extreme rays of some cones. Kunert [Kun] even describes the faces of the cone of non-negative forms in 3 homogeneous variables and degree 4 modulo linear coordinate changes completely.
The main motivation of this topic is purely theoretical: we want to understand the "phenomenon of non-negativity". However, clearly, it also has impact on the applied question of certificates of non-negative polynomials via sums of squares, which is a main topic in polynomial optimization. But let us return to the convex structure. When Kunert proved his classification in the case of 3 homogeneous variables and degree 4, an important concept was cones of locally non-negative forms and their faces. Thus, let us take a closer look at such cones but we will work in the polynomial setting.
We consider the cone of polynomials locally non-negative at the origin of $\mathbb{R}^n$ without any restriction on the degree. At first glance, this is a very strange object. It is nearly a half-space (as one can see if one looks at the absolute coefficient) but it is not closed in an appropriate sense and it lives in an infinite-dimensional vector space. However, if the absolute coefficient is 0, then belonging to this cone just means to have a local minimum at the origin. Also, one can show that the relative algebraic interiors of faces of this cone describe elements that behave, up to multiplication of some positive constant, similar near 0. This shows that this cone and its facial structure carry important geometric information about minima. In turn, such a property usually leads to applications as posed questions frequently respect the geometry.
The aim of this work is to give an introduction to these and related cones. We will even reduce our structure to convex cones in rings that are closed under multiplication and study these objects. We generalize to local non-negativity on arbitrary or rather general subsets of $\mathbb{R}^n$ and similar cones in the ring of convergent and formal power series. Examples of faces with a given Newton polyhedron will be studied as well as the classification of faces. Finally, we will give an application of our theory on a question on sums of squares representations in the formal power series setting arising from Scheiderer's local-global principle.
The dissertation is organized as follows. In the first chapter, we introduce some basic convexity theory, define cones of globally and locally non-negative polynomials or forms with respect to a given subset of $\mathbb{R}^n$ or $\mathbb{P}^n(\mathbb{R})$ and present the relationship between them. Furthermore, we give classifications of the faces of cones of globally non-negative forms in the cases of equality in Hilbert's theorem, which include the result from Kunert. The results of this chapter are

considered as basic because they illustrate important, not too complicated examples and show the connection between the local and global setting.
In the second chapter, we prepare and study results from convexity theory used later. We work without any topology, as it is not necessary to use it and replace the usual topological terms by algebraic alternatives. However, we also mention the topology that could be used. Then, we study faces of cones with non-empty relative algebraic interior and explain the situation in detail as literature on this topic is rare. After giving some nice descriptions of convex constructions and some more properties, we formalize the facial structure of a set and specify what we mean by a classification of faces of a cone. We consider the ascending chain condition of faces, which leads to the consequence that all faces will have non-empty relative algebraic interior. After some comment on the lattice structure, we study cones (in an $\mathbb{R}$-algebra) that are closed under multiplication. Using this structure, we can also define a multiplication of faces and study it. If we additionally assume the cone to be Archimedean and the ring to be Noetherian, we get some beautiful properties of these cones. Finally, we apply these concepts from the abstract setting to our cones of globally and locally non-negative polynomials and give geometric interpretations.
The third chapter starts with constructions parallel to the cone of locally non-negative polynomials with respect to some subset of $\mathbb{R}^n$ in the rings of convergent and formal power series. We are rather general here, study the relationship between the three cones and we consider faces of finite codimension. Then, we reduce to the special situation that the set on which locally non-negativity is considered contains a neighbourhood of the origin or, equivalently, is $\mathbb{R}^n$ itself. We transfer results from Vassiliev on local isolated minima of analytic functions to our setting, which gives us the possibility to describe the structure of monomial faces completely. These are the $\subseteq$-maximal faces such that the support of its elements is contained in a given Newton polyhedron. We also study proper faces of such faces. This leads to a rich family of examples of faces of our cones. The next problem we consider, is the classification problem modulo appropriate automorphisms. We transfer definitions and results from singularity theory and obtain the important splitting lemma in the case of faces, which is crucial for the classification problem. We can also show that faces have a unique $\subseteq$-maximal proper face with larger corank up to such automorphisms. Finally, we apply the previous general results on the cases $n = 2$ and $n = 3$. If $n = 2$, monomial faces have a very nice canonical factorization and we obtain the facial structure modulo automorphisms in the cases of convergent and formal power series up to the 8th layer. In the case $n = 3$, the corresponding answer is given up to the 7th layer.
Finally, in the fourth and final chapter, we study a question motivated by Scheiderer's local-global principle. To be more specific, we want to know for a face of the cone of non-negative formal power series if its relative interior consists of sums of squares. The general theory developed in the second chapter also gives us some nice properties for the sums of squares cone and using knowledge about stable equivalence, we can reduce the problem drastically. Then, we can give a positive answer to our problem in the case $n = 3$ for all faces up to layer 7. We also give further positive and negative answers.

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1 Basic facts and notations

In this first chapter, we will present some of the most important objects and concepts of this work and illustrate them in the easiest cases. To give a faster introduction, notations will be partially introduced within the text if they are commonly used in mathematics - some of them will be defined later (see also our list of notations at p.122). In finite-dimensional vector spaces over $\mathbb{R}$ and in $\mathbb{P}^n(\mathbb{R})$, we will use the Euclidean topology (subsets will be given the subspace topology) if not mentioned differently.

1.1 Cones of non-negative forms and sums of squares

We assume throughout this work that $n \in \mathbb{N} := \{1, 2, \dots\}$ (we use $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$). We set $\underline{X} := (X_0, \dots, X_n)$ and we have $\mathbb{R}[\underline{X}] = \mathbb{R}[X_0, \dots, X_n]$. The homogeneous polynomials of degree d (including 0) in $\mathbb{R}[\underline{X}]$ are denoted by $\mathbb{R}[\underline{X}]_d$. We consider some subset $S \subseteq \mathbb{P}^n(\mathbb{R})$.

Definition 1.1.1. Let $d \in \mathbb{N}_0$. We set

$$\mathcal{P}_{n,2d} := \{f \in \mathbb{R}[\underline{X}]_{2d} \,|\, f \geq 0 \text{ on } S\},$$
$$\Sigma_{n,2d} := \left\{ \sum_{i=1}^{k} g_i^2 \;\middle|\; k \in \mathbb{N}_0, g_i \in \mathbb{R}[\underline{X}]_d \right\} \subseteq \mathbb{R}[\underline{X}]_{2d}$$

and call them the *cone of non-negative forms* on S and the *cone of sums of squares* (in $n+1$ homogeneous variables and degree $2d$). We also denote them by $\mathcal{P}_{2d}$ and Σ_{2d} as the considered number of variables is usually $n+1$.

Remark 1.1.2. Frequently, we will use the cones $\mathcal{P}_{n,2d}$ in the case $S = \mathbb{P}^n(\mathbb{R})$. If that is clear from the context, then we do not mention it explicitly.

Remark 1.1.3. Note that replacing S by its closure $\overline{S}$ in $\mathbb{P}^n(\mathbb{R})$ does not change $\mathcal{P}_{n,2d}$ since $f \geq 0$ on S also implies $f \geq 0$ on $\overline{S}$.

Clearly, we have $\Sigma_{n,2d} \subseteq \mathcal{P}_{n,2d}$ for all choices of n, d and S. We present the famous theorem of Hilbert.

Theorem 1.1.4. ***(Hilbert)*** *Let $d \in \mathbb{N}$ and $S = \mathbb{P}^n(\mathbb{R})$. Then, $\Sigma_{n,2d} = \mathcal{P}_{n,2d}$ holds if and only if we are in one of the following cases*

- $n = 1$
- $2d = 2$
- $(n, 2d) = (2, 4)$.

Proof. [Hil] □

In the cases of equality in Hilbert's theorem, the facial structure of the cones is completely known. We will discuss them later in this chapter.

1.2 Convex sets and convex cones

Let V be a vector space over $\mathbb{R}$. We introduce basic notations (usually as in [Gue]) and some facts about convex sets and convex cones.

Notation 1.2.1. Let $x, y \in V$. We set

$$[x,y] := \{(1-\lambda)x + \lambda y \,|\, 0 \leq \lambda \leq 1\}, \qquad [x,y[:= \{(1-\lambda)x + \lambda y \,|\, 0 \leq \lambda < 1\},$$
$$]x,y] := \{(1-\lambda)x + \lambda y \,|\, 0 < \lambda \leq 1\}, \qquad]x,y[:= \{(1-\lambda)x + \lambda y \,|\, 0 < \lambda < 1\}.$$

Let $K \subseteq V$. We denote the affine hull of K and the linear span of K by $\mathrm{aff}(K)$ and $\mathrm{span}(K)$ (it is $\mathrm{aff}(\emptyset) = \emptyset$).
For affine subspaces $A \subseteq V$, we denote its dimension by $\dim(A)$ and we set $\dim(\emptyset) := -1$. If $B \subseteq A$ is another affine subspace, we denote its codimension in A by $\mathrm{codim}_A(B)$ and we set $\mathrm{codim}_A(\emptyset) := \dim(A) + 1$. We also write $\mathrm{codim}(B)$ if A is clear from the context (usually if $A = V$).

Definition 1.2.2. Let $C \subseteq V$. We call C *convex*, if $[x,y] \subseteq C$ for all $x, y \in C$. We call C a *(convex) cone* if additionally $C \neq \emptyset$ and $\{\lambda x \,|\, \lambda \geq 0, x \in C\} \subseteq C$.

Remark 1.2.3. We will assume all cones to be convex cones. There is also the concept of cones over projective varieties (see [Hart, Exercise I.2.10.]), which leads to affine varieties $X \subseteq \mathbb{A}^n_K$ over some field K with $0 \in X$ and

$$\{\lambda x \in K \,|\, \lambda \in K, x \in X(K)\} \subseteq X(K).$$

As we do not use such cones within this work, there is no possibility of confusion.

Let us make some definitions and statements on convex sets and cones at the same time and put the statement on cones in brackets.

Definition 1.2.4. Let $C \subseteq V$ be a convex set (a cone).
We call $F \subseteq C$ a *face of* C if F is convex (a cone) and if $]x,y[\cap F \neq \emptyset$ with $x, y \in C$ implies $x, y \in F$. We write $F \trianglelefteq C$.
Let us assume that F is a face of C. We call F an *exposed face* if $F = \emptyset$, $F = C$ or $F = C \cap H$ for some hyperplane H in V. We set $\dim(F) := \dim(\mathrm{aff}(F))$ and $\mathrm{codim}(F) := \mathrm{codim}_V(\mathrm{aff}(F))$ and call them the *dimension and codimension of* F. Especially, C has codimension 0 if and only if $\mathrm{aff}(C) = V$. If $\dim(F) = 0$, then F is called *extreme point of* C. If C is a cone and $\dim(F) = 1$, then F is called *extreme ray of* C.

Warning 1.2.5. Sometimes the term "face" is used for the more special concept of exposed faces in the literature (e.g. in [BaFo], [BaFo2]). Alternative terms for "face" as defined above are "facet" ([BaFo], [BaFo2], fr. "facette") or "extremal set" (see [Val]).

Example 1.2.6. The sets $\mathcal{P}_{n,2d}$ and $\Sigma_{n,2d}$ as defined before are cones in $\mathbb{R}[\underline{X}]_{2d}$ for all choices of n, d and S. It is well-known that they are closed as subsets of $\mathbb{R}[\underline{X}]_{2d}$.

We add some remarks with well-known facts.

Remark 1.2.7. The point 0 is contained in each cone and therefore also in each face of a cone. Hence, if C is a cone, then the set of faces of C considered as a convex set consists of the faces of C considered as a cone together with $\emptyset$. However, if we are speaking about cones, we will always speak about faces of this set considered as a cone.

Remark 1.2.8. ([BaFo, III.4.1.2.a)]) If $F \trianglelefteq C$, then $F = \operatorname{aff}(F) \cap C$ for convex C and $F = \operatorname{span}(F) \cap C$ for cones C.

Remark 1.2.9. Let $F \subseteq C$ be convex sets (cones). If one wants to prove that F is a face of C, it is enough to show that $\frac{x+y}{2} \in F$ ($x + y \in F$) with $x, y \in C$ implies $x, y \in F$. Indeed, if F is no face, then there are $x, y \in C$ such that the interval $I := [x, y] \cap F$ is non-empty and strictly contained in $[x, y]$. We may assume $x \notin I$. Now, points $z \in I \subseteq F$ close to the endpoint of I that is closer to x may be represented as $z = \frac{x'+y'}{2}$ with $x', y' \in [x, y] \subseteq C$ and $x' \notin I$, so especially $x' \notin F$.

Remark 1.2.10. If C is a cone, $F \trianglelefteq C$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R} \setminus \{0\}$, $c_1, \ldots, c_k \in C$ ($k \geq 2$) with $\sum_{i=1}^{k} \lambda_i c_i \in F$, then $c_1, \ldots, c_k \in F$. Indeed, as F and C are cones, we may reduce to the case $\lambda_1 = \cdots = \lambda_k = 1$, and, by induction, we may reduce to the case $k = 2$. Now, $c_1 + c_2 \in F$ implies $\frac{1}{2}(c_1 + c_2) \in F$ and thus $c_1, c_2 \in F$ by definition of faces.

Remark 1.2.11. If C is a cone, then we have $\operatorname{span}(C) = C + (-C)$, where $+$ denotes the Minkowski sum, as any linear combination of elements of C may be written as a sum of a linear combination with positive coefficients and one with negative coefficients.

Remark 1.2.12. ([Gue, 4.9.(a)], [BaFo2, II.4.1.a)]) The intersection of a family of convex sets (cones) in V is a convex set (cone) in V. Similarly, the intersection of a family of faces of a convex set (or a cone) is a face of this convex set (cone).

This remark motivates the following definition.

Definition 1.2.13. Let $K \subseteq V$. The smallest convex set (cone) C with $K \subseteq C$ is called the *convex hull (conic hull) of* K and is denoted by $\operatorname{conv}(K)$ ($\operatorname{cone}(K)$).
Let $C \subseteq V$ be a convex set (a cone) and $K \subseteq C$. We call the smallest face F of C with $K \subseteq F$ the *carrier face of* K and we denote it by $\operatorname{carr}_C(K)$. We also write $\operatorname{carr}(K)$ if C is clear from the context and $\operatorname{carr}(x)$ if $K = \{x\}$.
Let C be a cone. We call $L_C := C \cap (-C)$ the *lineality space of* C. We call C *pointed* if $L_C = \{0\}$.

Remark 1.2.14. Carrier faces were usually studied for one point, but this generalization to arbitrary sets seems useful for us. The notation F_x [BaFo, III.4.1.2.] instead of $\operatorname{carr}(x)$ [ABKLM, 4.2.] is more common, but not appropriate for our purposes because we will work with several cones and the letter F is already frequently used. Not long ago, the author also called them "supporting faces". However, this may be irritating as the same term could also be used for (exposed) faces cut out by a supporting hyperplane. Furthermore, for some cones, we will call the lineality space the support of the cone and denote it by $\operatorname{supp}(C)$ (see 2.4.1), which is also close to the term "supporting face" and could lead to confusion.

We sum up some easy properties.

Remark 1.2.15. ([Gue, 4.10.,4.11.,4.19.,4.20.]) Let $K \subseteq V$, then

$$\operatorname{conv}(K) = \left\{ \sum_{i=1}^{k} \lambda_i x_i \;\middle|\; k \in \mathbb{N}, \lambda_i \geq 0, x_i \in K \text{ for each } i = 1, \ldots, k, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

$$\text{and} \quad \operatorname{cone}(K) = \left\{ \sum_{i=1}^{k} \lambda_i x_i \;\middle|\; k \in \mathbb{N}_0, \lambda_i \geq 0, x_i \in K \text{ for each } i = 1, \ldots, k \right\}.$$

Remark 1.2.16. Let C be a convex set (a cone) and $K \subseteq C$, then

$$\operatorname{conv}(K) \subseteq \operatorname{carr}(K) \quad (\operatorname{cone}(K) \subseteq \operatorname{carr}(K)).$$

If C is a cone, then $L_C = \operatorname{carr}(0)$ is the smallest face of C.

If V is finite-dimensional, it is useful to use the Euclidean topology to describe elements that lead to the same carrier face (see also 1.2.19 below).

Definition 1.2.17. Let V be finite-dimensional and $K \subseteq V$. Then, we call the interior of K considered as a subset of $\operatorname{aff}(K)$ the *relative interior of* K and we denote it by $\operatorname{ri}(K)$. The *relative boundary of* K is by definition $\operatorname{rbd}(K) := \overline{K} \setminus \operatorname{ri}(K)$.

Remark 1.2.18. [Gue, 5.22.] If $K \subseteq V$ is convex in finite-dimensional V and $K \neq \emptyset$, then $\operatorname{ri}(K) \neq \emptyset$.

The following remark gives some well-known statements of convexity theory, which will be presented later in a more general setting (see 2.1.14, 2.1.4, 2.2.3, 2.2.7(a)). We already need them in the next section.

Remark 1.2.19. Let V be finite-dimensional, C be a cone, $f, g \in V$ and $F \trianglelefteq C$. Then, we have $\operatorname{ri}(F) \neq \emptyset$ (by 1.2.18) and

$$\begin{array}{lrcl}
\text{given } f \in \operatorname{ri}(F): & g \in \operatorname{span}(F) & \Leftrightarrow & \exists\, \epsilon > 0 : f - \epsilon g \in F, \\
\text{given } f, g \in C: & \exists\, \epsilon > 0 : f - \epsilon g \in C & \Leftrightarrow & \operatorname{carr}_C(g) \subseteq \operatorname{carr}_C(f), \\
\text{given } f \in C: & f \in \operatorname{ri}(F) & \Leftrightarrow & F = \operatorname{carr}_C(f).
\end{array}$$

1.3 Cones of locally non-negative forms

Let $S \subseteq \mathbb{P}^n(\mathbb{R})$ as before. We want to describe the relation between global and local behaviour of non-negativity. Clearly, a form $f \in \mathbb{R}[\underline{X}]_{2d}$ with $d \in \mathbb{N}_0$ is non-negative on S if and only if for every $P \in \mathbb{P}^n(\mathbb{R})$, there is some neighbourhood U of P such that f is non-negative on $U \cap S$. This obvious starting point leads, together with compactness of $\mathbb{P}^n(\mathbb{R})$, to the following definitions and results. They were given for $S = \mathbb{P}^n(\mathbb{R})$ in [Kun, Section 1.6]. The statements are also slightly more general here, though the proofs are rather similar to those presented there.

Definition 1.3.1. Let $d \in \mathbb{N}_0$ and $P \in \mathbb{P}^n(\mathbb{R})$. We set

$$\mathcal{P}^{Loc}_{n,2d}(P) := \{f \in \mathbb{R}[\underline{X}]_{2d} \mid f \geq 0 \text{ on } S \cap U \text{ for some neighbourhood } U \text{ of } P \text{ in } \mathbb{P}^n(\mathbb{R})\}$$

and call it the *cone of forms locally non-negative at P (with respect to S)* or for short the *local (psd-)cone at P* (in $n+1$ homogeneous variables and degree $2d$). As before, we also write $\mathcal{P}^{Loc}_{2d}(P)$ as the considered number of variables is usually $n+1$.

Remark 1.3.2. As in the case $\mathcal{P}_{2d}$, one may replace S by its closure $\overline{S}$ and this does not change the set $\mathcal{P}^{Loc}_{2d}(P)$ for $P \in S$. Furthermore, $\mathcal{P}^{Loc}_{2d}(P)$ does only depend on the germ of S at P. Especially, $\mathcal{P}^{Loc}_{2d}(P) = \mathbb{R}[\underline{X}]_{2d}$ if $P \notin \overline{S}$.

Remark 1.3.3. $\mathcal{P}^{Loc}_{2d}(P)$ is a cone inside $\mathbb{R}[\underline{X}]_{2d}$ for all choices of n, d, S and $P \in \mathbb{P}^n(\mathbb{R})$. However, it is not closed in $\mathbb{R}[\underline{X}]_{2d}$ unless $d = 0$ or $S \cap U \subseteq \{P\}$ for some neighbourhood U of P. This may be seen from the easy facts that the interior of $\mathcal{P}^{Loc}_{2d}(P)$ consists of all $f \in \mathcal{P}^{Loc}_{2d}(P)$ with $f(P) > 0$ if $P \in \overline{S}$ and that there is some $g \notin \mathcal{P}^{Loc}_{2d}(P)$ with $g(P) = 0$ if $d \neq 0$ and $S \cap U \not\subseteq \{P\}$ for all neighbourhoods U of P (e.g. $g = -h^2$ with $h \in \mathbb{R}[\underline{X}]_d$ not vanishing on $S \cap U$ for any neighbourhood U of P).

Remark 1.3.4. As already described above, we have $\mathcal{P}_{2d} = \bigcap_{P\in\mathbb{P}^n(\mathbb{R})} \mathcal{P}_{2d}^{Loc}(P)$.

The following theorem shows how to write faces of $\mathcal{P}_{2d}$ as intersections of faces of $\mathcal{P}_{2d}^{Loc}(P)$ for $P \in \mathbb{P}^n(\mathbb{R})$. In fact, we will get representations with faces of $\mathcal{P}_{2d}^{Loc}(P)$ that are as small as possible.

Theorem 1.3.5. *Let $S \subseteq \mathbb{P}^n(\mathbb{R})$ and $F \trianglelefteq \mathcal{P}_{2d}$. Now, for each $f \in \mathrm{ri}(F)$ and $P \in \mathbb{P}^n(\mathbb{R})$ we have some face*

$$F^{Loc}(P) := \mathrm{carr}_{\mathcal{P}_{2d}^{Loc}(P)}(f) \trianglelefteq \mathcal{P}_{2d}^{Loc}(P),$$

which does not depend on the choice of f. Furthermore, we have

$$F = \bigcap_{P\in\mathbb{P}^n(\mathbb{R})} F^{Loc}(P)$$
$$\textit{and} \quad \mathrm{span}(F) = \bigcap_{P\in\mathbb{P}^n(\mathbb{R})} \mathrm{span}\left(F^{Loc}(P)\right).$$

Proof. We will frequently use 1.2.19. Note that $\mathrm{ri}(F) \neq \emptyset$. Let us assume that $f_1, f_2 \in \mathrm{ri}(F)$. Then, there exists some $\epsilon > 0$ with $f_1 - \epsilon f_2 \in F \subseteq \mathcal{P}_{2d}$, so $f_1 - \epsilon f_2 \geq 0$ on S and therefore $f_1 - \epsilon f_2 \in \mathcal{P}_{2d}^{Loc}(P)$ for $P \in \mathbb{P}^n(\mathbb{R})$. This shows

$$\mathrm{carr}_{\mathcal{P}_{2d}^{Loc}(P)}(f_2) \subseteq \mathrm{carr}_{\mathcal{P}_{2d}^{Loc}(P)}(f_1).$$

for $P \in \mathbb{P}^n(\mathbb{R})$. With an analogous argument, we obtain the opposite inclusion and this shows that the considered faces $F^{Loc}(P)$ do not depend on the choice of f.
The next statement, we show, is the statement on linear spans. Let $f \in \mathrm{ri}(F)$ and therefore $f \in \mathrm{ri}(F^{Loc}(P))$ for $P \in \mathbb{P}^n(\mathbb{R})$. If $g \in \mathrm{span}(F)$, then there exists some $\epsilon > 0$ with $f - \epsilon g \in F \subseteq \mathcal{P}_{2d}$. Hence, we have $f - \epsilon g \in \mathcal{P}_{2d}^{Loc}(P)$ for each $P \in \mathbb{P}^n(\mathbb{R})$ and therefore $g \in \mathrm{span}(F^{Loc}(P))$ for $P \in \mathbb{P}^n(\mathbb{R})$. Since $g \in \mathrm{span}(F)$ was chosen arbitrarily, this shows $\mathrm{span}(F) \subseteq \bigcap_{P\in\mathbb{P}^n(\mathbb{R})} \mathrm{span}\left(F^{Loc}(P)\right)$.
Let us now assume that $g \in \bigcap_{P\in\mathbb{P}^n(\mathbb{R})} \mathrm{span}\left(F^{Loc}(P)\right)$. Then, for each $P \in \mathbb{P}^n(\mathbb{R})$, there is some $\epsilon_P > 0$ such that $f - \epsilon_P g \in \mathcal{P}_{2d}^{Loc}(P)$. Thus, for each $P \in \mathbb{P}^n(\mathbb{R})$, we may choose some neighbourhood U_P of P such that $f - \epsilon_P g \geq 0$ on $S \cap U_P$ and the sets U_P cover $\mathbb{P}^n(\mathbb{R})$. Since $\mathbb{P}^n(\mathbb{R})$ is compact, there are finitely many $P_1, \ldots, P_k \in \mathbb{P}^n(\mathbb{R})$ with $\bigcup_{i=1}^k U_{P_i} = \mathbb{P}^n(\mathbb{R})$. We set

$$\epsilon := \min(\epsilon_{P_1}, \ldots, \epsilon_{P_k}) > 0.$$

We obtain $f - \epsilon g \in \mathcal{P}_{2d}$ and hence $g \in \mathrm{span}(F)$. Summing up, this gives our claim $\mathrm{span}(F) = \bigcap_{P\in\mathbb{P}^n(\mathbb{R})} \mathrm{span}\left(F^{Loc}(P)\right)$.
The claim on the faces themselves follows using 1.2.8 and 1.3.4 from

$$\bigcap_{P\in\mathbb{P}^n(\mathbb{R})} F^{Loc}(P) = \bigcap_{P\in\mathbb{P}^n(\mathbb{R})} \mathrm{span}\left(F^{Loc}(P)\right) \cap \mathcal{P}_{2d}^{Loc}(P) = \bigcap_{P\in\mathbb{P}^n(\mathbb{R})} \mathrm{span}(F) \cap \mathcal{P}_{2d} = F.$$

□

We have seen that each $F \trianglelefteq \mathcal{P}_{2d}$ may be written as an intersection of faces of the local cones. The following proposition shows that each intersection of faces of the local cones also gives some face of $\mathcal{P}_{2d}$.

Proposition 1.3.6. *Let $S \subseteq \mathbb{P}^n(\mathbb{R})$ and let $F_P \trianglelefteq \mathcal{P}_{2d}^{Loc}(P)$ for each $P \in \mathbb{P}^n(\mathbb{R})$. Then,*

$$F := \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} F_P \trianglelefteq \mathcal{P}_{2d}.$$

Proof. We have $\mathcal{P}_{2d} = \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} \mathcal{P}_{2d}^{Loc}(P) \supseteq \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} F_P = F$ and F is a cone since it is an intersection of cones. Let us assume that $f, g \in \mathcal{P}_{2d}$ with $\frac{f+g}{2} \in F$. Then, $f, g \in \mathcal{P}_{2d}^{Loc}(P)$ and $\frac{f+g}{2} \in F_P$ for each $P \in \mathbb{P}^n(\mathbb{R})$. Since F_P is a face of $\mathcal{P}_{2d}^{Loc}(P)$ for $P \in \mathbb{P}^n(\mathbb{R})$, this shows $f, g \in F_P$ for $P \in \mathbb{P}^n(\mathbb{R})$ and hence $f, g \in F$. We obtain our claim $F \trianglelefteq \mathcal{P}_{2d}$. □

Remark 1.3.7. In the intersections of the theorem and the propositions one just has to intersect over points $P \in \overline{S}$. All other points are superfluous since then $\mathcal{P}_{2d}^{Loc}(P) = \mathbb{R}[\underline{X}]_{2d}$, which is itself its only face.

Remark 1.3.8. In the situation of the proposition, we have

$$F = \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} F_P = \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} \big(\operatorname{span}(F_P) \cap \mathcal{P}_{2d}^{Loc}(P)\big) = \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} \operatorname{span}(F_P) \cap \mathcal{P}_{2d}.$$

Since $\mathbb{R}[\underline{X}]_{2d}$ is finite-dimensional, there are finitely many $P_1, \ldots, P_k \in \mathbb{P}^n(\mathbb{R})$ such that

$$\bigcap_{P \in \mathbb{P}^n(\mathbb{R})} \operatorname{span}(F_P) = \bigcap_{i=1}^{k} \operatorname{span}(F_{P_i})$$

$$\text{and hence} \qquad F = \bigcap_{i=1}^{k} \operatorname{span}(F_{P_i}) \cap \mathcal{P}_{2d} = \bigcap_{i=1}^{k} F_{P_i} \cap \bigcap_{P \notin \{P_1, \ldots, P_k\}} \mathcal{P}_{2d}^{Loc}(P).$$

This shows that in the representation of F as in the proposition, just a finite number of F_P have to be chosen different from $\mathcal{P}_{2d}^{Loc}(P)$.

1.4 The case $n = 1$

Let us study the structure of the cones $\mathcal{P}_{1,2d}$ for $S = \mathbb{P}^1(\mathbb{R})$ (though the case of arbitrary S is not much harder). We will use the variables X and Y instead of X_0 and X_1. First of all, we study the corresponding cones of locally non-negative forms.

Proposition 1.4.1. *Let $n = 1$, $S = \mathbb{P}^1(\mathbb{R})$, $P := (0 : 1)$ and $F \trianglelefteq \mathcal{P}_{2d}^{Loc}(P)$. We denote the ideal generated by X in $\mathbb{R}[X, Y]$ by $\mathfrak{m}$ - it is the ideal of polynomials vanishing at P.*
Then, either $F = \{0\}$ or there exists a unique $m \in \mathbb{N}_0$ with $m \leq d$ such that

$$F = F_m := \left\{ X^{2m} f \;\middle|\; f \in \mathcal{P}_{2(d-m)}^{Loc}(P) \right\} = \mathfrak{m}^{2m} \cap \mathcal{P}_{2d}^{Loc}(P).$$

Especially, the faces of $\mathcal{P}_{2d}^{Loc}(P)$ are linearly ordered by

$$\mathcal{P}_{2d}^{Loc}(P) = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_d \supseteq \{0\}.$$

Proof. Again, we will use 1.2.19. Let $f_1, f_2 \in \mathcal{P}_{2d}^{Loc}(P) \setminus \{0\}$, so the vanishing orders of f_1 and f_2 at P are even. Therefore, there are $m_1, m_2 \in \mathbb{N}_0$ such that $f_i = X^{2m_i} g_i$ with $g_i \in \mathbb{R}[X, Y]_{2(d-m_i)}$ and $g_i(P) > 0$ for $i = 1, 2$. If $m_1 \geq m_2$, then there is some $\epsilon > 0$ with

$$g_1(P) - \epsilon X^{2(m_1 - m_2)} g_2(P) > 0,$$

hence $f_1 - \epsilon f_2 \in \mathcal{P}_{2d}^{Loc}(P)$, i.e. $\operatorname{carr}(f_1) \subseteq \operatorname{carr}(f_2)$ in $\mathcal{P}_{2d}^{Loc}(P)$. If $m_1 < m_2$, then

$$f_1 - \epsilon f_2 = X^{2m_2}\left(X^{2(m_1-m_2)}g_1 - \epsilon g_2\right)$$

and $\left(X^{2(m_1-m_2)}g_1 - g_2\right)(P) < 0$ for all $\epsilon > 0$ imply $f_1 - \epsilon f_2 \notin \mathcal{P}_{2d}^{Loc}(P)$, i.e. $\operatorname{carr}(f_1) \nsubseteq \operatorname{carr}(f_2)$ in $\mathcal{P}_{2d}^{Loc}(P)$. Especially, we have $\operatorname{carr}(f_1) = \operatorname{carr}(f_2)$ if and only if $m_1 = m_2$. We have shown that for each vanishing order $2m \leq 2d$ there exists exactly one face of $\mathcal{P}_{2d}^{Loc}(P)$, they are sorted decreasingly by these orders and besides $\{0\}$ there is no other face. Therefore, the claims are clear. □

Remark 1.4.2. In the notation of the proposition, if $d > 0$, then

$$\begin{aligned} \operatorname{ri}(F_m) &= \left\{X^{2m}f \mid f \in \mathbb{R}[X,Y]_{2(d-m)}, f(P) > 0\right\} \\ \text{and} \quad \overline{F_m} &= \left\{X^{2m}f \mid f \in \mathbb{R}[X,Y]_{2(d-m)}, f(P) \geq 0\right\}. \end{aligned}$$

Remark 1.4.3. The proposition transfers to other points P of $\mathbb{P}^1(\mathbb{R})$. One replaces X in the description of F_m by some non-zero linear form $\lambda \in \mathbb{R}[X,Y]_1$ vanishing at P and $\mathfrak{m}$ by the ideal generated by λ - it is still the ideal of polynomials vanishing at P.

Let us now study the cones $\mathcal{P}_{1,2d}$ themselves.

Proposition 1.4.4. *Let $n = 1$, $S = \mathbb{P}^1(\mathbb{R})$ and $F \trianglelefteq \mathcal{P}_{2d}$. Then, either $F = \{0\}$ or there is a unique representation*

$$F = \bigcap_{P \in \mathbb{P}^1(\mathbb{R})} F_P$$

as in 1.3.6 and we have $\operatorname{codim}(F) = \sum_{P \in \mathbb{P}^1(\mathbb{R})} \operatorname{codim}(F_P)$.

Proof. Suppose $F = \bigcap_{P \in \mathbb{P}^1(\mathbb{R})} F_P$ and $f \in \operatorname{ri}(F)$. If $f = 0$, then $F = \{0\}$. Let $f \neq 0$. Then f has finitely many zeros and therefore, there are just finitely many $P \in \mathbb{P}^1(\mathbb{R})$ with $F_P \neq \mathcal{P}_{2d}^{Loc}(P)$, which we denote by $P_1, \ldots, P_k$ ($k \in \mathbb{N}_0$). We notice that $F_P \neq \{0\}$ for $P \in \mathbb{P}^1(\mathbb{R})$. Let $\lambda_1, \ldots, \lambda_k \in \mathbb{R}[X,Y]_1 \setminus \{0\}$ be given such that λ_i vanishes at P_i and let $\mathfrak{m}_i$ be the ideal generated by λ_i for $i = 1, \ldots, k$. Furthermore, let $m_1, \ldots, m_k \in \mathbb{N}$ such that $F_{P_i} = \mathfrak{m}_i^{2m_i} \cap \mathcal{P}_{2d}^{Loc}(P_i)$ for $i = 1, \ldots, k$ (compare with 1.4.1). We set $m := m_1 + \cdots + m_k$. Since $\lambda_i^{2m_i} \mid f$ for $i = 1, \ldots, k$ and λ_i irreducible, we obtain $m \leq d$. Now,

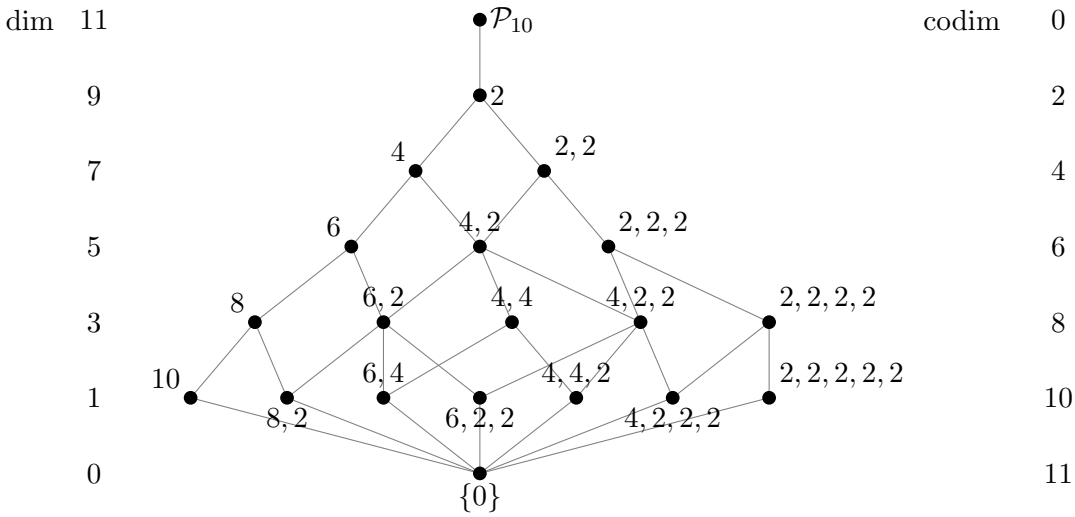
$$h := \lambda_1^{2m_1} \cdots \lambda_k^{2m_k}(X^2 + Y^2)^{d-m} \in \mathbb{R}[X,Y]_{2d}$$

fulfils $h \in \operatorname{ri}(F_P)$ for $P \in \mathbb{P}^1(\mathbb{R})$. An application of 1.3.5 gives $\operatorname{carr}_{\mathcal{P}_{2d}}(h) = \bigcap_{P \in \mathbb{P}^1(\mathbb{R})} F_P = F$ and also $h \in \operatorname{ri}(F^{Loc}(P))$ for $P \in \mathbb{P}^1(\mathbb{R})$ with $F^{Loc}(P)$ as in 1.3.5. This shows $F_P = F^{Loc}(P)$ for $P \in \mathbb{P}^1(\mathbb{R})$ (use 1.2.19) and hence the claimed uniqueness of the representation.
Clearly, we have $\operatorname{codim}(F_{P_i}) = 2m_i$ for $i = 1, \ldots k$. Additionally, by 1.3.5 and 1.4.1, $g \in \operatorname{span}(F)$ holds if and only if g is of the form $\lambda_1^{2m_1} \cdots \lambda_k^{2m_k} g'$ with $g' \in \mathbb{R}[X,Y]_{2(d-m)}$, thus $\operatorname{codim}(F) = 2m$. This shows the statement on codimensions. □

Remark 1.4.5. The proposition motivates classifications of the faces of $\mathcal{P}_{2d}$. One possibility is to assign the multi-set (i.e. elements are counted with multiplicity) $\{m_1, \ldots, m_k\}$ with the notation from the proof of 1.4.4 to each face and to sort faces with the same image in the same class of the classification.

One obtains an even more detailed finite classification of the faces of $\mathcal{P}_{2d}$ if one additionally remembers the order of $P_1, \ldots, P_k$ on $\mathbb{P}^1(\mathbb{R})$. This means that a cycle graph with labelled vertices is assigned to each face: one takes a cycle graph with k vertices and the vertices are labelled with the multiplicities $2m_i$ in the same order as the corresponding points are appearing on $\mathbb{P}^1(\mathbb{R})$. Again, faces with the same image are sorted in one class.
Such classifications provide more structure than just the equivalence classes described - it is possible to construct a graph from them. For each class of the studied classification, there is one vertex of the graph. Furthermore, we draw an edge between two classes if there are representatives of the classes such that one representative is an $\subseteq$-maximal proper face of the other one. We will study classifications abstractly in subsection 2.3.1 and these graphs will be the Hasse diagrams of corresponding partial orders on the classes of the classification. However, we will see that the construction of the graph described here does not work in the general setting (see 2.3.16).

Example 1.4.6. We present the classification of the faces of $\mathcal{P}_{10}$ - here the described classifications of 1.4.5 do not differ as the order of $P_1, \ldots, P_k$ on $\mathbb{P}^1(\mathbb{R})$ is irrelevant. The non-trivial classes of faces are labelled by the numbers $m_1, \ldots, m_k$ in decreasing order.



It is not hard to show that any face of some class with multiplicities $m_1 + m_2, m_3, \ldots, m_k$ is a "limit of faces" from the class with multiplicities $m_1, m_2, m_3, \ldots, m_k$ (in the sense that given a face F of the first class, then there are faces G_i for each $i \in \mathbb{N}$ of the second class such that for any $f \in F$ there are $g_i \in G_i$ for each $i \in \mathbb{N}$ with $g_i \to f$ for $i \to \infty$) by "moving the zeros with multiplicities m_1 and m_2 to one point". From this observation, one sees easily that the union of the set of extreme rays of the kind $2, 2, 2, 2, 2$ is dense in the union of all extreme rays of $\mathcal{P}_{10}$. This generalizes to higher (even) degrees.
The first example when the order of the zeros on $\mathbb{P}^1(\mathbb{R})$ is relevant and the corresponding classifications are different, is given in degree $2d = 12$. Indeed, one has to distinguish between $4, 2, 4, 2$ and $4, 4, 2, 2$.

1.5 The case $2d = 2$

We assume $2d = 2$ and $S = \mathbb{P}^n(\mathbb{R})$ within this section and we will study the cones $\mathcal{P}_{n,2}$. First of all, let us recall the connection between quadratic forms and symmetric matrices, which

leads to the notion of rank and corank of quadratic forms.

Remark 1.5.1. Let $f = \sum_{0 \leq i \leq j \leq n} a_{ij} X_i X_j \in \mathbb{R}[\underline{X}]_2$ and $B := (b_{ij})_{i,j} \in \mathrm{Sym}(n+1, \mathbb{R})$ where $\mathrm{Sym}(n+1, \mathbb{R})$ are the symmetric $(n+1) \times (n+1)$ matrices over $\mathbb{R}$. Then, there is a linear isomorphism between $\mathbb{R}[\underline{X}]_2$ and $\mathrm{Sym}(n+1, \mathbb{R})$ given by

$$\begin{aligned} &f \mapsto B \quad \text{with } b_{ii} := a_{ii} \text{ for } 0 \leq i \leq n, \quad b_{ij} = b_{ji} := \frac{1}{2} a_{ij} \text{ for all } 0 \leq i < j \leq n, \\ &B \mapsto f := \underline{X} B \underline{X}^T. \end{aligned}$$

The image of $\mathcal{P}_2$ under this isomorphism are the positive semidefinite (symmetric) matrices. Furthermore, the group action of the general linear group $\mathrm{GL}(n+1, \mathbb{R})$ on $\mathrm{Sym}(n+1, \mathbb{R})$ via $B \mapsto CBC^T$ for $C = (c_{ij})_{i,j} \in \mathrm{GL}(n+1, \mathbb{R})$ is compatible with the action of $\mathrm{GL}(n+1, \mathbb{R})$ on $\mathbb{R}[\underline{X}]$ induced by the linear coordinate change $X_j \mapsto \sum_{i=0}^{n} c_{ij} X_i$ for $j = 0, \ldots, n$ restricted to $\mathbb{R}[\underline{X}]_2$.

Definition 1.5.2. Let $f \in \mathbb{R}[\underline{X}]_2$. We call the rank and corank of the associated symmetric matrix also the *rank of* f and the *corank of* f and we denote them by $\mathrm{rk}(f)$ and $\mathrm{cork}(f)$.

Let us recall the theorem that is well-known under the name *Sylvester's theorem* or *Sylvester's law of inertia* and apply it to non-negative quadratic forms.

Theorem 1.5.3. ***(Sylvester's law of inertia)*** *Let $f \in \mathbb{R}[\underline{X}]_2$. Then, there exists some linear coordinate change that sends f to*

$$\sum_{i=0}^{k-1} X_i^2 - \sum_{i=k}^{k+l-1} X_i^2$$

where $k, l \in \mathbb{N}_0$ are uniquely determined and $k + l = \mathrm{rk}(f)$.

Proof. By [Syl], one may send f to some form like above where some positive coefficients a_i appear in front of the X_i^2 $(i = 0, \ldots, k+l-1)$. Clearly, the coordinate change $X_i \mapsto \frac{1}{\sqrt{a_i}} X_i$ for $i = 0, \ldots, k+l-1$ and $X_i \mapsto X_i$ for $i = k+l, \ldots, n$ gives the desired result. □

Corollary 1.5.4. *Let $S = \mathbb{P}^n(\mathbb{R})$, $f \in \mathcal{P}_2$ and $r := \mathrm{rk}(f)$. Then, there exists some linear coordinate change that sends f to*

$$X_0^2 + \cdots + X_{r-1}^2.$$

Especially, the set $\mathcal{V}_{\mathbb{R}}(f) := \{P \in \mathbb{P}^n(\mathbb{R}) \mid f(P) = 0\}$ is a linear subspace of $\mathbb{P}^n(\mathbb{R})$ of projective dimension $\mathrm{cork}(f) - 1$.

These preliminary remarks allow us to study the cones $\mathcal{P}_{n,2}$.

Proposition 1.5.5. *The faces F of $\mathcal{P}_2$ ($S = \mathbb{P}^n(\mathbb{R})$) are in bijection to the linear subspaces of $\mathbb{P}^n(\mathbb{R})$. The bijection is given by*

$$F \mapsto \mathcal{V}_{\mathbb{R}}(f) \text{ with } f \in \mathrm{ri}(F).$$

The relation of inclusion (of faces and subspaces) is reversed under this bijection.

Proof. We will make use of 1.2.19. Suppose $f_1, f_2 \in \mathcal{P}_2$ such that $f_1 - \epsilon f_2 \in \mathcal{P}_2$ for some $\epsilon > 0$, i.e. $\operatorname{carr}(f_2) \subseteq \operatorname{carr}(f_1)$. Then, clearly $\mathcal{V}_{\mathbb{R}}(f_2) \supseteq \mathcal{V}_{\mathbb{R}}(f_1)$. Suppose conversely that $f_1, f_2 \in \mathcal{P}_2$ and $\mathcal{V}_{\mathbb{R}}(f_2) \supseteq \mathcal{V}_{\mathbb{R}}(f_1)$ and we want to show that there exists some $\epsilon > 0$ with $f_1 - \epsilon f_2 \in \mathcal{P}_2$. Since these properties are invariant under linear coordinate changes, we may assume by 1.5.4 that

$$f_1 = X_0^2 + \cdots + X_{r-1}^2$$

where $r = \operatorname{rk}(f_1)$. Since $\mathcal{V}_{\mathbb{R}}(f_2) \supseteq \mathcal{V}_{\mathbb{R}}(f_1)$, f_2 vanishes whenever $X_0, \ldots, X_{r-1}$ are set to 0, so it may not contain monomials $X_i X_j$ with $i, j \geq r$. Let B_1, B_2 be the symmetric matrices associated to f_1, f_2 (see 1.5.1). So written as block matrices where the blocks are separated after the first r rows and columns, we have

$$B_1 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} C & D \\ D^T & 0 \end{pmatrix},$$

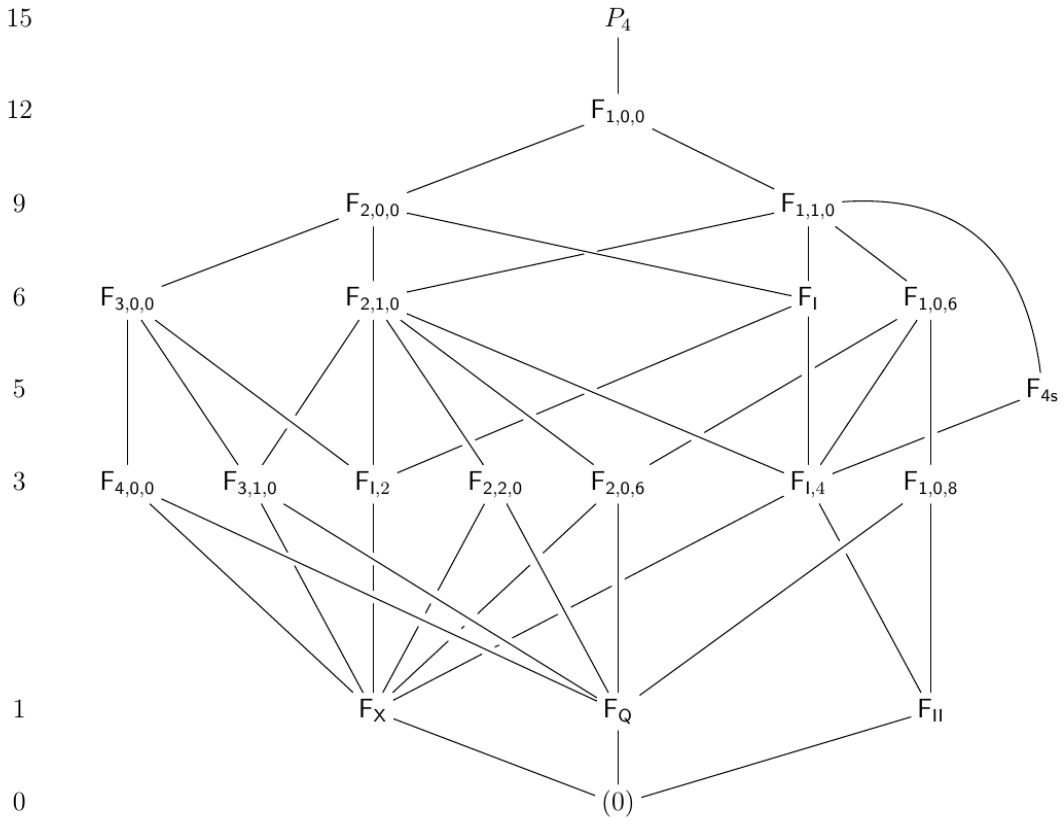
where I_r denotes the identity matrix of size r, C is symmetric of size $r \times r$ and D is an $r \times (n+1-r)$ matrix. However, B_2 is positive semidefinite, so $D = 0$. Now, $I_r - \epsilon C$ is positive definite for sufficiently small $\epsilon > 0$, which may be easily seen from the criterion of positive semidefiniteness that all principal leading minors are positive (and the minors depend continuously on ϵ). This shows that $B_1 - \epsilon B_2$ is positive semidefinite for some $\epsilon > 0$, hence $f_1 - \epsilon f_2 \in \mathcal{P}_2$ for some $\epsilon > 0$, i.e. $\operatorname{carr}(f_2) \subseteq \operatorname{carr}(f_1)$.
We have seen that $\operatorname{carr}(f_2) \subseteq \operatorname{carr}(f_1)$ is equivalent to $\mathcal{V}_{\mathbb{R}}(f_2) \supseteq \mathcal{V}_{\mathbb{R}}(f_1)$ for $f_1, f_2 \in \mathcal{P}_2$. This shows that $f_1, f_2 \in \operatorname{ri}(F)$ for $F \trianglelefteq \mathcal{P}_2$ implies $\mathcal{V}_{\mathbb{R}}(f_1) = \mathcal{V}_{\mathbb{R}}(f_2)$, so the map of the claim is well-defined. The converse, i.e. $\mathcal{V}_{\mathbb{R}}(f_2) = \mathcal{V}_{\mathbb{R}}(f_1)$ implies $F := \operatorname{carr}(f_2) = \operatorname{carr}(f_1)$ and $f_1, f_2 \in \operatorname{ri}(F)$, shows injectivity. Surjectivity follows from the fact that each proper subspace of $\mathbb{P}^n(\mathbb{R})$ (note that $\mathbb{P}^n(\mathbb{R})$ itself is the zero-set of 0) is an intersection of hyperplanes, since hyperplanes are the zero-sets of squares of linear forms and addition of elements of $\mathcal{P}_2$ results in the intersection of their zero-sets. □

Remark 1.5.6. The faces F of $\mathcal{P}_2$ such that $\mathcal{V}_{\mathbb{R}}(f)$ has projective dimension $n-r$ for $f \in \operatorname{ri}(F)$, i.e. f has rank r, have dimension r^2. This follows from the correspondence to the cone of positive semidefinite matrices.

Remark 1.5.7. A possible finite classification of the faces of $\mathcal{P}_2$ (compare with 1.4.5) is given by sorting faces F with the same dimension of $\mathcal{V}_{\mathbb{R}}(f)$ ($f \in \operatorname{ri}(F)$) in the same class. We have seen that this is also the classification of faces up to linear coordinate changes. However, if one draws the corresponding graph (compare with 1.4.5, 1.4.6) one obtains only a chain of classes.

1.6 The case $(n, 2d) = (2, 4)$

The facial structure of $\mathcal{P}_{2,4}$ (for $S = \mathbb{P}^2(\mathbb{R})$) was studied by Aaron Kunert in his dissertation (see [Kun]). The results from section 1.3 played an important role in his work. We will just give a rough description of his results but it will be sufficient for our purposes. His main result on the faces of $\mathcal{P}_{2,4}$ was a description of the facial structure of $\mathcal{P}_{2,4}$ modulo the group action of linear coordinate changes in [Kun, 2.32]. We include the corresponding diagram below together with the dimensions of the representatives of each class as presented in Kunert's work. See 2.3.8 for a more detailed abstract description of what will be explained here.



Facial structure of $\mathcal{P}_{2,4}$ modulo linear coordinate changes (Source: [Kun, 2.32])

We have seen in 1.3.5 that faces F of $\mathcal{P}_{2,4}$ may be described by faces F_P of the local cones of the points $P \in \mathbb{P}^n(\mathbb{R})$. These faces F_P already encode information like the vanishing order or, in the case of double zeros, the corank of the Hessian in local coordinates (see 3.4.15) of elements in $\mathrm{ri}(F_P)$ (and thus also of elements in $\mathrm{ri}(F)$). Thus, we can speak about zeros and their vanishing order as well as the corank of Hessians in the zeros *of a face*. As linear coordinate changes preserve such properties (clearly, one has to consider these properties in corresponding points), they are useful to give a description of the equivalence classes. Let us describe the equivalence classes to some extend (compare with [Kun, 2.16, 2.21]).
Clearly, P_4 and (0) are the equivalence classes $\{\mathcal{P}_{2,4}\}$ and $\{\{0\}\}$ consisting of just one face. Furthermore,

- faces in classes of the form $F_{i,j,k}$ $(i,j,k \in \mathbb{N}_0)$ have exactly i zeros,
- faces in classes of the form $F_{i,j,0}$ $(i,j \in \mathbb{N}_0)$ have $i-j$ double zeros with positive definite Hessian, j double zeros where the Hessian has corank 1 and no more zeros,
- faces in the class F_{4s} have exactly one zero of order 4,
- faces in the classes $F_{|}$, $F_{|,2}$ and $F_{|,4}$ vanish on a line to second order, faces in $F_{|,2}$ additionally have a double zero not on the line and faces in $F_{|,4}$ have one zero of order 4 on the line,
- faces in the class $F_{||}$ vanish on one line to fourth order,

- faces in the class F_{X} vanish on two different lines to second order,
- faces in the class F_{Q} vanish on one irreducible quadric to second order,
- faces in the classes $F_{1,0,6}$ and $F_{1,0,8}$ have one zero of order 2 with Hessian of corank 1, but they are of "higher vanishing" than the zero appearing for faces of type $F_{1,0,4}$,
- faces in the class $F_{2,0,6}$ have one zero as faces in the class of type $F_{1,0,6}$ and one double zero with positive definite Hessian.

Note that the faces in the classes $F_{||}$, F_{X} and F_{Q} are non-negative multiples of a square of a quadratic form.

1.7 Affine setting

Let us also take a look on the affine setting and study cones of non-negative polynomials and their faces. We set $\underline{x} := (x_1, \ldots, x_n)$, so we have $\mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \ldots, x_n]$. Furthermore, let $S \subseteq \mathbb{R}^n$.

Definition 1.7.1. We set

$$\begin{aligned} \mathcal{P} &:= \{f \in \mathbb{R}[\underline{x}] \,|\, f \geq 0 \text{ on } S\}, \\ \mathcal{P}^{Loc} &:= \{f \in \mathbb{R}[\underline{x}] \,|\, f \geq 0 \text{ on } S \cap U \text{ for some neighbourhood } U \text{ of } 0\} \end{aligned}$$

and call them the *cone of non-negative polynomials on* S and the *cone of locally non-negative polynomials on* S (in n variables).

Remark 1.7.2. As in the homogeneous setting, we may replace S by its closure $\overline{S}$ without changing $\mathcal{P}$ and $\mathcal{P}^{Loc}$. Also, $\mathcal{P}^{Loc}$ does just depend on the germ of S at 0.

Remark 1.7.3. $\mathcal{P}$ and $\mathcal{P}^{Loc}$ are cones in $\mathbb{R}[\underline{x}]$ as may be seen easily.

Now, we want to prove a theorem similar to 1.3.5 at the affine case. As it was already mentioned before, it is important to work on compact sets. For this reason, we consider the inclusion $\iota : \mathbb{R}^n \hookrightarrow \mathbb{P}^n(\mathbb{R}), (a_1, \ldots, a_n) \mapsto (1 : a_1 : \cdots : a_n)$ and we define local cones for all points in $\mathbb{P}^n(\mathbb{R})$.

Definition 1.7.4. Let $P \in \mathbb{P}^n(\mathbb{R})$. Then, we set

$$\mathcal{P}^{Loc}(P) := \{f \in \mathbb{R}[\underline{x}] \,|\, f \geq 0 \text{ on } S \cap \iota^{-1}(U) \text{ for some neighbourhood } U \text{ of } P \text{ in } \mathbb{P}^n(\mathbb{R})\}$$

and call it the *cone of locally non-negative polynomials at* P *(with respect to* S*).*

Remark 1.7.5. Clearly, we have $\mathcal{P} = \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} \mathcal{P}^{Loc}(P)$ (compare with 1.3.4) and $\mathcal{P}^{Loc} = \mathcal{P}^{Loc}(0)$. For $P \in \mathbb{R}^n$ the cones $\mathcal{P}^{Loc}(P)$ basically look all the same - to be more detailed, the translation $\underline{x} \mapsto \underline{x} - P^T$ or rather the corresponding linear map $f(\underline{x}) \mapsto f(\underline{x} - P^T)$ on $\mathbb{R}[\underline{x}]$ transfers $\mathcal{P}^{Loc}(P)$ in $\mathcal{P}^{Loc}$.

Let us prove a theorem parallel to 1.3.5.

Remark 1.7.6. In the next theorem, we will use the following statement, which will be proved in 2.2.8(a). Let C be a cone and $K \subseteq C$. Then,

$$\operatorname{carr}(K) = \{y \in C \,|\, \exists\, x \in \operatorname{cone}(K) : x - y \in C\}.$$

Theorem 1.7.7. *Let $S \subseteq \mathbb{R}^n$ and $F \trianglelefteq \mathcal{P}$. We set*

$$F^{Loc}(P) := \operatorname{carr}_{\mathcal{P}^{Loc}(P)}(F) \trianglelefteq \mathcal{P}^{Loc}(P).$$

for $P \in \mathbb{P}^n(\mathbb{R})$. Then, we have

$$F = \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} F^{Loc}(P) \quad \text{and} \quad \operatorname{span}(F) = \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} \operatorname{span}\left(F^{Loc}(P)\right).$$

Proof. Obviously, $F \subseteq F^{Loc}(P)$ for all $P \in \mathbb{P}^n(\mathbb{R})$, so we have

$$F \subseteq \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} F^{Loc}(P) \quad \text{and} \quad \operatorname{span}(F) \subseteq \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} \operatorname{span}\left(F^{Loc}(P)\right).$$

We prove the equality with the linear spans. Let $f \in \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} \operatorname{span}\left(F^{Loc}(P)\right)$ and we consider some $P \in \mathbb{P}^n(\mathbb{R})$, so especially $f \in \operatorname{span}\left(F^{Loc}(P)\right)$. Thus, by 1.2.11, we have $f = f_1 - f_2$ for some $f_1, f_2 \in F^{Loc}(P)$. By 1.7.6, we have

$$F^{Loc}(P) = \left\{ f \in \mathcal{P}^{Loc}(P) \,\middle|\, \exists\, g \in F : g - f \in \mathcal{P}^{Loc}(P) \right\}.$$

Hence, there are $g_1, g_2 \in F$ such that $g_1 - f_1, g_2 - f_2 \in \mathcal{P}^{Loc}(P)$. Setting $g_P := g_1 + g_2 \in F$, we obtain $g_P + f = g_1 + f_1 + (g_2 - f_2) \in \mathcal{P}^{Loc}(P)$ and $g_P - f = (g_1 - f_1) + g_2 + f_2 \in \mathcal{P}^{Loc}(P)$. Thus, we find some neighbourhood U_P of P with $g_P \pm f \geq 0$ on $S \cap \iota^{-1}(U_P)$. Since $\mathbb{P}^n(\mathbb{R})$ is compact, we find finitely many $P_1, \ldots, P_k \in \mathbb{P}^n(\mathbb{R})$ such that $\bigcup_{i=1}^k U_{P_i} = \mathbb{P}^n(\mathbb{R})$. Hence, we obtain $g \pm f \geq 0$ on S and thus $g \pm f \in \mathcal{P}$ with $g := \sum_{i=1}^k g_{P_i} \in F$. Now,

$$\frac{1}{2}(g + f) + \frac{1}{2}(g - f) = g \in F.$$

By the definition of faces, we have $g \pm f \in F$, so $f \in \operatorname{span}(F)$. This shows $\operatorname{span}(F) = \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} \operatorname{span}\left(F^{Loc}(P)\right)$ since f was arbitrary. The other claim follows as in 1.3.5 using 1.2.8 and 1.7.5

$$F = \operatorname{span}(F) \cap \mathcal{P} = \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} \operatorname{span}\left(F^{Loc}(P)\right) \cap \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} \mathcal{P}^{Loc}(P) = \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} F^{Loc}(P).$$

□

We also have a statement as in 1.3.6 and we can use exactly the same proof.

Proposition 1.7.8. *Let $S \subseteq \mathbb{R}^n$ and $F_P \trianglelefteq \mathcal{P}^{Loc}(P)$ for each $P \in \mathbb{P}^n(\mathbb{R})$. Then,*

$$F := \bigcap_{P \in \mathbb{P}^n(\mathbb{R})} F_P \trianglelefteq \mathcal{P}.$$

Remark 1.7.9. Clearly, the cones $\mathcal{P}^{Loc}(P)$ do just depend on the germ of S at P and as before, one can replace S by its closure in $\mathbb{R}^n$. If P does not belong to the closure of S in $\mathbb{P}^n(\mathbb{R})$, then $\mathcal{P}^{Loc}(P) = \mathbb{R}[\underline{x}]$ and thus, in all intersections one can restrict to points P in the closure of S in $\mathbb{P}^n(\mathbb{R})$.

1.8 Further notations and conventions

Let $K_1, K_2 \subseteq V$ with a real vector space V, then we denote the *Minkowski sum of* K_1 *and* K_2 by

$$K_1 + K_2 := \{x + y \,|\, x \in K_1, y \in K_2\}.$$

If $K \subseteq V$, then we set $-K := \{-x \,|\, x \in K\}$. Also, if R is a ring, M some R-module and $K_1 \subseteq R$, $K_2 \subseteq M$, then we set

$$K_1 \cdot K_2 := \{x \cdot y \,|\, x \in K_1, y \in K_2\} \subseteq M.$$

Note that this may lead to confusion if K_1 and K_2 are ideals of a ring. However, there is just one situation where a misunderstanding is possible and there, we mention explicitly how to interpret it. We will usually write the ideal product without any dot in between the ideals.
If V_1, V_2 are real vector spaces and $K_1 \subseteq V_1$ as well as $K_2 \subseteq V_2$, then we have the *direct sum of* K_1 *and* K_2 given by

$$K_1 \oplus K_2 := \{(x, y) \,|\, x \in K_1, y \in K_2\} \subseteq V_1 \oplus V_2.$$

If A is a commutative ring with $\frac{1}{2} \in A$, we will denote the sums of squares in this ring by $\Sigma\, A^2$ and the set of non-negative elements (where non-negativity is considered on all points of the real spectrum) by A_+. Thus, A_+ is the saturation of $\Sigma\, A^2$ (see 3.1.1) and we have $\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x \geq 0\}$. If an affine $\mathbb{R}$-variety and a regular function f are given, then we denote the set of real points of the variety where f is vanishing by $\mathcal{V}_{\mathbb{R}}(f)$. We use the same notation for homogeneous polynomials $f \in \mathbb{R}[\underline{X}]_d$ and the points of $\mathbb{P}^n(\mathbb{R})$ where f is vanishing.
We denote the endomorphisms of a structure S (e.g. vector space, ring, $\mathbb{R}$-algebra etc.) by $\mathrm{End}(S)$ and the automorphisms by $\mathrm{Aut}(S)$. The type of structure will be clear from the context and we will not mention it explicitly.
In the univariate case, we will usually consider the Big O notation at 0 and not for the behaviour to ∞. Similarly, the leading term of a polynomial or power series will be the term of lowest degree and not the term of largest degree.

2 Convexity

2.1 Algebraic interior and closure and topology

We have defined the two cones $\mathcal{P}$ and $\mathcal{P}^{Loc}$, which live in an infinite-dimensional vector space and will be studied later. Now, we have no canonical topology to define the relative interior of a face of these cones since the Euclidean topology is just available on finite-dimensional vector spaces. However, it is possible to study faces of cones without using any topology via the algebraic interior and the algebraic closure, which we will introduce now.

2.1.1 Definition and properties

Let V be a real vector space as before.

Definition 2.1.1. ([BaFo, I.1.3.,I.1.4.,I.1.5.] [Gue, 5.1.,5.34.])
Let $K \subseteq V$. We call

$$\mathrm{ai}(K) := \{x \in V \,|\, \forall v \in V\, \exists \epsilon > 0 :]x - \epsilon v, x + \epsilon v[\subseteq K\}$$

the *algebraic interior of* K and

$$\mathrm{ac}(K) := \{x \in V \,|\, \exists y \in K : [y, x[\subseteq K\}$$

the *algebraic closure of* K.

Remark 2.1.2. We do not require $x \neq y$ in the definition of algebraic closure, so $K \subseteq \mathrm{ac}(K)$. However, if the set is convex and has more than one element, it makes no difference to require $x \neq y$ (see [BaFo, I.4.1.a)]).

Remark 2.1.3. The algebraic interior may also be defined on affine spaces. Let W be an affine subspace of V and $K \subseteq W$. Let U be the linear subspace associated to W, i.e. $W = a + U$ for $a \in W$. Now, each such $a \in W$ gives W the structure of a vector space over $\mathbb{R}$ by carrying the structure from U over to W via $x \mapsto x + a$ (especially, a is the origin of W). The algebraic interior of K considered in one of these vector spaces does not depend on the choice of a and thus may be considered as the algebraic interior of K in W. However, if $W \neq \mathrm{aff}(K)$, the algebraic interior of K in W is empty.

The remark motivates the following definition, which gives an appropriate replacement of the relative interior in infinite-dimensional vector spaces.

Definition 2.1.4. Let $K \subseteq V$. If $K \neq \emptyset$, then the algebraic interior of K in $\mathrm{aff}(K)$ (see 2.1.3) is called the *relative algebraic interior of* K. We denote it by $\mathrm{rai}(K)$ and we set $\mathrm{rai}(\emptyset) := \emptyset$. Furthermore, we set

$$\mathrm{rabd}(K) := \mathrm{ac}(K) \setminus \mathrm{rai}(K)$$

and call it the *relative algebraic boundary of* K.

Remark 2.1.5. If C is convex, then $\mathrm{ai}(C)$, $\mathrm{ac}(C)$ and $\mathrm{rai}(C)$ are also convex [BaFo, I.3.2.]. If C is a cone, then $\mathrm{ai}(C) \cup \{0\}$, $\mathrm{ac}(C)$ and $\mathrm{rai}(C) \cup \{0\}$ are cones. We will use the definitions of algebraic interior, algebraic closure and relative algebraic closure mainly for convex sets.

However, algebraic interior and algebraic closure do not transfer all properties from the case of convex sets in finite-dimensional vector spaces.

Example 2.1.6. (a) We consider $\mathcal{P}$ for $n = 1$ and $S = \mathbb{R}$ and denote x_1 by x. Let $f \in \mathcal{P} \setminus \{0\}$ and let $2d$ be the degree of f ($d \in \mathbb{N}_0$). Then, clearly for each $\epsilon > 0$ we have $(f - \epsilon x^{2d+2})(a) < 0$ for sufficiently large a. Since $x^{2d+2} \in \mathcal{P}$, this shows that $f \notin \mathrm{rai}(\mathcal{P})$. Obviously $0 \notin \mathrm{rai}(\mathcal{P})$, so $\mathrm{rai}(\mathcal{P}) = \emptyset$. This shows that non-empty convex sets may have empty relative algebraic interior.

(b) We consider the univariate polynomial ring $\mathbb{R}[x]$ and

$$\mathcal{P}_\infty := \{f \in \mathbb{R}[x] \mid \exists M \in \mathbb{R} : \forall a \geq M : f(a) \geq 0\}.$$

It is easy to see that $\mathcal{P}_\infty$ is a cone in $\mathbb{R}[x]$. Now, if $f \in \mathbb{R}[x]$ and $2d > \deg(f)$, then $[x^{2d}, f[\subseteq \mathcal{P}_\infty$ since the leading terms of elements in $[x^{2d}, f[$ are positive multiples of x^{2d}. This shows $\mathrm{ac}(\mathcal{P}_\infty) = \mathbb{R}[x]$ though $\mathcal{P}_\infty \neq \mathbb{R}[x]$.

(c) There exist convex sets C with $\mathrm{ac}(\mathrm{ac}(C)) \neq \mathrm{ac}(C)$ (see e.g. [Koe, §16, 2(4)]).

Now, we list some properties, which also hold in the infinite-dimensional case.

Remark 2.1.7. Let C be a convex set (a cone).

(a) If $x \in \mathrm{rai}(C)$ and $y \in \mathrm{ac}(C)$, then $[x, y[\subseteq \mathrm{rai}(C)$. [BaFo, I.3.1.]

(b) We have $\mathrm{rai}(\mathrm{rai}(C)) = \mathrm{rai}(C)$ and $\mathrm{aff}(\mathrm{ac}(C)) = \mathrm{aff}(C)$. [BaFo, I.5.2.], [BaFo, I.4.1.]

(c) If $\mathrm{rai}(C) \neq \emptyset$, then

$$\begin{aligned}
\mathrm{aff}(C) &= \mathrm{aff}(\mathrm{rai}(C)) = \mathrm{aff}(\mathrm{ac}(C)) && \text{[BaFo, I.5.1.,I.4.1.]}\\
\mathrm{rai}(C) &= \mathrm{rai}(\mathrm{rai}(C)) = \mathrm{rai}(\mathrm{ac}(C)) && \text{[Gue, 5.9.]}\\
\mathrm{ac}(C) &= \mathrm{ac}(\mathrm{rai}(C)) = \mathrm{ac}(\mathrm{ac}(C)) && \text{[Gue, 5.9.]}\\
\mathrm{rabd}(C) &= \mathrm{rabd}(\mathrm{rai}(C)) = \mathrm{rabd}(\mathrm{ac}(C)).
\end{aligned}$$

Remark 2.1.8. Due to their good properties, convex sets with non-empty algebraic interior are also called *convex algebraic bodies* (see [Koe, §16, 4.], [Gue, 5.1.]) Convex sets with non-empty relative algebraic interior are named *convex cells* in [BaFo2] (see [BaFo2, p.1], fr. "cellule convexe").

2.1.2 Connection to topology

Though we are usually not using any topology, we will present how the algebraic definitions are related to topology and define the topology that could be used in our work without any problems.

Definition 2.1.9. [BaFo, V] A real vector space V together with some topology on V is called *topological vector space*, if $V \times V \to V, (x, y) \mapsto x + y$ and $\mathbb{R} \times V \mapsto V, (\alpha, x) \mapsto \alpha x$ are continuous maps where $\mathbb{R}$ carries the Euclidean topology and $V \times V$ as well as $\mathbb{R} \times V$ are considered with the product topologies.

Theorem 2.1.10. *[BaFo, V.1.1., V.1.3.] Let V together with some topology be a topological vector space and $K \subseteq V$. Then, we have*

$$\mathring{K} \subseteq \operatorname{rai}(K), \quad \operatorname{ac}(K) \subseteq \overline{K} \quad \textit{and} \quad \operatorname{rabd}(K) \subseteq \partial K.$$

If C is convex and $\mathring{C} \neq \emptyset$, then

$$\operatorname{rai}(C) = \mathring{C}, \quad \operatorname{ac}(C) = \overline{C} \quad \textit{and} \quad \operatorname{rabd}(C) = \partial C.$$

Remark 2.1.11. In [BaFo, V.1.1., V.1.3.] even a weaker condition than being a topological vector space is required for the topology.

The topology of the following definition is probably the most important topology in this context as one can see from the subsequent remarks and it is commonly used in this area.

Definition 2.1.12. [BaFo, V.3.1.] Let V be a real vector space. The set of convex sets $C \subseteq V$ with $C = \operatorname{ai}(C)$ is a basis of some topology, which is called *natural topology on V.*

Remark 2.1.13. [BaFo, V.3.2.] V together with its natural topology is a topological vector space.

Remark 2.1.14. [BaFo, V.3.2.][Gue, 5.22.,5.23.] Let V be finite-dimensional. Then, the natural topology coincides with the Euclidean topology. If $\emptyset \neq C \subseteq V$ is convex, then we have $\operatorname{ri}(C) \neq \emptyset$ and thus

$$\operatorname{rai}(C) = \operatorname{ri}(C) \text{ (especially } \operatorname{ai}(C) = \mathring{C}), \quad \operatorname{ac}(C) = \overline{C} \quad \text{and} \quad \operatorname{rabd}(C) = \operatorname{rbd}(C).$$

Remark 2.1.15. The natural topology is also called *convex core topology* or *finest/strongest locally convex topology* (see [Scha, p.56], [HoWr, p.168]). It was used in [CiMaNe] to study closures of cones that are connected to those presented in this work. Another topology, the *core topology*, was also studied in [BaFo, V.2.].

2.2 Statements on convex sets and faces

We have already studied convex sets in section 1.2. We will use now the just defined concepts of algebraic interior and algebraic closure to get some more advanced results.

2.2.1 Klee classes and Klee faces

We have seen in 2.1.7 that convex sets with non-empty relative algebraic interior are of special interest. Therefore, faces of convex sets (cones) C with this property will play a special role as well. This leads us to the term *Klee class*, which are the non-empty relative algebraic interiors of faces. They build a partition of C and appear for instance in [BaFo, III.4.2.1.] and [Roc, 18.2.]. However, the study of faces with non-empty relative algebraic interior seems to be very rare in mathematics. We could not find any proper term for such faces such that we introduce the term *Klee faces* here.

Definition 2.2.1. [BaFo, III.4.2.1.] Let $C \subseteq V$ be convex (a cone). We define a quasiorder $\leq$ on C via

$$\begin{aligned} y \leq x \quad &:\Leftrightarrow \quad x = y \text{ or } \exists\, z \in C : [y,x] \subseteq [y,z) \\ &\Leftrightarrow \quad \exists\, \epsilon > 0 : x - \epsilon(y-x) \in C. \end{aligned}$$

The equivalence classes induced by this quasiorder, given by

$$x \sim y \quad :\Leftrightarrow \quad x \leq y \text{ and } y \leq x,$$

are called *Klee classes of* C.
Furthermore, faces F of C with $\operatorname{rai}(F) \neq \emptyset$ are called *Klee faces of* C. We denote the set of faces of C by $\mathcal{F} = \mathcal{F}(C)$ and the set of Klee faces by $\mathcal{K} = \mathcal{K}(C)$.

Remark 2.2.2. We consider the quasiorder $\leq$ with respect to C and the quasiorder $\leq_F$ with respect to $F \trianglelefteq C$ as defined in 2.2.1. Since $\operatorname{aff}(F) \cap C = F$ (see 1.2.8), $\leq_F$ coincides with the restriction of $\leq$ on F.

We sum up the main properties concerning relative algebraic interior, carrier faces and Klee faces in the following remark.

Remark 2.2.3. Let C be convex (a cone) and let us use the notation from 2.2.1.

(a) Let $x, y \in C$, then

$$y \leq x \quad \Leftrightarrow \quad \operatorname{carr}(y) \subseteq \operatorname{carr}(x). \quad \text{[BaFo, III.4.2.2.a)]}$$

Especially, two elements of C belong to the same Klee class if and only if they have the same carrier face.

(b) If $x \in C$, then the corresponding Klee class is $\operatorname{rai}(\operatorname{carr}(x))$ [BaFo, III.4.2.2.b)]. Especially, we have $x \in \operatorname{rai}(\operatorname{carr}(x))$ (see also [BaFo, III.4.1.2.c)]) and carrier faces of one point are Klee faces.

(c) If F is a face of C and $x \in \operatorname{rai}(F)$, then $F = \operatorname{carr}(x)$ [BaFo, III.4.1.2.d)]. This shows that Klee faces are carrier faces of single points. Now, (b) implies for $F \trianglelefteq C$

$$F \text{ is a Klee face} \quad \Leftrightarrow \quad \operatorname{rai}(F) \neq \emptyset \quad \Leftrightarrow \quad \exists\, x \in C : F = \operatorname{carr}(x).$$

Furthermore, we have for $x, y \in C$

$$x \sim y \quad \Leftrightarrow \quad \operatorname{carr}(x) = \operatorname{carr}(y) \quad \Leftrightarrow \quad \exists F \in \mathcal{F} : x, y \in \operatorname{rai}(F).$$

Indeed, the first equivalence follows from (a). Now, (b) shows that $x \sim y$ implies $x, y \in \operatorname{rai}(F)$ where $F = \operatorname{carr}(x)$. Finally, we have seen for $F \in \mathcal{F}$ and $x, y \in \operatorname{rai}(F)$ that $\operatorname{carr}(x) = F = \operatorname{carr}(y)$.

(d) From (a), (b) and (c) we obtain bijections between the set of Klee faces and the set of Klee classes given by (see [BaFo, III.4.2.2.Cor.b)])

$$\mathcal{K} \to \{\text{Klee classes of } C\}, \quad F \mapsto \operatorname{rai}(F)$$
$$\{\text{Klee classes of } C\} \to \mathcal{K}, \quad O \mapsto \operatorname{carr}(O) \text{ (or } O \mapsto \operatorname{carr}(x) \text{ with some } x \in O)$$

With respect to this bijection, the semiorder induced by $\leq$ on the Klee classes is compatible with the semiorder given by the inclusion on $\mathcal{K}$.

(e) Since any face is obviously the union of elements whose carrier face is contained in the given face, (a) implies that for $x \in C$ (see also [BaFo, III.4.1.2.c)])

$$\operatorname{carr}(x) = \{y \in C \,|\, y \leq x\} = \{y \in C \,|\, \exists\, \epsilon > 0 : x - \epsilon(y - x) \in C\}\,.$$

Especially, C is a Klee face if and only if there is some $x \in C$ with $y \leq x$ for all $y \in C$ and $\operatorname{rai}(C)$ consists exactly of such x by (b). If one applies this idea on a face $F \trianglelefteq C$, then one obtains

$$\operatorname{rai}(F) = \{x \in F \,|\, \forall\, y \in F : y \leq x\} = \{x \in F \,|\, \forall\, y \in F\, \exists\, \epsilon > 0 : x - \epsilon(y - x) \in F\}\,.$$

2.2.2 Carrier faces, relative algebraic interior and lineality space

We have just seen descriptions of the carrier face (of one point) and the relative algebraic interior of a face in the case of convex sets. Due to their importance in this work and for later reference, we will include more descriptions of them - also in the case of cones - and we generalize to carrier faces of sets. Additionally, we will describe the lineality space of the algebraic closure of a cone and present some further property.
The following lemma allows a more conceptual proof of the subsequent proposition. It may be found for ascending chains of faces instead of directed sets in [BaFo2, II.4.8.] and is stated in [Gue, 5.26] without proof.

Lemma 2.2.4. *Let $C \subseteq V$ be a convex set (a cone) and $M \subseteq \mathcal{F}(C)$ be a set of faces of C that (together with $\subseteq$) is a directed set, i.e. $M \neq \emptyset$ and for $F_1, F_2 \in M$ there exists some $F_3 \in M$ with $F_1 \cup F_2 \subseteq F_3$. Then, $\bigcup_{F \in M} F$ is a face of C.*

Proof. Let $G := \bigcup_{F \in M} F$. If C is a cone, then $0 \in G$ since $M \neq \emptyset$.
We show that G is convex. Let $x, y \in G$. Then, there exist $F_1, F_2 \in M$ with $x \in F_1$ and $y \in F_2$. Choose $F_3 \in M$ with $F_1 \cup F_2 \subseteq F_3$. Since F_3 is convex, we have $[x, y] \subseteq F_3 \subseteq G$. Thus, G is convex.
Suppose now that $x, y \in C$ with $\frac{x+y}{2} \in G$. Hence, there is some $F \in M$ with $\frac{x+y}{2} \in F$. Since F is a face of C, we have $x, y \in F \subseteq G$, which shows that G is a face. □

Proposition 2.2.5. *Let $C \subseteq V$ be convex.*

(a) For $K \subseteq C$, we have

$$\operatorname{carr}(K) = \bigcup_{x \in \operatorname{conv}(K)} \operatorname{carr}(x) = \{y \in C \,|\, \exists\, x \in \operatorname{conv}(K)\, \exists\, \epsilon > 0 : x - \epsilon(y - x) \in C\}\,.$$

(b) Let F be a face of C. Then, we have

$$\begin{aligned}\operatorname{rai}(F) &= \{x \in F \,|\, \forall\, y \in \operatorname{aff}(F)\, \exists\, \epsilon > 0 : x - \epsilon(y - x) \in F\} \\ &= \{x \in F \,|\, \forall\, y \in F\, \exists\, \epsilon > 0 : x - \epsilon(y - x) \in C\}\,.\end{aligned}$$

Proof. (a) The claim is clear if $K = \emptyset$. So let $K \neq \emptyset$. Obviously, we have

$$\operatorname{carr}(K) \supseteq \bigcup_{x \in \operatorname{conv}(K)} \operatorname{carr}(x).$$

For the opposite inclusion, we just have to show that the right-hand side is a face of C. Due to the previous lemma, it is enough to show that

$$M := \{\operatorname{carr}(x) \mid x \in \operatorname{conv}(K)\}$$

is a directed set. Since $K \neq \emptyset$, we have $M \neq \emptyset$. Let $F_1 = \operatorname{carr}(x_1) \in M$ and $F_2 = \operatorname{carr}(x_2) \in M$ with $x_1, x_2 \in \operatorname{conv}(K)$. Then, $\frac{x_1+x_2}{2} \in]x_1, x_2[$ and therefore $\frac{x_1+x_2}{2} \geq x_1$ and $\frac{x_1+x_2}{2} \geq x_2$ using the quasiorder of 2.2.1 (with respect to C). This shows (see 2.2.3(a)) that

$$F_3 := \operatorname{carr}\left(\frac{x_1 + x_2}{2}\right) \supseteq F_1, F_2.$$

Obviously, $F_3 \in M$, which shows the claim. The representation of $\operatorname{carr}(K)$ in set-builder notation follows from 2.2.3(e) where the claim for one-point sets K is stated.

(b) The first representation of $\operatorname{rai}(F)$ follows from definition or [BaFo, I.7.4.a)]. The second representation is implied by 2.2.3(e) and $\operatorname{aff}(F) \cap C = F$.

□

Remark 2.2.6. On the first view, 2.2.5(a) seems to follow from [Dub, (3.3)]. However, they use an unusual concept of faces.

The following lemma is useful to translate the result to cones.

Lemma 2.2.7. *Let C be a cone.*

(a) Let $x \in C$ and $y \in V$. Then, there exists some $\epsilon > 0$ such that $x - \epsilon(y - x) \in C$ if and only if there exists some $\epsilon > 0$ with $x - \epsilon y \in C$.

(b) Let $x, y \in C$. Then, using the quasiorder $\leq$ from 2.2.1, we have $y \leq x$ if and only if there is some $\epsilon > 0$ with $x - \epsilon y \in C$.

Proof. (a) For $\epsilon > 0$ and $x - \epsilon(y - x) \in C$, we have

$$x - \frac{\epsilon}{1+\epsilon} y = \frac{1}{1+\epsilon}((1+\epsilon)x - \epsilon y) = \frac{1}{1+\epsilon}(x - \epsilon(y - x)) \in C,$$

which shows the implication "⇒".
Conversely, let $\epsilon > 0$ and $x - \epsilon y \in C$. Then, we have $x - \frac{\epsilon}{1+\epsilon} y = \frac{\epsilon}{1+\epsilon} x + \frac{1}{1+\epsilon}(x - \epsilon y) \in C$. Thus, we may choose $0 < \epsilon < 1$ with $x - \epsilon y \in C$ and we have

$$x - \frac{\epsilon}{1-\epsilon}(y - x) = \frac{1}{1-\epsilon}((1-\epsilon)x - \epsilon(y - x)) = \frac{1}{1-\epsilon}(x - \epsilon y) \in C.$$

This shows the implication "⇐".

(b) This follows immediately from (a) and 2.2.1.

□

Let us state the result for cones. Later, the statement of 2.2.8(c) will be used frequently.

Proposition 2.2.8. *Let C be a cone.*

(a) Let $K \subseteq C$. Then, we have

$$\begin{aligned}&\operatorname{carr}(K) = L_C, \text{ if } K = \emptyset \text{ and}\\ &\operatorname{carr}(K) = \{y \in C \mid \exists\, x \in \operatorname{conv}(K)\, \exists\, \epsilon > 0 : x - \epsilon y \in C\} \text{ else.}\end{aligned}$$

We always have

$$\operatorname{carr}(K) = \{y \in C \mid \exists\, x \in \operatorname{cone}(K) : x - y \in C\}.$$

and in the case $K = \{x\}$, we have

$$\operatorname{carr}(x) = \{y \in C \mid \exists\, \epsilon > 0 : x - \epsilon y \in C\}.$$

(b) Let F be a face of C. Then, we have

$$\begin{aligned}\operatorname{rai}(F) &= \{x \in F \mid \forall\, y \in \operatorname{aff}(F)\, \exists\, \epsilon > 0 : x - \epsilon y \in F\}\\ &= \{x \in F \mid \forall\, y \in F\, \exists\, \epsilon > 0 : x - \epsilon y \in C\}.\end{aligned}$$

(c) Let $x \in \operatorname{rai}(C)$. Then, we have

$$\begin{aligned}L_{\operatorname{ac}(C)} &= \{y \in V \mid \forall\, \alpha \in \mathbb{R} : x + \alpha y \in C\}\\ &= \{y \in V \mid \forall\, \alpha \in \mathbb{R} : \alpha y + \operatorname{rai}(C) \subseteq \operatorname{rai}(C)\}.\end{aligned}$$

Thus, $L_{\operatorname{ac}(C)}$ is the largest linear subspace W of V with $W + \operatorname{rai}(C) = \operatorname{rai}(C)$.

Proof. (a) First of all, we consider the case $K \neq \emptyset$. Now, the carrier face with respect to C as a cone coincides with the carrier face of C as a convex set. Thus, using 2.2.7(a) and $\mathbb{R}_+ \cdot C \subseteq C$, 2.2.5(a) implies

$$\operatorname{carr}(K) = \{y \in C \mid \exists\, x \in \operatorname{conv}(K)\, \exists\, \epsilon > 0 : x - \epsilon y \in C\}.$$

The other representation of $\operatorname{carr}(K)$ follows from another application of $\mathbb{R}_+ \cdot C \subseteq C$ and the case $K = \{x\}$ is also clear.
It remains the case $K = \emptyset$. Then, $\operatorname{carr}(K)$ is the smallest face of C and since every face contains 0, we have $\operatorname{carr}(\emptyset) = \operatorname{carr}(0)$. As just shown, we have

$$\begin{aligned}\operatorname{carr}(0) &= \{y \in C \mid \exists\, \epsilon > 0 : -\epsilon y \in C\} = \{y \in C \mid -y \in C\}\\ &= C \cap (-C) = L_C.\end{aligned}$$

This proves the claim.

(b) This follows from 2.2.5(b) and 2.2.7(a) (applied to F and C).

(c) We denote the terms on the right-hand side by R_1 and R_2, so

$$\begin{aligned}&R_1 := \{y \in V \mid \forall\, \alpha \in \mathbb{R} : x + \alpha y \in C\}\\ \text{and}\quad &R_2 := \{y \in V \mid \forall\, \alpha \in \mathbb{R} : \alpha y + \operatorname{rai}(C) \subseteq \operatorname{rai}(C)\}.\end{aligned}$$

Obviously, we have $R_1 \supseteq R_2$. Let us show that $L_{\mathrm{ac}(C)} \subseteq R_2$. So let $y \in L_{\mathrm{ac}(C)}$ and $\alpha \in \mathbb{R}$. Now, $\pm y \in \mathrm{ac}(C)$ and 2.1.7(a) imply that

$$\frac{1}{1+|\alpha|}(x+\alpha y) = \frac{x+|\alpha|\mathrm{sgn}(\alpha)y}{1+|\alpha|} \in [x, \mathrm{sgn}(\alpha)y[\subseteq \mathrm{rai}(C),$$

so $x + \alpha y \in \mathrm{rai}(C)$ for $\alpha \in \mathbb{R}$ using 2.1.5. Since $x \in \mathrm{rai}(C)$ was chosen arbitrarily, we have $\alpha y + \mathrm{rai}(C) \subseteq \mathrm{rai}(C)$.
It remains to show that $R_1 \subseteq L_{\mathrm{ac}(C)}$. Let $y \in V$ with $x + \alpha y \in C$ for all $\alpha \in \mathbb{R}$, thus $\frac{1}{1+|\alpha|}(x+\alpha y) \in C$. We obtain

$$[x, y[= \left\{ \frac{1}{1+|\alpha|}(x+\alpha y) \,\middle|\, \alpha \geq 0 \right\} \subseteq C$$
$$\text{and} \quad [x, -y[= \left\{ \frac{1}{1+|\alpha|}(x+\alpha y) \,\middle|\, \alpha \leq 0 \right\} \subseteq C.$$

This shows $y \in \mathrm{ac}(C)$ and $-y \in \mathrm{ac}(C)$ and hence $y \in L_{\mathrm{ac}(C)}$.

□

We end with a remark and a corollary, which are easy but useful.

Remark 2.2.9. Let $C_2 \subseteq C_1$ be convex sets (cones). The propositions 2.2.5(a) and 2.2.8(a) imply that $\mathrm{carr}_{C_2}(K) \subseteq \mathrm{carr}_{C_1}(K)$ for all $K \subseteq C_2$.

Corollary 2.2.10. *Let $C \subseteq V$ be convex (a cone) and $K \subseteq C$ with $x \in \mathrm{rai}(K)$. Then, we have $x \in \mathrm{rai}(\mathrm{carr}(K))$.*

Proof. Since $K \neq \emptyset$, it is enough to show it for convex sets. Let $x \in \mathrm{rai}(K)$ and $y \in \mathrm{carr}(K)$. Now, by 2.2.5(a), there is some $z \in \mathrm{conv}(K)$ and some $\epsilon > 0$ with $z - \epsilon(y - z) \in C$. Since $z \in \mathrm{conv}(K) \subseteq \mathrm{aff}(K)$ and $x \in \mathrm{rai}(K)$, there is some $\mu > 0$ with $x - \mu(z - x) \in K$. Hence, we have

$$x - \frac{\epsilon\mu}{1+\epsilon+\mu}(y-x) = \frac{1+\epsilon}{1+\epsilon+\mu}(x-\mu(z-x)) + \frac{\mu}{1+\epsilon+\mu}(z-\epsilon(y-z)) \in C.$$

As $y \in \mathrm{carr}(K)$ was chosen arbitrarily, 2.2.5(b) shows that $x \in \mathrm{rai}(\mathrm{carr}(K))$. □

2.2.3 Further properties

We add some remark on the behaviour of convex sets under affine maps, which will be used later.

Remark 2.2.11. Let $C \subseteq V$ be a convex set (a cone) and W be a vector space over $\mathbb{R}$. If $\varphi : V \to W$ is an affine map, then $\varphi(C)$ is convex and $\varphi(\mathrm{rai}(C)) \subseteq \mathrm{rai}(\varphi(C))$. If C is a cone and $\varphi : V \to W$ is linear, then $\varphi(C)$ is also a convex cone. [Gue, 4.9.(c), 5.16.]

The following proposition is probably well-known (maybe with exception of the statement on the Klee faces).

Proposition 2.2.12. *Let $C \subseteq V$ be a cone, V_0 a linear subspace of V and $C_0 := C \cap V_0$. Then, we have an inclusion*

$$\iota : \mathcal{F}(C_0) \hookrightarrow \mathcal{F}(C), \quad F_0 \mapsto \mathrm{carr}_C(F_0).$$

We have $\mathrm{carr}_C(F_0) \cap C_0 = F_0$ for each $F_0 \in \mathcal{F}(C_0)$ and $\iota(\mathcal{K}(C_0)) \subseteq \mathcal{K}(C)$.

Proof. Suppose first that $F_0 \in \mathcal{K}(C_0)$ and $x \in \operatorname{rai}(F_0)$. Then, by 2.2.10, we have $x \in \operatorname{rai}(\operatorname{carr}_C(F_0))$. This shows $\iota(\mathcal{K}(C_0)) \subseteq \mathcal{K}(C)$.
It remains to show that $\operatorname{carr}_C(F_0) \cap C_0 = F_0$ since it implies that ι is an inclusion. By 2.2.8(a), we have

$$\operatorname{carr}_C(F_0) = \{y \in C \mid \exists\, x \in F_0 : x - y \in C\}.$$

Now, using that $C_0 = C \cap V_0$ and since $x, y \in V_0$ implies $x - y \in V_0$, we obtain

$$\begin{aligned}\operatorname{carr}_C(F_0) \cap C_0 &= \{y \in C_0 \mid \exists\, x \in F_0 : x - y \in C\} \\ &= \{y \in C_0 \mid \exists\, x \in F_0 : x - y \in C_0\} \\ &= \operatorname{carr}_{C_0}(F_0) = F_0.\end{aligned}$$

□

The following proposition will be relevant as it shows that a cone $C \subseteq V$ is essentially determined by the image of the projection on $V/L_{\operatorname{ac}(C)}$.

Proposition 2.2.13. *Let $C \subseteq V$ be a cone, $V = L_{\operatorname{ac}(C)} \oplus W$ with some linear subspace W of V and let $p : V \to W$ be the projection. Then, we have*

$$\operatorname{ac}(C) = L_{\operatorname{ac}(C)} \oplus p(\operatorname{ac}(C)) \quad \text{and} \quad \operatorname{rai}(C) = L_{\operatorname{ac}(C)} \oplus p(\operatorname{rai}(C)).$$

If $\operatorname{rai}(C) \neq \emptyset$*, then* $p(.)$ *commutes with* rai(.), ac(.) *and* rabd(.) *applied on* C *such that*

$$\begin{aligned}&\operatorname{ac}(C) = L_{\operatorname{ac}(C)} \oplus \operatorname{ac}(p(C)), \qquad \operatorname{rai}(C) = L_{\operatorname{ac}(C)} \oplus \operatorname{rai}(p(C)) \\ \text{and} \quad &\operatorname{rabd}(C) = L_{\operatorname{ac}(C)} \oplus \operatorname{rabd}(p(C)).\end{aligned}$$

Proof. Clearly, we have

$$\operatorname{ac}(C) \subseteq L_{\operatorname{ac}(C)} \oplus p(\operatorname{ac}(C)) \quad \text{and} \quad \operatorname{rai}(C) \subseteq L_{\operatorname{ac}(C)} \oplus p(\operatorname{rai}(C)).$$

Furthermore, $L_{\operatorname{ac}(C)} + \operatorname{ac}(C) = \operatorname{ac}(C)$ and we have seen in 2.2.8(c) that $L_{\operatorname{ac}(C)} + \operatorname{rai}(C) = \operatorname{rai}(C)$. This shows

$$\begin{aligned}L_{\operatorname{ac}(C)} \oplus p(\operatorname{ac}(C)) &\subseteq L_{\operatorname{ac}(C)} + (\operatorname{ac}(C) + L_{\operatorname{ac}(C)}) = \operatorname{ac}(C) \\ \text{and} \quad L_{\operatorname{ac}(C)} \oplus p(\operatorname{rai}(C)) &\subseteq L_{\operatorname{ac}(C)} + (\operatorname{rai}(C) + L_{\operatorname{ac}(C)}) = \operatorname{rai}(C)\end{aligned}$$

and hence the first two equalities hold.
Now, we show that $p(.)$ commutes with rai(.) and ac(.) with respect to C if $\operatorname{rai}(C) \neq \emptyset$. The inclusions $p(\operatorname{rai}(C)) \subseteq \operatorname{rai}(p(C))$ and $p(\operatorname{ac}(C)) \subseteq \operatorname{ac}(p(C))$ are obvious anyway (see also [BaFo, I.10.3., I.10.7.]).
Let us assume $\operatorname{rai}(C) \neq \emptyset$. We will show that $p(\operatorname{rai}(C)) \supseteq \operatorname{rai}(p(C))$ and use that $V = L_{\operatorname{ac}(C)} \oplus W$. Let $y \in \operatorname{rai}(p(C))$ and $(s_1, x) \in \operatorname{rai}(C) \neq \emptyset$ with $s_1 \in L_{\operatorname{ac}(C)}$ and $x \in W$. Since $y \in \operatorname{rai}(p(C))$ and $x \in p(C)$, there exists some $\epsilon > 0$ with $y - \epsilon x \in p(C)$. Hence, we will find some $s_2 \in L_{\operatorname{ac}(C)}$ with $(s_2, y - \epsilon x) \in C$. Now, 2.1.7(a) implies

$$\frac{1}{1+\epsilon}(s_2 + \epsilon s_1, y) = \frac{1}{1+\epsilon}\left((s_2, y - \epsilon x) + \epsilon(s_1, x)\right) \in \operatorname{rai}(C)$$

and therefore $(s_2 + \epsilon s_1, y) \in \mathrm{rai}(C)$, so $y \in p(\mathrm{rai}(C))$. We obtain $p(\mathrm{rai}(C)) = \mathrm{rai}(p(C))$.
For the proof of $p(\mathrm{ac}(C)) = \mathrm{ac}(p(C))$, we use in the following chain of equalities 2.1.7(c), commutativity of ac(.) and $\oplus$ (see [BaFo, I.8.3.b)]), $p(\mathrm{rai}(C)) = \mathrm{rai}(p(C))$ and the fact that $L_{\mathrm{ac}(C)}$, as a linear space, coincides with its algebraic closure. We have

$$\begin{aligned} p(\mathrm{ac}(C)) &= p(\mathrm{ac}(\mathrm{rai}(C))) = p(\mathrm{ac}(L_{\mathrm{ac}(C)} \oplus p(\mathrm{rai}(C)))) = p(\mathrm{ac}(L_{\mathrm{ac}(C)}) \oplus \mathrm{ac}(p(\mathrm{rai}(C)))) \\ &= p(L_{\mathrm{ac}(C)} \oplus \mathrm{ac}(\mathrm{rai}(p(C)))) = \mathrm{ac}(\mathrm{rai}(p(C))) = \mathrm{ac}(p(C)). \end{aligned}$$

Finally, we have

$$\begin{aligned} \mathrm{rabd}(C) &= \mathrm{ac}(C) \setminus \mathrm{rai}(C) = L_{\mathrm{ac}(C)} \oplus (p(\mathrm{ac}(C)) \setminus p(\mathrm{rai}(C))) \\ &= L_{\mathrm{ac}(C)} \oplus (\mathrm{ac}(p(C))) \setminus \mathrm{rai}(p(C))) = L_{\mathrm{ac}(C)} \oplus \mathrm{rabd}(p(C)) \end{aligned}$$

which also implies $p(\mathrm{rabd}(C)) = \mathrm{rabd}(p(C))$. □

2.3 Facial structure of convex sets and cones

We have seen properties of convex sets (cones) and their faces. Now, let us study the structure of all faces or at least some part of it.

2.3.1 The facial structure

It is usual to consider the facial structure of a cone as the set of faces together with inclusion as a partially ordered set. We will be interested in classifications of faces and especially such classifications that arise by symmetries from our cone as we have already seen in chapter 1. Let us explain these ideas in more detail. The following definition contains well-known definitions on partially ordered sets (see e.g. [DaPr, p.23]) and some more rather straightforward definitions.

Definition 2.3.1. We call a set M together with a partial order $\leq$ a *partially ordered set* or a *poset*. We call a poset $(M, \leq)$ *Noetherian* if every chain $a_1 \leq a_2 \leq \ldots$ with $a_1, a_2, \cdots \in M$ becomes stationary, which means that there is some $m \in \mathbb{N}$ with $a_m = a_{m+1} = \ldots$.
A map $\varphi : M \to N$ between posets $(M, \leq_M)$ and $(N, \leq_N)$ is called

$$\begin{aligned} \textit{order-preserving:} &\text{ if } \forall\, a, b \in M : a \leq_M b \Rightarrow \varphi(a) \leq_N \varphi(b), \\ \text{an } \textit{order-embedding:} &\text{ if } \forall\, a, b \in M : a \leq_M b \Leftrightarrow \varphi(a) \leq_N \varphi(b). \end{aligned}$$

Order-embeddings are always injective. If φ is a bijective order-embedding, then φ is called an *order-isomorphism* and $(M, \leq_M)$ and $(N, \leq_N)$ are called *isomorphic as posets*.
Let $(M, \leq_M)$ be a poset, G a group and $\varphi : G \times M \to M$ an *order-preserving group action* on M, this means that for all $g \in G$ the map $M \to M, a \mapsto ga$ is order-preserving (and automatically even an order-isomorphism). Let $\sim$ be the equivalence relation given by the orbits of φ.
If M (with $\leq_M$) is Noetherian, then there is a partial order (see below) on the set of equivalence classes ${}^M/_\sim$ given by $A \leq_\sim B$ for $A, B \in {}^M/_\sim$ if and only if $a \leq_M b$ for some $a \in A$ and $b \in B$. We call this poset *the quotient of M by G* and we also denote it by ${}^M/_G$.

Remark 2.3.2. The relation $\leq_\sim$ defined above is indeed a partial order on ${}^M/_\sim$. It is obviously reflexive. If $A \leq_\sim B$ and $B \leq_\sim C$ $(A, B, C \in {}^M/_\sim)$, then there are $a \in A$, $b, b' \in B$, $c \in C$ and

$g \in G$ such that $a \leq_M b$, $b' \leq_M c$, $b = gb'$ and thus $a \leq_M b = gb' \leq_M gc \in C$, which shows that $\leq_\sim$ is transitive. Let us show that $\leq_\sim$ is antisymmetric, so let $A \leq_\sim B$ and $B \leq_\sim A$ $(A, B \in {}^M\!/\!\sim)$. Then, there are $a, a' \in A$, $b, b' \in B$, $g, g' \in G$ such that $a \leq_M b$, $b' \leq_M a'$, $b = gb'$ and $a' = g'a$. Thus,

$$a \leq_M b = gb' \leq_M ga' = gg'a.$$

An iteration of this argument gives $a \leq_M b \leq_M (gg')^k a$ for $k \in \mathbb{N}$. If $(gg')^k a = a$ for some $k \in \mathbb{N}$, then we also get $a = b$ and thus $A = B$. Else, we have an infinite chain (note that one can cancel powers of gg' from the left)

$$a \leq_M (gg')^1 a \leq_M (gg')^2 a \leq_M \dots$$

which contradicts that M is Noetherian. (The property that M is Noetherian is important as it is shown by the example $M = \mathbb{Z}$ with the usual order and $G = 2\mathbb{Z}$ acting by taking the sum.)
Note that the quotient map $M \to {}^M\!/\!\sim$ is by definition order-preserving. The quotient construction presented here was also studied in [Hal] (see [Hal, 2.1.1.]) but different assumptions on the poset $(M, \leq)$ were used.

The following lemma shows that elements of the same equivalence class inherit the same structure on the elements smaller than them.

Lemma 2.3.3. *Let $(M, \leq_M)$ be a poset, G a group acting order-preserving on M with induced equivalence relation $\sim$ (see 2.3.1). Let $a, b \in M$ with $a \sim b$, so $ga = b$ for some $g \in G$. Then, $x \mapsto gx$ induces an order isomorphism between*

$$M_a := \{x \in M \mid x \leq_M a\} \quad \text{and} \quad M_b := \{x \in M \mid x \leq_M b\}$$

which both carry the poset structure induced by $\leq_M$.

Proof. We consider the order isomorphism

$$\varphi : M \to M, \quad x \mapsto gx.$$

It restricts to an order isomorphism $M_a \to gM_a$. Thus, we just have to show that $gM_a = M_b$. Since φ is order-preserving, we have $gM_a \subseteq M_b$. However, since $\varphi^{-1} : M \to M, x \mapsto g^{-1}x$ is also order-preserving, it follows that $g^{-1}M_b \subseteq M_a$ and thus $M_b = gg^{-1}M_b \subseteq gM_a$. This proves the claim. □

As announced before, we define the facial structure and similarly the Klee face structure. Furthermore, we introduce the layers of a face.

Definition 2.3.4. Let C be a convex set (a cone). We call the poset $(\mathcal{F}, \subseteq)$ the *facial structure of C* and $(\mathcal{K}, \subseteq)$ the *Klee face structure of C*.
Let $F \in \mathcal{F}$. Then, we say that F *belongs to the i-th layer*, if there is some chain of faces

$$C =: F_0 \supsetneq F_1 \supsetneq \dots \supsetneq F_i = F$$

such that F_{i+1} is an $\subseteq$-maximal proper face of F_i for $i = 0, \dots, i-1$.

Remark 2.3.5. Faces may belong to several layers as it is for instance shown by 2.3.16.

The following theorem sums up main results about the facial structure of cones invariant under some group action and the corresponding equivalence classes.

Theorem 2.3.6. *Let $C \subseteq V$ be a cone with Noetherian face structure and G be a subgroup of $\mathrm{Aut}(V)$ with $\varphi(C) \subseteq C$ for all $\varphi \in G$ (we have automatically $\varphi(C) = C$). Then, the mapping $F \mapsto \varphi(F)$ induces an order-preserving group action on $(\mathcal{F}, \subseteq)$ and thus a quotient ${}^{\mathcal{F}}/_{G}$. Furthermore, faces F in the same equivalence class of ${}^{\mathcal{F}}/_{G}$ share the properties of*

- *having the same facial structure up to isomorphism*
- *having the same dimensions and codimensions of F and $L_{\mathrm{ac}(F)}$*
- *having the same dimension of ${}^{\mathrm{span}(F)}/_{L_{\mathrm{ac}(F)}}$.*

Proof. Let $F \in \mathcal{F}$ and $\varphi \in G$. By 2.2.11, $\varphi(F)$ is again a convex cone. Given $x \in \varphi(F)$ and $a, b \in C$ with $x = \frac{a+b}{2}$, then $\varphi^{-1}(x) \in F$, $\varphi^{-1}(a), \varphi^{-1}(b) \in C$,

$$\varphi^{-1}(x) = \frac{\varphi^{-1}(a) + \varphi^{-1}(b)}{2}$$

and thus, $\varphi^{-1}(a), \varphi^{-1}(b) \in F$. This shows $a, b \in \varphi(F)$, so $\varphi(F) \in \mathcal{F}$ by 1.2.9. Now, it follows easily from the definition that we have an order-preserving group action on $(\mathcal{F}, \subseteq)$ as $\varphi \in G$ respects inclusion.
The faces in the same equivalence class ${}^{\mathcal{F}}/_{G}$ have the same facial structure by 2.3.3. Since φ is linear, we have $\mathrm{span}(\varphi(F)) = \varphi(\mathrm{span}(F))$ and we also have $L_{\mathrm{ac}(\varphi(F))} = \varphi(L_{\mathrm{ac}(F)})$ since

$$L_{\mathrm{ac}(F)} = \{x \in \mathrm{span}(F) \mid \exists\, y, z \in F : [y, x[\subseteq F, [z, -x[\subseteq F\}$$

and this description "is linear". Thus, the statements on dimensions and codimensions are reduced to well-known statements on dimensions and codimensions of linear spaces under automorphisms. □

Remark 2.3.7. We will see in 2.3.18 that cones C with Noetherian Klee face structure have the property $\mathcal{F} = \mathcal{K}$ and thus also have a Noetherian facial structure.

Let us give some examples for this theorem.

Example 2.3.8. Linear coordinate changes of $\mathbb{P}^n(\mathbb{R})$, represented by elements of $\mathrm{GL}(n+1, \mathbb{R})$, induce a subgroup G of $\mathrm{Aut}(\mathbb{R}[\underline{X}]_{2d})$ for each $d \in \mathbb{N}_0$. The action is induced by the map $\underline{X}^T \mapsto A \cdot \underline{X}^T$ for $A \in \mathrm{GL}(n+1, \mathbb{R})$. Clearly, we have $\varphi(\mathcal{P}_{2d}) \subseteq \mathcal{P}_{2d}$ where $\mathcal{P}_{2d}$ are the globally non-negative forms (i.e. $S = \mathbb{P}^n(\mathbb{R})$). Thus, the previous theorem may be applied to these cases. Indeed, we have already seen the corresponding quotients ${}^{\mathcal{F}}/_{G}$ in the cases $2d = 2$ (in 1.5.7) and $(n, 2d) = (2, 4)$ (in section 1.6) represented by Hasse diagrams as classifications of faces.

Example 2.3.9. We have seen classifications of faces of $\mathcal{P}_{2d}$ (with $S = \mathbb{P}^n(\mathbb{R})$) for $n = 1$ in 1.4.5. One may now ask if there is some subgroup G of $\mathrm{Aut}(\mathbb{R}[\underline{X}]_{2d})$ as in 2.3.6 such that one of these classifications corresponds to the quotient ${}^{\mathcal{F}}/_{G}$. If this was the case, then any $\varphi \in G$ has to send the set of $f \in \mathcal{P}_{2d}$ with a zero of multiplicity $2d$ to itself as these are the representatives of one class of extreme rays in the classification. However, one can show that this already restricts the set of possible φ to the subgroup G induced by linear coordinate

changes from the previous example. Furthermore, if $2d \geq 8$, then linear coordinate changes may not transfer some extreme ray with d double zeros into any other such extreme ray as the cross-ratio of 4 points is an invariant under such transformations (thus, $\mathcal{F}/G$ would even be infinite). This shows that no such subgroup G exists if $2d \geq 8$.

This motivates the following definition.

Definition 2.3.10. Let C be a cone, $\mathcal{G} \subseteq \mathcal{F}$, M a partition of $\mathcal{G}$ with corresponding map $\varphi : \mathcal{G} \to M$ and $\leq$ a partial order on M such that φ is order-preserving. Then, φ or M is called a *regular classification of the faces in* $\mathcal{G}$ if the following three properties are satisfied.

- If $A \in M$, $F_1, F_2 \in A$ and $F_2 \subseteq F_1$, then $F_1 = F_2$.
- If $A, B \in M$ with $B \leq A$, then for all $F_1 \in A$ there is some $F_2 \in B$ such that $F_2 \subseteq F_1$.
- If $A, B \in M$ with $B \leq A$, then for all $F_2 \in B$ there is some $F_1 \in A$ such that $F_2 \subseteq F_1$.

If $\mathcal{G} = \mathcal{F}$, then we also call it a *regular classification of the faces of* C. Furthermore, we say that it *preserves dimensions (codimensions)* if the dimension (codimension) does not depend on the representative $F \in A$ for any given $A \in M$.

Remark 2.3.11. There is always the trivial regular classification of faces in $\mathcal{G}$ given by the map that assigns $F \mapsto \{F\}$. It preserves dimensions and codimensions.
If $\varphi : \mathcal{G} \to M$ is a regular classification of faces of $\mathcal{G}$ and $\mathcal{G}' \subseteq \mathcal{G}$ such that for all $A \in M$ either $A \subseteq \mathcal{G}'$ or $A \cap \mathcal{G}' = \emptyset$, then φ restricts to a regular classification of faces of $\mathcal{G}'$. Especially, every regular classification of faces of a cone C preserving codimensions induces a regular classification of faces of finite codimension or codimension $\leq c$ for any $c \in \mathbb{N}_0$.

Example 2.3.12. The classifications of $\mathcal{F}(\mathcal{P}_{1,2d})$ ($S = \mathbb{P}^1(\mathbb{R})$) presented in 1.4.5 (see also 2.3.9) are regular classifications in the sense of the definition.

Let us show that this new concept contains the classification induced by group actions as in 2.3.6.

Lemma 2.3.13. *In the situation of 2.3.6,* $\mathcal{F}/G$ *gives a regular classification of the faces of* C*, which preserves dimensions and codimensions.*

Proof. To show this, we have to check the three properties from the definition - the other requirements are satisfied obviously. If $A \in \mathcal{F}/G$, $F_1, F_2 \in A$ and $F_2 \subseteq F_1$, then there is some $g \in G$ with $gF_2 = F_1$. Thus, we have

$$F_2 \subseteq F_1 = gF_2 \subseteq gF_1 = g^2F_2 \subseteq \dots$$

This sequence will become stationary as C has Noetherian facial structure and thus $F_1 = F_2$ as we may cancel out powers of g (from the left) by applying the inverse power.
Now, let $A, B \in \mathcal{F}/G$ with $B \leq A$ and $F_1 \in A$. As $B \leq A$, there are $F_1' \in A$ and $F_2' \in B$ such that $F_2' \subseteq F_1'$. Furthermore, $F_1, F_1' \in A$ implies the existence of some $g \in G$ with $F_1 = gF_1'$. Thus, $F_1 = gF_1' \supseteq gF_2' \in B$. This shows the second property and the third property is proved in the same way. □

Now, one may ask if this concept is maybe too general. The following example shows that even the easiest regular classifications preserving dimensions (as in 2.3.10) may inherit quite some structure of the cone.

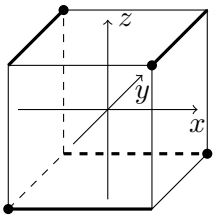
Example 2.3.14. The classification of faces of $\mathcal{P}_{2,4}$ ($S = \mathbb{P}^n(\mathbb{R})$) presented in section 1.6 is the coarsest regular classification of faces of $\mathcal{P}_{2,4}$ that preserves dimensions. We will sketch the proof. In the following arguments, we will just use that the classification from section 1.6 is regular and the knowledge about the dimensions of the faces in the corresponding classes (both represented in the diagram in section 1.6).
Let $\varphi : \mathcal{F}(\mathcal{P}_{2,4}) \to M$ be any classification preserving dimensions. We have to show that each equivalence class of faces does not contain faces of several classes of the diagram in section 1.6. This is obvious for classes with faces of dimension 0, 5, 12 or 15 as there is just one class with faces of this dimension in the diagram.
Let us suppose the existence of $A \in M$ with faces of dimension 9 containing elements of $F_{1,1,0}$ as well as $F_{2,0,0}$. Then, as each face $F_1 \in A \cap F_{1,1,0}$ contains some face $F_2 \in F_{4s}$ and as φ is order-preserving, we have $B := \varphi(F_2) \leq A$ and $B \subseteq F_{4s}$ by above. By the second property of 2.3.10, each face $F_1 \in A$ contains some face $F_2 \in B \subseteq F_{4s}$. However, this implies $F_1 \in F_{1,1,0}$ in contradiction to $A \cap F_{2,0,0} \neq \emptyset$. This shows that there is no such class A.
Now, let us suppose the existence of some class $B \in M$ with faces of dimension 6 containing elements of $F_{3,0,0}$ as well as some face contained in a face in $F_{1,1,0}$ (i.e. an element of $F_{2,1,0} \cup F_{|} \cup F_{1,0,6}$). Since φ is order-preserving, given some face $F_2 \in B$ with $F_2 \subseteq F_1$ for some $F_1 \in F_{1,1,0}$, we obtain $A := \varphi(F_1) \in M$ containing faces of dimension 9 with $B \leq A$ and $A \subseteq F_{1,1,0}$ by above. This contradicts the third property of 2.3.10 as there is no $F_1 \in A$ with $F_2 \subseteq F_1$ for $F_2 \in B \cap F_{3,0,0}$. Thus, there is no such class B. Analogously, one can argue that there is no class $B \in M$ with faces of dimension 6 containing elements of $F_{1,0,6}$ as well as some face contained in a face in $F_{2,0,0}$.
We give some more arguments without details. Using the third property of 2.3.10, one can show that there is no class $B \in M$ containing faces of $F_{|,4}$ and faces of a different class of the diagram as these are the only faces of dimension 3 contained in some face of F_{4s}. Analogous statements hold for $F_{1,0,8}$, as these are the only faces of dimension 3 not contained in some face of $F_{2,0,0}$, and $F_{2,0,6}$, as these are the faces of dimension 3 contained in faces of $F_{1,0,6}$ that do not belong to $F_{|,4}$ or $F_{1,0,8}$. Thus, there is no class $A \in M$ containing faces of $F_{2,1,0}$ and $F_{|}$ as just the first type of faces contains faces of $F_{2,0,6}$. Continuing in this way, one obtains the claim.

Finally, we want to point to a situation that is maybe very unexpected in such classifications but appears even in classifications induced by groups as in 2.3.6.

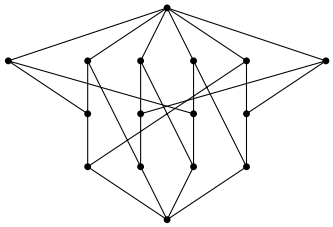
Warning 2.3.15. Let $\varphi : \mathcal{F} \to M$ be a regular classification of the faces of C and G an $\subseteq$-maximal proper face of F for some $F, G \in \mathcal{F}$. Then, we have $\varphi(G) \leq \varphi(F)$ but in general $\varphi(G)$ is not maximal under the classes $\lneq \varphi(F)$ even if the classification arises as in 2.3.6. We present a counterexample in the setting of convex sets below. However, if $A, B \in M$ and B is maximal with $B \lneq A$, $F \in A$, $G \in B$ with $G \subseteq F$, then there is no $H \in \mathcal{F}$ with $G \subsetneq H \subsetneq F$ as φ is order-preserving.

Example 2.3.16. We construct a convex subset of $\mathbb{R}^3$. We take the cube $[-1,1]^3$ and we remove all edges and vertices to obtain an object C_0. Then, we add one edge with only one endpoint to C_0, namely $[-1,1[\times\{-1\} \times \{-1\}$ to obtain C_1. We consider the isometry ψ of $\mathbb{R}^3$ given by the composition of a rotation by 90° around the z-axis and the reflection at the x-y-plane. ψ generates a group G of order 4. Let $C_2 := \bigcup_{i=0}^{3} \psi^i(C_1)$. It is a convex set, which is the union of C_0 and 4 edges each with only one endpoint and it is obviously invariant under the action of G (see picture below).

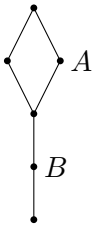
So what is the structure that arises if one considers the faces of C_2 modulo G? G acts transitive on vertices, on edges and clearly also on $\{C_2\}$ and on $\{\emptyset\}$, so there is one class of faces for each of the dimensions -1, 0, 1 and 3. However, there are two classes of dimension 2, namely the 2 faces perpendicular to the z-axis (represented by the most left and most right point in the second row in the facial structure below) and the 4 faces parallel to the z-axis (represented by the four points in between). One class is smaller (w.r.t. the induced partial order) than another class if and only if it contains faces of lower dimension. Let us call the last described class of 2-dimensional faces A and the class of vertices B. Then, each $F \in A$ has one edge that has one vertex $G \in B$ that is no $\subseteq$-maximal proper face of F and one vertex $G' \in B$ that is an $\subseteq$-maximal proper face of F. This gives the desired example.



C_2



facial structure of C_2



facial structure modulo G

The definition of regular classifications shall be a suggestion for an appropriate concept for such classifications. We will not study it further but the examples above seem promising that it is well-behaved.

2.3.2 Noetherian facial structures

We have already seen that we sometimes need the assumption that the studied structures fulfil the ascending chain condition, i.e. they are Noetherian. Let us show that the Noetherian property for the Klee face structure implies that every non-empty face has non-empty relative algebraic interior. We start with a lemma, which implies directly the desired result of the subsequent theorem.

Lemma 2.3.17. *Let C be a convex set (a cone). If $F \in \mathcal{F} \setminus \mathcal{K}$, then there is no $\subseteq$-maximal face $G \in \mathcal{K}$ with $G \subseteq F$.*

Proof. We give an indirect proof. Let $F \in \mathcal{F} \setminus \mathcal{K}$, $G \in \mathcal{K}$ and $G \subseteq F$ an $\subseteq$-maximal face with these properties. Then, there is some $x \in \operatorname{rai}(G)$ and some $y \in F \setminus G$. Now, we have $\frac{x+y}{2} \in \operatorname{rai}([x,y])$ and by 2.2.10, we have $\frac{x+y}{2} \in \operatorname{rai}(\operatorname{carr}([x,y]))$, so $\operatorname{carr}(\frac{x+y}{2}) = \operatorname{carr}([x,y])$. Thus, we have $\operatorname{carr}([x,y]) \in \mathcal{K}$ and $G = \operatorname{carr}(x) \subsetneq \operatorname{carr}([x,y]) \subseteq F$, which leads to a contradiction. □

Theorem 2.3.18. *Let C be a convex set (cone) and suppose that C has a Noetherian Klee face structure. Then, $\mathcal{F} \setminus \{\emptyset\} = \mathcal{K}$ ($\mathcal{F} = \mathcal{K}$), this means that every face $F \neq \emptyset$ of C has the property* $\operatorname{rai}(F) \neq \emptyset$.

Proof. Suppose $F \in \mathcal{F} \setminus \{\emptyset\}$ and $F \notin \mathcal{K}$. Then, the set

$$\{G \in \mathcal{K} \mid G \subseteq F\} \subseteq \mathcal{K}$$

is non-empty (it contains $\operatorname{carr}(x)$ for any $x \in F$) and has no maximal element by 2.3.17. It is well-known for posets that being Noetherian is equivalent to the property that the non-empty subsets have maximal elements. This gives a contradiction. □

We illustrate this theorem with an example.

Example 2.3.19. Assume that C is a cone and ${}^{\operatorname{span}(C)}\!/_{L_C}$ is finite-dimensional. For any $F \in \mathcal{F}(C)$ we have $F \supseteq L_C$ since L_C is the smallest face of C and thus F has finite codimension in $\operatorname{span}(C)$. Hence, each ascending chain of faces becomes stationary. The theorem shows that every face has a point in the relative algebraic interior.
However, this may also be shown starting from the observation that the projection of C in ${}^{\operatorname{span}(C)}\!/_{L_C}$ is a finite-dimensional cone, which thus has faces with non-empty relative interior (see 1.2.18).

Let us conclude by the following trivial but important remark.

Remark 2.3.20. Let C be convex with Noetherian Klee face structure or equivalent Noetherian facial structure. Then, every non-empty set of faces has a maximal element (w.r.t. $\subseteq$).

2.3.3 Lattice structure and sums of faces

Let us complete section 2.3 with some remarks on the lattice structure, some new notation and a remark on its properties. It is well-known that the set of faces of a non-empty convex set (cone) is a complete lattice induced by the semiorder $\subseteq$ (see [BaFo2, II.4.2.]). The corresponding operation $\wedge$ (infimum) is given by $\cap$. However, there seems to be no similar notation for the operation $\vee$ (supremum). We will see that, in the case of cones, this operation meets with the sum of faces. We introduce $\boxplus$ as its symbol since $+$ (Minkowski sum) and $\oplus$ (direct sum) are already occupied.

Definition 2.3.21. Let F_1, F_2 be faces of a convex set (a cone) C. We set

$$F_1 \boxplus F_2 := \operatorname{carr}(F_1 \cup F_2),$$

and call it the *sum of* F_1 *and* F_2 and the operation $\boxplus$ *addition of faces.*
Let $(F_i)_{i\in I}$ be a family of faces of C. Then, we set

$$\bigboxplus_{i\in I} F_i := \operatorname{carr}\left(\bigcup_{i\in I} F_i\right),$$

and call it the *sum of* $(F_i)_{i\in I}$. We denote the identity with respect to $\boxplus$ by 0.

Remark 2.3.22. Let $F_1, \ldots, F_k$ be Klee faces of a cone C and $x_j \in \operatorname{rai}(F_j)$ for $j = 1, \ldots, k$. Then, as $\frac{x_1+\cdots+x_k}{k} \in \operatorname{rai}(\operatorname{conv}(\{x_1, \ldots, x_k\}))$, 2.2.10 implies

$$\begin{aligned}\bigboxplus_{i=1}^{k} F_i &= \operatorname{carr}\left(\bigcup_{i=1}^{k} F_i\right) = \operatorname{carr}\left(\bigcup_{i=1}^{k} \operatorname{carr}(x_i)\right) = \operatorname{carr}(\{x_1, \ldots, x_k\}) \\ &= \operatorname{carr}(\operatorname{conv}(\{x_1, \ldots, x_k\})) = \operatorname{carr}\left(\frac{x_1 + \cdots + x_k}{k}\right) = \operatorname{carr}(x_1 + \cdots + x_k).\end{aligned}$$

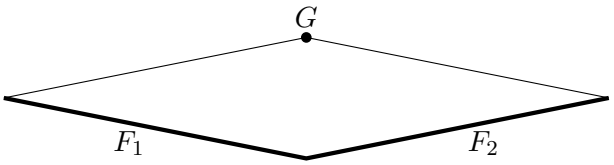
This motivates the notation.

The statements of the following remark are well-known.

Remark 2.3.23. As the addition is the supremum of the lattice, it is commutative and associative. Furthermore, we have $F \boxplus C = C$ for $F \trianglelefteq C$. The identity element is $\emptyset$ if C is a convex set and L_C if C is a cone.
In general, $\cap$ is not distributive over $\boxplus$. However, if $(F_i)_{i\in I}$ is a family of faces of C and $G \trianglelefteq C$, then it is easy to see that

$$\left(\bigboxplus_{i\in I} F_i\right) \cap G \supseteq \bigboxplus_{i\in I} (F_i \cap G)\,.$$

On the other hand, the following example with $I = \{1, 2\}$ shows that in general no equality holds.

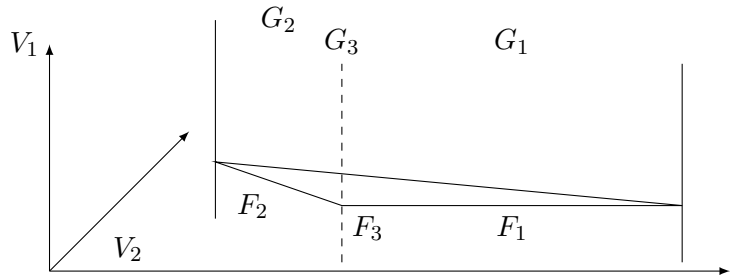


Warning 2.3.24. The lattice of all faces $\mathcal{F}$ does in general not restrict to a lattice on the set of Klee faces $\mathcal{K}$. Though the supremum of two Klee faces is always a Klee face, the intersection is in general not a Klee face and by 2.3.17 there is no infimum at all. Clearly, the easiest example are two faces with empty intersection, but we give an example that one can carry over to the setting of cones.
Let V_1 and V_2 be real vector spaces, $C_1 \subseteq V_1$ be non-empty convex with $\operatorname{rai}(C_1) = \emptyset$ (see 2.1.6(a)) and $C_2 \subseteq V_2$ be convex with Klee faces F_1, F_2 and $F_3 := F_1 \cap F_2 \notin \{\emptyset, F_1, F_2\}$ (e.g. F_1, F_2 are sides of a triangle $C_2 \subseteq V_2 := \mathbb{R}^2$). Then, we may consider the following convex sets in $V_1 \oplus V_2$:

$$\begin{aligned} C :=& (V_1 \oplus (C_2 \setminus F_3)) \cup (C_1 \oplus F_3) \subseteq V_1 \oplus C_2 \\ G_i :=& C \cap (V_1 \oplus F_i), \quad i = 1, 2, 3. \end{aligned}$$

G_1 and G_2 coincide with the sets $V_1 \oplus F_1$ and $V_1 \oplus F_2$ except on $V_1 \oplus F_3$ where we have

$$G_1 \cap (V_1 \oplus F_3) = G_2 \cap (V_1 \oplus F_3) = G_3 = C_1 \oplus F_3.$$



Now, G_1, G_2 are Klee faces of C, but $G_3 = G_1 \cap G_2$ is no Klee face.

2.4 Multiplicative cones

The cones $\mathcal{P}$ and $\mathcal{P}^{Loc}$ presented in section 1.7 have the property that they are closed under multiplication. So let us consider such cones in rings in more generality and see what this tells us about $\mathcal{P}$ and $\mathcal{P}^{Loc}$. In the following, let A be a commutative, integral algebra over $\mathbb{R}$ with 1 (and $0 \neq 1$) and $C \subseteq A$ a cone.

Definition 2.4.1. We call C *multiplicative* if $C \cdot C \subseteq C$. Furthermore, C is called *divisible* if C is multiplicative and $ab \in C$ implies $b \in C$ for all $a \in C \setminus \{0\}$ and $b \in A$.
We call C *Noetherian* if A is Noetherian. In such algebras A, we also write $\operatorname{supp}(C)$ for L_C and $\operatorname{csupp}(C)$ for $\operatorname{supp}(\operatorname{ac}(C))$ if $C \subseteq A$ is any cone.

Remark 2.4.2. A multiplicative cone with $1 \in C$ is the same as a semiring over $\mathbb{R}_+$. Indeed, we will have $1 \in C$ in all applications. Furthermore, the cones studied will even be preorderings (so especially containing all squares).

2.4.1 Multiplication and division of faces

In multiplicative cones one has an induced multiplication on the set of faces $\mathcal{F}$. Let us study this structure.

Definition 2.4.3. Let C be multiplicative and let $F_1, F_2 \in \mathcal{F}$. We define the *product of the faces* F_1 *and* F_2 by

$$F_1 \boxdot F_2 := \operatorname{carr}(F_1 \cdot F_2).$$

We call the operation $\boxdot$ *multiplication of faces*. As usual, multiplication $\boxdot$ precedes addition $\boxplus$ in calculations (if there are no parentheses).

The following proposition shows that this definition is very well-behaved.

Proposition 2.4.4. *Let C be multiplicative. The multiplication on $\mathcal{F}$ is commutative, associative and distributive over $\boxplus$. Furthermore, given $F_1, F_2 \in \mathcal{K}$ and $a_1 \in \operatorname{rai}(F_1)$, $a_2 \in \operatorname{rai}(F_2)$, we have*

$$F_1 \boxdot F_2 = \operatorname{carr}(a_1 \cdot a_2) \in \mathcal{K}.$$

Proof. Commutativity follows directly from $F_1 \cdot F_2 = F_2 \cdot F_1$ for all $F_1, F_2 \in \mathcal{F}$.
We consider three faces F_1, F_2, F_3 of C. Now, obviously

$$(F_1 \cdot F_2) \cdot F_3 = F_1 \cdot (F_2 \cdot F_3) = F_1 \cdot F_2 \cdot F_3 := \{f_1 \cdot f_2 \cdot f_3 \mid f_1 \in F_1, f_2 \in F_2, f_3 \in F_3\}.$$

We show that

$$(F_1 \boxdot F_2) \boxdot F_3 := \operatorname{carr}(\operatorname{carr}(F_1 \cdot F_2) \cdot F_3) = \operatorname{carr}(F_1 \cdot F_2 \cdot F_3)$$

which implies associativity of $\boxdot$. The inclusion "$\supseteq$" is obvious. Now, let us show that $\operatorname{carr}(F_1 \cdot F_2) \cdot F_3 \subseteq \operatorname{carr}(F_1 \cdot F_2 \cdot F_3)$, which implies "$\subseteq$". Let $f_3 \in F_3$ and $g \in \operatorname{carr}(F_1 \cdot F_2)$. By 2.2.8(a), we have some $\tilde{g} \in \operatorname{cone}(F_1 \cdot F_2)$ with $\tilde{g} - g \in C$. Hence,

$$\underbrace{gf_3}_{\in C} + \underbrace{(\tilde{g} - g)f_3}_{\in C} = \tilde{g}f_3 \in \operatorname{cone}(F_1 \cdot F_2 \cdot F_3) \subseteq \operatorname{carr}(F_1 \cdot F_2 \cdot F_3)$$

By the definition of faces and as we are working in a cone, we have $gf_3 \in \operatorname{carr}(F_1 \cdot F_2 \cdot F_3)$, which shows the claimed equality.
Let us show that $\boxdot$ is distributive over $\boxplus$, this means

$$F_1 \boxdot (F_2 \boxplus F_3) = (F_1 \boxdot F_2) \boxplus (F_1 \boxdot F_3).$$
$$[:\Leftrightarrow \quad \operatorname{carr}(F_1 \cdot \operatorname{carr}(F_2 \cup F_3)) = \operatorname{carr}(\operatorname{carr}(F_1 \cdot F_2) \cup \operatorname{carr}(F_1 \cdot F_3))]$$

where $F_1, F_2, F_3 \trianglelefteq C$. The inclusion "$\supseteq$" is obvious (one may leave out the carr(.) within the outer carr(.) on the right hand side without changing that side). Now, we show that $F_1 \cdot (F_2 \boxplus F_3)$ is contained in the right hand side, which implies the other inclusion "$\subseteq$". Let $f_1 \in F_1$ and $g \in F_2 \boxplus F_3$. Again, by 2.2.8(a), we find some $\tilde{g} \in \operatorname{cone}(F_2 \cup F_3)$ with $\tilde{g} - g \in C$. Since $\operatorname{cone}(F_2 \cup F_3) = F_2 + F_3$, there are $f_2 \in F_2$ and $f_3 \in F_3$ such that $\tilde{g} = f_2 + f_3$. Now, we have

$$\underbrace{f_1(\tilde{g} - g)}_{\in C} + \underbrace{f_1 g}_{\in C} = f_1 f_2 + f_1 f_3 \in (F_1 \boxdot F_2) \boxplus (F_1 \boxdot F_3).$$

By the definition of face and as we are working in a cone, we have $f_1 g \in (F_1 \boxdot F_2) \boxplus (F_1 \boxdot F_3)$, which shows equality and thus the claimed distributivity.
Finally, let $F_1, F_2 \in \mathcal{K}$ and $a_1 \in \operatorname{rai}(F_1)$, $a_2 \in \operatorname{rai}(F_2)$. Thus, we have

$$F_1 \boxdot F_2 = \operatorname{carr}(F_1 \cdot F_2) \supseteq \operatorname{carr}(a_1 \cdot a_2).$$

We show that $F_1 \cdot F_2 \subseteq \operatorname{carr}(a_1 \cdot a_2)$, which implies the other inclusion. Let $b_1 \in F_1$ and $b_2 \in F_2$. Then, by 2.2.8(b), we find some $\epsilon > 0$ such that $a_1 - \epsilon b_1 \in F_1 \subseteq C$ and $a_2 - \epsilon b_2 \in F_2 \subseteq C$. Now, we have

$$\underbrace{\frac{1}{2}\left((a_1 + \epsilon b_1)(a_2 - \epsilon b_2) + (a_1 - \epsilon b_1)(a_2 + \epsilon b_2)\right)}_{\in C} + \underbrace{\epsilon^2 b_1 b_2}_{\in C} = a_1 a_2 \in \operatorname{carr}(a_1 a_2).$$

As before, we obtain $b_1 b_2 \in \operatorname{carr}(a_1 \cdot a_2)$ and thus the claim. □

Let us also take a look at division of faces in the following definition.

Definition 2.4.5. Let C be a multiplicative cone. We say that C or $\boxdot$ is *cancellative* if

$$F_1 \boxdot F_2 = F_1 \boxdot F_3 \Rightarrow F_2 = F_3 \qquad \forall F_1, F_2, F_3 \in \mathcal{F},\ F_1 \neq 0\,(= \operatorname{supp}(C)).$$

When this property is just required for faces in $\mathcal{K}$ instead of $\mathcal{F}$, then we say that C or $\boxdot$ is *Klee cancellative* or *cancellative on Klee faces*.
Given $F_1, F_2 \in \mathcal{F}$, we write $F_2 \,|\, F_1$ if $F \boxdot F_2 = F_1$ for some $F \in \mathcal{F}$ and we say that F_2 *divides* F_1. If such F is unique, we set $\frac{F_1}{F_2} := F$ and call it the *quotient of* F_1 *by* F_2. We call this operation *division of faces*.

The following proposition shows that division of faces is at least well-behaved on Klee faces if the cone itself is well-behaved with respect to division. However, this is not the case for all faces as it is demonstrated by the subsequent remark.

Proposition 2.4.6. *If C is divisible, then $\boxdot$ is cancellative on Klee faces.*

Proof. Let $F_1, F_2, F_3 \in \mathcal{K}$ and $F_1 \neq 0$ such that $F_1 \boxdot F_2 = F_1 \boxdot F_3$ and $a_1 \in \operatorname{rai}(F_1) \setminus \{0\}$, $a_2 \in \operatorname{rai}(F_2)$ as well as $a_3 \in \operatorname{rai}(F_3)$. Then, by 2.4.4, we have

$$a_1 \cdot a_2 \in \operatorname{rai}(F_1 \boxdot F_2), \quad a_1 \cdot a_3 \in \operatorname{rai}(F_1 \boxdot F_3) = \operatorname{rai}(F_1 \boxdot F_2),$$

thus $a_1 \cdot a_2 - \epsilon_1 a_1 \cdot a_3 \in C$ and $a_1 \cdot a_3 - \epsilon_2 a_1 \cdot a_2 \in C$ with some $\epsilon_1, \epsilon_2 > 0$. Since $a_1 \in C \setminus \{0\}$ and C is divisible, we also obtain $a_2 - \epsilon_1 a_3 \in C$ and $a_3 - \epsilon_2 a_2 \in C$, hence $F_2 = \operatorname{carr}(a_2) = \operatorname{carr}(a_3) = F_3$. □

Remark 2.4.7. Not every divisible C has the property that $\boxdot$ is cancellative (on all faces) as it is shown by cones of globally non-negative polynomials $\mathcal{P}$ (i.e. $S = \mathbb{R}^n$, n chosen arbitrary). Indeed, $\mathcal{P}$ is divisible, but for any $d \in \mathbb{N}_0$ we have the face $\mathcal{P} \cap \mathbb{R}[\underline{x}]_{\leq 2d}$ of $\mathcal{P}$ and

$$(\mathcal{P} \cap \mathbb{R}[\underline{x}]_{\leq 2d}) \boxdot \mathcal{P} \supseteq 1 \cdot \mathcal{P} = \mathcal{P}$$

which implies $(\mathcal{P} \cap \mathbb{R}[\underline{x}]_{\leq 2d}) \boxdot \mathcal{P} = \mathcal{P}$ for any $d \in \mathbb{N}_0$. This argument works for each multiplicative cone C with $1 \in C \setminus \operatorname{rai}(C)$ as $C \boxdot C \supseteq \operatorname{carr}(1) \boxdot C \supseteq 1 \cdot C = C$ and thus $C \boxdot C = \operatorname{carr}(1) \boxdot C$.

Let us end with a summary of calculation rules that automatically hold in such situations.

Remark 2.4.8. Let C be multiplicative and let $F_1, F_2, F_3, F_4 \in \mathcal{F}$. If $F_3 \,|\, F_2$ and $F_2 \,|\, F_1$, then $F_3 \,|\, F_1$ as may be easily seen from the definition.
Let us assume that C is cancellative. Then, one may deduce the following calculation rules formally (i.e. just using the semiring structure and the cancellation property). If all expressions are defined (i.e. divisors are not 0 and divide the corresponding dividends), we have

$$\frac{F_1}{F_2} \boxplus \frac{F_3}{F_4} = \frac{F_1 \boxdot F_4 \boxplus F_2 \boxdot F_3}{F_2 \boxdot F_4}, \quad \frac{F_1}{F_2} \boxdot \frac{F_3}{F_4} = \frac{F_1 \boxdot F_3}{F_2 \boxdot F_4}, \quad \frac{\frac{F_1}{F_2}}{F_3} = \frac{F_1}{F_2 \boxdot F_3}, \quad \frac{F_1}{\frac{F_2}{F_3}} = \frac{F_1 \boxdot F_3}{F_2}.$$

To prove these expressions, one just has to multiply with all divisors until one obtains an obvious equality. Then, the claim follows from the cancellation property.
Furthermore, if $F_1 \boxdot F_3 \subseteq F_2 \boxdot F_3$ and $F_3 \neq 0$, then $(F_1 \boxplus F_2) \boxdot F_3 = F_2 \boxdot F_3$ and since C is cancellative, we have $F_1 \boxplus F_2 = F_2$ and thus $F_1 \subseteq F_2$.

2.4.2 Archimedean and Noetherian cones, ideals and factorization

One can even show more interesting properties of multiplicative cones if one gives them some structure as being Archimedean or Noetherian. This will be done in the following.

Definition 2.4.9. We call a cone C *Archimedean* if for all $x \in A$ there exists some $n \in \mathbb{N}$ such that $n - x \in C$.

Remark 2.4.10. A cone is Archimedean if and only if $1 \in \operatorname{ai}(C)$. A different formulation is that C has codimension 0 and $1 \in \operatorname{rai}(C)$.

Now, we will discuss some property for faces that holds if the considered multiplicative cone C is Archimedean. However, if we assume in one of the following results that a cone is Archimedean, then we just use this weaker property and that C has codimension 0.

Remark 2.4.11. Let C be a multiplicative cone in A. We consider faces $F \trianglelefteq C$ and the property $C \cdot F \subseteq F$. This property holds, if

- $1 \in \operatorname{rai}(C)$ (e.g. C Archimedean), since then $f \in F$ implies

$$C \cdot f \subseteq \operatorname{carr}(C \cdot \operatorname{carr}(f)) = C \boxdot \operatorname{carr}(f) = \operatorname{carr}(1 \cdot f) = \operatorname{carr}(f) \subseteq F,$$

- $F = C \boxdot G$ for some $G \trianglelefteq C$, since then

$$C \cdot F \subseteq C \boxdot F = C \boxdot (C \boxdot G) = (C \boxdot C) \boxdot G \subseteq C \boxdot G = F$$

- $F = C \cap I$ for some ideal I of A, since then

$$C \cdot F = C \cdot (C \cap I) \subseteq C \cap (C \cdot I) \subseteq C \cap I = F.$$

Note, that if $1 \in C$, then $C \cdot F \subseteq F$ is equivalent to $C \cdot F = F$ and $C \boxdot F = F$. Especially, C is a multiplicative identity and every face F with $C \cdot F \subseteq F$ is of the form $F = C \boxdot G$ for some $G \trianglelefteq C$ (namely $G = F$).
We will see in the next proposition that if C has codimension 0, then each F with $F \cdot C \subseteq C$ may be written as $F = C \cap I$ for some ideal I of A (indeed with $I = \operatorname{span}(F)$).

The following proposition, which shows that some sets constructed from a face are ideals, still uses the property from the remark.

Proposition 2.4.12. *Let C be multiplicative and of codimension 0 and F a face of C with $C \cdot F \subseteq F$ (e.g. C Archimedean). Then,* $\operatorname{span}(F)$, $\operatorname{supp}(F)$ *and* $\operatorname{csupp}(F)$ *are ideals of A.*

Proof. Let $I \in \{\operatorname{span}(F), \operatorname{supp}(F), \operatorname{csupp}(F)\}$. Since I is a linear subspace of A, we just have to show that $ax \in I$ for each $x \in I$ and $a \in A$. However, C has codimension 0, so each $a \in A$ is a linear combination of elements in C and thus a sum of elements in $C \cup (-C)$. Again, because I is linear, this reduces our claim to show that $ax \in I$ for each $x \in I$ and $a \in C \cup (-C)$.
Suppose $I = \operatorname{span}(F)$ and $x \in \operatorname{span}(F)$. Then, there are $k \in \mathbb{N}_0$, $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ and $f_1, \dots, f_k \in F$ such that $x = \sum_{i=1}^{l} \lambda_i f_i$. Now, if $a \in C \cup (-C)$, then either $af_i \in C \cdot F \subseteq F$ or $af_i \in (-C) \cdot F \subseteq -F$ for $i = 1, \dots, k$ and hence

$$ax = \sum_{i=1}^{l} \lambda_i (af_i) \in \operatorname{span}(F).$$

Now, suppose G is a cone in A with $C \cdot G \subseteq G$ and $x \in \operatorname{supp}(G) = G \cap (-G)$. So, we have

$$\begin{aligned} \text{if } a \in C: \quad & ax \in C \cdot G \subseteq G, && ax = -a(-x) \in -(C \cdot G) \subseteq -G \\ \text{if } -a \in C: \quad & ax = (-a)(-x) \in C \cdot G \subseteq G, && ax = -(-a)x \in -(C \cdot G) \subseteq -G. \end{aligned}$$

This shows $ax \in \operatorname{supp}(G)$ if $a \in C \cup (-C)$ and $x \in \operatorname{supp}(G)$. Particularly, the case $G = F$ shows that $\operatorname{supp}(F)$ is an ideal of A. However, if $a \in C$ and $x \in \operatorname{ac}(F)$, then there is some $y \in F$ with $[y, x[\subseteq F$, hence $[ay, ax[= a \cdot [y, x[\subseteq C \cdot F \subseteq F$, so $ax \in \operatorname{ac}(F)$. Thus, we have $C \cdot \operatorname{ac}(F) \subseteq \operatorname{ac}(F)$ and we may choose $G = \operatorname{ac}(F)$, which shows that $\operatorname{csupp}(F)$ is an ideal of A. □

Corollary 2.4.13. *Let C be multiplicative and of codimension 0. Then,* $\operatorname{supp}(C)$ *and* $\operatorname{csupp}(C)$ *are ideals of A.*

Proof. We have $C \cdot C \subseteq C$ since C is multiplicative. 2.4.12 shows the claim. □

The following corollary generalizes 2.4.6.

Corollary 2.4.14. *Let C be multiplicative and of codimension 0 and let $C/\operatorname{supp}(C) \subseteq A/\operatorname{supp}(C)$ be divisible (as a cone). Then, C is cancellative on Klee faces.*

Proof. By 2.4.6, we know that ${}^{C}/_{\mathrm{supp}(C)}$ is cancellative on Klee faces. We will show that the Klee faces of C have an isomorphic multiplicative structure. We consider some isomorphism

$$A \to {}^{A}/_{\mathrm{supp}(C)} \oplus \mathrm{supp}(C)$$

of vector spaces given by the quotient map $\varphi : A \to {}^{A}/_{\mathrm{supp}(C)}$ and some projection $A \to \mathrm{supp}(C)$. Under this isomorphism, C is identified with ${}^{C}/_{\mathrm{supp}(C)} \oplus \mathrm{supp}(C)$. This induces a bijection ψ from faces of C to faces of ${}^{C}/_{\mathrm{supp}(C)}$ (by forgetting the second component). If $F \trianglelefteq C$, then for $x \in A$ we have $x \in \mathrm{rai}(F)$ if and only if $\varphi(x) \in \mathrm{rai}(\psi(F))$ since $F = \psi(F) \oplus \mathrm{supp}(C)$. This shows that ψ preserves the property of being a Klee face (or not being a Klee face) and since φ is a ring homomorphism, 2.4.4 implies that ψ is compatible with multiplication on Klee faces. This shows that the multiplicative structures on Klee faces of C and ${}^{C}/_{\mathrm{supp}(C)}$ are isomorphic and thus C is cancellative on Klee faces. □

The following theorem is crucial for the study of the faces of such cones.

Theorem 2.4.15. *Let C be multiplicative, Archimedean and Noetherian. Then, C has Noetherian Klee face structure, so every face of C is a Klee face by 2.3.18.*

Proof. Let $F_1 \subsetneq F_2 \subsetneq \dots$ be a strictly ascending chain of Klee faces $F_1, F_2, \dots \in \mathcal{K}$ (we do not need the property that they are Klee faces in the following). By 2.4.12 and 1.2.8, this induces a strictly ascending chain of ideals

$$\mathrm{span}(F_1) \subsetneq \mathrm{span}(F_2) \subsetneq \dots$$

This is not possible since A is Noetherian. Hence, there is no strictly ascending chain of Klee faces and C has a Noetherian Klee face structure. □

Let us consider the product structure of $\mathcal{F}$ for Archimedean cones. It has some special properties as one can see in the following proposition.

Proposition 2.4.16. *Let C be a multiplicative Archimedean cone. If $F_1, F_2 \in \mathcal{F}$, then*

$$F_1 \boxdot F_2 \subseteq F_1 \cap F_2.$$

Especially, $G \mid F$ with $F, G \in \mathcal{F}$ implies $F \subseteq G$. If C is also cancellative (on all faces) and $F, G \in \mathcal{F}$ with $F \neq 0$, then $F = F \boxdot G$ implies $G = C$.

Proof. We use 2.4.11. Let $F_1, F_2 \in \mathcal{F}$. We have $F_1 \cdot F_2 \subseteq F_1 \cdot C \subseteq F_1$, so $F_1 \boxdot F_2 \subseteq F_1$. An analogous argument gives $F_1 \boxdot F_2 \subseteq F_2$ and hence $F_1 \boxdot F_2 \subseteq F_1 \cap F_2$.
If $G, F \in \mathcal{F}$ with $G \mid F$, then there is some $H \in \mathcal{F}$ with $F = G \boxdot H$. We obtain $F = G \boxdot H \subseteq G \cap H \subseteq G$.
Let C be cancellative and $F, G \in \mathcal{F}$ with $F = F \boxdot G$. Then,

$$F = G \boxdot F \subseteq C \boxdot F \subseteq C \cap F = F,$$

so $G \boxdot F = C \boxdot F$. The cancellative property implies $G = C$. □

Now, let us consider factorizations of faces. We define irreducibility with respect to multiplication and have a corresponding theorem of factorization into irreducible faces. The proof of the theorem is analogous to the well-known arguments on factorizations (e.g. of ideals into prime ideals in Dedekind domains).

Definition 2.4.17. Let C be a multiplicative cone and $F \in \mathcal{F}\backslash\{0\}$. We call F *(multiplicative) irreducible*, if $F = F_1 \boxdot F_2$ with $F_1, F_2 \in \mathcal{F}$ implies $F \in \{F_1, F_2\}$. Otherwise, we call F *reducible*.

Theorem 2.4.18. *Let C be a multiplicative, Archimedean, Noetherian cone. Then, for each $F \in \mathcal{F}\backslash\{0\}$, we have some factorization $F = F_1 \boxdot \cdots \boxdot F_k$ with irreducible faces $F_1, \ldots, F_k \in \mathcal{F}$.*

Proof. By 2.4.15, C has Noetherian Klee face structure and every face of C is a Klee face. Suppose, the claim was wrong. Then, there is an $\subseteq$-maximal face $F \neq 0$ of C that does not have a factorization as claimed. Especially, F is reducible and there are $G, H \in \mathcal{F}$ with $F = G \boxdot H$, $F \notin \{G, H\}$. Furthermore, either G or H has no factorization as claimed, since else we would have such a factorization for F. Without loss of generality, let us assume that G has no such representation. Now, by 2.4.16, we have $F = G \boxdot H \subseteq G \cap H \subseteq G$ and since $F \notin \{G, H\}$, we have even $F \subsetneq G$. This contradicts the maximality of F. □

The following proposition is very interesting as it shows (together with 2.2.13) that the structure of faces is essentially determined by a convex set of a finite-dimensional vector space. It may be applied in "local situations" as a maximal ideal is involved.

Proposition 2.4.19. *Let C be a multiplicative, Noetherian and Archimedean cone and F some face of C. Then,*

$$\operatorname{csupp}(C)\operatorname{span}(F) \subseteq \operatorname{csupp}(F)$$

where the left hand side is the ideal product. Especially, if A is a ring with a maximal ideal $\mathfrak{m}$ with residue field $\mathbb{R}$ and $1 + \mathfrak{m} \subseteq C$ (with C as above), we have $\operatorname{csupp}(C) \supseteq \mathfrak{m}$ and ${}^{\operatorname{span}(F)}/_{\operatorname{csupp}(F)}$ has finite dimension.

Proof. By 2.4.15, F is a Klee face and we find some $f \in \operatorname{rai}(F)$. Each element of the ideal product $\operatorname{csupp}(C)\operatorname{span}(F)$ may be written in the form $\sum_{i=1}^{k} h_i g_i$ with $k \in \mathbb{N}_0$, $h_i \in \operatorname{csupp}(C)$ and $g_i \in F$ as $\operatorname{span}(F) = F + (-F)$ (see 1.2.11). Let $\epsilon > 0$ with $f - \epsilon \sum_{i=1}^{k} g_i \in F$. We have $1 \in \operatorname{rai}(C)$, so

$$f - \alpha \sum_{i=1}^{k} h_i g_i = f - \epsilon \sum_{i=1}^{k} g_i + \epsilon \sum_{i=1}^{k} \left(1 - \frac{\alpha}{\epsilon} h_i\right) g_i \in F$$

for any $\alpha \in \mathbb{R}$ by 2.2.8(c) and since $C \cdot F \subseteq F$ by 2.4.11. 2.2.8(c) also shows that $\sum_{i=1}^{k} h_i g_i \in \operatorname{csupp}(F)$, so the first claim $\operatorname{csupp}(C)\operatorname{span}(F) \subseteq \operatorname{csupp}(F)$.
Suppose now that A is a ring with maximal ideal $\mathfrak{m}$ with residue field $\mathbb{R}$ and $1 + \mathfrak{m} \subseteq C$. Now, if $g \in \mathfrak{m}$, then $-\alpha g \in \mathfrak{m}$ and $1 - \alpha g \in C$ for any $\alpha \in \mathbb{R}$. This shows $g \in \operatorname{csupp}(C)$ (use 2.2.8(c)) and thus $\operatorname{csupp}(C) \supseteq \mathfrak{m}$. Since A is Noetherian, $\operatorname{span}(F)$ is a finitely generated ideal (see 2.4.12) and we find generators $f_1, \ldots, f_k$. Then, using that $\mathfrak{m} \cdot f_i$ is closed under addition and thus $\mathfrak{m} \cdot f_i = (\mathfrak{m} \cdot A) \cdot f_i = \mathfrak{m} \cdot (A \cdot f_i)$ equals the ideal product $\mathfrak{m}(A \cdot f_i)$ for $i = 1, \ldots k$, we obtain

$$\begin{aligned}
\operatorname{span}(F) &= \sum_{i=1}^{k} A \cdot f_i = \sum_{i=1}^{k} (\mathbb{R} + \mathfrak{m}) \cdot f_i = \sum_{i=1}^{k} \mathbb{R} \cdot f_i + \sum_{i=1}^{k} \mathfrak{m} \cdot f_i \\
&= \operatorname{span}(f_1, \ldots, f_k) + \sum_{i=1}^{k} \mathfrak{m}(A \cdot f_i) = \operatorname{span}(f_1, \ldots, f_k) + \mathfrak{m} \sum_{i=1}^{k} A \cdot f_i \\
&= \operatorname{span}(f_1, \ldots, f_k) + \mathfrak{m} \operatorname{span}(F).
\end{aligned}$$

This shows that $\mathfrak{m}\,\mathrm{span}(F)$ has codimension $\leq k$ in $\mathrm{span}(F)$ and thus $\mathrm{csupp}(F)$ has finite codimension in $\mathrm{span}(F)$. □

We conclude with an overview of the results, which will be applied to several cones later.

Summary 2.4.20. Let C be a multiplicative cone of codimension 0. Then, for each face $F \trianglelefteq C$ with $F \cdot C \subseteq C$ we have that $\mathrm{span}(F)$, $\mathrm{supp}(F)$ and $\mathrm{csupp}(F)$ are ideals of A and $\mathrm{supp}(C)$ and $\mathrm{csupp}(C)$ are always ideals of A. In the case that ${}^{C}/_{\mathrm{supp}(C)}$ is divisible, C is cancellative on Klee faces.
Let us assume from now on that C is also Archimedean. Then,

- $\mathrm{span}(F)$, $\mathrm{supp}(F)$ and $\mathrm{csupp}(F)$ are ideals of A for each $F \trianglelefteq C$,
- $C \boxdot F = F$, $F \boxdot G \subseteq F \cap G$ and $G \,|\, F \Rightarrow F \subseteq G$ for all $F, G \trianglelefteq C$.

Let us assume from now on that C is also Noetherian. Then,

- every face of C is a Klee face and C has Noetherian facial structure,
- each $F \trianglelefteq C$ with $F \neq 0$ has a factorization into irreducible faces.

If A is a ring with maximal ideal $\mathfrak{m}$ with residue field $\mathbb{R}$ and $1 + \mathfrak{m} \subseteq C$, then

- $\mathfrak{m}\,\mathrm{span}(F) \subseteq \mathrm{csupp}(F)$ and ${}^{\mathrm{span}(F)}/_{\mathrm{csupp}(F)}$ has finite dimension for every face F of C.

2.4.3 Application to cones of (locally) non-negative polynomials

Let us apply the results of chapter 2 to the cones $\mathcal{P}$ and $\mathcal{P}^{Loc}$. We start with some straightforward geometric interpretations, which are also motivations to study the convex structure of these cones.

Proposition 2.4.21. *Let $S \subseteq \mathbb{R}^n$.*

(a) Let $f, g \in \mathcal{P}$. Then, we have

$$\mathrm{carr}(g) \subseteq \mathrm{carr}(f) \quad \Leftrightarrow \quad \exists\, \epsilon > 0 : f \geq \epsilon g \text{ on } S$$

and if the statements are true, we have $\mathcal{V}_{\mathbb{R}}(f) \cap S \subseteq \mathcal{V}_{\mathbb{R}}(g) \cap S$. Especially, it follows

$$\begin{aligned} \mathrm{carr}(g) = \mathrm{carr}(f) \quad &\Leftrightarrow \quad \exists F \in \mathcal{F}(\mathcal{P}) : f, g \in \mathrm{rai}(F) \\ &\Leftrightarrow \quad \exists\, M, \epsilon > 0 : \epsilon g \leq f \leq M g \text{ on } S \end{aligned}$$

and if these statements are true, then $\mathcal{V}_{\mathbb{R}}(f) \cap S = \mathcal{V}_{\mathbb{R}}(g) \cap S$. Furthermore, we have $\mathrm{supp}(\mathcal{P}) = \mathcal{I}(S)$.

(b) Let $f, g \in \mathcal{P}^{Loc}$. Then, we have

$$\mathrm{carr}(g) \subseteq \mathrm{carr}(f) \quad \Leftrightarrow \quad \exists \text{ neighbourhood } U \text{ of } 0, \exists\, \epsilon > 0 : f \geq \epsilon g \text{ on } U \cap S$$

and if the statements are true then $\mathcal{V}_{\mathbb{R}}(f) \cap U \cap S \subseteq \mathcal{V}_{\mathbb{R}}(g) \cap U \cap S$ for some neighbourhood U of 0. Especially, we have

$$\begin{aligned} \mathrm{carr}(g) = \mathrm{carr}(f) \quad &\Leftrightarrow \quad \exists F \in \mathcal{F}(\mathcal{P}^{Loc}) : f, g \in \mathrm{rai}(F) \\ &\Leftrightarrow \quad \exists \text{ neighbourhood } U \text{ of } 0, \exists\, M, \epsilon > 0 : \epsilon g \leq f \leq M g \text{ on } U \cap S. \end{aligned}$$

and $\mathcal{V}_{\mathbb{R}}(f) \cap U \cap S = \mathcal{V}_{\mathbb{R}}(g) \cap U \cap S$ for some neighbourhood U of 0 if the equivalent statements are true. Furthermore, we have $\mathrm{supp}(\mathcal{P}^{Loc}) = \mathcal{I}(S \cap U)$ for some sufficiently small neighbourhood U of 0.

(c) For $F \in \mathcal{K}(\mathcal{P})$, $f \in \operatorname{rai}(F)$ and $g \in \mathbb{R}[\underline{x}]$, we have

$$\begin{aligned} g \in \operatorname{span}(F) &\Leftrightarrow \exists\, \epsilon > 0 : f \pm \epsilon g \in \mathcal{P} \\ &\Leftrightarrow \exists\, \epsilon > 0 : f \geq \epsilon |g| \text{ on } S \end{aligned}$$

Especially, if $\sum_{i=1}^{k} |g_i| \geq \epsilon |g|$ on S for some $g_1, \ldots, g_k \in \operatorname{span}(F), g \in \mathbb{R}[\underline{x}]$ and $\epsilon > 0$, then $g \in \operatorname{span}(F)$.

(d) Let $F \in \mathcal{K}(\mathcal{P}^{Loc})$, $f \in \operatorname{rai}(F)$ and $g \in \mathbb{R}[\underline{x}]$. Then, we have

$$\begin{aligned} g \in \operatorname{span}(F) &\Leftrightarrow \exists\, \epsilon > 0 : f \pm \epsilon g \in \mathcal{P}^{Loc} \\ &\Leftrightarrow \exists \text{ neighbourhood } U \text{ of } 0, \exists\, \epsilon > 0 : f \geq \epsilon |g| \text{ on } U \cap S, \\ g \in \operatorname{csupp}(F) &\Leftrightarrow \forall\, \alpha \in \mathbb{R} : f + \alpha g \in \mathcal{P}^{Loc} \\ &\Leftrightarrow \forall\, \alpha \in \mathbb{R}, \exists \text{ neighbourhood } U \text{ of } 0 : f \geq \alpha g \text{ on } U \cap S. \end{aligned}$$

Especially, if $\sum_{i=1}^{k} |g_i| \geq \epsilon |g|$ for some neighbourhood U of 0, $\epsilon > 0$, $g_1, \ldots, g_k \in \operatorname{span}(F)$ ($g_1, \ldots, g_k \in \operatorname{csupp}(F)$) and $g \in \mathbb{R}[\underline{x}]$, then $g \in \operatorname{span}(F)$ ($g \in \operatorname{csupp}(F)$).

Proof. (a) The first equivalence follows from the definition of $\mathcal{P}$ and 2.2.3(a) and 2.2.7(b). The statement on the vanishing sets is clear. The special case of equality of the carrier faces follows with 2.2.3(c). Finally, $\operatorname{supp}(\mathcal{P}) = \mathcal{I}(S)$ follows from definition.

(b) The proof is completely analogous to the proof of (a) except of the statement on the support. However, by definition, we have $\operatorname{supp}(\mathcal{P}^{Loc}) = \bigcup_U \mathcal{I}(S \cap U)$ where the union is taken over all neighbourhoods U of 0. Since $\mathbb{R}[\underline{x}]$ is Noetherian, it follows that one may replace $\bigcup_U \mathcal{I}(S \cap U)$ by one ideal $\mathcal{I}(S \cap U)$ for sufficiently small U.

(c) The first implication follows from 2.2.8(b). The reverse implication follows since $f \pm \epsilon g \in \mathcal{P}$ and $\frac{1}{2}(f + \epsilon g) + \frac{1}{2}(f - \epsilon g) = f \in F$ implies $f \pm \epsilon g \in F$ by definition of a face and from this we have $g = \frac{1}{2\epsilon}((f + \epsilon g) - (f - \epsilon g)) \in \operatorname{span}(F)$. The second equivalence follows by definition of $\mathcal{P}$. For the last claim, let $\epsilon_i > 0$ for $i = 1, \ldots, k$ such that $f \geq \epsilon_i |g_i|$. Then, we have

$$f \geq \frac{1}{k} \sum_{i=1}^{k} \epsilon_i |g_i| \geq \frac{1}{k} \min\{\epsilon_i \,|\, i = 1, \ldots, k\} \sum_{i=1}^{k} |g_i| \geq \frac{\epsilon}{k} \min\{\epsilon_i \,|\, i = 1, \ldots, k\} |g|$$

which shows the claim.

(d) The proof is basically the same as in (c) but we use the definition of $\mathcal{P}^{Loc}$ instead of the definition of $\mathcal{P}$ and 2.2.8(c) in the third equivalence.

□

The following remark sums up the obvious consequences of the previous subsection 2.4.2 (see 2.4.20).

Remark 2.4.22. Let $C = \mathcal{P}$ or $C = \mathcal{P}^{Loc}$ for arbitrary $S \subseteq \mathbb{R}^n$ or $C = \Sigma\, \mathbb{R}[\underline{x}]^2$. Then, C is a multiplicative Noetherian cone of codimension 0. Especially, for each face $F \trianglelefteq C$ with $F \cdot C \subseteq C$ we have that $\operatorname{span}(F)$, $\operatorname{supp}(F)$ and $\operatorname{csupp}(F)$ are ideals of $\mathbb{R}[\underline{x}]$ and $\operatorname{supp}(C)$ and $\operatorname{csupp}(C)$ are always ideals of $\mathbb{R}[\underline{x}]$.
Let $S \subseteq \mathbb{R}^n$ and $C = \mathcal{P}$ with compact S or $C = \mathcal{P}^{Loc}$. Then, C is Archimedean since $1 \in \operatorname{ai}(C)$. Especially, we have

- $\operatorname{span}(F)$, $\operatorname{supp}(F)$ and $\operatorname{csupp}(F)$ are ideals of $\mathbb{R}[\underline{x}]$ for each $F \trianglelefteq C$
- $C \boxdot F = F$, $F \boxdot G \subseteq F \cap G$ and $G \,|\, F \Rightarrow F \subseteq G$ for all $F, G \trianglelefteq C$
- every face of C is a Klee face and C has Noetherian facial structure
- each $F \trianglelefteq C$ with $F \neq 0$ has a factorization into irreducible faces
- if $C = \mathcal{P}^{Loc}$ and $F \trianglelefteq \mathcal{P}^{Loc}$, then $\mathfrak{m} \operatorname{span}(F) \subseteq \operatorname{csupp}(F)$ and ${}^{\operatorname{span}(F)}/_{\operatorname{csupp}(F)}$ has finite dimension where $\mathfrak{m} = (x_1, \ldots, x_n)$.

The last property also has a nice geometric interpretation, which we present in the following corollary.

Corollary 2.4.23. *Let $f \in \mathcal{P}^{Loc}$ ($S \subseteq \mathbb{R}^n$ arbitrary). Then, there exists some $k \in \mathbb{N}$ such that for arbitrary $g_1, \ldots, g_{k+1} \in \mathbb{R}[\underline{x}]$ one of the following cases holds.*

- *There is some $i \in \{1, \ldots, k+1\}$ such that for all $\epsilon > 0$ we have $f + \epsilon g_i \notin \mathcal{P}^{Loc}$ or $f - \epsilon g_i \notin \mathcal{P}^{Loc}$ (i.e. $g_i \notin \operatorname{span}(F)$)*
- *There is some $g \in \mathbb{R}[\underline{x}]$ that is a non-trivial linear combination of $g_1, \ldots, g_{k+1}$ such that $f + \alpha g \in \mathcal{P}^{Loc}$ for all $\alpha \in \mathbb{R}$ (i.e. $g \in \operatorname{csupp}(F)$).*

Furthermore, we see that the finite dimension of ${}^{\operatorname{span}(F)}/_{\operatorname{csupp}(F)}$ for faces F of $\mathcal{P}^{Loc}$ induces a finite-dimensional cone in ${}^{\operatorname{span}(F)}/_{\operatorname{csupp}(F)}$, which describes F almost completely.

Proposition 2.4.24. *Let F be a face of $\mathcal{P}^{Loc}$ ($S \subseteq \mathbb{R}^n$ arbitrary) and $\varphi : \operatorname{span}(F) \to {}^{\operatorname{span}(F)}/_{\operatorname{csupp}(F)}$ be the projection. We set $G := \varphi(F)$. Then, $\operatorname{rai}(G) \neq \emptyset$ and*

$$\varphi^{-1}(\operatorname{rai}(G)) \subseteq F \subseteq \varphi^{-1}(\operatorname{ac}(G)).$$

Thus, G determines the geometric form of F completely except on its relative algebraic boundary.

Proof. This follows from 2.2.13 and since all faces are Klee faces. □

Finally, let us study how the cones $\mathcal{P}$ and $\mathcal{P}^{Loc}$ behave with respect to division of faces. The following example shows that some premises of the subsequent proposition on the cancellation property are necessary.

Example 2.4.25. (a) Let $n = 2$. We use x and y as variables instead of x_1 and x_2 and we set

$$S = \{x = 0\} \cup \{y = 0\}.$$

Furthermore, we set $f := x^2$, $g := y^2$ and $F := \operatorname{carr}(f)$, $G := \operatorname{carr}(g)$ in $\mathcal{P}$ or $\mathcal{P}^{Loc}$. Then, we have $fg = 0$ on S, so $F \boxdot G = 0 = F \boxdot 0$. Clearly, we have $F, G \neq 0$ and they are Klee faces. So $\mathcal{P}$ and $\mathcal{P}^{Loc}$ are not cancellative on Klee faces.

(b) We consider $n = 1$ and $S = \{0\} \cup [1, 2] \subseteq \mathbb{R}$. S is irreducible in the Zariski topology. We set $f := (x-1)(2-x)$ and $g := x$. Then, we have $fg \geq 0$ on S and $g \geq 0$ on S, so

$fg, g \in \mathcal{P}$. However, $f(0) < 0$ and thus $f \notin \mathcal{P}$. This shows that $\mathcal{P}$ is not divisible.
Let $G := \operatorname{carr}(x) \trianglelefteq \mathcal{P}$. Since $1 \in \operatorname{rai}(\mathcal{P})$ and $\frac{1}{2}x^2 \leq x \leq x^2$ on S, we have

$$\begin{aligned} G \boxdot G &= \operatorname{carr}(x) \boxdot \operatorname{carr}(x) = \operatorname{carr}(x^2) = \operatorname{carr}(x) \\ &= \operatorname{carr}(x \cdot 1) = \operatorname{carr}(x) \boxdot \operatorname{carr}(1) = G \boxdot \mathcal{P}. \end{aligned}$$

We have $G \neq \mathcal{P}$, $G \neq 0$ and $G, \mathcal{P} \in \mathcal{K}(\mathcal{P})$ and thus, $\mathcal{P}$ is not cancellative on Klee faces.

(c) Let $n = 2$ and we denote the variables by x and y. We set

$$S := \{x = 0, y \geq 0\} \cup \{0 \leq y \leq x \leq 2y\}, \quad g := x, \quad h := y.$$

S is irreducible in the Zarisiki topology. Now, let $G := \operatorname{carr}(g)$ and $H := \operatorname{carr}(h)$ in $\mathcal{P}^{Loc}$. We have $\frac{1}{2}x^2 \leq xy \leq x^2$ on S, so

$$G \boxdot G = \operatorname{carr}(x) \boxdot \operatorname{carr}(x) = \operatorname{carr}(x^2) = \operatorname{carr}(xy) = \operatorname{carr}(x) \boxdot \operatorname{carr}(y) = G \boxdot H.$$

Again, we have $G \neq H$, $G \neq 0$, $G, H \in \mathcal{K}(\mathcal{P}^{Loc})$, so $\mathcal{P}^{Loc}$ is not cancellative on Klee faces.

Proposition 2.4.26. *Let $S \subseteq \mathbb{R}^n$ be irreducible in the Zariski topology and of locally constant dimension. Then, ${}^{\mathcal{P}}/_{\operatorname{supp}(\mathcal{P})}$ and ${}^{\mathcal{P}^{Loc}}/_{\operatorname{supp}(\mathcal{P}^{Loc})}$ are divisible and thus, $\mathcal{P}$ and $\mathcal{P}^{Loc}$ are cancellative on Klee faces by 2.4.14.*

Remark 2.4.27. The property of $S \subseteq \mathbb{R}^n$ being irreducible in the Zariski topology and of locally constant dimension may be reformulated in the following way. If $x \in S$ and U is a neighbourhood of x in $\mathbb{R}^n$, then we have $\overline{U \cap S}^{zar} \supseteq S$.

Proof of 2.4.26. By the remark, we have $\mathcal{I}(S) = \mathcal{I}(U \cap S)$ for open U with $U \cap S \neq \emptyset$ and thus $\mathcal{I}(\mathcal{S}) = \operatorname{supp}(\mathcal{P}^{Loc})$ (2.4.21(b)). Note, that $\mathcal{I}(S) = \operatorname{supp}(\mathcal{P})$ anyway (2.4.21(a)).
Let $f, g \in A := {}^{\mathbb{R}[\underline{x}]}/_{\mathcal{I}(S)}$ with $f, fg \in {}^{\mathcal{P}}/_{\operatorname{supp}(\mathcal{P})}$ and $g \notin {}^{\mathcal{P}}/_{\operatorname{supp}(\mathcal{P})}$. Then, there is some $x \in S$ with $g(x) < 0$ and so we will find some neighbourhood U of x with $g(y) < 0$ for $y \in U \cap S$ (note, that g is well-defined on S). Now, $fg \in {}^{\mathcal{P}}/_{\operatorname{supp}(\mathcal{P})}$, so $fg \geq 0$ on $U \cap S$ and $f \geq 0$ on $U \cap S$, so $f(y) = 0$ for all $y \in U \cap S$. Since $U \cap S$ is Zariski dense in S, we have $f = 0$ on S and thus $f = 0$ in A. It follows that ${}^{\mathcal{P}}/_{\operatorname{supp}(\mathcal{P})}$ is divisible.
Now, let $f, g \in A = {}^{\mathbb{R}[\underline{x}]}/_{\mathcal{I}(S)}$ with $f, fg \in {}^{\mathcal{P}^{Loc}}/_{\operatorname{supp}(\mathcal{P}^{Loc})}$ and $g \notin {}^{\mathcal{P}^{Loc}}/_{\operatorname{supp}(\mathcal{P}^{Loc})}$. We choose some open neighbourhood V of 0 with $fg \geq 0$ on $V \cap S$. Since $g \notin {}^{\mathcal{P}^{Loc}}/_{\operatorname{supp}(\mathcal{P}^{Loc})}$, there is some $x \in V \cap S$ with $g(x) < 0$. Thus, we find some neighbourhood $U \subseteq V$ of x with $g(y) < 0$ for $y \in U$. Now, since $f \geq 0$ on $U \cap S$, we have $f(y) = 0$ for $y \in U \cap S$. As above, this implies $f = 0$ on S and $f = 0$ in A. Hence, ${}^{\mathcal{P}^{Loc}}/_{\operatorname{supp}(\mathcal{P}^{Loc})}$ is divisible. □

3 Cones of local positivity

3.1 Polynomials, convergent and formal power series

We have defined the cone of locally non-negative polynomials in 1.7.1. Now, we will generalize this construction to convergent and formal power series and discuss their relationships. For this purpose, we recall the definitions of real radicals and saturations of preorderings and some of their basic properties in the subsequent remark.

Definition 3.1.1. [KneSch, III.§2, Def. 2, III. §10, Def. 2] Let A be a commutative ring with $\frac{1}{2} \in A$ and I some ideal of A.
A is called *real*, if $f_1^2 + \cdots + f_k^2 = 0$ with $f_1, \ldots, f_k \in A$ implies $f_1 = \cdots = f_k = 0$. I is called *real*, if A/I is real. The *real radical* $\sqrt[re]{I}$ of I is defined as the intersection of all real prime ideals $\mathfrak{p}$ with $\mathfrak{p} \supseteq I$.
Let T be a preordering of A. The *saturation of* T is the intersection of all orderings P of A with $P \supseteq T$ and is denoted by $\mathrm{Sat}(T)$.

Remark 3.1.2. (a) [Sch, III.§2, Bem.10.] We have

$$\sqrt[re]{I} = \{f \in A \,|\, \exists N \in \mathbb{N}, \sigma \in \Sigma A^2 : f^{2N} + \sigma \in I\}.$$

(b) [Sch, III.§9, Satz 3., III.§10, Def.2.] It is

$$\mathrm{Sat}(T) = \{f \in A \,|\, \exists\, t_1, t_2 \in T, N \in \mathbb{N}_0 : ft_1 = f^{2N} + t_2\}.$$

(c) By [Sch, 1.5] and [Sch2, 3.12.], we have

$$\mathrm{supp}(\mathrm{Sat}(T)) = \sqrt{\mathrm{supp}(T)} = \sqrt[re]{\mathrm{supp}(T)}.$$

Now, we have a straightforward definition to transfer the cone of locally non-negative polynomials to the setting of convergent power series.

Definition 3.1.3. Let $S \subseteq \mathbb{R}^n$. We set

$$\tilde{\mathcal{P}}^{Loc} := \{f \in \mathbb{R}\{\underline{x}\} \,|\, f \geq 0 \text{ on } S \cap U \text{ for some neighbourhood } U \text{ of } 0\}.$$

and call it the *cone of locally non-negative convergent power series (on S).*

Remark 3.1.4. Clearly, $\tilde{\mathcal{P}}^{Loc}$ does only depend on the germ of S at 0.

Remark 3.1.5. The statements of 2.4.21 (b),(d) carry over to $\tilde{\mathcal{P}}^{Loc}$ with the obvious changes and the same proofs.

However, for certain sets S, we also have an algebraic description of $\tilde{\mathcal{P}}^{Loc}$. The following proposition is probably well-known to the experts.

Proposition 3.1.6. *Let T be a finitely generated preordering of $\mathbb{R}\{\underline{x}\}$ such that its corresponding basic closed semi-analytic set germ coincides with the germ of S. Then,*

$$\tilde{\mathcal{P}}^{Loc} = \mathrm{Sat}(T).$$

Proof. Let $T_0 := \{g_1, \ldots, g_s\} \subseteq \mathbb{R}\{\underline{x}\}$ be generators of T as preordering in $\mathbb{R}\{\underline{x}\}$ and U some neighbourhood of 0 where $g_1, \ldots, g_s$ are defined. We may assume that

$$S \cap U = \{x \in U \mid g_1(x) \geq 0, \ldots, g_s(x) \geq 0\}.$$

Let $f \in \tilde{\mathcal{P}}^{Loc}$ and w.l.o.g. let f be defined on U and $f \geq 0$ on $U \cap S$. Thus, we have

$$\{x \in U \mid f(x) \neq 0, -f(x) \geq 0, g_1(x) \geq 0, \ldots, g_s(x) \geq 0\} = \emptyset.$$

By some Positivstellensatz from [AnBrRu, VIII.2.8 a)], there are $N, d \in \mathbb{N}$ and $a_{\underline{m}} \in \sum \mathbb{R}\{\underline{x}\}^2$ for each $(s+1)$-tuple $\underline{m} = (m_0, \ldots, m_s) \in \mathbb{N}_0^{s+1}$ with $|\underline{m}| \leq d$ such that all $a_{\underline{m}}$ converge on some neighbourhood W of 0 (w.l.o.g. $W = U$) and

$$f^{2N} + \sum_{|\underline{m}| \leq d} a_{\underline{m}} (-f)^{m_0} g_1^{m_1} \cdots g_s^{m_s} = 0$$

on U. Thus, we also have

$$f^{2N} + \underbrace{\sum_{|\underline{m}| \leq d, 2 | m_0} a_{\underline{m}} f^{m_0} g_1^{m_1} \cdots g_s^{m_s}}_{\in T} = \underbrace{\left(\sum_{|\underline{m}| \leq d, 2 \nmid m_0} a_{\underline{m}} f^{m_0 - 1} g_1^{m_1} \cdots g_s^{m_s} \right)}_{\in T} f.$$

By 3.1.2(b), we have $f \in \mathrm{Sat}(T)$. This shows $\tilde{\mathcal{P}}^{Loc} \subseteq \mathrm{Sat}(T)$.
Let us assume now that $f \in \mathrm{Sat}(\tilde{T})$. So, by 3.1.2(b), there are $t_1, t_2 \in T$ and $N \in \mathbb{N}_0$ with $f t_1 = f^{2N} + t_2$. Let U be some neighbourhood of 0 where t_1, t_2 and f are defined. Now, if $x \in U \cap S$, we have either $f(x) = 0$ or $f^{2N}(x) > 0$. In the second case, $t_1(x) \geq 0$ and $t_2(x) \geq 0$ imply $f(x) > 0$. Thus, $f \geq 0$ on $U \cap S$, so $f \in \tilde{\mathcal{P}}^{Loc}$. This shows $\tilde{\mathcal{P}}^{Loc} \supseteq \mathrm{Sat}(T)$ and hence the claim. □

We use this proposition as motivation for the following definition, which generalizes $\mathcal{P}^{Loc}$ to the setting of formal power series.

Definition 3.1.7. Let T be a preordering in $\mathbb{R}[[\underline{x}]]$. We set

$$\hat{\mathcal{P}}^{Loc} := \mathrm{Sat}(T)$$

and call it the *cone of non-negative (formal) power series (corresponding to T).*

The following theorem shows that the property of local non-negativity of some element does not change if we work in a larger ring. As a corollary, we obtain canonical inclusions of the corresponding sets of faces of these cones.

Theorem 3.1.8. *Let $S \subseteq \mathbb{R}^n$.*

(a) We have $\tilde{\mathcal{P}}^{Loc} \cap \mathbb{R}[\underline{x}] = \mathcal{P}^{Loc}$.

(b) Suppose that S has the same germ at 0 as the basic closed semi-analytic set defined by some finite $T_0 \subseteq \mathbb{R}\{\underline{x}\}$. Let T be the preordering defined by T_0 in $\mathbb{R}[[\underline{x}]]$ and $\hat{\mathcal{P}}^{Loc}$ the corresponding cone of non-negative power series. Then,

$$\hat{\mathcal{P}}^{Loc} \cap \mathbb{R}\{\underline{x}\} = \tilde{\mathcal{P}}^{Loc}.$$

Proof. (a) is obvious by definition. Suppose now that T_0, S, T and $\hat{\mathcal{P}}^{Loc}$ are given as in (b). Let $\tilde{T}$ be the preordering generated by T_0 in $\mathbb{R}\{\underline{x}\}$ and $f \in \tilde{\mathcal{P}}^{Loc} = \mathrm{Sat}(\tilde{T})$ (by 3.1.6). Now, 3.1.2(b) shows together with $\tilde{T} \subseteq T$ that $f \in \mathrm{Sat}(T) = \hat{\mathcal{P}}^{Loc}$. This shows $\tilde{\mathcal{P}}^{Loc} \subseteq \hat{\mathcal{P}}^{Loc}$.
Let us show that $f \in \mathbb{R}\{\underline{x}\}$ and $f \notin \tilde{\mathcal{P}}^{Loc}$ imply $f \notin \hat{\mathcal{P}}^{Loc}$, which shows the claim. We consider $\mathbb{R}^n$ with 2-norm $||.||$ and some $\epsilon > 0$ such that f converges on the ϵ-neighbourhood of 0 and S is semi-analytic on this ϵ-neighbourhood of 0. Now, we have the semi-analytic set

$$S^- := \{x \in \mathbb{R}^n \,|\, x \in S, ||x|| < \epsilon, f(x) < 0\}\,.$$

We note that T_0 may not contain elements with negative absolute coefficient as this would imply that the germ of S at the origin is empty and this would contradict $f \notin \tilde{\mathcal{P}}^{Loc}$. Therefore, T and $\hat{\mathcal{P}}^{Loc} := \mathrm{Sat}(T)$ may also not contain such elements. Hence, the claim follows if $f(0) < 0$ and we may reduce to the case $f(0) \geq 0$. Then, we have $0 \in \overline{S^-} \setminus (S^-)$. So, by the curve selecting lemma (see [Loj, p.94]), there is a so-called semi-analytic arc in S^- with endpoint 0. By the theorem on the parametrisation of such arcs (see [Loj, p.94]), there is some $\delta > 0$ and a non-constant analytic map $\psi : (-\delta, \delta) \to \mathbb{R}^n$ with $\psi(0) = 0$, $\psi(t) \in S$ and $f(\psi(t)) < 0$ for $0 < t < \delta$. W.l.o.g., we may choose δ such that ψ is a convergent power series on $(-\delta, \delta)$ and we may identify ψ with this power series. We set

$$T_\psi := \{g \in \mathbb{R}[[\underline{x}]] \,|\, g(\psi(t)) = 0 \text{ or the leading coefficient of } g(\psi(t)) \text{ is positive}\}.$$

Clearly, this is a preordering of $\mathbb{R}[[\underline{x}]]$ with $T_\psi \cup (-T_\psi) = \mathbb{R}[[\underline{x}]]$. If $g, h \in \mathbb{R}[[\underline{x}]]$ with $g \cdot h \in \mathrm{supp}(T_\psi)$, then $g(\psi(t)) \cdot h(\psi(t)) = (g \cdot h)(\psi(t)) = 0$ and as $\mathbb{R}[[t]]$ has no zero-divisors, we have $g(\psi(t)) = 0$ or $h(\psi(t)) = 0$. Therefore, $\mathrm{supp}(T_\psi)$ is a prime ideal and T_ψ is an ordering of $\mathbb{R}[[\underline{x}]]$. We have $f \notin T_\psi$ since $-f(\varphi(t)) < 0$ for $t \in (0, \delta)$ and $T \subseteq T_\psi$ since $\varphi(t) \in S$ and hence $g(\varphi(t)) \geq 0$ for $t \in (0, \delta)$ and $g \in T_0$. This shows $f \notin \mathrm{Sat}(T) = \hat{\mathcal{P}}^{Loc}$. □

Corollary 3.1.9. *In the situation of theorem 3.1.8(b), we have canonical inclusions*

$$\mathcal{F}(\mathcal{P}^{Loc}) \hookrightarrow \mathcal{F}(\tilde{\mathcal{P}}^{Loc}) \hookrightarrow \mathcal{F}(\hat{\mathcal{P}}^{Loc})$$

given by carr(.). *The inverse maps (defined on the images) are given by intersection with the smaller ring.*

Proof. This follows from 3.1.8 and 2.2.12. □

Remark 3.1.10. The inclusions of the corollary are usually no bijections. Let us consider the case that $0 \in \mathring{S}$. Then, we will find some $f \in \mathbb{R}\{\underline{x}\}$ that locally at 0 has a zero set that is not describable as the zero set of a polynomial. Now, it is rather clear that $\mathrm{carr}_{\tilde{\mathcal{P}}^{Loc}}(f^2)$ may not arise from a face of $\mathcal{P}^{Loc}$. A similar argument works to show that $\mathcal{F}(\tilde{\mathcal{P}}^{Loc}) \hookrightarrow \mathcal{F}(\hat{\mathcal{P}}^{Loc})$ is not surjective. However, the corresponding argument is rather long and we do not include it here.

We will still assume that $S \subseteq \mathbb{R}^n$ and some preordering $T \subseteq \mathbb{R}[[\underline{x}]]$ are given, so we can use $\mathcal{P}^{Loc}$, $\tilde{\mathcal{P}}^{Loc}$ and $\hat{\mathcal{P}}^{Loc}$. Let us see which results for $\mathcal{P}^{Loc}$ carry over to $\tilde{\mathcal{P}}^{Loc}$ and $\hat{\mathcal{P}}^{Loc}$.

Remark 3.1.11. $C = \tilde{\mathcal{P}}^{Loc}$ and $C = \hat{\mathcal{P}}^{Loc}$ are multiplicative, Noetherian and Archimedean cones. The statement that the corresponding rings are Noetherian may be found in [GrLoSh, I.1.15]. The Archimedean property holds as each element with positive absolute coefficient of these rings is already a square und thus belongs to the corresponding cone C. Thus, we have (see 2.4.20 and compare with 2.4.22)

- $\operatorname{span}(F)$, $\operatorname{supp}(F)$ and $\operatorname{csupp}(F)$ are ideals for each $F \trianglelefteq C$
- $C \boxdot F = F$, $F \boxdot G \subseteq F \cap G$ and $G \,|\, F \Rightarrow F \subseteq G$ for all $F, G \trianglelefteq C$
- every face of C is a Klee face and C has Noetherian facial structure
- each $F \trianglelefteq C$ with $F \neq 0$ has a factorization into irreducible faces
- $\mathfrak{m} \operatorname{span}(F) \subseteq \operatorname{csupp}(F)$ and ${}^{\operatorname{span}(F)}/_{\operatorname{csupp}(F)}$ has finite dimension for each $F \trianglelefteq C$ (where $\mathfrak{m}$ is the maximal ideal of the ring)

The geometric interpretation of the last property in the case $\mathcal{P}^{Loc}$ (see 2.4.23) carries over to these cones (though, one would not call it "geometric" anymore if $C = \hat{\mathcal{P}}^{Loc}$). Also, the statement that the essential geometry of F (with $F \trianglelefteq C$) is determined by its projection on ${}^{\operatorname{span}(F)}/_{\operatorname{csupp}(F)}$ (see 2.4.24) carries over. The proofs are completely analogous.
Finally, one can also transfer proposition 2.4.26 to show that ${}^{\tilde{\mathcal{P}}^{Loc}}/_{\operatorname{supp}(\tilde{\mathcal{P}}^{Loc})}$ is divisible and thus $\tilde{\mathcal{P}}^{Loc}$ is cancellative on Klee faces for appropriate sets S. Indeed, the only property used is that one can find some neighbourhood U of 0 such that there is no $f \in \mathbb{R}\{\underline{x}\} \setminus \operatorname{supp}(\tilde{\mathcal{P}}^{Loc})$ that is defined and vanishes on $S \cap V$ for some open $V \subseteq U$ with $S \cap V \neq \emptyset$ (one can reformulate it similar to 2.4.26 using the analytic Zariski topology).

3.2 Faces of finite codimension

We assume that $S \subseteq \mathbb{R}^n$ is locally semi-analytic and $T \subseteq \mathbb{R}[[\underline{x}]]$ is a preordering of $\mathbb{R}[[\underline{x}]]$ generated by finitely many elements of $\mathbb{R}\{\underline{x}\}$ describing S locally. We have the corresponding cones $\mathcal{P}^{Loc}$, $\tilde{\mathcal{P}}^{Loc}$ and $\hat{\mathcal{P}}^{Loc}$.

Lemma 3.2.1. *Let $h := x_1^2 + \cdots + x_n^2$, $d \in \mathbb{N}_0$, $H := \operatorname{carr}_{\mathcal{P}^{Loc}}(h^d)$ and $\mathfrak{m} = (x_1, \ldots, x_n)$. Then, we have $\mathfrak{m}^{2d} \subseteq \operatorname{span}(H)$ and $\mathfrak{m}^{2d+1} \subseteq \operatorname{csupp}(H)$. Similar statements hold for $\tilde{\mathcal{P}}^{Loc}$ and $\hat{\mathcal{P}}^{Loc}$.*

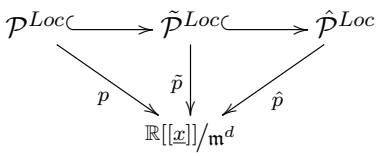
Proof. The proof is the same in all three cases. Let m_1, m_2 be monomials of degree d. Then, clearly $m_1^2, m_2^2 \in H$ as they appear in the monomials of h^d. Furthermore,

$$(m_1 + m_2)^2 + (m_1 - m_2)^2 = 2m_1^2 + 2m_2^2 \in H$$

and as H is a face, we also have that $(m_1 + m_2)^2 \in H$. Now, we observe that $m_1 m_2 = \frac{1}{2}\left((m_1 + m_2)^2 - m_1^2 - m_2^2\right) \in \operatorname{span}(H)$. As $\operatorname{span}(H)$ is an ideal, this shows $\mathfrak{m}^{2d} \subseteq \operatorname{span}(H)$ and as $\mathfrak{m} \operatorname{span}(H) \subseteq \operatorname{csupp}(H)$, we also get $\mathfrak{m}^{2d+1} \subseteq \operatorname{csupp}(H)$. □

In the following proposition, we show that $\mathcal{P}^{Loc}$, $\tilde{\mathcal{P}}^{Loc}$ and $\hat{\mathcal{P}}^{Loc}$ "coincide on d-jets".

Proposition 3.2.2. *Let $d \in \mathbb{N}_0$ and $\mathfrak{m}$ be the maximal ideal of $\mathbb{R}[[\underline{x}]]$. We consider the following commutative diagram of sets where $\hat{p}$ is induced by the quotient map $\mathbb{R}[[\underline{x}]] \to {}^{\mathbb{R}[[\underline{x}]]}/_{\mathfrak{m}^d}$.*



Then, we have $p(\mathcal{P}^{Loc}) = \tilde{p}(\tilde{\mathcal{P}}^{Loc}) = \hat{p}(\hat{\mathcal{P}}^{Loc})$.

Proof. We have obviously $p(\mathcal{P}^{Loc}) \subseteq \tilde{p}(\tilde{\mathcal{P}}^{Loc}) \subseteq \hat{p}(\hat{\mathcal{P}}^{Loc})$. Thus, we just have to show that $p(\mathcal{P}^{Loc}) \supseteq \hat{p}(\hat{\mathcal{P}}^{Loc})$. So let us consider some $f \in \hat{\mathcal{P}}^{Loc}$. Let $h := x_1^2 + \cdots + x_n^2$ and $m := \lceil \frac{d}{2} \rceil$, so especially $h^m \in \mathfrak{m}^d$. Let $H := \operatorname{carr}_{\hat{\mathcal{P}}^{Loc}}(h^m)$. Hence, setting

$$F := \operatorname{carr}_{\hat{\mathcal{P}}^{Loc}}(f + h^m) \supseteq \operatorname{carr}_{\hat{\mathcal{P}}^{Loc}}(h^m) = H$$

we obtain $\operatorname{csupp}(F) \supseteq \operatorname{csupp}(H) \supseteq \mathfrak{m}^{2m+1}$ by 3.2.1. By 2.2.8(c), we have $f + h^m + g \in \hat{\mathcal{P}}^{Loc}$ for each $g \in \mathfrak{m}^{2m+1}$ and we set g in such a way that $f + h^m + g \in \mathbb{R}[\underline{x}]_{\leq 2m}$. We obtain $f + h^m + g \in \mathcal{P}^{Loc}$ as $\mathcal{P}^{Loc} = \hat{\mathcal{P}}^{Loc} \cap \mathbb{R}[\underline{x}]$ by 3.1.8. Consequently, $\hat{p}(f) = \hat{p}(f + h^m + g) = p(f + h^m + g) \in p(\mathcal{P}^{Loc})$ which shows the claim as $f \in \hat{\mathcal{P}}^{Loc}$ was chosen arbitrarily. □

Let us consider faces of finite codimension.

Lemma 3.2.3. *Let F be a face of finite codimension of $\mathcal{P}^{Loc}$. Then, there is some $d \in \mathbb{N}_0$ such that $F \supseteq \mathfrak{m}^{2d} \cap \mathcal{P}^{Loc}$ (with $\mathfrak{m} = (x_1, \ldots, x_n)$). Similar statements hold for $\tilde{\mathcal{P}}^{Loc}$ and $\hat{\mathcal{P}}^{Loc}$.*

Proof. Let $h := x_1^2 + \cdots + x_n^2$. We consider the sequence of faces

$$F \boxplus \operatorname{carr}(h) \supseteq F \boxplus \operatorname{carr}(h^2) \supseteq F \boxplus \operatorname{carr}(h^3) \supseteq \ldots$$

which contain F. As F has finite codimension, we can find some d with $F \boxplus \operatorname{carr}(h^d) = F \boxplus \operatorname{carr}(h^{d+1})$. Especially, there is some $\epsilon > 0$ such that $(f + h^{d+1}) - 2\epsilon(f + h^d) \in \mathcal{P}^{Loc}$. However, we also have $\epsilon h^d - h^{d+1} \in \mathcal{P}^{Loc}$, which implies $(1 - 2\epsilon)f - \epsilon h^d \in \mathcal{P}^{Loc}$. Hence, $\operatorname{carr}(f) \supseteq \operatorname{carr}(h^d)$. 3.2.1 shows the claim. □

We have the following nice corollary.

Corollary 3.2.4. *The faces of finite codimension of $\mathcal{P}^{Loc}$, $\tilde{\mathcal{P}}^{Loc}$ and $\hat{\mathcal{P}}^{Loc}$ are in canonical bijection to each other. Furthermore, corresponding faces of finite codimension share the values* $\operatorname{codim}(F)$, $\operatorname{codim}(\operatorname{csupp}(F))$, *they have canonically isomorphic spaces* $\mathrm{span}(F)/\mathrm{csupp}(F)$ *and under these isomorphisms the images of the faces are in bijection.*

Proof. Let $C \in \{\mathcal{P}^{Loc}, \tilde{\mathcal{P}}^{Loc}, \hat{\mathcal{P}}^{Loc}\}$, F be a face of finite codimension of C and $\mathfrak{m} = (x_1, \ldots, x_n)$ in the corresponding ring. By 3.2.3 and 3.2.1, there is some sufficiently high d such that $\mathfrak{m}^{2d} \subseteq \operatorname{csupp}(F)$. Let p be the canonical map from our considered ring to $\mathbb{R}[[\underline{x}]]/\hat{\mathfrak{m}}^{2d}$ ($\hat{\mathfrak{m}}$ maximal ideal of $\mathbb{R}[[\underline{x}]]$) and $\mathcal{P}_{2d}^{\mathrm{jets}} := p(C)$, which does not depend on the ring by 3.2.2. $p(F)$ is convex by 2.2.11 and we will show that it is a face of $p(C)$. So let $f_1, f_2 \in C$ with $p(f_1) + p(f_2) = p(g)$ and $g \in F$. We have

$$\delta := f_1 + f_2 - g \in \mathfrak{m}^{2d},$$

so $h - \delta \in C$ for some $h \in \mathfrak{m}^{2d} \cap C \subseteq F$ by 3.2.1. This gives

$$f_1 + f_2 + (h - \delta) = g + h \in F$$

and thus $f_1, f_2 \in F$ as F is a face of C. Hence, $p(F)$ is a face of $p(C) = \mathcal{P}_{2d}^{\mathrm{jets}}$. Note that $\operatorname{rai}(F) = p^{-1}(\operatorname{rai}(p(F)))$ by 2.2.13, so F may be recovered from $p(F)$. Furthermore, $\operatorname{rai}(F)$ contains some polynomial by 2.2.8(c) and $\mathfrak{m}^{2d} \subseteq \operatorname{csupp}(F)$. Such a polynomial defines

carrier faces in all three cones, which are independent of the choice of the polynomial and they project to the same face of $\mathcal{P}_{2d}^{\text{jets}}$. These faces shall correspond to each other under the bijections and we have just seen that this is well-defined. The other statements are obvious as this information is encoded in $p(F)$. □

We conclude with a geometric interpretation of faces of finite codimension.

Lemma 3.2.5. *Let F be a proper face of $\mathcal{P}^{Loc}$ ($\tilde{\mathcal{P}}^{Loc}$). Then, F has finite codimension if and only if for (one or equivalently every) $f \in \operatorname{rai}(F)$ there is some neighbourhood U of the origin such that the vanishing set of f on $S \cap U$ is just $\{0\}$.*

Proof. We consider the case $\tilde{\mathcal{P}}^{Loc}$, which implies the case $\mathcal{P}^{Loc}$ using 3.2.4. Let $h := x_1^2 + \cdots + x_n^2$. If F has finite codimension, then $\operatorname{span}(F) \supseteq \mathfrak{m}^{2d}$ for some $d \in \mathbb{N}$ by 3.2.3, especially $h^d \in F$. Hence, for $f \in \operatorname{rai}(F)$, we find some $\epsilon > 0$ such that $f - \epsilon h^d \in \tilde{\mathcal{P}}^{Loc}$. Let U be a neighbourhood of the origin where f converges and $f - \epsilon h^d \geq 0$ on $S \cap U$. Then, the vanishing set of f on $S \cap U$ is just $\{0\}$.
Now, suppose that the vanishing set of f on $S \cap U$ is just $\{0\}$ for some $f \in \operatorname{rai}(F)$ and some neighbourhood U of the origin. Let $T_0 := \{g_1, \ldots, g_s\} \subseteq \mathbb{R}\{\underline{x}\}$ be germs of analytic functions describing S locally and T the preordering generated by T_0 in $\mathbb{R}\{\underline{x}\}$. Then, the semianalytic set germ defined by $f = 0$, $g_1, \ldots, g_s \geq 0$ and $h \neq 0$ is empty. By some Positivstellensatz (see [AnBrRu, VIII.2.8 a)]), we find $d \in \mathbb{N}$, $t \in T \subseteq \tilde{\mathcal{P}}^{Loc}$ and $f' \in f \cdot \mathbb{R}\{\underline{x}\} \subseteq \operatorname{span}(F)$ such that $h^{2d} + t = f'$. As $h^{2d}, t \in \tilde{\mathcal{P}}^{Loc}$, we have $f' \in F$ and $h^{2d} \in F$ and F has finite codimension by 3.2.1. □

Remark 3.2.6. A similar lemma may be shown in the case $\hat{\mathcal{P}}^{Loc}$, but one has to introduce the real spectrum of a ring to formulate it properly. The proof basically carries over to this case. (Use 3.2.1 to show that h^d has an isolated zero in the first implication.)

3.3 Faces and their Newton diagram

From now on, we will assume that $S = \mathbb{R}^n$, so especially $\tilde{\mathcal{P}}^{Loc} = \mathbb{R}\{\underline{x}\}_+$ and $\hat{\mathcal{P}}^{Loc} = \mathbb{R}[[\underline{x}]]_+$. The aim of this section is to study faces (of one of the cones $\mathcal{P}^{Loc}$, $\mathbb{R}\{\underline{x}\}_+$ and $\mathbb{R}[[\underline{x}]]_+$) and Newton polyhedra and Newton diagrams and the relations between them.

3.3.1 Transfer of results of Vassiliev

In [Vas], Vassiliev studied analytic functions f that have a local isolated minimum at $0 \in \mathbb{R}^n$, so especially all elements in the relative interior of faces of $\mathbb{R}\{\underline{x}\}_+$ of finite codimension. He obtains results that relate the property of f having an isolated minimum at 0 to properties of the coefficients of f on the corresponding Newton diagram. We transfer his results to our situation and thus, arbitrary elements of the cones $\mathcal{P}^{Loc}$, $\mathbb{R}\{\underline{x}\}_+$ and $\mathbb{R}[[\underline{x}]]_+$. This has the advantage that we can formulate the statements in full generality and we do not need to restrict to faces of finite codimension. Though, on the first sight, this may be only a very slight improvement in application, it may be very useful to construct faces of infinite codimension with certain properties that induce faces of finite codimension with the same properties (e.g. one can apply it on the examples in 4.3.8(b),(c)).
We start with some definitions (see [Vas] or [Sek]).

Definition 3.3.1. Let $\Delta \subseteq \mathbb{R}^n_+$. We call Δ a *Newton polyhedron*, if Δ is a polyhedron with $\Delta + \mathbb{R}^n_+ = \Delta$ and each vertex of Δ lies in $\mathbb{N}_0^n$. We call the union of compact faces $\Gamma = \Gamma(\Delta)$ of some Newton polyhedron Δ the *Newton diagram of* Δ. As Δ is uniquely determined by Γ (see 3.3.2), we also write $\Delta(\Gamma)$ for the Newton polyhedron with Newton diagram Γ.
Faces (*vertices*) of Γ are by definition non-empty, compact faces (vertices) of Δ (Γ is usually not convex). If M is the set of vertices of Γ, then $g(\Gamma) := \sum_{\alpha \in M} x^\alpha$ is called the *canonical polynomial* of Γ.
We call a Newton diagram Γ a *psd-diagram* if all vertices of Γ have just even coordinates. We call Γ a *min-diagram* if Γ is a psd-diagram and if Γ intersects each coordinate axis.
Let $f \in \mathbb{R}[[\underline{x}]]$. We define the *support* $\operatorname{supp}(f)$ of $f = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha x^\alpha$ as

$$\operatorname{supp}(f) := \{\alpha \in \mathbb{N}_0^n \mid a_\alpha \neq 0\}.$$

Furthermore, we set

$$\Delta(f) := \operatorname{conv}\left(\operatorname{supp}(f) + \mathbb{R}^n_+\right)$$

and call it *Newton polyhedron of* f. It is a polyhedron by Dickson's lemma (see [Dic, Lemma A]) and thus, it is indeed a Newton polyhedron. We denote the Newton diagram of $\Delta(f)$ by $\Gamma(f)$ and call it the *Newton diagram of* f.
For any $\tau \subseteq \mathbb{R}^n$ we set (with $f = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha x^\alpha$)

$$f^\tau := \sum_{\alpha \in \operatorname{supp}(f) \cap \tau} a_\alpha x^\alpha.$$

Let us take a look at the convex structure of Newton polyhedra in the following remark.

Remark 3.3.2. By the Minkowski-Weyl theorem (see [Gue, 7.13.]), every polyhedron is of the form

$$\operatorname{conv}(\alpha_1, \ldots, \alpha_s) + \operatorname{cone}(v_1, \ldots, v_t)$$

with $s, t \in \mathbb{N}_0$, $\alpha_1, \ldots, \alpha_s, v_1, \ldots, v_t \in \mathbb{R}^n$. If one applies it on some Newton polyhedron Δ, then one has $\Delta \subseteq \mathbb{R}^n_+$ and $\Delta + \mathbb{R}^n_+ = \Delta$, so $\mathbb{R}^n_+ = \operatorname{cone}(v_1, \ldots, v_t)$. After removing superfluous elements from $\alpha_1, \ldots, \alpha_s$, one obtains $\Delta = \operatorname{conv}(\bigcup_{i=1}^s \alpha_i + \mathbb{R}^n_+)$ where the α_i are the vertices of Δ. One has $\alpha_1, \ldots, \alpha_s \in \mathbb{N}_0^n$. This shows that for every Newton polyhedron Δ with corresponding Newton diagram Γ we have $\Delta = \Delta(g)$ and $\Gamma = \Gamma(g)$ where $g := g(\Gamma)$ is the canonical polynomial of Γ. Especially, Δ is uniquely determined by Γ.

We have an intuitive lemma and a corollary, which are of importance for us to transfer results of Vassiliev to our general setting.

Lemma 3.3.3. *Let Δ be a Newton polyhedron. Then, there is some $d_0 \in \mathbb{N}_0$ such that for all $d \geq d_0$ and faces τ of $\Gamma(\Delta)$, the Newton diagram $\Gamma(\Delta')$ corresponding to the Newton polyhedron*

$$\Delta' := \operatorname{conv}\left(\Delta \cup \bigcup_{i=1}^n \left(de_i + \mathbb{R}^n_+\right)\right)$$

has τ as a face. Especially, we have $\Gamma(\Delta) \subseteq \Gamma(\Delta')$. Here, e_i denotes the i-th vector in the standard basis of $\mathbb{R}^n$.

Proof. Let $\alpha_1, \dots, \alpha_s$ be the vertices of Δ and let us assume without loss of generality that the vertices of a considered face τ of $\Gamma(\Delta)$ are $\alpha_1, \dots, \alpha_t$ with $1 \leq t \leq s$. Let us consider first some fixed $d \in \mathbb{N}_0$ with $d > |\alpha|$ for all $\alpha \in \tau$. By 3.3.2, each $\alpha \in \Delta'$ may be written as

$$\alpha = \sum_{j=1}^{s} \lambda_j \cdot \alpha_j + \sum_{k=1}^{n} \mu_k \cdot de_k + r \in \operatorname{conv}(\alpha_1, \dots, \alpha_s, de_1, \dots, de_k) + \mathbb{R}_+^n \qquad (*)$$

with $\lambda_j \geq 0, \mu_k \geq 0$ for $j = 1, \dots, s, k = 1, \dots, n$, $\sum_{j=1}^{s} \lambda_j + \sum_{k=1}^{n} \mu_k = 1$ and $r \in \mathbb{R}_+^n$.
Let us show that τ is a face of Δ' if and only if $\sum_{j=1}^{t} \lambda_j = 1$ for each representation of each $\alpha \in \tau$. Suppose first that τ is a face of Δ' and $\alpha \in \tau$. Then, $\alpha = \frac{1}{2}(\alpha - r) + \frac{1}{2}(\alpha + r)$ with $\alpha - r, \alpha + r \in \Delta'$ implies $\alpha - r \in \tau$ and thus, $\alpha_j \in \tau$ if $\lambda_j > 0$ and $de_k \in \tau$ if $\mu_k > 0$ $(j = 1, \dots, s, k = 1, \dots, n)$, so $\sum_{j=1}^{t} \lambda_j = 1$ as $\alpha_{t+1}, \dots, \alpha_s, de_1, \dots, de_n \notin \tau$. On the other hand, if τ is no face of Δ', then we find $\alpha', \alpha'' \in \Delta' \setminus \tau$ with $\alpha = \frac{\alpha' + \alpha''}{2} \in \tau$. Now, representations of α', α'' as in $(*)$ lead to such a representation for α. Suppose that α', α'' are represented as $\sum_{j=1}^{t} \lambda_j' \cdot \alpha_j + r'$ and $\sum_{j=1}^{t} \lambda_j'' \cdot \alpha_j + r''$ as in $(*)$ such that we have representations with $\sum_{j=1}^{t} \lambda_j' = \sum_{j=1}^{t} \lambda_j'' = 1$. This is not possible as we would have $\alpha = \beta + \frac{r' + r''}{2}$ with $\alpha, \beta \in \tau$, which implies $\frac{r' + r''}{2} = 0$ (as τ is a face of $\Gamma(\Delta)$) and thus $r' = r'' = 0$, so $\alpha', \alpha'' \in \tau$. Hence, we find some representation of α with $\sum_{j=1}^{t} \lambda_j < 1$ as claimed.
Let us call representations of $\alpha \in \tau$ with $\sum_{j=1}^{t} \lambda_j < 1$ *bad representations* as they certify that τ is not a face of Δ'. Given a bad representation of $\alpha \in \tau$, we set $\nu := \sum_{j=1}^{t} \lambda_j < 1$ and $\beta := \frac{1}{\nu} \sum_{j=1}^{t} \lambda_j \cdot \alpha_j \in \tau$. Now, if $\frac{1}{1-\nu'} (\alpha - \nu'\beta) \in \tau$ for some $0 < \nu' \leq \nu$, then we get another bad representation

$$\frac{1}{1-\nu'} \big(\alpha - \nu'\beta\big) = \frac{\nu - \nu'}{1-\nu'}\beta + \sum_{j=t+1}^{s} \frac{\lambda_j}{1-\nu'} \cdot \alpha_j + \sum_{k=1}^{n} \frac{\mu_k}{1-\nu'} \cdot de_k + \frac{1}{1-\nu'} \cdot r$$

since $\nu = 1$ if and only if $\frac{\nu - \nu'}{1-\nu'} = 1$. Thus, if they exist, there are bad representation with $\nu = 0$ (if one can choose $\nu' = \nu$) or $\frac{1}{1-\epsilon} (\alpha - \epsilon\beta) \notin \tau$ for small $\epsilon > 0$ (if $\nu' = \nu$ is not possible and one chooses ν' maximal). Note also that a bad representation for some $d \in \mathbb{N}_0$ gives bad representations for smaller d by fixing ν, β, μ_k for $k = 1, \dots, n$ and changing r appropriately.
Let us give an indirect proof of the claim. So let us assume that there is some face τ of $\Gamma(\Delta)$ such that it is no face of Δ' for some arbitrary high $d \in \mathbb{N}$ and we choose an $\subseteq$-minimal such τ. Thus, we find $0 \leq \nu_d < 1$, $\beta_d \in \tau$, $\mu_{k,d} \geq 0$ for $k = 1, \dots, n$, $r \in \mathbb{R}_+^n$ such that $\nu_d + \sum_{k=1}^{n} \mu_{k,d} = 1$, which give bad representations of some elements $\alpha_d \in \tau$ for arbitrary high d. By the properties of bad representations shown above, we may assume that these bad representations exist for each $d \in \mathbb{N}_0$ and $\nu_d = 0$ as $\alpha_d \in \operatorname{rai}(\tau)$ by minimality of τ (for $d \in \mathbb{N}_0$).
Since $[0, 1]$ and τ are compact, we may assume that (some subsequences and without loss of generality the whole sequences) α_d, $\lambda_{j,d}$, $\mu_{k,d}$ converge for $d \to \infty$ to α, λ_j and μ_k (for $j = t + 1, \dots, s$, $k = 1, \dots, n$). Since $|\alpha_d|$ is bounded, we necessarily have $\mu_k = 0$ for $k = 1, \dots, n$. This gives

$$\alpha \geq \sum_{j=t+1}^{s} \lambda_j \cdot \alpha_j$$

with $\alpha \in \tau$, $\lambda_j \geq 0$ for $j = t + 1, \dots, s$ and $\sum_{j=t+1}^{s} \lambda_j = 1$. This contradicts that τ is a face of Δ with vertices $\alpha_1, \dots, \alpha_t$ and thus shows the claim. $\square$

Corollary 3.3.4. *Let $f \in \mathbb{R}[[x]]_+$ with $f(0) = 0$, d_0 as in lemma 3.3.3 with respect to $\Delta(f)$ and $h := x_1^2 + \cdots + x_n^2$. Then, for all $d \in \mathbb{N}_0$ with $2d > d_0$, the unique polynomial $g \in \mathbb{R}[\underline{x}]_{\leq 2d}$ with $f + h^d - g \in \mathfrak{m}^{2d+1}$ has an isolated local minimum at 0, $\Gamma(f) \subseteq \Gamma(g)$, each face of $\Gamma(f)$ is a face of $\Gamma(g)$ and $f^{\Gamma(f)} = g^{\Gamma(f)}$.*

Proof. Let $g' := g - \frac{1}{2}h^d$, so $f + \frac{1}{2}h^d - g' \in \mathfrak{m}^{2d+1}$. We have $\operatorname{carr}(f + h^d) \supseteq \operatorname{carr}(h^d)$, so

$$\operatorname{csupp}(\operatorname{carr}(f + h^d)) \supseteq \mathfrak{m}^{2d+1}$$

by 3.2.1. Clearly, $\operatorname{carr}(f + h^d) = \operatorname{carr}\left(f + \frac{1}{2}h^d\right)$ and thus $g' \in \operatorname{rai}\left(\operatorname{carr}\left(f + \frac{1}{2}h^d\right)\right) \subseteq \mathbb{R}[[x]]_+$ by 2.2.8(c). By 3.1.8, we have $g' \in \mathcal{P}^{Loc}$ and $g = g' + \frac{1}{2}h^d$ has an isolated local minimum at 0.
The claims that $\Gamma(f) \subseteq \Gamma(g)$ and that each face of $\Gamma(f)$ is a face of $\Gamma(g)$ follow directly from lemma 3.3.3. $f^{\Gamma(f)} = g^{\Gamma(f)}$ follows since each vertex α of $\Gamma(f)$ and thus each element α of $\Gamma(f) \cap \mathbb{Z}^n$ has $|\alpha| \leq d_0 < 2d$ by choice of d_0. □

We have now the following proposition with part (a) from [Vas] and the corresponding statement in our setting in (b).

Proposition 3.3.5. *(a) Let $\Gamma \neq \{0\}$ be a Newton diagram. Then, there exists some $f \in \mathbb{R}[\underline{x}]$ (resp. $f \in \mathbb{R}\{\underline{x}\}$) with $\Gamma(f) = \Gamma$ with an isolated local minimum at 0 if and only if Γ is a min-diagram.*
Furthermore, for any $f \in \mathbb{R}\{\underline{x}\}$ with isolated local minimum at 0 and $\Gamma(f) = \Gamma$ and all vertices α of Γ, the coefficient of the monomial $\underline{x}^\alpha$ in f is positive.

(b) Let Γ be a Newton diagram. Then, there exists some $f \in \mathcal{P}^{Loc}$ (resp. $f \in \mathbb{R}\{\underline{x}\}_+$ or $f \in \mathbb{R}[[\underline{x}]]_+$) with $\Gamma(f) = \Gamma$ if and only if Γ is a psd-diagram.
For any $f \in \mathbb{R}[[\underline{x}]]_+$ with $\Gamma(f) = \Gamma$ and all vertices α of Γ, the coefficient of the monomial $\underline{x}^\alpha$ in f is positive.

Proof. (a) The statement on equivalence for $f \in \mathbb{R}\{\underline{x}\}$ may be found in [Vas, 1.2.]. The case $f \in \mathbb{R}[\underline{x}]$ follows since the canonical polynomial $g(\Gamma)$ has isolated minimum in 0 if Γ is a min-diagram (see the proof of [Vas, 1.2.]). The statement on the coefficients corresponding to vertices may be found in [Vas, p.164] (see also [Sek, 3.1.]).

(b) If Γ is a psd-diagram, then clearly $g(\Gamma) \in \mathcal{P}^{Loc}$. This shows the implication "⇐" of the equivalence in all three cases. It remains to show that for all $f \in \mathbb{R}[[\underline{x}]]_+$ and vertices α of $\Gamma(f)$, we have that α has just even coordinates and the coefficient of the monomial $\underline{x}^\alpha$ is positive since this implies all other claims. By 3.3.4, we may reduce to show this claim for some $g \in \mathbb{R}[\underline{x}]$ with isolated local minimum at 0. The claim follows from (a). □

As a consequence, we see that Δ is well-behaved under addition of locally non-negative elements.

Corollary 3.3.6. *Let $f, g \in \mathcal{P}^{Loc}$ ($\mathbb{R}\{\underline{x}\}_+$ or $\mathbb{R}[[\underline{x}]]_+$). Then,*

$$\Delta(f + g) = \operatorname{conv}(\Delta(f) \cup \Delta(g)).$$

If $\operatorname{carr}(g) \subseteq \operatorname{carr}(f)$, then $\Delta(g) \subseteq \Delta(f)$. Especially, if F is a face of $\mathcal{P}^{Loc}$ ($\mathbb{R}\{\underline{x}\}_+$ or $\mathbb{R}[[\underline{x}]]_+$) and $f, g \in \operatorname{rai}(F)$, then $\Delta(f) = \Delta(g)$.

Proof. We just have to show the case $\mathbb{R}[[\underline{x}]]_+$ (see 3.1.8). Let $f, g \in \mathbb{R}[[\underline{x}]]_+$. The inclusion $\Delta(f+g) \subseteq \operatorname{conv}(\Delta(f) \cup \Delta(g))$ is obvious. Let

$$\Delta(f) = \operatorname{conv}(\alpha_1, \ldots, \alpha_s) + \mathbb{R}_+^n \quad \text{and} \quad \Delta(g) = \operatorname{conv}(\beta_1, \ldots, \beta_t) + \mathbb{R}_+^n$$

with $s, t \in \mathbb{N}_0$ and $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t \in \mathbb{N}_0^n$ the vertices (see 3.3.2). Then,

$$\operatorname{conv}(\Delta(f) \cup \Delta(g)) = \operatorname{conv}(\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t) + \mathbb{R}_+^n.$$

Let $C := \operatorname{conv}(\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t)$. We just have to show that any vertex γ of C is contained in $\Delta(f+g)$ to show the other inclusion. As $\gamma \in \{\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t\}$, γ is a vertex of $\Delta(f)$ or $\Delta(g)$ and if it is not a vertex of both, then it is even not contained in one of the Newton polyhedra. Using 3.3.5(b), this shows that the coefficient of $\underline{x}^\gamma$ in $f+g$ is positive and thus $\gamma \in \Delta(f+g)$. This shows the first claim.
Now, let us assume that $\operatorname{carr}(g) \subseteq \operatorname{carr}(f)$. Then, we find some $\epsilon > 0$ with $f - \epsilon g \in \mathbb{R}[[\underline{x}]]_+$. This gives

$$\Delta(f) = \Delta(f - \epsilon g + \epsilon g) = \operatorname{conv}(\Delta(f - \epsilon g) \cup \Delta(g)),$$

so $\Delta(f) \supseteq \Delta(g)$. □

This motivates the following definition.

Definition 3.3.7. Let F be a face of $\mathcal{P}^{Loc}$ ($\mathbb{R}\{\underline{x}\}_+$ or $\mathbb{R}[[\underline{x}]]_+$). We set $\Delta(F) := \Delta(f)$ and $\Gamma(F) := \Gamma(f)$ for any $f \in \operatorname{rai}(F)$ and call them the *Newton polyhedron of* F and the *Newton diagram of* F.
We define the *order of* F, denoted by $\operatorname{ord}(F)$, as the largest $m \in \mathbb{N}_0 \cup \{\infty\}$ such that $|\alpha| \geq m$ for $\alpha \in \Delta(F)$.

We prove some easy lemma, which shows that a property considered in 3.3.9 and 3.3.10 holds for small perturbations of $g(\Gamma)$.

Lemma 3.3.8. *Let Γ be a psd-diagram and $f \in \mathbb{R}[[\underline{x}]]$. Then, there exists some $\epsilon > 0$ such that*

$$(g(\Gamma) - \epsilon f)^\tau(x) > 0$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_1 \cdots x_n \neq 0$ and faces τ of Γ.

Proof. We have $g(\Gamma)^\tau(y) > 0$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_1 \cdots x_n \neq 0$ and faces τ of Γ. Thus, it is enough to show the claim with ≥ 0 instead of > 0. Furthermore, we just have to solve the problem for one arbitrary face τ of Γ since there are just finitely many of them. The claim is clear if $f^\tau = 0$, so we may assume $f^\tau \neq 0$. Let $\alpha_1, \ldots, \alpha_t$ be the vertices (with even coordinates) of τ and $\alpha \in \tau \cap \mathbb{Z}^n$. Then, there are $\lambda_1, \ldots, \lambda_t \geq 0$ with $\lambda_1 + \cdots + \lambda_t = 1$ such that $\alpha = \sum_{i=1}^t \lambda_i \alpha_i$. By the arithmetic-geometric inequality, we have

$$g(\tau) = \sum_{i=1}^t y^{\alpha_i} \geq \sum_{i=1}^t \lambda_i y^{\alpha_i} \geq |y^\alpha|$$

for all $y \in \mathbb{R}^n$. This implies $(g(\tau) \pm \underline{x}^\alpha)(y) \geq 0$ for all $y \in \mathbb{R}^n$. Let ϵ be the reciprocal of the sum of the absolute values of the coefficients of f^τ. Then, the inequality implies $(g(\tau) - \epsilon f)^\tau(y) \geq 0$ for all $y \in \mathbb{R}^n$ and thus the claim as $g(\Gamma)(y) \geq g(\tau)(y)$ for all $y \in \mathbb{R}^n$. □

We will also transfer the following theorem to our situation in the subsequent corollary.

Theorem 3.3.9. *[Vas, 1.5.] Let $f \in \mathbb{R}\{\underline{x}\}$ with $f(0) = 0$ and $\Gamma := \Gamma(f)$ a min-diagram.*

(a) If f has an isolated local minimum at 0, then $f^\tau \geq 0$ on $\mathbb{R}^n$ for each face τ of Γ.

(b) The following statements are equivalent:

(b_1) f has an isolated local minimum at 0 and there is some $\epsilon > 0$ with $f - \epsilon g(\Gamma) \in \mathbb{R}\{\underline{x}\}_+$

(b_2) $f^\tau(x) > 0$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_1 \cdots x_n \neq 0$ and faces τ of Γ

Corollary 3.3.10. *(a) Let $f \in \mathbb{R}[[\underline{x}]]_+$. Then, $f^\tau \geq 0$ on $\mathbb{R}^n$ for all faces τ of $\Gamma := \Gamma(f)$.*

(b) We consider $f \in \mathbb{R}[\underline{x}]$ ($\mathbb{R}\{\underline{x}\}$ or $\mathbb{R}[[\underline{x}]]$) such that $\Gamma := \Gamma(f)$ is a psd-diagram and the corresponding cone $\mathcal{P}^{Loc}$ ($\mathbb{R}\{\underline{x}\}_+$ or $\mathbb{R}[[\underline{x}]]_+$). The following statements are equivalent:

(b_1) $f \in \mathrm{rai}(\mathrm{carr}(g(\Gamma)))$

(b_2) $f^\tau(x) > 0$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_1 \cdots x_n \neq 0$ and faces τ of Γ

Proof. (a) The statement is clear if $f(0) > 0$. Let $f(0) = 0$. Then 3.3.4 reduces the problem to some g with isolated minimum at 0. The claim follows from 3.3.9(a).

(b) We just have to show the claim in the case $\mathbb{R}[[\underline{x}]]$ since it implies the other cases (see 3.1.8). If $\Gamma = \{0\}$, then the claim is clear since both statements are equivalent to $f(0) > 0$. Let us assume $\Gamma \neq \{0\}$ and let $f \in \mathrm{rai}(\mathrm{carr}(g(\Gamma)))$. 3.3.4 reduces the problem to some polynomial with isolated local minimum at 0 and the statement of (b_2) follows from 3.3.9(b).
Let us assume now that $\Gamma \neq \{0\}$ and the statement of (b_2) holds for f. We will first show that $f \in \mathbb{R}[[\underline{x}]]_+$. Let $d \in \mathbb{N}_0$ with $2d > d_0$ with d_0 as in 3.3.3 with respect to $\Delta = \Delta(f)$. Furthermore, let $\alpha \in (\Delta \setminus \Gamma) \cap \mathbb{Z}^n$, $c \in \mathbb{R}$, $h := x_1^2 + \cdots + x_n^2$ and $f' \in \mathbb{R}[\underline{x}]_{\leq 2d}$ be the unique polynomial with $f - f' \in \mathfrak{m}^{2d+1}$. By 3.3.9(b),

$$f' + c\underline{x}^\alpha + h^d$$

has an isolated local minimum at 0. Since this holds for arbitrary large d, we have $f' + c\underline{x}^\alpha \in \mathbb{R}\{\underline{x}\}_+$. (Indeed, this follows as if $f' + c\underline{x}^\alpha \notin \mathbb{R}\{\underline{x}\}_+$, then there was some semi-analytic curve that can be parametrized such that $f' + c\underline{x}^\alpha$ gets a negative leading coefficient in this parametrization and elements of $\mathfrak{m}^{2d}$ for large degree will get leading coefficients of large degree such that the sum may not be in $\mathbb{R}\{\underline{x}\}_+$.) Hence, $\underline{x}^\alpha \in \mathrm{csupp}(\mathrm{carr}_{\mathbb{R}\{\underline{x}\}_+}(f'))$ by 2.2.8(c) as c was arbitrary. Now, 2.2.8(c) and 3.1.8 show that $\underline{x}^\alpha \in \mathrm{csupp}(\mathrm{carr}_{\mathbb{R}[[\underline{x}]]_+}(f'))$. By construction, $f - f'$ lies in the ideal generated by the monomials $\underline{x}^\alpha$ with $\alpha \in (\Delta \setminus \Gamma) \cap \mathbb{Z}^n$, so $f - f' \in \mathrm{csupp}(\mathrm{carr}_{\mathbb{R}[[\underline{x}]]_+}(f'))$ and hence

$$f = f' + f - f' \in \mathrm{rai}(\mathrm{carr}_{\mathbb{R}[[\underline{x}]]_+}(f')) \subseteq \mathbb{R}[[\underline{x}]]_+.$$

By 3.3.9(b), $f' - \epsilon g(\Gamma) \in \mathbb{R}\{\underline{x}\}_+ \subseteq \mathbb{R}[[\underline{x}]]_+$ for some $\epsilon > 0$ and thus $\mathrm{carr}_{\mathbb{R}[[\underline{x}]]_+}(f') \supseteq \mathrm{carr}_{\mathbb{R}[[\underline{x}]]_+}(g(\Gamma))$. By 3.3.8, we find some $\epsilon > 0$ such that (b_2) holds for $g(\Gamma) - \epsilon f'$. Applying the arguments from before on $g(\Gamma) - \epsilon f'$ instead of f, we obtain $g(\Gamma) - \epsilon f' \in \mathbb{R}[[\underline{x}]]_+$, so $\mathrm{carr}_{\mathbb{R}[[\underline{x}]]_+}(f') \subseteq \mathrm{carr}_{\mathbb{R}[[\underline{x}]]_+}(g(\Gamma))$ and thus

$$\mathrm{carr}_{\mathbb{R}[[\underline{x}]]_+}(f) = \mathrm{carr}_{\mathbb{R}[[\underline{x}]]_+}(f') = \mathrm{carr}_{\mathbb{R}[[\underline{x}]]_+}(g(\Gamma)).$$

□

3.3.2 Monomial faces and their subfaces

We will now study monomial faces and proper faces of them. Monomial faces are faces that may be characterized by the three equivalent conditions of the following theorem. Each subface G of a monomial face F induces a face of the cone of non-negative polynomials with support in a given face of $\Gamma(F)$. This observation will be used later to construct subfaces G and to characterize $\subseteq$-maximal proper faces G of F. Let us start with the announced characterizations of monomial faces.

Theorem 3.3.11. *Let F be a face of $\mathcal{P}^{Loc}$ ($\mathbb{R}\{\underline{x}\}_+$ or $\mathbb{R}[[\underline{x}]]_+$). Then, the following are equivalent:*

(a) F is the $\subseteq$-maximal face with its Newton diagram (or Newton polyhedron)

(b) $F = \operatorname{carr}(g(\Gamma(F)))$

(c) There is some family m_i of monomials indexed by some set I such that

$$F = \operatorname{carr}(\{m_i^2 \mid i \in I\})$$

Proof. The proof is the same in all three cases. Clearly, we have $\Gamma(F) = \Gamma(\operatorname{carr}(g(\Gamma(F))))$. Thus, the equivalence between (a) and (b) follows from the claim that $G := \operatorname{carr}(g(\Gamma(F)))$ is the $\subseteq$-maximal face with its Newton diagram. Let $G_1 \supseteq G$ with $\Gamma := \Gamma(G) = \Gamma(G_1)$, $g := g(\Gamma) \in \operatorname{rai}(G)$ and $g_1 \in \operatorname{rai}(G_1)$. Let τ be a face of Γ. Then, 3.3.10(a) shows $g_1^\tau \geq 0$ on $\mathbb{R}^n$. By 3.3.10(b), $g \in \operatorname{rai}(G)$ implies $g + g_1 \in \operatorname{rai}(G)$. This shows $G \boxplus G_1 = G$, thus $G_1 = G$ and the equivalence (a)$\Leftrightarrow$(b).
The implication (b)$\Rightarrow$(c) is obvious since $g(\Gamma(F))$ is a sum of squares of monomials. Let us assume that (c) holds. Since the considered cones have a Noetherian facial structure, there is a finite subset $J \subseteq I$ with

$$F = \operatorname{carr}\left(\{m_i^2 \mid i \in J\}\right) = \operatorname{carr}\left(\sum_{i \in J} m_i^2\right).$$

We have $\Gamma(F) = \Gamma(\sum_{i\in J} m_i^2)$ and thus, $\{m_i^2 \mid i \in J\}$ contains all monomials that represent vertices of $\Gamma(F)$. Hence, $\left(\sum_{i\in J} m_i^2\right) - g(\Gamma(F)) \in \mathcal{P}^{Loc}$. This shows $F \supseteq \operatorname{carr}(g(\Gamma(F)))$. As (a)$\Leftrightarrow$(b), we have always $F \subseteq \operatorname{carr}(g(\Gamma(F)))$, which implies (b). □

Definition 3.3.12. We call faces of $\mathcal{P}^{Loc}$ ($\mathbb{R}\{\underline{x}\}_+$ or $\mathbb{R}[[\underline{x}]]_+$) that fulfil the three equivalent conditions of 3.3.11 *monomial faces.*

We observe that monomial faces have nice properties with respect to all the operations in the following remark.

Remark 3.3.13. 3.3.11(c) shows that monomial faces are closed under the operations $\boxplus$ and $\boxdot$. 3.3.11(a) shows that monomial faces are closed under intersection.

Now, we may even characterize the relative algebraic interior, the algebraic closure, the linear span and the support of the algebraic closure of monomial faces.

Theorem 3.3.14. *Let F be a monomial face of $\mathcal{P}^{Loc}$ ($\mathbb{R}\{\underline{x}\}_+$ or $\mathbb{R}[[\underline{x}]]_+$) and f some element of the considered ring. We have $f \in \operatorname{rai}(F)$ if and only if*

- $\Gamma(f) = \Gamma(F)$
- *and* $f^\tau(x) > 0$ *for all* $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *with* $x_1 \cdots x_n \neq 0$ *and faces* τ *of* $\Gamma(F)$.

We have $f \in \mathrm{ac}(F)$ *if and only if*

- $\Delta(f) \subseteq \Delta(F)$
- *and* $f^\tau(x) \geq 0$ *for all* $x \in \mathbb{R}^n$ *and faces* τ *of* $\Gamma(F)$.

Furthermore, we have

$$\begin{aligned} f \in \mathrm{span}(F) &\quad\Leftrightarrow\quad \mathrm{supp}(f) \subseteq \Delta(F), \\ f \in \mathrm{csupp}(F) &\quad\Leftrightarrow\quad \mathrm{supp}(f) \subseteq \Delta(F) \setminus \Gamma(F). \end{aligned}$$

Proof. The characterization of $f \in \mathrm{rai}(F)$ follows from 3.3.10(b). From this, it is obvious that each $f \in \mathrm{ac}(F)$ fulfils the stated conditions. Let us consider now some f with $\Delta(f) \subseteq \Delta(F)$ and $f^\tau(x) \geq 0$ for all $x \in \mathbb{R}^n$ and faces τ of $\Gamma(F)$. Then, by the characterization of elements in $\mathrm{rai}(F)$, we have $f + \epsilon g(\Gamma(F)) \in \mathrm{rai}(F)$ for each $\epsilon > 0$. This shows $f \in \mathrm{ac}(F)$ and thus the characterization of elements from $\mathrm{ac}(F)$.
$f \in \mathrm{span}(F)$ implies $\mathrm{supp}(f) \subseteq \Delta(F)$ by 2.1.7(c). We have $g := g(\Gamma(F)) \in \mathrm{rai}(F)$. Let f be given with $\mathrm{supp}(f) \subseteq \Delta(F)$. By 3.3.8 and 3.3.10(b), we have $g - \epsilon f \in \mathrm{rai}(F)$ for some $\epsilon > 0$ and thus $f \in \mathrm{span}(F)$. This shows the characterization of $f \in \mathrm{span}(F)$.
By 2.2.8(c), $\mathrm{supp}(f) \subseteq \Delta(F) \setminus \Gamma(F)$ implies $f \in \mathrm{csupp}(F)$. Let $f \in \mathrm{csupp}(F)$. Then, $f \in \mathrm{span}(F)$, so $\mathrm{supp}(f) \subseteq \Delta(F)$. If $\alpha \in \mathrm{supp}(f) \cap \Gamma(F)$ and $\alpha \in \tau$ for some face τ of Γ, then $f^\tau \neq 0$ and it is not possible that $f^\tau \geq 0$ on $\mathbb{R}^n$ and $(-f)^\tau \geq 0$ on $\mathbb{R}^n$. This shows $f \notin \mathrm{csupp}(F)$. Hence, we have $\mathrm{supp}(f) \subseteq \Delta(F) \setminus \Gamma(F)$ and thus the claim holds. □

Let us discuss proper faces of monomial faces as announced before. We need the following definition and we have two easy observations in the subsequent remarks.

Definition 3.3.15. Let $\tau \subseteq \mathbb{R}^n_+$ and $M \subseteq \mathbb{R}[[\underline{x}]]$. We set

$$M^\tau := \{g^\tau \mid g \in M\} \subseteq \{f \in \mathbb{R}[[\underline{x}]] \mid \mathrm{supp}(f) \subseteq \tau\}.$$

Let Γ be a psd-diagram with corresponding monomial face F (of $\mathcal{P}^{Loc}$, $\mathbb{R}\{\underline{x}\}_+$ or $\mathbb{R}[[\underline{x}]]_+$) and $\mathcal{P} \subseteq \mathbb{R}[\underline{x}]$ be the cone of globally non-negative polynomials (see 1.7.1). If $G \subseteq F$ and τ is some face of Γ, then we call the face (see 3.3.10(a))

$$\tau(G) := \mathrm{carr}_{\mathcal{P}_\tau}(G^\tau)$$

of the cone $\mathcal{P}_\tau := \mathcal{P} \cap \mathbb{R}[\underline{x}]^\tau$ the *face induced by* G *on* τ. Furthermore, we call the mapping $\tau \mapsto \tau(G)$ (defined on the set of faces of Γ) the *face-wise* Γ*-information of* G.

Remark 3.3.16. $\mathbb{R}[\underline{x}]^\tau$ consists of quasi-homogeneous polynomials. Indeed, each hyperplane separating τ from $\Delta(F)$ defined by an equation over the integers induces corresponding weights, which automatically have the same sign as $\Delta(F) = \Delta(F) + \mathbb{R}^n_+$ (and τ is a compact face) and so they even may be chosen positive.

Remark 3.3.17. We will use this definition especially if G is a face of F, which means that G is a face of the considered cone and $\Delta(G) \subseteq \Delta(\Gamma)$. Note that for any face τ of Γ each face G_τ of $\mathcal{P}_\tau$ appears as $\tau(G)$ for some appropriate face G of F. Indeed, if one takes $g \in \mathrm{rai}(G_\tau)$ and $G := \mathrm{carr}_F(g)$, then $g \in \mathrm{rai}(\tau(G))$ as we will see in 3.3.18(b) and thus $\tau(G) = G_\tau$.

The following lemma sums up easy but crucial properties of the previous definition.

Lemma 3.3.18. *Let F be a monomial face of $\mathcal{P}^{Loc}$ ($\mathbb{R}\{\underline{x}\}_+$ or $\mathbb{R}[[\underline{x}]]_+$) with Newton diagram $\Gamma := \Gamma(F)$.*

(a) If $\tau_1, \tau_2 \subseteq \mathbb{R}^n_+$ and $M \subseteq \mathbb{R}[[\underline{x}]]$, then $(M^{\tau_1})^{\tau_2} = M^{\tau_1 \cap \tau_2}$

(b) If $G \subseteq F$, τ is some face of Γ and $g \in \operatorname{rai}(G)$, then $g^\tau \in \operatorname{rai}(G^\tau)$ and $g^\tau \in \operatorname{rai}(\tau(G))$.

(c) For any two faces $\tau_1 \subseteq \tau_2$ of Γ and $G \subseteq F$ with $\operatorname{rai}(G) \neq \emptyset$, we have

$$\tau_1(G) = \tau_1(\tau_2(G)).$$

Especially, the face-wise Γ-information of a face G of F is determined by $\tau(G)$ for $\subseteq$-maximal faces of Γ.

(d) For any two faces G_1 and G_2 of F and faces τ of Γ, we have

$$\tau(G_1 \boxplus G_2) = \tau(G_1) \boxplus \tau(G_2).$$

(e) If G is a proper face of F, then there is some face τ of Γ with $\tau(G) \neq \mathcal{P}_\tau$.

Proof. (a) This is obvious.

(b) 2.2.11 implies $g^\tau \in \operatorname{rai}(G^\tau)$. The other claim follows from 2.2.10.

(c) Let $g \in \operatorname{rai}(G)$. We have $g^{\tau_1} \in \operatorname{rai}(\tau_1(G))$ and $g^{\tau_2} \in \operatorname{rai}(\tau_2(G))$ by (b). Another application of (b) gives $g^{\tau_1} = (g^{\tau_2})^{\tau_1} \in \operatorname{rai}(\tau_1(\tau_2(G)))$. Hence, g^{τ_1} is in the relative interior of both sides, which are both faces of $\mathcal{P}_{\tau_1}$ and thus equal.

(d) This follows from (b) (and 2.3.22).

(e) Let $g \in \operatorname{rai}(G)$. By 3.3.14, there is some face τ of Γ such that $g^\tau(y) = 0$ for some $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ with $y_1 \cdots y_n \neq 0$. Thus, we have $g^\tau \notin \operatorname{rai}(\mathcal{P}_\tau)$ and hence $\tau(G) \neq \mathcal{P}_\tau$.

□

In the following remark, we see that the faces $\tau(G)$ are already determined if one considers the projection of G on ${}^{\operatorname{span}(F)}/_{\operatorname{csupp}(F)}$ or, differently stated, if one considers G inside $\operatorname{ac}(F)$.

Remark 3.3.19. Let Γ, F, G be as in definition 3.3.15, τ some face of Γ and $g \in \operatorname{rai}(G)$, so $g^\tau \in \operatorname{rai}(\tau(G))$ by 3.3.18(b). Then, we also have $g \in \operatorname{rai}(\operatorname{carr}_{\operatorname{ac}(F)}(G))$ by 2.2.10 and thus, $g^\tau \in \operatorname{rai}((\operatorname{carr}_{\operatorname{ac}(F)}(G))^\tau)$ by 2.2.11. Thus, we have

$$\tau(G) = \operatorname{carr}_{\mathcal{P}_\tau}\left((\operatorname{carr}_{\operatorname{ac}(F)}(G))^\tau\right).$$

and the right hand side is well-defined since $(\operatorname{carr}_{\operatorname{ac}(F)}(G))^\tau \subseteq (\operatorname{ac}(F))^\tau = \mathcal{P}_\tau$ by 3.3.14. This shows that the faces $\tau(G)$ come from faces of $\operatorname{ac}(F)$, which is a way simpler cone than F (see 2.2.13, 2.4.24, 3.1.11).

Let us illustrate the construction of faces $\tau(G)$ by two examples.

Example 3.3.20. Let $n = 2$. We use the variables x and y. Let F be the monomial face $F = \operatorname{carr}(x^6, x^2y^2, y^4)$,

$$g_1 := x^6 - 2x^4y + x^2y^2 + y^4, \quad g_2 := x^6$$

and $G_i := \operatorname{carr}(g_i)$ for $i = 1, 2$. Then, $\Gamma := \Gamma(F)$ has 5 faces.

- If $\tau = \{(6, 0)\}$, then $\tau(G_1) = \tau(G_2) = \mathbb{R}_+ \cdot x^6$.
- If $\tau = \{(2, 2)\}$, then $\tau(G_1) = \mathbb{R}_+ \cdot x^2y^2$ and $\tau(G_2) = \{0\}$.
- If $\tau = \{(0, 4)\}$, then $\tau(G_1) = \mathbb{R}_+ \cdot y^4$ and $\tau(G_2) = \{0\}$.
- If $\tau = \operatorname{conv}((6, 0), (2, 2))$, then $\tau(G_1) = \mathbb{R}_+ \cdot (x^3 - xy)^2$ and $\tau(G_2) = \mathbb{R}_+ \cdot x^6$.
- If $\tau = \operatorname{conv}((2, 2), (0, 4))$, then $\tau(G_1) = \{ax^2y^2 + bxy^3 + cy^4 \mid 4ac - b^2 \geq 0, a \geq 0, c \geq 0\}$ and $\tau(G_2) = \{0\}$.

Example 3.3.21. Let $n = 3$. We use the variables x, y and z. Let F be the monomial face $F = \operatorname{carr}(x^6, y^6, z^6)$. We have the Motzkin form

$$g := x^4y^2 + x^2y^4 - 3x^2y^2z^2 + z^6$$

and $G := \operatorname{carr}(g)$. Then, $\Gamma := \Gamma(F)$ has 7 faces.

- If $\tau = \{(6, 0, 0)\}$ or $\tau = \{(0, 6, 0)\}$, then $\tau(G) = \{0\}$.
- If $\tau = \{(0, 0, 6)\}$, then $\tau(G) = \mathbb{R}_+ \cdot z^6$.
- If $\tau = \operatorname{conv}((6, 0, 0), (0, 6, 0))$, then

$$\tau(G) = \{ax^4y^2 + bx^3y^3 + cx^2y^4 \mid 4ac - b^2 \geq 0, a \geq 0, c \geq 0\}.$$

- If $\tau = \operatorname{conv}((6, 0, 0), (0, 0, 6))$ or $\tau = \operatorname{conv}((0, 6, 0), (0, 0, 6))$, then $\tau(G) = \mathbb{R}_+ \cdot z^6$.
- If $\tau = \Gamma$, then $\tau(G) = \mathbb{R}_+ \cdot g$ as one can show that this is an extreme ray of the cone $\mathcal{P}_\tau$ (see 3.3.22 and [ChLa2, 3.4.]).

In the following example, we consider the case $F = \mathfrak{m}^{2d} \cap \mathcal{P}^{Loc}$, so locally non-negative polynomials vanishing to $2d$-th order.

Example 3.3.22. Let $n \geq 2$, $d \in \mathbb{N}_0$, $F := \mathfrak{m}^{2d} \cap \mathcal{P}^{Loc}$ with $\mathfrak{m} = (x_1, \ldots, x_n)$, $\Gamma := \Gamma(F)$ and $\tau := \Gamma$. We identify $\mathcal{P}_\tau$ with the cone of globally non-negative forms $\mathcal{P}_{n-1,2d}$ via the isomorphism $\mathbb{R}[\underline{x}] \to \mathbb{R}[X_0, \ldots, X_{n-1}]$ that maps $x_i \mapsto X_{i-1}$ for $i = 1, \ldots, n$. Thus, any face G of F gives some face $\tau(G)$ of $\mathcal{P}_{n-1,2d}$. Furthermore, this gives a bijection between faces G of F with $\mathfrak{m}^{2d+2} \cap \mathcal{P}^{Loc} \subseteq G$ and faces of $\mathcal{P}_{n-1,2d}$.
To show surjectivity, let G_0 be some face of $\mathcal{P}_{n-1,2d}$ and $g_0 \in \operatorname{ri}(G)$. Then, $G := \operatorname{carr}(g_0 + h^{d+1})$ with $h := x_1^2 + \cdots + x_n^2$ fulfils $\mathfrak{m}^{2d+2} \cap \mathcal{P}^{Loc} \subseteq G$ and $\tau(G) = G_0$ by 3.3.18(b).
To show injectivity, let G be some face of F with $\mathfrak{m}^{2d+2} \cap \mathcal{P}^{Loc} \subseteq G$ and $g \in \operatorname{rai}(G)$ with $g \in \mathbb{R}[\underline{x}]_{\leq 2d+2}$ (this exists since $\mathfrak{m}^{2d+3} \subseteq \operatorname{csupp}(G)$). Then, $g = g_{2d} + g_{2d+1} + g_{2d+2}$ with g_i homogeneous of degree i. We have $g_{2d} \in \mathcal{P}_{n-1,2d}$ and we may assume that $g_{2d+2} \in \operatorname{rai}(\mathcal{P}_{n-1,2d+2})$

(using the identification from above in degree $2d+2$) by adding some appropriate element of $\mathcal{P}_{n-1,2d+2} \subseteq \mathfrak{m}^{2d+2} \cap \mathcal{P}^{Loc}$ to g if necessary (e.g. a multiple of h^{d+1}). We have now

$$g' := g_{2d} + 2g_{2d+1} + 4g_{2d+2} = \frac{1}{2^{2d}} g(2\underline{x}) \in \mathcal{P}^{Loc}.$$

Since $g' + g_{2d} = 2g + 2g_{2d+2} \in G$ and $g_{2d} \in \mathcal{P}^{Loc}$, we have $g' \in G$ as G is a face of $\mathcal{P}^{Loc}$. However, we also have $g + 3g_{2d+2} \in G$ and thus $g_{2d+1} = g' - (g + 3g_{2d+2}) \in \operatorname{span}(G)$. As $g_{2d+2} \in \operatorname{span}(G)$ (and $g \in \operatorname{span}(G)$), this also gives $g_{2d} \in \operatorname{span}(G)$ and

$$g'' := g_{2d} + g_{2d+2} \in \operatorname{span}(G) \cap \mathcal{P}^{Loc} = G.$$

We have $g'' = g(\underline{x}) + g(-\underline{x})$ with $g(-\underline{x}) \in \mathcal{P}^{Loc}$ and thus $\operatorname{carr}_{\mathcal{P}^{Loc}}(g'') \supseteq \operatorname{carr}_{\mathcal{P}^{Loc}}(g) = G$, so $\operatorname{carr}_{\mathcal{P}^{Loc}}(g'') = G$. Now, $g_{2d} \in \operatorname{rai}(\tau(G)) \subseteq \mathcal{P}^{Loc}$ (see 3.3.18(b)) and $g_{2d+2} \in \operatorname{rai}(\mathcal{P}_{n-1,2d+2}) \subseteq \mathcal{P}^{Loc}$ show that

$$G = \operatorname{carr}_{\mathcal{P}^{Loc}}(g'') = \operatorname{carr}_{\mathcal{P}^{Loc}}(g_{2d} + g_{2d+2}) = \operatorname{carr}_{\mathcal{P}^{Loc}}(\tau(G) \cup \mathcal{P}_{n-1,2d+2}).$$

This shows injectivity. Clearly, this example carries over to the cones $\mathbb{R}\{\underline{x}\}_+$ and $\mathbb{R}[[\underline{x}]]_+$ instead of $\mathcal{P}^{Loc}$.

We have seen how to construct faces $\tau(G)$ of $\mathcal{P}_\tau$ from some face G of a monomial face F (for each face τ of $\Gamma(F)$). Now, we want to reverse this process and construct faces G of F from faces of the cones $\mathcal{P}_\tau$ for each ($\subseteq$-maximal) face τ of $\Gamma(F)$ (remember 3.3.18(c) and 3.3.22). We will also ask how to construct $\subseteq$-maximal proper faces of faces G of F.

Proposition 3.3.23. *Let F be a monomial face of $\mathcal{P}^{Loc}$ ($\mathbb{R}\{\underline{x}\}_+$ or $\mathbb{R}[[\underline{x}]]_+$) with Newton diagram $\Gamma := \Gamma(F)$.*

(a) Let $\tau_1 \subseteq \tau_2$ be faces of Γ and G_{τ_1} be a face of $\mathcal{P}_{\tau_1}$. Then,

$$H := \{g \in \mathcal{P}_{\tau_2} \mid g^{\tau_1} \in G_{\tau_1}\}$$

is a face of $\mathcal{P}_{\tau_2}$ and $\tau_1(H) = G_{\tau_1}$.

(b) Let G_τ be some face of $\mathcal{P}_\tau$ for each face τ of Γ. Then,

$$G := \{g \in F \mid g^\tau \in G_\tau \text{ for all faces } \tau \text{ of } \Gamma\}$$

is the $\subseteq$-maximal face of F with $\tau(G) \subseteq G_\tau$ for all faces τ of Γ. Especially, within all mappings $\tau \mapsto G_\tau$ that induce a given face G, there is a smallest one (with respect to inclusion for each τ) and this is given by $\tau \mapsto \tau(G)$.

(c) We consider some face G_2 of F. If G_1 is an $\subseteq$-maximal proper face of G_2, then exactly one of the following cases holds

- *$\tau(G_1) = \tau(G_2)$ for all faces τ of Γ*
- *there is some $\subseteq$-maximal face τ_0 of Γ and an $\subseteq$-maximal proper face H of $\tau_0(G_2)$ such that $G_1 = G_2 \cap G$ where G is given via the mapping $\tau_0 \mapsto H$ and $\tau \mapsto \tau(G_2)$ for $\tau \neq \tau_0$ as described in (b).*

Proof. (a) Clearly, H is a convex cone in $\mathcal{P}_{\tau_2}$. Furthermore, if $f_1, f_2 \in \mathcal{P}_{\tau_2}$ with $\frac{f_1^{\tau_1}+f_2^{\tau_1}}{2} = \left(\frac{f_1+f_2}{2}\right)^{\tau_1} \in G_{\tau_1}$, then $f_1^{\tau_1}, f_2^{\tau_1} \in G_{\tau_1}$ since G_{τ_1} is a face of $\mathcal{P}_{\tau_1}$. Thus, H is a face of $\mathcal{P}_{\tau_2}$. We have $H^{\tau_1} = G_{\tau_1}$ by definition and since $G_{\tau_1} \subseteq H$. This implies $\tau_1(H) = G_{\tau_1}$.

(b) We show first that G is indeed a face of F. Let $p_\tau : \mathbb{R}[\underline{x}] \to \mathbb{R}[\underline{x}]^\tau, f \mapsto f^\tau$ be the projection for faces τ of Γ. G is the intersection of F and the cones $p_\tau^{-1}(G_\tau)$ for faces τ of Γ and thus itself a cone in F. Let $f_1, f_2 \in F$ with $\frac{f_1+f_2}{2} \in G$ and τ be some face of Γ. Then, $\frac{f_1^\tau+f_2^\tau}{2} = \left(\frac{f_1+f_2}{2}\right)^\tau \in G_\tau$ by definition of G. Since G_τ is a face of $\mathcal{P}_\tau$ and $f_1, f_2 \in \mathcal{P}_\tau$ (see 3.3.10(a)), we have $f_1, f_2 \in G_\tau$. Since this holds for all faces τ, we get $f_1, f_2 \in G$. Thus, G is a face of F.
Let τ be a face of Γ. By definition of G, we have $G^\tau \subseteq G_\tau$. Since G_τ is a face of $\mathcal{P}_\tau$, we also have $\tau(G) := \mathrm{carr}_{\mathcal{P}_\tau}(G^\tau) \subseteq G_\tau$. Let H be a face of F with $\tau(H) \subseteq G_\tau$ for all faces τ of G and $g \in H$. Then, $g^\tau \in H^\tau \subseteq \tau(H) \subseteq G_\tau$ for all faces τ of Γ, so $g \in G$. This shows $H \subseteq G$ and the claimed maximality of G.

(c) Clearly, $\tau(G_1) \subseteq \tau(G_2)$ for all faces τ of Γ. Let us assume that there is some face τ_0 of Γ with $\tau_0(G_1) \subsetneq \tau_0(G_2)$. Let H be an $\subseteq$-maximal face of $\tau_0(G_2)$ with $\tau_0(G_1) \subseteq H \subsetneq \tau_0(G_2)$ and G be the face of F determined by the mapping $\tau_0 \mapsto G_{\tau_0} := H$ and $\tau \mapsto G_\tau := \tau(G_2)$ for $\tau \neq \tau_0$ as described in (b). We have $G_1 \subseteq G$ since G is the $\subseteq$-maximal face of F with $\tau(G) \subseteq G_\tau$ for all faces τ of Γ by (b). Now, $G_1 \subseteq G_2$ implies $G_1 \subseteq G \cap G_2$ and $\tau_0(G) \subseteq G_{\tau_0} \subsetneq \tau_0(G_2)$ implies $G \cap G_2 \subsetneq G_2$. Since G_1 is an $\subseteq$-maximal proper face of G_2, we get $G_1 = G \cap G_2$, which shows the claim.

□

We illustrate the proposition by some example.

Example 3.3.24. Let $n = 2$ and $F := \mathrm{carr}(x^2, y^2)$, $\Gamma := \Gamma(F)$. Γ has three faces, namely Γ itself and its endpoints. $G_d := \mathrm{carr}(x^{2d}, y^2)$ is a face of F for each $d \geq 1$ and G_{d+1} is an $\subseteq$-maximal proper face of G_d as we will see later (see 3.5.6). However, for each face τ of Γ, $\tau(G_d)$ does not depend on d for $d \geq 2$. The $\subseteq$-maximal proper faces G of G_d are uniquely determined by $\tau(G)$ with $\tau := \Gamma(G_d)$, which may attain the faces $\mathbb{R}_+ \cdot (ax + by^d)^2$ with $(a, b) \neq (0, 0)$ where non-zero multiples of (a, b) give the same face.

In the following remark we give an algorithm how to obtain the property 3.3.18(c) from given faces G_τ.

Remark 3.3.25. Let us consider the situation of 3.3.23(b) with the background of 3.3.18(c). Let $\tau_1 \subseteq \tau_2$ be faces of Γ. Then, we may replace G_{τ_2} by $H \cap G_{\tau_2}$ with

$$H := \{g \in \mathcal{P}_{\tau_2} \mid g^{\tau_1} \in G_{\tau_1}\}$$

as in 3.3.23(a) and the construction of 3.3.23(b) will still give G. Indeed, if $g \in G$, then $(g^{\tau_2})^{\tau_1} = g^{\tau_1} \in G_{\tau_1}$, which shows that intersecting G_{τ_2} with H does not change G. We may also replace G_{τ_1} by $\tau_1(G_{\tau_2}) \cap G_{\tau_1}$ without changing G by 3.3.18(c) (and 3.3.23(b)).
By a repetition of these constructions for all combinations $\tau_1 \subseteq \tau_2$ of Γ (possibly multiple times), we will achieve that $\tau_1(G_{\tau_2}) = G_{\tau_1}$ holds for all faces $\tau_1 \subseteq \tau_2$ of Γ, which is a necessary condition for the property $\tau(G) = G_\tau$ for all faces τ of Γ. However, for $n \geq 3$, it seems to be a difficult question to determine the appearing mappings $\tau \mapsto \tau(G)$ (defined on the faces of Γ) for faces G of F (see 3.3.27).

Let us also illustrate the algorithm from the remark with two examples.

Example 3.3.26. Let $n = 3$, $F := \operatorname{carr}(x^2y^2, x^2z^2, y^2z^2, x^6, y^6, z^6)$ and $\Gamma := \Gamma(F)$. We set

$$\tau_0 := \operatorname{conv}((2,2,0),(2,0,2),(0,2,2)), \qquad \tau_x := \operatorname{conv}((6,0,0),(2,2,0),(2,0,2)),$$
$$\tau_y := \operatorname{conv}((0,6,0),(2,2,0),(0,2,2)\}), \qquad \tau_z := \operatorname{conv}((0,0,6),(2,0,2),(0,2,2)).$$

These are the $\subseteq$-maximal faces of Γ. Note that the elements of $\mathbb{R}[\underline{x}]^\tau$ for $\subseteq$-maximal faces τ of Γ are quadratic forms in the monomials corresponding to the elements of $\frac{1}{2}\tau \cap \mathbb{Z}^3$ and thus the cones $\mathcal{P}_\tau$ may be described as preimages of $\mathcal{P}_{2,2}$. We set $G_\tau := \mathcal{P}_\tau$ if $\tau \notin \{\tau_x, \tau_y, \tau_z\}$ and

$$G_{\tau_x} := \operatorname{carr}\left(x^2(x^2+y+\alpha z)^2\right), \qquad G_{\tau_y} := \operatorname{carr}\left(y^2(x+y^2+z)^2\right),$$
$$G_{\tau_z} := \operatorname{carr}\left(z^2(x+y+z^2)^2\right),$$

for some $\alpha \in \mathbb{R}$ where the carrier faces are taken in the corresponding cone $\mathcal{P}_\tau$ ($\tau \in \{\tau_x, \tau_y, \tau_z\}$). One can show that these faces are extreme rays (similarly to squares of linear forms in $\mathcal{P}_{2,2}$). We have the corresponding face G as described in 3.3.23(b). Now, by definition and 3.3.18(b),

$$(\tau_x \cap \tau_0)(G_{\tau_x}) = \operatorname{carr}\left(x^2(y+\alpha z)^2\right), \qquad (\tau_y \cap \tau_0)(G_{\tau_y}) = \operatorname{carr}\left(y^2(x+z)^2\right),$$
$$(\tau_z \cap \tau_0)(G_{\tau_z}) = \operatorname{carr}\left(z^2(x+y)^2\right),$$

with carrier faces again in the corresponding cone $\mathcal{P}_\tau$ ($\tau \in \{\tau_x \cap \tau_0, \tau_y \cap \tau_0, \tau_z \cap \tau_0\}$). Thus, in the sense of 3.3.25, we may replace $G_{\tau_x \cap \tau_0}$, $G_{\tau_y \cap \tau_0}$ and $G_{\tau_z \cap \tau_0}$ by the faces $(\tau_x \cap \tau_0)(G_{\tau_x})$, $(\tau_y \cap \tau_0)(G_{\tau_y})$ and $(\tau_z \cap \tau_0)(G_{\tau_z})$. By the first construction of 3.3.25, we may also replace G_{τ_0}. For instance, if one sets $\tau_1 := \tau_x \cap \tau_0 \subseteq \tau_2 := \tau_0$ with $G_{\tau_1} = \operatorname{carr}\left(x^2(y+\alpha z)^2\right) = \mathbb{R}_+ \cdot x^2(y+\alpha z)^2$, then

$$H_x := H = \{g \in \mathcal{P}_{\tau_0} \mid g^{\tau_x \cap \tau_0} \in \mathbb{R}_+ \cdot x^2(y+\alpha z)^2\}.$$

One gets similarly

$$H_y = \{g \in \mathcal{P}_{\tau_0} \mid g^{\tau_y \cap \tau_0} \in \mathbb{R}_+ \cdot y^2(x+z)^2\},$$
$$H_z = \{g \in \mathcal{P}_{\tau_0} \mid g^{\tau_z \cap \tau_0} \in \mathbb{R}_+ \cdot z^2(x+y)^2\}.$$

We may replace G_{τ_0} by $H_x \cap H_y \cap H_z$, which is $\{0\}$ if $|\alpha| \neq 1$, as may be seen easily from the coefficients of x^2y^2, x^2z^2 and y^2z^2. Though, it may be seen directly from these coefficients that $\tau_0(G) = \{0\}$, it illustrates the constructions from 3.3.25.

Example 3.3.27. Let $F := \operatorname{carr}(x^8, x^4y^4, x^4z^4, y^8z^8)$ and $\Gamma := \Gamma(F)$. The $\subseteq$-maximal faces of Γ are

$$\tau_1 := \operatorname{conv}((8,0,0),(4,0,4),(4,4,0)), \qquad \tau_2 := \operatorname{conv}((0,8,8),(4,0,4),(4,4,0)).$$

The elements of $\mathbb{R}[\underline{x}]^{\tau_1}$ are forms of degree 4 in $a = x^2$, $b = xy$ and $c = xz$ such that $\mathcal{P}_{\tau_1}$ may be identified by $\mathcal{P}_{2,4}$ by dividing by x^4. The elements of $\mathbb{R}[\underline{x}]^{\tau_2}$ are forms of degree 4 in $X_0 = y^2z^2$, $X_1 = xy$ and $X_2 = xz$ and the cone $\mathcal{P}_{\tau_2}$ may be described as a preimage of $\mathcal{P}_{2,4}$ under the corresponding linear map $\mathbb{R}[\underline{x}]^{\tau_2} \to \mathbb{R}[X_0, X_1, X_2]_4$. We set

$$G_{\tau_1} := \operatorname{carr}\left((x^2y^2 + x^2z^2 - x^4)^2\right), \qquad G_{\tau_2} := \operatorname{carr}\left((x^2y^2 + 2x^2z^2 - y^4z^4)^2\right).$$

with carrier faces in the respective cone $\mathcal{P}_\tau$ for $\tau \in \{\tau_1, \tau_2\}$. It is not hard to see that these faces are extreme rays (compare section 1.6). Furthermore, we set $G_\tau := \tau(G_{\tau_1})$ if $\tau \subseteq \tau_1$ and $G_\tau := \tau(G_{\tau_2})$ if $\tau \subseteq \tau_2$. This is well-defined since

$$(\tau_1 \cap \tau_2)(G_{\tau_1}) = \mathcal{P}_{\tau_1 \cap \tau_2} = (\tau_1 \cap \tau_2)(G_{\tau_2}).$$

We have $\tau(G_{\tau'}) = G_\tau$ for all faces $\tau \subseteq \tau'$ of Γ and the algorithm of 3.3.25 does not change anything. However, the face G defined by 3.3.18(b) has $\tau_1(G) = \tau_2(G) = \{0\}$ since $(G_{\tau_1})^{\tau_1 \cap \tau_2} \cap (G_{\tau_2})^{\tau_1 \cap \tau_2} = \{0\}$.

3.4 Facial structure modulo automorphisms

Let us now study the facial structure modulo automorphism as already done more abstractly in 2.3.6. However, here we will restrict to automorphisms of the rings and in the case of polynomials, we also have to fix $0 \in \mathbb{R}^n$. These considerations are parallel to constructions of singularity theory. We will transfer results from there to our setting later in this section. Let us start with the notation in the polynomial case.

Notation 3.4.1. We define a submonoid of the monoid of endomorphisms and a subgroup of the group of automorphisms of $\mathbb{R}[\underline{x}]$ via

$$\begin{aligned} \mathrm{End}_0(\mathbb{R}[\underline{x}]) &:= \{\varphi \in \mathrm{End}(\mathbb{R}[\underline{x}]) \,|\, \varphi(x_i)(0) = 0 \text{ for } i = 1, \ldots, n\}, \\ \mathrm{Aut}_0(\mathbb{R}[\underline{x}]) &:= \{\varphi \in \mathrm{Aut}(\mathbb{R}[\underline{x}]) \,|\, \varphi(x_i)(0) = 0 \text{ for } i = 1, \ldots, n\}. \end{aligned}$$

In the next remark, we will discuss the geometric transformation assigned to automorphisms in the polynomial case and the case of convergent power series. After that we characterize automorphisms of $\mathbb{R}\{\underline{x}\}$ and $\mathbb{R}[[\underline{x}]]$ and see the relation between the automorphism groups.

Remark 3.4.2. Real polynomials in $\mathbb{R}[\underline{x}]$ may be identified with polynomial functions $\mathbb{R}^n \to \mathbb{R}$. Then, endomorphisms φ of $\mathbb{R}[\underline{x}]$ may be identified with polynomial functions $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ via

$$\begin{array}{ccc} \varphi & \mapsto & (a \mapsto (\varphi(x_1)(a), \ldots, \varphi(x_n)(a))) \\ \alpha & \mapsto & (f \mapsto f(\alpha^*(x_1), \ldots, \alpha^*(x_n))) \end{array}$$

where $\alpha^*(x_i)$ denotes the pullback of x_i by α. Note that composition is reversed under this identification. Furthermore, elements of $\mathrm{End}_0(\mathbb{R}[\underline{x}])$ correspond to polynomial functions $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ with $\alpha(0) = 0$. From this, it is clear that $\mathrm{Aut}_0(\mathbb{R}[\underline{x}])$ consists of the invertible elements in $\mathrm{End}_0(\mathbb{R}[\underline{x}])$.
Note that, in the same vein, $\mathrm{End}(\mathbb{R}\{\underline{x}\})$ may be identified with germs of analytic functions $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ at 0 with the property $\alpha(0) = 0$. Indeed, to be able to set the image of an analytic germ $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ as described in the polynomial case, it is necessary that $f(\alpha^*(x_1), \ldots, \alpha^*(x_n))$ is defined at 0 for any $f \in \mathbb{R}\{\underline{x}\}$. Clearly, this implies $\alpha^*(x_i)(0) = 0$ for $i = 1, \ldots, n$ and thus $\alpha(0) = 0$.

Remark 3.4.3. Let φ be an element of $\mathrm{End}_0(\mathbb{R}[\underline{x}])$, $\mathrm{End}(\mathbb{R}\{\underline{x}\})$ or $\mathrm{End}(\mathbb{R}[[\underline{x}]])$ and $\varphi_i := \varphi(x_i)$ for $i = 1, \ldots, n$. There are unique linear forms λ_i such that $\varphi_i - \lambda_i \in \mathfrak{m}^2$ for $i = 1, \ldots, n$ (here, $\mathfrak{m}$ is the maximal ideal of $\mathbb{R}[[\underline{x}]]$). We have an induced linear endomorphism ψ of $\mathrm{span}(x_1, \ldots, x_n)$ with $\psi(x_i) := \lambda_i$ for $i = 1, \ldots, n$, which we call the *linear part of* φ. If φ

is an automorphism, then ψ is an isomorphism as one may easily see that the linear part of φ^{-1} is an inverse of the linear part ψ of φ. Note that the Jacobian matrix of φ corresponds to the matrix representation of ψ with respect to the basis $x_1, \ldots, x_n$ (see also [GrLoSh, I.1.19.1.(1)]). In the cases of convergent and formal power series, the converse is true as well: every endomorphism such that its linear part is invertible is itself an automorphism (see [GrLoSh, I.1.21]). However, in the polynomial case, there are already plenty counterexamples in the case $n = 1$ - for instance the endomorphism $x \mapsto x + x^2$ has no inverse in $\mathrm{End}_0(\mathbb{R}[x])$. Note also that one has canonical inclusions $\mathrm{End}_0(\mathbb{R}[\underline{x}]) \subseteq \mathrm{End}(\mathbb{R}\{\underline{x}\}) \subseteq \mathrm{End}(\mathbb{R}[[\underline{x}]])$ and hence $\mathrm{Aut}_0(\mathbb{R}[\underline{x}]) \subseteq \mathrm{Aut}(\mathbb{R}\{\underline{x}\}) \subseteq \mathrm{Aut}(\mathbb{R}[[\underline{x}]])$ as well.

The following lemma shows that the considered cones are indeed mapped into themselves by elements of the corresponding set of endomorphisms and thus, we may apply 2.3.6, which leads to the subsequent corollary.

Lemma 3.4.4. *The natural action of the monoid* $\mathrm{End}_0(\mathbb{R}[\underline{x}])$ *on* $\mathbb{R}[\underline{x}]$ *has the property that* $\varphi(\mathcal{P}^{Loc}) \subseteq \mathcal{P}^{Loc}$ *for all* $\varphi \in \mathrm{End}_0(\mathbb{R}[\underline{x}])$. *Analogous statements hold for the action of* $\mathrm{End}(\mathbb{R}\{\underline{x}\})$ *on* $\mathbb{R}\{\underline{x}\}$ *and* $\mathbb{R}\{\underline{x}\}_+$ *and the action of* $\mathrm{End}(\mathbb{R}[[\underline{x}]])$ *on* $\mathbb{R}[[\underline{x}]]$ *and* $\mathbb{R}[[\underline{x}]]_+$.

Proof. The statement in the cases of polynomials and convergent power series easily follows "geometrically" from the identification of endomorphisms with corresponding continuous maps (or map germs) $\mathbb{R}^n \to \mathbb{R}^n$ with $0 \mapsto 0$ as explained in 3.4.2. In the cases $\mathbb{R}\{\underline{x}\}_+$ and $\mathbb{R}[[\underline{x}]]_+$ the claim follows from the fact that images of non-negative elements under ring homomorphisms stay non-negative (the fact itself follows from 3.1.2(b) and as non-negative elements are the elements in the saturation of sums of squares). The polynomial case is also implied by the case of convergent power series as $\mathbb{R}\{\underline{x}\}_+ \cap \mathbb{R}[\underline{x}] = \mathcal{P}^{Loc}$. □

Corollary 3.4.5. *The group* $\mathrm{Aut}_0(\mathbb{R}[\underline{x}])$ *acts on* $\mathcal{P}^{Loc}$ *and induces by 2.3.6 an action on* $\mathcal{F}(\mathcal{P}^{Loc})$ *preserving inclusion and a quotient* $\mathcal{F}(\mathcal{P}^{Loc})/\mathrm{Aut}_0(\mathbb{R}[\underline{x}])$. *Faces* F *of the same equivalence class share*

- *the same facial structure up to isomorphism*
- *the same dimensions and codimensions of* F *and* $\mathrm{csupp}(F)$
- *the same dimension of* $\mathrm{span}(F)/\mathrm{csupp}(F)$.

Analogous statements hold for $\mathrm{Aut}(\mathbb{R}\{\underline{x}\})$ *and* $\mathbb{R}\{\underline{x}\}_+$ *as well as* $\mathrm{Aut}(\mathbb{R}[[\underline{x}]])$ *and* $\mathbb{R}[[\underline{x}]]_+$.

Remark 3.4.6. There are also other properties invariant under such automorphisms. For instance, one easily sees that the possible numbers of factors of irreducible representations as in 2.4.18 is invariant (see 2.4.4). The layers in which a face lies are not changed as well (see 2.3.3 and apply it on the reversed partial order).

The following lemma is a direct consequence of the fact (see 2.3.13) that classifications arising from 2.3.6 are regular classifications in the sense of 2.3.10.

Lemma 3.4.7. *Let* F *be a face of* $\mathcal{P}^{Loc}$ *(*$\mathbb{R}\{\underline{x}\}_+$, $\mathbb{R}[[\underline{x}]]_+$*) and* φ *an element of* $\mathrm{Aut}_0(\mathbb{R}[\underline{x}])$ *(*$\mathrm{Aut}(\mathbb{R}\{\underline{x}\})$, $\mathrm{Aut}(\mathbb{R}[[\underline{x}]])$*) with* $\varphi(F) \subseteq F$. *Then, we have* $\varphi(F) = F$.

As already mentioned, the consideration of faces modulo the action of the automorphism group is related to singularity theory where the same consideration is made for singularities (at least in the setting of formal or convergent power series). Let us transfer results from there to our setting. We present the theory as in [MaSt] and start with the definition of the corank and right equivalence of elements.

Definition 3.4.8. Let $f \in \mathbb{R}[[\underline{x}]]$ and $H(f)$ be the Hessian matrix of f (at 0). Then, we set

$$\operatorname{cork}(f) := \operatorname{cork}(H(f))$$

and also call it the *corank of* f (see [MaSt, Def. 10.] and 1.5.1, 1.5.2).
If f, g are elements of $\mathbb{R}\{\underline{x}\}$ ($\mathbb{R}[[\underline{x}]]$), then we call f and g *right equivalent* (denoted by $f \overset{r}{\sim} g$) if there is some $\varphi \in \operatorname{Aut}(\mathbb{R}\{\underline{x}\})$ ($\varphi \in \operatorname{Aut}(\mathbb{R}[[\underline{x}]])$) such that $\varphi(f) = g$ (see [MaSt, Def. 1.], [GrLoSh, I.2.9.]).

We have now the *splitting lemma* (also called *generalized Morse lemma*), which is the important result of singularity theory we will use later.

Theorem 3.4.9. ***(Splitting Lemma)*** *Let $f \in \mathbb{R}\{\underline{x}\}$ ($\mathbb{R}[[\underline{x}]]$) with $f \in \mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal of the considered ring. If f has corank c, then*

$$f \overset{r}{\sim} g(x_1, \ldots, x_c) - \sum_{i=c+1}^{c+\lambda} x_i^2 + \sum_{i=c+\lambda+1}^{n} x_i^2,$$

with some $\lambda \in \mathbb{N}_0$, $0 \leq \lambda \leq n - c$ and $g \in \mathfrak{m}^3 \cap \mathbb{R}[[x_1, \ldots, x_c]]$.

Proof. The case of isolated singularities and the ring $\mathbb{R}[[\underline{x}]]$ may be found in [MaSt, Theorem 11]. The case of converging power series over the complex numbers was proved in [GrLoSh, I.2.47] (though they omitted the proof of convergence). However, the algorithm used in both of these results works for $\mathbb{R}\{\underline{x}\}$ and $\mathbb{R}[[\underline{x}]]$ without any further assumption to give the desired result. □

Remark 3.4.10. The element g in the splitting lemma is also called *residual part of* f. Usually, the statements that the equivalence classes of g modulo $\overset{r}{\sim}$ and λ are uniquely determined by f are also presented as part of the splitting lemma. We do not include them here, as it is not obvious if this part of the theorem holds in the generality as the lemma was stated above.

Remark 3.4.11. The corresponding claim in $\mathbb{R}[\underline{x}]$ is clearly wrong as one may see for example from the case $n = 1$ and $f = x^2 + x^4$. There is no $\varphi \in \operatorname{Aut}_0(\mathbb{R}[\underline{x}])$ such that $\varphi(f) = x^2$.

The splitting lemma leads to the following definition (see [MaSt, 11.], [Arn, p.91]).

Definition 3.4.12. Two elements $f \in \mathbb{R}\{x_1, \ldots, x_{n_1}\}$ and $g \in \mathbb{R}\{x_1, \ldots, x_{n_2}\}$ for some $n_1, n_2 \in \mathbb{N}$ are called *stably equivalent*, denoted by $\overset{s}{\sim}$, if there is some $n \geq n_1, n_2$ such that

$$f \pm x_{n_1+1}^2 \pm \cdots \pm x_n^2 \quad \overset{r}{\sim} \quad g \pm x_{n_2+1}^2 \pm \cdots \pm x_n^2.$$

for some choice of signs. We use an analogous definition for the rings $\mathbb{R}[[\underline{x}]]$.

The following remark clarifies the definition.

Remark 3.4.13. As stated in 3.4.10, in other settings of the splitting lemma the residual part g is uniquely determined up to right equivalence. Then, an application of the splitting lemma shows that stably equivalence may be rephrased as having right equivalent residual parts.
Note that it is important to know in which rings the elements f and g live. For instance $f \in \mathbb{R}\{x_1, \ldots, x_{n_1}\}$ is not stably equivalent to $f \in \mathbb{R}\{x_1, \ldots, x_{n_1}, x_{n+1}\}$ but it is stably equivalent to $f + x_{n+1}^2 \in \mathbb{R}\{x_1, \ldots, x_{n_1}, x_{n+1}\}$.
Over $\mathbb{C}$, everything works analogously but one does not need any signs. If we are just working with locally non-negative elements, then we also just have to consider positive signs.

Let us transfer the theory for singularities to faces of the cones. We start with a lemma that shows that the corank does not change under automorphisms.

Lemma 3.4.14. *Let F be a proper face of $\mathcal{P}^{Loc}$ ($\mathbb{R}\{\underline{x}\}_+$, $\mathbb{R}[[\underline{x}]]_+$). Then,* $\operatorname{cork}(f)$ *does not depend on the choice of $f \in \operatorname{rai}(F)$. Furthermore, if φ is an element of* $\operatorname{Aut}_0(\mathbb{R}[\underline{x}])$ *(*$\operatorname{Aut}(\mathbb{R}\{\underline{x}\})$, $\operatorname{Aut}(\mathbb{R}[[\underline{x}]])$*), then* $\operatorname{cork}(\varphi(f)) = \operatorname{cork}(f)$.

Proof. F is a proper face of $\mathcal{P}^{Loc}$, so $F \subseteq \mathfrak{m}^2$. We consider

$$\tau := \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n \mid \alpha_1 + \cdots + \alpha_n = 2\},$$

which is a psd-diagram with $\Delta(F) \subseteq \Delta(\tau)$ and itself its only $\subseteq$-maximal face such that we may consider $\tau(F)$. We have $f^\tau \in \operatorname{rai}(\tau(F))$ for $f \in \operatorname{rai}(F)$ and clearly, the corank of f is the corank of f^τ. However, the corank of elements of $\operatorname{rai}(\tau(F))$ is uniquely determined by 1.5.4 and 1.5.5. The linear part of φ (see 3.4.3) gives a linear coordinate change on the homogeneous part of lowest degree of f that preserves coranks (see 1.5.1). Thus, $\operatorname{cork}(\varphi(f)) = \operatorname{cork}(f)$. □

This leads us to the following definition where we also define right equivalence of faces.

Definition 3.4.15. Let F be a proper face of $\mathcal{P}^{Loc}$ ($\mathbb{R}\{\underline{x}\}_+$, $\mathbb{R}[[\underline{x}]]_+$). We call

$$\operatorname{cork}(F) := \operatorname{cork}(f) \text{ with } f \in \operatorname{rai}(F)$$

the *corank of F*. We also give the equivalence classes in ${}^{\mathcal{F}(\mathcal{P}^{Loc})}/_{\operatorname{Aut}_0(\mathbb{R}[\underline{x}])}$ (${}^{\mathbb{R}\{\underline{x}\}_+}/_{\operatorname{Aut}(\mathbb{R}\{\underline{x}\})}$, ${}^{\mathbb{R}[[\underline{x}]]_+}/_{\operatorname{Aut}(\mathbb{R}[[\underline{x}]])}$) the corresponding corank (if the class is different from the class that just contains the cone itself).
Let F and G be faces of $\mathcal{P}^{Loc}$ ($\mathbb{R}\{\underline{x}\}_+$, $\mathbb{R}[[\underline{x}]]_+$). Then we call F and G *right equivalent* (notated by $f \overset{r}{\sim} g$) if there is some $\varphi \in \operatorname{Aut}_0(\mathbb{R}[\underline{x}])$ ($\operatorname{Aut}(\mathbb{R}\{\underline{x}\})$, $\operatorname{Aut}(\mathbb{R}[[\underline{x}]])$) such that $\varphi(F) = G$.

Before we prove a corresponding splitting lemma for faces, we will study the faces that shall be our normal form with respect to right equivalence (i.e. the faces that shall replace the expression $g(x_1, \ldots, x_c) - \sum_{i=c+1}^{c+\lambda} x_i^2 + \sum_{i=c+\lambda+1}^{n} x_i^2$ in the splitting lemma). We need some notation and lemma, which we formulate a bit more general than our setting, before we can prove the subsequent proposition on the faces.

Notation 3.4.16. Let A be a ring (e.g. $\mathbb{R}[\underline{x}]$, $\mathbb{R}\{\underline{x}\}$, $\mathbb{R}[[\underline{x}]]$). For any $M \subseteq A$, we set

$$\sqrt[2]{M} := \{f \in A \mid f^2 \in M\}.$$

We will need the following lemma just in the cases $\mathcal{P}^{Loc}$, $\mathbb{R}\{\underline{x}\}_+$ and $\mathbb{R}[[\underline{x}]]_+$.

Lemma 3.4.17. *Let A be a Noetherian, commutative, integral algebra over $\mathbb{R}$ with 1 and C an Archimedean preordering. For any face F of C, the sets $\sqrt[2]{F}$ and $\sqrt[2]{\operatorname{csupp}(F)}$ are ideals of A.*

Proof. Let $f_1, f_2 \in \sqrt[2]{F}$, so $f_1^2, f_2^2 \in F$. Then, $\frac{1}{2}(f_1 + f_2)^2 + \frac{1}{2}(f_1 - f_2)^2 = f_1^2 + f_2^2 \in F$ with $(f_1 \pm f_2)^2 \in C$. Since F is a face, this gives $(f_1 + f_2)^2 \in F$ and thus $f_1 + f_2 \in \sqrt[2]{F}$. Now, let $f \in \sqrt[2]{F}$ and $g \in A$. Then, $f^2 \in F$ and $g^2 \in C$. C is Archimedean and thus 2.4.11 implies $(fg)^2 \in F$, hence $fg \in \sqrt[2]{F}$. This shows that $\sqrt[2]{F}$ is an ideal of A. To show the other claim, we use $\sqrt[2]{\operatorname{csupp}(F)} = \sqrt[2]{\operatorname{csupp}(F) \cap C}$ and reduce to showing that $G := \operatorname{csupp}(F) \cap C$ is a

face of C. It is clearly convex and non-empty. Let $f \in \operatorname{rai}(F)$ (see 2.4.20 for existence) and let $g_1, g_2 \in C$ with $g := \frac{g_1+g_2}{2} \in G$. By 2.2.8(c), we have $f - \alpha g \in C$ for all $\alpha \in \mathbb{R}$. Clearly, this implies $f - \alpha g_2 = f - 2\alpha g + \alpha g_1 \in C$ for all $\alpha \geq 0$ and thus, $f - \alpha g_2 \in C$ for all $\alpha \in \mathbb{R}$. This shows $g_2 \in \operatorname{csupp}(F) \cap C$ and analogously we get $g_1 \in \operatorname{csupp}(F) \cap C$. This shows the claim. □

Now, we describe the structure of the faces appearing in the splitting lemma for faces.

Proposition 3.4.18. *Let $0 \leq c < n$. Given a proper face G of $\mathcal{P}^{Loc} \cap \mathbb{R}[x_1, \ldots, x_c]$, then*

$$F := \operatorname{carr}(G) \boxplus \operatorname{carr}(x_{c+1}^2, \ldots, x_n^2)$$

is the $\subseteq$-largest face of $\mathcal{P}^{Loc}$ such that $p(f) \in G$ for $f \in F$ where p is the projection

$$p : \mathbb{R}[\underline{x}] \to \mathbb{R}[x_1, \ldots, x_c], \quad f \mapsto f(x_1, \ldots, x_c, 0, \ldots, 0).$$

Now, let F be a proper face of $\mathcal{P}^{Loc}$ such that $x_{c+1}^2, \ldots, x_n^2 \in F$. Then,

$$F = \operatorname{carr}(G) \boxplus \operatorname{carr}(x_{c+1}^2, \ldots, x_n^2)$$

with $G := F \cap \mathbb{R}[x_1, \ldots, x_c]$. Furthermore, let $\mathfrak{m} := (x_1, \ldots, x_n)$ and $\mathfrak{m}_{n-c} := (x_{c+1}, \ldots, x_n)$. Then, the decomposition

$$\mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \ldots, x_c] \oplus \bigoplus_{i=c+1}^{n} x_i \cdot \mathbb{R}[x_1, \ldots, x_c] \oplus \mathfrak{m}_{n-c}^2$$

induces decompositions

$$\operatorname{span}(F) = \operatorname{span}(G) \oplus \bigoplus_{i=c+1}^{n} x_i \cdot \sqrt[2]{G} \oplus \mathfrak{m}_{n-c}^2$$

$$\text{and } \operatorname{csupp}(F) = \operatorname{csupp}(G) \oplus \bigoplus_{i=c+1}^{n} x_i \cdot \sqrt[2]{\operatorname{csupp}(G)} \oplus \mathfrak{m}\mathfrak{m}_{n-c}^2.$$

Especially, we have

$$\operatorname{codim}(F) = \operatorname{codim}(G) + (n-c) \cdot \operatorname{codim}\left(\sqrt[2]{G}\right)$$

$$\text{and } \operatorname{codim}(\operatorname{csupp}(F)) = \operatorname{codim}(\operatorname{csupp}(G)) + (n-c) \cdot \operatorname{codim}\left(\sqrt[2]{\operatorname{csupp}(G)}\right) + (n-c)^2.$$

Analogous statements hold for $\mathbb{R}\{\underline{x}\}$ and $\mathbb{R}[[\underline{x}]]$.

Proof. The proofs of all three cases are completely analogous and we will write down the case $\mathcal{P}^{Loc}$. We will start with a proper face F of $\mathcal{P}^{Loc}$ with $F = \operatorname{carr}(G) \boxplus \operatorname{carr}(x_{c+1}^2, \ldots, x_n^2)$ where $G := F \cap \mathbb{R}[x_1, \ldots, x_c]$ and show that the claimed equations for $\operatorname{span}(F)$ and $\operatorname{csupp}(F)$ hold. We fix some $g \in \operatorname{rai}(G)$, so $f := g + x_{c+1}^2 + \cdots + x_n^2 \in \operatorname{rai}(F)$ and for $h \in \mathbb{R}[\underline{x}]$, we will consider the corresponding decomposition

$$h = h_G + \sum_{i=c+1}^{n} x_i h_i + h_R$$

with $h_G, h_{c+1}, \ldots, h_n \in \mathbb{R}[x_1, \ldots, x_c]$ and $h_R \in \mathfrak{m}^2_{n-c}$.
We have $x_i^2, x_j^2 \in F$, so $x_i x_j \in \operatorname{span}(F)$ (proof as in 4.2.2(a)) for $c+1 \leq i, j \leq n$ and thus $\mathfrak{m}^2_{n-c} \subseteq \operatorname{span}(F)$ since $\operatorname{span}(F)$ is an ideal. Clearly, we have $\operatorname{span}(G) \subseteq \operatorname{span}(F)$ as $G \subseteq F$. Conversely, if $h \in \operatorname{span}(F)$, then $f - \epsilon h \in \mathcal{P}^{Loc}$ for some $\epsilon > 0$ and thus $g - \epsilon h_G \in \mathcal{P}^{Loc} \cap \mathbb{R}[x_1, \ldots, x_c]$ by setting $x_{c+1}, \ldots, x_n$ to 0, so $h_G \in \operatorname{span}(G)$. These observations show that for any $h \in \mathbb{R}[\underline{x}]$ the statement $h \in \operatorname{span}(F)$ is equivalent to $h_G \in \operatorname{span}(G)$ and $\sum_{i=c+1}^n x_i h_i \in \operatorname{span}(F)$. Hence, we may reduce to the case $h = \sum_{i=c+1}^n x_i h_i$ to prove the claimed representation of $\operatorname{span}(F)$.
Let us assume first that $\sum_{i=c+1}^n x_i h_i \in \operatorname{span}(F)$, so $f - \epsilon \sum_{i=c+1}^n x_i h_i \in \mathcal{P}^{Loc}$ for some $\epsilon > 0$. Clearly, $h_i(0) = 0$ for $i = c+1, \ldots, n$. For all $c+1 \leq j \leq n$, we obtain by setting $x_i := 0$ for all $c+1 \leq i \leq n$ with $i \neq j$ and $x_j := \frac{1}{2}\epsilon h_j$ in $f - \epsilon \sum_{i=c+1}^n x_i h_i$ that

$$g - \frac{1}{4}\epsilon^2 h_j^2 = g + x_j^2 - \epsilon x_j h_j \in \mathcal{P}^{Loc}$$

and thus $h_j^2 \in \operatorname{span}(G)$. This shows $h_j \in \sqrt[2]{G}$ for all $c+1 \leq j \leq n$. Let us assume now that $h_j \in \sqrt[2]{G}$ for $c+1 \leq j \leq n$. Then, $x_i^2, h_i^2 \in F$ imply that $x_i h_i \in \operatorname{span}(F)$ for $c+1 \leq i \leq n$ (proof as in 4.2.2(a)). Thus, the claimed representation of $\operatorname{span}(F)$ holds.
Let us prove the representation of $\operatorname{csupp}(F)$. Clearly, $\mathfrak{m}^2_{n-c} \subseteq \operatorname{span}(F)$ and $G \subseteq F$ imply $\mathfrak{m}\mathfrak{m}^2_{n-c} \subseteq \operatorname{csupp}(F)$ (see 2.4.22) and $\operatorname{csupp}(G) \subseteq \operatorname{csupp}(F)$. Let h be decomposed as above. Now, $h \in \operatorname{csupp}(F)$ and thus $f + \alpha h \in F$ for all $\alpha \in \mathbb{R}$ by 2.2.8(c) implies $g + \alpha h_G \in \mathcal{P}^{Loc} \cap \mathbb{R}[x_1, \ldots, x_c]$ by setting $x_i = 0$ for $i = c+1, \ldots, n$ and hence $h_G \in \operatorname{csupp}(G)$. As $F \subseteq \mathfrak{m}^2 \cap \mathcal{P}^{Loc}$, we have $h \in \operatorname{csupp}(F) \subseteq \mathfrak{m}^3$ by 3.3.14, so $h_R \in \mathfrak{m}\mathfrak{m}^2_{n-c}$. Analogous to the proof of the decomposition of $\operatorname{span}(F)$, we may reduce to the consideration of the case $h = \sum_{i=c+1}^n x_i h_i$.
Let us assume first that $\sum_{i=c+1}^n x_i h_i \in \bigoplus_{i=c+1}^n x_i \cdot \sqrt[2]{\operatorname{csupp}(G)}$. For any $j \in \{c+1, \ldots, n\}$, we have $x_j^2 \in \operatorname{span}(F)$ and $h_j^2 \in \operatorname{csupp}(G) \subseteq \operatorname{csupp}(F)$ and thus $x_j h_j \in \operatorname{csupp}(F)$ (see 4.2.2(b)). This shows $\sum_{i=c+1}^n x_i h_i \in \operatorname{csupp}(F)$. Let us consider now $h := \sum_{i=c+1}^n x_i h_i \in \operatorname{csupp}(F)$, so $f - \alpha h \in \mathcal{P}^{Loc}$ for all $\alpha \in \mathbb{R}$. For any $j \in \{c+1, \ldots, n\}$, setting $x_i := 0$ for $c+1 \leq i \leq n$ and $i \neq j$ as well as $x_j := \frac{1}{2}\alpha h_j$ in $f - \alpha h$ gives

$$g - \frac{1}{4}\alpha^2 h_j^2 = g + x_j^2 - \alpha x_j h_j \in \mathcal{P}^{Loc}$$

for all $\alpha \in \mathbb{R}$. This shows $h_j^2 \in \operatorname{csupp}(G)$ and thus $h_j \in \sqrt[2]{\operatorname{csupp}(G)}$. Hence, the claimed representation of $\operatorname{csupp}(F)$ holds.
The formulas on the codimensions directly follow from the decompositions by summation of the codimensions in each direct summand.
Let us show now that $F = \operatorname{carr}(G) \boxplus \operatorname{carr}(x_{c+1}^2, \ldots, x_n^2)$ does hold if F is a proper face of $\mathcal{P}^{Loc}$ with $x_{c+1}^2, \ldots, x_n^2 \in F$ and $G := F \cap \mathbb{R}[x_1, \ldots, x_c]$. The inclusion "$\supseteq$" is obvious. Let $h \in \operatorname{rai}(F)$ be decomposed as before and let us show that h is in the linear span of the right-hand side, which would prove this equality. Using the inclusion "$\supseteq$", we see that $\operatorname{csupp}(F) \supseteq \operatorname{csupp}(\operatorname{carr}(x_{c+1}^2, \ldots, x_n^2)) \supseteq \mathfrak{m}\mathfrak{m}^2_{n-c}$ such that we may assume that h_R is homogeneous of degree 2 (see 2.2.8(c)). Since any such form may be bounded above by some multiple of $x_{c+1}^2 + \cdots + x_n^2$ (see 1.5.5), we may replace h_R by this multiple and after some scaling with a positive number we may assume $h_R = x_{c+1}^2 + \cdots + x_n^2$ for some $h \in \operatorname{rai}(F)$. Setting $x_i := -\frac{1}{2}h_i$

for $i = c+1, \ldots, n$ (note that $h_i(0) = 0$) in h gives $h_G - \frac{1}{4}\sum_{i=c+1}^{n} h_i^2 \in \mathcal{P}^{Loc}$. Now, we have

$$\frac{3}{4}h_G + \frac{1}{4}\left(h_G - \frac{1}{4}\sum_{i=c+1}^{n} h_i^2\right) + \sum_{i=c+1}^{n}\left(2x_i + \frac{1}{4}h_i\right)^2 = h + 3(x_{c+1}^2 + \cdots + x_n^2) \in F,$$

so $h_G \in F$ and $h_G - \frac{1}{4}\sum_{i=c+1}^{n} h_i^2 \in F$. Thus, $h_G \in G$ and $h_i^2 \in G$ for $i = c+1, \ldots, n$, which gives $h_i \in \sqrt[2]{G}$ for $i = c+1, \ldots, n$. This implies $h \in \operatorname{span}(\operatorname{carr}(G) \boxplus \operatorname{carr}(x_{c+1}^2, \ldots, x_n^2))$ and thus, it shows the claimed equality.
Finally, let G be a proper face of $\mathcal{P}^{Loc} \cap \mathbb{R}[x_1, \ldots, x_c]$, F' a face of $\mathcal{P}^{Loc}$ such that $p(f) \in G$ for all $f \in F'$ and we want to show that $F' \subseteq F := \operatorname{carr}(G) \boxplus \operatorname{carr}(x_{c+1}^2, \ldots, x_n^2)$. We set $F'' := F' \boxplus \operatorname{carr}(x_{c+1}^2, \ldots, x_n^2)$ and $G'' := F'' \cap \mathbb{R}[x_1, \ldots, x_c]$. Now, if $f' \in \operatorname{rai}(F')$, then $f'' := f' + (x_{c+1}^2 + \cdots + x_n^2) \in \operatorname{rai}(F'')$, so $p(f'') \in \operatorname{rai}(G'')$ (by 2.2.11) and $p(f'') = p(f') \in G$. This shows $G'' \subseteq G$ and we obtain $F' \subseteq F'' \subseteq F$. □

In the following warning, we show that some properties of faces as in 3.4.18 are not satisfied for arbitrary faces.

Warning 3.4.19. If F is a proper face of $\mathcal{P}^{Loc}$, $f \in F$, $1 \leq c < n$, $\mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \ldots, x_c] \oplus \mathfrak{m}_{n-c}\mathbb{R}[\underline{x}]$ ($\mathfrak{m}_{n-c} := (x_{c+1}, \ldots, x_n)$) and $f = f_1 \oplus f_2$ is the corresponding composition, then it is not necessarily $f_1 \in F$. Indeed, $n = 2$, $c = 1$, $F = \operatorname{carr}((x-y)^2)$ and $f = (x-y)^2$ already gives a counterexample since $(x-y)^2 - \epsilon x^2 \notin \mathcal{P}^{Loc}$ for all $\epsilon > 0$.
However, it is easy to see that $\mathbb{R}[x_1, \ldots, x_c] \cap F$ is a face of $\mathbb{R}[x_1, \ldots, x_c] \cap \mathcal{P}^{Loc}$ (see 2.2.12). In contrast to this, $p(F)$ with the projection p as in 3.4.18 is not always a face of $\mathbb{R}[x_1, \ldots, x_c] \cap \mathcal{P}^{Loc}$ though the relative interior points belong all to the same face of $\mathbb{R}[x_1, \ldots, x_c] \cap \mathcal{P}^{Loc}$. A counterexample is for instance given by the carrier face G of the Motzkin form as studied in 3.3.21 and $c = 2$. Indeed, $p(G)$ contains $x^4y^2 + x^2y^4$ but not $(x^2y \pm xy^2)^2$ though $(x^2y \pm xy^2)^2 \in \mathbb{R}[x, y] \cap \mathcal{P}^{Loc}$ and $(x^2y + xy^2)^2 + (x^2y - xy^2)^2 = 2(x^4y^2 + x^2y^4)$.

We also obtain the immediate corollary of 3.4.18.

Corollary 3.4.20. *For any $1 \leq c \leq n$, there is a canonical bijection between faces F of $\mathcal{P}^{Loc}$ such that $x_{c+1}^2, \ldots, x_n^2 \in F$ and faces G of $\mathcal{P}^{Loc} \cap \mathbb{R}[x_1, \ldots, x_c]$. The statement holds with the obvious changes for $\mathbb{R}\{\underline{x}\}_+$ and $\mathbb{R}[[\underline{x}]]_+$.*

Now, we can prove the splitting lemma for faces. The first part of the statement directly follows from the usual splitting lemma. However, we also include the statement corresponding to the uniqueness of the residual part (compare with 3.4.13) and here it is not clear that it immediately carries over to faces.

Theorem 3.4.21. ***(Splitting lemma for faces)*** *Let F be a proper face of $\mathbb{R}[[\underline{x}]]_+$. If F has corank c, then*

$$F \overset{r}{\sim} G \boxplus \operatorname{carr}(x_{c+1}^2, \ldots, x_n^2),$$

with $G = \operatorname{carr}(g) \trianglelefteq \mathbb{R}[[\underline{x}]]_+$ for some $g \in \mathbb{R}[[\underline{x}]]_+ \cap \mathbb{R}[[x_1, \ldots, x_c]] \cap \mathfrak{m}_c^4$ where $\mathfrak{m}_c := (x_1, \ldots, x_c)$. The equivalence class of $G \cap \mathbb{R}[[x_1, \ldots, x_c]]$ modulo $\overset{r}{\sim}$ is uniquely determined by F.
The statement also holds for the cone $\mathbb{R}\{\underline{x}\}_+$ with the obvious changes.

Proof. We show the claim in the case $\mathbb{R}[[\underline{x}]]_+$ and the proof for $\mathbb{R}\{\underline{x}\}_+$ is completely analogous. Let $f \in \operatorname{rai}(F)$ and $\mathfrak{m} = (x_1, \ldots, x_n)$, so $f \in \mathfrak{m}^2$ since F is a proper face. By the splitting lemma 3.4.9, we have

$$f \overset{r}{\sim} g(x_1, \ldots, x_c) - \sum_{i=c+1}^{c+\lambda} x_i^2 + \sum_{i=c+\lambda+1}^{n} x_i^2,$$

with some $\lambda \in \mathbb{N}_0$, $0 \leq \lambda \leq n - c$ and $g \in \mathfrak{m}_c^3$. Now, we have $f \in \mathbb{R}[[\underline{x}]]_+$, so $\lambda = 0$ and $g \in \mathbb{R}[[x_1, \ldots, x_c]]_+$, which implies $g \in \mathfrak{m}_c^4$. This shows that a representation of F as claimed exists.
Let us consider now $\varphi_1, \varphi_2 \in \operatorname{Aut}(\mathbb{R}[[\underline{x}]])$ such that $\varphi_i(F) = G_i \boxplus \operatorname{carr}(x_{c+1}^2, \ldots, x_n^2)$ and $G_i = \operatorname{carr}(g_i) \trianglelefteq \mathbb{R}[[\underline{x}]]_+$ with $g_i \in \mathbb{R}[[x_1, \ldots, x_c]]_+ \cap \mathfrak{m}_c^4$ for $i = 1, 2$. We set $G_i' := G_i \cap \mathbb{R}[[x_1, \ldots, x_c]]$ for $i = 1, 2$, which are the carrier faces of g_i in $\mathbb{R}[[x_1, \ldots, x_c]]_+$, and we want to show that $G_1' \overset{r}{\sim} G_2'$. We may assume without loss of generality that φ_1 is the identity and $\psi := \varphi_2$ is any element of $\operatorname{Aut}(\mathbb{R}[[\underline{x}]])$ sending $F_1 := F = G_1 \boxplus \operatorname{carr}(x_{c+1}^2, \ldots, x_n^2)$ to $F_2 := G_2 \boxplus \operatorname{carr}(x_{c+1}^2, \ldots, x_n^2)$. Let $f_1 := g_1 + x_{c+1}^2 + \cdots + x_n^2 \in \operatorname{rai}(F_1)$, so

$$\psi(f_1) = g_1(\psi(x_1), \ldots, \psi(x_c)) + \psi(x_{c+1})^2 + \cdots + \psi(x_n)^2 \in F_2.$$

We consider $\psi' \in \operatorname{End}(\mathbb{R}[[\underline{x}]])$ given by

$$\begin{aligned} \psi'(x_i) &:= (\psi(x_i))(x_1, \ldots, x_c, 0, \ldots, 0) \quad \text{for} \quad i = 1, \ldots, c, \\ \psi'(x_i) &:= (\psi(x_i))(0, \ldots, 0, x_{c+1}, \ldots, x_n) \quad \text{for} \quad i = c+1, \ldots, n. \end{aligned}$$

We claim that $\psi' \in \operatorname{Aut}(\mathbb{R}[[\underline{x}]])$ and to show this, we just have to prove that the linear part ψ_1' of ψ' is an automorphism of $\operatorname{span}(x_1, \ldots, x_n)$ (see 3.4.3). We set $V_1 := \operatorname{span}(x_1, \ldots, x_c)$ and $V_2 := \operatorname{span}(x_{c+1}, \ldots, x_n)$. The linear part ψ_1 of ψ induces a linear transformation on the leading term and thus on the the homogeneous part of degree 2 of f_1. Since $\psi(f_1) \in F_2$, we have $\psi_1(x_i) \in V_2$ for $i = c+1, \ldots, n$ (as the linear part of ψ induces a linear coordinate change on the homogeneous part of degree 2 of f_1 and by 1.5.5) and thus ψ_1 induces a linear monomorphism and thus an automorphism of V_2. Furthermore, ψ_1' (the linear part of ψ') acts on V_2 as ψ_1 and thus $V_2 \subseteq \operatorname{im}(\psi_1')$. Since ψ_1 is an automorphism of $V_1 \oplus V_2$ fixing V_2, it induces an automorphism on $V_1 \oplus V_2 / V_2$ and this automorphism coincides with the automorphism induced by ψ_1'. Especially, $V_1 \subseteq \operatorname{im}(\psi_1') + V_2$. Summing up, we get $\operatorname{span}(x_1, \ldots, x_n) = V_1 \oplus V_2 \subseteq \operatorname{im}(\psi_1')$, which shows $\psi' \in \operatorname{Aut}(\mathbb{R}[[\underline{x}]])$. Let $h_i := (\psi(x_i))(x_1, \ldots, x_c, 0, \ldots, 0)$ for $i = c+1, \ldots, n$, so

$$\begin{aligned} \psi'(f_1)(x_1, \ldots, x_c, 0, \ldots, 0) + \sum_{i=c+1}^{n} h_i^2 &= g_1(\psi'(x_1), \ldots, \psi'(x_c)) + \sum_{i=c+1}^{n} h_i^2 \\ &= \psi(f_1)(x_1, \ldots, x_c, 0, \ldots, 0) \in G_2' \end{aligned}$$

and as G_2' is a face and all summands on the left hand side are in $\mathbb{R}[[x_1, \ldots, x_c]]_+$, we obtain that $\psi'(f_1)(x_1, \ldots, x_c, 0, \ldots, 0) \in G_2'$. We consider the projection

$$p : \mathbb{R}[[\underline{x}]] \to \mathbb{R}[[x_1, \ldots, x_c]], \quad f \mapsto f(x_1, \ldots, x_c, 0, \ldots, 0).$$

In the rest of the proof, we consider all carrier faces with respect to $\mathbb{R}[[x_1, \ldots, x_c]]_+$. We have $p(\psi'(F_1)) \subseteq p(\mathbb{R}[[\underline{x}]]_+) = \mathbb{R}[[x_1, \ldots, x_c]]_+$ (compare with 3.4.4). We have $p(\psi'(f_1)) \in$

$\operatorname{rai}(p(\psi'(F_1)))$ by 2.2.11 and $p(\psi'(f_1)) \in \operatorname{rai}(\operatorname{carr}(p(\psi'(F_1))))$ by 2.2.10. We have seen before that $p(\psi'(f_1)) \in G_2'$, so

$$p(\psi'(F_1)) \subseteq \operatorname{carr}(p(\psi'(F_1))) = \operatorname{carr}(p(\psi'(f_1))) \subseteq G_2'.$$

3.4.18 implies that $\psi'(F_1) \subseteq F_2$. 3.4.7 shows $\psi'(F_1) = F_2$, so

$$\psi'(g_1) = p(\psi'(f_1)) \in \operatorname{rai}(p(\psi'(F_1))) = \operatorname{rai}(G_2')$$

and thus $\psi'(G_1') = G_2'$. This shows that the equivalence class of $G \cap \mathbb{R}[[x_1, \ldots, x_c]]$ modulo $\overset{r}{\sim}$ is uniquely determined. □

We have two remarks on the splitting lemma.

Remark 3.4.22. The splitting lemma for faces is compatible with $\subseteq$ in the following sense. Let $F_1 \subseteq F_2$ be faces of $\mathbb{R}[[\underline{x}]]_+$ and $\varphi \in \operatorname{Aut}(\mathbb{R}[[\underline{x}]])$ with $\varphi(F_1) = G_1 \boxplus \operatorname{carr}(x_{c+1}^2, \ldots, x_n^2)$ as in 3.4.21 with $c := \operatorname{cork}(F_1)$ and $G_1 := \operatorname{carr}(g_1) \trianglelefteq \mathbb{R}[[\underline{x}]]_+$, $g_1 \in \mathbb{R}[[x_1, \ldots, x_c]] \cap \mathfrak{m}_c^4$. Then, we have $\varphi(F_2) = G_2 \boxplus \operatorname{carr}(x_{c+1}^2, \ldots, x_n^2)$ with $G_2 = \operatorname{carr}(g_2)$ for some $g_2 \in \mathbb{R}[[x_1, \ldots, x_c]]_+$ and $\operatorname{carr}(g_1) \subseteq \operatorname{carr}(g_2)$ in $\mathbb{R}[[x_1, \ldots, x_c]]_+$. This is an immediate consequence of 3.4.18 and $\operatorname{carr}(x_{c+1}^2, \ldots, x_n^2) \subseteq \varphi(F_1) \subseteq \varphi(F_2)$.

Remark 3.4.23. There is no corresponding statement to the splitting lemma in the polynomial case. For example, in the case $n = 2$, the only face of the form $G \boxplus \operatorname{carr}(x_2^2)$ of $\mathcal{P}^{Loc}$ as in 3.4.21 with infinite zero-set in each neighbourhood of 0 is $\operatorname{carr}(x_2^2)$ itself. However, one can show that $\operatorname{carr}((x_2 + x_1^2 + x_2^2)^2) = (x_2 + x_1^2 + x_2^2)^2 \cdot \mathcal{P}^{Loc}$ and it is not possible to transfer this face to $\operatorname{carr}(x_2^2) = x_2^2 \cdot \mathcal{P}^{Loc}$ (as faces of $\mathcal{P}^{Loc}$) using some element of $\operatorname{Aut}_0(\mathbb{R}[x_1, x_2])$. Indeed, one may not transfer $x_2 + x_1^2 + x_2^2$ to x_2 as $\varphi_2 + \varphi_1^2 + \varphi_2^2$ has even degree for $\varphi_1, \varphi_2 \in \mathbb{R}[x_1, x_2]$. This shows that the consideration of $\mathcal{P}^{Loc}$ modulo $\operatorname{Aut}_0(\mathbb{R}[\underline{x}])$ is not as well-behaved as the setting of convergent or formal power series.

Now, let us carry over the definition of stable equivalence to faces.

Definition 3.4.24. Let $F \trianglelefteq \mathbb{R}[[x_1, \ldots, x_{n_1}]]_+$ and $G \trianglelefteq \mathbb{R}[[x_1, \ldots, x_{n_2}]]_+$. Then, F and G are called *stably equivalent*, denoted by $F \overset{s}{\sim} G$ if there are some $n \geq n_1, n_2$ such that

$$\operatorname{carr}(F) \boxplus \operatorname{carr}(x_{n_1+1}^2, \ldots, x_n^2) \quad \overset{r}{\sim} \quad \operatorname{carr}(G) \boxplus \operatorname{carr}(x_{n_2+1}^2, \ldots, x_n^2)$$

in the cone $\mathbb{R}[[\underline{x}]]_+$. We make the analogous definition in the case $\mathbb{R}\{\underline{x}\}_+$.

Again, we make some remark to clarify this definition.

Remark 3.4.25. We note that two faces are stably equivalent if and only if the corresponding faces $G \cap \mathbb{R}[[x_1, \ldots, x_c]]$ from the splitting lemma for faces 3.4.21 are right equivalent (see also 3.4.13). Especially, they need to have the same corank.
Furthermore, if F and G are faces as in 3.4.24 and $n_1 \leq n_2$, then they are stably equivalent if and only if $\operatorname{carr}(F) \boxplus \operatorname{carr}(x_{n_1+1}^2, \ldots, x_{n_2}^2)$ and G are right equivalent (in $\mathbb{R}[[x_1, \ldots, x_{n_2}]]_+$). Especially, one can reach G by the steps

$$\operatorname{carr}(F) \boxplus \operatorname{carr}\left(x_{n_1+1}^2, \ldots, x_m^2\right) \quad \to \quad \operatorname{carr}(F) \boxplus \operatorname{carr}\left(x_{n_1+1}^2, \ldots, x_m^2\right) \boxplus \operatorname{carr}\left(x_{m+1}^2\right)$$

from a face of the cone $\mathbb{R}[[x_1, \ldots, x_m]]_+$ to a face of $\mathbb{R}[[x_1, \ldots, x_{m+1}]]_+$ for $m = n_1, \ldots, n_2 - 1$ and some right equivalence at the end. Thus, if some property is invariant under such steps, it is invariant under stable equivalence.

Finally, given any face F of $\mathcal{P}^{Loc}$ ($\mathbb{R}\{\underline{x}\}_+$, $\mathbb{R}[[\underline{x}]]_+$), let us study the $\subseteq$-maximal proper faces of F with higher corank.

Proposition 3.4.26. *Let $F = \operatorname{carr}(G) \boxplus \operatorname{carr}(x_{c+1}^2, \ldots, x_n^2)$ be as in 3.4.18 a face of $\mathcal{P}^{Loc}$ ($\mathbb{R}\{\underline{x}\}_+$, $\mathbb{R}[[\underline{x}]]_+$) with $c = \operatorname{cork}(F) < n$. If F' is an $\subseteq$-maximal proper face of F with $\operatorname{cork}(F') \neq c$, then $\operatorname{cork}(F') = c+1$ and*

$$F' \overset{r}{\sim} H := \operatorname{carr}(G) \boxplus \operatorname{carr}\left(x_1^2 \cdot x_{c+1}^2, \ldots, x_c^2 \cdot x_{c+1}^2, x_{c+1}^4, x_{c+2}^2, \ldots, x_n^2\right).$$

Furthermore, such $\subseteq$-maximal proper faces of F exist.

Proof. The proof is the same in all three cases and we use the notation of the case $\mathcal{P}^{Loc}$. Clearly, $\operatorname{cork}(F') \neq c$ implies $c' := \operatorname{cork}(F') > c$. We may assume without loss of generality (using some linear coordinate change in the variables $x_{c+1}, \ldots, x_n$ that does not change F) that the only variables appearing in monomials of degree 2 of elements of F' are $x_{c'+1}, \ldots, x_n$. Let $F'' := F' \boxplus H$. We have $F' \subseteq F'' \subsetneq F$ since $x_{c+1}^2 \notin F''$ and thus $F' = F''$ since F' is an $\subseteq$-maximal proper face of F. This shows $\operatorname{cork}(F') = c+1$ and we reduce the problem to the case $c+1 = n$ using 3.4.20. Furthermore, it is enough to show that H is an $\subseteq$-maximal proper face of F since this implies $F' = H$ and shows the existence of $\subseteq$-maximal proper faces of F as claimed.
Let $\tilde{F}$ be a face of F with $H \subseteq \tilde{F} \subseteq F$ and $f \in \operatorname{rai}(\tilde{F})$. We write $f = f_G + f_{c+1}x_{c+1} + f_R x_{c+1}^2$ as in the proof of 3.4.18 with $f_G, f_{c+1} \in \mathbb{R}[x_1, \ldots, x_c]$. We set $\mathfrak{m} := (x_1, \ldots, x_{c+1})$ and we have $f_G \in \mathfrak{m}^4$ and $f_{c+1} \in \mathfrak{m}^2$. Let us assume that $f_R(0) \neq 0$. Then, clearly $f_R(0) > 0$ and $f_R \in \mathcal{P}^{Loc}$. We have

$$f\left(x_1, \ldots, x_c, \frac{1}{2}x_{c+1}\right) + \frac{1}{4}f_R x_{c+1}^2 = \frac{1}{2}f + \frac{1}{2}f_G \in \tilde{F}$$

and thus $f_R x_{c+1}^2 \in \tilde{F}$. We have $(f_R x_{c+1}^2 + \epsilon x_{c+1}^2) + (f_R x_{c+1}^2 - \epsilon x_{c+1}^2) = 2f_R x_{c+1}^2 \in \tilde{F}$ and thus $f_R x_{c+1}^2 \pm \epsilon x_{c+1}^2 \in \tilde{F}$ for $0 < \epsilon < f_R(0)$. Hence, $x_{c+1}^2 \in \operatorname{span}(\tilde{F}) \cap \mathcal{P}^{Loc} = \tilde{F}$, so $\tilde{F} \supseteq \operatorname{carr}(G) \boxplus \operatorname{carr}(x_{c+1}^2) = F$ and thus $\tilde{F} = F$.
Let us assume now that $f_R(0) = 0$ and we want to show that $f \in \operatorname{span}(H)$, which would imply $H = \tilde{F}$. We have $f_G \in G \subseteq H \subseteq \operatorname{span}(H)$. Furthermore, $f_R(0) = 0$ gives $f \in \mathfrak{m}^3$ and thus $f \in \mathfrak{m}^4$, so $f_R \in \mathfrak{m}^2$. Now, we have (using that polynomials doubly vanishing on the hyperplane $x_{c+1} = 0$ in a neighbourhood of the origin are also divisible by x_{c+1}^2)

$$\operatorname{carr}(x_1^2 \cdot x_{c+1}^2, \ldots, x_c^2 \cdot x_{c+1}^2, x_{c+1}^4) = x_{c+1}^2 \cdot (\mathfrak{m}^2 \cap \mathcal{P}^{Loc}),$$

so its linear span is $x_{c+1}^2 \cdot \mathfrak{m}^2$, which shows $f_R \in \operatorname{span}(H)$. It remains to show that $f_{c+1}x_{c+1} \in \operatorname{span}(H)$. Since $\tilde{f}_R := x_1^2 + \cdots + x_{c+1}^2 \in \operatorname{rai}(\mathfrak{m}^2 \cap \mathcal{P}^{Loc})$, there is some $M \in \mathbb{N}$ such that $M\tilde{f}_R - f_R \in H$. We may add $(M\tilde{f}_R - f_R)x_{c+1}^2$ to f and assume without loss of generality $f_R = M\tilde{f}_R$ to show $f_{c+1}x_{c+1} \in \operatorname{span}(H)$. Let $g \in \operatorname{rai}(G)$, so $h := g + x_{c+1}^2 \cdot (x_1^2 + \cdots + x_c^2) \in \operatorname{rai}(H)$. We have for sufficiently small $\epsilon > 0$ that

$$h \pm \epsilon f_{c+1}x_{c+1} = (g - \epsilon f_G) + (1 - M\epsilon)x_{c+1}^2(x_1^2 + \cdots + x_c^2) + \epsilon f(x_1, \ldots, x_c, \pm x_{c+1}) \in \mathcal{P}^{Loc}$$

which shows $f_{c+1}x_{c+1} \in \operatorname{span}(H)$. Summing up, we get $H = \tilde{F}$ if $f_R(0) = 0$ for every choice of $f \in \operatorname{rai}(\tilde{F})$ and $\tilde{F} = F$ else. Thus, H is an $\subseteq$-maximal proper face of F. $\square$

3.5 The case $n = 2$

Let us study the case $n = 2$. We will first study monomial faces and their factorizations. Then, we take a look at the classification of the faces modulo automorphisms. Note that this also gives a classification up to stable equivalence of the corresponding faces of corank at most 2 (but just in the cases of convergent and formal power series).

3.5.1 Monomial faces

The description of monomial faces is very easy in the case $n = 2$. We need the following description of their Newton diagram.

Remark 3.5.1. Let $n = 2$ and $F \neq \{0\}$ be a monomial face of $\mathcal{P}^{Loc}$ $(\mathbb{R}\{\underline{x}\}_+, \mathbb{R}[[\underline{x}]]_+)$. Let $g := g(\Gamma(F)) = \sum_{i=0}^{r} x^{2a_i}y^{2b_i}$ with $r \in \mathbb{N}_0$, $a_0 > \cdots > a_r$, $b_0 < \cdots < b_r$ be the canonical polynomial of $\Gamma(F)$. We set $k_i := a_{i-1} - a_i$, $l_i := b_i - b_{i-1}$ for $i = 1, \ldots, r$. Now, $\frac{k_1}{l_1} > \frac{k_2}{l_2} > \cdots > \frac{k_r}{l_r}$ are the negatives of the slopes of the one-dimensional compact faces of $\Gamma(F)$. Furthermore, F is uniquely determined by $r, a_r, b_0, k_1, \ldots, k_r, l_1, \ldots, l_r$. Note that $a_r = b_0 = 0$ if F has finite codimension.

From this, we can easily obtain a multiplicative representation of monomial faces and the Newton diagram of each factor is either a point or a line segment. We present it in the following proposition.

Proposition 3.5.2. *Let $n = 2$ and $F \neq \{0\}$ be a monomial face of $\mathcal{P}^{Loc}$ $(\mathbb{R}\{\underline{x}\}_+, \mathbb{R}[[\underline{x}]]_+)$. We consider $r, a_r, b_0, k_1, \ldots, k_r, l_1, \ldots, l_r$ as in 3.5.1 and we set $F_i := \operatorname{carr}(x^{2k_i} + y^{2l_i})$ for $i = 1, \ldots, r$. Then,*

$$F = \operatorname{carr}(x^{2a_r}) \boxdot \operatorname{carr}(y^{2b_0}) \boxdot F_1 \boxdot F_2 \boxdot \cdots \boxdot F_r.$$

Proof. We show the claim by induction on r. The case $r = 0$ is obvious as $x^{2a_r}y^{2b_0}$ is the canonical polynomial of $\Gamma(F)$ (see 2.4.4). If $r = 1$, then F has vertices $(2a_0, 2b_0) = (2a_1, 2b_0) + (2k_1, 0)$ und $(2a_1, 2b_1) = (2a_1, 2b_0) + (0, 2l_1)$ and thus $x^{2a_1}y^{2b_0}(x^{2k_1} + y^{2l_1})$ is the canonical polynomial of $\Gamma(F)$, which shows the claim.
Let $r \geq 2$ and suppose the claim holds in the case $r - 1$. We set

$$F' := \operatorname{carr}(x^{2a_r}) \boxdot \operatorname{carr}(y^{2b_0}) \boxdot F_1 \boxdot F_2 \boxdot \cdots \boxdot F_{r-1}$$

By induction hypothesis, F' is the monomial face with respect to $r - 1, a_r, b_0$, $k_1, \ldots, k_{r-1}$, $l_1, \ldots, l_{r-1}$ in terms of 3.5.1. From these data one calculates the vertices $(2a'_i, 2b'_i)$ $(i = 0, \ldots, r-1$, $a'_0 > a'_1 > \cdots > a'_{r-1}$, $b'_0 < b'_1 < \cdots < b'_{r-1})$ of $\Gamma(F')$ via 3.5.1 and we have

$$\begin{aligned}
a'_{r-1} &= a_r = a_{r-1} - k_r, & b'_0 &= b_0,\\
a'_{r-2} &= a'_{r-1} + k_{r-1} = (a_{r-1} - k_r) + k_{r-1} = a_{r-2} - k_r, & b'_1 &= b_0 + l_1 = b_1,\\
a'_{r-3} &= a'_{r-2} + k_{r-3} = (a_{r-3} - k_r) + k_{r-2} = a_{r-3} - k_r, & b'_2 &= b_1 + l_2 = b_2,\\
&\cdots & &\cdots\\
a'_1 &= a'_2 + k_1 = (a_1 - k_r) + k_2 = a_1 - k_r, & b'_{r-2} &= b_{r-3} + l_{r-2} = b_{r-2},\\
a'_0 &= a'_1 + k_0 = (a_0 - k_r) + k_1 = a_0 - k_r, & b'_{r-1} &= b_{r-2} + l_{r-1} = b_{r-1}.
\end{aligned}$$

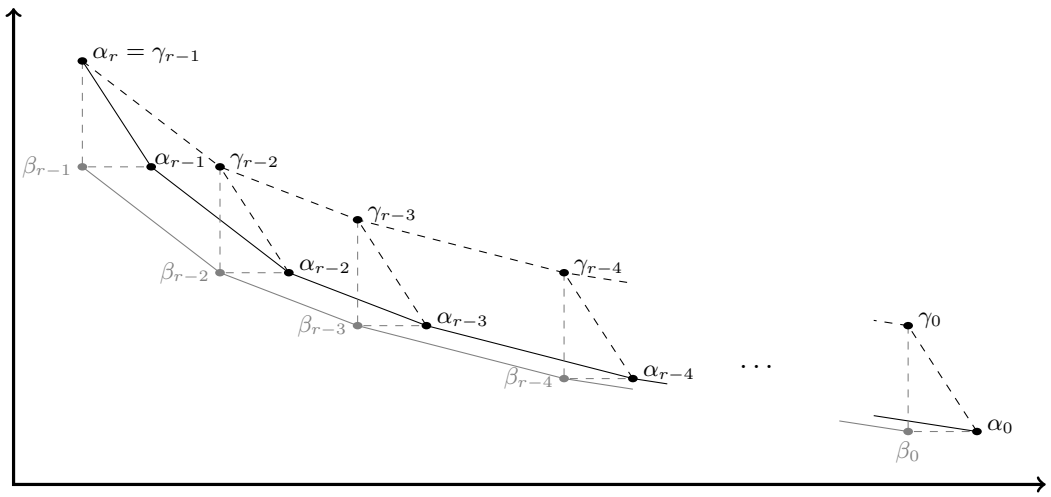
This shows that $\Gamma(F')$ has canonical polynomial $\sum_{i=0}^{r-1} x^{2(a_i-k_r)}y^{2b_i}$. We have to show that $F = F' \boxdot F_r$. Now, we have

$$\sum_{i=0}^{r} x^{2a_i}y^{2b_i} + \sum_{i=0}^{r-2} x^{2(a_i-k_r)}y^{2(b_i+l_r)} = \left(\sum_{i=0}^{r-1} x^{2(a_i-k_r)}y^{2b_i}\right) \cdot \left(x^{2k_r} + y^{2l_r}\right) \in \mathrm{rai}(F' \boxdot F_r).$$

As the first sum is the canonical polynomial of $\Gamma(F)$, we see $F \subseteq F' \boxdot F_r$ and it remains to show that $x^{2(a_i-k_r)}y^{2(b_i+l_r)} \in F$, which means $\gamma_i := (2(a_i - k_r), 2(b_i + l_r)) \in \Delta(F)$, for $i = 1, \ldots, r-2$ by 3.3.14. We name the vertices of $\Gamma(F)$ and $\Gamma(F')$ by $\alpha_i := (2a_i, 2b_i)$ $(i = 1, \ldots, r)$ and $\beta_i := (2(a_i - k_r), 2b_i)$ $(i = 1, \ldots, r-1)$ (see drawing below). Especially, we have $\alpha_i = \beta_i + (2k_r, 0)$ and $\gamma_i = \beta_i + (0, 2l_r)$ for $i = 1, \ldots, r-1$. We have $\gamma_{r-1} = \alpha_r$ and

$$\gamma_i - \gamma_{i-1} = (2(a_i - a_{i-1}), 2(b_i - b_{i-1})) = \alpha_i - \alpha_{i-1}$$

implies that α_{i-1}, α_i, γ_i and γ_{i-1} are vertices of parallelograms for each $i = 1, \ldots, r-1$. As $\frac{l_r}{k_r} > \frac{l_i}{k_i}$, the absolute value of the slope between γ_i and α_i is larger than the corresponding value between α_i and α_{i-1} and both slopes are negative. Thus, $\gamma_{i-1} \in \mathrm{conv}(\alpha_i, \alpha_{i-1}, \gamma_i) + \mathbb{R}_+^2$ and especially, $\gamma_i \in \Delta(F)$ implies $\gamma_{i-1} \in \Delta(F)$. Repeated application for $i = r-1, r-2, \ldots, 1$ shows the claim.



□

The proposition motivates the following definition.

Definition 3.5.3. Let $n = 2$ and F be a monomial face of $\mathcal{P}^{Loc}$ $(\mathbb{R}\{\underline{x}\}_+, \mathbb{R}[[\underline{x}]]_+)$. We call $r, a_r, b_0, k_1, \ldots, k_r, l_1, \ldots, l_r$ as in 3.5.1 the *multiplicative data corresponding to* F. If F has finite codimension ($a_r = b_0 = 0$), then we also write $(0 : 0)$ if $F = \mathcal{P}^{Loc}$ and

$$(2k_1 : 2l_1) \boxdot (2k_2 : 2l_2) \boxdot \cdots \boxdot (2k_r : 2l_r)$$

if $r > 0$ and we call it the *canonical multiplicative representation of* F.

In the following remark, we describe how to multiply monomial faces with respect to their canonical multiplicative representation. We illustrate it by a subsequent example.

Remark 3.5.4. Let $(2k : 2l)$ and $(2k' : 2l')$ be monomial faces with $k, l, k', l' \neq 0$ and $\frac{k}{l} = \frac{k'}{l'}$. Then, we have $(2k : 2l) \boxdot (2k' : 2l') = (2(k + k') : 2(l + l'))$. Indeed,

$$\left(x^{2k} + y^{2l}\right)\left(x^{2k'} + y^{2l'}\right) = x^{2(k+k')} + x^{2k}y^{2l'} + x^{2k'}y^{2l} + y^{2(l+l')}$$

and all monomials correspond to points on the Newton diagram of $(2(k+k') : 2(l+l'))$, which implies the claim.
Thus, we get easy rules for the multiplication of monomial faces. Let us consider two monomial faces different from $\mathcal{P}^{Loc}$ in canonical multiplicative representation

$$\begin{aligned}(2k_1 : 2l_1) \boxdot (2k_2 : 2l_2) \boxdot \cdots \boxdot (2k_r : 2l_r)\\ \text{and } (2k'_1 : 2l'_1) \boxdot (2k'_2 : 2l'_2) \boxdot \cdots \boxdot (2k'_s : 2l'_s)\end{aligned}$$

and calculate the canonical multiplicative representation of the product. Then, we order the factors $(2k_i, 2l_i)$ and $(2k'_i, 2l'_i)$ such that the quotients $\frac{k_i}{l_i}$ and $\frac{k'_i}{l'_i}$ are monotonically decreasing and we apply the rule from above on factors with the same quotient.

Example 3.5.5. Let us consider the product of the monomial faces

$$(4 : 2) \boxdot (2 : 2) \boxdot (4 : 6) \boxdot (2 : 4) \quad \text{and} \quad (6 : 2) \boxdot (4 : 4) \boxdot (2 : 4).$$

Then, we have

$$\begin{aligned}&((4 : 2) \boxdot (2 : 2) \boxdot (4 : 6) \boxdot (2 : 4)) \boxdot ((6 : 2) \boxdot (4 : 4) \boxdot (2 : 4))\\ =&\,(6 : 2) \boxdot (4 : 2) \boxdot (2 : 2) \boxdot (4 : 4) \boxdot (4 : 6) \boxdot (2 : 4) \boxdot (2 : 4)\\ =&\,(6 : 2) \boxdot (4 : 2) \boxdot (6 : 6) \boxdot (4 : 6) \boxdot (4 : 8)\end{aligned}$$

3.5.2 Faces modulo automorphisms

The following proposition is important for the classification of faces up to right equivalence.

Proposition 3.5.6. *Let $n = 2$ and F be a proper monomial face of $\mathcal{P}^{Loc}$ ($\mathbb{R}\{x, y\}_+$, $\mathbb{R}[[x, y]]_+$) of finite codimension with canonical multiplicative representation*

$$(2k_1 : 2l_1) \boxdot (2k_2 : 2l_2) \boxdot \cdots \boxdot (2k_r : 2l_r)$$

and vertices $(2a_i, 2b_i)$ of $\Gamma := \Gamma(F)$ ($i = 0, \ldots, r$, $a_0 > a_1 > \cdots > a_r$). Furthermore, let G be an $\subseteq$-maximal proper face of F, so by 3.3.18(e) , there is some face τ of Γ with $\tau(G) \subsetneq \tau(F)$ and we choose a $\subseteq$-minimal such face.

(a) If τ is a vertex, then G is the monomial face corresponding to the $\subseteq$-maximal Newton polyhedron Δ' such that it has even vertices, $\Delta' \subseteq \Delta(F)$ and $\tau \nsubseteq \Delta'$.

(b) If τ is one-dimensional with endpoints $(2a_0, 2b_0)$ and $(2a_1, 2b_1)$ and $l_1 \,|\, k_1$, then $G \overset{r}{\sim} G'$ where G' is the monomial face as it appears in (a) with respect to $\tau' = \{(2a_0, 2b_0)\}$.

(c) Analogously, if τ is one-dimensional with endpoints $(2a_{r-1}, 2b_{r-1})$ and $(2a_r, 2b_r)$ such that $k_r \,|\, l_r$, then $G \overset{r}{\sim} G'$ where G' is the monomial face as it appears in (a) with respect to $\tau' = \{(2a_r, 2b_r)\}$.

Proof. Let us prove (a). We have $\tau(F) = \mathbb{R}_+ \cdot x^{2a_i} y^{2b_i}$ for some $i = 0, \dots, r$, so $\tau(G) = \{0\}$ and thus, $(2a_i, 2b_i) \notin \Delta(G)$. This shows the claim.
Let us prove (b). We show the polynomial case but the other cases are analogous. We set $\lambda := \frac{k_1}{l_1}$ and we have $b_0 = 0$ and

$$\mathbb{R}[\underline{x}]^\tau = x^{2a_1} \cdot \left\{ \sum_{i=0}^{2l_1} c_i x^{\lambda i} y^{2l_1 - i} \,\middle|\, c_0, \dots, c_{2l_i} \in \mathbb{R} \right\}.$$

Now, we have $f \in \mathcal{P}_\tau$ if and only if $\frac{f}{x^{2a_1}}$ is non-negative on the unit circle S^1 as f is quasi-homogeneous. Furthermore, given $f \in \mathbb{R}[\underline{x}]^\tau$, we have $f \in \operatorname{rai}(\mathcal{P}_\tau)$ if and only if $\frac{f}{x^{2a_1}}$ is strictly positive on S^1. Indeed, we have $f_0 := x^{2a_1}(x^{2k_1} + y^{2l_1}) \in \mathcal{P}_\tau$, so $f \in \operatorname{rai}(\mathcal{P}_\tau)$ implies $f - \epsilon f_0 \in \mathcal{P}_\tau$ for some $\epsilon > 0$, so $\frac{1}{x^{2a_1}}(f - \epsilon f_0) \in \mathcal{P}_\tau$. Hence, $\frac{f}{x^{2a_1}}$ is strictly positive on S^1. Conversely, if $f \in \mathbb{R}[\underline{x}]^\tau$ and $\frac{f}{x^{2a_1}}$ is strictly positive on S^1, then $g \in \mathbb{R}[\underline{x}]^\tau$ implies $f - \epsilon g \in \mathcal{P}_\tau$ for some $\epsilon > 0$ since $\frac{f}{x^{2a_1}}$ and $\frac{g}{x^{2a_1}}$ may be identified with polynomials, which are continuous functions, and as S^1 is compact.
Hence, if $g \in \operatorname{rai}(\tau(G))$ and $g' := \frac{g}{x^{2a_1}}$, then $g'(x, y) = 0$ for some $(x, y) \in S^1$. As τ was chosen $\subseteq$-minimal in the premise, which implies that the monomials x^{2k_1} and y^{2l_1} appear in g', we have $(x, y) \notin \{(\pm 1, 0), (0, \pm 1)\}$. By quasi-homogeneity of g', we also find $\alpha \in \mathbb{R} \setminus \{0\}$ with $g'(1, \alpha) = 0$. Thus, if $h \in \tau(G)$, then $h'(1, \alpha) = 0$ with $h' := \frac{h}{x^{2a_1}}$ as $g - \epsilon h \in \mathcal{P}_\tau$ for small $\epsilon > 0$. This shows that

$$\tau(G) \subseteq x^{2a_1} \cdot \left\{ h'(x^\lambda, y) \,\middle|\, h' \in \mathbb{R}[x, y]_{2l_1}, h'(1, \alpha) = 0 \right\}.$$

Let us consider $\varphi \in \operatorname{Aut}_0(\mathbb{R}[\underline{x}])$ given by $\varphi(x) = x$ and $\varphi(y) = y - \alpha x^\lambda$. Then, as $\Delta(F) = \Delta(F) + \mathbb{R}_+ \cdot (\lambda, 1)$ (here, we use that $\frac{k_1}{l_1}$ is maximal under the quotients $\frac{k_i}{l_i}$ for $i = 1, \dots, r$), we have $\operatorname{supp}(\varphi(f)) \subseteq \Delta$ for each $f \in \mathbb{R}[\underline{x}]$ with $\operatorname{supp}(f) \subseteq \Delta$ and thus $\varphi(F) \subseteq F$. This implies $\varphi(F) = F$ by 3.4.7. Furthermore, we have $\operatorname{supp}(\varphi(f)) \subseteq \Delta \setminus \tau$ for each $f \in \mathbb{R}[\underline{x}]$ with $\operatorname{supp}(f) \subseteq \Delta \setminus \tau$ using that $\Delta(F) \setminus \tau = \Delta(F) \setminus \tau + \mathbb{R}_+ \cdot (\lambda, 1)$. This implies

$$\varphi(G)^\tau \subseteq \varphi(G^\tau) \subseteq \varphi(\tau(G)) \subseteq x^{2a_1} \cdot \left\{ h(x^\lambda, y) \,\middle|\, h \in \mathbb{R}[x, y]_{2l_1}, h(1, 0) = 0 \right\}$$

and thus, we have $(2a_0, 2b_0) \notin \Delta(\varphi(G))$. Now, $\varphi(G)$ is an $\subseteq$-maximal proper face of $\varphi(F) = F$ and we are in the situation of (a). This shows the claim (b) and the proof of (c) is analogous with interchanged roles of x and y. □

The previous proposition has solved the classification problem of $\subseteq$-maximal proper faces up to right equivalence for some monomial faces. However, we also have to consider some $\subseteq$-maximal proper faces of other faces. For this purpose, we need some weighted degree.

Definition 3.5.7. Let $k \in \mathbb{N}$ and $\lambda := (2, 2k + 1)$. We define a monomial gradation on $\mathbb{R}[[\underline{x}]]$ by the map $d_\lambda(x^i y^j) := 2i + (2k + 1)j$ for $i, j \in \mathbb{N}_0$.
If $d \in \mathbb{N}_0$ and $f \in \mathbb{R}[[\underline{x}]]$, then we define the *homogeneous part of λ-degree d* of $f = \sum_{i,j} a_{i,j} x^i y^j$ via

$$(f)_{\lambda, d} := \sum_{d_\lambda(x^i y^j) = d} a_{i,j} x^i y^j.$$

We will need the following lemma for the decomposition of power series with respect to the λ-degree.

Lemma 3.5.8. *(a) Let $f \in \mathbb{R}[x,y]$ be homogeneous in λ-degree d. Then, either $f = 0$ or there is some $i \in \mathbb{N}_0$ with $f = (x^{2k+1} - y^2)^i g$ and $g(1,1) \neq 0$.*

(b) Let $f \in \mathbb{R}[[x,y]]$. Then, there are $f_0, f_1, f_2 \in \mathbb{R}[[x,y]]$ such that

$$f = (x^{2k+1} - y^2)^2 f_2 + (x^{2k+1} - y^2) f_1 + f_0,$$

for each $d \in \mathbb{N}_0$ we have $((x^{2k+1} - y^2)^i f_i)_{\lambda,d} \neq 0$ for at most one $i \in \{0,1,2\}$ and if this inequality holds in some degree $d \in \mathbb{N}_0$ with $i \in \{0,1\}$, then $(f_i)_{\lambda, d-2(2k+1)i}(1,1) \neq 0$.

Proof. (a) Let $f \neq 0$. Since x^{2k+1} is not a square, $x^{2k+1} - y^2$ is irreducible. Thus, there is some $i \in \mathbb{N}_0$ with $f = (x^{2k+1} - y^2)^i g$ and $x^{2k+1} - y^2 \nmid g$. We may divide g by $x^{2k+1} - y^2$ with respect to lexicographic order with $x < y$ and we have $g = (x^{2k+1} - y^2)h(x,y) + r(x,y)$ with λ-homogeneous $h, r \in \mathbb{R}[x,y]$ and $r \neq 0$ by the choice of i. Now, as r has degree ≤ 1 in y, the formula for the λ-degree implies that r is a term (consider the parity). This shows $g(1,1) = r(1,1) \neq 0$.

(b) This is an immediate consequence of (a). Indeed, one just has to solve it in each λ-degree and sum up the solutions. For elements f homogeneous with respect to some λ-degree, one sets $f_i = g$ if $i \in \{0,1\}$ and $f_i = (x^{2k+1} - y^2)^{i-2} g$ if $i \geq 2$ where i is defined as in (a). (All the other f_i are set to 0.)

□

Now, we can characterize some faces with the property that their Newton diagram has the form $\Gamma = \mathrm{conv}((4k+2, 0), (0, 4))$.

Proposition 3.5.9. *Let $k \in \mathbb{N}$, $\lambda := (2, 2k+1)$ and $x^{i_0} y^{j_0}$ a monomial of even λ-degree $2d_1 := d_\lambda(x^{i_0} y^{j_0})$ with $d_1 > 2(2k+1)$. Then, $(x^{2k+1} - y^2)^2 + x^{i_0} y^{j_0} \in \mathcal{P}^{Loc}$ and we set*

$$F := \mathrm{carr}\left((x^{2k+1} - y^2)^2 + x^{i_0} y^{j_0}\right).$$

Let us consider $f = (x^{2k+1} - y^2)^2 f_2 + (x^{2k+1} - y^2) f_1 + f_0 \in \mathbb{R}[\underline{x}]$ decomposed as in 3.5.8(b). Then, we have

$$\begin{aligned} f \in \mathrm{rai}(F) \quad \Leftrightarrow \quad & a := f_2(0,0) > 0, \quad (4a f_0 - f_1^2)_{\lambda, 2d_1}(1,1) > 0, \\ & f_1 \text{ does not contain monomials of } \lambda\text{-degree } < d_1, \\ & f_0 \text{ does not contain monomials of } \lambda\text{-degree } < 2d_1. \end{aligned}$$

Furtermore, $f \in \mathrm{ac}(F)$ holds if and only if the conditions on the right hand side hold with > 0 replaced by ≥ 0 in the first two inequalities.
Analogous statements hold in the cones $\mathbb{R}\{x,y\}_+$ and $\mathbb{R}[[x,y]]_+$.

Proof. Let M be the set of f described on the right hand side above and let us show that $M \subseteq \mathcal{P}^{Loc}$, so especially $(x^{2k+1} - y^2)^2 + x^{i_0} y^{j_0} \in \mathcal{P}^{Loc}$. We choose some $0 < b < a$ such that

$(4bf_0 - f_1^2)_{\lambda,2d_1}(1,1) > 0$. Now, we have

$$f = \left(\sqrt{b}(x^{2k+1} - y^2) + \frac{1}{2\sqrt{b}}f_1\right)^2 + \underbrace{(x^{2k+1} - y^2)^2(f_2 - b) + f_0 - \frac{1}{4b}f_1^2}_{r(x,y)}$$

$$\text{with } \left(f_0 - \frac{1}{4b}f_1^2\right)_{\lambda,2d_1}(1,1) \geq \left(f_0 - \frac{1}{4a}f_1^2\right)_{\lambda,2d_1}(1,1) > 0.$$

Thus, $r(x,y) \in M$ and since $f - r \in \mathcal{P}^{Loc}$, we reduce to the problem to show that $r \in \mathcal{P}^{Loc}$. Hence, we may assume $f_1 = 0$ (and rename f_0 and f_2 such that f is represented as before). We define some polynomials via

$$\tilde{f_2}(u,v,t) = f_2(ut^2, vt^{2k+1}) - a \in O(t), \quad g(u,v) := t^{-2d_1}(f_0)_{\lambda,2d_1}(ut^2, vt^{2k+1})$$
$$\text{and} \quad \tilde{f_0}(u,v,t) = f_0(ut^2, vt^{2k+1}) - g(u,v)t^{2d_1} \in O(t^{2d_1+1})$$

and the relations on the Big O Notation hold in the sense that one considers $\tilde{f_2}$ and $\tilde{f_0}$ as functions in t at 0 after fixing u and v. Now, let $(x,y) \in \mathbb{R}^2 \setminus \{0\}$ and we consider the functions $t := \sqrt[2(2k+1)]{\frac{1}{2}(y^2 + |x|^{2k+1})}$, $u := \frac{x}{t^2}$ and $v := \frac{y}{t^{2k+1}}$ depending on x, y, so especially $|u|^{2k+1} + v^2 = 2$. Then, we have

$$f(x,y) = (u^{2k+1} - v^2)^2 t^{2(2k+1)} \left(a + \tilde{f_2}(u,v,t)\right) + g(u,v)t^{2d_1} + \tilde{f_0}(u,v,t).$$

Let us consider the right-hand side and ignore x and y for the moment. We have $g(1,\pm 1) > 0$ since $(f_0)_{\lambda,2d_1}(1,1) > 0$. Thus, we find compact sets K_1 and K_2 such that $(1,\pm 1) \notin K_1$, $\{(u,v) \,|\, |u|^{2k+1} + v^2 = 2\} = K_1 \cup K_2$ and $g(u,v) > \epsilon$ for some $\epsilon > 0$ and all $(u,v) \in K_2$.
Let us fix some $\mu > 0$ such that $(u^{2k+1} - v^2)^2 > \mu$ for $(u,v) \in K_1$. Then, the coefficient of $t^{2(2k+1)}$ in the presentation of f above, is at least $\mu \cdot \frac{a}{2} > 0$ for $0 < |t| < t_1$ and $(u,v) \in K_1$ by choosing $t_1 > 0$ such that $\tilde{f_2}(u,v,t) > -\frac{a}{2}$ holds. As the term of order $t^{2(2k+1)}$ dominates in the representation for small $|t|$, we may choose t_1 such that the whole representation is non-negative for $0 < t < t_1$ and all $(u,v) \in K_1$.
Similarly, let us choose t_2 such that $\tilde{f_2}(u,v,t) > -\frac{a}{2}$ holds for $0 < |t| < t_2$ and $(u,v) \in K_2$ and thus, the first summand of the representation of f gets non-negative. As $g(u,v) > \epsilon$ and the term of order t^{2d_1} dominates $g(u,v)t^{2d_1} + \tilde{f_0}(u,v,t)$ for small $|t| \neq 0$, we may choose $t_2 > 0$ such that the whole representation gets non-negative for all $(u,v) \in K_2$ and $0 < |t| < t_2$.
We set $t_0 := \min(t_1, t_2) > 0$. Now, using $x := ut^2$, $y := vt^{2k+1}$ with u and v satisfying $|u|^{2k+1} + v^2 = 2$,

$$\{(0,0)\} \cup \{(x,y) \,|\, 0 < t < t_0\}$$

is a neighbourhood of 0 in $\mathbb{R}^2$. This shows $f \in \mathcal{P}^{Loc}$.
Let $f \in M$ as above and $g = (x^{2k+1} - y^2)^2 g_2 + (x^{2k+1} - y^2)g_1 + g_0 \in M$ represented as f. Then, for sufficiently small $\epsilon > 0$, we also have

$$f - \epsilon g = (x^{2k+1} - y^2)^2(f_2 - \epsilon g_2) + (x^{2k+1} - y^2)(f_1 - \epsilon g_1) + (f_0 - \epsilon g_0) \in M,$$

as $a > 0$ and $(4af_0 - f_1^2)_{\lambda,2d_1}(1,1) > 0$ are "open conditions". This shows $M = \mathrm{rai}(M)$ and thus all $f \in M$ belong to the same face of $\mathcal{P}^{Loc}$ (see 2.2.10), hence $M \subseteq \mathrm{rai}(F)$.

Let us suppose that $f \in \mathrm{rai}(F)$ with corresponding polynomials f_0, f_1, f_2. Now, we have $\Gamma(f) = \Gamma((x^{2k+1} - y^2)^2 + x^{i_0}y^{j_0})$ such that f_i does not contain monomials of λ-degree smaller than $(2-i)2(2k+1)$ for $i = \{0, 1\}$. Let $\tau := \Gamma(F)$. Then,

$$\tau(F) = \mathrm{carr}_{\mathcal{P}_\tau}\left((x^{2k+1} - y^2)^2\right) = \mathbb{R}_+ \cdot (x^{2k+1} - y^2)^2.$$

Thus, $f^\tau = a(x^{2k+1} - y^2)^2$ with some $a > 0$, $(f_2)_{\lambda,0} = a$, $(f_1)_{\lambda,2(2k+1)} = 0$ and $(f_0)_{\lambda,4(2k+1)} = 0$. Let $g := ((x^{2k+1} - y^2)^2 + x^{i_0}y^{j_0}) \in \mathrm{rai}(F)$ and $\epsilon > 0$ with $g - \epsilon f \in \mathcal{P}^{Loc}$. Then, setting $x = t^2$ and $y = t^{2k+1}$, we get $t^{2d_1} - \epsilon f_0(t^2, t^{2k+1}) \geq 0$ for t of small (positive) absolute value. We have

$$f_0\left(t^2, t^{2k+1}\right) = \sum_{d \in \mathbb{N}_0} (f_0)_{\lambda,d}\left(t^2, t^{2k+1}\right) = \sum_{d \in \mathbb{N}_0} ((f_0)_{\lambda,d}(1,1))t^d$$

which implies $(f_0)_{\lambda,d} = 0$ for $d < 2d_1$.
Let d_2 be minimal with $(f_1)_{\lambda,d_2} \neq 0$, if such a $d_2 \leq d_1$ exists and $d_2 := d_1$ else. We have seen that $d_2 > 2(2k+1)$ and $(f_0)_{\lambda,d} = 0$ for $d < 2d_2 \leq 2d_1$. Let us consider $s \in \mathbb{R}$, some small $t > 0$, $c := \sqrt{1 - st^{d_2 - 2(2k+1)}}$, $x := t^2$ and $y := ct^{2k+1}$. We have

$$\begin{aligned} f(x,y) \in\ & (1-c^2)^2 t^{4(2k+1)}(a + O(t)) + (1-c^2)t^{2(2k+1)}\left((f_1)_{\lambda,d_2}(1,c)t^{d_2} + O\left(t^{d_2+1}\right)\right) \\ & + \left((f_0)_{\lambda,2d_2}(1,c)t^{2d_2} + O\left(t^{2d_2+1}\right)\right) \\ =\ & \left(as^2 + (f_1)_{\lambda,d_2}(1,c)s + (f_0)_{\lambda,2d_2}(1,c)\right)t^{2d_2} + O\left(t^{2d_2+1}\right). \end{aligned}$$

As $f \in \mathcal{P}^{Loc}$, the limit $t \to 0$ leads to the observation that $as^2 + (f_1)_{\lambda,d_2}(1,1)s + (f_0)_{\lambda,2d_2}(1,1)$ is a non-negative quadratic polynomial in s. This shows $(4af_0 - f_1^2)_{\lambda,2d_2}(1,1) \geq 0$. Especially, we have $(f_0)_{\lambda,2d_2}(1,1) > 0$ if $(f_1)_{\lambda,2d_2}(1,1) \neq 0$ and thus $d_1 = d_2$.
We have shown all properties to show $f \in M$ except $(4af_0 - f_1^2)_{\lambda,2d_2}(1,1) \neq 0$. However, we get at least $f \in \mathrm{ac}(M)$ as g and thus also $f + \epsilon g$ for some $\epsilon > 0$ induce strictly positive quadratic polynomials in s as above, which indeed implies $f + \epsilon g \in \mathrm{rai}(M)$. We have shown $\mathrm{rai}(F) \subseteq \mathrm{ac}(M)$ and seen before that $M \subseteq \mathrm{rai}(F)$. Now, M is a convex set (as the two describing inequalities correspond to strictly positive quadratic polynomials) such that 2.1.7(c) implies

$$\mathrm{rai}(F) = \mathrm{rai}(\mathrm{rai}(F)) \subseteq \mathrm{rai}(\mathrm{ac}(M)) = \mathrm{rai}(M) \subseteq M \subseteq \mathrm{rai}(F)$$

and thus $M = \mathrm{rai}(F)$. From the considerations before, the description of $\mathrm{ac}(F)$ is obvious. The statements in the other cones follow directly from 3.1.8. □

Remark 3.5.10. Note that the proof of 3.5.9 shows us that all $f \in \mathcal{P}^{Loc}$ with $\Gamma(f) = \Gamma(x^{2(2k+1)} + y^4)$ such that $f(t^2, t^{2k+1})$ has leading term of degree $\geq 2d_1$ (i.e. f_0 contains just monomials of λ-degree $\geq 2d_1$) automatically belong to F (using the notation from 3.5.9).

We need the following lemma to apply automorphisms to the faces that were just characterized.

Lemma 3.5.11. *Let $k, l \in \mathbb{N}$ and $f(x,y) \in \mathbb{R}[[x,y]]$ of the form*

$$f(x,y) = x^{2k+1} - y^2 + g(x,y).$$

with $\Delta(g(x,y)) \subseteq \Delta(x^{2k+1} - y^2) \setminus \Gamma(x^{2k+1} - y^2)$. Then, there is some $\varphi \in \mathrm{Aut}(\mathbb{R}[[\underline{x}]])$ such that the image of any monomial does just contain monomials of the same or higher λ-degree and $\varphi(f) = x^{2k+1} - y^2 + g'(x,y)$ with $g' \in \mathfrak{m}^l$ where $\mathfrak{m} = (x,y)$.

Proof. We consider the decomposition

$$g(x,y) = g_0(x) + g_1(x)y + \dots,$$

so especially $g_0 \in \mathfrak{m}^{2k+2}$, $g_1 \in \mathfrak{m}^{k+1}$ and $g_2 \in \mathfrak{m}$. An application of $\varphi \in \mathrm{Aut}(\mathbb{R}[[x,y]])$ given by $\varphi(x) = x$ and $\varphi(y) = y + \frac{1}{2}g_1(x)$ leads to

$$\varphi(f) = x^{2k+1} - y^2 + \left(g_0(x) + \frac{1}{4}g_1(x)^2\right) + g_2(x)y^2 + g_3(x)y^3 + \dots$$

such that we may assume (without loss of generality) that $g_1(x) = 0$. Suppose now that there are $i, m \in \mathbb{N}_0$ with $i \geq 2$ and $i + m \leq l - 1$ such that $g_i(x) \in ax^m + O(x^{m+1})$ with $a \neq 0$. Here, we use the Big O Notation in the limit to 0 (and in the generalized setting of power series as well), so $O(x^{m+1}) = x^{m+1} \cdot \mathbb{R}[[x,y]]$. If there is such an i, we choose i minimal and we apply $\varphi \in \mathrm{Aut}(\mathbb{R}[[x,y]])$ given by $\varphi(x) = x$ and $\varphi(y) = y + \frac{1}{2}ax^m y^{i-1}$ such that

$$\varphi(f) \in x^{2k+1} - y^2 + g_0(x) + O(x^{m+1}) \cdot y^i + g'_{i+1}y^{i+1} + \dots + g'_{l-1}y^{l-1} + \mathfrak{m}^l$$

for appropriate $g'_{i+1}, \dots, g'_{l-1} \in \mathbb{R}[[x,y]]$. Repeated applications of such automorphisms reduce our problem to

$$f \in x^{2k+1} - y^2 + g_0(x) + \mathfrak{m}^l.$$

If $g_0(x) \in \mathfrak{m}^l$, then the claim follows. Suppose now that $g_0(x) \in ax^m + O(x^{m+1})$ with $a \neq 0$ and $m < l$. Then, we apply $\varphi \in \mathrm{Aut}(\mathbb{R}[[x,y]])$ given by $\varphi(x) = x - \frac{1}{2k+1}ax^{m-2k}$ and $\varphi(y) = y$ such that

$$f \in x^{2k+1} - y^2 + g'_0(x) + \mathfrak{m}^l.$$

with $g'_0(x) \in O(x^{m+1})$. Repeating this argument, we obtain our claim. Note that each of the applied automorphisms really increases λ-degrees (i.e. the image of each monomial contains just monomials of the same or higher λ-degree) and this property is preserved under compositions of such automorphisms. □

Remark 3.5.12. The statement and proof of 3.5.11 directly carry over to $\mathbb{R}\{x,y\}$. However, this is not the case for the ring $\mathbb{R}[x,y]$ as we use an automorphism φ of the form $\varphi(x) = x - \frac{1}{2k+1}ax^{m-2k}$ and $\varphi(y) = y$.

Proposition 3.5.13. *Let $k \in \mathbb{N}$ and $\lambda = (2, 2k+1)$ and F be a face of $\mathbb{R}\{x,y\}_+$ or $\mathbb{R}[[x,y]]_+$ of finite codimension with $\Gamma(F) = \Gamma(x^{2(2k+1)} + y^4)$ as in 3.5.9. Let G be an $\subseteq$-maximal proper face of F with $\Gamma(G) = \Gamma(F)$.*

(a) If F is the monomial face $\mathrm{carr}(x^{2(2k+1)} + y^4)$, then

$$G \overset{r}{\sim} \mathrm{carr}\left((x^{2k+1} - y^2)^2 + x^{i_0}y^{j_0}\right),$$

with $i_0, j_0 \in \mathbb{N}_0$ such that $d_\lambda(x^{i_0}y^{j_0}) = 4(2k+1) + 2$.

(b) If $F = \mathrm{carr}((x^{2k+1} - y^2)^2 + x^{i_0}y^{j_0})$ with $2d_1 := d_\lambda(x^{i_0}y^{j_0}) > 4(2k+1)$, then

$$G \overset{r}{\sim} \mathrm{carr}\left((x^{2k+1} - y^2)^2 + x^{i'_0}y^{j'_0}\right)$$

with $i'_0, j'_0 \in \mathbb{N}_0$ such that $d_\lambda(x^{i'_0}y^{j'_0}) = 2d_1 + 2$.

Especially, for each such F, we have either

$$F = \operatorname{carr}(x^{2(2k+1)} + y^4) \quad \text{or} \quad F \overset{r}{\sim} \operatorname{carr}((x^{2k+1} - y^2)^2 + x^{i_0}y^{j_0})$$

with $i_0, j_0 \in \mathbb{N}_0$ such that $d_\lambda(x^iy^j)$ is even and $d_\lambda(x^iy^j) > 4(2k+1)$.

Proof. (a) Let $g \in \operatorname{rai}(G)$ and $\tau := \Gamma(F)$. By 3.3.18(e), $\tau(G)$ is a proper face of

$$\begin{aligned}\tau(F) &= \operatorname{carr}_{\mathcal{P}_\tau}\left(x^{2(2k+1)} + y^4\right) \\ &= \left\{ax^{2(2k+1)} + bx^{2k+1}y^2 + cy^4 \,\middle|\, a \geq 0, c \geq 0, 4ac - b^2 \geq 0\right\}.\end{aligned}$$

Thus, we have $g^\tau = ax^{2(2k+1)} + bx^{2k+1}y^2 + cy^4$ with $4ac - b^2 = 0$. Since $\Gamma(G) = \Gamma(F)$, we have $a \neq 0$ and $c \neq 0$, so $a > 0$ and $c > 0$. The automorphism φ with $x \mapsto \frac{-\operatorname{sgn}(b)}{2(2k+1)\sqrt{a}}x$, $y \mapsto \frac{1}{\sqrt[4]{c}}x$ has the property $\varphi(F) = F$ and gives $\tilde{g}^\tau = (x^{2k+1} - y^2)^2$ for $\tilde{g} := \varphi(g) \in \operatorname{rai}(\tilde{G})$ with $\tilde{G} := \varphi(G)$. We set $d_1 := 2(2k+1) + 1$ and we will show that $\tilde{G} = G'$ with $G' := \operatorname{carr}((x^{2k+1} - y^2)^2 + x^{i_0}y^{j_0})$ where $i_0, j_0 \in \mathbb{N}_0$ with $2d_1 := d_\lambda(x^{i_0}y^{j_0}) = 4(2k+1)+2$. We just have to show that $\tilde{g} \in G'$ as this implies $\tilde{G} \subseteq G' \subsetneq F$ and thus $\tilde{G} = G'$ as $\tilde{G}$ is also an $\subseteq$-maximal proper face of $\varphi(F) = F$. We decompose

$$\tilde{g} = (x^{2k+1} - y^2)^2\tilde{g}_2 + (x^{2k+1} - y^2)\tilde{g}_1 + \tilde{g}_0.$$

as in 3.5.8. Since $\tilde{g}^\tau = (x^{2k+1} - y^2)^2$, each monomial in $\tilde{g}_1$ has λ-degree $> 2(2k+1)$ (i.e. $\geq d_1$) and each monomial in $\tilde{g}_0$ has λ-degree $> 4(2k+1)$. We set $x = t^2$ and $y = t^{2k+1}$ and obtain that $\tilde{g}_0(t^2, t^{2k+1})$ has leading term of degree $> 4(2k+1)$ and as $\tilde{g}_0(t^2, t^{2k+1}) = \tilde{g}(t^2, t^{2k+1})$ is non-negative for small (positive) absolute value of t, the leading term has even degree $\geq 4(2k+1) + 2 = 2d_1$. 3.5.10 shows $\tilde{g} \in G'$ and thus the claim.

(b) Let $g \in \operatorname{rai}(G)$. We decompose g as

$$g = (x^{2k+1} - y^2)^2 g_2 + (x^{2k+1} - y^2)g_1 + g_0.$$

as in 3.5.8. Since $g \notin \operatorname{rai}(F)$ and $\Gamma(g) = \Gamma(F)$, we have $(4ag_0 - g_1^2)_{\lambda,2d_1}(1,1) = 0$ with $a := g_2(0,0)$. Let $g_1' := (g_1)_{\lambda,d_1}$. We have

$$\begin{aligned}g = &\left(x^{2k+1} - y^2 + \frac{1}{2a}g_1'\right)^2 g_2 + \left(x^{2k+1} - y^2 + \frac{1}{2a}g_1'\right)\left(g_1 - \frac{1}{a}g_2g_1'\right) \\ &+ \underbrace{g_0 - \frac{1}{2a}g_1'\left(g_1 - \frac{1}{a}g_2g_1'\right) - \frac{1}{4a^2}g_1'^2g_2}_{r(x,y)}.\end{aligned}$$

There are no monomials of λ-degree $< d_1$ in g_1 such that $\left(\frac{1}{2a}g_1'\left(g_1 - \frac{1}{a}g_2g_1'\right)\right)_{\lambda,2d_1} = 0$ and thus $(r)_{\lambda,2d_1}(1,1) = 0$ as $(4ag_0 - g_1^2)_{\lambda,2d_1}(1,1) = 0$. We note that there are no monomials of λ-degree $< 2d_1$ in r and that $(r)_{\lambda,2d_1}(1,1) = 0$ implies $x^{2k+1} - y^2 \mid (r)_{\lambda,2d_1}$. Let us choose $l \in \mathbb{N}$ such that $\mathfrak{m}^l \subseteq \operatorname{csupp}(G)$. By 3.5.11, we find some automorphism $\varphi \in \operatorname{Aut}(\mathbb{R}[[\underline{x}]])$ that just increases the λ-degree of monomials with the property

$\varphi(x^{2k+1}-y^2+\frac{1}{2a}g_1') = x^{2k+1}-y^2+h(x,y)$ and $h(x,y)\in\mathfrak{m}^l$. We set $\tilde{G}:=\varphi(G)$, $\tilde{g}_2:=\varphi(g_2)$, $\tilde{g}_1=\varphi\left(g_1-\frac{1}{a}g_2g_1'\right)$ and $\tilde{g}_0:=\varphi(r)$, so

$$\varphi(g)=(x^{2k+1}-y^2+h(x,y))^2\tilde{g}_2+(x^{2k+1}-y^2+h(x,y))\tilde{g}_1+\tilde{g}_0.$$

Let $\tilde{g}':=(x^{2k+1}-y^2)^2\tilde{g}_2+(x^{2k+1}-y^2)\tilde{g}_1+\tilde{g}_0$. We also have $\tilde{G}=\operatorname{carr}(\tilde{g}')$, as $h(x,y)\in\mathfrak{m}^l\subseteq\operatorname{csupp}(G)$ (use 2.2.8(c)). Now, $x^{2k+1}-y^2\,|\,(r)_{\lambda,2d_1}$ implies $x^{2k+1}-y^2\,|\,(\tilde{g}_0)_{\lambda,2d_1}$ such that $\tilde{g}'(t^2,t^{2k+1})$ has leading term of degree $>2d_1$. By the same argument as in (a), $\tilde{g}'(t^2,t^{2k+1})$ has leading term of degree $\geq 2d_1+2$ and using 3.5.10, we obtain

$$\varphi(G)=\tilde{G}=\operatorname{carr}(\tilde{g}')\subseteq G':=\operatorname{carr}((x^{2k+1}-y^2)^2+x^{i_0'}y^{j_0'})\subsetneq F$$

with $i_0',j_0'\in\mathbb{N}_0$ such that $d_\lambda(x^{i_0'}y^{j_0'})=2d_1+2$.
Let $f\in F$, so we have a decomposition $f=(x^{2k+1}-y^2)^2f_2+(x^{2k+1}-y^2)f_1+f_0$ such that the homogeneous part of lowest λ-degree of f_0 has λ-degree $\geq 2d_1$ and the homogeneous part of lowest λ-degree of f_1 has λ-degree $\geq d_1$ by 3.5.9. Then, we have

$$\varphi(x^{2k+1}-y^2)=x^{2k+1}-y^2+h(x,y)-\frac{1}{2a}\varphi(g_1')$$

and we find some f' such that

$$\begin{aligned}&\varphi(f)-(x^{2k+1}-y^2)f'\\&=\left(h(x,y)-\frac{1}{2a}\varphi(g_1')\right)^2\varphi(f_2)+\left(h(x,y)-\frac{1}{2a}\varphi(g_1')\right)\varphi(f_1)+\varphi(f_0)=:f_0'.\end{aligned}$$

The homogeneous part of lowest λ-degree of f_0' has degree $\geq 2d_1$ by construction. As $\varphi(f)(t^2,t^{2k+1})=f_0'(t^2,t^{2k+1})$, we obtain $\varphi(f)\in F$ by 3.5.10. This shows $\varphi(F)\subseteq F$ and thus $\varphi(F)=F$ by 3.4.7. As G is an $\subseteq$-maximal proper face of F, $\varphi(G)$ is an $\subseteq$-maximal proper face of $\varphi(F)$ and $\varphi(G)\subseteq G'\subsetneq F$ gives $\varphi(G)=G'$ as claimed.

□

We have now the following theorem on the classification of faces modulo right equivalence in the case $n=2$.

Theorem 3.5.14. *We consider the regular classification*

$$\varphi:\mathcal{F}(\mathbb{R}[[x,y]]_+)\to{}^{\mathcal{F}(\mathbb{R}[[x,y]]_+)}\!/_{\operatorname{Aut}(\mathbb{R}[[x,y]])}.$$

It induces a regular classification M of all faces of $\mathcal{F}(\mathbb{R}[[x,y]]_+)$ up to layer 8. We claim that M is represented by the Hasse diagram on page 86. The labels are faces that represent the equivalence class, they are chosen as monomial faces if possible and their codimension is indicated.
The statement transfers to the ring $\mathbb{R}\{x,y\}$ with the obvious changes.

Proof. We consider the faces that are labels of one vertex v of the Hasse diagram on page 86 that does not belong to the last row. As a first step, we will show that their $\subseteq$-maximal proper faces are right equivalent to one face that is a label of a vertex covered by the original vertex v. This follows directly from 3.5.6 for all vertices v except those covering some vertex that is not labelled by a monomial face (i.e. $v\notin\{B_6,E_{14},E_{16},H_6\}$).

In the cases of B_6, E_{14} and E_{16}, 3.5.13 reduces the problem to the determination of the $\subseteq$-maximal proper faces with smaller Newton polyhedron. Clearly, G and B_8 are the unique $\subseteq$-maximal proper faces of B_6 with smaller Newton polyhedron (use 3.5.6(a)). Let $\tau := \mathrm{conv}((6,0),(0,4))$, then $\tau(E_{2k}) = \mathbb{R}_+ \cdot (x^3 - y^2)^2$ $(k \geq 7)$, so 3.3.23(c) implies that there is a unique $\subseteq$-maximal proper face X_{2k} of E_{2k} with a smaller Newton polyhedron, which is determined by $X_{2k} = \{f \in E_{2k} \mid f^\tau = 0\}$. As J_4 is the largest monomial face with $J_4 \subseteq B_6$ and $\tau \cap \Delta(J_4) = \emptyset$ and as one can easily check that $J_4 \subseteq E_{14}$ (use 3.5.9), we have $X_{14} = J_4$. Now, $\Delta(X_{16}) \subseteq \Delta(X_{14})$ and $X_{16} \subseteq E_{16}$ imply $(4,2) \notin \Delta(X_{16})$ (use 3.5.9) and thus, $X_{16} \subseteq L$. As $L \subseteq E_{16}$ by 3.5.9, this shows $X_{16} = L$.

Finally, let us consider $\subseteq$-maximal proper faces of H_6. Now, 3.5.6 reduces the problem to the case of $\subseteq$-maximal proper faces X of H_6 with $\Delta(X) = \Delta(H_6)$ and $\tau(X) \neq \tau(H_6)$ with $\tau := \mathrm{conv}((0,4),(4,2))$ $(\mathcal{P}_\tau = \{ay^4 + by^3x^2 + cy^2x^4 \mid a \geq 0, c \geq 0, 4ac - b^2 \geq 0\})$. The $\subseteq$-maximal proper faces of $\mathcal{P}_\tau$ supported on whole τ are $Y_\lambda := \mathbb{R}_+ \cdot y^2(y - \lambda x^2)^2$ with $\lambda \in \mathbb{R} \setminus \{0\}$, so X is of the form $X = \{f \in G_6 \mid f^\tau \in Y_\lambda\}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$ by 3.3.23(c). We have to prove that $X \overset{r}{\sim} K$.

Let us show that $f := y^2(y - \lambda x^2)^2 + x^{10} \in \mathrm{rai}(X)$. For this purpose, we consider $X' := \mathrm{carr}(f)$ and its linear span. We claim that $x^8 y \in \mathrm{span}(X')$. Indeed, let $\epsilon := \frac{1}{2|\lambda|}$, then $|\epsilon x^8 y| \leq x^{10}$ for $|y| \leq \frac{1}{\epsilon}x^2$ and $|\epsilon x^8 y| \leq y^2(y - \lambda x^2)^2$ for $|y| \geq \frac{1}{\epsilon}x^2$ and small $|x| \geq 0$. Thus, we have $x^k y \in \mathrm{span}(X')$ for $k \geq 8$. We also note that $x^k \in \mathrm{span}(X')$ for $k \geq 10$. As $y^2(y - \lambda x^2)^2 \in \mathrm{span}(X')$, we have $y(y - \lambda x^2)x^k y^l \in \mathrm{span}(X')$ (especially, $x^k y^{l+2} \in \mathrm{span}(X')$ if and only if $x^{k+2} y^{l+1} \in \mathrm{span}(X')$) for all $x^{2k} y^{2l} \in \mathrm{span}(X')$ by the same argument as in 4.2.2(a). Since X' has finite codimension (3.2.5 and 3.2.4), a successive application of this observation (small calculation necessary) implies that $\mathrm{span}(X')$ contains the linear space of elements f with $\Delta(f) \subseteq \Delta(H_6)$, $f^\tau \in \mathbb{R} \cdot y^2(y - \lambda x^2)^2$ and $(y - \lambda x^2) \mid f^{\tau'}$ where $\tau' = \{(a,b) \in \mathbb{R}_+^2 \mid a + 2b = 9\}$. This linear space is contained in $\mathrm{span}(X)$ because the first two properties are obviously satisfied by elements of $\mathrm{span}(X)$ and the consideration of the leading term of $f(t, \lambda t^2)$ with $f \in X$ gives the third property. This shows $X' = X$ and $f := y^2(y - \lambda x^2)^2 + x^{10} \in \mathrm{rai}(X)$.

Using this observation, it is easy to prove that $X \overset{r}{\sim} J$. The automorphism given by $x \mapsto x$ and $y \mapsto y + \lambda x^2$ sends $X = \mathrm{carr}(y^2(y - \lambda x^2)^2 + x^{10})$ to the face $\mathrm{carr}(y^2(y + \lambda x^2)^2 + x^{10})$, so we may reduce to the case $\lambda > 0$. However, then the automorphism given by $x \mapsto \frac{1}{\sqrt{\lambda}}x$ and $y \mapsto y$ sends X to K as the size of the positive coefficient of x^{10} is irrelevant. This completes the first step of the proof.

It is an easy task to calculate the codimensions of the faces and the data shown on page 87 as all linear hulls are known: for monomial faces by 3.3.14, for faces of the form E_{2k} by 3.5.9 and for K by the consideration above.

In a second step, we show that all equivalence classes induced by the labels at the vertices are pairwise distinct. For this purpose, one can use the following four properties of representing faces F (see page 87), which are invariant under automorphisms: the codimension of F, the order of F, the minimimal d such that $\mathfrak{m}^d \subseteq \mathrm{span}(F)$ and the equivalence class of $\tau(F) \trianglelefteq \mathcal{P}_\tau$ modulo linear transformations with $\tau = \{(a,b) \in \mathbb{R}_+^2 \mid a + b = \mathrm{ord}(F)\}$ (so especially the number and multiplicities of the zeros of the homogeneous part of lowest degree of $f \in \mathrm{rai}(F)$). (An alternative way to distinguish two classes is using arguments as in 2.3.14.)

One observes that just the distinction of the faces K and B_{10} fails using these four properties above. Suppose that there would be some automorphism φ with $\varphi(B_{10}) = K$. As the homogeneous parts of degree 4 of B_{10} and K consist of multiples of y^4, we see that $\varphi(y)$ equals ay plus terms of order at least 2 (with $a \in \mathbb{R} \setminus \{0\}$). Furthermore, the coefficient

of x^2 in $\varphi(y)$ has to be zero as we would have $(8,0) \in \mathrm{supp}(\varphi(x^{10}+y^4)) \subseteq \Delta(\varphi(B_{10}))$ otherwise and $(8,0) \notin \Delta(K)$. However, this coefficient may not be zero as we would have $(4,2) \notin \Delta(\varphi(x^{10}+y^4)) = \Delta(\varphi(B_{10}))$, but $(4,2) \in \Delta(K)$. Hence, the equivalence classes of K and B_{10} are distinct.
Finally, we have to treat the problem mentioned in 2.3.15, which says that it is possible that $\subseteq$-maximal proper faces are not representing maximal smaller equivalence classes. If we could have made a complete diagram for all codimensions using the procedure of listing the equivalence classes of $\subseteq$-maximal proper faces as above, we would have to identify classes appearing multiple times and remove redundant edges to obtain the desired Hasse diagram for equivalence classes. As codimension is a strictly monotonic decreasing quantity on the equivalence classes, no class with faces of codimension ≤ 24 may appear another time below the classes shown (in such a complete diagram of equivalence classes of $\subseteq$-maximal proper faces). Hence, we may reduce to the proof of the statement that no two classes of the last layer are comparable. To show this statement, we also use that the other three properties behave monotonic. The order and the minimimal d such that $\mathfrak{m}^d \subseteq \mathrm{span}(.)$ behave monotonic decreasing and if the orders coincide, then the orders of the zeros of $\tau(.)$ (with $\tau = \{(a,b) \in \mathbb{R}^2_+ \mid a+b = \mathrm{ord}(.)\}$ as above) behave monotonic increasing (using one of the orders of 1.4.5 - see also 1.4.6). In this way, we can reduce to the proof of the statement that the equivalence classes represented by H_8, B_{10} and K are not smaller than the class represented by E_{18}. However, by the considerations above on $\subseteq$-maximal faces of E_{2k} ($k \geq 7$), classes smaller than E_{18} are either classes equal to or smaller than E_{20} or they are smaller than L (which was already ruled out). We obtain our claim as the codimension of E_{20} is larger than the codimensions of H_8, B_{10} and K. □

Remark 3.5.15. The result partially carries over to the polynomial case and faces of $\mathcal{P}^{Loc}$ modulo right equivalence. Indeed, if one looks at the automorphisms used, one just needs automorphism not in $\mathrm{Aut}_0(\mathbb{R}[\underline{x}])$ in the consideration of the faces of the chain $(x^3-y^2)^2+x^4y^2$, $(x^3-y^2)^2+x^8$, $(x^3-y^2)^2+y^6$. However, the result for $\mathbb{R}[[x,y]]_+$ still gives us a regular classification of all faces of finite codimension of $\mathcal{P}^{Loc}$ in the abstract sense as defined in 2.3.10.

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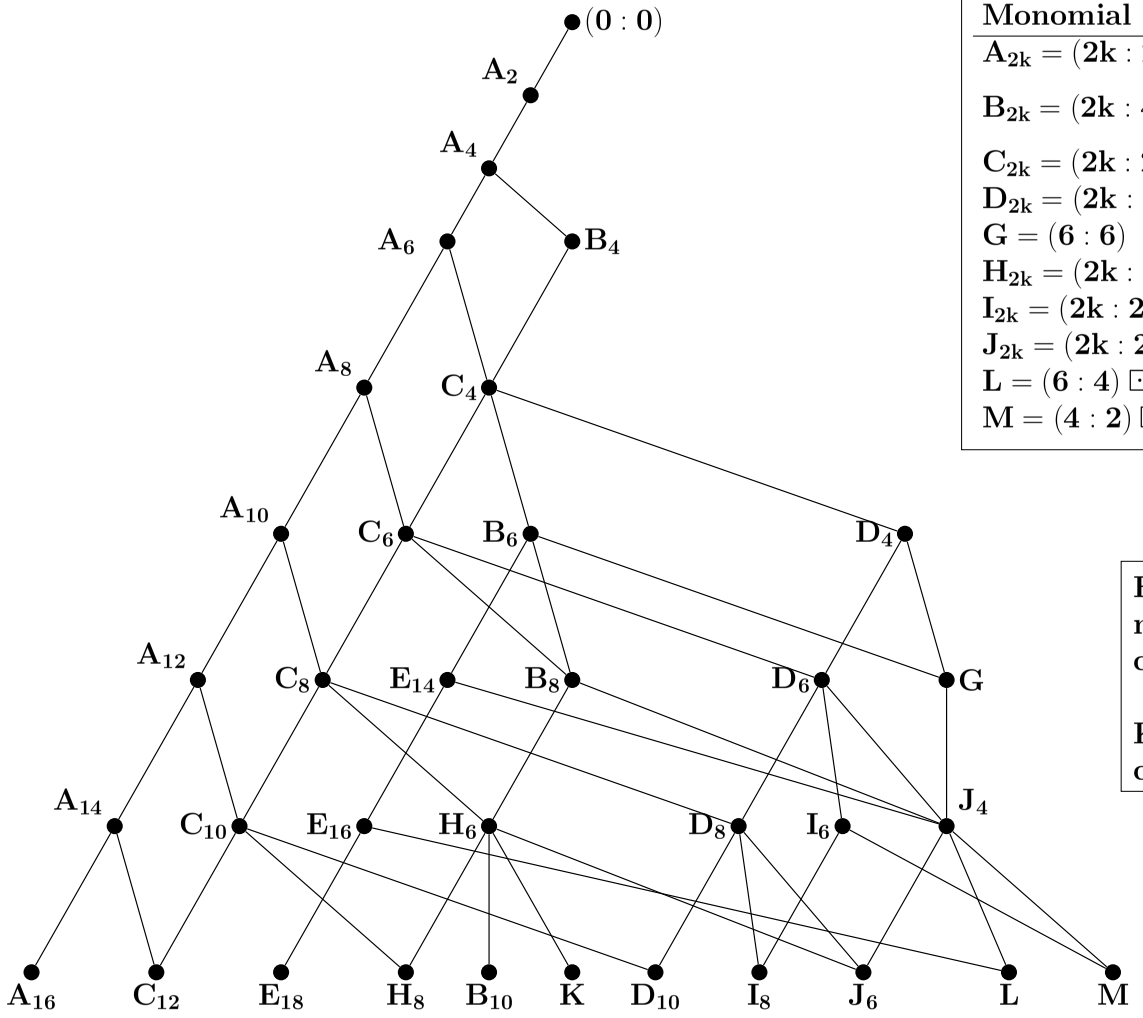


Monomial face	Codimension
$\mathbf{A_{2k} = (2k : 2),\ k \geq 1}$	$\mathbf{3k}$
$\mathbf{B_{2k} = (2k : 4),\ k \geq 2}$	$\mathbf{5k}$, k even $\mathbf{5k+1}$, k odd
$\mathbf{C_{2k} = (2k : 2) \boxdot (2 : 2),\ k \geq 2}$	$\mathbf{3k+7}$
$\mathbf{D_{2k} = (2k : 2) \boxdot (2 : 4),\ k \geq 2}$	$\mathbf{3k+10}$
$\mathbf{G = (6 : 6)}$	**21**
$\mathbf{H_{2k} = (2k : 2) \boxdot (4 : 2),\ k \geq 3}$	$\mathbf{3k+14}$
$\mathbf{I_{2k} = (2k : 2) \boxdot (2 : 6),\ k \geq 3}$	$\mathbf{3k+13}$
$\mathbf{J_{2k} = (2k : 2) \boxdot (4 : 4),\ k \geq 2}$	$\mathbf{3k+18}$
$\mathbf{L = (6 : 4) \boxdot (2 : 2)}$	**27**
$\mathbf{M = (4 : 2) \boxdot (2 : 2) \boxdot (2 : 4)}$	**27**

$\mathbf{E_{2k} = \mathrm{carr}((x^3 - y^2)^2 + m),\ k \geq 7}$
m monomial of $(2,3)$-degree 2k
codimension 3k − 2

$\mathbf{K = \mathrm{carr}(y^2(y - x^2)^2 + x^{10})}$
codimension 26

See 3.5.14

Face	ord(.)	Minimal d with $\mathfrak{m}^d \subseteq$ spn(.)	Orders of zeros of $\tau(.)$
$A_{2k} = carr(x^{2k} + y^2)$	2	2k	2, if $k > 1$
$B_{2k} = carr(x^{2k} + y^4)$	4	2k	4, if $k > 2$
$C_{2k} = carr(x^{2k+2} + x^2y^2 + y^4)$	4	$2k + 2$	2
$D_{2k} = carr(x^{2k+2} + x^2y^2 + y^6)$	4	$2k + 2$	2, 2
$G = carr(x^6 + y^6)$	6	6	-
$H_{2k} = carr(x^{2k+4} + x^4y^2 + y^4)$	4	$2k + 4$	4
$I_{2k} = carr(x^{2k+2} + x^2y^2 + y^8)$	4	$2k + 2$	2, 2
$J_{2k} = carr(x^{2k+4} + x^4y^2 + y^6)$	6	$2k + 4$	2
$L = carr(x^8 + x^2y^4 + y^6)$	6	8	4
$M = carr(x^8 + x^4y^2 + x^2y^4 + y^8)$	6	8	2, 2
$E_{2k} = carr((x^3 - y^2)^2 + m)$	4	k	4
$K = carr(y^2(y - x^2)^2 + x^{10})$	4	10	4

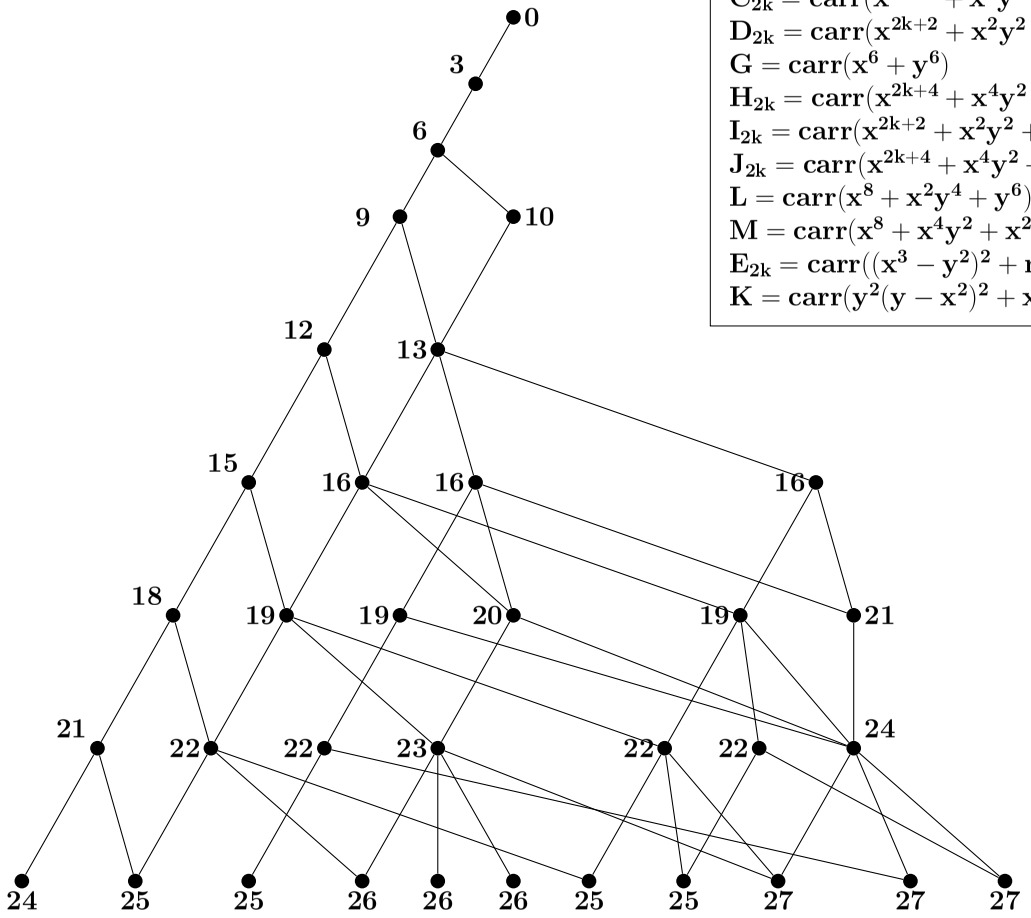


Diagram labelled by codimensions

See page before and 3.5.14

3.6 The case $n = 3$

We will now treat the case $n = 3$ and study faces modulo right equivalence. The following proposition is important to study them.

Proposition 3.6.1. *Let $n = 3$, F a monomial face of $\mathcal{P}^{Loc}$ of finite codimension, $\Delta := \Delta(F)$, $\Gamma := \Gamma(F)$ and G an $\subseteq$-maximal proper face of F. Suppose further that there is some face τ of Γ with $\tau(G) \neq \mathcal{P}_\tau$ such that there are $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}^3_+$ with $2\alpha_1 \in \Delta(G)$,*

$$\tau \cap \mathbb{Z}^3 = \{\alpha_i + \alpha_j \,|\, i, j \in \{1, 2, 3\}\},$$

$\alpha_2, \alpha_3 \geq \alpha_1 - (1, 0, 0)$ and $(\Delta + \mathbb{R}_+ \cdot (\alpha_i - \alpha_1)) \cap \mathbb{R}^3_+ = \Delta$ for $i = \{2, 3\}$. Then, $G \overset{r}{\sim} G'$ where G' is an $\subseteq$-maximal proper face of F such that $\tau'(G') \neq \mathcal{P}_{\tau'}$ with $\tau' := \mathrm{conv}(2\alpha_2, 2\alpha_3)$.

Proof. We set $\gamma := \alpha_1 - (1, 0, 0)$, $\beta_2 := \alpha_2 - \gamma$ and $\beta_3 := \alpha_3 - \gamma$. Let us consider the case of a 1-dimensional face τ, so $\alpha_2 = \alpha_3$. Then, we have

$$\mathbb{R}[\underline{x}]^\tau = \underline{x}^{2\gamma} \cdot \left\{ a_1 x_1^2 + a_2 x_1 \underline{x}^{\beta_2} + a_3 \underline{x}^{2\beta_2} \,\middle|\, a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

Note that $\mathcal{P}_\tau$ is given by the inequalities $a_1 \geq 0$, $a_3 \geq 0$ and $4a_1a_3 - a_2^2 \geq 0$. As $(\Delta + \mathbb{R}_+ \cdot (\alpha_2 - \alpha_1)) \cap \mathbb{R}^3_+ = \Delta$, we obtain by intersecting with the line through $2\alpha_1$ and $2\alpha_2$ that $(2\alpha_2 + \mathbb{R}_+ \cdot (\alpha_2 - \alpha_1)) \cap \mathbb{R}^3_+ \subseteq [2\alpha_1, 2\alpha_2]$ such that $2\alpha_2 + \epsilon(\alpha_2 - \alpha_1) \notin \mathbb{R}^3_+$ for $\epsilon > 0$. Since $\alpha_2 - \alpha_1 = \beta_2 - (1, 0, 0)$, this shows that the first coordinate of α_2 is 0. Hence, the first coordinates of β_2 and γ are 0.
Let $g \in \mathrm{rai}(G)$, so $g^\tau \in \mathrm{rai}(\tau(G))$. We may write $g^\tau = \underline{x}^{2\gamma} \cdot h(x_1, \underline{x}^{\beta_2})$ with some $h \in \mathcal{P}_{1,2}$. As $\tau(G)$ is a proper face of $\mathcal{P}_\tau$ and $2\alpha_1 \in \Delta(G)$, there is some $c \in \mathbb{R}$ with $h(c, 1) = 0$, so $h(y_1, y_2) \in \mathbb{R}_+ \cdot (y_1 - cy_2)^2$. Let us consider $\varphi \in \mathrm{Aut}_0(\mathbb{R}[\underline{x}])$ given by $\varphi(x_1) = x_1 + c\underline{x}^{\beta_2}$, $\varphi(x_2) = x_2$ and $\varphi(x_3) = x_3$. As in 3.5.6, we observe from $(\Delta + \mathbb{R}_+ \cdot (\alpha_2 - \alpha_1)) \cap \mathbb{R}^3_+ = \Delta$ that $\varphi(F) = F$ and $\varphi(G)^\tau \subseteq \varphi(\tau(G)) \subseteq \mathbb{R}_+ \cdot x_1^2$. Thus, $G' := \varphi(G)$, which is an $\subseteq$-maximal proper face of $\varphi(F) = F$, has the property $2\alpha_2 \notin \Delta(G')$, so $\tau'(G') = \{0\}$. This shows the claim in the 1-dimensional case.
Suppose now that τ has dimension 2, so $\alpha_2 \neq \alpha_3$. Then, we have

$$\mathbb{R}[\underline{x}]^\tau = \underline{x}^{2\gamma} \cdot \left\{ h(x_1, \underline{x}^{\beta_2}, \underline{x}^{\beta_3}) \,\middle|\, h \in \mathbb{R}[y_1, y_2, y_3]_2 \right\}.$$

As above, one can see that the first coordinates of β_2, β_3 and γ are 0. Furthermore, by the description of $\tau \cap \mathbb{Z}^3$, it is not possible that $\beta_2 + \beta_3 \in 2\mathbb{Z}^3$ as this would imply $\frac{3}{2}\alpha_2 + \frac{1}{2}\alpha_3 \in \tau \cap \mathbb{Z}^3$. From this and as $\beta_2 \neq \beta_3$, it is easy to see that the set of $(z_1 : z^{\beta_2} : z^{\beta_3}) \in \mathbb{P}^2(\mathbb{R})$ for some $z \in \mathbb{R}^3$ with $z_1 z_2 z_3 \neq 0$ is dense in $\mathbb{P}^2(\mathbb{R})$. This shows

$$\mathcal{P}_\tau = \underline{x}^{2\gamma} \cdot \left\{ h(x_1, \underline{x}^{\beta_2}, \underline{x}^{\beta_3}) \,\middle|\, h \in \mathcal{P}_{2,2} \right\}.$$

Let $g \in \mathrm{rai}(G)$, so $g^\tau \in \mathrm{rai}(\tau(G))$. We may write $g^\tau = \underline{x}^{2\gamma} \cdot h(x_1, \underline{x}^{\beta_2}, \underline{x}^{\beta_3})$ with $h \in \mathcal{P}_{2,2}$. As $\tau(G)$ is a proper face of $\mathcal{P}_\tau$, we find $(c_1 : c_2 : c_3) \in \mathbb{P}^2(\mathbb{R})$ with $h(c_1, c_2, c_3) = 0$ and we have $c_2 \neq 0$ or $c_3 \neq 0$ as $2\alpha_1 \in \Delta(G)$. Without loss of generality, let $c_2 \neq 0$. We define $\varphi \in \mathrm{Aut}_0(\mathbb{R}[\underline{x}])$ by $\varphi(x_1) = x_1 + \frac{c_1}{c_2}\underline{x}^{\beta_2}$, $\varphi(x_2) = x_2$ and $\varphi(x_3) = x_3$. As before, we observe $\varphi(F) = F$ and

$$\varphi(G)^\tau \subseteq \varphi(\tau(G)) \subseteq \underline{x}^{2\gamma} \cdot \left\{ h(x_1, \underline{x}^{\beta_2}, \underline{x}^{\beta_3}) \,\middle|\, h \in \mathcal{P}_{2,2},\, h(0, c_2, c_3) = 0 \right\}.$$

We set $G' := \varphi(G)$. Now, $\tau \cap \mathbb{Z}^3 = \{\alpha_i + \alpha_j \mid i,j \in \{1,2,3\}\}$ implies

$$\mathcal{P}_{\tau'} = \underline{x}^{2\gamma} \cdot \left\{ a_1 \underline{x}^{2\beta_2} + a_2 \underline{x}^{\beta_2+\beta_3} + a_3 \underline{x}^{2\beta_3} \,\middle|\, a_1, a_3, 4a_1a_3 - a_2^2 \geq 0 \right\}.$$

As seen above, we have $g'^{\tau}(0, c_2, c_3) = 0$ for each $g' \in G'$. Thus, we obtain the claim as $\tau'(G') \neq \mathcal{P}_{\tau'}$. □

Before we prove the theorem, we introduce some formulation to describe faces of $\mathcal{P}_{2,4}$ in the following remark.

Remark 3.6.2. Let F be a face of $\mathcal{P}_{2,4}$, $f \in \mathrm{ri}(F)$ and $P \in \mathbb{P}^n(\mathbb{R})$ such that $f(P) = 0$ and $f(Q) > 0$ for $Q \in U \setminus \{P\}$ and a neighbourhood U of P in $\mathbb{P}^n(\mathbb{R})$. After some linear coordinate change, we may assume that $P = (1:0:0)$ and after dehomogenization with X_0 and using lower case letters, we get an element g of $\mathcal{P} \cap \mathbb{R}[x_1, x_2]_{\leq 4}$ where we consider $\mathcal{P}$ for $n = 2$. Clearly, $g \in \mathcal{P}^{Loc}$ and we may consider the corresponding class of $\mathrm{carr}_{\mathcal{P}^{Loc}}(g)$ in the classification of the previous section. This class does not depend on the linear coordinate change we chose and also not on the choice of $f \in \mathrm{ri}(F)$. Thus, we can say that P is a zero of the type of this class.
For instance, faces of $\mathcal{P}_{2,4}$ of the class $F_{i,0,0}$ (see section 1.6) will have i zeros of type $(2:2)$. Faces of the class $F_{1,1,0}$ have one zero of type $(4:2)$. (A priori, it is not clear if zeros of type $(4:2)$ exist at all. But as there are examples of faces with one zero of type $(4:2)$ and no other zeros, they necessarily have to describe a face in the class $F_{1,1,0}$.) Similarly, faces of the class $F_{1,0,6}$ have one zero of type $(6:2)$ and so on.
Note also that the zero $P = (1:0:0)$ of f above may be recognized in the Newton polytope of f. Furthermore, if one has a zero of type $(4:2)$ (or higher) and one does an appropriate linear coordinate change fixing P, then one may also recognize it in the Newton polytope of f (choose the kernel of the Hessian in that way that it corresponds to the line through P and $(0:1:0)$ or $(0:0:1)$).

Now, we may prove the corresponding theorem.

Theorem 3.6.3. *We consider the regular classification*

$$\varphi : \mathcal{F}(\mathbb{R}[[x,y,z]]_+) \to {}^{\mathcal{F}(\mathbb{R}[[x,y,z]]_+)}\!/_{\mathrm{Aut}(\mathbb{R}[[x,y,z]])}.$$

It induces a regular classification M of all faces of $\mathcal{F}(\mathbb{R}[[x,y,z]]_+)$ up to layer 7. We claim that the equivalence classes with faces of corank at least 2 of M are represented by the Hasse diagram on page 92. The classes are labelled as the corresponding class on page 86 if they contain faces of corank 2 (recall 3.4.21), else they are labelled by some face representing the equivalence class. For the latter, we also give the codimensions and a characterization of the homogeneous part of degree 4 modulo linear coordinate changes.
An analogous result holds in the ring $\mathbb{R}\{x,y,z\}$ for the cone $\mathbb{R}\{x,y,z\}_+$.

Proof. Let T be the set of faces of $\mathbb{R}[[x,y,z]]_+$ with $\mathrm{ord}(F) = 4$ and $F \supseteq \mathfrak{m}^6 \cap \mathbb{R}[[x,y,z]]_+$ (i.e. faces with $\mathfrak{m}^4 \cap \mathbb{R}[[x,y,z]]_+ \supseteq F \supseteq \mathfrak{m}^6 \cap \mathbb{R}[[x,y,z]]_+$). By an application of 3.3.22 to the case $n = 2$ and $2d = 4$, they are in bijection to the faces of $\mathcal{P}_{2,4}$. Now, let us consider the action of $\psi \in \mathrm{Aut}(\mathbb{R}[[x,y,z]])$ on faces $F \in T$. As ψ sends the faces $\mathfrak{m}^4 \cap \mathbb{R}[[x,y,z]]_+$ and $\mathfrak{m}^6 \cap \mathbb{R}[[x,y,z]]_+$ to themselves, we also have $\psi(T) = T$. Furthermore, faces $F \in T$ are completely determined by $\tau(F)$ with $\tau := \Gamma(\mathfrak{m}^4 \cap \mathbb{R}[[x,y,z]]_+)$ and ψ acts by its linear part on $\tau(F)$. This shows that the facial structure of $\mathcal{P}_{2,4}$ modulo linear coordinate changes studied

by Kunert in [Kun] (see section 1.6) equals the structure of $\mathcal{F}(\mathbb{R}[[x,y,z]]_+)/\mathrm{Aut}(\mathbb{R}[[x,y,z]])$ restricted to the faces in T such that it appears as full subgraph in $\mathcal{F}(\mathbb{R}[[x,y,z]]_+)/\mathrm{Aut}(\mathbb{R}[[x,y,z]])$. We have the following correspondences (we write the face labelling the equivalence class on page 92 on the right-hand side)

$$P_4 \leftrightarrow B_4', \qquad F_{1,0,0} \leftrightarrow C_4', \qquad F_{1,1,0} \leftrightarrow B_6', \qquad F_{2,0,0} \leftrightarrow D_4',$$
$$F_| \leftrightarrow G', \qquad F_{1,0,6} \leftrightarrow N, \qquad F_{4s} \leftrightarrow O, \qquad F_{2,1,0} \leftrightarrow P, \qquad F_{3,0,0} \leftrightarrow Q.$$

All correspondences with a monomial face on the right-hand side (i.e. all except the case $F_{1,0,6} \leftrightarrow N$) follow from the easy observations that these faces lie in T and that they have the correct Newton polyhedron such that $\tau(.)$ induces the corresponding left-hand side (see 3.6.2). The face $\tau(N)$ has exactly one zero in $\mathbb{P}^3$ and it is of type $(6:2)$ (set $x=1$ in $(xz-y^2)^2+z^4+y^2z^2$ and apply the coordinate change $z \mapsto z+y^2$). Then, 4.2.15(b) shows the claimed correspondence $F_{1,0,6} \leftrightarrow N$.
In the following part in parentheses, we argue similarly to the case of two variables (see 3.5.14). We determine (up to right equivalence) the $\subseteq$-maximal proper faces of all faces of corank 3 that label vertices that are not in the last row of the diagram on page 92. This part nearly shows the claim, it illustrates 3.6.1 and it is algorithmic but there is a more elegant proof of the statement below such that it may be omitted.
(We apply 3.3.23(c) and 3.6.1 to the faces labelling the equivalence classes to classify their $\subseteq$-maximal proper faces. Using the correspondences above, we may reduce to the consideration of $\subseteq$-maximal proper faces not in T. Thus, if we start with some monomial face F_0 in T and we apply 3.3.23(c), then we are in the second case of 3.3.23(c) and we just have to consider faces τ_0 of $\Gamma(F)$ not contained in $\tau := \Gamma(\mathfrak{m}^4 \cap \mathbb{R}[[x,y,z]]_+)$. Hence, in the case of B_4', we will not have other $\subseteq$-maximal proper faces.
Furthermore, the determination of the $\subseteq$-maximal proper faces F of C_4' is reduced to those with $\tau_0(F) \neq \mathcal{P}_{\tau_0}$ and $\tau_1(F) = \mathcal{P}_{\tau_1}$ where $\tau_0 := \mathrm{conv}((6,0,0),(2,2,0),(2,0,2))$ and $\tau_1 := \mathrm{conv}((2,2,0),(2,0,2))$. Setting $\alpha_1 := (1,1,0)$, $\alpha_2 := (1,0,1)$ and $\alpha_3 := (3,0,0)$, we are in the situation of 3.6.1 with x_1 and x_2 exchanged (i.e. we have to use $(0,1,0)$ instead of $(1,0,0)$ in $\alpha_2, \alpha_3 \geq \alpha_1 - (1,0,0)$) and we may reduce to the case $\tau_2(F) \neq \mathcal{P}_{\tau_2}$ with $\tau_2 := \mathrm{conv}((2,0,2),(6,0,0))$. Another application of 3.6.1 with x_1 and x_3 exchanged and $\alpha_1 := (1,0,1)$, $\alpha_2 := \alpha_3 := (3,0,0)$ reduces the problem to the case $\tau_3(F) \neq \mathcal{P}_{\tau_3}$ with $\tau_3 := \{(6,0,0)\}$, which leads to C_6'.
Analogously, one proves in the cases C_6' and D_4' that the $\subseteq$-maximal proper faces with the same face $\tau(.)$ are right equivalent to C_8' and D_6' respectively. This completes the classification of $\subseteq$-maximal proper faces of $D_4' \in T$. In the case of C_6' ($\notin T$), the $\subseteq$-maximal proper faces F with smaller $\tau(.)$ are divided in those with $\tau_1(F) \neq \mathcal{P}_{\tau_1}$ that are right equivalent to B_8' via some linear change of coordinates in y and z and those with $\tau_1(F) = \mathcal{P}_{\tau_1}$ that are right equivalent to D_6' via some linear change of coordinates in y and z and one of the form $x \mapsto x + ay$ ($a \in \mathbb{R}$) as one can check easily.
In the determination of $\subseteq$-maximal proper faces F of $B_6' \in T$ with the same face $\tau(.)$, one has to consider faces $F \trianglelefteq B_6'$ with $\tau_4(F) \neq \mathcal{P}_{\tau_4}$ with $\tau_4 := \mathrm{conv}((6,0,0),(0,4,0),(2,0,2))$. One may consider elements of $\tau_4(B_6')$ as quadratic forms $f(t,u,v)$ with $t = x^3$, $u = y^2$ and $v = xz$. Since $\{(x^3 : y^2 : xz) \,|\, x,y,z \in \mathbb{R}, xy \neq 0\}$ is dense in $\mathbb{P}^3$, the $\subseteq$-maximal proper faces of $\tau_4(B_6')$ are those such that the quadratic form $f(t,u,v)$ has some zero in $\mathbb{P}^3$. We may assume that this zero does not lie on $t=0$ as $\tau(F) = \tau(B_6')$. Now, a linear transformation of the form $v \mapsto v + at$ ($a \in \mathbb{R}$) sends this zero on the line $v=0$. Thus, in the original setting,

the transformation $z \mapsto z + ax^2$ will reduce the problem to faces F with $\tau_5(F) \neq \mathcal{P}_{\tau_5}$ with $\tau_5 := \mathrm{conv}((6,0,0),(0,4,0))$ (note that this transformation sends B_6' to itself by 3.4.7). As $\mathcal{P}_{\tau_5} = \{ax^6 + bx^3y^2 + cy^4 \,|\, a, c, 4ac - b^2 \geq 0\}$ and as $(0,4,0) \in \Delta(F)$, a scaling of x or y reduces our problem to the cases $\tau_5(F) = \{ax^6 \,|\, a \geq 0\}$ or $\tau_5(F) = \{a(x^3 - y^2)^2 \,|\, a \geq 0\}$. The first case leads directly to B_8' and the second case results in E_{14}' by 3.5.14 and 3.4.26 and using that $\Delta(F) = \Delta(B_6')$ and $\tau_5(F) = \{a(x^3 - y^2)^2 \,|\, a \geq 0\}$ determine a unique $\subseteq$-maximal proper face F of B_6'.)

Let us return to the actual proof. Let S be the set of faces of $\mathbb{R}[[x,y,z]]_+$ of corank 3 that are $\subseteq$-maximal proper faces of faces of corank 2. We consider two equivalence classes (modulo right equivalence) A and B of faces of S with $B \leq A$. Hence, using 3.4.26, we find some $G_B \trianglelefteq \mathbb{R}[[x,y]]_+$ such that $F_B := \mathrm{carr}(G_B) \boxplus \mathrm{carr}(x^2z^2, y^2z^2, z^4)$ represents B. Let $F_A \in A$ with $F_A \supseteq F_B$. As $\mathrm{carr}(x^2z^2, y^2z^2, z^4) \subseteq F_A$, there is some $f = f_1 + f_2 z + f_3 z^2 \in \mathrm{rai}(F_A)$ with $f_1, f_2 \in \mathbb{R}[[x,y]]$ and $f_3 = x^2 + y^2 + z^2$ (compare proof of 3.4.26). Hence, $f_3 z^2 \in \mathrm{span}(F_A)$ implies $2f + 14 f_3 z^2 \in \mathrm{rai}(F_A)$ and

$$2f + 14f_3z^2 - f_1 = f_1 + f_2 \cdot (2z) + f_3(x,y,2z) \cdot (2z)^2 + 12(x^2+y^2)z^2 \in \mathbb{R}[[x,y,z]]_+$$

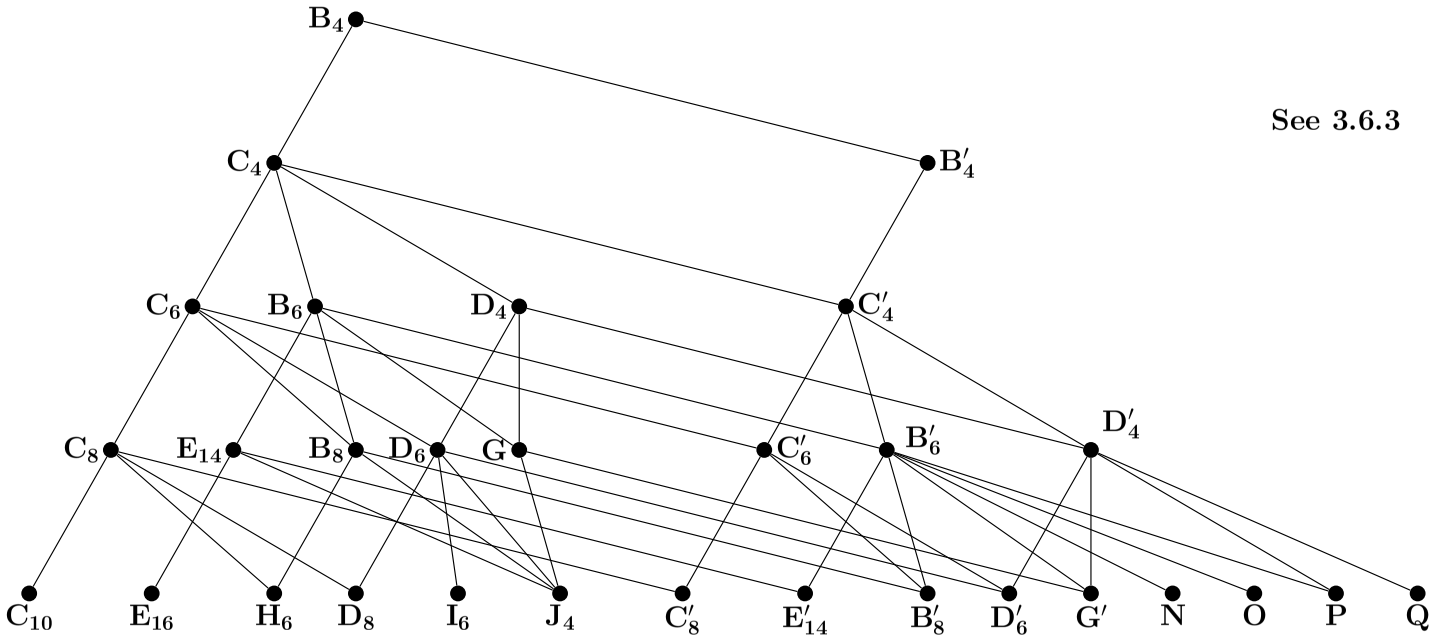
implies $f_1 \in \mathrm{span}(F_A)$, so $f_1 \in F_A$. Now, 3.4.18 and 2.2.11 imply $F_A \subseteq \mathrm{carr}(G_A) \boxplus \mathrm{carr}(z^2)$ with $G_A := F_A \cap \mathbb{R}[[x,y]] \supseteq G_B$. We also have $F_A \supseteq \mathrm{carr}(G_A) \boxplus \mathrm{carr}(x^2z^2, y^2z^2, z^4)$. Since $\mathrm{carr}(G_A) \boxplus \mathrm{carr}(x^2z^2, y^2z^2, z^4)$ is an $\subseteq$-maximal proper face of $\mathrm{carr}(G_A) \boxplus \mathrm{carr}(z^2)$ by 3.4.26 and as F_A has corank 3, we obtain $F_A = \mathrm{carr}(G_A) \boxplus \mathrm{carr}(x^2z^2, y^2z^2, z^4)$. Thus, considering stable equivalence, A and B are the classes of corank 3 that are maximal smaller than the classes represented by G_A and G_B and the latter classes are ordered accordingly as $G_B \subseteq G_A$. This shows that the full subgraph of the Hasse diagram of $\mathcal{F}(\mathbb{R}[[x,y,z]]_+)/\mathrm{Aut}(\mathbb{R}[[x,y,z]])$ with vertices corresponding to classes with faces of S is a copy of the full subgraph of classes of corank 2. By 3.4.26, the only edges in the Hasse diagram of $\mathcal{F}(\mathbb{R}[[x,y,z]]_+)/\mathrm{Aut}(\mathbb{R}[[x,y,z]])$ connecting classes of faces of corank 2 and corank 3 are those arising from corresponding vertices.

Using 3.4.26, one can see easily that the vertices labelled by faces with apostrophe represent the equivalence classes of corank 3 that are maximal smaller than the equivalence classes of corank 2 represented by the face without apostrophe. We have seen how the vertices, their labels and the edges on page 92 arise and it is an easy task to calculate the corresponding classes of $\tau(.)$. The codimensions of the monomial faces of corank 3 may be calculated using 3.3.14 and the codimensions of N and E_{14}' may be read off from their linear spans, which will be calculated in 4.2.15(b),(c).

Let us show that there are no more vertices, so let F be one of the faces B_4', C_4', C_6', B_6' or D_4' and X be an $\subseteq$-maximal proper face of F. Then, either $\tau(X) = \tau(F)$ or $\tau(X)$ is an $\subseteq$-maximal proper face of $\tau(F)$ and $X = \{f \in F \,|\, f^\tau \in \tau(X)\}$. If $\tau(X)$ is either in P_4, $F_{1,0,0}$, $F_{1,1,0}$ or $F_{2,0,0}$, then $X \in S$. Furthermore, if $F \in T$ and $\tau(X)$ is an $\subseteq$-maximal proper face of $\tau(F)$, then we have also $X \in T$. It is easy to check that one of both cases always holds such that there are indeed no more vertices.

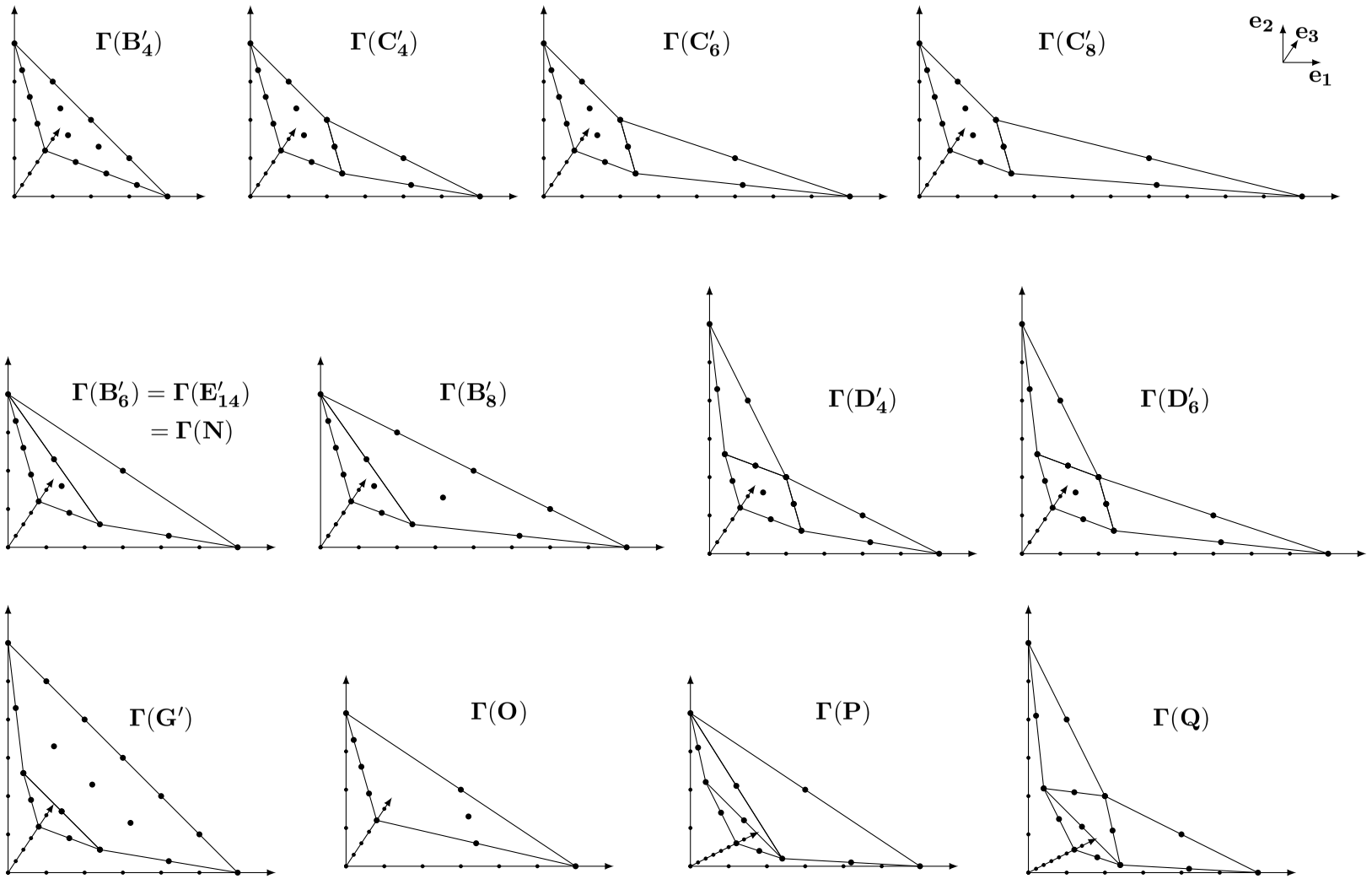
Finally, we have to show that there are no more edges. By the prior observations, this is clear between classes that have either corank at most 2 or classes with faces of S and between classes with faces of T. Thus, we just have to consider relations between one class A with faces of corank 2 or with faces in S and one class B with faces in $T \setminus S$ (both in the first 7 layers of $\mathcal{F}(\mathbb{R}[[x,y,z]]_+)/\mathrm{Aut}(\mathbb{R}[[x,y,z]])$). It is not possible that $A \leq B$ as both classes contain faces of order at most 4 and the faces in B have order 4 and smaller faces $\tau(.)$. If $A \geq B$, then we may reduce to the case that A is of corank 3, but this implies that A consists of faces in T and we obtain one of the edges already drawn. $\square$

Face	Codimension	Class of $\tau(.)$
$\mathbf{B'_4 = carr(x^4 + y^4 + z^4)}$	**20**	$\mathbf{P_4}$
$\mathbf{C'_{2k} = carr(x^{2k+2} + y^4 + z^4 + x^2y^2 + x^2z^2), k \geq 2}$	$\mathbf{4k + 16}$	$\mathbf{F_{1,0,0}}$
$\mathbf{B'_{2k} = carr(x^{2k} + y^4 + z^4 + x^2z^2), k > 2}$	$\mathbf{\frac{1}{2}(13k + 14), k}$ even $\mathbf{\frac{1}{2}(13k + 17), k}$ odd	$\mathbf{F_{1,1,0}}$
$\mathbf{D'_{2k} = carr(x^{2k+2} + y^6 + z^4 + x^2y^2 + x^2z^2 + y^2z^2), k \geq 2}$	$\mathbf{4k + 20}$	$\mathbf{F_{2,0,0}}$
$\mathbf{E'_{14} = carr((x^3 - y^2)^2 + x^4y^2 + x^2z^2 + y^2z^2 + z^4)}$	**32**	$\mathbf{F_{1,1,0}}$
$\mathbf{G' = carr(x^6 + y^6 + x^2z^2 + y^2z^2 + z^4)}$	**35**	$\mathbf{F_{|}}$
$\mathbf{N = carr((xz - y^2)^2 + z^4 + y^2z^2 + x^6)}$	**32**	$\mathbf{F_{1,0,6}}$
$\mathbf{O = carr(x^6 + y^4 + z^4)}$	**33**	$\mathbf{F_{4s}}$
$\mathbf{P = carr(x^6 + y^4 + x^2z^2 + y^2z^2 + z^6)}$	**32**	$\mathbf{F_{2,1,0}}$
$\mathbf{Q = carr(x^6 + y^6 + z^6 + z^4 + x^2y^2 + x^2z^2 + y^2z^2)}$	**32**	$\mathbf{F_{3,0,0}}$



See 3.6.3

Newton diagrams



4 Application to Scheiderer's local-global principle

This chapter studies the representability of locally non-negative elements as sums of squares in the setting of formal power series and relaxed to faces of the considered cone of non-negative power series. It is motivated by the typical geometric application of Scheiderer's local-global principle. There, the existence of such representations are important to ensure that the premises are satisfied to apply the local-global principle. We will just study the case $\mathbb{R}[[\underline{x}]]_+$, so non-negative power series with respect to the preordering $\Sigma\,\mathbb{R}[[\underline{x}]]^2$.
We start with a motivation, which explains the relation to the local-global principle in more detail. Then, we show that our problem is invariant under stable equivalence, which (together with some well-known results) reduces the faces to be studied drastically. Our main goal is then to study large faces that are not contained in a face with a negative answer. We divide this problem in the study of a simplified question on the support of the algebraic closure of the face and a remaining step where especially sum of squares representations on the Newton diagram are important. We get positive results in the studied large cases but we also present some cases where the problem has a negative answer.

4.1 Motivation and invariance under stable equivalence

Let us state a special case of Scheiderer's local-global principle (see [Sch3]) in the following theorem.

Theorem 4.1.1. ***(Scheiderer's local-global principle)***
Let V be a non-singular n-dimensional affine irreducible $\mathbb{R}$-variety and M a finitely generated Archimedean quadratic module in the coordinate ring $\mathbb{R}[V]$. Let $S \subseteq V(\mathbb{R})$ be the corresponding basic closed semi-algebraic set.
We consider $f \in \mathbb{R}[V]$ with $f \geq 0$ on S such that $\mathcal{V}_{\mathbb{R}}(f) \cap U$ is finite for some neighbourhood U of S on $V(\mathbb{R})$. If f is contained in the quadratic module $\hat{M}_P$ generated by M in the completion $\hat{\mathcal{O}}_{V,P}$ for all $P \in \mathcal{V}_{\mathbb{R}}(f) \cap S$, then $f \in M$.

Proof. A more general version may be found in [Sch3, 3.4.]. □

Since V is non-singular, we have $\hat{\mathcal{O}}_{V,P} \cong \mathbb{R}[[\underline{x}]]$ for all $P \in V(\mathbb{R})$ (see [Hart, p.34]). Given some finite set of non-zero generators of M and some general $P \in \mathring{S}$, none of the generators of M will vanish in P. Then, it is easy to see that $\hat{M}_P = \Sigma\,\hat{\mathcal{O}}_{V,P}^2$ (see 4.1.2(a)) and we have $f \in (\hat{\mathcal{O}}_{V,P})_+$ for each $f \geq 0$ on S. Thus, for the application of the theorem, it is an important question by how much the cones $\mathbb{R}[[\underline{x}]]_+$ and $f \in \Sigma\,\mathbb{R}[[\underline{x}]]^2$ differ from each other. We have the following well-known positive results.

Proposition 4.1.2. *Let $f \in \mathbb{R}[[\underline{x}]]_+$. We have $f \in \Sigma\,\mathbb{R}[[\underline{x}]]^2$ if one of the following conditions holds*

(a) $f(0) > 0$

(b) $f(0) = 0$ *and* $\mathrm{Hess}(f)$ *is positive definite*

(c) $n = 2$.

Proof. (a) holds since every such f is even a square. (b) may be found in [Sch2, 3.18.]. (c) was proved in [Sch4, 4.1.] □

More positive results on this question were found by Sekiguchi in [Sek]. We present results based on his work later. However, we will study the question in the following weaker version.

Problem 4.1.3. Let F be a face of $\mathbb{R}[[\underline{x}]]_+$. When does it hold that $\mathrm{rai}(F) \subseteq \Sigma\,\mathbb{R}[[\underline{x}]]^2$? What are the $\subseteq$-maximal faces of $\mathbb{R}[[\underline{x}]]_+$ such that it does not hold?

The second question of this problem is relevant since these faces will quantify the set of $f \in \mathbb{R}[[\underline{x}]]_+$ that are not sums of squares.
Let us first give a necessary condition for $\mathrm{rai}(F) \subseteq \Sigma\,\mathbb{R}[[\underline{x}]]^2$ (with $F \trianglelefteq \mathbb{R}[[\underline{x}]]_+$). We need the following notation (corresponding to $\mathcal{P}_\tau$ defined in 3.3.15).

Notation 4.1.4. Let τ be a face of a psd-diagram Γ. Then, we set

$$\Sigma_\tau := \left\{ \sum_{i=1}^{k} f_i^2 \;\middle|\; f_i \in \mathbb{R}[\underline{x}]^{\frac{\tau}{2}} \right\}.$$

Lemma 4.1.5. *Let $f \in \Sigma\,\mathbb{R}[[\underline{x}]]^2$ and $\Gamma := \Gamma(f)$. Then, there are $g_1, \ldots, g_k \in \mathbb{R}[\underline{x}]$ such that $\mathrm{supp}(g_i) \subseteq \frac{1}{2}\Gamma$ and $f^\Gamma = \sigma^\Gamma$ with $\sigma := g_1^2 + \cdots + g_k^2$. Especially, we have $f^\tau \in \Sigma_\tau$ for all faces τ of Γ.*

Proof. Let $f = \sum_{i=1}^k f_i^2$ with $f_i \in \mathbb{R}[[\underline{x}]]$. For any $h \in \mathbb{R}[[\underline{x}]]$ one has $\Delta(h) \subseteq \frac{1}{2}\Delta(h^2)$. Indeed, each vertex α of $\Delta(h)$ (so $\alpha \in \mathrm{supp}(h)$) leads to $h^{\{\alpha\}} = cx^\alpha$ for some $c \neq 0$ and thus, $h^{\{2\alpha\}} = c^2x^{2\alpha}$ with $c^2 \neq 0$ since $2\alpha = \alpha + \alpha$ is the unique decomposition of 2α in a sum of two elements of $\Delta(h)$. Hence, $2\alpha \in \mathrm{supp}(h^2) \subseteq \Delta(h^2)$. Now, using 3.3.6, we obtain

$$\mathrm{supp}(f_i) \subseteq \frac{1}{2}\mathrm{supp}\left(f_i^2\right) \subseteq \frac{1}{2}\mathrm{supp}\left(\sum_{i=1}^{k} f_i^2\right) = \frac{1}{2}\mathrm{supp}(f) = \frac{1}{2}\Delta$$

for $i = 1, \ldots, k$ (compare also with [Rez, Theorem 1]) and using $\Delta := \Delta(f)$. We set $g_i := (f_i)^{\frac{\Gamma}{2}}$ for $i = 1, \ldots, k$. Let m_1, m_2 be monomials with $\mathrm{supp}(m_1) = \{\alpha\}$, $\mathrm{supp}(m_2) = \{\beta\}$ and $\alpha, \beta \in \frac{1}{2}\Delta$ such that $\alpha + \beta = \frac{2\alpha+2\beta}{2} \in \tau$ for some face τ of Γ. As τ is a face of Δ, we have $2\alpha, 2\beta \in \tau$ and thus $\mathrm{supp}(m_1), \mathrm{supp}(m_2) \subseteq \frac{1}{2}\tau \subseteq \frac{1}{2}\Gamma$. As $\alpha + \beta \in \tau$ is equivalent to $\mathrm{supp}(m_1 \cdot m_2) \subseteq \tau$, these considerations show that each term of f^τ arising in the calculation of $(\sum_{i=1}^k f_i^2)^\tau$ also arises in σ^τ with $\sigma := g_1^2 + \cdots + g_k^2$. Thus, $f^\tau = \sigma^\tau$ for all faces τ of Γ. This shows the claim $f^\Gamma = \sigma^\Gamma$. Setting $h_i := (f_i)^{\frac{\tau}{2}}$, we similarly have $f^\tau = \sum_{i=1}^k h_i^2 \in \Sigma_\tau$ for all faces τ of Γ. □

Corollary 4.1.6. *Let F be a monomial face of $\mathbb{R}[[\underline{x}]]_+$ with $\mathrm{rai}(F) \subseteq \Sigma\,\mathbb{R}[[\underline{x}]]^2$, then $\mathcal{P}_\tau = \Sigma_\tau$ for faces τ of $\Gamma(F)$.*

From this corollary we get immediately some negative answers to our posed problem 4.1.3.

Remark 4.1.7. By Hilbert's theorem 1.1.4, this corollary implies that $\mathrm{rai}(F) \not\subseteq \Sigma\,\mathbb{R}[[\underline{x}]]^2$ in the cases $F = \mathbb{R}[[\underline{x}]]_+ \cap \mathfrak{m}^6$ for $n = 3$ and $F = \mathbb{R}[[\underline{x}]]_+ \cap \mathfrak{m}^4$ if $n = 4$ (where $\mathfrak{m}$ denotes the maximal ideal of $\mathbb{R}[[\underline{x}]]$) and all other cases with higher n or higher even exponents of $\mathfrak{m}$. This idea to construct counterexamples is well-known (with polynomials instead of faces) and goes under the name "bad points" (see [Del, V.A.]).

Clearly, the classification of faces F of $\mathbb{R}[[\underline{x}]]_+$ with $\mathrm{rai}(F) \subseteq \Sigma\,\mathbb{R}[[\underline{x}]]^2$ is invariant under automorphisms of $\mathbb{R}[[\underline{x}]]$. The following theorem shows that problem 4.1.3 is even invariant under stable equivalence (see 3.4.24 for definition).

Theorem 4.1.8. *Let $F \trianglelefteq \mathbb{R}[[x_1,\dots,x_{n_1}]]_+$ and $G \trianglelefteq \mathbb{R}[[x_1,\dots,x_{n_2}]]_+$ with $F \overset{s}{\sim} G$, then*

$$\operatorname{rai}(F) \subseteq \Sigma\,\mathbb{R}[[x_1,\dots,x_{n_1}]]^2 \quad \Leftrightarrow \quad \operatorname{rai}(G) \subseteq \Sigma\,\mathbb{R}[[x_1,\dots,x_{n_2}]]^2.$$

Proof. The claim is obvious if $F = \mathbb{R}[[x_1,\dots,x_{n_1}]]_+$ or $F = \mathbb{R}[[x_1,\dots,x_{n_1}]]_+ \cap \mathfrak{m}^2$ ($\mathfrak{m}$ is the maximal ideal of $\mathbb{R}[[x_1,\dots,x_{n_1}]]$, use 4.1.2(a),(b)). Thus, without loss of generality, we may assume that $n_1 = \operatorname{cork}(F) \geq 1$ and $n_1 < n_2$.
"$\Leftarrow$" Let $\operatorname{rai}(G) \subseteq \Sigma\,\mathbb{R}[[x_1,\dots,x_{n_2}]]^2$ and $f \in \operatorname{rai}(F)$. Then,

$$\sigma := f + x_{n_1+1}^2 + \cdots + x_{n_2}^2 \in \operatorname{rai}(G) \subseteq \Sigma\,\mathbb{R}[[x_1,\dots,x_{n_2}]]^2.$$

Hence, $f = \sigma(x_1,\dots,x_{n_1},0,\dots,0) \in \Sigma\,\mathbb{R}[[x_1,\dots,x_{n_1}]]^2$, which shows one implication.
"$\Rightarrow$" Let $\operatorname{rai}(F) \subseteq \Sigma\,\mathbb{R}[[x_1,\dots,x_{n_1}]]^2$ and $g \in \operatorname{rai}(G)$. By the Splitting lemma 3.4.9, g is stably equivalent to some $f' \in \mathbb{R}[[x_1,\dots,x_{n_1}]]$ and by the Splitting lemma for faces 3.4.21, f' is right equivalent to some $f \in \operatorname{rai}(F)$. Thus, $f, f' \in \Sigma\,\mathbb{R}[[x_1,\dots,x_{n_1}]]^2$ and $f' + x_{n_1+1}^2 + \cdots + x_{n_2}^2, g \in \Sigma\,\mathbb{R}[[x_1,\dots,x_{n_2}]]^2$. This shows the claim. □

This theorem, 4.1.2 and 4.1.7 reduce the problem of determining the $\subseteq$-maximal faces F of $\mathbb{R}[[\underline{x}]]_+$ with $\operatorname{rai}(F) \not\subseteq \Sigma\,\mathbb{R}[[\underline{x}]]^2$ as posed in 4.1.3 to the case of faces F with $n = 3$ and $\operatorname{ord}(F) = 4$. We will show that some large faces F of $\mathbb{R}[[\underline{x}]]_+$ with $n = 3$ and $\operatorname{ord}(F) = 4$ fulfil $\operatorname{rai}(F) \subseteq \Sigma\,\mathbb{R}[[\underline{x}]]^2$ and give some more positive and negative answers on the question if the studied property is satisfied. However, we will first study the weaker condition (see 2.2.8(c)) if $\operatorname{csupp}(F \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2) = \operatorname{csupp}(F)$ holds.

4.2 Equality of the supports of the algebraic closures

Let us start with an application of the results of chapter 2 to $\mathbb{R}[[\underline{x}]]^2$.

Remark 4.2.1. The cone $\Sigma\,\mathbb{R}[[\underline{x}]]^2$ is multiplicative, Noetherian and Archimedean. Thus, the statements of 2.4.20 hold. Especially, if G is a face of $\Sigma\,\mathbb{R}[[\underline{x}]]^2$, then

- $\operatorname{span}(G)$ and $\operatorname{csupp}(G)$ are ideals of $\mathbb{R}[[\underline{x}]]$
- G is a Klee face
- $\mathfrak{m}\operatorname{span}(G) \subseteq \operatorname{csupp}(G)$ and ${}^{\operatorname{span}(G)}/_{\operatorname{csupp}(G)}$ has finite dimension (where $\mathfrak{m}$ is the maximal ideal of $\mathbb{R}[[\underline{x}]]$)

Furthermore, using 2.2.8(c), if $f \in \operatorname{rai}(G)$, then

$$\operatorname{csupp}(G) = \{g \in \mathbb{R}[[\underline{x}]] \mid \forall\, \alpha \in \mathbb{R} : f + \alpha g \in G\}.$$

Especially, this holds for all faces G of the form $G = F \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2$ with some face F of $\mathbb{R}[[\underline{x}]]_+$.

The following lemma is important to calculate the support of the algebraic closure of faces of $\Sigma\,\mathbb{R}[[\underline{x}]]^2$, which is applied in the subsequent example.

Lemma 4.2.2. *Let G be a face of $\Sigma\,\mathbb{R}[[\underline{x}]]^2$.*

(a) If $g_1^2, g_2^2 \in G$, then $g_1 g_2 \in \operatorname{span}(G)$.

(b) If $g_1^2 \in \operatorname{csupp}(G)$ and $g_2^2 \in G$, then $g_1 g_2 \in \operatorname{csupp}(G)$.

(c) *If* $g \in \operatorname{span}(G)$ *and* $h \in \mathfrak{m}$*, then* $hg \in \operatorname{csupp}(G)$*. Especially, if* $g_1^2, g_2^2 \in G$ *and* $h \in \mathfrak{m}$*, then* $hg_1g_2 \in \operatorname{csupp}(G)$*.*

Proof. (a) We have

$$\frac{1}{2}\left((g_1+g_2)^2+(g_1-g_2)^2\right) = g_1^2+g_2^2 \in G.$$

By the definition of faces, we have $(g_1+g_2)^2 \in G$ and thus

$$g_1g_2 = \frac{1}{2}\left((g_1+g_2)^2 - g_1^2 - g_2^2\right) \in \operatorname{span}(G).$$

(b) Let $f \in \operatorname{rai}(G)$ and $\epsilon > 0$ with $f - \epsilon g_2^2 \in \operatorname{rai}(G)$. Furthermore, let $\alpha \in \mathbb{R}$. The claim follows from (a), 2.2.8(c) and

$$f + \alpha g_1g_2 = f - \epsilon g_2^2 - \frac{\alpha^2}{4\epsilon}g_1^2 + \left(\frac{\alpha}{2\sqrt{\epsilon}}g_1 + \sqrt{\epsilon}g_2\right)^2 \in \Sigma\,\mathbb{R}[[\underline{x}]]^2 \cap \operatorname{span}(G) = G.$$

(c) This follows from 4.2.1 and (a).

□

Example 4.2.3. Let $F = \mathfrak{m}^{2d} \cap \mathbb{R}[[\underline{x}]]_+$ and $G := F \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2$ with the maximal ideal $\mathfrak{m}$ of $\mathbb{R}[[\underline{x}]]$. Now, let $m \in \mathbb{R}[[\underline{x}]]$ be a monomial of degree $\geq 2d+1$ and $m = m_1m_2m_3$ with monomials m_1, m_2, m_3 and $\deg(m_1) = \deg(m_2) = d$, $\deg(m_3) > 0$. Then, lemma 4.2.2 (c) applied on $m_1^2, m_2^2 \in G$ and $m_3 \in \mathfrak{m}$ shows that $m \in \operatorname{csupp}(G)$. Thus, we have $\operatorname{csupp}(G) = \mathfrak{m}^{2d+1} = \operatorname{csupp}(F)$.

We modify a result of Sekiguchi to obtain more information about $\operatorname{csupp}(G)$ for appropriate faces G of $\Sigma\,\mathbb{R}[[\underline{x}]]^2$. We need the following definition (compare with [Sek, pp. 8-9]).

Definition 4.2.4. Let Γ be a psd-diagram. We set

$$\Delta_E(\Gamma) := \bigcup\{\alpha + \mathbb{R}_+^n \mid \alpha \in \Gamma \cap 2\mathbb{Z}^n\}.$$

Let $\Delta_E \subseteq \mathbb{R}_+^n$. We consider finite sequences $\alpha_1, \ldots, \alpha_N \in \Delta_E \cap 2\mathbb{Z}^n$ with $N \in \mathbb{N}$ and we set

$$\beta_1 := \alpha_1, \quad \beta_{i+1} := \frac{\beta_i + \alpha_{i+1}}{2} \quad \text{for} \quad i = 1, \ldots, N-1.$$

We call the set of $\beta_N \in \mathbb{Z}^n$ that arise in this way the *bisectional convex hull* of Δ_E and we denote it by $\operatorname{bconv}(\Delta_E)$.
Furthermore, we define the *relaxed bisectional convex hull* of Δ_E, denoted by $\operatorname{rbconv}(\Delta_E)$, as

$$\operatorname{rbconv}(\Delta_E) := \bigcap_{d\in\mathbb{N}} \operatorname{bconv}\left(\Delta_E \cup \{\alpha \in \mathbb{Z}_+^n \mid |\alpha| \geq d\}\right) \supseteq \operatorname{bconv}(\Delta_E).$$

Remark 4.2.5. Let us look a little closer on the definitions of $\operatorname{bconv}(\Delta_E)$ and $\operatorname{rbconv}(\Delta_E)$ in the case $\Delta_E = \Delta_E(\Gamma)$ for some psd-diagram Γ, which is the only case we are interested in. Now, in the case of $\operatorname{bconv}(.)$, the α_i may be chosen as elements of $2\mathbb{Z}^n$ that are larger (in each component) than one element of $\Gamma \cap 2\mathbb{Z}^n$. In the case of $\operatorname{rbconv}(.)$, one may additionally use elements α_i of $2\mathbb{Z}_+^n$ "with large absolute value".

We have the following nice properties of bconv(.) and rbconv(.).

Remark 4.2.6. Let $\beta_N \in \mathbb{Z}^n$ be as in the definition 4.2.4. Then, we have $\beta_1, \ldots, \beta_{N-1} \in 2\mathbb{Z}^n$ (see [Sek, 4.10.]). Furthermore, for any psd-diagram Γ, we have

$$\mathrm{bconv}(\Delta_E(\Gamma)) + \mathbb{Z}^n_+ = \mathrm{bconv}(\Delta_E(\Gamma)) \quad \text{and} \quad \mathrm{rbconv}(\Delta_E(\Gamma)) + \mathbb{Z}^n_+ = \mathrm{rbconv}(\Delta_E(\Gamma))$$

since for any $\beta_N \in \mathrm{bconv}(\Delta_E(\Gamma))$ ($\beta_N \in \mathrm{rbconv}(\Delta_E(\Gamma))$) given by $\alpha_1, \ldots, \alpha_N$, one obtains $\beta_N + \gamma$ for any $\gamma \in \mathbb{Z}^n_+$ using the sequence $\alpha_1, \ldots, \alpha_{N-1}, \alpha_N + 2\gamma$ and the considered sets inside bconv(.) are invariant under addition of elements $2\gamma \in 2\mathbb{Z}^n_+$.

These concepts directly apply to the calculation of $\mathrm{csupp}(G)$ if $G := F \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2$ and F is a monomial face of $\mathbb{R}[[\underline{x}]]_+$ of finite codimension as we see in the following proposition and the subsequent corollary.

Proposition 4.2.7. *Let F be a monomial face of $\mathbb{R}[[\underline{x}]]_+$ of finite codimension, τ a face of $\Gamma(F)$ and $G := F \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2$. Now, if $\gamma \in \mathrm{rbconv}(\Delta_E(\tau)) \setminus \tau$, then $\underline{x}^\gamma \in \mathrm{csupp}(G)$.*

Proof. The statement for $\gamma \in \mathrm{bconv}(\Delta_E(\tau)) \setminus \tau$ follows directly from [Sek, 4.14.]. Let us prove the more general claim. As F has finite codimension, we have $\mathfrak{m}^{2d'} \cap \mathbb{R}[[\underline{x}]]_+ \subseteq F$ and thus $\mathfrak{m}^{2d'+1} \subseteq \mathrm{csupp}(G)$ for some $d' \in \mathbb{N}$ by 4.2.3. We choose such a d' and we set $d := 2d' + 1$. As $\gamma \in \mathrm{rbconv}(\Delta_E(\tau))$, we also have

$$\gamma \in \mathrm{bconv}\left(\Delta_E(\tau) \cup \{\alpha \in \mathbb{Z}^n_+ \mid |\alpha| \geq d\}\right).$$

Let $\alpha_1, \ldots, \alpha_N \in \left(\Delta_E(\tau) \cup \{\alpha \in \mathbb{Z}^n_+ \mid |\alpha| \geq d\}\right) \cap 2\mathbb{Z}^n$ be given such that $\gamma = \beta_N$ holds with the notation of 4.2.4, which gives us corresponding $\beta_1, \ldots, \beta_N$. Since $\gamma \notin \tau$, we find some $k \in \{1, \ldots, N\}$ with $\alpha_k \notin \tau$. Then, we have $|\alpha_k| \geq d$ or we find some $\delta \in \mathbb{R}^n_+ \setminus \{0\}$ with $\alpha_k - \delta \in \tau \cap 2\mathbb{Z}^n$ (so $\delta \in 2\mathbb{Z}^n_+ \setminus \{0\}$). Now, $\mathfrak{m}^d \subseteq \mathrm{csupp}(G)$ and 4.2.2(c) (with $g^2 = \underline{x}^{\alpha_k - \delta}$, $h = \underline{x}^\delta$) respectively show

$$\underline{x}^{\alpha_k} \in \mathrm{csupp}(G).$$

We have $\underline{x}^{\beta_1} = \underline{x}^{\alpha_1} \in G$. Since $\underline{x}^{\alpha_i} \in G$ for $i = 1, \ldots, N$, we obtain (by induction on i) using 4.2.2(a) (with $g_1^2 = \underline{x}^{\beta_{i-1}}$ and $g_2^2 = \underline{x}^{\alpha_i}$ if $i > 1$) and 4.2.6 (to show $\beta_{i-1} \in 2\mathbb{Z}^n$) that $\underline{x}^{\beta_i} \in G$ for $i = 1, \ldots, k-1$. 4.2.2(b) with $g_1^2 = \underline{x}^{\alpha_k}$ and $g_2^2 = \underline{x}^{\beta_{k-1}}$ (use 4.2.6) implies

$$\underline{x}^{\beta_k} \in \mathrm{csupp}(G).$$

Another induction, using 4.2.2(b) (again with $g_1^2 = \underline{x}^{\beta_{i-1}}$ and $g_2^2 = \underline{x}^{\alpha_i}$) and 4.2.6, shows

$$\underline{x}^{\beta_i} \in \mathrm{csupp}(G)$$

for $i = k, \ldots, N$ and thus the claim. □

Corollary 4.2.8. *Let F be a monomial face of $\mathbb{R}[[\underline{x}]]_+$ of finite codimension and $G := F \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2$. Let us assume that $\Gamma(F)$ may be covered by lattice polytopes with even vertices $\tau \subseteq \Gamma(F)$ that satisfy*

$$(\Delta(\tau) \setminus \tau) \cap \mathbb{Z}^n \subseteq \mathrm{rbconv}(\Delta_E(\tau)).$$

Then, we have $\mathrm{csupp}(G) = \mathrm{csupp}(F)$.

Proof. csupp(F) is generated as an ideal by the monomials $\underline{x}^\alpha$ with $\alpha \in (\Delta(F) \setminus \Gamma(F)) \cap \mathbb{Z}^n$ by 3.3.14. For any such α, there is some $\beta \in \Gamma(F)$ with $\alpha \in \beta + \mathbb{R}^n_+$ and thus a polytope with even vertices $\tau \subseteq \Gamma(F)$ with $\beta \in \tau$ such that

$$\alpha \in (\Delta(\tau) \setminus \tau) \cap \mathbb{Z}^n \subseteq \mathrm{rbconv}(\Delta_E(\tau)).$$

Let $F' \subseteq F$ be a monomial face of $\mathbb{R}[[\underline{x}]]_+$ of finite codimension such that τ is a face of $\Gamma(F')$ (see 3.3.3 with $\Delta := \Delta(\tau)$ for existence of F'). Then, by 4.2.7, we have $\underline{x}^\alpha \in \mathrm{csupp}(F' \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2)$. Now, $F' \subseteq F$ implies $\underline{x}^\alpha \in \mathrm{csupp}(F \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2) = \mathrm{csupp}(G)$ and thus the claim. □

In the following example, we give some construction for monomial faces F such that $\mathrm{csupp}(F \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2) \neq \mathrm{csupp}(F)$ (and thus $\mathrm{rai}(F) \nsubseteq \Sigma\,\mathbb{R}[[\underline{x}]]^2$). It complements the previous positive results.

Example 4.2.9. We set $n = 3$ and we will use variables x, y, z instead of x_1, x_2, x_3. Let $F := \mathrm{carr}(x^{2i} + y^{2j} + z^{2k}) \trianglelefteq \mathbb{R}[[x, y, z]]_+$ with pairwise coprime $i, j, k \in \mathbb{N}$. We define a monomial gradation on $\mathbb{R}[[x, y, z]]$ by assigning the degree of a monomial $x^a y^b z^c$ as

$$\deg_\lambda(x^a y^b z^c) := ajk + bik + cij \quad (\lambda := (jk, ik, ij))$$

Thus, span(F) is generated as an ideal by the monomials m with $\deg_\lambda(m) \geq 2ijk$ and csupp(F) is generated by the monomials m with $\deg_\lambda(m) > 2ijk$ (see 3.3.14).
Let us assume that there is some monomial $m := x^a y^b z^c$ of λ-degree $2ijk + 1$ with $i > a$, $j > b$ and $k > c$. Now, let $g^2 \in F$ such that m appears in g^2 (with non-zero coefficient). Then, there are monomials m_1 and m_2 appearing in g with $m = m_1 m_2$. Furthermore, $\deg_\lambda(m_1), \deg_\lambda(m_2) \geq ijk$, since else, there would be some monomial of λ-degree $< 2ijk$ appearing in g^2 (see proof of 4.1.5). Without loss of generality, we may assume

$$\deg_\lambda(m_1) = ijk, \quad \deg_\lambda(m_2) = ijk + 1.$$

Let $m_1 = x^{a_1} y^{b_1} z^{c_1}$. Then, evaluation of its λ-degree gives us $i \,|\, a_1$, $j \,|\, b_1$ and $k \,|\, c_1$. Now, since $i > a \geq a_1$, $j > b \geq b_1$ and $k > c \geq c_1$ we have $a_1 = b_1 = c_1 = 0$ and thus a contradiction since $\deg_\lambda(m_1) = ijk$. This shows that m does not appear in any square $g^2 \in F$ and thus also in no sum of squares in F. Especially, $m \notin \mathrm{csupp}(F \cap \Sigma\,\mathbb{R}[[x, y, z]]^2)$ though $m \in \mathrm{csupp}(F)$. The described situation really exists as it is shown by the examples

$$F = \mathrm{carr}(x^4 + y^6 + z^{14}), \quad m = xy^2z^6$$
$$\text{or} \quad F = \mathrm{carr}(x^6 + y^8 + z^{10}), \quad m = x^2y^3z^3.$$

Note that the first case is our first example of a face F of $\mathbb{R}[[\underline{x}]]_+$ with $n = 3$, $\mathrm{ord}(F) = 4$ such that $\mathrm{rai}(F) \nsubseteq \Sigma\,\mathbb{R}[[\underline{x}]]^2$.

Now, let us study the property that is necessary in 4.2.8 to imply $\mathrm{csupp}(F \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2) = \mathrm{csupp}(F)$ to be able to really show it for some monomial faces. We start with some remark.

Remark 4.2.10. (a) If $(\Delta(\tau) \setminus \tau) \cap \mathbb{Z}^n \subseteq \mathrm{rbconv}(\Delta_E(\tau))$ for some face τ of a psd-diagram, then it also holds for the situation translated by 2γ for some $\gamma \in \mathbb{Z}^n_+$. Indeed, this follows from $2\gamma + (\Delta(\tau) \setminus \tau) \cap \mathbb{Z}^n = (\Delta(\tau + 2\gamma) \setminus (\tau + 2\gamma)) \cap \mathbb{Z}^n$ and

$$\begin{aligned} 2\gamma + \mathrm{rbconv}(\Delta_E(\tau)) &= 2\gamma + \bigcap_{d \in \mathbb{N}} \mathrm{bconv}\left(\Delta_E(\tau) \cup \{\alpha \in \mathbb{Z}^n_+ \,|\, |\alpha| \geq d\}\right) \\ &\subseteq \bigcap_{d \in \mathbb{N}} \mathrm{bconv}\left(\Delta_E(\tau + 2\gamma) \cup \{\alpha \in \mathbb{Z}^n_+ \,|\, |\alpha| \geq d + |2\gamma|\}\right) \\ &= \mathrm{rbconv}(\Delta_E(\tau + 2\gamma)). \end{aligned}$$

(b) Let τ be a face of a psd-diagram and $V := \tau \cap 2\mathbb{Z}^n$. Now, if $\varphi \in (\Delta(\tau) \setminus \tau) \cap \mathbb{Z}^n$ with corresponding $\delta \in \mathbb{R}^n_+ \setminus \{0\}$ such that $\varphi - \delta \in \tau$ and $v \in V$ such that $(2\varphi - v) - 2\delta \in \tau$ (so especially $\varphi' := 2\varphi - v \in (\Delta(\tau) \setminus \tau) \cap 2\mathbb{Z}^n$), then

$$\varphi = \frac{1}{2}\left(v + (2\varphi - v)\right) \text{ with } v \in V.$$

Thus, if $\varphi' := 2\varphi - v \in \mathrm{bconv}(\Delta_E(\tau))$ holds, then we also have $\varphi \in \mathrm{bconv}(\Delta_E(\tau))$ by the definition of the bisectional convex hull. If $\varphi' \in \mathrm{bconv}(\Delta_E(\tau))$ is not obvious, we can iterate this process if we find some appropriate $v \in V$ in each step. As δ is getting doubled and τ consists of elements of bounded absolute value, we will reach vectors with arbitrary high absolute value. This gives a strategy to show $\varphi \in \mathrm{rbconv}(\Delta_E(\tau))$.

(c) In the situation of (b), $\varphi' \in \mathrm{rbconv}(\Delta_E(\tau))$ also implies $\varphi \in \mathrm{rbconv}(\Delta_E(\tau))$ as the definition of bconv(.) is recursive.

We will use the following lemma just in the case $n = 3$ (even just in the proof of 4.2.13) but the proof does not change when we prove it for general n and one can still use it in 4.2.7 to obtain information about $\mathrm{csupp}(G)$ (with G as in 4.2.7).

Lemma 4.2.11. *Let $\tau = \mathrm{conv}(2\alpha, 2\beta)$ with $\alpha, \beta \in \mathbb{N}_0^n$ be a face of a psd-diagram. Then,*

$$(\Delta(\tau) \setminus \tau) \cap \mathbb{Z}^n \subseteq \mathrm{rbconv}(\Delta_E(\tau)).$$

Proof. Let $\varphi \in (\Delta(\tau) \setminus \tau) \cap \mathbb{Z}^n$. Then, there is some $\lambda \in [0,1]$ and some $\delta \in \mathbb{R}^n_+ \setminus \{0\}$ with

$$\varphi - \delta = \lambda \cdot 2\alpha + (1-\lambda) \cdot 2\beta \in \tau.$$

Now,

$$\text{if } \lambda \geq \frac{1}{2}, \text{ then } \quad 2\varphi - v - 2\delta = (2\lambda - 1) \cdot 2\alpha + (2 - 2\lambda) \cdot 2\beta \in \tau \quad \text{with } v := 2\alpha,$$
$$\text{and if } \lambda < \frac{1}{2}, \text{ then } \quad 2\varphi - v - 2\delta = 2\lambda \cdot 2\alpha + (1 - 2\lambda) \cdot 2\beta \in \tau \quad \text{with } v := 2\beta.$$

The claim follows from the considerations in 4.2.10(b). □

In the following example, we see that we really have to use the relaxed bisectional hull to obtain the result of the lemma above.

Example 4.2.12. The statement of 4.2.11 does not hold if one replaces $\mathrm{rbconv}(\Delta_E(\tau))$ by $\mathrm{bconv}(\Delta_E(\tau))$. This is shown by the example with $\alpha := (1,3,0)^T$, $\beta := (0,0,3)^T$ and $\varphi := (1,1,5)^T \notin \mathrm{bconv}(\Delta_E(\tau))$. Indeed, each element of $\Delta_E(\tau)$ is either $\geq 2\alpha$ or $\geq 2\beta$. Thus, if $\varphi \in \mathrm{bconv}(\Delta_E(\tau))$, then φ would have to be a convex combination of such elements such that each coefficient of the combination is a quotient of a non-negative integer by some power of 2. However, $\varphi \geq \lambda \cdot 2\alpha + (1-\lambda) \cdot 2\beta$ already implies $\lambda = \frac{1}{6}$, which shows $\varphi \notin \mathrm{bconv}(\Delta_E(\tau))$. The algorithm of 4.2.11 does not terminate in this case.

Now, we can prove the lemma that will include all relevant cases to ensure the premise of 4.2.8 in the application to the monomial faces studied later.

Lemma 4.2.13. *Let $n = 3$, $\tau = \mathrm{conv}(2\alpha, 2\beta, 2\gamma)$ with $\alpha, \beta \in \mathbb{N}_0^3$, $\gamma \in \{0,1\}^3$ such that τ is a psd-diagram. Then,*

$$(\Delta(\tau) \setminus \tau) \cap \mathbb{Z}^3 \subseteq \mathrm{rbconv}(\Delta_E(\tau)).$$

Proof. Let $\varphi \in (\Delta(\tau) \setminus \tau) \cap \mathbb{Z}^3$. We consider the set

$$M_\varphi := \left\{ \psi \in \tau \,\middle|\, \varphi - \psi \in \mathbb{R}^3_+ \right\} = \tau \cap \left(\varphi + \left(-\mathbb{R}^3_+\right)\right).$$

This set is non-empty by definition of $\Delta(\tau) \setminus \tau$ and a subset of the triangle τ. If there is some $\psi \in M_\varphi \cap \partial\tau$, then the claim follows from 4.2.11 since $\varphi \in (\Delta(\tau') \setminus \tau') \cap \mathbb{Z}^3$ for some edge τ' of τ with $\psi \in \tau'$. Let us assume from now on that there is no such ψ. Then, τ is a non-degenerated triangle, $M_\varphi \cap \mathrm{ri}(\tau) \neq \emptyset$ and $M_\varphi = \mathrm{aff}(\tau) \cap \left(\varphi + \left(-\mathbb{R}^3_+\right)\right)$ since the intersection of convex sets is convex. Now, M_φ is the intersection of a translated octant and a plane, it is compact and non-empty and it does not contain the origin φ of the translated octant. Thus, M_φ is a triangle in the interior of τ. Hence, there are $\lambda, \mu \in (0,1)$ with $\lambda + \mu < 1$ and $\delta \in \mathbb{R}^3_+$ such that

$$\varphi - \delta = \lambda \cdot 2\alpha + \mu \cdot 2\beta + (1 - \lambda - \mu) \cdot 2\gamma.$$

We may choose λ, μ in such a way that $\delta_i = 0$ for the corresponding δ and some arbitrary $i \in \{1,2,3\}$ (just put $\varphi - \delta$ on the edge of M_φ that corresponds to i). By 4.2.10(a), the property $(\Delta(\tau) \setminus \tau) \cap \mathbb{Z}^3 \subseteq \mathrm{rbconv}(\Delta_E(\tau))$ does still hold after translations of τ with some element of $2\mathbb{Z}^3_+$. Hence, we may reduce to proving the case with $\alpha_i\beta_i\gamma_i = 0$ for $i = 1,2,3$, which means that each coordinate plane contains at least one vector from $\{\alpha, \beta, \gamma\}$.
Let us fix some $i \in \{1,2,3\}$ with $\gamma_i = 1$ (this exists as τ is not degenerated as a point) and we choose λ, μ such that $\delta_i = 0$. In the case $\alpha_i = \beta_i = 0$, we have $\varphi_i = 2(1 - \lambda - \mu)$, so $\varphi_i = 1$ and $1 - \lambda - \mu = \frac{1}{2}$. Hence,

$$2\varphi - 2\gamma - 2\delta = 2\lambda \cdot 2\alpha + 2\mu \cdot 2\beta \in \tau$$

and 4.2.10(c) reduces the problem to the proof of $2\varphi - 2\gamma \in \mathrm{rbconv}(\Delta_E(\tau))$. However,

$$2\varphi - 2\gamma - 2\delta \in M_{2\varphi - 2\gamma} \cap \partial\tau$$

and thus $2\varphi - 2\gamma \in \mathrm{rbconv}(\Delta_E(\tau))$ as above. This shows the claim in the case $\alpha_i = \beta_i = 0$. Let us assume $\alpha_i \neq 0$ (the case $\beta_i \neq 0$ is analogous), so $\beta_i = 0$ (as $\alpha_i\beta_i\gamma_i = 0$). Then, we have $\varphi_i = \lambda \cdot 2\alpha_i + 2(1 - \lambda - \mu) > 0$. If $\varphi_i = 1$, then

$$\mu = \frac{1}{2}\left(1 + 2(\alpha_i - 1)\lambda\right) \geq \frac{1}{2},$$

so $\lambda < \frac{1}{2}$ and

$$2\varphi - 2\beta - 2\delta = 2\lambda \cdot 2\alpha + (2\mu - 1) \cdot 2\beta + 2(1 - \lambda - \mu) \cdot 2\gamma \in \tau$$

which, using remark 4.2.10 (c), reduces the problem to the proof of $2\varphi - 2\beta \in \mathrm{rbconv}(\Delta_E(\tau))$. Now, obviously $2\varphi_j - 2\beta_j \neq 1$ for $j = 1,2,3$. We reduce to the case $\varphi_i > 1$ for all $i \in \{1,2,3\}$ with $\gamma_i = 1$. However, then M_φ may not be in the interior of τ since $\varphi - 2\gamma \in \mathbb{R}^3_+$. We get a contradiction and thus, the claim follows. □

The considerations from before result in the following theorems.

Theorem 4.2.14. *Let $n = 3$ and F a monomial face of $\mathbb{R}[[x,y,z]]_+$ of finite codimension with $\mathrm{ord}(F) = 4$. Let $\tau := \{\alpha \in \mathbb{R}^3_+ \mid |\alpha| = 4\} = \Gamma(\mathbb{R}[[x,y,z]]_+ \cap \mathfrak{m}^4)$ and $\tau(F)$ be the corresponding face of $\mathcal{P}_{2,4}$ (see 3.3.15, 3.3.22). Let us assume that $\tau(F)$ is a face of $\mathcal{P}_{2,4}$ such that its zeros are described by (see section 1.6 and 3.6.2)*

(a) up to three double zeros with positive definite Hessian

(b) one zero of type $(4:2)$ *and possibly another double zero with positive definite Hessian*

(c) a double line.

Then, $\operatorname{csupp}(\Sigma\,\mathbb{R}[[x,y,z]]^2 \cap F) = \operatorname{csupp}(F)$.

Proof. (a) Since F is a monomial face, it is possible to extract the zeros of $\tau(F)$ from $\Gamma(F)$. Now, it is easy to see that F is of the form

$$F = \operatorname{carr}(x^2y^2 + x^2z^2 + y^2z^2 + x^{2n_1} + y^{2n_2} + z^{2n_3}) \text{ with } n_1, n_2, n_3 \geq 2.$$

$\Gamma(F)$ may be covered by the triangles (they are the faces of $\Gamma(F)$ if $n_1, n_2, n_3 > 2$)

$$\begin{aligned} &\operatorname{conv}((2,2,0),(2,0,2),(0,2,2)), && \operatorname{conv}((2n_1,0,0),(2,2,0),(2,0,2)),\\ &\operatorname{conv}((0,2n_2,0),(2,2,0),(0,2,2)), && \operatorname{conv}((0,0,2n_3),(2,0,2),(0,2,2)). \end{aligned}$$

The claim follows from 4.2.8 and 4.2.13.

(b) As in (a), it is easy to see that the face F is up to permutation of x, y and z one of the faces (here, the zero of type $(4:2)$ of $\tau(F)$ is at $(1:0:0)$ and the Hessian at this point is vanishing in the direction of $(0:1:0)$)

$$\begin{aligned} F &= \operatorname{carr}\left(y^4 + x^2z^2 + y^2z^2 + z^{2n_2} + x^{2n_1}\right) \text{ with } n_1 > 2, n_2 \geq 2\\ \text{or } F &= \operatorname{carr}\left(y^4 + x^2z^2 + y^2z^2 + x^{2n_1}y^2 + x^{2n_2} + z^{2n_3}\right) \text{ with } n_2 > 2n_1 > 2, n_3 \geq 2. \end{aligned}$$

As in (a) the claim follows by a covering of $\Gamma(F)$ by triangles and application of 4.2.8 and 4.2.13. In the first case, we use

$$\begin{aligned} &\operatorname{conv}((0,4,0),(2,0,2),(0,2,2)), && \operatorname{conv}((0,0,2n_2),(2,0,2),(0,2,2)),\\ &\operatorname{conv}((0,4,0),(2,0,2),(2n_1,0,0)) \end{aligned}$$

and in the second case we may use

$$\begin{aligned} &\operatorname{conv}((0,4,0),(2,0,2),(0,2,2)), && \operatorname{conv}((0,0,2n_3),(2,0,2),(0,2,2)),\\ &\operatorname{conv}((0,4,0),(2,0,2),(2n_1,2,0)), && \operatorname{conv}((2,0,2),(2n_1,2,0),(2n_2,0,0)). \end{aligned}$$

(c) Without loss of generality, we may assume that $\tau(F)$ is doubly vanishing on the z-axis. Then,

$$F = \operatorname{carr}\left(z^4 + x^2z^2 + y^2z^2 + g\right),$$

with the canonical polynomial $g = g(\Gamma(F)))$ (see 3.3.1) of a monomial face G of $\mathbb{R}[[x,y]]_+$ of finite codimension and $\operatorname{ord}(G) \geq 6$. By 3.5.1, there are $r \in \mathbb{N}$, $a_0 > a_1 > \cdots > a_r = 0$, $0 = b_0 < b_1 < \cdots < b_r$ with monotonic decreasing sequence $\frac{a_{i-1}-a_i}{b_i-b_{i-1}}$ for $i = 1, \ldots, r$ and

$$g = \sum_{i=0}^{r} x^{2a_i}y^{2b_i}.$$

We cover $\Gamma(F)$ by the following triangles where i takes the values $1, \ldots, r$ and j is chosen maximal with $\frac{a_{j-1}-a_j}{b_j-b_{j-1}} \geq 1$:

$$\begin{aligned}
&\operatorname{conv}((0,0,4),(2,0,2),(0,2,2)),\\
&\operatorname{conv}((2a_{i-1},2b_{i-1},0),(2a_i,2b_i,0),(2,0,2)) \quad \text{if} \quad \frac{a_{i-1}-a_i}{b_i-b_{i-1}} \geq 1,\\
&\operatorname{conv}((2a_{i-1},2b_{i-1},0),(2a_i,2b_i,0),(0,2,2)) \quad \text{if} \quad \frac{a_{i-1}-a_i}{b_i-b_{i-1}} < 1,\\
&\operatorname{conv}((2a_j,2b_j,0),(2,0,2),(0,2,2)).
\end{aligned}$$

(These triangles coincide with the faces of $\Gamma(F)$ if there is no i such that $\frac{a_{i-1}-a_i}{b_i-b_{i-1}} = 1$. However, if such an i exists, then one face of $\Gamma(F)$ is a trapezoid that is cut by one diagonal into two triangles.)
The claim follows again by 4.2.8 and 4.2.13.

□

In part (b) and (c) of the following theorem, we have to calculate $\operatorname{csupp}(G)$ "by hand" using 4.2.2.

Theorem 4.2.15. *Let $n = 3$ and $\tau := \Gamma(\mathbb{R}[[x,y,z]]_+ \cap \mathfrak{m}^4)$ (with $\mathfrak{m} := (x,y,z)$). We consider the following faces F of $\mathbb{R}[[x,y,z]]_+$ with $\operatorname{ord}(F) = 4$ and $G := F \cap \Sigma\,\mathbb{R}[[x,y,z]]^2$ (see 3.6.2).*

(a) $F = \operatorname{carr}(x^6+y^4+z^4)$ (i.e. the face O on page 92)

(b) Let F be the $\subseteq$-maximal face F with $\tau(F) = \operatorname{carr}((xz-y^2)^2+z^4+y^2z^2)$ ($\tau(F)$ has one zero of type $(6:2)$). We claim that it is the face N on page 92, so

$$F = \operatorname{carr}((xz-y^2)^2+z^4+y^2z^2+x^6).$$

(c) $F = \operatorname{carr}((x^3-y^2)^2+x^4y^2+x^2z^2+y^2z^2+z^4)$ (i.e. the face E_{14}' on page 92)

In all three cases, we have $\operatorname{csupp}(G) = \operatorname{csupp}(F)$.

Proof. (a) F is monomial. The claim $\operatorname{csupp}(G) = \operatorname{csupp}(F)$ follows again from 4.2.8 and 4.2.13 by covering $\Gamma(F)$ with the triangles

$$\operatorname{conv}((6,0,0),(0,4,0),(0,2,2)) \quad \text{and} \quad \operatorname{conv}((6,0,0),(0,0,4),(0,2,2)).$$

(b) Let F' be the monomial face of $\mathbb{R}[[x,y,z]]_+$ with $\Gamma(F') = \Gamma(F)$. Then,

$$F' = \operatorname{carr}(y^4+x^2z^2+z^4+x^6).$$

Thus, the monomials x^4, x^3y, x^2y^2, xy^3, x^3z, x^2yz, x^5, x^4y do not appear in any $f \in \operatorname{span}(F) \subseteq \operatorname{span}(F')$.
Since $F \supseteq \mathfrak{m}^6 \cap \mathbb{R}[[x,y,z]]_+$, we have $\mathfrak{m}^7 \subseteq \operatorname{csupp}(G)$. For the following considerations, we recommend to the reader to write down all 47 monomials $m \in \operatorname{span}(F')$ with $\deg(m) \in \{5,6\}$ ($x^5, x^4y \notin \operatorname{span}(F')$) and to cross them out whenever $m \in \operatorname{csupp}(G)$ is shown. Since $z^4 \in G$ and $y^2z^2 \in G$ by 4.2.2(c) (with $g \in \{z^2, yz\}$) we have

$$\begin{aligned}
&z^5, xz^4, yz^4, y^3z^2, y^2z^3, xy^2z^2 \in \operatorname{csupp}(G),\\
&z^6, xz^5, yz^5, x^2z^4, xyz^4, y^2z^4, xy^2z^3, x^2y^2z^2, y^3z^3, xy^3z^2, y^4z^2 \in \operatorname{csupp}(G).
\end{aligned}$$

Application of 4.2.2(b) with $g_2^2 = x^6$ and $g_1^2 \in \{x^2z^4, x^2y^2z^2, y^4z^2\}$ shows

$$x^4z^2, x^4yz, x^3y^2z \in \operatorname{csupp}(G).$$

Another application of 4.2.2(b) with $g_1^2 = x^4z^2$ and $g_2^2 \in \{x^6, x^2y^2z^2, y^2z^4, x^2z^4\}$ shows

$$x^5z, x^3yz^2, x^2yz^3, x^3z^3 \in \operatorname{csupp}(G).$$

Again, 4.2.2(b) with $g_2 = xz - y^2$ and $g_1 \in \{x^2yz, xy^2z, y^3z, x^4, x^3y, x^2y^2, xy^3, y^4\}$ implies

$$x^3yz^2 - x^2y^3z, x^2y^2z^2 - xy^4z, xy^3z^2 - y^5z, x^5z - x^4y^2, x^4yz - x^3y^3, x^3y^2z - x^2y^4,$$
$$x^2y^3z - xy^5, xy^4z - y^6 \in \operatorname{csupp}(G)$$

and since $\operatorname{csupp}(G)$ is closed under addition and negation, we have

$$x^2y^3z, xy^4z, y^5z, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6 \in \operatorname{csupp}(G).$$

Now, by 4.2.2(b) with $g_2^2 = x^6$ and $g_1^2 = x^4y^2$, we also have $x^5y \in \operatorname{csupp}(G)$. This shows that $\operatorname{csupp}(G)$ contains all monomials of degree 6 except of x^6.
Lemma 4.2.2(b) with $g_2^2 = y^2z^2$ and $g_1^2 \in \{x^4y^2, x^4z^2\}$ gives

$$x^2y^2z, x^2yz^2 \in \operatorname{csupp}(G)$$

and choosing $g_2 = xz - y^2$ and $g_1 \in \{x^2z, xz^2, yz^2, xyz, y^2z, xy^2, y^3, x^2y\}$, we also have

$$x^3z^2 - x^2y^2z, x^2z^3 - xy^2z^2, xyz^3 - y^3z^2, x^2yz^2 - xy^3z, xy^2z^2 - y^4z, x^2y^2z - xy^4,$$
$$xy^3z - y^5, x^3yz - x^2y^3 \in \operatorname{csupp}(G).$$

Since $\operatorname{csupp}(G)$ is closed under addition and negation, we have

$$x^3z^2, x^2z^3, xyz^3, xy^3z, y^4z, xy^4, y^5, x^3yz - x^2y^3 \in \operatorname{csupp}(G).$$

Let I be the vector space (I is even an ideal) generated by $\mathfrak{m}^7$, the monomials m for which we have shown that $m \in \operatorname{csupp}(G)$ and $x^3yz - x^2y^3$. We have seen that

$$I \subseteq \operatorname{csupp}(G) \subseteq \operatorname{csupp}(F).$$

Let $f \in \operatorname{span}(F)$. Since $f \in \operatorname{span}(F')$, we may express f as

$$f \in a_1x^2z^2 + a_2xy^2z + a_3y^4 + a_4xyz^2 + a_5y^3z + a_6y^2z^2 + a_7xz^3 + a_8yz^3 + a_9z^4$$
$$+ b_1x^4z + b_2x^3y^2 + b_3x^3yz + cx^6 + i(x, y, z),$$

where the monomials have real coefficients and $i \in I$.
Let us assume for the moment that $f \in \operatorname{rai}(F)$ and $i = 0$ (such f exists since $f \in \operatorname{rai}(F)$ if and only if $f - i \in \operatorname{rai}(F)$). Now, $\tau(F) = \operatorname{carr}((xz - y^2)^2 + z^4 + y^2z^2)$ implies $a_1 = a_3$ and $a_2 = -2a_1$ since for $\tau' := \operatorname{conv}((0, 4, 0), (2, 0, 2))$ we have $\tau'(F) = \operatorname{carr}((xz - y^2)^2)$ and $\tau'(\tau(F)) = \tau'(F)$ (see 3.3.18(b),(c)).
We set $x = t^3s$, $y = t^4s$ and $z = t^5s$ ($s, t \in \mathbb{R}$) and we obtain some expression of the form

$$(a_4s^4 + a_5s^4 + b_1s^5 + b_2s^5)t^{17} + (a_6s^4 + a_7s^4 + b_3s^5 + cs^6)t^{18} + O(t^{19}).$$

Since $f \in \mathcal{P}^{Loc}$ and by the behaviour for $t \to 0$ we obtain $a_4 + a_5 = 0$ and $b_1 + b_2 = 0$ as well as $a_6 + a_7 + b_3 s + cs^2 \geq 0$ for $s \in \mathbb{R}$. This inequality does not hold for every $s \in \mathbb{R}$ if $b_3^2 > 4(a_6 + a_7)c$ so $x^3yz \notin \operatorname{csupp}(F)$.
Now, let us consider some $f \in \operatorname{csupp}(F) \subseteq \operatorname{span}(F)$. As the monomials x^2z^2, xy^2z, y^4, xyz^2, y^3z, y^2z^2, xz^3, yz^3, z^4, x^4z, x^3y^2, x^6 correspond to points on $\Gamma(F)$, they do not appear in elements of $\operatorname{csupp}(F)$ (see 2.2.8(c), 3.3.10). Thus, if $f \in \operatorname{csupp}(F)$, then $f = b_3x^3yz + i$ with $i \in I$. Since $x^3yz \notin \operatorname{csupp}(F)$, we obtain $b_3 = 0$ and thus $\operatorname{csupp}(F) = I$, which shows the claim $\operatorname{csupp}(F) = \operatorname{csupp}(G)$.
Let $h := (xz - y^2)^2 + z^4 + y^2z^2 + x^6$. Easy calculations show that $h \pm \epsilon g \in \mathcal{P}^{Loc}$ for sufficiently small $\epsilon > 0$ and

$$g \in \{(xz - y^2)^2, (xz - y^2)yz, y^2z^2, yz^3, z^4, (xz - y^2)x^3, x^3yz, x^6\}.$$

Furthermore, this also holds for $g = xz^3$ since for any $0 < \epsilon \leq 1$

$$h \pm \epsilon xz^3 = \frac{\epsilon}{2}(xz - y^2 \pm z^2)^2 + \Big(h - \frac{\epsilon}{2}(xz - y^2)^2 \pm \epsilon y^2z^2 - \frac{\epsilon}{2}z^4\Big) \in \mathcal{P}^{Loc}.$$

Since (as shown above) there is some $f \in \operatorname{rai}(F)$ that is in the linear span of such polynomials g, we obtain $h - \epsilon f \in \mathcal{P}^{Loc}$ for sufficiently small $\epsilon > 0$ and thus $h \in \operatorname{rai}(F)$.

(c) Let F' be the monomial face with $\Gamma(F') = \Gamma(F)$, which is the same monomial face as in (b). Thus, the monomials x^4, x^3y, x^2y^2, xy^3, x^3z, x^2yz, x^5, x^4y do not appear in $f \in \operatorname{span}(F) \subseteq \operatorname{span}(F')$. We have seen in the proof of 3.6.3 that F is the $\subseteq$-maximal face with its Newton diagram and $\tau'(F) = \{a \cdot (x^3 - y^2)^2 \,|\, a \geq 0\}$ where $\tau' := \operatorname{conv}((6,0,0),(0,4,0))$. We have $F \cap \mathbb{R}[[x,y]]_+ = \operatorname{carr}((x^3 - y^2)^2 + x^4y^2)$ by 3.5.13(a). $F \supseteq \mathfrak{m}^8 \cap \mathbb{R}[[x,y,z]]_+$ implies $\mathfrak{m}^9 \supseteq \operatorname{csupp}(G)$. An application of 4.2.2(c) with

$$g \in \{z^4, x^2z^2, y^2z^2\} \subseteq G,$$

shows that $m \in \operatorname{csupp}(G)$ for all monomials m with $\deg(m) \geq 5$ and $z^2 \,|\, m$. We recommend to the reader to write down all 37 monomials $m \in \operatorname{span}(F')$ with $5 \leq \deg(m) \leq 7$ ($x^5, x^4y \notin \operatorname{span}(F')$) and $z^2 \nmid m$ and to cross them out whenever $m \in \operatorname{csupp}(G)$ is shown.
Since $F \cap \mathbb{R}[[x,y]]_+ = \operatorname{carr}((x^3 - y^2)^2 + x^4y^2)$ is an $\subseteq$-maximal proper face of the face $\operatorname{carr}(x^6 + y^4)$ in $\mathbb{R}[[x,y]]_+$, we have $x^8, x^4y^2, x^2y^4, y^6 \in G$. Now, lemma 4.2.2(c) with $g \in \{x^4y^2, x^2y^4, y^6\}$ shows that

$$x^5y^2, x^4y^2z, x^4y^3, x^3y^4, x^2y^4z, x^2y^5, xy^6, y^6z, y^7 \in \operatorname{csupp}(G)$$

and thus $m \in \operatorname{csupp}(G)$ for all monomials m with $\deg(m) = 8$, $y^2 \,|\, m$ and $z^2 \nmid m$.
Let us apply 4.2.2(b) with $g_2^2 = x^8$ and $g_1^2 \in \{x^6y^2, x^4y^2z^2, x^6z^2\}$, so

$$x^7y, x^6yz, x^7z \in \operatorname{csupp}(G).$$

Another application with $g_2^2 = (x^3 - y^2)^2$ and $g_1^2 \in \{x^{10}, x^6y^2z^2, x^2y^6z^2, x^4y^4, x^2y^4z^2, x^2y^6, y^6z^2, y^8\}$ gives

$$x^8 - x^5y^2, x^6yz - x^3y^3z, x^4y^3z - xy^5z, x^5y^2 - x^2y^4, x^4y^2z - xy^4z, x^4y^3 - xy^5,$$
$$x^3y^3z - y^5z, x^3y^4 - y^6 \in \operatorname{csupp}(G)$$

and thus by addition and negation of above elements of csupp(G) also

$$x^8, x^3y^3z, xy^5z, x^2y^4, xy^4z, xy^5, y^5z, y^6 \in \mathrm{csupp}(G).$$

Especially, we have $\mathfrak{m}^8 \subseteq \mathrm{csupp}(G)$. Furthermore, we have $m \in \mathrm{csupp}(G)$ for all monomials m with $\deg(m) = 7$ and $y^2 \mid m$ or $\deg(m) = 6$ and $y^4 \mid m$.
Now, 4.2.2(b) with $g_2^2 = x^6$ and $g_1^2 \in \{x^4z^2, x^2y^2z^2, y^4z^2, y^6\}$ or $g_2^2 = x^2y^4$ and $g_1^2 = x^2y^2z^2$ or $g_2^2 = y^2z^2$ and $g_1^2 \in \{x^2y^4, y^6\}$ shows that

$$x^5z, x^4yz, x^3y^2z, x^3y^3, x^2y^3z, xy^3z, y^4z \in \mathrm{csupp}(G).$$

Again, 4.2.2(b) with $g_2^2 = (x^3 - y^2)^2$ and $g_1^2 \in \{x^6y^2, x^6z^2, x^4y^2z^2, x^4z^2, y^6, x^8, x^2y^4\}$ gives

$$\begin{aligned} &x^6y - x^3y^3, x^6z - x^3y^2z, x^5yz - x^2y^3z, x^5z - x^2y^2z, x^3y^3 - y^5, \\ &x^7 - x^4y^2, x^4y^2 - xy^4 \in \mathrm{csupp}(G) \end{aligned}$$

and thus by appropriate addition and negation of terms also

$$x^6y, x^6z, x^5yz, x^2y^2z, y^5, x^7 - x^4y^2, x^4y^2 - xy^4 \in \mathrm{csupp}(G).$$

Let I be the vector space (I is even an ideal) spanned by $\mathfrak{m}^9$, the monomials m for which we have shown that $m \in \mathrm{csupp}(G)$ as well as $x^7 - x^4y^2$ and $x^4y^2 - xy^4$. We have seen that

$$I \subseteq \mathrm{csupp}(G) \subseteq \mathrm{csupp}(F).$$

Let $f \in \mathrm{span}(F)$. Since $f \in \mathrm{span}(F')$, we may express f as

$$\begin{aligned} f \in\, &a_1x^2z^2 + a_2xy^2z + a_3y^4 + a_4xyz^2 + a_5y^3z + a_6y^2z^2 + a_7xz^3 + a_8yz^3 + a_9z^4 \\ &+ b_1x^4z + b_2x^3y^2 + b_3x^3yz + b_4x^2y^3 + c_1x^6 + c_2x^5y + c_3x^4y^2 + i(x, y, z), \end{aligned}$$

with monomials with real coefficients and $i \in I$.
Let us assume for the moment that $f \in \mathrm{rai}(F)$ and $i = 0$ (as in (b)). Now, $f(x, y, 0) \in H := \mathrm{carr}_{\mathbb{R}[[x,y]]_+}((x^3 - y^2)^2 + x^4y^2)$ and thus $a_3 = c_1 > 0$, $b_2 = -2a_3$, $b_4 + c_2 = 0$ and $4a_3c_3 - b_4^2 > 0$ (see 3.5.9).
We set $x = t^2$, $y = t^3$ and $z = st^5$ ($s, t \in \mathbb{R}$) and we obtain

$$(a_2 + b_1)st^{13} + (a_1s^2 + a_5s + b_3s + c_3)t^{14} + O(t^{15}).$$

Now, $f \in \mathcal{P}^{Loc}$ and the behaviour $t \to 0$ gives $a_2 + b_1 = 0$ and $4a_1c_3 \geq (a_5 + b_3)^2$. Note that these equalities and inequalities with $\geq$ instead of $>$ also hold for elements $f \in \mathrm{ac}(F)$ and thus for all $f \in \mathrm{csupp}(F)$.
The monomials $x^2z^2, xy^2z, y^4, xyz^2, y^3z, y^2z^2, xz^3, yz^3, z^4, x^4z, x^3y^2, x^6$ lie on $\Gamma(F)$, so they do not appear in elements of csupp(F) (see 2.2.8(c), 3.3.10). Let us assume now that $f \in \mathrm{csupp}(F)$. Then,

$$f = b_3x^3yz + b_4(x^2y^3 - x^5y) + c_3x^4y^2 + i(x, y, z)$$

with $i \in I$. Now, $4a_3c_3 - b_4^2 \geq 0$ and $4a_1c_3 \geq (a_5 + b_3)^2$ imply $-b_4^2 \geq 0$ and $0 \geq b_3^2$, so $b_3 = b_4 = 0$. Since $f(x, y, 0) \in \mathrm{csupp}(H)$, $x^4y^2 \notin \mathrm{csupp}(H)$ and $i(x, y, 0) \in \mathrm{csupp}(H)$, we have $c_3 = 0$. This shows $I = \mathrm{csupp}(F)$ and thus $\mathrm{csupp}(G) = \mathrm{csupp}(F)$.

□

4.3 Relative algebraic interior and sums of squares

Now, let us show that the faces F of the previous two theorems also have the property that $\operatorname{rai}(F) \subseteq \sum \mathbb{R}[[\underline{x}]]^2$. In the case of the monomial faces, we will need the following graph to connect sums of squares representations with partially non-negative matrices (similar to [BlSiVe, Section 6.1]).

Definition 4.3.1. Let Γ be a psd-diagram. We have a set of vertices and edges

$$V := \frac{1}{2}\Gamma \cap \mathbb{Z}^n \quad \text{and} \quad E := \{\{\alpha, \beta\} \subseteq V \mid \alpha + \beta \in \Gamma\}$$

and call the undirected graph $G(\Gamma) := (V, E)$ (with loops) the *graph corresponding to* Γ.

The following remark already indicates the connection to sums of squares.

Remark 4.3.2. Let F be a monomial face of $\mathbb{R}[[\underline{x}]]_+$ and $\Gamma := \Gamma(F)$. Furthermore, let $\sigma = \sum_{i=1}^k g_i^2 \in F \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2$ for some $g_i \in \mathbb{R}[[\underline{x}]]$. Then, by 4.1.5, V consists exactly of the exponent vectors of monomials in g_i that contribute to σ^Γ, so V is $\subseteq$-minimal with $\sigma^\Gamma = \left(\sum_{i=1}^k ((g_i)^V)^2\right)^\Gamma$ for all such σ. Note also that each face τ of Γ induces the clique $\frac{1}{2}\tau \cap \mathbb{Z}^n$ in $G(\Gamma)$.

A result that is important in this context will be 4.3.4, which gives a criterion when it is possible to complete partially filled matrices to some positive semidefinite matrix. We need the following definition, which is based on definitions in [GJSW, p.111, p.118].

Definition 4.3.3. Let $G = (V, E)$ be a finite undirected graph that may have loops. Then, G is called *chordal* if each cycle of length ≥ 4 has a chord.
Corresponding to G, we define the set

$$\mathcal{D}(G) := \{(i, j) \in V \times V \mid \{i, j\} \in E\}.$$

and call them the *indices given by* G. A *real G-partial matrix* is a map

$$\begin{aligned} A(G) : \mathcal{D}(G) &\to \mathbb{R} \\ (i, j) &\mapsto A(G)_{i,j}. \end{aligned}$$

We call $A(G)$ *G-partial symmetric* if $A(G)_{i,j} = A(G)_{j,i}$ for $(i, j) \in \mathcal{D}(G)$ and G-partial non-negative if it is G-partial symmetric and for each clique $C \subseteq V$ and $\mathcal{D}(C) := \mathcal{D}(G) \cap (C \times C)$ the matrix $A(G)|_{\mathcal{D}(C)}$ (indexed by C) is positive semidefinite.
A *completion* of $A(G)$ is a map

$$A : V \times V \to \mathbb{R} \quad \text{with} \quad A|_{\mathcal{D}(G)} = A(G).$$

We call A a *non-negative completion* of $A(G)$ if A is a completion and positive semidefinite as square matrix indexed by V (especially it is symmetric). We call G *nonnegative-completable* if each G-partial non-negative matrix has a non-negative completion.

Theorem 4.3.4. *Let G be a finite undirected graph (with loops). Then, G is nonnegative-completable if and only if G is chordal.*

Proof. It was proved in [GJSW, Proposition 2, Theorem 7] for Hermitian matrices. As the counterexample of a not nonnegative-completable matrix for given non-chordal G is a real matrix (see [GJSW, proof of Theorem 7]), the result carries over to real (symmetric) matrices. □

The following theorem and its proof show our strategy how to establish our goal $\mathrm{rai}(F) \subseteq \Sigma\,\mathbb{R}[[\underline{x}]]^2$ for monomial faces F. However, we have to study the premises later.

Theorem 4.3.5. *Let F be a monomial face of $\mathbb{R}[[\underline{x}]]_+$ with $\mathrm{csupp}(F \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2) = \mathrm{csupp}(F)$. We set $\Gamma := \Gamma(F)$ and let $G := G(\Gamma) = (V, E)$ be the corresponding graph. Furthermore, let us assume*

(a) $\mathcal{P}_\tau = \Sigma_\tau$ for all $\subseteq$-maximal faces τ of Γ.

(b) G is chordal.

(c) For every clique $C \subseteq V$ of G there is a face τ of Γ such that $C \subseteq V_\tau$ with $V_\tau := \frac{1}{2}\tau \cap \mathbb{Z}^n$.

(d) Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in V$ with $\{\alpha_1, \alpha_2\} \neq \{\alpha_3, \alpha_4\}$ and $\beta := \alpha_1 + \alpha_2 = \alpha_3 + \alpha_4$. Then, there is at most one $\subseteq$-maximal face τ of Γ with $\beta \in \tau$.

Then, we have $\mathrm{rai}(F) \subseteq \Sigma\,\mathbb{R}[[\underline{x}]]^2$.

Proof. Let $f_0 \in \mathrm{rai}(F)$ and $\sigma_0 \in \mathrm{rai}(F \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2)$. We choose $\epsilon > 0$ such that

$$f := f_0 - \epsilon\sigma_0 \in \mathrm{rai}(F).$$

Let M be the set of $\subseteq$-maximal faces of Γ and $\tau \in M$. Then, $f^\tau \in \mathcal{P}_\tau = \Sigma_\tau$ by 3.3.10(a) and (a) and we may choose a representation as sums of squares $f^\tau = \sum_{i=1}^{k_\tau} (g_{\tau,i})^2$ with $k_\tau \in \mathbb{N}_0$ and $g_{\tau,i} \in \mathbb{R}[\underline{x}]^{\frac{\tau}{2}}$ for $i = 1, \ldots, k_\tau$ and each $\tau \in M$. We have a corresponding positive semidefinite Gram matrix $A_\tau : V_\tau \times V_\tau \to \mathbb{R}$.
We define a G-partial matrix $A(G)$ via $A(G)|_{V_\tau \times V_\tau} := A_\tau$ for $\tau \in M$ and we show that it is well-defined. We show first that $E = \bigcup_{\tau \in M} E_\tau$ with

$$E_\tau = \{\{\alpha, \beta\} \mid \alpha, \beta \in V_\tau\}.$$

(One may consider it as definition of E_τ but it coincides with the definition if one considers τ as psd-diagram.)
If $\{\alpha, \beta\} \in E$ such that $\alpha + \beta \in \Gamma$, then there is some $\tau \in M$ with $\alpha + \beta \in \tau$. Since $\alpha + \beta = \frac{1}{2}(2\alpha + 2\beta)$ with $2\alpha, 2\beta \in \Delta(F)$ and since τ is a face of $\Delta(F)$, we have $2\alpha, 2\beta \in \tau$ and thus $\alpha, \beta \in V_\tau$. This shows $E = \bigcup_{\tau \in M} E_\tau$ (compare also with 4.1.5 and 4.3.2). Now, it is easy to see that $\mathcal{D}(G) = \bigcup_{\tau \in M} V_\tau \times V_\tau$.
This shows that $A(G)$ assigns at least one value to each $(\alpha, \beta) \in \mathcal{D}(G)$. Let us show uniqueness of the value at $(\alpha, \beta) \in \mathcal{D}(G)$, so independence of $\tau \in M$ with $\{\alpha, \beta\} \in V_\tau$. Let us assume that there are distinct $\tau_1, \tau_2 \in M$ with $\{\alpha, \beta\} \in V_{\tau_1} \cap V_{\tau_2}$. Then,

$$\delta := \alpha + \beta \in \tau_1 \cap \tau_2,$$

and by (d) we see that α and β are uniquely determined by $\delta = \alpha + \beta$ and $\alpha, \beta \in V$. Let $\tau \in \{\tau_1, \tau_2\}$, $a_{\tau,i}$ and $b_{\tau,i}$ be the coefficients of $\underline{x}^\alpha$ and $\underline{x}^\beta$ in $g_{\tau,i}$ and c the coefficient of $\underline{x}^\delta$ in

f for $i = 1, \dots, k_\tau$. Then, uniqueness of the representation $\delta = \alpha + \beta$ and $f^\tau = \sum_{i=1}^{k_\tau} (g_{\tau,i})^2$ show that

$$c = \begin{cases} \sum_{i=1}^{k_\tau} (a_{\tau,i})^2 & \text{if } \alpha = \beta \\ \sum_{i=1}^{k_\tau} 2a_{\tau,i}b_{\tau,i} & \text{if } \alpha \neq \beta. \end{cases}$$

The entry of A_τ at (α, β) is by definition of the Gram matrix $\sum_{i=1}^{k_\tau} (a_{\tau,i})^2 = c$ if $\alpha = \beta$ and $\sum_{i=1}^{k_\tau} a_{\tau,i}b_{\tau,i} = \frac{1}{2}c$ if $\alpha \neq \beta$ independently of the choice $\tau \in \{\tau_1, \tau_2\}$. This shows that $A(G)$ does just depend on f and thus, it is well-defined.
By definition and (c), $A(G)$ is a G-partial non-negative matrix and by (b) and 4.3.4 it has a non-negative completion $A : V \times V \to \mathbb{R}$. Now, A is a Gram matrix corresponding to some sum of squares $\sigma := \sum_{i=1}^{k} g_i^2$ with $g_i \in \mathbb{R}[\underline{x}]^V$. By definition, $f^\Gamma = \sigma^\Gamma$, so $\operatorname{supp}(f - \sigma) \subseteq \Delta(F) \setminus \Gamma$ and thus, by 3.3.14 and assumption,

$$f - \sigma \in \operatorname{csupp}(F) = \operatorname{csupp}(F \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2).$$

We obtain (using 2.2.8(c))

$$f_0 = f + \epsilon\sigma_0 = (\epsilon\sigma_0 + (f - \sigma)) + \sigma \in (F \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2) + \Sigma\,\mathbb{R}[[\underline{x}]]^2 \subseteq \Sigma\,\mathbb{R}[[\underline{x}]]^2.$$

Since $f_0 \in \operatorname{rai}(F)$ was chosen arbitrarily, this shows the claim $\operatorname{rai}(F) \subseteq \Sigma\,\mathbb{R}[[\underline{x}]]^2$. □

The following remark says that the conditions (b)-(d) are easily verified if Γ has just one $\subseteq$-maximal face (which is itself).

Remark 4.3.6. If Γ is a polytope, then conditions (b)-(d) in 4.3.5 automatically hold. Indeed, then G is a clique graph (see 4.3.2), so (b), (c) and (d) follow as there is just one $\subseteq$-maximal face of Γ.

The idea of the proof of 4.3.5 was to construct a matrix A by glueing positive semidefinite matrices and completing it. If we already know that our considered element f is a sum of squares, then such a matrix A automatically exists as the next remark explains.

Remark 4.3.7. Let F be a monomial face of $\mathbb{R}[[\underline{x}]]_+$ and $f \in F \cap \Sigma\,\mathbb{R}[[\underline{x}]]^2$ with $f = \sum_{i=1}^{k} g_i^2$ where $g_i \in \mathbb{R}[[\underline{x}]]$ and $k \in \mathbb{N}$. Using the notation of 4.3.5 and setting $k_\tau := k$ and $g_{\tau,i} := (g_i)^{\frac{1}{2}\tau}$ for $\subseteq$-maximal faces τ of Γ and $i = 1, \dots, k$, we always (without using (d)) get a corresponding well-defined G-partial matrix $A(G)$ by construction. Indeed, it is even non-negative completable as it is the Gram matrix corresponding to $\sum_{i=1}^{k}((g_i)^V)^2$ restricted to $\mathcal{D}(G)$ (see also 4.1.5 and 4.3.2). Especially, given $f \in F$ and that each G-partial matrix $A(G)$ constructed as in 4.3.5 is not non-negative completable, then $f \notin \Sigma\,\mathbb{R}[[\underline{x}]]^2$.

Let us take a look at the premises (a)-(d) of the theorem. Clearly, property (a) is a necessary condition to imply $\operatorname{rai}(F) \subseteq \Sigma\,\mathbb{R}[[\underline{x}]]^2$ (see 4.1.6 and note that Σ_τ is closed in $\mathcal{P}_\tau$). In the following example, we provide counterexamples F (i.e. $\operatorname{rai}(F) \not\subseteq \Sigma\,\mathbb{R}[[\underline{x}]]^2$) for which one of the premises (b)-(d) fails (we do not check the other properties). All counterexamples are in the case $n = 3$ and the first and the third example have even $\operatorname{ord}(F) = 4$ as desired.

Example 4.3.8. (a) We give an example where (c) does not hold and also $\operatorname{rai}(F) \not\subseteq \Sigma\,\mathbb{R}[[x, y, z]]^2$. Let $F := \operatorname{carr}(x^4 + y^8 + x^2y^2z^2 + z^{10})$. Let us consider the following graph

$$V := \{(2,0,0), (1,2,0), (0,4,0), (1,1,1), (0,0,5)\}$$
$$\text{and} \quad E := \{\{\alpha, \beta\} \mid \alpha, \beta \in V\} \setminus \{\{(1,2,0), (0,0,5)\}\}.$$

After visualizing the graph interpreting the vertices in $\mathbb{R}^3$, it is easy to see that it is the graph corresponding to $\Gamma(F)$. We have (see 3.3.14)

$$f := 2x^4 + 3y^8 + 4z^{10} + x^2y^2z^2 + 4x^2z^5 + 6y^4z^5 \in \operatorname{rai}(F).$$

It is easy to see that each G-partial matrix $A(G)$ corresponding to f as in the proof of 4.3.5 is of the following form for some $a \geq 0$ (the entries are ordered in the same order as V is defined above).

$$A(G) = \begin{pmatrix} 2 & 0 & -a & 0 & 2 \\ 0 & 2a & 0 & 0 & \\ -a & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & & 3 & 0 & 4 \end{pmatrix}$$

$G(\Gamma)$ has another clique (not corresponding to any face of Γ) with vertices $(2,0,0)$, $(0,4,0)$ and $(0,0,5)$, so (c) does not hold. The corresponding submatrix, which results from eliminating 2nd and 4th row and column, has determinant $-6-12a-4a^2 < 0$ and thus, is not positive semi-definite. This shows that there is no sum of squares σ with $\operatorname{supp}(\sigma) \subseteq \Delta(F)$ with $f^\Gamma = \sigma^\Gamma$ as $A(G)$ is not nonnegative-completable for any $a \in \mathbb{R}$. Especially, $f \notin \Sigma\,\mathbb{R}[[x,y,z]]^2$ by 4.3.7.
A similar example where the matrix is completely determined without parameter (indeed it coincides with the matrix $A(G)$ above after eliminating 2nd and 4th row and column and setting $a = 0$) is given by

$$\begin{aligned} & F := \operatorname{carr}(x^8 + y^{10} + x^2y^2z^2 + z^{14}) \\ \text{and} \quad & f := 2x^8 + 3y^{10} + 4z^{14} + x^2y^2z^2 + 4x^4z^7 + 6y^2z^5. \end{aligned}$$

(b) Let us give an example with non-chordal $G(\Gamma(F))$ and $\operatorname{rai}(F) \nsubseteq \Sigma\,\mathbb{R}[[x,y,z]]^2$. We set

$$F := \operatorname{carr}(x^6y^2 + x^2y^8 + y^2z^6 + x^2z^6 + x^2y^2z^2).$$

As in (a), we have the following graph that is the graph corresponding to $\Gamma(F)$. We have

$$\begin{aligned} & V = \{(3,1,0),(1,4,0),(0,1,3),(1,0,3),(1,1,1)\} \\ \text{and} \quad & E = \{\{\alpha,\beta\} \mid \alpha,\beta \in V\} \setminus \{\{(3,1,0),(0,1,3)\},\{(1,4,0),(1,0,3)\}\}. \end{aligned}$$

This graph is not chordal since the 4-cycle with vertices $(3,1,0)$, $(1,4,0)$, $(0,1,3)$, $(1,0,3)$ has no chord. By 4.3.4, there is a G-partial non-negative matrix that may not be completed to a positive semi-definite matrix. One may choose for instance (see [GJSW, proof of Theorem 7])

$$A(G) = \begin{pmatrix} 1 & 1 & & -1 & 0 \\ 1 & 1 & 1 & & 0 \\ & 1 & 1 & 1 & 0 \\ -1 & & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us complete $A(G)$ by filling the matrix with reals a and b to achieve a symmetric matrix A. We consider $A + \epsilon I_5$ with the 5×5 identity matrix I_5 and say $1 \geq \epsilon \geq 0$. Clearly, if $a \notin [-2, 2]$ or $b \notin [-2, 2]$, then $A + \epsilon I_5$ may not be positive semi-definite since the corresponding principal minor of size 2×2 is negative. We set

$$\varphi(a, b) := \min\{\epsilon \in \mathbb{R} \mid A + \epsilon I_5 \text{ is positive semi-definite}\}.$$

By construction, we have $\varphi(a, b) > 0$ for all $a, b \in \mathbb{R}$. Since $\varphi(a, b)$ is continuous (use for instance the description of semi-definiteness by non-negative principal minors), it attains its minimum $\epsilon_0 > 0$ on $[-2, 2] \times [-2, 2]$. We choose some $0 < \epsilon < \min\{\epsilon_0, 1\}$ such that $A(G) + \epsilon I_5$ is not non-negative completable and we set

$$\begin{aligned} f :=& x^6y^2 + x^2y^8 + y^2z^6 + x^2z^6 + 2x^4y^5 + 2xy^5z^3 + 2xyz^6 - 2x^4yz^3 \\ &+ \epsilon(x^6y^2 + x^2y^8 + y^2z^6 + x^2z^6 + x^2y^2z^2). \end{aligned}$$

Now, $A(G) + \epsilon I_5$ is the only G-partial matrix corresponding to f as in 4.3.5, which implies $f \notin \Sigma\,\mathbb{R}[[x, y, z]]^2$ by 4.3.7. One may show $f \in \operatorname{rai}(F)$ using 3.3.14 (note that the monomials with coefficient ± 2 correspond to the midpoints of the edges of the non-chordal 4-cycle and visualize this in $\Gamma(F)$).

(c) Finally, we give an example for which (d) does not hold. Let $F := \operatorname{carr}(z^4 + x^4z^2 + y^4z^2 + x^6y^4)$. As in (a), we have the graph corresponding to $\Gamma(F)$ given by

$$\begin{aligned} & V = \{(0, 0, 2), (2, 0, 1), (1, 1, 1), (0, 2, 1), (3, 2, 0)\} \\ \text{and} \quad & E = \{\{\alpha, \beta\} \mid \alpha, \beta \in V\} \setminus \{\{(0, 0, 2), (3, 2, 0)\}\}. \end{aligned}$$

We have $(2, 0, 1) + (0, 2, 1) = (1, 1, 1) + (1, 1, 1) = (2, 2, 2) \in \Gamma(F)$ and (d) does not hold. We consider

$$f = z^4 + 2x^4z^2 + 2y^4z^2 + x^6y^4 + 2\sqrt{2}xyz^3 + \frac{14}{5}x^5y^2z.$$

We have the corresponding G-partial matrix

$$\begin{pmatrix} 1 & 0 & \sqrt{2} & 0 & \\ 0 & 2 & 0 & -a & 1.4 \\ \sqrt{2} & 0 & 2a & 0 & 0 \\ 0 & -a & 0 & 2 & 0 \\ & 1.4 & 0 & 0 & 1 \end{pmatrix}.$$

The upper-left 4×4 matrix is positive semi-definite if and only if $1 \leq a \leq 2$ and the lower-right 4×4 matrix is positive semi-definite if and only if $0 \leq a \leq \frac{\sqrt{2}}{5}$. Now, $f^\tau \in \operatorname{rai}(\Sigma_\tau)$ for the two faces τ of Γ and thus $f \in \operatorname{rai}(F)$. On the other hand, it is not possible to find any a such that one gets a G-partial non-negative matrix, so $f \notin \Sigma\,\mathbb{R}[[x, y, z]]^2$ by 4.3.7.

In the following, we will prepare theorem 4.3.11, which is important to check if condition (a) of 4.3.5 holds. These statements are well-known but we state them in our terms.

Definition 4.3.9. Let $P \subseteq \mathbb{R}^s$ and $Q \subseteq \mathbb{R}^t$ be lattice polytopes. We call P and Q *equivalent* if there is some $n \in \mathbb{N}$ with $n \geq s, t$ such that P and Q considered as lattice polytopes in $\mathbb{R}^n = \mathbb{R}^s \times \mathbb{R}^{n-s} = \mathbb{R}^t \times \mathbb{R}^{n-t}$ may be transferred into each other via affine integral automorphisms of $\mathbb{R}^n$.

Remark 4.3.10. (a) Affine integral automorphisms are automorphisms of $\mathbb{R}^n$ as an affine space of the form $z \mapsto Az + b$ with some integral unimodular $n \times n$ matrix A and some $b \in \mathbb{Z}^n$.

(b) Let $\mathbb{R}[\underline{x}, \underline{x}^{-1}] := \mathbb{R}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ be the ring of Laurent polynomials in n indeterminates. Given any $f = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha \underline{x}^\alpha \in \mathbb{R}[\underline{x}, \underline{x}^{-1}]$ and $\tau \subseteq \mathbb{R}^n$, we set

$$\operatorname{supp}(f) := \{\alpha \in \mathbb{Z}^n \mid a_\alpha \neq 0\} \text{ and } \mathbb{R}[\underline{x}, \underline{x}^{-1}]^\tau := \{f \in \mathbb{R}[\underline{x}, \underline{x}^{-1}] \mid \operatorname{supp}(f) \subseteq \tau\}.$$

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n, z \mapsto Az + b$ be an affine integral automorphism and furthermore, let $\varphi' : \mathbb{R}^n \to \mathbb{R}^n, z \mapsto Az$ be the corresponding linear map. Then, we have an induced map on monomials of $\mathbb{R}[\underline{x}, \underline{x}^{-1}]$ via $\underline{x}^\alpha \mapsto \underline{x}^{\varphi'(\alpha)}$ and a unique induced ring automorphism ψ' of $\mathbb{R}[\underline{x}, \underline{x}^{-1}]$. If $\tau \subseteq \mathbb{R}^n$, then

$$\begin{aligned} & f \in \mathbb{R}[\underline{x}, \underline{x}^{-1}]^{2\tau} && \text{is a sum of squares of elements of } \mathbb{R}[\underline{x}, \underline{x}^{-1}]^\tau \\ \Leftrightarrow \quad & \psi'(f) \in \mathbb{R}[\underline{x}, \underline{x}^{-1}]^{2\varphi'(\tau)} && \text{is a sum of squares of elements of } \mathbb{R}[\underline{x}, \underline{x}^{-1}]^{\varphi'(\tau)} \\ \Leftrightarrow \quad & \underline{x}^{2b} \cdot \psi'(f) \in \mathbb{R}[\underline{x}, \underline{x}^{-1}]^{2\varphi(\tau)} && \text{is a sum of squares of elements of } \mathbb{R}[\underline{x}, \underline{x}^{-1}]^{\varphi(\tau)}. \end{aligned}$$

Note that $\psi''(f) = \underline{x}^{2b} \cdot \psi'(f)$ where ψ'' is the vector space automorphism of $\mathbb{R}[\underline{x}, \underline{x}^{-1}]$ induced by $\underline{x}^\alpha \mapsto \underline{x}^{\varphi''(\alpha)}$ with $\varphi'' : \mathbb{R}^n \to \mathbb{R}^n, z \mapsto Az + 2b$.
Furthermore, we have an induced group automorphism $\chi : (\mathbb{R}^*)^n \to (\mathbb{R}^*)^n$ (with respect to multiplication) given by $(x_1, \ldots, x_n) \mapsto (x_1^{\varphi'(e_1)}, \ldots, x_n^{\varphi'(e_n)})$ with the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. Especially, we have $\psi''(f)(a) = (\underline{x}^b(a))^2 f(\chi(a))$ for $f \in \mathbb{R}[\underline{x}, \underline{x}^{-1}]$ and $a \in (\mathbb{R}^*)^n$. Now, for any $f \in \mathbb{R}[\underline{x}, \underline{x}^{-1}]$ we have $f \geq 0$ on $(\mathbb{R}^*)^n$ if and only if $\psi''(f) \geq 0$ on $(\mathbb{R}^*)^n$.
Let P and Q be equivalent lattice polytopes with induced affine integral automorphism φ. Then, by the discussion above, the following two statements are equivalent:

- every $f \in \mathbb{R}[\underline{x}, \underline{x}^{-1}]^{2P}$ with $f \geq 0$ on $(\mathbb{R}^*)^n$ is a sum of squares of elements from $\mathbb{R}[\underline{x}, \underline{x}^{-1}]^P$
- every $f \in \mathbb{R}[\underline{x}, \underline{x}^{-1}]^{2Q}$ with $f \geq 0$ on $(\mathbb{R}^*)^n$ is a sum of squares of elements from $\mathbb{R}[\underline{x}, \underline{x}^{-1}]^Q$.

Note that for any $f \in \mathbb{R}[\underline{x}]$ we have $f \geq 0$ on $(\mathbb{R}^*)^n$ if and only if $f \geq 0$ on $\mathbb{R}^n$.

Now, here is the theorem that is important to verify condition (a) of 4.3.5. However, in our application we will just need case (b) of the following theorem.

Theorem 4.3.11. *Let $P \subseteq \mathbb{R}^n$ be a lattice polytope such that $P \cap \mathbb{Z}^n$ generates the group $\mathbb{Z}^n$. Then, every $f \in \mathbb{R}[\underline{x}, \underline{x}^{-1}]^{2P}$ with $f \geq 0$ on $(\mathbb{R}^*)^n$ is a sum of squares (with elements from $\mathbb{R}[\underline{x}, \underline{x}^{-1}]^P$) if and only if there is some equivalent lattice polytope Q that is contained in one of the following polytopes:*

(a) the standard simplex $\operatorname{conv}(0, e_1, \ldots, e_m) \subseteq \mathbb{R}^m$

(b) the Cayley polytope of m lines $[0, d_i]$ ($d_i \in \mathbb{N}_0$, $i = 1, \ldots, m$):

$$\operatorname{conv}(([0, d_1] \times e_1) \cup \cdots \cup ([0, d_n] \times e_n)) \subseteq \mathbb{R} \times \mathbb{R}^m$$

(c) the simplex $\operatorname{conv}((0,0), (0,2), (2,0)) \subseteq \mathbb{R}^2$

(d) a polytope of the form

$$\operatorname{conv}((Q \times \{0\}) \cup (\{0\} \times \Delta^{m_2})) \subseteq \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$$

with some polytope Q as in (a)-(c) and $\Delta^{m_2} := \operatorname{conv}(e_1, \ldots, e_{m_2})$.

As usual, we denote the standard basis vectors of $\mathbb{R}^m$ by $e_1, \ldots, e_m$.

Proof. [CPSV, Theorem 2.1] □

The following lemma is important to obtain 4.3.13, which is the consequence of 4.3.11 of importance for the faces we study. However, 4.3.13 will just be used in the case $n = 3$ (but the proof is the same for general n such that we include it here).

Lemma 4.3.12. *Let $\alpha \in \mathbb{Z}^n$ with coprime entries. Then, there exists some unimodular integral $n \times n$ matrix A such that α is the first column of A.*

Proof. See [New, Theorem II.1.]. □

Corollary 4.3.13. *We consider a two-dimensional lattice polytope $P \subseteq \mathbb{R}^n$ with the vertices $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}^n$, $\alpha_1 \neq \alpha_2$ such that $\alpha_4 - \alpha_3 \in \mathbb{R} \cdot (\alpha_2 - \alpha_1)$ (i.e. a trapezoid or a triangle) and*

$$\operatorname{conv}(\alpha_1, \alpha_2, \alpha_3) \cap \mathbb{Z}^n \subseteq \{\alpha_3\} \cup \operatorname{conv}(\alpha_1, \alpha_2).$$

Then, $\mathcal{P}_{2P} = \Sigma_{2P}$.

Proof. We will show that P is equivalent to a polytope that is contained in a polytope as in 4.3.11(b). Then, 4.3.11 and 4.3.10(b) show the claim. We will repeatedly restrict to special cases by passing on to some equivalent polytopes or enlarge our polytope until we obtain the desired result. Note that the property $\operatorname{conv}(\alpha_1, \alpha_2, \alpha_3) \cap \mathbb{Z}^n \subseteq \{\alpha_3\} \cup \operatorname{conv}(\alpha_1, \alpha_2)$ is preserved under affine integral automorphisms.
We may assume that $\alpha_1 = 0$ since translation by $-\alpha_1$ is an affine integral automorphism on $\mathbb{R}^n$. Let k be the greatest common divisor of the entries of α_2 and we set $\beta := \frac{1}{k}\alpha_2$. By 4.3.12, there is an integral unimodular $n \times n$ matrix A with β as its first column. Thus, the affine integral automorphism $x \mapsto A^{-1}x$ sends β to $(1, 0, \ldots, 0)$. Hence, we may assume without loss of generality that $\alpha_2 = (k, 0, \ldots, 0)$. Let $\alpha_3' \in \mathbb{Z}^{n-1}$ the vector that arises from α_3 by crossing out the first entry, l the greatest common divisor of the entries of α_3' ($l > 0$ since P is two-dimensional) and $\gamma := \frac{1}{l}\alpha_3'$. Using 4.3.12, we obtain an integral unimodular $(n-1) \times (n-1)$ matrix B' that has γ as its first column and thus, we have an integral unimodular $n \times n$ block matrix

$$B := \begin{pmatrix} 1 & 0 \\ 0 & B' \end{pmatrix}.$$

The affine integral automorphism $x \mapsto B^{-1}x$ sends α_1, α_2 on themselves and α_3 on an element of the form $(k', l, 0, \ldots, 0)$ with $k' \in \mathbb{Z}$ and $l \in \mathbb{N}$ (and α_4 is also of that form with possibly a different first entry). We may reduce to the case $n = 2$. Using the unimodular automorphism $x \mapsto Cx$ with

$$C := \begin{pmatrix} 1 & -\left\lfloor \frac{k'}{l} \right\rfloor \\ 0 & 1 \end{pmatrix},$$

we reduce to $0 \leq k' < l$. This implies $k' = 0$ and $l = 1$ as in all other cases we have that $(1,1)$ (in the case $k' > 0$) or $(0,1)$ (in the case $k' = 0$) is contained in $\mathrm{conv}(\alpha_1, \alpha_2, \alpha_3)$ and not contained in $\{\alpha_3\} \cup \mathrm{conv}(\alpha_1, \alpha_2)$. After some translation in the second coordinate, we see that our polytope is contained in $Q := [0,d] \times [0,1]$ for some $d \in \mathbb{N}$. Clearly, the polytope $Q \times \{1\}$ in $\mathbb{R}^3$ is equivalent to Q as it arises by some translation of Q considered as polytope in $\mathbb{R}^3$. An application of the affine integral map $x \mapsto Dx$ with

$$D := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

gives the desired Cayley polytope $\mathrm{conv}(([0,d] \times e_2) \cup ([0,d] \times e_3)) \subseteq \mathbb{R}^3$ and thus shows the claim. □

The following lemma just simplifies the proof of the subsequent theorem. It gives some inductive argument how to show the conditions (a)-(d) of 4.3.5.

Lemma 4.3.14. *Let $F_2 \subsetneq F_1$ be monomial faces of $\mathbb{R}[[x,y,z]]_+$ such that $\Gamma(F_1) = \Gamma(F_2) \cup \gamma_0$ with some two-dimensional face γ_0 of $\Gamma(F_1)$ with vertices $2\alpha_1, 2\alpha_2 \in \Gamma(F_2)$, $2\alpha_3, 2\alpha_4 \in \Gamma(F_1) \setminus \Gamma(F_2)$ such that $\alpha_4 - \alpha_3 \in \mathbb{R} \cdot (\alpha_2 - \alpha_1)$ and $\mathrm{conv}(\alpha_1, \alpha_2) \cap \mathbb{Z}^3 = \{\alpha_1, \alpha_2\}$. Furthermore, let us assume that either*

- *γ_0 is a triangle (with $\alpha_3 = \alpha_4$) and $\mathrm{conv}(\alpha_1, \alpha_2, \alpha_3) \cap \mathbb{Z}^3 \subseteq \{\alpha_1\} \cup \mathrm{conv}(\alpha_2, \alpha_3)$ or*
- *γ_0 is a trapezoid and $\mathrm{conv}(\alpha_1, \alpha_3, \alpha_4) \cap \mathbb{Z}^n \subseteq \{\alpha_1\} \cup \mathrm{conv}(\alpha_3, \alpha_4)$.*

Finally, let us assume that the conditions 4.3.5(a)-(d) hold for F_2. Then, 4.3.5(a)-(d) also hold for F_1.

Proof. Let us assume that the conditions 4.3.5(a)-(d) hold for F_2. Then, by 4.3.13, condition (a) of 4.3.5 also holds for F_1 since γ_0 is the unique $\subseteq$-maximal face of $\Gamma(F_1)$ that is no $\subseteq$-maximal face of $\Gamma(F_2)$. We set

$$G_i = (V_i, E_i) := G(\Gamma(F_i)) \text{ for } i = 1, 2.$$

Furthermore, let $V_{\gamma_0} := \frac{1}{2}\gamma_0 \cap \mathbb{Z}^n$. Let us consider condition 4.3.5(d), so as 4.3.5(d) holds for F_2, we consider $\beta \in \Gamma(F_1)$ that belongs to two $\subseteq$-maximal faces of $\Gamma(F_1)$ and may be written as $\beta = \alpha + \alpha'$ with $\alpha \in V_{\gamma_0}$, $\alpha' \in V_1$. As γ_0 is an $\subseteq$-maximal proper face of $\Delta(F_1)$, $\alpha \in V_{\gamma_0}$ and $\beta \in \Gamma(F_1)$, we also have $\alpha' \in V_{\gamma_0}$. Hence, $\beta \in \gamma_0$ and as it belongs to another $\subseteq$-maximal face of $\Gamma(F_1)$, we have $\beta \in \mathrm{conv}(2\alpha_1, 2\alpha_2)$ and $\alpha, \alpha' \in \mathrm{conv}(\alpha_1, \alpha_2) \cap \mathbb{Z}^3 = \{\alpha_1, \alpha_2\}$. This implies 4.3.5(d) for F_1.
Let $V_{\gamma_0} = \{\alpha_1, \ldots, \alpha_l\}$ with $l \geq 3$. Then, $V_1 = V_2 \dot\cup \{\alpha_3, \ldots, \alpha_l\}$ and

$$E_1 = E_2 \cup \{\{\alpha, \alpha'\} \,|\, \alpha, \alpha' \in \{\alpha_1, \ldots, \alpha_l\}\}.$$

Thus, G_1 is the clique sum of G_2 and the graph corresponding to the clique V_{γ_0} (glued along the clique $\{\alpha_1, \alpha_2\}$). It is well-known that this construction preserves chordal graphs and indeed every chordal graph may be constructed by such glueing of clique graphs along a clique starting from a clique graph (see [Dir, Theorem 2]). Thus, as G_2 is chordal, G_1 is also

chordal and condition 4.3.5(b) holds for F_1.
Finally, let C be a clique of G_1 with $\alpha_i \in C$ for some $i \in \{3, \dots, l\}$. We have

$$C \subseteq \{\alpha_1, \dots, \alpha_l\} = V_{\gamma_0}$$

since these are all neighbours of α_i. Again, since 4.3.5(c) holds for F_2, this also implies 4.3.5(c) for F_1. □

Finally, we prove our desired results.

Theorem 4.3.15. *Let $n = 3$ and F be a monomial face of $\mathbb{R}[[x, y, z]]_+$ with $\operatorname{ord}(F) = 4$ as in 4.2.14. We recall the different cases using the description of the zeros of $\tau(F)$ (as in 4.2.14), so we have either*

(a) up to three double zeros with positive definite Hessian

(b) one zero of type $(4 : 2)$ and possibly another double zero with positive definite Hessian

(c) a double line.

Then, $\operatorname{rai}(F) \subseteq \Sigma\,\mathbb{R}[[x, y, z]]^2$.

Proof. We use theorem 4.3.5 in all three cases. Now, $\operatorname{csupp}(F \cap \Sigma\,\mathbb{R}[[x, y, z]]^2) = \operatorname{csupp}(F)$ holds by 4.2.14 and thus, we reduce to the proof of the conditions 4.3.5(a)-(d). For this purpose, we can use 4.3.14 (repeatedly). Indeed, in each of the three cases, this reduces the problem of showing 4.3.5(a)-(d) to the case of the monomial face F' with $\Gamma(F') = \tau \cap \Gamma(F)$ where $\tau := \{\alpha \in \mathbb{R}^3_+ \mid |\alpha| = 4\} = \Gamma(\mathbb{R}[[x, y, z]]_+ \cap \mathfrak{m}^4)$ as one may easily from the descriptions of $\Gamma(F)$ in the proof of 4.2.14. However, then 4.3.5(a) holds by Hilbert's theorem 1.1.4 (see 3.3.22) and 4.3.5(b)-(d) hold by 4.3.6. □

The idea for the proof of the next theorem is the same as in the theorem before. However, in (b) and (c) it is not possible to apply 4.3.5 since these faces are not monomial and we have to carry out the same idea "by hand". The considered graph for the partial matrix will be slightly larger than the graph corresponding to $\Gamma(F)$ in these two cases.

Theorem 4.3.16. *Let $n = 3$. We consider the faces F of $\mathbb{R}[[x, y, z]]_+$ from 4.2.15, this means the faces*

(a) $F = \operatorname{carr}(x^6 + y^4 + z^4)$ (i.e. the face O on page 92)

(b) $F = \operatorname{carr}((xz - y^2)^2 + z^4 + y^2z^2 + x^6)$ (i.e. the face N on page 92)

(c) $F = \operatorname{carr}((x^3 - y^2)^2 + x^4y^2 + x^2z^2 + y^2z^2 + z^4)$ (i.e. the face E'_{14} on page 92)

In all three cases, we have $\operatorname{rai}(F) \subseteq \Sigma\,\mathbb{R}[[x, y, z]]^2$.

Proof. (a) The claim follows from 4.3.5. Indeed, $\operatorname{csupp}(F \cap \Sigma\,\mathbb{R}[[x, y, z]]^2) = \operatorname{csupp}(F)$ follows from 4.2.15(a). Condition (a) of 4.3.5 follows from 4.3.13 and conditions (b)-(d) follow from 4.3.6.

(b) Let $f_0 \in \operatorname{rai}(F)$, $\sigma_0 \in \operatorname{rai}(\Sigma\,\mathbb{R}[[x,y,z]]^2 \cap F)$ and $\epsilon > 0$ with $f := f_0 - \epsilon\sigma_0 \in \operatorname{rai}(F)$. We have seen in the proof of 4.2.15(b) that f is of the form

$$\begin{aligned} f =& a_1(xz-y^2)^2 + a_4(xz-y^2)yz + a_6y^2z^2 + a_7xz^3 + a_8yz^3 + a_9z^4 \\ &+ b_1(xz-y^2)x^3 + b_3x^3yz + cx^6 + i(x,y,z) \end{aligned}$$

with real coefficients and $i(x,y,z) \in \operatorname{csupp}(F)$. By 4.2.15(b), we have $\operatorname{csupp}(F) = \operatorname{csupp}(\Sigma\,\mathbb{R}[[x,y,z]]^2 \cap F)$. Since our intention is to show that $f_0 \in \operatorname{rai}(\Sigma\,\mathbb{R}[[x,y,z]]^2 \cap F)$, we may assume that $i(x,y,z) = 0$ by 2.2.8(c). We define a graph $G = (V,E)$ via

$$V = \{xz-y^2, z^2, yz, x^3\}, \quad E = \{\{g_1,g_2\} \subseteq V\} \setminus \{z^2, x^3\}$$

and we consider the following G-partial matrix

$$A(G) := \begin{pmatrix} a_1 & \frac{a_7}{2} & \frac{a_4}{2} & \frac{b_1}{2} \\ \frac{a_7}{2} & a_9 & \frac{a_8}{2} & \\ \frac{a_4}{2} & \frac{a_8}{2} & a_6+a_7 & \frac{b_3}{2} \\ \frac{b_1}{2} & & \frac{b_3}{2} & c \end{pmatrix}.$$

The upper-left 3×3 matrix is positive semi-definite since it is the Gram matrix (with respect to $\{xz-y^2, z^2, yz\}$) of the homogeneous part in degree 4 of f (by 3.3.10(a) and Hilbert's theorem 1.1.4).
We set $x = (1+rt)t^3s$, $y = t^4s$, $z = t^5s$ with $r,s,t \in \mathbb{R}$ and f is of the form

$$(a_1r^2 + a_4r + a_6 + a_7 + b_1rs + b_3s + cs^2)s^4t^{18} + O(t^{19}).$$

Since $f \in \mathcal{P}^{Loc}$, the behaviour $t \to 0$ shows that

$$a_1r^2 + a_4r + a_6 + a_7 + b_1rs + b_3s + cs^2$$

is a non-negative polynomial in r,s and thus the submatrix of $A(G)$ given by eliminating second row and column is positive semi-definite. Hence, $A(G)$ is G-partial non-negative matrix. Since G is chordal (there is just one edge missing), by 4.3.4, we may complete $A(G)$ to a positive semi-definite 4×4 matrix A and we denote the double (the sum) of the inserted entries by e.
Thus, there is some sum of squares $\sigma = \sum_{i=1}^{l} g_i^2$ where the g_i are linear combinations from elements of V such that A is the corresponding Gram matrix. We have $f = \sigma - ex^3z^2$. This shows the claim since $x^3z^2 \in \operatorname{csupp}(F \cap \Sigma\,\mathbb{R}[[x,y,z]]^2)$ and thus (by 2.2.8(c))

$$\begin{aligned} f_0 = f + \epsilon\sigma_0 &= \sigma + (\epsilon\sigma_0 - ex^3z^2) \in \Sigma\,\mathbb{R}[[x,y,z]]^2 + \operatorname{rai}(F \cap \Sigma\,\mathbb{R}[[x,y,z]]^2) \\ &\subseteq \Sigma\,\mathbb{R}[[x,y,z]]^2. \end{aligned}$$

(c) Let $f_0 \in \operatorname{rai}(F)$, $\sigma_0 \in \operatorname{rai}(\Sigma\,\mathbb{R}[[x,y,z]]^2 \cap F)$ and $\epsilon > 0$ with $f := f_0 - \epsilon\sigma_0 \in \operatorname{rai}(F)$. We have seen in the proof of 4.2.15(c) that f is of the form

$$\begin{aligned} f =& c_1(x^3-y^2)^2 + c_2(x^3-y^2)x^2y + b_1(x^3-y^2)xz + a_1x^2z^2 + a_4xyz^2 + a_5y^3z \\ &+ a_6y^2z^2 + a_7xz^3 + a_8yz^3 + a_9z^4 + b_3x^3yz + c_3x^4y^2 + i(x,y,z) \end{aligned}$$

with real coefficients and $i(x,y,z) \in \operatorname{csupp}(F)$. As in (b), we may assume $i(x,y,z) = 0$ using 4.2.15(c). We define a graph $G = (V, E)$ by

$$V = \{z^2, yz, xz, x^3 - y^2, x^2y\}, \quad E = \{\{g_1, g_2\} \subseteq V\} \setminus \{\{z^2, x^2y\}, \{yz, x^2y\}\}.$$

and we have the following G-partial matrix

$$A(G) := \begin{pmatrix} a_9 & \frac{a_8}{2} & \frac{a_7}{2} & -\frac{e_0}{2} & \\ \frac{a_8}{2} & a_6 - e_0 & \frac{a_4}{2} & -\frac{a_5}{2} & \\ \frac{a_7}{2} & \frac{a_4}{2} & a_1 & \frac{b_1}{2} & \frac{a_5+b_3}{2} \\ -\frac{e_0}{2} & -\frac{a_5}{2} & \frac{b_1}{2} & c_1 & \frac{c_2}{2} \\ & & \frac{a_5+b_3}{2} & \frac{c_2}{2} & c_3 \end{pmatrix}$$

for any $e_0 \in \mathbb{R}$. The homogeneous part of degree 4 of f is non-negative (see 3.3.10(a)) and thus a sum of squares by Hilbert's theorem 1.1.4. Every corresponding Gram matrix equals the matrix B that arises if one eliminates the last row and column of $A(G)$ and chooses an appropriate e_0. We fix some $e_0 \in \mathbb{R}$ such that B is positive semi-definite. We set $x = (\sqrt[3]{1+rt})t^2$, $y = t^3$ and $z = t^5 s$ with $r, s, t \in \mathbb{R}$ such that f is of the form

$$(c_1r^2 + c_2r + b_1rs + a_1s^2 + a_5s + b_3s + c_3)t^{14} + O(t^{15}).$$

Since $f \in \mathcal{P}^{Loc}$, the behaviour for $t \to 0$ shows that $c_1r^2+c_2r+b_1rs+a_1s^2+a_5s+b_3s+c_3$ is a non-negative polynomial in r, s. Thus, the submatrix of $A(G)$ that arises by eliminating the first two rows and columns is a positive semi-definite matrix. Hence, $A(G)$ is a G-partial non-negative matrix. It is easy to see that G is chordal (see 4.3.18(c)). By 4.3.4, we may complete $A(G)$ to a positive semi-definite matrix A. We denote the double value at the previous empty entries $(1,5)$ and $(2,5)$ with e_1 and e_2.
Now, we find some sum of squares $\sigma = \sum_{i=1}^{l} g_i^2$ such that g_i are linear combinations of the polynomials in V and A is the corresponding Gram matrix. We have $f = \sigma + e_0x^3z^2 - e_1x^2yz^2 - e_2x^2y^2z$. This shows the claim since $x^3z^2, x^2yz^2, x^2y^2z \in \operatorname{csupp}(F) = \operatorname{csupp}(F \cap \Sigma\,\mathbb{R}[[x,y,z]]^2)$ (by 4.2.15(c)) and thus (see 2.2.8(c))

$$\begin{aligned} f_0 = f + \epsilon\sigma_0 &= \sigma + (\epsilon\sigma_0 + e_0x^3z^2 - e_1x^2yz^2 - e_2x^2y^2z) \\ &\in \Sigma\,\mathbb{R}[[x,y,z]]^2 + \operatorname{rai}(F \cap \Sigma\,\mathbb{R}[[x,y,z]]^2) \subseteq \Sigma\,\mathbb{R}[[x,y,z]]^2. \end{aligned}$$

□

Corollary 4.3.17. *All faces F of $\mathbb{R}[[x,y,z]]_+$ belonging to one of the first 7 layers (classified in 3.6.3 up to right equivalence) have the property* $\operatorname{rai}(F) \subseteq \Sigma\,\mathbb{R}[[x,y,z]]^2$.

Proof. If F is a face of corank at most 2, then the claim follows from 4.1.2(c) and 4.1.8. Hence, we may reduce to show the claim for the faces representing the classes of corank 3 in 3.6.3 (see also page 92). In the case of the faces B_4', C_4', C_6', C_8', D_4', D_6' and Q, this is implied by 4.3.15(a). The faces B_6', B_8' and P have this property by 4.3.15(b) and G' by 4.3.15(c). Finally, the statement for the faces O, N and E_{14}' follows from 4.3.16(a),(b),(c). □

Remark 4.3.18. (a) We consider faces F as in 4.3.15(a),(b), so monomial faces F of finite codimension with $\operatorname{ord}(F) = 4$ such that (in the notation of 4.2.14) $\tau(F)$ has either up to three double zeros or one zero of type $(4:2)$ and possibly one further double zero.

We have seen that $\operatorname{rai}(F) \subseteq \Sigma\,\mathbb{R}[[x,y,z]]^2$. The condition that F is monomial is not necessary. Indeed, one can show using 3.6.1 that each such face is equivalent to such a monomial face by some automorphism of $\mathbb{R}[[x,y,z]]$ (see part in parantheses in the proof of 3.6.3).

(b) The graph (without loops) in the proof of 4.3.16(c) is well-known and it is called "Haus vom Nikolaus" (The house of the Saint Nicolas) in German. It is a common exercise for children to find an Eulerian path in this graph. In the future, we suggest to add the question if this graph is chordal as it was used in the proof.

Remark 4.3.19. It is possible to formulate versions of Scheiderer's local-global principle such that one can also deal with zeros on the boundary of the considered semi-algebraic set (at least at sufficiently regular points of the boundary). Then, again, one is interested in elements that lie in the corresponding quadratic module $\hat{M}_P$ in the completion $\hat{\mathcal{O}}_{V,P}$. In this more general setting, $\hat{M}_P$ will be different from the cone of sums of squares in $\hat{\mathcal{O}}_{V,P}$. However, in [Mar, 2.2.], Murray Marshall observed that elements satisfying some boundary Hessian condition are contained in $\hat{M}_P$. As this condition is an open condition, it is rather easy to see that these elements are exactly the elements in the relative algebraic interior of the $\subseteq$-maximal proper face of $\hat{M}_P$. This gives some positive answer to problem 4.1.3 in this more general setting. It may be applied if elements with boundary Hessian condition exist, which means that the boundary conditions intersect transversely.

Concluding remarks and perspective

In this work, we have studied local non-negativity in the setting of polynomials, convergent and formal power series. Of course, there are also other interesting settings. The case of the local ring $\mathbb{R}[\underline{x}]_{\mathfrak{m}}$ with $\mathfrak{m} = (x_1, \ldots, x_n)$ will behave very similar to the polynomial case. For other people, rings given by the set of continuous or smooth functions may be of interest. These rings behave differently as they are not Noetherian and one can easily construct examples of infinite increasing chains of faces. However, faces of finite codimension should look the same in the case of smooth functions. The case of continuous functions will be very different as one may not expand them into a Taylor series.
The classification of the faces of this work is very similar to the classification of minimal points in [Vas]. In fact, he classifies enough singularities such that for each class of our classification in the case $n = 2$ (see p.86) there is some face in this class that contains one of his classified singularities in the relative algebraic interior. Thus, our classification is just a small improvement as we show the preorder structure between the classes. However, in his classification, he obtains different classes $x_1^4 + B_0 x_1^2 x_2^2 + x_2^4$ ($B_0 > -2$, $B_0 \neq 2$) and $(x_1^2 + x_2^2)^2 + B_0 x_2^{4+r}$ ($B_0 \neq 0$) in [Vas, p.168], which clearly both represent elements in the relative algebraic interior of $(4:4)$. In contrast to the case $n = 2$, there is just one type of singularities of corank 3 in [Vas] such that we are more detailed.
There is another possibility to study singularities over $\mathbb{C}$ via topological equivalence, which is related to so-called Enriques diagrams in the case $n = 2$. Especially, the process of blowing up plays an important role there. This theory may be transferred to our setting of faces of cones of locally non-negative polynomials (for $n = 2$) and there are promising results as unique factorization of faces, that each face of finite codimension belongs to only one layer and easy descriptions of the data as codimensions, a classification of faces and so on. However, this is work for the future. A quick overview of these results and the results of the thesis was already published in [HKSV][pp. 679-680].

Zusammenfassung auf Deutsch

Wir untersuchen den konvexen Kegel $\mathcal{P}^{Loc}$ der Polynome, die nichtnegativ in einer Umgebung des Ursprung von $\mathbb{R}^n$ sind. Die Motivation für das Studium dieser Kegel stammt aus der Charakterisierung von Seiten global nichtnegativer Polynome mittels Seiten von Kegeln lokal nichtnegativer Polynome bezüglich eines Punktes. Desweiteren sind die Elemente $f \in \mathcal{P}^{Loc}$ mit $f(0) = 0$ gerade die Polynome, die in $0 \in \mathbb{R}^n$ ein lokales Minimum haben und dort verschwinden. Außerdem hat auch die Zugehörigkeit zweier Elemente zum relativen algebraischen Inneren einer Seite eine geometrische Interpretation. Es bedeutet im Wesentlichen, dass sich diese Polynome gleichartig im Grenzwert gegen 0 verhalten.
Zu Beginn dieser Arbeit stellen wir die Beziehung zwischen Kegeln solcher lokal nichtnegativer Polynome und global nichtnegativen Polynomen dar und beantworten diese Frage nun auch bezüglich Nichtnegativität auf einer Menge. Wir führen grundlegende Dinge aus der Konvexitätstheorie ein und betrachten Klassifikationen von Seiten von Kegeln global nichtnegativer Formen in den Fällen, wo solche bekannt sind.
Im zweiten Kapitel gehen wir auf die Konvexitätstheorie ein, die notwendig ist. Da wir in unendlich-dimensionalen Vektorräumen arbeiten, ersetzen wir die Topologie durch ähnliche algebraische Konzepte und studieren, wie sich Seiten von Kegeln bezüglich dieser verhalten. Wir betrachten die Seitenstruktur dieser Kegel abstrakt und präzisieren, was wir unter Klassifikationen von Seiten verstehen. Desweiteren betrachten wir aufsteigende Ketten von Seiten und Kegel, die abgeschlossen unter einer Multiplikation sind, in einem recht allgemeinen Setting. Setzt man voraus, dass die Kegel auch archimedisch sind und der umgebende Ring noethersch, so haben diese Kegel schon viele interessante Eigenschaften. Wir wenden diese Resultate dann auf Kegel lokal nichtnegativer Polynome und global nichtnegativer Polynome an.
Im dritten Kapitel führen wir zunächst ähnliche Kegel wie den Kegel lokal nichtnegativer Polynome im Setting von konvergenten und formalen Potenzreihen ein und können wieder die allgemeineren Resultate von vorher übertragen. Auch hier arbeiten wir in einem allgemeinen Setting bezüglich der Nichtnegativität auf einer Menge und setzen die drei Kegel zueinander in Beziehung. Diese wird insbesondere bei Seiten endlicher Kodimension deutlich, die wir danach studieren. Um weitere Resultate zu erhalten, beschränken wir uns jedoch auf den Fall von Nichtnegativität einer kompletten Umgebung des Urprungs. Wir übertragen Resultate von Vassiliev und können sogenannte monomiale Seiten und deren Seiten untersuchen. Dadurch erhalten wir viele Beispiele für Seiten der betrachteten Kegel. Darauffolgend betrachten wir die Seitenstruktur modulo Automorphismen der Ringe, wobei wir uns im polynomiellen Fall auf solche beschränken, deren zugehöriger Morphismus auf dem $\mathbb{R}^n$ den Nullpunkt fixiert. Wir übertragen Definitionen aus der Singularitätentheorie und erhalten als wichtiges Resultat insbesondere ein Splitting Lemma für Seiten der Kegel im Setting konvergenter und formaler Potenzreihen. Wir wenden die Betrachtungen aus dem Kapitel auf die Fälle $n = 2$ und $n = 3$ an. Im Fall $n = 2$ haben die monomialen Seiten eine sehr schöne multiplikative Zerlegung. Desweiteren können wir die ersten acht Schichten des Kegeln modulo Automorphismen im Setting von Potenzreihen klassifizieren. Im Fall $n = 3$ ist dies bis zur siebenten Schicht möglich.
Im vierten Kapitel wenden wir uns Quadratsummen im Ring formaler Potenzreihen zu. Wir behandeln eine Frage, die vom Lokal-Global-Prinzip von Scheiderer motiviert ist, betrachten jedoch diese bezüglich einer Seite des Kegel und nicht bezüglich einzelner Elemente. Durch das zuvor nachgewiesene Splitting Lemma und einem induktiven Argument können wir die

Frage auf recht wenige Fälle zurückführen. Wir erhalten außerdem eine positive Antwort in den Fällen von Seiten, die wir zuvor im Fall $n = 3$ klassifizieren konnten. Desweiteren geben wir weitere positive und negative Antworten.
Insgesamt behandelt die Arbeit verschiedene allgemeine Aspekte der beschriebenen Kegel. Die wichtigsten Ergebnisse sind mit Sicherheit die abstrakten Betrachtungen der archimedischen, noetherschen, multiplikativen Kegel, die Übertragung des Splitting Lemmas, die Klassifikationen in den Fällen $n = 2$ und $n = 3$ und letztlich die Anwendung auf die Frage zu den Quadratsummen. Desweiteren sind auch die monomialen Seiten und deren Seiten sowie die dazugehörigen Konstruktionen zu erwähnen, da sie die Theorie fassbarer machen und zu vielen Beispielen führen.

List of notations

$\mathbb{N}$, $\mathbb{N}_0$	$\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$	p.4
$\underline{X}$	tuple of (homogeneous) variables $(X_0, \dots, X_n)$	p.4
$\mathbb{R}[\underline{X}]_d$	forms of degree d in $\mathbb{R}[\underline{X}]$	p.4
$\mathcal{P}_{n,2d}$, $\mathcal{P}_{2d}$	forms in $\mathbb{R}[\underline{X}]_{2d}$ non-negative on $S \subseteq \mathbb{P}^n(\mathbb{R})$	1.1.1
$\Sigma_{n,2d}$, Σ_{2d}	sums of squares in $\mathbb{R}[\underline{X}]_{2d}$	1.1.1
$[x, y], [x, y[,]x, y],]x, y[$	line segment between x and y with/without endpoints	1.2.1
aff(.), span(.)	affine hull and linear span	1.2.1
dim(.), codim$_.$(.)	dimension and codimension of affine sets	1.2.1
$F \trianglelefteq C$	F is a face of the convex set or cone C	1.2.4
dim(F), codim(F)	dimension and codimension of a face F	1.2.4
conv(.), cone(.)	convex hull and conic hull	1.2.13
carr$_C(K)$, carr$_C(x)$	carrier face of a set K or a point x in C	1.2.13
L_C	lineality space of C	1.2.13
ri(.), rbd(.)	relative interior and relative boundary	1.2.17
$\mathcal{P}^{Loc}_{n,2d}(P)$, $\mathcal{P}^{Loc}_{2d}(P)$	forms in $\mathbb{R}[\underline{X}]_{2d}$ locally non-negative on $S \subseteq \mathbb{P}^n(\mathbb{R})$ at P	1.3.1
Sym$(n, \mathbb{R})$	symmetric matrices of size $n \times n$ over $\mathbb{R}$	1.5.1
GL$(n, \mathbb{R})$	general linear group of the matrices of size $n \times n$ over $\mathbb{R}$	1.5.1
rk(.), cork(.)	rank and corank of a quadratic form	1.5.2
$\underline{x}$	tuple of variables $(x_1, \dots, x_n)$	p.15
$\mathcal{P}$, $\mathcal{P}^{Loc}$	non-negative and locally non-negative polynomials on $S \subseteq \mathbb{R}^n$	1.7.1
$\mathcal{P}^{Loc}(P)$	polynomials locally non-negative on $S \subseteq \mathbb{R}^n$ at $P \in \mathbb{P}^n(\mathbb{R})$	1.7.4
$+$	Minkowski sum	Sec. 1.8
$\cdot$	set of pairwise product of elements	Sec. 1.8
$\oplus$	direct sum of element	Sec. 1.8
ΣA^2	sum of squares in the ring A	Sec. 1.8
A_+	non-negative elements on the real spectrum	Sec. 1.8
$\mathcal{V}_{\mathbb{R}}(f)$	real vanishing set of some element f	Sec. 1.8
End(.), Aut(.)	the endomorphisms and automorphisms	Sec. 1.8
ai(.), ac(.)	algebraic interior and algebraic closure	2.1.1
rai(.), rabd(.)	relative algebraic interior and relative algebraic boundary	2.1.4
$\mathcal{F} = \mathcal{F}(C)$, $\mathcal{K} = \mathcal{K}(C)$	faces and Klee faces of C	2.2.1, 2.3.4
sgn(.)	sign of a real number	2.2.8(c)
M/G	quotient of a poset M by a group G acting on M	2.3.1
$\boxplus$, 0	sum of faces and its neutral element	2.3.21
supp(.), csupp(.)	support and support of algebraic closure	2.4.1
$\boxdot$	product of faces	2.4.3
$F_2 \,\vert\, F_1$, $\frac{F_1}{F_2}$	divisibility, quotient of faces	2.4.5
$\mathcal{I}(.)$	ideal of polynomials in $\mathbb{R}[\underline{x}]$ vanishing on the set	2.4.21

$\sqrt[re]{I}$	real radical of I	3.1.1
$\mathrm{Sat}(T)$	saturation of the preordering T	3.1.1
$\tilde{P}^{Loc}$	cone of locally non-negative convergent power series on S	3.1.3
$\hat{\mathcal{P}}^{Loc}$	cone of non-negative formal power series	3.1.7
$\Delta(\Gamma)$, $\Delta(f)$	Newton polyhedron corresponding to Newton diagram or formal power series	3.3.1
$\Gamma(\Delta)$, $\Gamma(f)$	Newton diagram corresponding to Newton polyhedron or formal power series	3.3.1
$g(\Gamma)$	canonical polynomial of Newton diagram	3.3.1
$\mathrm{supp}(.)$	support of a power series	3.3.1
f^τ	part of f with support in τ	3.3.1
$\Delta(F)$, $\Gamma(F)$, $\mathrm{ord}(F)$	Newton polyhedron, Newton diagram and order of a face	3.3.7
M^τ	projection on set of power series with support in τ	3.3.15
$\mathcal{P}_\tau$	non-negative polynomials with support in τ	3.3.15
$\tau(G)$	face induced by G on τ	3.3.15
$\mathrm{End}_0(\mathbb{R}[\underline{x}])$, $\mathrm{Aut}_0(\mathbb{R}[\underline{x}])$	endomorphisms and automorphisms φ with $\varphi(x_i)(0) = 0$ for $i = 1, \ldots, n$	3.4.1
φ_1	linear part of an endomorphism	3.4.3
$\mathrm{cork}(.)$, $\overset{r}{\sim}$	corank and right equivalence for elements	3.4.8
$\overset{s}{\sim}$	stable equivalence of elements	3.4.12
$\mathrm{cork}(.)$, $\overset{r}{\sim}$	corank and right equivalence for faces	3.4.15
$\sqrt[2]{M}$	elements with its square in M	3.4.16
$\overset{s}{\sim}$	stable equivalence of faces	3.4.24
$(2k : 2l)$	type of monomial face for $n = 2$	3.5.3
$\mathbb{R}[V]$	coordinate ring of an affine $\mathbb{R}$-variety	4.1.1
$\mathcal{O}_{V,P}$	local ring of a variety V at P	4.1.1
Σ_τ	sums of squares with support in τ	4.1.4
$\Delta_E(\Gamma)$	set of $\alpha \in \mathbb{R}^n$ with $\alpha \geq \beta$ for some $\beta \in \Gamma \cap 2\mathbb{Z}^n$	4.2.4
$\mathrm{bconv}(.)$, $\mathrm{rbconv}(.)$	bisectional convex hull and relaxed bisectional convex hull	4.2.4
$G(\Gamma)$	graph corresponding to Γ	4.3.1
$\mathcal{D}(G)$	indices defined by the graph G	4.3.3
I_n	identity matrix of size $n \times n$	4.3.8(b)
$\mathbb{R}[\underline{x}, \underline{x}^{-1}]$	ring of Laurent polynomials	4.3.10(b)
$\mathrm{supp}(f)$	support of Laurent polynomial	4.3.10(b)
$\mathbb{R}[\underline{x}, \underline{x}^{-1}]^\tau$	Laurent polynomial with support in τ	4.3.10(b)
$\mathbb{R}^*$	$\mathbb{R} \setminus \{0\}$	4.3.10(b)

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