AN INVERSE GAUSS CURVATURE FLOW FOR HYPERSURFACES EXPANDING IN A CONE

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Abstract. We consider hypersurfaces which are graphs over a sphere evolving in a cone, driven by the $(-1/n)$-th power of the Gauß curvature and subject to a Neumann boundary condition. We show existence for all times and convergence after rescaling, to a subset of a sphere.

1. Introduction

In this paper we study the parabolic initial value problem describing the evolution of a strictly convex hypersurface evolving inside a solid convex cone in $\mathbb{R}^{n+1}$, that is perpendicular to the boundary of this cone at all points of intersection of the two hypersurfaces. As in [4] we only want to consider hypersurfaces, that may be written as a graph over the unit sphere.

Let $\mathcal{C}^{n+1} \subset \mathbb{R}^{n+1}$ be a $(n+1)$-dimensional closed convex cone, such that $\mathcal{C}^{n+1} \subset \left\{\left(z^1, \ldots, z^{n+1}\right) \in \mathbb{R}^{n+1} \middle| \, z^{n+1} > 0\right\} \cup \{0\} \subset \mathbb{R}^{n+1}$, assume $\mathcal{C}^{n+1} \setminus \{0\}$ to be smooth and define $\Omega := \operatorname{int}\left(\mathcal{C}^{n+1} \cap \mathrm{S}^n\right)$, where S^n is the n-dimensional unit sphere. We use standard notation as explained below and will prove the following statement.

Theorem 1.1. *Let $M_0 \subset \mathcal{C}^{n+1}$ be a strictly convex hypersurface (meaning that all eigenvalues of its second fundamental form are strictly positive), such that $\partial M_0 \subset \partial \mathcal{C}^{n+1} \setminus \{0\}$, $M_0 \perp \partial \mathcal{C}^{n+1}$ and $M_0 = \operatorname{graph}_{\mathrm{S}^n} u_{0|\Omega}$ for a positive map $u_0 : \Omega \to \mathbb{R}$ with $u_0 \in C^4(\bar{\Omega})$. Then there is a family of hypersurfaces $(M_t)_{0 \le t < \infty}$ with*

$$\begin{cases} \dfrac{d}{dt}\tilde{X} = \dfrac{1}{\tilde{K}^{1/n}} \cdot \tilde{\nu}, \\ \partial M_t \subset \partial \mathcal{C}^{n+1} \setminus \{0\}, M_t \perp \partial \mathcal{C}^{n+1} \quad \text{for all } t \in [0, \infty), \\ M_t|_{t=0} = M_0, \end{cases} \tag{1.1}$$

where the unit normal vector $\tilde{\nu}(y,t)$ to M_t satisfies $\left\langle \tilde{\nu}(y,t), \tilde{X}(y,t) \right\rangle > 0$ for all (y,t) and where we write $M_t \perp \partial \mathcal{C}^{n+1}$ $(0 \le t < \infty)$ for $\langle \tilde{\nu}(y,t), \bar{\nu}(p) \rangle = 0$ for all $p = \tilde{X}(y,t) \in \partial M_t$, with $\bar{\nu}$ being the outward pointing unit normal vector to $\partial \mathcal{C}^{n+1}$ at the point p.

Moreover $M_t e^{-t}$ converges to Ω for $t \to \infty$.

Working with coordinates on the sphere, we may equivalently formulate the problem locally by the following initial value problem with a Neumann boundary condition

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(this will be proved in the next section). We suppose furthermore the abstract n-dimensional unit sphere to be embedded into $\mathbb{R}^{n+1}$ via the map

$$\iota : \mathrm{S}^n \to \mathrm{S}^n \subset \mathbb{R}^{n+1},\ x \mapsto (\iota^1(x), \dots, \iota^{n+1}(x)).$$

and we will denote the image of Ω under ι by $\hat{\Omega} := \iota(\Omega)$. Theorem 1.1 becomes equivalent to:

Theorem 1.2. *For a scalar map $\varphi_0 \in C^4(\bar{\Omega})$, $\varphi_0(x) = \ln u_0(x)$, with*

$$\sigma_{ij} - \varphi_{0;ij} + \varphi_{0,i}\varphi_{0,j} > 0$$

up to the boundary $\partial\Omega$, there exist a function $\varphi : \Omega \times [0, \infty) \to \mathbb{R}$,

$$\varphi \in C^{2,\alpha;1,\frac{\alpha}{2}}(\bar{\Omega} \times [0, \infty)) \cap C^\infty(\bar{\Omega} \times (0, \infty)),$$

which solves

$$\begin{cases} \dot{\varphi} = \left(1 + |D\varphi|^2\right)^{\frac{n+1}{n}} \cdot \dfrac{[\det(\sigma_{ij})]^{1/n}}{[\det(\sigma_{ij} - \varphi_{;ij} + \varphi_{,i}\varphi_{,j})]^{1/n}} & \text{in } \Omega \times [0, \infty), \\ D_{\bar{\nu}}\varphi := \bar{\nu}^i \varphi_{,i} = 0 & \text{on } \partial\Omega \times [0, \infty), \\ \varphi(x, 0) = \varphi_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

with $w_{ij} := \sigma_{ij} - \varphi_{;ij} + \varphi_{,i}\varphi_{,j} > 0$, up to the boundary $\partial\Omega$ for all $(x, t) \in \Omega \times [0, \infty)$ and where $\bar{\nu}$ is the outward pointing unit normal vector to $\partial\hat{\Omega}$ relative to $\iota(S^n)$.

Moreover $\varphi(x, t) - t$ converges to a real number r for $t \to \infty$.

We say that a hypersurface M may be represented as a graph over the unit sphere if there exists a (possibly time-dependent) strictly positive map $\bar{u} : \Upsilon \to \mathbb{R}$, for an open subset $\Upsilon \subseteq \mathrm{S}^n$, such that

$$M = \mathrm{graph}_{\mathrm{S}^n}\, \bar{u}_{|\Upsilon} := \left\{ \iota(x) \cdot \bar{u}(x) \in \mathbb{R}^{n+1} |\, x \in \Upsilon \right\}.$$

We consequently define the embedding vector of M into $\mathbb{R}^{n+1}$ by $\bar{X} : \Upsilon \to \mathbb{R}^{n+1}$, $\bar{X}(x) = \iota(x) \cdot \bar{u}(x)$.

Let $M_0 \subset \mathcal{C}^{n+1}$ be such a hypersurface, i.e. $M_0 = \mathrm{graph}_{\mathrm{S}^n}\, u_{0|\Omega}$, for a strictly positive map $u_0 : \Omega \to \mathbb{R}$. Since we work with a normal velocity related to the Gauß curvature, if we make a suitable convexity assumption we obtain the existence of the flow for (at least) small times. We will denote by $X(x, t) = \iota(x) \cdot u(x, t)$ the time-dependent embedding vector of the hypersurface M_0 into the $\mathbb{R}^{n+1}$, defined on a maximal time interval $[0, t^*)$, with $t^* > 0$, and set $M_t := X(\Omega, t)$. We will often identify the embedded manifold with its image without indicating it explicitly.

To denote a time-dependent local parametrization of the abstract unit sphere we will use

$$\begin{aligned} &\eta : \mathbb{R}^n \times [0, \infty) \to \mathrm{S}^n, \\ &\quad (y^1, \dots, y^n, t) \mapsto (\eta^1(y^1, \dots, y^n, t), \dots, \eta^n(y^1, \dots, y^n, t)), \end{aligned}$$

where $y := (y^1, \dots, y^n) \in \mathbb{R}^n$, with inverse at the corresponding time t given by

$$\begin{aligned} &\zeta : \mathrm{S}^n \times [0, \infty) \to \mathbb{R}^n, \\ &\quad (x, t) \mapsto (\zeta^1(x, t), \dots, \zeta^n(x, t)). \end{aligned}$$

It holds therefore $\eta(\zeta(x, t), t) = x$ as well as $\zeta(\eta(y, t), t) = y$. It is here implicit that η and ζ are (possibly) defined on subsets of $\mathbb{R}^n$ respectively S^n. The corresponding embedding vector in coordinates $\tilde{X} : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^{n+1}$ of M_t will be described by

$$\tilde{X}(y,t) := X(\eta(y,t),t) = u(\eta(y,t),t) \cdot \iota(\eta(y,t)).$$

Throughout this paper K will represent the induced Gauß curvature of M_t, while ν the outward pointing normal unit vector field on it. The corresponding quantities expressed in the local parametrization of the unit sphere will be $\tilde{K}$ and $\tilde{\nu}$. Indices preceded by a semicolon will indicate covariant derivatives with respect to σ, the standard metric on the sphere, induced by the inclusion into the $\mathbb{R}^{n+1}$; those preceded by a comma, partial derivatives. Furthermore we use the Einstein summation convention, summing over indices which appear twice, as a lower and an upper index. Normal latin indices and latin indices in the Fraktur hand range from 1 to n and refer to geometric quantities on the sphere respectively in $\mathbb{R}^n$, Greek indices range from 1 to $n+1$ and refer to components in the ambient space $\mathbb{R}^{n+1}$, which is endowed with the Euclidean scalar product $\langle \cdot, \cdot \rangle$. Finally we use $c, c_1, c_2, \dots$ to denote estimated constants, which may change their value from line to line.

The existence for small times follows from the general parabolic partial differential equations theory. The long time existence will follow from this together with $\dot{\varphi}$, C^0, C^1 and C^2 bounds and application of Krylov-Safonov estimates and Schauder theory. Convergence to a translating solution can then be derived, as in [11], from the a priori estimates. Except for the C^1 case we are going to work directly on the sphere, without an explicit choice of coordinates. The C^0-estimates and time derivative estimates employ straightforward maximum principle methods. These will also play a significant role in the C^2-estimates. To obtain C^1-estimates we apply the Ice-cream cone theorem of [12], which generalizes a result of [8]. For the C^2-estimates we use ideas from [8], without always mentioning this explicitly in that section.

The convergence, after rescaling, to round spheres for strictly convex initial surfaces moving by the Gauß curvature flow in $\mathbb{R}^3$ is due to Andrews [1]. Expanding surfaces without boundary were studied for instance in the works of Gerhardt [4] and Urbas [13, 14], mean curvature flow respectively Gauß curvature flow with boundary conditions in those of Huisken [6] and of Schnürer and Schwetlick [12]. The elliptic Neumann boundary problem for this kind of equation, of Monge-Ampère type, has been explored by Lions, Trudinger and Urbas in [8]. Marquardt proved a result similar to the one discussed in this paper for star-shaped hypersurfaces evolving in a convex cone under inverse mean curvature flow [9].

In the next section we derive the formulation of the problem as stated in Theorem (1.2). The following chapters are devoted to the respective estimates.

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2. The corresponding initial value problem

We now describe the evolution problem in local coordinates and compute some geometric quantities induced by the embedding of the hypersurface into $\mathbb{R}^{n+1}$.

It is possible to find most of the results of this section for instance in [4] or in [9]. We nevertheless summarize them here, with all calculations.

Lemma 2.1. *Let $\Omega \subseteq \mathrm{S}^n$, $t \geq 0$ a fixed time, and $u(\cdot, t) : \Omega \to \mathbb{R}$ be positive and smooth. Then $\operatorname{graph}_{\mathrm{S}^n} u(\cdot, t)_{|\Omega}$ is a n-dimensional submanifold in $\mathbb{R}^{n+1}$. The metric g_{ij}, the outward unit normal vector ν, the second fundamental form h_{ij}, and the Gauß curvature K are given in graph coordinates by*

$$\begin{aligned} g_{ij} &= u^2\sigma_{ij} + u_{,i}u_{,j}, \\ g^{ij} &= u^{-2}\left(\sigma^{ij} - \frac{u^i u^j}{u^2 + |Du|^2}\right), \\ \nu &= \frac{1}{\sqrt{u^2 + |Du|^2}}\left(u\iota - \sigma^{lk}u_{,l}\iota_{,k}\right), \\ h_{ij} &= \frac{1}{\sqrt{u^2 + |Du|^2}}\left(u^2\sigma_{ij} - uu_{;ij} + 2u_{,i}u_{,j}\right), \\ K &= \frac{1}{(u^2 + |Du|^2)^{n/2}} \cdot \frac{\det\left(u^2\sigma_{ij} - uu_{;ij} + 2u_{,i}u_{,j}\right)}{\det\left(u^2\sigma_{ij} + u_{,i}u_{,j}\right)}, \end{aligned}$$

where we lifted the indices with respect to the metric of the sphere, i. e. $u^i := \sigma^{il}u_{,l}$, and write $|Du|^2 := \sigma^{ij}u_{,i}u_{,j}$.

Proof. $\operatorname{graph}_{\mathrm{S}^n} u(\cdot, t)_{|\Omega}$ is an embedded n-dimensional submanifold of the $\mathbb{R}^{n+1}$, since it corresponds to the 0-level set of the map, expressed in spherical coordinates,

$$\Omega \times \mathbb{R} \to \mathbb{R},\ (x, r) \mapsto r - u(x, t),$$

whose differential $-u_{,i}dx^i + dr$ is surjective for every point of $\mathbb{R}^{n+1}$.

We then check the expressions for the geometric quantities.

(i) We use the embedding vector $X(\cdot, t) = \iota(\cdot) \cdot u(\cdot, t)$, $X(\cdot, t) : \Omega \to \mathbb{R}^{n+1}$. The induced metric is the pull-back of the metric in the Euclidean $\mathbb{R}^{n+1}$, $g := X^*\delta_{\mathbb{R}^{n+1}}$. We have $X_{,i} = u_{,i}\iota + u\iota_{,i}$. Hence

$$\begin{aligned} g_{ij} &= X^\alpha_{,i}\delta_{\alpha\beta}X^\beta_{,j} = (u_{,i}\iota^\alpha + u\iota^\alpha_{,i})\delta_{\alpha\beta}(u_{,j}\iota^\beta + u\iota^\beta_{,j}) \\ &= u_{,i}u_{,j}|\iota|^2 + 2\iota^\alpha_{,i}\delta_{\alpha\beta}\iota^\beta + u^2\iota^\alpha_{,i}\delta_{\alpha\beta}\iota^\beta_{,j} = u_{,i}u_{,j} + u^2\sigma_{ij}, \end{aligned}$$

because of $\iota^\alpha_{,i}\delta_{\alpha\beta}\iota^\beta = 0$ and $\sigma_{ij} = \iota^\alpha_{,i}\delta_{\alpha\beta}\iota^\beta_{,j}$.

(ii) We check, that g^{ij} is the inverse of g_{ij}:

$$\begin{aligned} g_{ij}g^{jk} &= \left(u^2\sigma_{ij} + u_{,i}u_{,j}\right)u^{-2}\left(\sigma^{jk} - \frac{u^j u^k}{u^2 + |Du|^2}\right) \\ &= \delta^k_i + u^{-2}\sigma^{jk}u_{,i}u_{,j} - \frac{\sigma_{ij}u^j u^k}{u^2 + |Du|^2} - u^{-2}|Du|^2 \cdot \frac{u_{,i}u^k}{u^2 + |Du|^2} \\ &= \delta^k_i + \frac{u_{,i}u^k}{u^2 + |Du|^2}\left(u^{-2}(u^2 + |Du|^2) - 1 - u^{-2}|Du|^2\right) = \delta^k_i. \end{aligned}$$

(iii) The vectors $X_{,i} = u_{,i}\iota + u\iota_{,i}$ are tangent to $\operatorname{graph}_{\mathrm{S}^n} u$. We therefore look for a normal vector of the form $a\iota + b^k\iota_{,k}$, with a and (b^k) to be appropriately determined. The inner product

$$\begin{aligned}\left\langle u_{,i}\iota + u\iota_{,i}, a\iota + b^k\iota_{,k}\right\rangle &= au_{,i}\left\langle \iota,\iota\right\rangle + u_{,i}b^k\left\langle \iota,\iota_{,k}\right\rangle \\ &\quad + au\left\langle \iota_{,i},\iota\right\rangle + ub^k\left\langle \iota_{,i},\iota_{,k}\right\rangle \\ &= au_{,i} + ub^k\sigma_{ik}\end{aligned}$$

should vanish. We can reach this by taking $a = u$ and $b^k = -u_{,l}\sigma^{lk}$ and obtaining a normal vector, which we normalize by setting

$$\nu = \frac{u\iota - \sigma^{lk}u_{,l}\iota_{,k}}{\sqrt{u^2+|Du|^2}}.$$

because of

$$\begin{aligned}&\left\langle u\iota - \sigma^{lk}u_{,l}\iota_{,k}\,,\, u\iota - \sigma^{pq}u_{,p}\iota_{,q}\right\rangle \\ &\qquad = u^2\left\langle \iota,\iota\right\rangle - u\sigma^{lk}u_{,l}\left\langle \iota_{,k},\iota\right\rangle \\ &\qquad\quad - u\sigma^{pq}u_{,p}\left\langle \iota,\iota_{,q}\right\rangle + \sigma^{lk}u_{,l}\sigma^{pq}u_{,p}\left\langle \iota_{,k},\iota_{,q}\right\rangle \\ &\qquad = u^2 + \sigma^{lk}\sigma^{pq}\sigma_{kq}u_{,l}u_{,p} = u^2 + |Du|^2.\end{aligned}$$

This is in fact an outward pointing vector:

$$\begin{aligned}\left\langle \nu, X\right\rangle &= \frac{\left\langle u\iota - \sigma^{lk}u_{,l}\iota_{,k}, u\iota\right\rangle}{\sqrt{u^2+|Du|^2}} \\ &= \frac{u^2\left\langle \iota,\iota\right\rangle - u\sigma^{lk}u_{,l}\left\langle \iota_{,k},\iota\right\rangle}{\sqrt{u^2+|Du|^2}} = \frac{u^2}{\sqrt{u^2+|Du|^2}} > 0.\end{aligned}$$

(iv) We use the Gauß formula for hypersurfaces $X^\alpha_{;ij} = -h_{ij}\nu^\alpha$ and compute the scalar product with ν to get

$$\begin{aligned}h_{ij} &= -\left\langle X_{;ij},\nu\right\rangle \\ &= -\left\langle u_{;ij}\iota + u_{,i}\iota_{,j} + u_{,j}\iota_{,i} + u\iota_{;ij}, \frac{u\iota - \sigma^{lk}u_{,l}\iota_{,k}}{\sqrt{u^2+|Du|^2}}\right\rangle \\ &= \frac{-1}{\sqrt{u^2+|Du|^2}}\left(uu_{;ij} - 2u_{,i}u_{,j} + u^2\left\langle \iota_{;ij},\iota\right\rangle - u\sigma^{lk}u_{,l}\left\langle \iota_{;ij},\iota_{,k}\right\rangle\right) \\ &= \frac{1}{\sqrt{u^2+|Du|^2}}\left(-uu_{;ij} + 2u_{,i}u_{,j} + u^2\sigma_{ij}\right),\end{aligned}$$

since for a sphere we have $\iota_{;ij} = -\sigma_{ij}\cdot\iota$, as a direct consequence of the Gauß formula.

(v) From the defining equation for the principal curvatures, we obtain

$$\begin{aligned}K &= \prod_{i=1}^n \lambda_i = \det\left(g^{ij}h_{jk}\right) = \det g^{ij}\cdot\det h_{ij} = \frac{\det h_{ij}}{\det g_{ij}} \\ &= \frac{1}{(u^2+|Du|^2)^{n/2}}\cdot\frac{\det\left(u^2\sigma_{ij} - uu_{;ij} + 2u_{,i}u_{,j}\right)}{\det\left(u^2\sigma_{ij} + u_{,i}u_{,j}\right)}.\end{aligned}$$

□

Remark 2.2. We are not going to compute the corresponding formulas in the coordinates induced by the local parametrization η of the sphere. However, since we will use this fact, let us only mention that one would get that $\nu(x,t)$ and $\tilde{\nu}(y,t)$ have the same form.

As u is assumed to be strictly positive, it is possible to slightly simplify the calculations by setting $\varphi = \log u$. We obtain in terms of φ:

$$\begin{aligned} u_{,i} &= e^{\varphi}\varphi_{,i}, \\ u_{;ij} &= e^{\varphi}(\varphi_{;ij} + \varphi_{,i}\varphi_{,j}), \end{aligned}$$

and for the normal vector

$$\nu = \frac{1}{\sqrt{1+|D\varphi|^2}}\left(\iota - \sigma^{lk}\varphi_{,l}\iota_{,k}\right) \tag{2.1}$$

as well as for the Gauß curvature

$$\begin{aligned} K &= \frac{1}{(e^{2\varphi} + e^{2\varphi}|D\varphi|^2)^{n/2}} \cdot \frac{\det\left(e^{2\varphi}\sigma_{ij} - e^{2\varphi}(\varphi_{;ij} + \varphi_{,i}\varphi_{,j}) + 2e^{2\varphi}\varphi_{,i}\varphi_{,j}\right)}{\det(e^{2\varphi}\sigma_{ij} + e^{2\varphi}\varphi_{,i}\varphi_{,j})} \\ &= \frac{e^{-n\varphi}}{(1+|D\varphi|^2)^{n/2}} \cdot \frac{\det\left(\sigma_{ij} - \varphi_{;ij} + \varphi_{,i}\varphi_{,j}\right)}{\det(\sigma_{ij}) \cdot (1+|D\varphi|^2)} \\ &= \frac{e^{-n\varphi}}{(1+|D\varphi|^2)^{(n+2)/2}} \cdot \frac{\det\left(\sigma_{ij} - \varphi_{;ij} + \varphi_{,i}\varphi_{,j}\right)}{\det(\sigma_{ij})}. \end{aligned} \tag{2.2}$$

Let us now compute the evolution equation in local coordinates.

Lemma 2.3. *Let $\Omega \subseteq \mathrm{S}^n$ and $\varphi : \Omega \times [0,\infty) \to \mathbb{R}$ be a smooth function such that $\operatorname{graph}_{\mathrm{S}^n} e^{\varphi}{}_{|\Omega}$ evolves according to $\frac{d}{dt}\tilde{X} = \frac{1}{\tilde{K}^{1/n}} \cdot \tilde{\nu}$. Then*

$$\dot{\varphi} = \sqrt{1+|D\varphi|^2} \cdot e^{-\varphi} \cdot \frac{1}{\tilde{K}^{1/n}}.$$

Proof. The vectors $\tilde{X}_{,1}, \ldots, \tilde{X}_{,n}, \tilde{\nu}$ are orthogonal to each other and hence form a basis of the $\mathbb{R}^{n+1}$. This implies that

$$\frac{d}{dt}\tilde{X} = \frac{1}{\tilde{K}^{1/n}} \cdot \tilde{\nu}$$

is equivalent to the following condition:

$$\left\langle \tilde{X}_{,\mathfrak{a}}, \frac{d}{dt}\tilde{X} - \frac{1}{\tilde{K}^{1/n}} \cdot \tilde{\nu} \right\rangle = 0 \tag{2.3}$$

for all $\mathfrak{a} = 1, \ldots, n$, and

$$\left\langle \tilde{\nu}, \frac{d}{dt}\tilde{X} - \frac{1}{\tilde{K}^{1/n}} \cdot \tilde{\nu} \right\rangle = 0. \tag{2.4}$$

From (2.4) we obtain

$$0 = \left\langle \tilde{\nu}, \frac{d}{dt}\tilde{X} - \frac{1}{\tilde{K}^{1/n}} \cdot \tilde{\nu} \right\rangle = \left\langle \tilde{\nu}, \frac{d}{dt}\tilde{X} \right\rangle - \frac{1}{\tilde{K}^{1/n}}.$$

Consequently inserting the expression (2.1) of the normal vector in this last identity, provides

$$\begin{aligned}
\frac{1}{\tilde{K}^{1/n}} &= \left\langle \tilde{\nu}, \frac{d}{dt}\tilde{X} \right\rangle = \left\langle \frac{\iota - \sigma^{lk}\varphi_{,l}\iota_{,k}}{\sqrt{1+|D\varphi|^2}}, \iota_{,i} \cdot \frac{d\eta^i}{dt} \cdot e^{\varphi} + \iota \cdot (e^{\varphi})_{,j} \cdot \frac{d\eta^j}{dt} + \iota \cdot \frac{de^{\varphi}}{dt} \right\rangle \\
&= \frac{1}{\sqrt{1+|D\varphi|^2}} \cdot \left(\left\langle \iota, \iota e^{\varphi}\varphi_{,j} \cdot \frac{d\eta^j}{dt} \right\rangle + \left\langle \iota, \iota e^{\varphi} \cdot \frac{d\varphi}{dt} \right\rangle \right. \\
&\qquad \left. - \left\langle \sigma^{lk}\varphi_{,l}\iota_{,k}, \iota_{,i} \cdot \frac{d\eta^i}{dt} \cdot e^{\varphi} \right\rangle \right) \\
&= \frac{1}{\sqrt{1+|D\varphi|^2}} \cdot \left(e^{\varphi}\varphi_{,j} \cdot \frac{d\eta^j}{dt} + e^{\varphi} \cdot \frac{d\varphi}{dt} - e^{\varphi}\sigma^{lk}\sigma_{ki}\varphi_{,l} \cdot \frac{d\eta^i}{dt} \right) \\
&= \frac{1}{\sqrt{1+|D\varphi|^2}} \cdot \left(e^{\varphi}\varphi_{,j} \cdot \frac{d\eta^j}{dt} + e^{\varphi} \cdot \frac{d\varphi}{dt} - e^{\varphi}\varphi_{,i} \cdot \frac{d\eta^i}{dt} \right) \\
&= \frac{1}{\sqrt{1+|D\varphi|^2}} \cdot \left(e^{\varphi} \cdot \frac{d\varphi}{dt} \right),
\end{aligned}$$

which is the expected equation. Whereas from (2.3) we obtain

$$\begin{aligned}
0 &= \left\langle \tilde{X}_{,\mathfrak{a}}, \frac{d}{dt}\tilde{X} - \frac{1}{\tilde{K}^{1/n}} \cdot \tilde{\nu} \right\rangle = \left\langle \tilde{X}_{,\mathfrak{a}}, \frac{d}{dt}\tilde{X} \right\rangle \\
&= \left\langle \iota_{,i}\eta^i_{,\mathfrak{a}}e^{\varphi} + \iota \cdot (e^{\varphi})_{,j}\eta^j_{,\mathfrak{a}}, \iota_{,k} \cdot \frac{d\eta^k}{dt} \cdot e^{\varphi} + \iota \cdot (e^{\varphi})_{,l} \cdot \frac{d\eta^l}{dt} + \iota \cdot \frac{de^{\varphi}}{dt} \right\rangle \\
&= \left\langle \iota_{,i}\eta^i_{,\mathfrak{a}}e^{\varphi}, \iota_{,k} \cdot \frac{d\eta^k}{dt} \cdot e^{\varphi} \right\rangle + \left\langle \iota \cdot (e^{\varphi})_{,j}\eta^j_{,\mathfrak{a}}, \iota \cdot (e^{\varphi})_{,l} \cdot \frac{d\eta^l}{dt} \right\rangle \\
&\quad + \left\langle \iota \cdot (e^{\varphi})_{,j}\eta^j_{,\mathfrak{a}}, \iota \cdot \frac{de^{\varphi}}{dt} \right\rangle \\
&= e^{2\varphi}\sigma_{ik}\eta^i_{,\mathfrak{a}} \cdot \frac{d\eta^k}{dt} + (e^{\varphi})_{,j}(e^{\varphi})_{,l}\eta^j_{,\mathfrak{a}} \cdot \frac{d\eta^l}{dt} + \frac{de^{\varphi}}{dt} \cdot (e^{\varphi})_{,j}\eta^j_{,\mathfrak{a}} \\
&= g_{ik}\eta^i_{,\mathfrak{a}} \cdot \frac{d\eta^k}{dt} + \frac{de^{\varphi}}{dt} \cdot (e^{\varphi})_{,j}\eta^j_{,\mathfrak{a}},
\end{aligned}$$

for all $\mathfrak{a} = 1, \dots, n$; that is an equation for the evolution of η, which we don't need to take care of in the following sections. It namely describes a tangential motion, which only affects the parametrization and not the shape of the solution. □

Combining the results of this last lemma with the formula of the Gauß curvature (2.2), we get a representation of the above problem through the following partial differential equation

$$\begin{aligned}
\dot{\varphi} &= \sqrt{1+|D\varphi|^2} \cdot e^{-\varphi} \cdot \frac{1}{\tilde{K}^{1/n}} \\
&= \sqrt{1+|D\varphi|^2} \cdot e^{-\varphi} \cdot \frac{\left(1+|D\varphi|^2\right)^{(n+2)/2n}}{e^{-\varphi}} \cdot \frac{\det^{1/n}(\sigma_{ij})}{\det^{1/n}\left(\sigma_{ij} - \varphi_{;ij} + \varphi_{,i}\varphi_{,j}\right)} \\
&= \left(1+|D\varphi|^2\right)^{\frac{n+1}{n}} \cdot \frac{\det^{1/n}(\sigma_{ij})}{\det^{1/n}\left(\sigma_{ij} - \varphi_{;ij} + \varphi_{,i}\varphi_{,j}\right)}. \qquad (2.5)
\end{aligned}$$

We suppose furthermore the hypersurface to evolve staying perpendicularly to the cone. We hence obtain a Neumann boundary condition.

Lemma 2.4. *Let $\bar{\nu}(p)$ be the outward pointing unit normal vector to $\mathcal{C}^{n+1}$ at a point $p = X(x,t) \in \partial M_t$ and ν the unit normal vector field to M_t. Then*

$$\langle \bar{\nu}(p), \nu(x,t) \rangle = 0 \Longleftrightarrow D_{\bar{\nu}}\varphi(x,t) = \bar{\nu}^i(\iota(x))\varphi_{,i}(x,t) = 0,$$

where $(\bar{\nu}^i)_{1\le i\le n}$ are the components of $\bar{\nu}(\iota(x))$ in a coordinate system of the tangent space of $\hat{\Omega} = \iota(\Omega)$ at the point $\iota(x) \in \partial\hat{\Omega}$.

Proof. Let $(\hat{\bar{\nu}}^\alpha(p))_{1\le\alpha\le n+1}$ be the coordinates of $\bar{\nu}$ in the standard basis of the $\mathbb{R}^{n+1}$. Since $\mathcal{C}^{n+1}$ is a (convex) cone, we have $\bar{\nu}(p) = \bar{\nu}(X(x,t)) = \bar{\nu}(\iota(x)\varphi(x,t)) = \bar{\nu}(\iota(x))$ for $x \in \partial\Omega$.

The condition

$$\langle \bar{\nu}(p), \nu(x,t) \rangle = \langle \bar{\nu}(X(x,t)), \nu(x,t) \rangle = \langle \bar{\nu}(\iota(x)), \nu(x,t) \rangle = 0$$

is equivalent, using the representation of $\bar{\nu}$ in the coordinate system of the tangent space of $\hat{\Omega}$ at $\iota(x)$, to

$$\left\langle \bar{\nu}^i(\iota(x))\iota_{,i}(x), \nu(x,t) \right\rangle = 0$$

and hence to

$$\bar{\nu}^i(\iota(x)) \left\langle \iota_{,i}(x), \nu(x,t) \right\rangle = 0. \tag{2.6}$$

We now compute the inner product, inserting the expression (2.1) of the normal vector, and obtain

$$\begin{aligned}
\langle \iota_{,i}, \nu \rangle &= \left\langle \iota_{,i}, \frac{1}{\sqrt{1+|D\varphi|^2}} \left(\iota - \sigma^{lk}\varphi_{,l}\iota_{,k}\right) \right\rangle \\
&= \frac{1}{\sqrt{1+|D\varphi|^2}} \left(\langle \iota_{,i}, \iota \rangle - \left\langle \iota_{,i}, \sigma^{lk}\varphi_{,l}\iota_{,k} \right\rangle \right) \\
&= -\frac{\sigma^{lk}\varphi_{,l} \langle \iota_{,i}, \iota_{,k} \rangle}{\sqrt{1+|D\varphi|^2}} = -\frac{\sigma^{lk}\varphi_{,l}\sigma_{ik}}{\sqrt{1+|D\varphi|^2}} = \frac{-\varphi_{,i}}{\sqrt{1+|D\varphi|^2}}.
\end{aligned}$$

So this implies that (2.6) is the same as

$$\bar{\nu}^i(\iota(x))\varphi_{,i}(x,t) = 0$$

for all $x \in \partial\Omega$ and $t \in [0, t^*)$. □

From (2.5) and Lemma 2.4 we eventually get the formulation of the problem stated in Theorem 1.2, that is

$$\begin{cases}
\dot{\varphi} = \left(1 + |D\varphi|^2\right)^{\frac{n+1}{n}} \cdot \dfrac{[\det(\sigma_{ij})]^{1/n}}{\left[\det\left(\sigma_{ij} - \varphi_{;ij} + \varphi_{,i}\varphi_{,j}\right)\right]^{1/n}} & \text{in } \Omega \times [0,\infty), \\
D_{\bar{\nu}}\varphi = 0 & \text{on } \partial\Omega \times [0,\infty), \\
\varphi(x,0) = \varphi_0(x) & \text{in } \Omega.
\end{cases} \tag{2.7}$$

3. $\dot\varphi$-Estimates

In this section we treat the $\dot\varphi$-estimates: We will show that $\dot\varphi$ stays bounded during the flow.

Lemma 3.1. *Let φ be a solution of* (1.2)*, then its time derivative is bounded:*

$$m_1 := \min_{x'\in\bar\Omega} \dot\varphi(x',0) \le \dot\varphi(x,t) \le \max_{x'\in\bar\Omega} \dot\varphi(x',0) =: m_2$$

holds for all $t \in [0,t^*)$ and $x \in \bar\Omega$.

Proof. Let $0 < t' < t^*$. Assume first $w_{ij}(x,t) > 0$ in the sense of matrices for all $t \in [0,t']$ and $x \in \bar\Omega$ and let $w^{ij}(x,t)$ indicate the inverse of $w_{ij}(x,t) = \sigma_{ij}(x) - \varphi_{;ij}(x,t) + \varphi_{,i}(x,t)\varphi_{,j}(x,t)$. We compute the time derivative of $\dot\varphi$, differentiating the partial differential equation, to obtain

$$\begin{aligned}\ddot\varphi &= \frac{\dot\varphi}{n}\left[(n+1)\cdot\frac{2\sigma^{ij}\dot\varphi_{,i}\varphi_{,j}}{1+|D\varphi|^2} + w^{ij}\left(\dot\varphi_{;ij} - \dot\varphi_{,i}\varphi_{,j} - \varphi_{,i}\dot\varphi_{,j}\right)\right]\\ &= \frac{\dot\varphi}{n}\cdot w^{ij}\dot\varphi_{;ij} + \frac{2\dot\varphi}{n}\cdot\left(\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{ij}\varphi_{,j} - w^{ij}\varphi_{,j}\right)\dot\varphi_{,i},\end{aligned}$$

because of the symmetry of w^{ij}. Moreover the differentiated boundary condition is given by

$$\bar\nu^i\dot\varphi_{,i} = 0. \tag{3.1}$$

The parabolic maximum principle for $\dot\varphi$ implies therefore the inequalities.

If w_{ij} is not strictly positive for all times in the interval $[0,t']$, then there is a minimal time t_1, such that $w_{ij}(x_1,t_1)$ has a zero eigenvalue, and in particular we have $\det(w_{ij}(x_1,t_1)) = 0$ for a $x_1 \in \bar\Omega$.

With the same idea as above, we then would get

$$\min_{x'\in\bar\Omega} \dot\varphi(x',0) \le \dot\varphi(x,t) \le \max_{x'\in\bar\Omega} \dot\varphi(x',0)$$

for all $t \in [0,t_1-\varepsilon]$, $\varepsilon > 0$, and $x \in \bar\Omega$. Letting ε go to zero, would extend this inequalities to all $t \in [0,t_1]$ and this would be a contradiction to $\det(w_{ij}(x_1,t_1)) = 0$, because of the partial differential equation we are considering.

Finally, the differentiated boundary condition (3.1) shows that extrema of $\dot\varphi(x,t)$, for $0 < t < t^*$, cannot occur on the boundary, if φ is nonconstant, since Hopf's maximum principle would force $\bar\nu^i\dot\varphi_{,i}$ to have a sign in that point. □

Remark 3.2. It is clear from the equation (1.2) that $\dot\varphi$ is strictly positive for all $(x,t) \in \bar\Omega\times[0,t^*)$, so $m_1 = \min_{x\in\bar\Omega}\dot\varphi(x,0)$ has to be positive. Moreover this means that we are considering an expanding flow.

In the proof of the lemma we also showed the following.

Corollary 3.3. *Let φ be a solution of* (1.2) *with*

$$w_{ij}(x,0) = \sigma_{ij}(x) - \varphi_{;ij}(x,0) + \varphi_{,i}(x,0)\varphi_{,j}(x,0) > 0,$$

up to the boundary, in the sense of matrices. Then

$$w_{ij}(x,t) = \sigma_{ij}(x) - \varphi_{;ij}(x,t) + \varphi_{,i}(x,t)\varphi_{,j}(x,t) > 0,$$

up to the boundary, for all $(x,t) \in \bar\Omega\times[0,t^*)$.

This implies that the convexity of the starting hypersurface, which is the only admissibility condition we need to care about, is going to be preserved during the flow.

4. C^0-Estimates

We note first, that the bounds obtained integrating the inequalities of Lemma 3.1 wouldn't be sharp enough to ensure convergence.

We now prove a comparison principle using a standard maximum principle argument. This follows from an interpolation argument, a similar idea was applied in [15] to the Schouten equation.

Lemma 4.1. *Let φ and ψ be two solutions of* (1.2) *with* $\varphi(x,0) \le \psi(x,0)$ *for all* $x \in \bar{\Omega}$, *then*

$$\varphi(x,t) \le \psi(x,t)$$

holds for all $t \in (0,t^*)$ *and* $x \in \bar{\Omega}$.

Proof. We define

$$\chi(x,t) := \varphi(x,t) - \psi(x,t).$$

It is immediately clear that this new function is initially nonpositive,

$$\chi(x,0) = \varphi(x,0) - \psi(x,0) \le 0,$$

and satisfies the same boundary condition as φ and ψ:

$$\bar{\nu}^i \chi_{,i} = \bar{\nu}^i \varphi_{,i} - \bar{\nu}^i \psi_{,i} = 0.$$

Furthermore, for a real number $s \in [0,1]$, we set

$$\begin{aligned} v_{ij}(x,t)[s] :=\, & \sigma_{ij}(x) - s\varphi_{;ij}(x,t) + s\varphi_{,i}(x,t)\varphi_{,j}(x,t) \\ & - (1-s)\psi_{;ij}(x,t) + (1-s)\psi_{,i}(x,t)\psi_{,j}(x,t). \end{aligned}$$

Since the set of positive definite matrices is convex, it is possible to apply the main theorem of calculus to write

$$\begin{aligned} \dot{\chi} &= \dot{\varphi} - \dot{\psi} \\ &= \int_0^1 \frac{d}{ds} \left(\frac{\left(1 + |D(s\varphi + (1-s)\psi)|^2\right)^{\frac{n+1}{n}} \cdot \det^{1/n}(\sigma_{ij})}{\det^{1/n}\left(\sigma_{ij} - s\varphi_{;ij} + s\varphi_{,i}\varphi_{,j} - (1-s)\psi_{;ij} + (1-s)\psi_{,i}\psi_{,j}\right)} \right) ds. \end{aligned} \tag{4.1}$$

To compute the derivative in the integral the following calculations will be useful. First, using the formula for the derivative of the determinant, we have

$$\begin{aligned} \frac{d}{ds}\det^{1/n}(v_{ij}) &= \frac{1}{n}\left(\det(v_{ij})\right)^{1/n-1} \cdot \det(v_{ij}) \cdot v^{ji} \cdot \frac{d}{ds} v_{ij} \\ &= \frac{1}{n}\det^{1/n}(v_{ij}) \cdot v^{ij} \cdot (\psi_{;ij} - \varphi_{;ij} + \varphi_{,i}\varphi_{,j} - \psi_{,i}\psi_{,j}) \\ &= \frac{1}{n}\det^{1/n}(v_{ij}) \cdot v^{ij} \cdot \left(-\chi_{;ij} + \chi_{,i}\left(\varphi_{,j} + \psi_{,j}\right)\right), \end{aligned}$$

where v^{ij} denotes the inverse of v_{ij}, which is positive, and where the symmetry of v^{ij} yields

$$v^{ij}(\varphi_{,i}\varphi_{,j} - \psi_{,i}\psi_{,j}) = v^{ij}\left(\varphi_{,i} - \psi_{,i}\right)\left(\varphi_{,j} + \psi_{,j}\right) = v^{ij}\chi_{,i}\left(\varphi_{,j} + \psi_{,j}\right).$$

Secondly, the derivative of the first factor of the numerator of the argument of the integral is given by

$$
\begin{aligned}
&\frac{d}{ds}\left(1+|D(s\varphi+(1-s)\psi)|^2\right)^{\frac{n+1}{n}}\\
&\qquad=\frac{n+1}{n}\left(1+|D(s\varphi+(1-s)\psi)|^2\right)^{\frac{1}{n}}\cdot\frac{d}{ds}|D(s\varphi+(1-s)\psi)|^2\\
&\qquad=\frac{n+1}{n}\left(1+|D(s\varphi+(1-s)\psi)|^2\right)^{\frac{1}{n}}\\
&\qquad\quad\cdot\sigma^{ij}\left[(\varphi_{,i}-\psi_{,i})(s\varphi_{,j}+(1-s)\psi_{,j})+(s\varphi_{,i}+(1-s)\psi_{,i})(\varphi_{,j}-\psi_{,j})\right]\\
&\qquad=\frac{n+1}{n}\left(1+|D(s\varphi+(1-s)\psi)|^2\right)^{\frac{1}{n}}\cdot\left[2\sigma^{ij}\chi_{,i}(s\varphi_{,j}+(1-s)\psi_{,j})\right]
\end{aligned}
$$

and this implies that the derivative in (4.1) may be written as

$$
\begin{aligned}
&\frac{d}{ds}\left(\frac{\left(1+|D(s\varphi+(1-s)\psi)|^2\right)^{\frac{n+1}{n}}}{\det^{1/n}(v_{ij})}\right)\\
&=\frac{\frac{n+1}{n}\left(1+|D(s\varphi+(1-s)\psi)|^2\right)^{\frac{1}{n}}\cdot\left[2\sigma^{ij}\chi_{,i}(s\varphi_{,j}+(1-s)\psi_{,j})\right]}{\det^{1/n}(v_{ij})}\\
&\quad-\frac{\left(1+|D(s\varphi+(1-s)\psi)|^2\right)^{\frac{n+1}{n}}\cdot\frac{1}{n}\det^{1/n}(v_{ij})\cdot v^{ij}\cdot(-\chi_{;ij}+\chi_{,i}(\varphi_{,j}+\psi_{,j}))}{\det^{2/n}(v_{ij})}\\
&=\frac{\frac{n+1}{n}\left(1+|D(s\varphi+(1-s)\psi)|^2\right)^{\frac{1}{n}}\cdot\left[2\sigma^{ij}\chi_{,i}(s\varphi_{,j}+(1-s)\psi_{,j})\right]}{\det^{1/n}(v_{ij})}\\
&\quad-\frac{\left(1+|D(s\varphi+(1-s)\psi)|^2\right)^{\frac{n+1}{n}}\cdot\frac{1}{n}\cdot v^{ij}\cdot(-\chi_{;ij}+\chi_{,i}(\varphi_{,j}+\psi_{,j}))}{\det^{1/n}(v_{ij})}.
\end{aligned}
$$

Introducing the following notation for the positive definite coefficient matrix of the second derivatives

$$
A^{ij}:=\frac{1}{n}\cdot\det{}^{1/n}(\sigma_{kl})\cdot\int_0^1\frac{\left(1+|D(s\varphi+(1-s)\psi)|^2\right)^{\frac{n+1}{n}}\cdot v^{ij}}{\det^{1/n}(v_{ij})}\,ds
$$

and setting

$$
\begin{aligned}
B^i:=&-A^{ij}(\varphi_{,j}+\psi_{,j})+\frac{2(n+1)}{n}\cdot\det{}^{1/n}(\sigma_{kl})\cdot\sigma^{ij}\\
&\cdot\int_0^1\frac{\left(1+|D(s\varphi+(1-s)\psi)|^2\right)^{\frac{1}{n}}\cdot(s\varphi_{,j}+(1-s)\psi_{,j})}{\det^{1/n}(v_{ij})}\,ds,
\end{aligned}
$$

it follows, in view of (4.1) and of the last computations,

$$
\begin{cases}
\dot\chi-A^{ij}\chi_{,ij}+B^i\chi_{,i}=0, & \text{in } \Omega\times[0,t^*)\\
\chi(x,0)\le 0 & \text{in } \Omega\\
D_{\bar\nu}\chi=0 & \text{on } \partial\Omega\times[0,t^*).
\end{cases}
$$

Using the parabolic maximum principle, we can hence conclude that χ has to be nonpositive for all $t\in[0,t^*)$. □

It is easily seen that the maps $\psi(x,t) = t+r$ solve the problem with initial condition $\psi_0(x) = r$ for $r \in \mathbb{R}$. The following corollary, providing uniform bounds on $\varphi(x,t)-t$ for any solution φ of (1.2), is therefore an immediate consequence of the previous lemma.

Corollary 4.2. *Let φ be a solution of* (1.2) *with*

$$c_1 \le \varphi(x,0) \le c_2,$$

for all $x \in \bar{\Omega}$ and $c_1, c_2 \in \mathbb{R}$ then

$$t + c_1 \le \varphi(x,t) \le t + c_2$$

holds for all $t \in (0,t^*)$ and $x \in \bar{\Omega}$.

5. C^1-Estimates

In order to obtain C^1-estimates we are going to use a result proved in [12], the so-called *Ice-cream cone estimate.*

Theorem 5.1 (Ice-cream cone estimate)**.** *Let $U \subset \mathbb{R}^n$ be a smooth bounded domain, $\tilde{F} : \bar{U} \to \mathbb{R}$ a smooth strictly convex function with $|\tilde{F}_\gamma|$ uniformly bounded on ∂U, where γ is a unit vector field on ∂U such that $\langle \gamma, \nu_U \rangle \ge \tilde{c}_\gamma$ for a positive constant $\tilde{c}_\gamma > 0$ (where ν_U is the inner unit normal to ∂U). Then there is a uniform bound for* $\sup |D\tilde{F}|$*, independent of* $\sup |\tilde{F}|$.

To apply this theorem we need to use an explicit choice of a coordinate system to work in $\mathbb{R}^n$. Let us however first consider a (possibly time-dependent) smooth map $f : \mathrm{S}^n_+ \to \mathbb{R}$, $z = (z^1, \ldots, z^{n+1}) \mapsto f(z^1, \ldots, z^{n+1})$, let $F : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ be the positive and homogeneous of degree one map defined by

$$F(z) = f\left(\frac{z}{|z|}\right)|z| \tag{5.1}$$

and let

$$\tilde{F} : \mathbb{R}^n \to \mathbb{R}, (y^1, \ldots, y^n) \mapsto \tilde{F}(y^1, \ldots, y^n) := F(y^1, \ldots, y^n, 1) \tag{5.2}$$

be the restriction of F to $\mathbb{R}^n \times \{1\}$. We then choose coordinates like in [13], which allow us to prove the next lemma, that establishes a relation between the partial derivatives of $\tilde{F}$ and the covariant of F.

Lemma 5.2. *Let*

$$\bar{\eta}' : \mathbb{R}^n \times \{1\} \to \mathrm{S}^n_+ \subset \mathbb{R}^{n+1}$$

be the embedding given by

$$(y^1, \ldots, y^n, 1) \mapsto \frac{(y^1, \ldots, y^n, 1)}{\sqrt{1+|y|^2}}$$

and let

$$\bar{\eta} : \mathbb{R}^n \to \mathrm{S}^n_+ \subset \mathbb{R}^{n+1}, (y^1, \ldots, y^n) \mapsto \bar{\eta}(y^1, \ldots, y^n) := \bar{\eta}'(y^1, \ldots, y^n, 1).$$

Then for f, F and $\tilde{F}$ as above it follows

$$\frac{\tilde{F}_{,\mathfrak{ab}}(y)}{\sqrt{1+|y|^2}} = [F(\bar{\eta}(y))]_{;\mathfrak{ab}} + F(\bar{\eta}(y))\sigma_{\mathfrak{ab}} = [f(\bar{\eta}(y))]_{;\mathfrak{ab}} + f(\bar{\eta}(y))\sigma_{\mathfrak{ab}},$$

where $y = (y^1, \ldots, y^n) \in \mathbb{R}^n$.

Proof. Since $|\bar{\eta}(y)| = 1$, we have $F(\bar{\eta}(y)) = f(\bar{\eta}(y))$, which provides the second equality.

We can express the standard round metric σ of the sphere in the coordinates induced by the imbedding $\bar{\eta}$ computing the pullback $\bar{\eta}^*\delta_{\mathbb{R}^{n+1}}$ of the Euclidean metric in $\mathbb{R}^{n+1}$. We differentiate $\bar{\eta}$, for this purpose, obtaining

$$\bar{\eta}_{,\mathfrak{a}}(y) = \frac{(e_{\mathfrak{a}}, 0)}{\sqrt{1+|y|^2}} - \frac{(y,1)}{(\sqrt{1+|y|^2})^3} \cdot y_{\mathfrak{a}},$$

for $1 \leq \mathfrak{a} \leq n$, insert in the definition

$$\begin{aligned}\sigma_{\mathfrak{a}\mathfrak{b}} = \bar{\eta}^{\alpha}_{,\mathfrak{a}}\delta_{\alpha\beta}\bar{\eta}^{\beta}_{,\mathfrak{b}} = \frac{1}{1+|y|^2}\Bigg[&\sum_{\alpha,\beta<n+1}\delta_{\alpha\beta}\left(\delta^{\alpha}_{\mathfrak{a}} - \frac{y^{\alpha}y_{\mathfrak{a}}}{1+|y|^2}\right)\left(\delta^{\beta}_{\mathfrak{b}} - \frac{y^{\beta}y_{\mathfrak{b}}}{1+|y|^2}\right)\\
&+\sum_{\alpha<n+1}\delta_{\alpha n+1}\left(\delta^{\alpha}_{\mathfrak{a}} - \frac{y^{\alpha}y_{\mathfrak{a}}}{1+|y|^2}\right)\cdot\frac{(-y_{\mathfrak{b}})}{1+|y|^2}\\
&+\sum_{\beta<n+1}\delta_{n+1\beta}\cdot\frac{(-y_{\mathfrak{a}})}{1+|y|^2}\cdot\left(\delta^{\beta}_{\mathfrak{b}} - \frac{y^{\beta}y_{\mathfrak{b}}}{1+|y|^2}\right)\\
&+\delta_{n+1n+1}\cdot\frac{y_{\mathfrak{a}}y_{\mathfrak{b}}}{(1+|y|^2)^2}\Bigg]\end{aligned}$$

and, since the mixed terms vanish, we get

$$\begin{aligned}\sigma_{\mathfrak{a}\mathfrak{b}} &= \frac{1}{1+|y|^2}\Bigg[\sum_{\alpha,\beta<n+1}\Bigg(\delta^{\alpha}_{\mathfrak{a}}\delta_{\alpha\beta}\delta^{\beta}_{\mathfrak{b}} - \frac{\delta^{\alpha}_{\mathfrak{a}}\delta_{\alpha\beta}y^{\beta}y_{\mathfrak{b}}}{1+|y|^2} - \frac{\delta^{\beta}_{\mathfrak{b}}\delta_{\alpha\beta}y^{\alpha}y_{\mathfrak{a}}}{1+|y|^2}\\
&\qquad + y^{\alpha}\delta_{\alpha\beta}y^{\beta}\cdot\frac{y_{\mathfrak{a}}y_{\mathfrak{b}}}{(1+|y|^2)^2}\Bigg) + \frac{y_{\mathfrak{a}}y_{\mathfrak{b}}}{(1+|y|^2)^2}\Bigg]\\
&= \frac{1}{1+|y|^2}\left[\delta_{\mathfrak{a}\mathfrak{b}} - \frac{y_{\mathfrak{a}}y_{\mathfrak{b}}}{1+|y|^2}\right]. \end{aligned} \tag{5.3}$$

It is not difficult to verify that the corresponding inverse metric can be expressed by

$$\sigma^{\mathfrak{a}\mathfrak{b}} = \left(1+|y|^2\right)\left(\delta^{\mathfrak{a}\mathfrak{b}} + y^{\mathfrak{a}}y^{\mathfrak{b}}\right). \tag{5.4}$$

The partial derivatives of the metric are given by

$$\begin{aligned}\sigma_{\mathfrak{a}\mathfrak{b},\mathfrak{c}} &= \left[\frac{1}{1+|y|^2}\left(\delta_{\mathfrak{a}\mathfrak{b}} - \frac{y_{\mathfrak{a}}y_{\mathfrak{b}}}{1+|y|^2}\right)\right]_{,\mathfrak{c}}\\
&= \frac{-2y_{\mathfrak{c}}}{(1+|y|^2)^2}\left(\delta_{\mathfrak{a}\mathfrak{b}} - \frac{y_{\mathfrak{a}}y_{\mathfrak{b}}}{1+|y|^2}\right) - \frac{1}{1+|y|^2}\left(\frac{\delta_{\mathfrak{a}\mathfrak{c}}y_{\mathfrak{b}} + \delta_{\mathfrak{b}\mathfrak{c}}y_{\mathfrak{a}}}{1+|y|^2} - \frac{2y_{\mathfrak{a}}y_{\mathfrak{b}}y_{\mathfrak{c}}}{(1+|y|^2)^2}\right)\\
&= \frac{1}{(1+|y|^2)^2}\left(-2\delta_{\mathfrak{a}\mathfrak{b}}y_{\mathfrak{c}} + \frac{2y_{\mathfrak{a}}y_{\mathfrak{b}}y_{\mathfrak{c}}}{1+|y|^2} - (\delta_{\mathfrak{a}\mathfrak{c}}y_{\mathfrak{b}} + \delta_{\mathfrak{b}\mathfrak{c}}y_{\mathfrak{a}}) + \frac{2y_{\mathfrak{a}}y_{\mathfrak{b}}y_{\mathfrak{c}}}{(1+|y|^2)}\right)\\
&= \frac{1}{(1+|y|^2)^2}\left(-2\delta_{\mathfrak{a}\mathfrak{b}}y_{\mathfrak{c}} - \delta_{\mathfrak{a}\mathfrak{c}}y_{\mathfrak{b}} - \delta_{\mathfrak{b}\mathfrak{c}}y_{\mathfrak{a}} + \frac{4y_{\mathfrak{a}}y_{\mathfrak{b}}y_{\mathfrak{c}}}{1+|y|^2}\right).\end{aligned}$$

To get the connection coefficients with respect to $\sigma_{\mathfrak{ab}}$ we first compute

$$\begin{aligned}
\sigma_{\mathfrak{ab},\mathfrak{c}} + \sigma_{\mathfrak{ac},\mathfrak{b}} - \sigma_{\mathfrak{bc},\mathfrak{a}} = \frac{1}{(1+|y|^2)^2} \Bigg[& -2\delta_{\mathfrak{ab}} y_{\mathfrak{c}} - \delta_{\mathfrak{ac}} y_{\mathfrak{b}} - \delta_{\mathfrak{bc}} y_{\mathfrak{a}} + \frac{4 y_{\mathfrak{a}} y_{\mathfrak{b}} y_{\mathfrak{c}}}{1+|y|^2} \\
& + \left(-2\delta_{\mathfrak{ac}} y_{\mathfrak{b}} - \delta_{\mathfrak{ab}} y_{\mathfrak{c}} - \delta_{\mathfrak{cb}} y_{\mathfrak{a}} + \frac{4 y_{\mathfrak{a}} y_{\mathfrak{b}} y_{\mathfrak{c}}}{1+|y|^2} \right) \\
& - \left(-2\delta_{\mathfrak{bc}} y_{\mathfrak{a}} - \delta_{\mathfrak{ba}} y_{\mathfrak{c}} - \delta_{\mathfrak{ca}} y_{\mathfrak{b}} + \frac{4 y_{\mathfrak{a}} y_{\mathfrak{b}} y_{\mathfrak{c}}}{1+|y|^2} \right) \Bigg] \\
= \frac{1}{(1+|y|^2)^2} & \left[-2\delta_{\mathfrak{ab}} y_{\mathfrak{c}} - 2\delta_{\mathfrak{ac}} y_{\mathfrak{b}} + \frac{4 y_{\mathfrak{a}} y_{\mathfrak{b}} y_{\mathfrak{c}}}{1+|y|^2} \right] \\
= \frac{2}{(1+|y|^2)^2} & \left[-\delta_{\mathfrak{ab}} y_{\mathfrak{c}} - \delta_{\mathfrak{ac}} y_{\mathfrak{b}} + \frac{2 y_{\mathfrak{a}} y_{\mathfrak{b}} y_{\mathfrak{c}}}{1+|y|^2} \right]
\end{aligned}$$

and therefore, using the formula (5.4) for the inverse of the metric, it follows

$$\begin{aligned}
{}^{\sigma}\Gamma^{\mathfrak{d}}_{\mathfrak{bc}} &= \frac{1}{2} \sigma^{\mathfrak{da}} \left(\sigma_{\mathfrak{ab},\mathfrak{c}} + \sigma_{\mathfrak{ac},\mathfrak{b}} - \sigma_{\mathfrak{bc},\mathfrak{a}} \right) \\
&= \frac{1}{2} \sigma^{\mathfrak{da}} \frac{2}{(1+|y|^2)^2} \left[-\delta_{\mathfrak{ab}} y_{\mathfrak{c}} - \delta_{\mathfrak{ac}} y_{\mathfrak{b}} + \frac{2 y_{\mathfrak{a}} y_{\mathfrak{b}} y_{\mathfrak{c}}}{1+|y|^2} \right] \\
&= (1+|y|^2) \left(\delta^{\mathfrak{da}} + y^{\mathfrak{a}} y^{\mathfrak{d}} \right) \frac{1}{(1+|y|^2)^2} \left[-\delta_{\mathfrak{ab}} y_{\mathfrak{c}} - \delta_{\mathfrak{ac}} y_{\mathfrak{b}} + \frac{2 y_{\mathfrak{a}} y_{\mathfrak{b}} y_{\mathfrak{c}}}{1+|y|^2} \right] \\
&= \frac{1}{1+|y|^2} \left[-\delta^{\mathfrak{d}}_{\mathfrak{b}} y_{\mathfrak{c}} - \delta^{\mathfrak{d}}_{\mathfrak{c}} y_{\mathfrak{b}} + \frac{2 y^{\mathfrak{d}} y_{\mathfrak{b}} y_{\mathfrak{c}}}{1+|y|^2} - y_{\mathfrak{b}} y^{\mathfrak{d}} y_{\mathfrak{c}} - y_{\mathfrak{c}} y^{\mathfrak{d}} y_{\mathfrak{b}} + \frac{2|y|^2 y_{\mathfrak{b}} y_{\mathfrak{c}} y^{\mathfrak{d}}}{1+|y|^2} \right] \\
&= \frac{1}{1+|y|^2} \left[-\delta^{\mathfrak{d}}_{\mathfrak{b}} y_{\mathfrak{c}} - \delta^{\mathfrak{d}}_{\mathfrak{c}} y_{\mathfrak{b}} \right]. \qquad (5.5)
\end{aligned}$$

From this last expression we obtain

$$\begin{aligned}
(F(\bar{\eta}(y)))_{;\mathfrak{ab}} &= (F(\bar{\eta}(y)))_{,\mathfrak{ab}} - {}^{\sigma}\Gamma^{\mathfrak{c}}_{\mathfrak{ab}} (F(\bar{\eta}(y)))_{,\mathfrak{c}} \\
&= (F(\bar{\eta}(y)))_{,\mathfrak{ab}} - \frac{1}{1+|y|^2} \left[-\delta^{\mathfrak{c}}_{\mathfrak{a}} y_{\mathfrak{b}} - \delta^{\mathfrak{c}}_{\mathfrak{b}} y_{\mathfrak{a}} \right] (F(\bar{\eta}(y)))_{,\mathfrak{c}} \\
&= (F(\bar{\eta}(y)))_{,\mathfrak{ab}} + \frac{1}{1+|y|^2} \left[(F(\bar{\eta}(y)))_{,\mathfrak{a}} y_{\mathfrak{b}} + (F(\bar{\eta}(y)))_{,\mathfrak{b}} y_{\mathfrak{a}} \right]
\end{aligned}$$

for the covariant derivatives of $F(\bar{\eta})$ with respect to the standard metric of the sphere. On the other hand, because of the definition (5.2) and of the homogeneity of F,

$$\tilde{F}(y) = F\left(\bar{\eta}(y) \cdot \sqrt{1+|y|^2} \right) = F(\bar{\eta}(y)) \cdot \sqrt{1+|y|^2},$$

the partial derivatives of $\tilde{F}$ are given by

$$(\tilde{F}(y))_{,\mathfrak{a}} = (F(\bar{\eta}(y)))_{,\mathfrak{a}} \sqrt{1+|y|^2} + \frac{1}{\sqrt{1+|y|^2}} \left[(F(\bar{\eta}(y))) y_{\mathfrak{a}} \right]$$

respectively

$$\begin{aligned}
(\tilde{F}(y))_{,\mathfrak{ab}} = (F(\bar{\eta}(y)))_{,\mathfrak{ab}} \sqrt{1+|y|^2} &+ \frac{1}{\sqrt{1+|y|^2}} \left[(F(\bar{\eta}(y)))_{,\mathfrak{a}} y_{\mathfrak{b}} + (F(\bar{\eta}(y)))_{,\mathfrak{b}} y_{\mathfrak{a}} \right] \\
&+ \frac{(F(\bar{\eta}(y)))}{\sqrt{1+|y|^2}} \left(\delta_{\mathfrak{ab}} - \frac{y_{\mathfrak{a}} y_{\mathfrak{b}}}{1+|y|^2} \right).
\end{aligned}$$

This provides the statement of the lemma

$$\begin{aligned}\frac{(\tilde{F}(y))_{,\mathfrak{ab}}}{\sqrt{1+|y|^2}} &= (F(\bar{\eta}(y)))_{,\mathfrak{ab}} + \frac{1}{1+|y|^2}\left[(F(\bar{\eta}(y)))_{,\mathfrak{a}}y_{\mathfrak{b}} + (F(\bar{\eta}(y)))_{,\mathfrak{b}}y_{\mathfrak{a}}\right] \\ &\quad + (F(\bar{\eta}(y)))\sigma_{\mathfrak{ab}} \\ &= (F(\bar{\eta}(y)))_{;\mathfrak{ab}} + (F(\bar{\eta}(y)))\sigma_{\mathfrak{ab}}. \end{aligned}$$ □

If we now consider the subset $\hat{\Omega} = \iota(\Omega) \subset \mathrm{S}^n_+ \subset \mathbb{R}^{n+1}$ and restrict $\tilde{F}$ and F to $U = \bar{\eta}^{-1}(\hat{\Omega}) \subset \mathbb{R}^n$, we can obtain, applying this lemma with an appropriate choice of f, a function $\tilde{F}$ that is strictly convex on the compact set $\bar{U} = \overline{\bar{\eta}^{-1}(\hat{\Omega})}$: For a fixed time $t \in [0, t^*)$ we define $f(\cdot, t) : \hat{\Omega} \to \mathbb{R}$ to be the map given by

$$f(z,t) = e^{-\bar{\varphi}\left(\iota^{-1}(z),t\right)} - \frac{1}{\hat{c}} \cdot \left(z^{n+1} + 1\right),$$

where $z = (z^1, \dots, z^{n+1}) \in \hat{\Omega}$,

$$\bar{\varphi}(x,t) := \varphi(x,t) - t - \underline{m}, \quad \underline{m} := \min_{x' \in \bar{\Omega}} \varphi(x', 0),$$

and

$$\hat{c} \equiv \hat{c}(\hat{\Omega}) := \inf_{z \in \hat{\Omega}} z^{n+1}$$

is a constant, depending only on $\hat{\Omega}$, which fulfils $0 < \hat{c} < 1$ because $\hat{\Omega}$ is a subset of a sphere contained in a cone. We note that f is strictly positive and bounded, since Corollary 4.2 implies that

$$\underline{m} \le \varphi(x,t) - t$$

and that there is a real number c such that it holds

$$|\bar{\varphi}(x,t)| = |\varphi(x,t) - t - \underline{m}| \le c$$

for all $(x,t) \in \bar{\Omega} \times [0, t^*)$. In particular we have that $0 < e^{-\bar{\varphi}(x,t)} \le 1$ for all $(x,t) \in \bar{\Omega} \times [0, t^*)$.

We want now to show that $f_{;\mathfrak{ab}}(\bar{\eta}, t)$ is positive definite. Working with the coordinates induced by $\bar{\eta}$ as in the proof of the last lemma we can express the metric as

$$\sigma_{\mathfrak{ab}} = \frac{1}{1+|y|^2}\left(\delta_{\mathfrak{ab}} - \frac{y_{\mathfrak{a}} y_{\mathfrak{b}}}{1+|y|^2}\right) \tag{5.6}$$

and the connection coefficients by (5.5) as

$${}^{\sigma}\Gamma^{\mathfrak{d}}_{\mathfrak{bc}} = \frac{1}{1+|y|^2}\left[-\delta^{\mathfrak{d}}_{\mathfrak{b}} y_{\mathfrak{c}} - \delta^{\mathfrak{d}}_{\mathfrak{c}} y_{\mathfrak{b}}\right].$$

For the covariant derivative of the first term of $f(\bar\eta,t)$, we get

$$\begin{aligned}\left(e^{-\bar\varphi(\bar\eta(y),t)}\right)_{;\mathfrak{ab}} &= \left(e^{-\bar\varphi(\bar\eta(y),t)}\right)_{,\mathfrak{ab}} - {}^{\sigma}\Gamma^{\mathfrak{c}}_{\mathfrak{ab}}\left(e^{-\bar\varphi(\bar\eta(y),t)}\right)_{,\mathfrak{c}}\\ &= e^{-\bar\varphi(\bar\eta(y),t)}\Big(-\left(\bar\varphi(\bar\eta(y),t)\right)_{,\mathfrak{ab}} + \left(\bar\varphi(\bar\eta(y),t)\right)_{,\mathfrak{a}}\left(\bar\varphi(\bar\eta(y),t)\right)_{,\mathfrak{b}}\\ &\qquad -{}^{\sigma}\Gamma^{\mathfrak{c}}_{\mathfrak{ab}}\cdot\left(-\bar\varphi(\bar\eta(y),t)\right)_{,\mathfrak{c}}\Big)\\ &= e^{-\bar\varphi(\bar\eta(y),t)}\Big(-\left(\varphi(\bar\eta(y),t)\right)_{,\mathfrak{ab}} + \left(\varphi(\bar\eta(y),t)\right)_{,\mathfrak{a}}\left(\varphi(\bar\eta(y),t)\right)_{,\mathfrak{b}}\\ &\qquad -{}^{\sigma}\Gamma^{\mathfrak{c}}_{\mathfrak{ab}}\cdot\left(-\varphi(\bar\eta(y),t)\right)_{,\mathfrak{c}}\Big)\\ &= e^{-\bar\varphi(\bar\eta(y),t)}\Big(-\left(\varphi(\bar\eta(y),t)\right)_{;\mathfrak{ab}} + \left(\varphi(\bar\eta(y),t)\right)_{,\mathfrak{a}}\left(\varphi(\bar\eta(y),t)\right)_{,\mathfrak{b}}\Big),\end{aligned}$$

in local coordinates, because of the definition of $\bar\varphi$. Whereas for the second term it holds

$$\begin{aligned}\left(\bar\eta^{n+1}(y)\right)_{;\mathfrak{ab}} &= \left(\bar\eta^{n+1}(y)\right)_{,\mathfrak{ab}} - {}^{\sigma}\Gamma^{\mathfrak{c}}_{\mathfrak{ab}}\left(\bar\eta^{n+1}(y)\right)_{,\mathfrak{c}}\\ &= \left(\frac{1}{\sqrt{1+|y|^2}}\right)_{,\mathfrak{ab}} - \frac{1}{1+|y|^2}\left[-\delta^{\mathfrak{c}}_{\mathfrak{b}}y_{\mathfrak{a}} - \delta^{\mathfrak{c}}_{\mathfrak{a}}y_{\mathfrak{b}}\right]\left(\frac{1}{\sqrt{1+|y|^2}}\right)_{,\mathfrak{c}}\\ &= \frac{-\delta_{\mathfrak{ab}}}{(\sqrt{1+|y|^2})^3} + \frac{3y_{\mathfrak{a}}y_{\mathfrak{b}}}{(\sqrt{1+|y|^2})^5}\\ &\quad + \frac{1}{1+|y|^2}\left[\delta^{\mathfrak{c}}_{\mathfrak{b}}y_{\mathfrak{a}} + \delta^{\mathfrak{c}}_{\mathfrak{a}}y_{\mathfrak{b}}\right]\frac{-y_{\mathfrak{c}}}{(\sqrt{1+|y|^2})^3}\\ &= -\frac{\delta_{\mathfrak{ab}}}{(\sqrt{1+|y|^2})^3} + \frac{y_{\mathfrak{a}}y_{\mathfrak{b}}}{(\sqrt{1+|y|^2})^5}.\end{aligned}$$

Comparing with the metric (5.6) we have

$$\begin{aligned}\left(\bar\eta^{n+1}(y)\right)_{;\mathfrak{ab}} &= -\frac{\delta_{\mathfrak{ab}}}{(\sqrt{1+|y|^2})^3} + \frac{y_{\mathfrak{a}}y_{\mathfrak{b}}}{(\sqrt{1+|y|^2})^5}\\ &= -\frac{1}{(\sqrt{1+|y|^2})^3}\left(\delta_{\mathfrak{ab}} - \frac{y_{\mathfrak{a}}y_{\mathfrak{b}}}{1+|y|^2}\right)\\ &= -\frac{\sigma_{\mathfrak{ab}}}{\sqrt{1+|y|^2}}.\end{aligned}$$

Using $z^{n+1} \ge \hat c$ for all $z = (z^1,\dots,z^{n+1}) \in \hat\Omega$, which is equivalent to $1 \ge \hat c\cdot\sqrt{1+|y|^2}$ for all $y \in U$ in the coordinates induced by $\bar\eta$, and $0 < e^{-\bar\varphi(x,t)} \le 1$, for all $x \in \Omega$, it follows eventually

$$\begin{aligned}\left(f(\bar\eta,t)\right)_{;\mathfrak{ab}} &= e^{-\bar\varphi(\bar\eta,t)}\Big(-\left(\varphi(\bar\eta,t)\right)_{;\mathfrak{ab}} + \left(\varphi(\bar\eta,t)\right)_{;\mathfrak{a}}\left(\varphi(\bar\eta,t)\right)_{;\mathfrak{b}}\Big) + \frac{1}{\hat c}\cdot\frac{\sigma_{\mathfrak{ab}}}{\sqrt{1+|y|^2}}\\ &\ge e^{-\bar\varphi(\bar\eta,t)}\Big(-\left(\varphi(\bar\eta,t)\right)_{;\mathfrak{ab}} + \left(\varphi(\bar\eta,t)\right)_{;\mathfrak{a}}\left(\varphi(\bar\eta,t)\right)_{;\mathfrak{b}}\Big) + \sigma_{\mathfrak{ab}}\\ &\ge e^{-\bar\varphi(\bar\eta,t)}\Big(\sigma_{\mathfrak{ab}} - \left(\varphi(\bar\eta,t)\right)_{;\mathfrak{ab}} + \left(\varphi(\bar\eta,t)\right)_{;\mathfrak{a}}\left(\varphi(\bar\eta,t)\right)_{;\mathfrak{b}}\Big) > 0,\end{aligned}$$

in the sense of matrices.

Thus we could construct a function f, which possess the needed properties to obtain, applying Lemma 5.2, $\tilde F_{;\mathfrak{ab}} > 0$ on the set $\bar U$ for the corresponding map $\tilde F$.

We now address the question of how the normal vector to $\partial\Omega$ transforms under the inverse of the diffeomorphism $\bar{\eta}$. In particular how does the boundary condition translate to $\mathbb{R}^n$?

Like in the proof of Lemma 2.4 we consider the boundary condition,

$$\langle \bar{\nu}(X(x,t)), \nu(x,t)\rangle = \langle \bar{\nu}(\iota(x)), \nu(x,t)\rangle = 0,$$

at a point (x,t), with $x = \iota^{-1}(\bar{\eta}(y)) \in \Omega$.

Since $\bar{\eta}$ is a diffeomorphism there is a nowhere vanishing vector field $\gamma = (\gamma^{\mathfrak{a}})_{1\le \mathfrak{a}\le n}$ on ∂U such that

$$\hat{\bar{\nu}}^{\alpha}(\bar{\eta}(y)) = \gamma^{\mathfrak{a}}(y)\bar{\eta}^{\alpha}_{,\mathfrak{a}}(y),$$

where $\bar{\nu} = (\hat{\bar{\nu}}^{\alpha})_{1\le\alpha\le n+1}$, and then

$$0 = \langle \gamma^{\mathfrak{a}}(y)\bar{\eta}_{,\mathfrak{a}}(y), \nu'(y,t)\rangle$$

with $\nu'(y,t)$ being the normal vector in these coordinates, given by

$$\nu'(y,t) = \frac{1}{\sqrt{1+|D\varphi\left(\iota^{-1}(\bar{\eta}(y)),t\right)|^2}}\Big(\bar{\eta}(y) - \sigma^{\mathfrak{ab}}(y)\left[\varphi\left(\iota^{-1}(\bar{\eta}(y)),t\right)\right]_{,\mathfrak{a}}\bar{\eta}_{,\mathfrak{b}}(y)\Big).$$

We now compute the inner product, inserting the expression of the normal vector. We obtain

$$\begin{aligned}
\langle \gamma^{\mathfrak{a}}(y)\bar{\eta}_{,\mathfrak{a}}(y), \nu'(y,t)\rangle =& \gamma^{\mathfrak{a}}(y)\left\langle \bar{\eta}_{,\mathfrak{a}}(y), \frac{\bar{\eta}(y) - \sigma^{\mathfrak{bc}}(y)\left[\varphi\left(\iota^{-1}(\bar{\eta}(y)),t\right)\right]_{,\mathfrak{b}}\bar{\eta}_{,\mathfrak{c}}(y)}{\sqrt{1+|D\varphi\left(\iota^{-1}(\bar{\eta}(y)),t\right)|^2}}\right\rangle \\
=& \frac{\gamma^{\mathfrak{a}}(y)}{\sqrt{1+|D\varphi\left(\iota^{-1}(\bar{\eta}(y)),t\right)|^2}} \\
&\cdot \Big(\langle \bar{\eta}_{,\mathfrak{a}}(y), \bar{\eta}(y)\rangle \\
&\quad - \Big\langle \bar{\eta}_{,\mathfrak{a}}(y), \sigma^{\mathfrak{bc}}(y)\left[\varphi\left(\iota^{-1}(\bar{\eta}(y)),t\right)\right]_{,\mathfrak{b}}\bar{\eta}_{,\mathfrak{c}}(y)\Big\rangle\Big) \\
=& -\gamma^{\mathfrak{a}}(y)\cdot\frac{\sigma^{\mathfrak{bc}}(y)\left[\varphi\left(\iota^{-1}(\bar{\eta}(y)),t\right)\right]_{,\mathfrak{b}}\langle \bar{\eta}_{,\mathfrak{a}}(y), \bar{\eta}_{,\mathfrak{c}}(y)\rangle}{\sqrt{1+|D\varphi\left(\iota^{-1}(\bar{\eta}(y)),t\right)|^2}} \\
=& -\gamma^{\mathfrak{a}}(y)\cdot\frac{\sigma^{\mathfrak{bc}}(y)\left[\varphi\left(\iota^{-1}(\bar{\eta}(y)),t\right)\right]_{,\mathfrak{b}}\sigma_{\mathfrak{ac}}(y)}{\sqrt{1+|D\varphi\left(\iota^{-1}(\bar{\eta}(y)),t\right)|^2}} \\
=& \frac{-\gamma^{\mathfrak{a}}(y)\left[\varphi\left(\iota^{-1}(\bar{\eta}(y)),t\right)\right]_{,\mathfrak{a}}}{\sqrt{1+|D\varphi\left(\iota^{-1}(\bar{\eta}(y)),t\right)|^2}}
\end{aligned}$$

and this implies finally

$$0 = \gamma^{\mathfrak{a}}(y)\left[\varphi\left(\iota^{-1}(\bar{\eta}(y)),t\right)\right]_{,\mathfrak{a}}. \tag{5.7}$$

It follows immediately from the definition,

$$\tilde{F}(y,t) = f(\bar{\eta}(y),t)\cdot\sqrt{1+|y|^2},$$

given in (5.1) and (5.2), that the partial derivatives of this map are

$$\tilde{F}_{,\mathfrak{a}}(y,t) = f(\bar{\eta}(y),t)_{,\mathfrak{a}}\cdot\sqrt{1+|y|^2} + f(\bar{\eta}(y),t)\cdot\frac{y_{\mathfrak{a}}}{\sqrt{1+|y|^2}}$$

and, substituting for $f(\bar{\eta},t)$ in the first summand, we have

$$\begin{aligned}\tilde{F}_{,\mathfrak{a}}(y,t) &= \left(e^{-\bar{\varphi}\left(\iota^{-1}(\bar{\eta}(y)),t\right)} - \frac{1}{\hat{c}}\cdot\bar{\eta}^{n+1}(y)\right)_{,\mathfrak{a}}\cdot\sqrt{1+|y|^2} + f(\bar{\eta}(y),t)\cdot\frac{y_{\mathfrak{a}}}{\sqrt{1+|y|^2}}\\ &= \left(-e^{-\bar{\varphi}\left(\iota^{-1}(\bar{\eta}(y)),t\right)}\cdot\left(\bar{\varphi}\left(\iota^{-1}(\bar{\eta}(y)),t\right)\right)_{,\mathfrak{a}} + \frac{1}{\hat{c}}\cdot\frac{y_{\mathfrak{a}}}{(\sqrt{1+|y|^2})^3}\right)\cdot\sqrt{1+|y|^2}\\ &\quad + f(\bar{\eta}(y),t)\cdot\frac{y_{\mathfrak{a}}}{\sqrt{1+|y|^2}},\end{aligned}$$

for all $\mathfrak{a}=1,\ldots,n$. From (5.7) it follows

$$\begin{aligned}\left\langle D\tilde{F},\gamma\right\rangle &= \frac{1}{\hat{c}}\cdot\frac{\langle y,\gamma\rangle}{1+|y|^2} + \frac{f\left(\bar{\eta}\left(y\right),t\right)}{\sqrt{1+|y|^2}}\cdot\langle y,\gamma\rangle\\ &= \frac{\langle y,\gamma\rangle}{1+|y|^2}\cdot\left(\frac{1}{\hat{c}} + f\left(\bar{\eta}\left(y\right),t\right)\cdot\sqrt{1+|y|^2}\right).\end{aligned}$$

So the absolute value of this scalar product is bounded, as required for the application of Theorem 5.1, because U is a bounded domain. Furthermore γ is nowhere tangential to the normal ν_U to ∂U, since otherwise $\bar{\nu}$ would be somewhere tangential to $\partial\Omega$, since $\bar{\eta}$ is a diffeomorphism, that maps tangent vectors to tangent vectors.

We can now use the Ice-cream cone estimate and hence obtain that $\sup|D\tilde{F}|$ is uniformly bounded in U.

To deduce from this a bound for $D\varphi$, we need first of all to compute what the inverse of $\bar{\eta}$ is:

$$\begin{aligned}&\bar{\eta}^{-1}:\hat{\Omega}\to U,\\ &(z^1,\ldots,z^{n+1})\mapsto\left(\frac{z^1}{z^{n+1}},\ldots,\frac{z^n}{z^{n+1}}\right).\end{aligned}$$

Furthermore its partial derivatives are given by

$$\bar{\eta}^{-1}_{,\alpha}(z) = \frac{\left(\delta^1_\alpha,\ldots,\delta^n_\alpha\right)}{z^{n+1}} - \frac{\left(z^1,\ldots,z^n\right)}{\left(z^{n+1}\right)^2}\cdot\delta^{n+1}_\alpha,$$

whenever $\alpha=1,\ldots,n+1$.

From the definition of

$$\tilde{F}(y,t) = f(\bar{\eta}(y),t)\cdot\sqrt{1+|y|^2}$$

for a fixed time $t\in[0,t^*)$, we have

$$\tilde{F}(\bar{\eta}^{-1}(z),t) = f(\bar{\eta}(\bar{\eta}^{-1}(z)),t)\cdot\sqrt{1+|\bar{\eta}^{-1}(z)|^2} = f(z,t)\cdot\frac{1}{z^{n+1}},$$

substituting $y=\bar{\eta}^{-1}(z)$, where $z\in\hat{\Omega}$. It follows

$$f(z,t) = \tilde{F}(\bar{\eta}^{-1}(z),t)\cdot z^{n+1}$$

and

$$\begin{aligned} f_{,\alpha}(z,t) &= \tilde{F}_{,\mathfrak{a}}(\bar{\eta}^{-1}(z),t)(\bar{\eta}^{-1}(z))^{\mathfrak{a}}_{,\alpha} \cdot z^{n+1} + \tilde{F}(\bar{\eta}^{-1}(z),t) \cdot \delta^{n+1}_{\alpha} \\ &= \tilde{F}_{,\mathfrak{a}}(\bar{\eta}^{-1}(z),t) \left(\delta^{\mathfrak{a}}_{\alpha} - \frac{z^{\mathfrak{a}} \delta^{n+1}_{\alpha}}{z^{n+1}} \right) + \tilde{F}(\bar{\eta}^{-1}(z),t) \delta^{n+1}_{\alpha} \\ &= \tilde{F}_{,\alpha}(\bar{\eta}^{-1}(z),t) - \frac{\delta^{n+1}_{\alpha}}{z^{n+1}} \cdot \tilde{F}_{,\mathfrak{a}}(\bar{\eta}^{-1}(z),t) z^{\mathfrak{a}} + \tilde{F}(\bar{\eta}^{-1}(z),t) \delta^{n+1}_{\alpha}, \end{aligned}$$

for $1 \leq \alpha \leq n+1$. As we have seen, since $\overline{\hat{\Omega}}$ is contained in a cone, there is a constant $\hat{c} < 1$, such that

$$z^{n+1} \geq \hat{c} > 0$$

for all $z = (z^1, \ldots, z^{n+1}) \in \hat{\Omega}$. This implies

$$\begin{aligned} |f_{,\alpha}| &\leq \sup_U |D\tilde{F}| + c \cdot \sqrt{\sum_{\mathfrak{a}=1}^{n} \left[\tilde{F}_{,\mathfrak{a}}(\bar{\eta}^{-1}(z),t) \right]^2} \cdot \sqrt{\sum_{\beta=1}^{n} (z^{\beta})^2} + c \cdot \sup_U |\tilde{F}| \\ &\leq \sup_U |D\tilde{F}| + c \cdot \sup_U |D\tilde{F}| + c \cdot \sup_U |\tilde{F}| \end{aligned}$$

and yields

$$|Df| < c,$$

because of the estimate obtained above and of the boundedness of f and consequently of $\tilde{F}$.

Differentiating f and using this last estimate provides

$$f_{,\alpha}(z,t) = -e^{-\bar{\varphi}(\iota^{-1}(z),t)} \cdot \left[\bar{\varphi}(\iota^{-1}(z),t) \right]_{,\alpha} - \frac{1}{\hat{c}} \cdot \delta^{n+1}_{\alpha}$$

and

$$|\bar{\varphi}_{,i}(y,t)| \leq e^{\bar{\varphi}(y,t)} \left(|f_{,\alpha}(z,t)| + c \right) \leq c \left(|f_{,\alpha}(z,t)| + c \right) \leq c$$

with $y = \iota^{-1}(z)$. The desired estimate follows then from

$$|\varphi_{,i}(x,t)| = \left| \varphi_{,i}(x,t) - (t + \underline{m})_{,i} \right| = |\bar{\varphi}_{,i}(x,t)|$$

for all $x \in \Omega$.

Let us at this point state the principal result of this section.

Lemma 5.3. *Let $c > 0$ and $\Omega \subset \mathrm{S}^n$ be a domain such that $\iota^{n+1}(x) \geq c$ for all $x \in \Omega$ and $\varphi(\cdot,t) : \Omega \to \mathbb{R}$, $t \in [0,t^*)$, be a smooth map, fulfiling $\sigma_{ij} - \varphi_{;ij} + \varphi_{,i}\varphi_{,j} > 0$ in the sense of matrices as well as $D_{\bar{\nu}}\varphi(x,t) = 0$ for all $(x,t) \in \partial\Omega \times [0,t^*)$. Then $|D\varphi|$ is uniformly bounded on Ω.*

We conclude this part by remarking that the C^1-estimate implies an upper bound on $\det(w_{ij})$.

Corollary 5.4. *There exist a constant $c > 0$, such that* $\det(w_{ij}(x,t)) \leq c$ *for all $(x,t) \in \bar{\Omega} \times [0,t^*)$.*

Proof. Since φ is a solution of (1.2) it holds

$$\dot{\varphi} = \left(1 + |D\varphi|^2\right)^{\frac{n+1}{n}} \cdot \frac{\left[\det(\sigma_{ij})\right]^{1/n}}{\left[\det\left(\sigma_{ij} - \varphi_{;ij} + \varphi_{,i}\varphi_{,j}\right)\right]^{1/n}}$$

and this implies

$$\det(w_{ij}) = \det\left(\sigma_{ij} - \varphi_{;ij} + \varphi_{,i}\varphi_{,j}\right) = \left(1 + |D\varphi|^2\right)^{n+1} \cdot \frac{\det(\sigma_{ij})}{\dot{\varphi}^n}.$$

The positive lower bound of Lemma 3.1 together with that of Lemma 5.3 supplies then a constant $c > 0$ which fulfils

$$\det(w_{ij}) \le c \cdot \det(\sigma_{ij}),$$

and hence $\det\left(w_{ij}(x,t)\right) \le c$, for a fixed coordinate system, for all $(x,t) \in \bar{\Omega} \times [0, t^*)$. □

6. C^2-Estimates

We first of all note that $w_{ij} = \sigma_{ij} - \varphi_{;ij} + \varphi_{,i}\varphi_{,j} > 0$, together with the C^1-estimate, implies an upper bound on $\varphi_{;ij}$. We hence only need to control the second covariant derivatives of φ from below.

Tangential-normal C^2-estimates at the boundary. Let $x_0 \in \partial\Omega$ be fixed. We choose a boundary coordinate chart containing x_0, so that $\partial\Omega$ is represented locally as graph ω over its tangent plane at $x_0 = (\hat{x}_0, x_0^n)$ in order that locally $\Omega = \{(\hat{x}, x^n) |\, x^n > \omega(\hat{x})\}$ and $D\omega(\hat{x}_0) = 0$.

We differentiate the boundary condition (see Lemma 2.4)

$$\bar{\nu}^i(\hat{x})\varphi_{,i}(\hat{x}, \omega(\hat{x}), t) = 0,$$

in direction $j < n$,

$$\bar{\nu}^i_{;j}\varphi_{,i} + \bar{\nu}^i\varphi_{;ij} + \bar{\nu}^i\varphi_{;in}\omega_{,j} = 0.$$

Since in the point x_0 it holds $\omega_{,i} = 0$ for all $i = 1, \ldots, n-1$, this becomes there

$$\bar{\nu}^i_{;j}\varphi_{,i} + \bar{\nu}^i\varphi_{;ij} = 0.$$

From the C^1-estimate it follows that $\varphi_{;\bar{\nu}j}$ is bounded.

Double normal C^2-estimates at the boundary. Let $x_0 \in \partial\Omega$ be fixed. Like in the preceding section, we choose a boundary coordinate chart containing x_0, so that $\partial\Omega$ is represented locally as graph ω over its tangent plane at x_0 in order that locally $\Omega = \{(\hat{x}, x^n) |\, x^n > \omega(\hat{x})\}$. Moreover let $\delta > 0$ sufficiently small, that, first of all, the signed (positive if $x \in \Omega$, negative if $x \notin \bar{\Omega}$) distance function $d : \mathrm{S}^n \to \mathbb{R}$ of $\partial\Omega$ is smooth in $\Omega_\delta := \Omega \cap B_\delta(x_0)$ and $\|d\|_{C^3(\Omega_\delta)}$ is bounded. For a map $u' : \bar{\Omega}_\delta \times [0, \infty) \to \mathbb{R}$ we consider the operator

$$Lu' := -\dot{u'} + \frac{\dot{\varphi}}{n} \cdot w^{ij}u'_{;ij} + 2 \cdot \frac{\dot{\varphi}}{n} \cdot \frac{n+1}{1 + |D\varphi|^2} \cdot \sigma^{lk}\varphi_{,l}u'_{;k},$$

in which w^{ij} indicates the inverse of w_{ij} as before.

Defining the map $\vartheta : \Omega_\delta \to \mathbb{R}$,

$$\vartheta(x) := d(x) - \mu d^2(x),$$

where $\mu \gg 1$ denotes a constant to be chosen sufficiently large, we obtain

$$\vartheta_{,i} = d_{,i} - 2\mu d d_{,i}, \qquad \vartheta_{;ij} = d_{;ij} - 2\mu d d_{;ij} - 2\mu d_{,i}d_{,j}$$

and

$$L\vartheta = \frac{\dot{\varphi}}{n}\left[w^{ij}d_{;ij} - 2\mu d w^{ij}d_{;ij} - 2\mu w^{ij}d_{,i}d_{,j} + \frac{2(n+1)}{1+|D\varphi|^2}\cdot\sigma^{lk}\varphi_{,l}(d_{,k} - 2\mu d d_{,k})\right]. \tag{6.1}$$

Throughout the whole C^2-estimates we will make use of the following result.

Lemma 6.1. *Let $A = (a^{ij})$ be a (symmetric) positive semi-definite matrix and $\Lambda = (\Lambda_{ij})$ a bounded matrix (i.e. $\max_{i,j}|\Lambda_{ij}| \leq c < \infty$), then*

$$|a^{ij}\Lambda_{ij}| \leq c(\Lambda)\operatorname{tr} a^{ij}.$$

Proof. Let O be an orthogonal matrix, such that OAO^t is diagonal. Then it holds

$$a^{ij}\Lambda_{ij} = \operatorname{tr}(OAO^tO\Lambda O^t) = \operatorname{tr}(OA\Lambda O^t),$$

since the trace is invariant under change of basis. It follows hence

$$a^{ij}\Lambda_{ij} = \operatorname{tr}(OAO^tO\Lambda O^t) = \sum_i (OAO^t)^{ii}(O\Lambda O^t)_{ii},$$

because OAO^t is diagonal. This yields

$$\begin{aligned}\left|a^{ij}\Lambda_{ij}\right| &\leq \sum_i \left|(OAO^t)^{ii}(O\Lambda O^t)_{ii}\right| \leq \sum_i (OAO^t)^{ii}\cdot n^2\cdot\max_{k,l}|\Lambda_{kl}|\\ &= c(\Lambda)\cdot\sum_i (OAO^t)^{ii} = c(\Lambda)\operatorname{tr} a^{ij},\end{aligned}$$

since OAO^t has nonnegative diagonal entries and the absolute value of the entries of O and O^t is bounded by 1. □

It is a direct consequence of the lemma that

$$-2\mu d w^{ij}d_{;ij} \leq c\mu\delta\operatorname{tr} w^{ij}, \tag{6.2}$$

and of the C^1-estimate that

$$\frac{2(n+1)}{1+|D\varphi|^2}\cdot\sigma^{pk}\varphi_{,p}(d_{,k} - 2\mu d d_{,k}) \leq c(1+\mu\delta), \tag{6.3}$$

for all $(x,t) \in \Omega_\delta \times [0,t^*)$.

We then consider the difference of the first and the third term of (6.1) with the corresponding maps evaluated at x_0, i.e.

$$\frac{\dot{\varphi}}{n}\cdot w^{ij}\cdot\left(d_{;ij}(x) - d_{;ij}(x_0) - 2\mu d_{,i}(x)d_{,j}(x) + 2\mu d_{,i}(x_0)d_{,j}(x_0)\right)$$

and we estimate obtaining

$$w^{ij}\left(d_{;ij}(x) - d_{;ij}(x_0)\right) \leq c\|d\|_{C^3(\Omega_\delta)}\cdot|x-x_0|\cdot\operatorname{tr} w^{ij} \leq c\delta\operatorname{tr} w^{ij} \tag{6.4}$$

respectively

$$\begin{aligned}w^{ij}\left(d_{,i}(x_0)d_{,j}(x_0) - d_{,i}(x)d_{,j}(x)\right) &= w^{ij}\left(d_{,i}(x_0) - d_{,i}(x)\right)\left(d_{,j}(x_0) + d_{,j}(x)\right)\\ &\leq 2c\|d\|_{C^1(\Omega_\delta)}\cdot\|d\|_{C^2(\Omega_\delta)}\cdot|x-x_0|\cdot\operatorname{tr} w^{ij}\\ &\leq c\delta\operatorname{tr} w^{ij}\end{aligned} \tag{6.5}$$

for all $(x,t) \in \Omega_\delta \times [0,t^*)$. It remains therefore to handle with

$$w^{ij} d_{;ij}(x_0) - 2\mu w^{ij} d_{,i}(x_0) d_{,j}(x_0).$$

For this we need to exploit the geometry of $\partial\Omega$ and use that on the sphere $d_{;ij}$ is negative semi-definite (see for instance [2]). After a rotation of the first $n-1$ coordinates and remembering that $\bar{\nu}(x_0) = e_n$, we therefore have

$$d_{;ij}(x_0) = d_{,ij}(x_0) = \begin{pmatrix} -k_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & -k_{n-1} & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix},$$

where there is a constant $\tilde{\varepsilon} \equiv \tilde{\varepsilon}(\partial\Omega) > 0$ such that $-k_i \le -\tilde{\varepsilon}$ for all principal curvatures k_i, $i = 1, \dots, n-1$, of $\partial\Omega$.

In x_0 one consequently gets

$$w^{ij} d_{;ij} = -k_1 w^{11} - k_2 w^{22} - \dots - k_{n-1} w^{n-1n-1} \le -\tilde{\varepsilon}(w^{11} + w^{22} + \dots + w^{n-1n-1})$$

and

$$\mu w^{ij} d_{,i} d_{,j} = \mu w^{nn},$$

since the differential of the distance function coincide with the normal vector: $Dd = \bar{\nu} = e_n$. Thus in x_0 it holds

$$w^{ij} d_{;ij} - 2\mu w^{ij} d_{,i} d_{,j} \le -\tilde{\varepsilon}(w^{11} + w^{22} + \dots + w^{n-1n-1}) - 2\mu w^{nn}. \tag{6.6}$$

On one side it is now possible to choose μ, so that $2\mu \ge \tilde{\varepsilon}$, to get

$$\frac{1}{2}\left(w^{ij} d_{;ij} - 2\mu w^{ij} d_{,i} d_{,j}\right) \le -\frac{\tilde{\varepsilon}}{2}\left(w^{11} + w^{22} + \dots + w^{n-1n-1} + w^{nn}\right) = -\frac{\tilde{\varepsilon}}{2} \cdot \operatorname{tr} w^{ij}. \tag{6.7}$$

On the other side the inequality of arithmetic and geometric means,

$$\frac{\tilde{\varepsilon}(w^{11} + w^{22} + \dots + w^{n-1n-1}) + 2\mu w^{nn}}{n} \ge \tilde{\varepsilon}^{(n-1)/n} (2\mu)^{1/n} \left(\prod_{i=1}^n w^{ii}\right)^{1/n},$$

yields for the terms of (6.6), which we didn't already considered above,

$$\frac{1}{2}\left(w^{ij} d_{;ij} - 2\mu w^{ij} d_{,i} d_{,j}\right) \le -\frac{n}{2} \cdot \tilde{\varepsilon}^{(n-1)/n} (2\mu)^{1/n} \left(\prod_{i=1}^n w^{ii}\right)^{1/n}$$

and from the Hadamard's inequality for positive definite matrices (a proof can be found in [5]),

$$\det(w^{ij}) \le \prod_{i=1}^n w^{ii},$$

it follows

$$\frac{1}{2}\left(w^{ij} d_{;ij} - 2\mu w^{ij} d_{,i} d_{,j}\right) \le -n \left(\frac{\tilde{\varepsilon}}{2}\right)^{(n-1)/n} \mu^{1/n} \left(\det w^{ij}\right)^{1/n}.$$

Corollary 5.4 implies that there is a positive constant c such that $\det(w^{ij}) = \det^{-1}(w_{ij}) \ge \frac{1}{c} > 0$, it follows hence

$$\frac{1}{2}\left(w^{ij} d_{;ij} - 2\mu w^{ij} d_{,i} d_{,j}\right) \le -\frac{n}{c} \cdot \left(\frac{\tilde{\varepsilon}}{2}\right)^{(n-1)/n} \mu^{1/n}.$$

Let now $\varepsilon' > 0$ be a small constant, fulfiling $\varepsilon' < \frac{\tilde{\varepsilon}}{2}$ and $\varepsilon := \frac{\min_{x\in\bar{\Omega}} \dot{\varphi}(x,0)}{n} \cdot \varepsilon' > 0$. Fixing μ so large that $\mu^{1/n} \geq \frac{c + \frac{\tilde{\varepsilon}}{2} - \varepsilon'}{\frac{n}{c} \cdot \left(\frac{\tilde{\varepsilon}}{2}\right)^{(n-1)/n}}$ and then δ so small that $c\mu\delta \leq \frac{\tilde{\varepsilon}}{2} - \varepsilon'$ for appropriate constants c, we obtain

$$\frac{\dot{\varphi}}{n} \cdot \left[\frac{1}{2}\left(w^{ij} d_{;ij} - 2\mu w^{ij} d_{,i} d_{,j}\right) + \frac{2(n+1)}{1+|D\varphi|^2} \cdot \sigma^{pk} \varphi_{,p} (d_{,k} - 2\mu d d_{,k})\right] \leq 0,$$

for all $x \in \Omega_\delta$, from (6.3), and

$$\frac{\dot{\varphi}}{n} \cdot \left[\frac{1}{2}\left(w^{ij} d_{;ij} - 2\mu w^{ij} d_{,i} d_{,j}\right) - 2\mu d w^{ij} d_{;ij}\right] \leq -\varepsilon \operatorname{tr} w^{ij},$$

for all $x \in \Omega_\delta$, from (6.2), (6.4), (6.5) and (6.7). We can thus state the existence of a constant $\varepsilon > 0$ so that $L\vartheta \leq -\varepsilon \operatorname{tr} w^{ij}$ in Ω_δ.

We have furthermore $\vartheta = 0$ on $\partial\Omega$ and $\vartheta = (1-\mu\delta)\delta \geq 0$ in $\Omega \cap \partial B_\delta(x_0)$, if we choose $\delta \leq 1/\mu$, if necessary. This implies

$$\begin{cases} L\vartheta \leq -\varepsilon \operatorname{tr} w^{ij} & \text{in } \Omega_\delta \\ \vartheta \geq 0 & \text{on } \partial\Omega_\delta. \end{cases}$$

We now extend the normal vector $\bar{\nu}$ to $\partial\Omega$ smoothly to the interior of Ω and define for a fixed time $0 \leq t < t^*$ the barriers $\Theta(\cdot, t) : \Omega_\delta \to \mathbb{R}$,

$$\Theta(x,t) := A\vartheta(x) + B\sigma_{ij}(x^i - x_0^i)(x^j - x_0^j) - \bar{\nu}^l(\iota(x))\varphi_{,l}(x,t),$$

with $A, B \gg 1$. By choosing B sufficiently large we can achieve

$$\Theta \geq 0$$

on $\partial\Omega_\delta$, since the last term is bounded because of the C^1-estimate. In Ω_δ it holds

$$\sigma^{lk}\varphi_{,l} \cdot \left(\sigma_{ij}(x^i - x_0^i)(x^j - x_0^j)\right)_{,k} = \sigma^{lk}\varphi_{,l} \cdot 2\sigma_{ik}(x^i - x_0^i) \leq c$$

as well as

$$w^{kl}\left(\sigma_{ij}(x^i - x_0^i)(x^j - x_0^j)\right)_{;kl} = 2w^{kl}\sigma_{kl} - w^{kl} \cdot {}^{\sigma}\Gamma^m_{kl} \cdot 2\sigma_{im}(x^i - x_0^i) \leq c \operatorname{tr} w^{ij}$$

and therefore

$$L\left(\sigma_{ij}(x^i - x_0^i)(x^j - x_0^j)\right) \leq c \operatorname{tr} w^{ij} + c.$$

Remark 6.2. Here and in the following the trace is taken with respect to the metric σ_{ij}.

We then calculate

$$\begin{aligned} L(\bar{\nu}^l\varphi_{,l}) = & -\bar{\nu}^l\dot{\varphi}_{,l} + \frac{\dot{\varphi}}{n} \cdot w^{ij}(\bar{\nu}^l\varphi_{;lij} + \bar{\nu}^l_{;i}\varphi_{;lj} + \bar{\nu}^l_{;j}\varphi_{;li} + \bar{\nu}^l_{;ij}\varphi_{,l}) \\ & + \frac{2\dot{\varphi}}{n} \cdot \frac{n+1}{1+|D\varphi|^2} \cdot \sigma^{pk}\varphi_{,p}\left(\bar{\nu}^l_{;k}\varphi_{,l} + \bar{\nu}^l\varphi_{;lk}\right) \\ = & \,\bar{\nu}^l\left[-\dot{\varphi}_{,l} + \frac{\dot{\varphi}}{n} \cdot w^{ij}\varphi_{;lij} + \frac{\dot{\varphi}}{n} \cdot \frac{2(n+1)}{1+|D\varphi|^2} \cdot \sigma^{pk}\varphi_{;lk}\varphi_{,p}\right] \\ & + \frac{2\dot{\varphi}}{n} \cdot \bar{\nu}^l_{;i} w^{ij}\varphi_{;lj} + \frac{\dot{\varphi}}{n} \cdot w^{ij}\bar{\nu}^l_{;ij}\varphi_{,l} + \frac{2\dot{\varphi}}{n} \cdot \frac{n+1}{1+|D\varphi|^2} \cdot \sigma^{pk}\bar{\nu}^l_{;k}\varphi_{,l}\varphi_{,p}. \end{aligned} \tag{6.8}$$

At this point we make use of the partial differential equation (1.2) to get rid of the third derivatives of φ. Interchanging covariant derivatives

$$\varphi_{;ijl} = \varphi_{;ilj} + R^m{}_{ijl}\varphi_{,m} = \varphi_{;lij} + R^m{}_{ijl}\varphi_{,m}, \tag{6.9}$$

by means of the Riemannian curvature tensor of the sphere $R^m{}_{ijl}$, and using

$$w^{ij}\varphi_{;lj} = w^{ij}\left(\sigma_{lj} - w_{lj} + \varphi_{,l}\varphi_{,j}\right) = w^{ij}\sigma_{lj} - \delta^i_l + w^{ij}\varphi_{,l}\varphi_{,j}$$

it follows

$$\begin{aligned} w^{pq}w_{pq;k} &= w^{pq}\left(\sigma_{pq;k} - \varphi_{;pqk} + \varphi_{;pk}\varphi_{,q} + \varphi_{,p}\varphi_{;qk}\right) \\ &= -\,w^{pq}\varphi_{;pqk} + w^{pq}\varphi_{;pk}\varphi_{,q} + w^{pq}\varphi_{,p}\varphi_{;qk} \\ &= -\,w^{pq}\left(\varphi_{;kpq} + R^s{}_{pqk}\varphi_{,s}\right) \\ &\quad + \left(w^{pq}\sigma_{pk} - \delta^q_k + w^{pq}\varphi_{,p}\varphi_{,k}\right)\varphi_{,q} + \left(w^{pq}\sigma_{qk} - \delta^p_k + w^{pq}\varphi_{,q}\varphi_{,k}\right)\varphi_{,p} \\ &= -\,w^{pq}\varphi_{;kpq} - w^{pq}R^s{}_{pqk}\varphi_{,s} + 2w^{pq}\sigma_{pk}\varphi_{,q} - 2\varphi_{,k} + 2w^{pq}\varphi_{,p}\varphi_{,q}\varphi_{,k}. \end{aligned}$$

Differentiating the partial differential equation (1.2), as previously mentioned,

$$\dot{\varphi}_{;l} = \frac{1}{n}\cdot\dot{\varphi}\left[(n+1)\cdot\frac{2\sigma^{pq}\varphi_{;pl}\varphi_{,q}}{(1+|D\varphi|^2)} - w^{pq}w_{pq;l}\right],$$

we then obtain

$$\begin{aligned} \dot{\varphi}_{,l} = \frac{\dot{\varphi}}{n}\cdot\Bigg[&\frac{2(n+1)}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{;pl}\varphi_{,q} + w^{pq}\varphi_{;lpq} \\ &+w^{pq}R^s{}_{pql}\varphi_{,s} - 2w^{ij}\sigma_{il}\varphi_{,j} + 2\varphi_{,l} - 2w^{pq}\varphi_{,p}\varphi_{,q}\varphi_{,l}\Bigg] \end{aligned}$$

and, as an immediate consequence,

$$\begin{aligned} \dot{\varphi}_{,l} - \frac{\dot{\varphi}}{n}&\left(w^{pq}\varphi_{;lpq} + \frac{2(n+1)}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{;pl}\varphi_{,q}\right) \\ &= \frac{\dot{\varphi}}{n}\cdot\left(w^{pq}R^s{}_{pql}\varphi_{,s} - 2w^{ij}\sigma_{il}\varphi_{,j} + 2\varphi_{,l} - 2w^{pq}\varphi_{,p}\varphi_{,q}\varphi_{,l}\right). \end{aligned} \tag{6.10}$$

Let us now remark that, according to the Gauß equation, the unit sphere S^n as an hypersurface in $\mathbb{R}^{n+1}$ possess the (purely covariant) Riemannian curvature tensor

$$R_{ijkl} = h^{\mathrm{S}^n}_{ik}h^{\mathrm{S}^n}_{jl} - h^{\mathrm{S}^n}_{il}h^{\mathrm{S}^n}_{jk} = \sigma_{ik}\sigma_{jl} - \sigma_{il}\sigma_{jk}, \tag{6.11}$$

since its first and second fundamental form coincide. Therefore inserting the formula

$$R^m{}_{jkl} = \sigma^{mi}R_{ijkl} = \delta^m_k\sigma_{jl} - \delta^m_l\sigma_{jk} \tag{6.12}$$

of the Riemannian curvature tensor of the sphere in the first term on the right side of (6.10),

$$w^{jk}R^m{}_{jkl}\varphi_{,m} = w^{jk}(\delta^m_k\sigma_{jl} - \delta^m_l\sigma_{jk})\varphi_{,m} = w^{jm}\sigma_{jl}\varphi_{,m} - w^{jk}\sigma_{jk}\varphi_{,l},$$

we get

$$\begin{aligned} \dot{\varphi}_{,l} - \frac{\dot{\varphi}}{n}&\left(w^{pq}\varphi_{;lpq} + \frac{2(n+1)}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{;pl}\varphi_{,q}\right) \\ &= \frac{\dot{\varphi}}{n}\cdot\left(w^{jm}\sigma_{jl}\varphi_{,m} - w^{jk}\sigma_{jk}\varphi_{,l} - 2w^{ij}\sigma_{il}\varphi_{,j} + 2\varphi_{,l} - 2w^{ij}\varphi_{,i}\varphi_{,j}\varphi_{,l}\right) \\ &= \frac{\dot{\varphi}}{n}\cdot\left(-\varphi_{,l}\cdot\operatorname{tr}w^{ij} - w^{ij}\sigma_{il}\varphi_{,j} + 2\varphi_{,l} - 2w^{ij}\varphi_{,i}\varphi_{,j}\varphi_{,l}\right). \end{aligned} \tag{6.13}$$

Eliminating the derivatives of third order in (6.8), this provides

$$\begin{aligned}
L(\bar\nu^l\varphi_{,l}) = {}& \bar\nu^l\left[-\dot\varphi_{,l}+\frac{\dot\varphi}{n}\cdot w^{ij}\varphi_{;lij}+\frac{\dot\varphi}{n}\cdot\frac{2(n+1)}{1+|D\varphi|^2}\cdot\sigma^{pk}\varphi_{;lk}\varphi_{,p}\right]\\
&+\frac{2\dot\varphi}{n}\cdot\bar\nu^l_{;i}w^{ij}\varphi_{;lj}+\frac{\dot\varphi}{n}\cdot w^{ij}\bar\nu^l_{;ij}\varphi_{,l}+\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pk}\bar\nu^l_{;k}\varphi_{,l}\varphi_{,p}\\
= {}& \bar\nu^l\left[-\frac{\dot\varphi}{n}\cdot\left(-\varphi_{,l}\cdot\operatorname{tr}w^{ij}-w^{ij}\sigma_{il}\varphi_{,j}+2\varphi_{,l}-2w^{pq}\varphi_{,p}\varphi_{,q}\varphi_{,l}\right)\right]\\
&+\frac{2\dot\varphi}{n}\cdot\bar\nu^l_{;i}(w^{ij}\sigma_{lj}-\delta^i_l+w^{ij}\varphi_{,l}\varphi_{,j})+\frac{\dot\varphi}{n}\cdot w^{ij}\bar\nu^l_{;ij}\varphi_{,l}\\
&+\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pk}\bar\nu^l_{;k}\varphi_{,l}\varphi_{,p}\\
= {}& \frac{\dot\varphi}{n}\cdot\Big[\bar\nu^l\varphi_{,l}\cdot\operatorname{tr}w^{ij}+\bar\nu^l w^{ij}\sigma_{il}\varphi_{,j}-2\bar\nu^l\varphi_{,l}\left(1-w^{pq}\varphi_{,p}\varphi_{,q}\right)\\
&\quad+2\bar\nu^l_{;i}w^{ij}\sigma_{lj}-2\bar\nu^l_{;l}+2\bar\nu^l_{;i}w^{ij}\varphi_{,l}\varphi_{,j}+w^{ij}\bar\nu^l_{;ij}\varphi_{,l}\\
&\quad+\frac{2(n+1)}{1+|D\varphi|^2}\cdot\sigma^{pk}\bar\nu^l_{;k}\varphi_{,l}\varphi_{,p}\Big].
\end{aligned}\tag{6.14}$$

and therefore

$$\begin{aligned}
\left|L(\bar\nu^l\varphi_{,l})\right| \le{}& c\operatorname{tr}w^{ij}+c\operatorname{tr}w^{ij}+c(1+c\operatorname{tr}w^{ij})\\
&+c\operatorname{tr}w^{ij}+c+c\operatorname{tr}w^{ij}+c\operatorname{tr}w^{ij}\\
&+c\\
\le{}& c\left(1+\operatorname{tr}w^{ij}\right).
\end{aligned}\tag{6.15}$$

We hence obtain

$$\begin{aligned}
L\Theta &= AL\vartheta+BL|x-x_0|^2-L(\bar\nu^l\varphi_{,l})\\
&\le -A\varepsilon\operatorname{tr}w^{ij}+Bc(1+\operatorname{tr}w^{ij})+c\left(1+\operatorname{tr}w^{ij}\right)\\
&\le c(1+B)+(-A\varepsilon+c(B+1))\operatorname{tr}w^{ij}.
\end{aligned}$$

Corollary 5.4 and the inequality of arithmetic and geometric means provide a strictly positive lower bound for the trace of w^{ij}:

$$0<c\le\det{}^{1/n}(w^{ij})\le\frac{1}{n}\operatorname{tr}w^{ij}.$$

Thus it is possible to pick A sufficiently large, namely $A\ge\frac{c(1+B)}{\varepsilon}\cdot\left(1+\frac{1}{\operatorname{tr}w^{ij}}\right)$, to get $L\Theta\le 0$ in Ω_δ. Applying the maximum principle, it follows from

$$\begin{cases} L\Theta\le 0 & \text{in } \Omega_\delta\times[0,t^*),\\ \Theta\ge 0 & \text{on } \partial\Omega_\delta\times[0,t^*)\end{cases}$$

that Θ is nonnegative in Ω_δ. We have further that $\Theta(x_0,t)=0$ is a minimum. This implies

$$\Theta_{,i}(x_0,t)\bar\nu^i(\iota(x_0))\le 0$$

for the outward normal $\bar{\nu}$. We obtain

$$A\vartheta_{,i}(x_0)\bar{\nu}^i(\iota(x_0)) + B\left(\left(\sigma_{kl}(x^k - x_0^k)(x^l - x_0^l)\right)_{,i}\right)_{|x=x_0}\bar{\nu}^i(\iota(x_0)) - \left(\left(\bar{\nu}^l(\iota(x))\varphi_{,l}(x,t)\right)_{;i}\right)_{|x=x_0}\bar{\nu}^i(\iota(x_0)) \leq 0$$

and, because of $\left(\left(\sigma_{kl}(x^k - x_0^k)(x^l - x_0^l)\right)_{,i}\right)_{|x=x_0} = \left(2\sigma_{il}\left(x^l - x_0^l\right)\right)_{|x=x_0} = 0$,

$$A\vartheta_{,i}(x_0)\bar{\nu}^i(\iota(x_0)) - \bar{\nu}^l_{;i}(\iota(x_0))\bar{\nu}^i(\iota(x_0))\varphi_{,l}(x_0,t) - \varphi_{,\bar{\nu}\bar{\nu}}(x_0,t) \leq 0.$$

Finally, since $\vartheta_{,i}(x_0)$ and the first derivatives of φ are bounded:

$$-\varphi_{,\bar{\nu}\bar{\nu}}(x_0,t) \leq c.$$

Interior C^2-estimates. Assume $0 < t' < t^*$ to be fixed. Let $\bar{\nu}$ be a smooth extension to Ω of the outward unit normal vector field to $\partial\Omega$. Define $v' : \bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^n \times [0,t'] \to \mathbb{R}$ by

$$v'(x,\xi_1,\xi_2,t) := -\bar{\nu}^l_{;k}\varphi_{,l}\left(\langle\xi_1,\bar{\nu}\rangle_\sigma \xi_2'^k + \langle\xi_2,\bar{\nu}\rangle_\sigma \xi_1'^k\right), \tag{6.16}$$

where

$$\xi_i' := \xi_i - \langle\xi_i,\bar{\nu}\rangle_\sigma \bar{\nu}$$

indicates the tangential component of the vector ξ_i, with $i = 1,2$, and where we write $\langle\cdot,\cdot\rangle_\sigma$ for the inner product induced by σ, that means for instance $\langle\xi,\bar{\nu}\rangle_\sigma = \sigma_{ij}\xi^i\bar{\nu}^j$. Moreover let $v'_{ij} : \bar{\Omega} \times [0,t'] \to \mathbb{R}$, with $1 \leq i,j \leq n$, represent the component functions

$$v'_{ij}(x,t) := -\bar{\nu}^q_{;p}\varphi_{,q}\left[\sigma_{ki}\bar{\nu}^k\left(\delta^p_j - \sigma_{lj}\bar{\nu}^l\bar{\nu}^p\right) + \sigma_{kj}\bar{\nu}^k\left(\delta^p_i - \sigma_{li}\bar{\nu}^l\bar{\nu}^p\right)\right],$$

of the symmetric 2-tensor field v'.

For $\lambda > 0$ to be chosen sufficiently large we maximize the map $v : \bar{\Omega} \times \mathrm{S}^n \times [0,t'] \to \mathbb{R}$, given by

$$v(x,\xi(x),t) := \log\left(\frac{\left[w_{ij}(x,t) + v'_{ij}(x,t)\right]\xi^i(x)\xi^j(x)}{\sigma_{ij}(x)\xi^i(x)\xi^j(x)} + C\right) + \frac{1}{2}\lambda|D\varphi(x,t)|^2,$$

over all (x,ξ,t). The constant C is chosen such that the argument of the logarithm is greater than or equal to 1 and therefore positive. This is possible since $w_{ij}\xi^i\xi^j$ and $\sigma_{ij}\xi^i\xi^j$ are positive, ν and ξ are smooth vectors fields and the first derivatives of φ are bounded, as we have seen.

The continuity of the map v implies that it has to achieve its maximum on the compact set $\bar{\Omega} \times \mathrm{S}^n \times [0,t']$. We initially suppose that this is attained in an interior point $(x_0,\xi_0,t_0) \in \Omega \times \mathrm{S}^n \times (0,t']$ and, following an idea of [3], we choose Riemannian normal coordinates around x_0, such that

$$\sigma_{ij}(x_0) = \delta_{ij} \qquad \text{and} \qquad {}^\sigma\Gamma^k_{ij}(x_0) = 0,$$

whenever $1 \leq i,j,k \leq n$. For the sake of simplicity we further rotate the coordinate system at (x_0,t_0) in order that

$$w_{11} + v'_{11} = \sup_{\xi \in \mathrm{S}^n} \frac{\left(w_{ij} + v'_{ij}\right)\xi^i\xi^j}{\sigma_{ij}\xi^i\xi^j} \equiv \frac{\left(w_{ij} + v'_{ij}\right)\xi_0^i\xi_0^j}{\sigma_{ij}\xi_0^i\xi_0^j},$$

we indicate by ξ_1 the constant vector field given by $\xi_1(x) := (1, 0, \dots, 0)$ for all $x \in \Omega$, in these coordinates, so that $\xi_1(x_0) = \xi_0$, and we hence define

$$W(x,t) := v(x, \xi_1, t) = \log\left(w_{11}(x,t) + v'(x, \xi_1, \xi_1, t) + C\right) + \frac{1}{2}\lambda |D\varphi(x,t)|^2.$$

For a map $u' : \bar{\Omega} \times [0, \infty) \to \mathbb{R}$ we will need to consider the following operator

$$Pu' := \dot{u}' - \frac{\dot{\varphi}}{n} \cdot w^{ij} u'_{;ij} - 2 \cdot \frac{\dot{\varphi}}{n} \left(\frac{n+1}{1 + |D\varphi|^2} \cdot \sigma^{pk} - w^{pk} \right) \varphi_{,p} u'_{;k},$$

and compute $PW(x_0, t_0)$, which we will now show to be nonnegative as v is maximal at (x_0, ξ_0, t_0). From this inequality we then deduce an upper bound on $w_{11}(x_0, t_0)$ as well as on $v(x_0, \xi_0, t_0)$ and consequently on both maps $v(x, \xi, t)$ and $w_{ij}(x, t)$.

To prove that there is no loss of generality working with W instead of v, we claim that the covariant (at least up to the second order) and the first time derivatives of

$$\left(\frac{\left(w_{ij} + v'_{ij}\right) \xi^i \xi^j}{\sigma_{ij} \xi^i \xi^j} \right)_{|\xi = \xi_1} \qquad \text{and} \qquad w_{11} + v'_{11}$$

coincide at (x_0, t_0) (in normal coordinates). Taking into account that the differential of ξ_1 vanish at x_0 since it is a constant vector field and the partial and covariant derivatives coincide there, it holds namely for the first derivatives

$$\begin{aligned} \left(\frac{\left(w_{ij} + v'_{ij}\right) \xi^i \xi^j}{\sigma_{ij} \xi^i \xi^j} \right)_{;k} = & \frac{\left(w_{ij;k} + v'_{ij;k}\right) \xi^i \xi^j}{\sigma_{ij} \xi^i \xi^j} + \frac{2 \left(w_{ij} + v'_{ij}\right) \xi^i_{;k} \xi^j}{\sigma_{ij} \xi^i \xi^j} \\ & - \frac{\left(\left(w_{ij} + v'_{ij}\right) \xi^i \xi^j\right) \left(2 \sigma_{pq} \xi^p_{;k} \xi^q\right)}{\left(\sigma_{ij} \xi^i \xi^j\right)^2} \end{aligned} \tag{6.17}$$

and, hence in (x_0, t_0),

$$\left(\frac{\left(w_{ij} + v'_{ij}\right) \xi^i \xi^j}{\sigma_{ij} \xi^i \xi^j} \right)_{;k|_{\xi = \xi_1}} = w_{11;k} + v'_{11;k}.$$

Analogously for the time derivative we have

$$\begin{aligned} \frac{d}{dt} \left(\frac{\left(w_{ij} + v'_{ij}\right) \xi^i \xi^j}{\sigma_{ij} \xi^i \xi^j} \right) = & \frac{\frac{d}{dt} \left(w_{ij} + v'_{ij}\right) \xi^i \xi^j}{\sigma_{ij} \xi^i \xi^j} + \frac{2 \left(w_{ij} + v'_{ij}\right) \left(\frac{d}{dt} \xi^i\right) \xi^j}{\sigma_{ij} \xi^i \xi^j} \\ & - \frac{\left(\left(w_{ij} + v'_{ij}\right) \xi^i \xi^j\right) \left(2 \sigma_{pq} \left(\frac{d}{dt} \xi^p\right) \xi^q\right)}{\left(\sigma_{ij} \xi^i \xi^j\right)^2}, \end{aligned}$$

and, in (x_0, t_0) if $\xi = \xi_1$,

$$\frac{d}{dt} \left(\frac{\left(w_{ij} + v'_{ij}\right) \xi^i \xi^j}{\sigma_{ij} \xi^i \xi^j} \right) = \frac{d}{dt} \left(w_{11} + v'_{11}\right),$$

since ξ_1 is clearly constant in time. Differentiating the first term of the first derivatives (6.17) one gets

$$\left(\frac{\left(w_{ij;k}+v'_{ij;k}\right)\xi^i\xi^j}{\sigma_{ij}\xi^i\xi^j}\right)_{;l}=\frac{\left(w_{ij;kl}+v'_{ij;kl}\right)\xi^i\xi^j}{\sigma_{ij}\xi^i\xi^j}+\frac{2\left(w_{ij;k}+v'_{ij;k}\right)\xi^i_{;l}\xi^j}{\sigma_{ij}\xi^i\xi^j}$$
$$-\frac{\left(\left(w_{ij;k}+v'_{ij;k}\right)\xi^i\xi^j\right)\left(2\sigma_{pq}\xi^p_{;l}\xi^q\right)}{\left(\sigma_{ij}\xi^i\xi^j\right)^2},$$

and, doing the same for the second term of (6.17), it follows

$$\left(\frac{2\left(w_{ij}+v'_{ij}\right)\xi^i_{;k}\xi^j}{\sigma_{ij}\xi^i\xi^j}\right)_{;l}=\frac{2\left(w_{ij;l}+v'_{ij;l}\right)\xi^i_{;k}\xi^j}{\sigma_{ij}\xi^i\xi^j}+\frac{2\left(w_{ij}+v'_{ij}\right)\xi^i_{;kl}\xi^j}{\sigma_{ij}\xi^i\xi^j}$$
$$+\frac{2\left(w_{ij}+v'_{ij}\right)\xi^i_{;k}\xi^j_{;l}}{\sigma_{ij}\xi^i\xi^j}-\frac{\left(2\left(w_{ij}+v'_{ij}\right)\xi^i_{;k}\xi^j\right)\left(2\sigma_{pq}\xi^p_{;l}\xi^q\right)}{\left(\sigma_{ij}\xi^i\xi^j\right)^2}$$

and for the last

$$\left(\frac{\left(\left(w_{ij}+v'_{ij}\right)\xi^i\xi^j\right)\left(2\sigma_{pq}\xi^p_{;k}\xi^q\right)}{\left(\sigma_{ij}\xi^i\xi^j\right)^2}\right)_{;l}$$
$$=\frac{\left(\left(w_{ij}+v'_{ij}\right)\xi^i\xi^j\right)_{;l}\left(2\sigma_{pq}\xi^p_{;k}\xi^q\right)}{\left(\sigma_{ij}\xi^i\xi^j\right)^2}+\frac{\left(\left(w_{ij}+v'_{ij}\right)\xi^i\xi^j\right)\left(2\sigma_{pq}\xi^p_{;k}\xi^q\right)_{;l}}{\left(\sigma_{ij}\xi^i\xi^j\right)^2}$$
$$-\frac{\left(\left(w_{ij}+v'_{ij}\right)\xi^i\xi^j\right)\left(2\sigma_{pq}\xi^p_{;k}\xi^q\right)\left(\left(\sigma_{rs}\xi^r\xi^s\right)^2\right)_{;l}}{\left(\sigma_{ij}\xi^i\xi^j\right)^4}.$$

The terms of this last expression may be written as

$$\frac{\left(\left(w_{ij}+v'_{ij}\right)\xi^i\xi^j\right)_{;l}\left(2\sigma_{pq}\xi^p_{;k}\xi^q\right)}{\left(\sigma_{ij}\xi^i\xi^j\right)^2}$$
$$=\frac{\left(\left(w_{ij;l}+v'_{ij;l}\right)\xi^i\xi^j+2\left(w_{ij}+v'_{ij}\right)\xi^i_{;l}\xi^j\right)\left(2\sigma_{pq}\xi^p_{;k}\xi^q\right)}{\left(\sigma_{ij}\xi^i\xi^j\right)^2},$$

respectively

$$\frac{\left(\left(w_{ij}+v'_{ij}\right)\xi^i\xi^j\right)\left(2\sigma_{pq}\xi^p_{;k}\xi^q\right)_{;l}}{\left(\sigma_{ij}\xi^i\xi^j\right)^2}=\frac{\left(\left(w_{ij}+v'_{ij}\right)\xi^i\xi^j\right)\left(2\sigma_{pq}\xi^p_{;kl}\xi^q+2\sigma_{pq}\xi^p_{;k}\xi^q_{;l}\right)}{\left(\sigma_{ij}\xi^i\xi^j\right)^2}$$

and, last of all,

$$\begin{aligned}&\frac{\left(\left(w_{ij}+v'_{ij}\right)\xi^i\xi^j\right)\left(2\sigma_{pq}\xi^p_{;k}\xi^q\right)\left(\left(\sigma_{rs}\xi^r\xi^s\right)^2\right)_{;l}}{\left(\sigma_{ij}\xi^i\xi^j\right)^4}\\&\qquad=\frac{\left(\left(w_{ij}+v'_{ij}\right)\xi^i\xi^j\right)\left(2\sigma_{ij}\xi^i_{;k}\xi^j\right)\left(2\sigma_{ij}\xi^i\xi^j\right)\left(2\sigma_{ij}\xi^i_{;l}\xi^j\right)}{\left(\sigma_{ij}\xi^i\xi^j\right)^4}.\end{aligned}$$

As remarked above, since ξ_1 is constant and the partial and covariant derivatives coincide at x_0, all products containing first derivatives of ξ_1 vanish and from this last calculations we hence obtain

$$\begin{aligned}\left(\frac{\left(w_{ij}+v'_{ij}\right)\xi^i\xi^j}{\sigma_{ij}\xi^i\xi^j}\right)_{;kl|_{\xi=\xi_1}}&=\frac{\left(w_{ij;kl}+v'_{ij;kl}\right)\xi^i_1\xi^j_1}{\sigma_{ij}\xi^i_1\xi^j_1}+\frac{2\left(w_{ij}+v'_{ij}\right)\xi^i_{1;kl}\xi^j_1}{\sigma_{ij}\xi^i_1\xi^j_1}\\&\quad-\frac{\left(\left(w_{ij}+v'_{ij}\right)\xi^i_1\xi^j_1\right)\left(2\sigma_{pq}\xi^p_{1;kl}\xi^q_1\right)}{\left(\sigma_{ij}\xi^i_1\xi^j_1\right)^2}\end{aligned}\tag{6.18}$$

for the second covariant derivatives at (x_0,t_0). Furthermore the maximality of $w_{ij}+v'_{ij}$ in direction ξ_1 in (x_0,t_0) implies that this is an eigenvector of $w_{ij}+v'_{ij}$ with respect to σ, i.e. it holds $\left(w_{ij}+v'_{ij}\right)\xi^j_1=\hat\lambda\sigma_{ij}\xi^j_1$, for a $\hat\lambda\in\mathbb{R}$, and this provides

$$\frac{2\left(w_{ij}+v'_{ij}\right)\xi^i_{1;kl}\xi^j_1}{\sigma_{ij}\xi^i_1\xi^j_1}=\frac{2\hat\lambda\sigma_{ij}\xi^i_{1;kl}\xi^j_1}{\sigma_{ij}\xi^i_1\xi^j_1}$$

as well as

$$\frac{\left(\left(w_{ij}+v'_{ij}\right)\xi^i_1\xi^j_1\right)\left(2\sigma_{pq}\xi^p_{1;kl}\xi^q_1\right)}{\left(\sigma_{ij}\xi^i_1\xi^j_1\right)^2}=\frac{\left(\hat\lambda\sigma_{ij}\xi^i_1\xi^j_1\right)\left(2\sigma_{pq}\xi^p_{1;kl}\xi^q_1\right)}{\left(\sigma_{ij}\xi^i_1\xi^j_1\right)^2}=\frac{2\hat\lambda\left(\sigma_{pq}\xi^p_{1;kl}\xi^q_1\right)}{\sigma_{ij}\xi^i_1\xi^j_1}.$$

Equation (6.18) consequently becomes

$$\begin{aligned}\left(\frac{\left(w_{ij}+v'_{ij}\right)\xi^i\xi^j}{\sigma_{ij}\xi^i\xi^j}\right)_{;kl|_{\xi=\xi_1}}&=\frac{\left(w_{ij;kl}+v'_{ij;kl}\right)\xi^i_1\xi^j_1}{\sigma_{ij}\xi^i_1\xi^j_1}+\frac{2\hat\lambda\sigma_{ij}\xi^i_{1;kl}\xi^j_1}{\sigma_{ij}\xi^i_1\xi^j_1}-\frac{2\hat\lambda\sigma_{ij}\xi^i_{1;kl}\xi^j_1}{\sigma_{ij}\xi^i_1\xi^j_1}\\&=w_{11;kl}+v'_{11;kl}\end{aligned}$$

at (x_0,t_0) and this concludes the proof of the desired identities.

We now introduce the notation $\tilde v:=v'_{11}+C$ and we want to compute

$$\begin{aligned}PW&=\dot W-\frac{\dot\varphi}{n}\cdot w^{ij}W_{;ij}-2\cdot\frac{\dot\varphi}{n}\left(\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pk}-w^{pk}\right)\varphi_{,p}W_{;k}\\&=P(\log(w_{11}+\tilde v))+P\left(\frac{1}{2}\lambda|D\varphi|^2\right).\end{aligned}$$

On one side it is possible to rewrite the first term in the following form

$$
\begin{aligned}
P(\log(w_{11}+\tilde v)) = & \frac{\dot w_{11}+\dot{\tilde v}}{w_{11}+\tilde v} - \frac{\dot\varphi}{n}\cdot w^{kl}\left[\frac{w_{11;kl}+\tilde v_{;kl}}{w_{11}+\tilde v} - \frac{(w_{11;k}+\tilde v_{;k})(w_{11;l}+\tilde v_{;l})}{(w_{11}+\tilde v)^2}\right] \\
& - 2\cdot\frac{\dot\varphi}{n}\left(\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pk} - w^{pk}\right)\varphi_{,p}\left(\frac{w_{11;k}+\tilde v_{;k}}{w_{11}+\tilde v}\right) \\
= & \frac{\dot w_{11}}{w_{11}+\tilde v} + \frac{\dot{\tilde v}}{w_{11}+\tilde v} - \frac{\dot\varphi}{n}\cdot w^{kl}\cdot\frac{w_{11;kl}}{w_{11}+\tilde v} - \frac{\dot\varphi}{n}\cdot w^{kl}\cdot\frac{\tilde v_{;kl}}{w_{11}+\tilde v} \\
& + \frac{\dot\varphi}{n}\cdot w^{kl}\cdot\frac{(w_{11;k}+\tilde v_{;k})(w_{11;l}+\tilde v_{;l})}{(w_{11}+\tilde v)^2} \\
& - \frac{2\dot\varphi}{n}\left(\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pk} - w^{pk}\right)\varphi_{,p}\cdot\frac{w_{11;k}}{w_{11}+\tilde v} \\
& - \frac{2\dot\varphi}{n}\left(\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pk} - w^{pk}\right)\varphi_{,p}\cdot\frac{\tilde v_{;k}}{w_{11}+\tilde v} \\
= & \frac{Pw_{11}}{w_{11}+\tilde v} + \frac{P\tilde v}{w_{11}+\tilde v} + \frac{\dot\varphi}{n}\cdot w^{kl}\cdot\frac{(w_{11;k}+\tilde v_{;k})(w_{11;l}+\tilde v_{;l})}{(w_{11}+\tilde v)^2}.
\end{aligned}
$$

and on the other side

$$
\begin{aligned}
P\left(\frac12\lambda|D\varphi|^2\right) = & \lambda\sigma^{rs}\dot\varphi_{,r}\varphi_{,s} \\
& - \frac{\dot\varphi}{n}\cdot w^{kl}\left[\lambda\left(\sigma^{rs}\varphi_{;rkl}\varphi_{,s} + \sigma^{rs}\varphi_{;rk}\varphi_{;sl}\right)\right] \\
& - 2\cdot\frac{\dot\varphi}{n}\left(\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pk} - w^{pk}\right)\varphi_{,p}\left(\lambda\sigma^{rs}\varphi_{;rk}\varphi_{,s}\right) \\
= & \lambda\sigma^{rs}\cdot\frac1n\cdot\dot\varphi\left[(n+1)\cdot\frac{2\sigma^{pq}\varphi_{;pr}\varphi_{,q}}{(1+|D\varphi|^2)} - w^{pq}w_{pq;r}\right]\varphi_{,s} \\
& - \frac{\lambda\dot\varphi}{n}\cdot w^{kl}\sigma^{rs}\varphi_{;rkl}\varphi_{,s} - \frac{\lambda\dot\varphi}{n}\cdot w^{kl}\sigma^{rs}\varphi_{;rk}\varphi_{;sl} \\
& - \frac{2\lambda\dot\varphi}{n}\left(\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pk} - w^{pk}\right)\cdot\sigma^{rs}\varphi_{;rk}\varphi_{,p}\varphi_{,s} \\
= & \frac{2\lambda\dot\varphi}{n}\cdot\frac{n+1}{(1+|D\varphi|^2)}\cdot\sigma^{rs}\sigma^{pq}\varphi_{;pr}\varphi_{,q}\varphi_{,s} - \frac{\lambda\dot\varphi}{n}\cdot\sigma^{rs}w^{pq}w_{pq;r}\varphi_{,s} \\
& - \frac{\lambda\dot\varphi}{n}\cdot w^{kl}\sigma^{rs}\varphi_{;rkl}\varphi_{,s} - \frac{\lambda\dot\varphi}{n}\cdot w^{kl}\sigma^{rs}\varphi_{;rk}\varphi_{;sl} \\
& - \frac{2\lambda\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pk}\sigma^{rs}\varphi_{;rk}\varphi_{,p}\varphi_{,s} + \frac{2\lambda\dot\varphi}{n}\cdot w^{pk}\sigma^{rs}\varphi_{;rk}\varphi_{,p}\varphi_{,s}.
\end{aligned} \tag{6.19}
$$

We start the estimates considering this last equation, i.e. all the terms containing the parameter λ, and simplifying the above expression by adding the second term in the first line and the first in the second line obtaining

$$
-\frac{\lambda\dot\varphi}{n}\cdot\sigma^{rs}w^{pq}w_{pq;r}\varphi_{,s} - \frac{\lambda\dot\varphi}{n}\cdot w^{kl}\sigma^{rs}\varphi_{;rkl}\varphi_{,s} = -\frac{\lambda\dot\varphi}{n}\cdot\sigma^{rs}\varphi_{,s}\left(w^{pq}w_{pq;r} + w^{kl}\varphi_{;rkl}\right).
$$

From the definition of w_{pq} it follows

$$\begin{aligned} w^{pq}w_{pq;r} &= w^{pq}\left(\sigma_{pq;r} - \varphi_{;pqr} + \varphi_{;pr}\varphi_{,q} + \varphi_{,p}\varphi_{;qr}\right) \\ &= -\, w^{pq}\varphi_{;pqr} + w^{pq}\varphi_{;pr}\varphi_{,q} + w^{pq}\varphi_{,p}\varphi_{;qr} \\ &= -\, w^{pq}\left(\varphi_{;rpq} + R^m{}_{pqr}\varphi_{,m}\right) \\ &\quad + \left(w^{pq}\sigma_{pr} - \delta^q_r + w^{pq}\varphi_{,p}\varphi_{,r}\right)\varphi_{,q} + \left(w^{pq}\sigma_{qr} - \delta^p_r + w^{pq}\varphi_{,q}\varphi_{,r}\right)\varphi_{,p} \\ &= -\, w^{pq}\varphi_{;rpq} - w^{pq}R^m{}_{pqr}\varphi_{,m} + 2w^{pq}\sigma_{pr}\varphi_{,q} - 2\varphi_{,r} + 2w^{pq}\varphi_{,p}\varphi_{,q}\varphi_{,r} \end{aligned}$$

in view of

$$w^{ij}\varphi_{;lj} = w^{ij}\left(\sigma_{lj} - w_{lj} + \varphi_{,l}\varphi_{,j}\right) = w^{ij}\sigma_{lj} - \delta^i_l + w^{ij}\varphi_{,l}\varphi_{,j}$$

and of the rule for interchanging covariant derivatives, that we already cited in (6.9),

$$\varphi_{;pqr} = \varphi_{;prq} + R^m{}_{pqr}\varphi_{,m} = \varphi_{;rpq} + R^m{}_{pqr}\varphi_{,m}.$$

Employing this provides

$$\begin{aligned} -\frac{\lambda\dot\varphi}{n}&\cdot\sigma^{rs}w^{pq}w_{pq;r}\varphi_{,s} - \frac{\lambda\dot\varphi}{n}\cdot w^{kl}\sigma^{rs}\varphi_{;rkl}\varphi_{,s} \\ &= -\frac{\lambda\dot\varphi}{n}\cdot\sigma^{rs}\varphi_{,s}\left(-w^{pq}\varphi_{;rpq} - w^{pq}R^m{}_{pqr}\varphi_{,m} + 2w^{pq}\sigma_{pr}\varphi_{,q}\right. \\ &\qquad\qquad \left. -2\varphi_{,r} + 2w^{pq}\varphi_{,p}\varphi_{,q}\varphi_{,r} + w^{kl}\varphi_{;rkl}\right) \\ &= -\frac{\lambda\dot\varphi}{n}\cdot\sigma^{rs}\varphi_{,s}\left(-w^{pq}R^m{}_{pqr}\varphi_{,m} + 2w^{pq}\sigma_{pr}\varphi_{,q} - 2\varphi_{,r} + 2w^{pq}\varphi_{,p}\varphi_{,q}\varphi_{,r}\right). \end{aligned} \tag{6.20}$$

Inserting the formula (6.12) of the Riemannian curvature tensor of the sphere,

$$R^m{}_{jkl} = \sigma^{mi}R_{ijkl} = \delta^m_k\sigma_{jl} - \delta^m_l\sigma_{jk},$$

in the first term of (6.20),

$$-w^{pq}R^m{}_{pqr}\varphi_{,m} = -w^{pq}(\delta^m_q\sigma_{pr} - \delta^m_r\sigma_{pq})\varphi_{,m} = -w^{pm}\sigma_{pr}\varphi_{,m} + w^{pq}\sigma_{pq}\varphi_{,r},$$

we get

$$\begin{aligned} -\frac{\lambda\dot\varphi}{n}&\cdot\sigma^{rs}w^{pq}w_{pq;r}\varphi_{,s} - \frac{\lambda\dot\varphi}{n}\cdot w^{kl}\sigma^{rs}\varphi_{;rkl}\varphi_{,s} \\ &= -\frac{\lambda\dot\varphi}{n}\cdot\left(-\sigma^{rs}\varphi_{,s}w^{pq}R^m{}_{pqr}\varphi_{,m} + 2\sigma^{rs}\varphi_{,s}w^{pq}\sigma_{pr}\varphi_{,q}\right. \\ &\qquad\qquad \left. -2\sigma^{rs}\varphi_{,s}\varphi_{,r} + 2w^{pq}\varphi_{,p}\varphi_{,q}\sigma^{rs}\varphi_{,s}\varphi_{,r}\right) \\ &= -\frac{\lambda\dot\varphi}{n}\cdot\left(\sigma^{rs}\varphi_{,s}\left(-w^{pm}\sigma_{pr}\varphi_{,m} + w^{pq}\sigma_{pq}\varphi_{,r}\right) + 2w^{sq}\varphi_{,s}\varphi_{,q}\right. \\ &\qquad\qquad \left. -2|D\varphi|^2 + 2|D\varphi|^2w^{pq}\varphi_{,p}\varphi_{,q}\right) \\ &= -\frac{\lambda\dot\varphi}{n}\cdot\big(-w^{pm}\varphi_{,p}\varphi_{,m} + |D\varphi|^2w^{pq}\sigma_{pq} \\ &\qquad\qquad +2w^{sq}\varphi_{,s}\varphi_{,q} - 2|D\varphi|^2 + 2|D\varphi|^2w^{pq}\varphi_{,p}\varphi_{,q}\big) \\ &= -\frac{\lambda\dot\varphi}{n}\cdot\left(w^{pq}\varphi_{,p}\varphi_{,q} + |D\varphi|^2w^{pq}\sigma_{pq} - 2|D\varphi|^2 + 2|D\varphi|^2w^{pq}\varphi_{,p}\varphi_{,q}\right). \end{aligned}$$

We then note that the first term in the first line and the first in the last line of (6.19) sum to zero

$$\frac{2\lambda\dot{\varphi}}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{rs}\sigma^{pq}\varphi_{;pr}\varphi_{,q}\varphi_{,s}-\frac{2\lambda\dot{\varphi}}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pk}\sigma^{rs}\varphi_{;rk}\varphi_{,p}\varphi_{,s}=0.$$

Furthermore, we transform the second term in the second line as following

$$\begin{aligned}
-\frac{\lambda\dot{\varphi}}{n}\cdot w^{kl}\sigma^{rs}\varphi_{;rk}\varphi_{;sl} &= -\frac{\lambda\dot{\varphi}}{n}\cdot\sigma^{rs}\varphi_{;rk}w^{kl}\left(\sigma_{sl}-w_{sl}+\varphi_{,s}\varphi_{,l}\right)\\
&= -\frac{\lambda\dot{\varphi}}{n}\cdot\sigma^{rs}\varphi_{;rk}\left(w^{kl}\sigma_{sl}-\delta^k_s+w^{kl}\varphi_{,s}\varphi_{,l}\right)\\
&= -\frac{\lambda\dot{\varphi}}{n}\cdot\left(w^{rk}\varphi_{;rk}-\sigma^{rk}\varphi_{;rk}+\sigma^{rs}\varphi_{,s}\varphi_{,l}w^{kl}\varphi_{;rk}\right)\\
&= -\frac{\lambda\dot{\varphi}}{n}\cdot\big[w^{kr}\left(\sigma_{rk}-w_{rk}+\varphi_{,r}\varphi_{,k}\right)\\
&\qquad -\sigma^{rk}\left(\sigma_{rk}-w_{rk}+\varphi_{,r}\varphi_{,k}\right)\\
&\qquad +\sigma^{rs}\varphi_{,s}\varphi_{,l}w^{kl}\left(\sigma_{rk}-w_{rk}+\varphi_{,r}\varphi_{,k}\right)\big]\\
&= -\frac{\lambda\dot{\varphi}}{n}\cdot\big(w^{rk}\sigma_{rk}-n+w^{kr}\varphi_{,k}\varphi_{,r}-n+w_{rk}\sigma^{rk}-|D\varphi|^2\\
&\qquad +w^{kl}\varphi_{,k}\varphi_{,l}-|D\varphi|^2+|D\varphi|^2w^{kl}\varphi_{,k}\varphi_{,l}\big)\\
&= -\frac{\lambda\dot{\varphi}}{n}\cdot\big(w^{rk}\sigma_{rk}+w_{rk}\sigma^{rk}-2n\\
&\qquad +2w^{kl}\varphi_{,k}\varphi_{,l}-2|D\varphi|^2+|D\varphi|^2w^{kl}\varphi_{,k}\varphi_{,l}\big).
\end{aligned}$$

So far we therefore have

$$\begin{aligned}
&-\frac{\lambda\dot{\varphi}}{n}\cdot\sigma^{rs}w^{pq}w_{pq;r}\varphi_{,s}-\frac{\lambda\dot{\varphi}}{n}\cdot w^{kl}\sigma^{rs}\varphi_{;rkl}\varphi_{,s}-\frac{\lambda\dot{\varphi}}{n}\cdot w^{kl}\sigma^{rs}\varphi_{;rk}\varphi_{;sl}\\
&\quad= -\frac{\lambda\dot{\varphi}}{n}\cdot\left(w^{pq}\varphi_{,p}\varphi_{,q}+|D\varphi|^2w^{pq}\sigma_{pq}-2|D\varphi|^2+2|D\varphi|^2w^{pq}\varphi_{,p}\varphi_{,q}\right)\\
&\qquad -\frac{\lambda\dot{\varphi}}{n}\cdot\big(w^{rk}\sigma_{rk}+w_{rk}\sigma^{rk}-2n+2w^{kl}\varphi_{,k}\varphi_{,l}\\
&\qquad\qquad -2|D\varphi|^2+|D\varphi|^2w^{kl}\varphi_{,k}\varphi_{,l}\big)\\
&\quad= -\frac{\lambda\dot{\varphi}}{n}\cdot\big(|D\varphi|^2w^{pq}\sigma_{pq}+w^{rk}\sigma_{rk}+w_{rk}\sigma^{rk}-2n\\
&\qquad\qquad +3w^{pq}\varphi_{,p}\varphi_{,q}-4|D\varphi|^2+3|D\varphi|^2w^{pq}\varphi_{,p}\varphi_{,q}\big).
\end{aligned}$$

The last term in the last line of (6.19) is

$$\begin{aligned}
\frac{2\lambda\dot{\varphi}}{n}\cdot w^{pk}\sigma^{rs}\varphi_{;rk}\varphi_{,p}\varphi_{,s} &= \frac{2\lambda\dot{\varphi}}{n}\cdot\sigma^{rs}w^{pk}\left(\sigma_{rk}-w_{rk}+\varphi_{,r}\varphi_{,k}\right)\varphi_{,p}\varphi_{,s}\\
&= \frac{2\lambda\dot{\varphi}}{n}\cdot\sigma^{rs}\left(w^{pk}\sigma_{rk}-\delta^p_r+w^{pk}\varphi_{,r}\varphi_{,k}\right)\varphi_{,p}\varphi_{,s}\\
&= \frac{2\lambda\dot{\varphi}}{n}\cdot\left(w^{ps}\varphi_{,p}\varphi_{,s}-\sigma^{ps}\varphi_{,p}\varphi_{,s}+\sigma^{rs}w^{pk}\varphi_{,r}\varphi_{,k}\varphi_{,p}\varphi_{,s}\right)\\
&= \frac{2\lambda\dot{\varphi}}{n}\cdot\left(w^{ps}\varphi_{,p}\varphi_{,s}-|D\varphi|^2+|D\varphi|^2w^{pk}\varphi_{,p}\varphi_{,k}\right).
\end{aligned}$$

For the terms involving λ we consequently obtain the following evolution equation

$$\begin{aligned}
P\left(\frac{1}{2}\lambda|D\varphi|^2\right) =& -\frac{\lambda\dot\varphi}{n}\cdot\sigma^{rs}w^{pq}w_{pq;r}\varphi_{,s}-\frac{\lambda\dot\varphi}{n}\cdot w^{kl}\sigma^{rs}\varphi_{;rkl}\varphi_{,s}\\
&-\frac{\lambda\dot\varphi}{n}\cdot w^{kl}\sigma^{rs}\varphi_{;rk}\varphi_{;sl}+\frac{2\lambda\dot\varphi}{n}\cdot w^{pk}\sigma^{rs}\varphi_{;rk}\varphi_{,p}\varphi_{,s}\\
=& -\frac{\lambda\dot\varphi}{n}\cdot\big(|D\varphi|^2w^{pq}\sigma_{pq}+w^{rk}\sigma_{rk}+w_{rk}\sigma^{rk}-2n\\
&\qquad +3w^{pq}\varphi_{,p}\varphi_{,q}-4|D\varphi|^2+3|D\varphi|^2w^{pq}\varphi_{,p}\varphi_{,q}\big)\\
&+\frac{2\lambda\dot\varphi}{n}\cdot\big(w^{ps}\varphi_{,p}\varphi_{,s}-|D\varphi|^2+|D\varphi|^2w^{pk}\varphi_{,p}\varphi_{,k}\big)\\
=& -\frac{\lambda\dot\varphi}{n}\cdot\big(\big(1+|D\varphi|^2\big)w^{kl}\sigma_{kl}+w_{kl}\sigma^{kl}-2n\\
&\qquad +\big(1+|D\varphi|^2\big)w^{kl}\varphi_{,k}\varphi_{,l}-2|D\varphi|^2\big). \qquad (6.21)
\end{aligned}$$

Hence we can write

$$\begin{aligned}
PW =& \frac{Pw_{11}}{w_{11}+\tilde v}+\frac{P\tilde v}{w_{11}+\tilde v}+\frac{\dot\varphi}{n}\cdot w^{kl}\cdot\frac{(w_{11}+\tilde v)_{;k}(w_{11}+\tilde v)_{;l}}{(w_{11}+\tilde v)^2}\\
&-\frac{\lambda\dot\varphi}{n}\cdot\big(\big(1+|D\varphi|^2\big)w^{kl}\sigma_{kl}+w_{kl}\sigma^{kl}-2n\\
&\qquad +\big(1+|D\varphi|^2\big)w^{kl}\varphi_{,k}\varphi_{,l}-2|D\varphi|^2\big).
\end{aligned}$$

As we did before we denote by $\operatorname{tr} w^{ij}$ the trace of w^{ij} with respect to σ_{ij}, that is we write

$$\operatorname{tr} w^{ij} = w^{kl}\sigma_{kl}. \qquad (6.22)$$

Furthermore we have already shown that the time derivative of φ is uniformly bounded below, so it holds

$$\begin{aligned}
-\frac{\lambda\dot\varphi}{n}\cdot(1+|D\varphi|^2)\cdot w^{kl}\sigma_{kl} =& -\frac{\lambda\dot\varphi}{n}\cdot(1+|D\varphi|^2)\cdot\operatorname{tr} w^{ij}\\
\le& -\frac{\lambda\dot\varphi}{n}\cdot\operatorname{tr} w^{ij}\le -\varepsilon\lambda\operatorname{tr} w^{ij}, \qquad (6.23)
\end{aligned}$$

with a bounded constant ε, and

$$-\frac{\lambda\dot\varphi}{n}\cdot(-2n)\le c\lambda. \qquad (6.24)$$

It follows

$$\begin{aligned}
PW \le& \frac{Pw_{11}}{w_{11}+\tilde v}+\frac{P\tilde v}{w_{11}+\tilde v}+\frac{\dot\varphi}{n}\cdot w^{kl}\cdot\frac{(w_{11}+\tilde v)_{;k}(w_{11}+\tilde v)_{;l}}{(w_{11}+\tilde v)^2}\\
&-\varepsilon\lambda\operatorname{tr} w^{ij}+c\lambda-\frac{\lambda\dot\varphi}{n}\cdot\big(w_{kl}\sigma^{kl}+\big(1+|D\varphi|^2\big)w^{kl}\varphi_{,k}\varphi_{,l}-2|D\varphi|^2\big). \qquad (6.25)
\end{aligned}$$

The first step we undertake to compute Pw_{11} is to eliminate the covariant derivatives of fourth order, which appear in the time derivative as well as in the second derivatives of w_{11}:

$$\begin{aligned}
\dot{w}_{11} &= -\dot{\varphi}_{;11} + 2\dot{\varphi}_{;1}\varphi_{,1} \\
&= -\dot{\varphi}_{;11} + \frac{2\dot{\varphi}}{n} \cdot \left[\frac{2(n+1)}{1+|D\varphi|^2} \cdot \sigma^{pq}\varphi_{;p1}\varphi_{,q} - w^{pq}w_{pq;1} \right] \varphi_{,1} \\
&= -\dot{\varphi}_{;11} + \frac{4\dot{\varphi}}{n} \cdot \frac{n+1}{1+|D\varphi|^2} \cdot \sigma^{pq}\varphi_{;p1}\varphi_{,q}\varphi_{,1} - \frac{2\dot{\varphi}}{n} \cdot w^{pq}w_{pq;1}\varphi_{,1}.
\end{aligned}$$

We expand the first term in this last expression using the partial differential equation to get

$$\begin{aligned}
-\dot{\varphi}_{;11} &= -\left[\frac{\dot{\varphi}}{n} \cdot \left(\frac{2(n+1)}{1+|D\varphi|^2} \cdot \sigma^{pq}\varphi_{;p1}\varphi_{,q} - w^{pq}w_{pq;1} \right) \right]_{;1} \\
&= -\left[\frac{\dot{\varphi}_{;1}}{n} \cdot \left(\frac{2(n+1)}{1+|D\varphi|^2} \cdot \sigma^{pq}\varphi_{;p1}\varphi_{,q} - w^{pq}w_{pq;1} \right) \right. \\
&\qquad + \frac{\dot{\varphi}}{n} \cdot \left(\frac{2(n+1)}{1+|D\varphi|^2} \cdot (\sigma^{pq}\varphi_{;p11}\varphi_{,q} + \sigma^{pq}\varphi_{;p1}\varphi_{;q1}) \right. \\
&\qquad\qquad - \frac{4(n+1)}{(1+|D\varphi|^2)^2} \cdot (\sigma^{pq}\varphi_{;p1}\varphi_{,q})^2 \\
&\qquad\qquad \left.\left. + w^{pa}w^{bq}w_{ab;1}w_{pq;1} - w^{pq}w_{pq;11} \right) \right],
\end{aligned}$$

which clearly implies

$$\begin{aligned}
-\dot{\varphi}_{;11} = &- \frac{\dot{\varphi}_{;1}}{n} \cdot \left(\frac{2(n+1)}{1+|D\varphi|^2} \cdot \sigma^{pq}\varphi_{;p1}\varphi_{,q} - w^{pq}w_{pq;1} \right) \\
&- \frac{2\dot{\varphi}}{n} \cdot \frac{n+1}{1+|D\varphi|^2} \cdot (\sigma^{pq}\varphi_{;p11}\varphi_{,q} + \sigma^{pq}\varphi_{;p1}\varphi_{;q1}) \\
&+ \frac{4\dot{\varphi}}{n} \cdot \frac{n+1}{(1+|D\varphi|^2)^2} \cdot (\sigma^{pq}\varphi_{;p1}\varphi_{,q})^2 \\
&- \frac{\dot{\varphi}}{n} \cdot w^{pa}w^{bq}w_{ab;1}w_{pq;1} + \frac{\dot{\varphi}}{n} \cdot w^{pq}w_{pq;11},
\end{aligned}$$

and we obtain

$$\begin{aligned}
\dot{w}_{11} = &- \frac{\dot{\varphi}_{;1}}{n} \cdot \left(\frac{2(n+1)}{1+|D\varphi|^2} \cdot \sigma^{pq}\varphi_{;p1}\varphi_{,q} - w^{pq}w_{pq;1} \right) \\
&- \frac{2\dot{\varphi}}{n} \cdot \frac{n+1}{1+|D\varphi|^2} \cdot (\sigma^{pq}\varphi_{;p11}\varphi_{,q} + \sigma^{pq}\varphi_{;p1}\varphi_{;q1}) \\
&+ \frac{4\dot{\varphi}}{n} \cdot \frac{n+1}{(1+|D\varphi|^2)^2} \cdot (\sigma^{pq}\varphi_{;p1}\varphi_{,q})^2 \\
&- \frac{\dot{\varphi}}{n} \cdot w^{pa}w^{bq}w_{ab;1}w_{pq;1} + \frac{\dot{\varphi}}{n} \cdot w^{pq}w_{pq;11} \\
&+ \frac{4\dot{\varphi}}{n} \cdot \frac{n+1}{1+|D\varphi|^2} \cdot \sigma^{pq}\varphi_{;p1}\varphi_{,q}\varphi_{,1} - \frac{2\dot{\varphi}}{n} \cdot w^{pq}w_{pq;1}\varphi_{,1}.
\end{aligned}$$

Using the curvature tensor (6.11) of the unit sphere, whose covariant derivatives vanish, we can interchange (quadruple) covariant differentiation

$$\begin{aligned}\varphi_{;11kl} =& \varphi_{;kl11} + R^s{}_{k1l}\varphi_{;s1} + R^s{}_{k1l;1}\varphi_{;s} + R^s{}_{k1l}\varphi_{;1s} + R^s{}_{11l}\varphi_{;ks} \\ &+ R^s{}_{11k}\varphi_{;sl} + R^s{}_{11k;l}\varphi_{;s} \\ =& \varphi_{;kl11} + 2R^s{}_{k1l}\varphi_{;s1} + R^s{}_{k1l;1}\varphi_{;s} + R^s{}_{11l}\varphi_{;ks} + R^s{}_{11k}\varphi_{;sl} + R^s{}_{11k;l}\varphi_{;s}, \\ =& \varphi_{;kl11} + 2R^s{}_{k1l}\varphi_{;s1} + R^s{}_{11l}\varphi_{;ks} + R^s{}_{11k}\varphi_{;sl},\end{aligned}$$

because of $\varphi_{;ks} = \varphi_{;sk}$. Applying the symmetry of w^{kl} it follows

$$\begin{aligned}w^{kl}w_{11;kl} =& w^{kl}\left[-\varphi_{;11kl} + (\varphi_{;1}\varphi_{;1})_{;kl}\right] \\ =& w^{kl}\left[-\varphi_{;kl11} - 2R^s{}_{k1l}\varphi_{;s1} - R^s{}_{11l}\varphi_{;ks} - R^s{}_{11k}\varphi_{;sl}\right. \\ &\left. +2\varphi_{;1kl}\varphi_{;1} + 2\varphi_{;1k}\varphi_{;1l}\right] \\ =& w^{kl}\left[-\varphi_{;kl11} - 2R^s{}_{k1l}\varphi_{;s1} - 2R^s{}_{11l}\varphi_{;ks} + 2\varphi_{;1kl}\varphi_{;1} + 2\varphi_{;k1}\varphi_{;l1}\right. \\ &\left. + \varphi_{;k11}\varphi_{;l} - \varphi_{;k11}\varphi_{;l} + \varphi_{;l11}\varphi_{;k} - \varphi_{;l11}\varphi_{;k}\right] \\ =& w^{kl}\left[-\varphi_{;kl11} - 2R^s{}_{k1l}\varphi_{;s1} - 2R^s{}_{11l}\varphi_{;ks} + \varphi_{;k11}\varphi_{;l} + 2\varphi_{;k1}\varphi_{;l1} + \varphi_{;l11}\varphi_{;k}\right. \\ &\left. +2\varphi_{;1kl}\varphi_{;1} - \varphi_{;k11}\varphi_{;l} - \varphi_{;l11}\varphi_{;k}\right] \\ =& w^{kl}w_{kl;11} + w^{kl}\left(-2R^s{}_{k1l}\varphi_{;s1} - 2R^s{}_{11l}\varphi_{;ks} + 2\varphi_{;1kl}\varphi_{;1} - 2\varphi_{;k11}\varphi_{;l}\right).\end{aligned}$$

Combining the first two terms in the expression for Pw_{11} we have

$$\begin{aligned}\dot{w}_{11} - \frac{\dot{\varphi}}{n}\cdot w^{kl}w_{11;kl} =& -\frac{\dot{\varphi}_{;1}}{n}\cdot\left(\frac{2(n+1)}{(1+|D\varphi|^2)}\cdot\sigma^{pq}\varphi_{;p1}\varphi_{,q} - w^{pq}w_{pq;1}\right) \\ &- \frac{2\dot{\varphi}}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\left(\sigma^{pq}\varphi_{;p11}\varphi_{,q} + \sigma^{pq}\varphi_{;p1}\varphi_{;q1}\right) \\ &+ \frac{4\dot{\varphi}}{n}\cdot\frac{n+1}{(1+|D\varphi|^2)^2}\cdot\left(\sigma^{pq}\varphi_{;p1}\varphi_{,q}\right)^2 \\ &- \frac{\dot{\varphi}}{n}\cdot w^{pa}w^{bq}w_{ab;1}w_{pq;1} + \frac{\dot{\varphi}}{n}\cdot w^{pq}w_{pq;11} \\ &+ \frac{4\dot{\varphi}}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{;p1}\varphi_{,q}\varphi_{,1} - \frac{2\dot{\varphi}}{n}\cdot w^{pq}w_{pq;1}\varphi_{,1} \\ &- \frac{\dot{\varphi}}{n}\cdot w^{kl}w_{kl;11} \\ &+ \frac{2\dot{\varphi}}{n}\cdot w^{kl}\left(R^s{}_{k1l}\varphi_{;s1} + R^s{}_{11l}\varphi_{;ks} - \varphi_{;1kl}\varphi_{;1} + \varphi_{;k11}\varphi_{;l}\right)\end{aligned}$$

and consequently, since the sum of the terms containing the covariant derivatives of fourth order vanishes,

$$\begin{aligned}\dot{w}_{11} - \frac{\dot{\varphi}}{n} \cdot w^{kl} w_{11;kl} = &- \frac{\dot{\varphi}_{;1}}{n} \cdot \left(\frac{2(n+1)}{1+|D\varphi|^2} \cdot \sigma^{pq} \varphi_{;p1} \varphi_{,q} - w^{pq} w_{pq;1} \right) \\ &- \frac{2\dot{\varphi}}{n} \cdot \frac{n+1}{1+|D\varphi|^2} \cdot \left(\sigma^{pq} \varphi_{;p11} \varphi_{,q} + \sigma^{pq} \varphi_{;p1} \varphi_{;q1} \right) \\ &+ \frac{4\dot{\varphi}}{n} \cdot \frac{n+1}{\left(1+|D\varphi|^2\right)^2} \cdot \left(\sigma^{pq} \varphi_{;p1} \varphi_{,q} \right)^2 \\ &- \frac{\dot{\varphi}}{n} \cdot w^{pa} w^{bq} w_{ab;1} w_{pq;1} \\ &+ \frac{4\dot{\varphi}}{n} \cdot \frac{n+1}{1+|D\varphi|^2} \cdot \sigma^{pq} \varphi_{;p1} \varphi_{,q} \varphi_{,1} - \frac{2\dot{\varphi}}{n} \cdot w^{pq} w_{pq;1} \varphi_{,1} \\ &+ \frac{2\dot{\varphi}}{n} \cdot w^{kl} \left(R^s{}_{k1l} \varphi_{;s1} + R^s{}_{11l} \varphi_{;ks} - \varphi_{;1kl} \varphi_{;1} + \varphi_{;k11} \varphi_{;l} \right).\end{aligned}$$

We can simplify this further on, indeed

$$\begin{aligned}&-\frac{2\dot{\varphi}}{n} \cdot w^{pq} w_{pq;1} \varphi_{,1} - \frac{2\dot{\varphi}}{n} \cdot w^{kl} \left(\varphi_{;1kl} \varphi_{;1} \right) \\ &\quad = - \frac{2\dot{\varphi}}{n} \left(w^{pq} w_{pq;1} \varphi_{,1} + w^{kl} \varphi_{;1kl} \varphi_{;1} \right) \\ &\quad = - \frac{2\dot{\varphi}}{n} \cdot \varphi_{,1} \cdot \left(-w^{kl} \varphi_{;kl1} + w^{kl} \varphi_{;k1} \varphi_{,l} + w^{kl} \varphi_{,k} \varphi_{;l1} + w^{kl} \varphi_{;1kl} \right) \\ &\quad = - \frac{2\dot{\varphi}}{n} \cdot w^{kl} \varphi_{,1} \cdot \left(-\varphi_{;kl1} + 2\varphi_{;k1} \varphi_{,l} + \varphi_{;kl1} + R^s{}_{1kl} \varphi_{,s} \right) \\ &\quad = - \frac{4\dot{\varphi}}{n} \cdot \varphi_{,1} w^{kl} \varphi_{;k1} \varphi_{,l},\end{aligned}$$

where the symmetry of w^{kl} and the skew symmetry of the curvature tensor imply

$$w^{kl} R^s{}_{1kl} = w^{lk} R^s{}_{1lk} = w^{kl} R^s{}_{1lk} = -w^{kl} R^s{}_{1kl}$$

and then $w^{kl} R^s{}_{1kl} = 0$. It follows

$$\begin{aligned}\dot{w}_{11} - \frac{\dot{\varphi}}{n} \cdot w^{kl} w_{11;kl} = &- \frac{\dot{\varphi}_{;1}}{n} \cdot \left(\frac{2(n+1)}{1+|D\varphi|^2} \cdot \sigma^{pq} \varphi_{;p1} \varphi_{,q} - w^{pq} w_{pq;1} \right) \\ &- \frac{2\dot{\varphi}}{n} \cdot \frac{n+1}{1+|D\varphi|^2} \cdot \left(\sigma^{pq} \varphi_{;p11} \varphi_{,q} + \sigma^{pq} \varphi_{;p1} \varphi_{;q1} \right) \\ &+ \frac{4\dot{\varphi}}{n} \cdot \frac{n+1}{\left(1+|D\varphi|^2\right)^2} \cdot \left(\sigma^{pq} \varphi_{;p1} \varphi_{,q} \right)^2 \\ &- \frac{\dot{\varphi}}{n} \cdot w^{pa} w^{bq} w_{ab;1} w_{pq;1} \\ &+ \frac{4\dot{\varphi}}{n} \cdot \frac{n+1}{1+|D\varphi|^2} \cdot \sigma^{pq} \varphi_{;p1} \varphi_{,q} \varphi_{,1} \\ &+ \frac{2\dot{\varphi}}{n} \cdot w^{kl} \left(R^s{}_{k1l} \varphi_{;s1} + R^s{}_{11l} \varphi_{;ks} + \varphi_{;k11} \varphi_{;l} \right) \\ &- \frac{4\dot{\varphi}}{n} \cdot \varphi_{,1} w^{kl} \varphi_{;k1} \varphi_{,l}. \end{aligned} \tag{6.26}$$

The term of first order in the expression for Pw_{11} is given by

$$\begin{aligned} &-\frac{2\dot\varphi}{n}\left(\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pk}-w^{pk}\right)\varphi_{,p}w_{11;k}\\ &\quad=-\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pk}\varphi_{,p}w_{11;k}+\frac{2\dot\varphi}{n}\cdot w^{pk}\varphi_{,p}w_{11;k}. \end{aligned}\tag{6.27}$$

Adding the first part of this last expression to the first term in the second line and to the term in the fifth line of (6.26) yields

$$\begin{aligned} &-\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{;p11}\varphi_{,q}-\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pk}\varphi_{,p}w_{11;k}\\ &\quad+\frac{4\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{;p1}\varphi_{,q}\varphi_{,1}\\ &=\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\left(-\sigma^{pq}\varphi_{;p11}\varphi_{,q}-\sigma^{qp}\varphi_{,q}w_{11;p}+2\sigma^{pq}\varphi_{;p1}\varphi_{,q}\varphi_{,1}\right)\\ &=\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\left[-\sigma^{pq}\varphi_{;p11}\varphi_{,q}-\sigma^{pq}\varphi_{,q}\left(-\varphi_{;11p}+2\varphi_{;1p}\varphi_{,1}\right)+2\sigma^{pq}\varphi_{;p1}\varphi_{,q}\varphi_{,1}\right]\\ &=\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{,q}\cdot\left(-\varphi_{;p11}+\varphi_{;11p}-2\varphi_{;1p}\varphi_{,1}+2\varphi_{;p1}\varphi_{,1}\right)\\ &=\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{,q}\left(\varphi_{,1}\sigma_{1p}-\sigma_{11}\varphi_{,p}\right)\\ &=\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\Big((\varphi_{,1})^2-\sigma_{11}|D\varphi|^2\Big), \end{aligned}$$

where we interchanged the covariant derivatives using once again

$$\varphi_{;11p}=\varphi_{;p11}+R^s{}_{11p}\varphi_{,s},$$

and inserted the formula (6.12) of the Riemannian curvature tensor of the sphere to get

$$\varphi_{;11p}=\varphi_{;p11}+R^s{}_{11p}\varphi_{,s}=\varphi_{;p11}+\left(\delta^s_1\sigma_{1p}-\delta^s_p\sigma_{11}\right)\varphi_{,s}=\varphi_{;p11}+\varphi_{,1}\sigma_{1p}-\sigma_{11}\varphi_{,p}.$$

The second part of (6.27), which contains third derivatives can be used to eliminate a similar term in (6.26):

$$\begin{aligned} &\frac{2\dot\varphi}{n}\cdot w^{kl}\left(\varphi_{;k11}\varphi_{;l}\right)+\frac{2\dot\varphi}{n}\cdot w^{pk}\varphi_{,p}w_{11;k}\\ &\quad=\frac{2\dot\varphi}{n}\cdot\left(w^{kl}\varphi_{;k11}\varphi_{;l}+w^{pk}\varphi_{,p}w_{11;k}\right)\\ &\quad=\frac{2\dot\varphi}{n}\cdot\left(w^{kl}\varphi_{;k11}\varphi_{,l}+w^{lk}\varphi_{,l}w_{11;k}\right)\\ &\quad=\frac{2\dot\varphi}{n}\cdot w^{kl}\varphi_{,l}\cdot\left(\varphi_{;k11}+w_{11;k}\right)\\ &\quad=\frac{2\dot\varphi}{n}\cdot w^{kl}\varphi_{,l}\cdot\left(\varphi_{;k11}-\varphi_{;11k}+2\varphi_{;1k}\varphi_{,1}\right)\\ &\quad=\frac{2\dot\varphi}{n}\cdot w^{kl}\varphi_{,l}\cdot\left(\varphi_{;k11}-\varphi_{;k11}-R^s{}_{11k}\varphi_{,s}+2\varphi_{;1k}\varphi_{,1}\right)\\ &\quad=\frac{2\dot\varphi}{n}\cdot w^{kl}\varphi_{,l}\cdot\left(-\varphi_{,1}\sigma_{1k}+\sigma_{11}\varphi_{,k}+2\varphi_{;1k}\varphi_{,1}\right). \end{aligned}$$

We hence obtain

$$
\begin{aligned}
Pw_{11} = & \dot{w}_{11} - \frac{\dot{\varphi}}{n} \cdot w^{kl} w_{11;kl} - \frac{2\dot{\varphi}}{n} \left(\frac{n+1}{1+|D\varphi|^2} \cdot \sigma^{pk} - w^{pk} \right) \varphi_{,p} w_{11;k} \\
= & - \frac{\dot{\varphi}_{;1}}{n} \cdot \left(\frac{2(n+1)}{1+|D\varphi|^2} \cdot \sigma^{pq} \varphi_{;p1} \varphi_{,q} - w^{pq} w_{pq;1} \right) \\
& - \frac{2\dot{\varphi}}{n} \cdot \frac{n+1}{1+|D\varphi|^2} \cdot \sigma^{pq} \varphi_{;p1} \varphi_{;q1} \\
& + \frac{4\dot{\varphi}}{n} \cdot \frac{n+1}{\left(1+|D\varphi|^2\right)^2} \cdot \left(\sigma^{pq} \varphi_{;p1} \varphi_{,q}\right)^2 \\
& - \frac{\dot{\varphi}}{n} \cdot w^{pa} w^{bq} w_{ab;1} w_{pq;1} \\
& + \frac{2\dot{\varphi}}{n} \cdot w^{kl} \left(R^s{}_{k1l} \varphi_{;s1} + R^s{}_{11l} \varphi_{;ks} \right) \\
& - \frac{4\dot{\varphi}}{n} \cdot \varphi_{,1} w^{kl} \varphi_{;k1} \varphi_{,l} \\
& + \frac{2\dot{\varphi}}{n} \cdot \frac{n+1}{1+|D\varphi|^2} \cdot \left((\varphi_{,1})^2 - \sigma_{11} |D\varphi|^2 \right) \\
& + \frac{2\dot{\varphi}}{n} \cdot w^{kl} \varphi_{,l} \cdot \left(-\varphi_{,1} \sigma_{1k} + \sigma_{11} \varphi_{,k} + 2\varphi_{;1k} \varphi_{,1} \right)
\end{aligned}
$$

and therefore, because two terms sum up to zero,

$$
\begin{aligned}
Pw_{11} = & - \frac{\dot{\varphi}_{;1}}{n} \cdot \left(\frac{2(n+1)}{1+|D\varphi|^2} \cdot \sigma^{pq} \varphi_{;p1} \varphi_{,q} - w^{pq} w_{pq;1} \right) \\
& - \frac{2\dot{\varphi}}{n} \cdot \frac{n+1}{1+|D\varphi|^2} \cdot \sigma^{pq} \varphi_{;p1} \varphi_{;q1} \\
& + \frac{4\dot{\varphi}}{n} \cdot \frac{n+1}{\left(1+|D\varphi|^2\right)^2} \cdot \left(\sigma^{pq} \varphi_{;p1} \varphi_{,q}\right)^2 \\
& - \frac{\dot{\varphi}}{n} \cdot w^{pa} w^{bq} w_{ab;1} w_{pq;1} \\
& + \frac{2\dot{\varphi}}{n} \cdot w^{kl} \left(R^s{}_{k1l} \varphi_{;s1} + R^s{}_{11l} \varphi_{;ks} \right) \\
& + \frac{2\dot{\varphi}}{n} \cdot \frac{n+1}{1+|D\varphi|^2} \cdot \left((\varphi_{,1})^2 - \sigma_{11} |D\varphi|^2 \right) \\
& + \frac{2\dot{\varphi}}{n} \cdot \left(-\varphi_{,1} w^{kl} \sigma_{1k} \varphi_{,l} + \sigma_{11} w^{kl} \varphi_{,k} \varphi_{,l} \right).
\end{aligned}
$$

Using (6.12) we rewrite the terms containing the curvature tensor as following

$$
\begin{aligned}
w^{kl} \left(R^s{}_{k1l} \varphi_{;s1} + R^s{}_{11l} \varphi_{;ks} \right) = & w^{kl} \left(\left(\delta^s_1 \sigma_{kl} - \delta^s_l \sigma_{k1} \right) \varphi_{;s1} + \left(\delta^s_1 \sigma_{1l} - \delta^s_l \sigma_{11} \right) \varphi_{;ks} \right) \\
= & w^{kl} \left(\sigma_{kl} \varphi_{;11} - \sigma_{k1} \varphi_{;l1} + \sigma_{1l} \varphi_{;k1} - \sigma_{11} \varphi_{;kl} \right) \\
= & w^{kl} \sigma_{kl} \varphi_{;11} - \sigma_{11} w^{kl} \left(\sigma_{kl} - w_{kl} + \varphi_{,k} \varphi_{,l} \right) \\
= & w^{kl} \sigma_{kl} \varphi_{;11} - \sigma_{11} w^{kl} \sigma_{kl} + \sigma_{11} \delta^k_k - \sigma_{11} w^{kl} \varphi_{,k} \varphi_{,l} \\
= & w^{kl} \sigma_{kl} \left(\varphi_{;11} - \sigma_{11} \right) + n\sigma_{11} - \sigma_{11} w^{kl} \varphi_{,k} \varphi_{,l} \\
= & - w^{kl} \sigma_{kl} \left(w_{11} - (\varphi_1)^2 \right) + n\sigma_{11} - \sigma_{11} w^{kl} \varphi_{,k} \varphi_{,l}
\end{aligned}
$$

to get

$$
\begin{aligned}
Pw_{11} = &-\frac{\dot\varphi_{;1}}{n}\cdot\left(\frac{2(n+1)}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{;p1}\varphi_{,q}-w^{pq}w_{pq;1}\right)\\
&-\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{;p1}\varphi_{;q1}\\
&+\frac{4\dot\varphi}{n}\cdot\frac{n+1}{\left(1+|D\varphi|^2\right)^2}\cdot\left(\sigma^{pq}\varphi_{;p1}\varphi_{,q}\right)^2\\
&-\frac{\dot\varphi}{n}\cdot w^{pa}w^{bq}w_{ab;1}w_{pq;1}\\
&+\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\left((\varphi_{,1})^2-\sigma_{11}|D\varphi|^2\right)\\
&+\frac{2\dot\varphi}{n}\cdot\left(-\varphi_{,1}w^{kl}\sigma_{1k}\varphi_{,l}+\sigma_{11}w^{kl}\varphi_{,k}\varphi_{,l}\right)\\
&+\frac{2\dot\varphi}{n}\cdot\left(-w^{kl}\sigma_{kl}\left(w_{11}-(\varphi_1)^2\right)+n\sigma_{11}-\sigma_{11}w^{kl}\varphi_{,k}\varphi_{,l}\right).
\end{aligned}
$$

Consequently some terms cancel:

$$
\begin{aligned}
Pw_{11} = &-\frac{\dot\varphi_{;1}}{n}\cdot\left(\frac{2(n+1)}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{;p1}\varphi_{,q}-w^{pq}w_{pq;1}\right)\\
&-\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{;p1}\varphi_{;q1}\\
&+\frac{4\dot\varphi}{n}\cdot\frac{n+1}{\left(1+|D\varphi|^2\right)^2}\cdot\left(\sigma^{pq}\varphi_{;p1}\varphi_{,q}\right)^2\\
&-\frac{\dot\varphi}{n}\cdot w^{pa}w^{bq}w_{ab;1}w_{pq;1}\\
&+\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\left((\varphi_{,1})^2-\sigma_{11}|D\varphi|^2\right)\\
&-\frac{2\dot\varphi}{n}\cdot\varphi_{,1}w^{kl}\sigma_{1k}\varphi_{,l}\\
&+\frac{2\dot\varphi}{n}\cdot\left(-w^{kl}\sigma_{kl}\left(w_{11}-(\varphi_1)^2\right)+n\sigma_{11}\right).
\end{aligned}
\tag{6.28}
$$

The first term is of the form $-(\dot\varphi_{,1})^2/\dot\varphi$ and is therefore negative, since $\dot\varphi$ is strictly positive, whereas the last term is less than a constant by virtue of the bounds on the metric and the time derivative. For the second-last summand the C^1-estimate yields

$$
\frac{2\dot\varphi}{n}\cdot w^{kl}\sigma_{kl}\left((\varphi_{,1})^2-w_{11}\right)\le\frac{2\dot\varphi}{n}\cdot w^{kl}\sigma_{kl}\left(c-w_{11}\right)\le 0,
\tag{6.29}
$$

where we may assume w.l.o.g that the last inequality is satisfied, since otherwise w_{11} would be bounded from above by this constant c. Moreover, applying Lemma 6.1, it holds

$$
-\frac{2\dot\varphi}{n}\cdot\left(\varphi_{,1}w^{kl}\sigma_{1k}\varphi_{,l}\right)\le c\,\operatorname{tr}w^{ij}.
\tag{6.30}
$$

We estimate the second and the third term of (6.28), getting

$$
\begin{aligned}
-\frac{2\dot\varphi}{n}&\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{;p1}\varphi_{;q1}+\frac{4\dot\varphi}{n}\cdot\frac{n+1}{\left(1+|D\varphi|^2\right)^2}\cdot\left(\sigma^{pq}\varphi_{;p1}\varphi_{,q}\right)^2\\
&=\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\left(-\sigma^{pq}\varphi_{;p1}\varphi_{;q1}+\frac{2}{1+|D\varphi|^2}\cdot\left(\sigma^{pq}\varphi_{;1p}\varphi_{,q}\right)^2\right)\\
&\le\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\left(-\sigma^{pq}\varphi_{;p1}\varphi_{;q1}+\frac{2}{1+|D\varphi|^2}\cdot\left(\sigma^{pq}\varphi_{;1p}\varphi_{;1q}\right)\left(\sigma^{pq}\varphi_{,p}\varphi_{,q}\right)\right)\\
&\le\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\left(-\sigma^{pq}\varphi_{;p1}\varphi_{;q1}+2\sigma^{pq}\varphi_{;1p}\varphi_{;1q}\right)\\
&=\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{;p1}\varphi_{;q1},
\end{aligned}
$$

and because of

$$
\begin{aligned}
\sigma^{pq}\varphi_{;p1}\varphi_{;q1}&=\sigma^{pq}\left(\sigma_{p1}-w_{p1}+\varphi_{,p}\varphi_{,1}\right)\left(\sigma_{q1}-w_{q1}+\varphi_{,q}\varphi_{,1}\right)\\
&=\sigma^{pq}(\sigma_{p1}\sigma_{q1}-w_{p1}\sigma_{q1}+\sigma_{q1}\varphi_{,p}\varphi_{,1}\\
&\qquad-\sigma_{p1}w_{q1}+w_{p1}w_{q1}-w_{q1}\varphi_{,p}\varphi_{,1}\\
&\qquad+\sigma_{p1}\varphi_{,q}\varphi_{,1}-w_{p1}\varphi_{,q}\varphi_{,1}+(\varphi_{,1})^2\varphi_{,p}\varphi_{,q})\\
&=\sigma_{11}-2w_{11}+2(\varphi_{,1})^2+\sigma^{pq}w_{p1}w_{q1}-2\varphi_{,1}\sigma^{pq}w_{p1}\varphi_{,q}+(\varphi_{,1})^2|D\varphi|^2
\end{aligned}
$$

we obtain

$$
\begin{aligned}
-\frac{2\dot\varphi}{n}&\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{;p1}\varphi_{;q1}+\frac{4\dot\varphi}{n}\cdot\frac{n+1}{\left(1+|D\varphi|^2\right)^2}\cdot\left(\sigma^{pq}\varphi_{;p1}\varphi_{,q}\right)^2\\
&\le\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{;p1}\varphi_{;q1}\\
&=\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot(\sigma_{11}-2w_{11}+2(\varphi_{,1})^2\\
&\qquad\qquad+\sigma^{pq}w_{p1}w_{q1}-2\varphi_{,1}\sigma^{pq}w_{p1}\varphi_{,q}+(\varphi_{,1})^2|D\varphi|^2).
\end{aligned}
\tag{6.31}
$$

Here, and in some other cases later, we will use that for the (symmetric) positive definite matrix w_{ij}, by testing it with vectors of the form

$$(1,0,\dots,0,1,0,\dots,0)\qquad\text{and}\qquad(1,0,\dots,0,-1,0,\dots,0),$$

we have

$$w_{11}\pm 2w_{1j}+w_{jj}>0$$

and clearly

$$w_{11}+w_{jj}+v'_{jj}>\mp 2w_{1j}+v'_{jj}$$

for all $j=1,\dots,n$. Because of the maximality of $(w_{11}+v'_{11})(x_0,t_0)$ it follows

$$w_{11}>|w_{1j}|+\frac{1}{2}\left(v'_{jj}-v'_{11}\right)=|w_{j1}|+\frac{1}{2}\left(v'_{jj}-v'_{11}\right)\ge|w_{1j}|-c=|w_{j1}|-c,\tag{6.32}$$

in the point (x_0,t_0), for each $j=1,\dots,n$.

Applying hence (6.32) and considering that the second term is negative, since otherwise w_{11} would be bounded by zero, (6.31) becomes

$$\begin{aligned}-\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pq}\varphi_{;p1}\varphi_{;q1}+\frac{4\dot\varphi}{n}\cdot\frac{n+1}{(1+|D\varphi|^2)^2}\cdot\left(\sigma^{pq}\varphi_{;p1}\varphi_{,q}\right)^2\\ \le\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot(\sigma_{11}-0+2(\varphi_{,1})^2+\sigma^{pq}w_{p1}w_{q1}+cw_{11}+c+(\varphi_{,1})^2|D\varphi|^2).\end{aligned}$$

Inserting this last inequality as well as (6.29) and (6.30) in (6.28), it follows

$$\begin{aligned}Pw_{11}\le&\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot(\sigma_{11}+2(\varphi_{,1})^2+\sigma^{pq}w_{p1}w_{q1}+cw_{11}+c+(\varphi_{,1})^2|D\varphi|^2)\\&-\frac{\dot\varphi}{n}\cdot w^{pa}w^{bq}w_{ab;1}w_{pq;1}\\&+\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\Big((\varphi_{,1})^2-\sigma_{11}|D\varphi|^2\Big)\\&+c\operatorname{tr}w^{ij}\\&+0+c\\=&\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot(\sigma_{11}+3(\varphi_{,1})^2-\sigma_{11}|D\varphi|^2\\&\qquad\qquad+\sigma^{pq}w_{p1}w_{q1}+cw_{11}+(\varphi_{,1})^2|D\varphi|^2)\\&-\frac{\dot\varphi}{n}\cdot w^{pa}w^{bq}w_{ab;1}w_{pq;1}\\&+c\operatorname{tr}w^{ij}+c.\end{aligned}$$

Because the metric, the time derivative and the first derivatives of φ are bounded, it is possible to estimate further, to get

$$\begin{aligned}Pw_{11}\le&\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot(c+\sigma^{kl}w_{k1}w_{l1}+cw_{11})\\&-\frac{\dot\varphi}{n}\cdot w^{pa}w^{bq}w_{ab;1}w_{pq;1}\\&+c\operatorname{tr}w^{ij}+c\\\le&\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\left(\sigma^{kl}w_{k1}w_{l1}\right)+c+cw_{11}\\&-\frac{\dot\varphi}{n}\cdot w^{pa}w^{bq}w_{ab;1}w_{pq;1}+c\operatorname{tr}w^{ij}+c\\\le&\frac{2\dot\varphi}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\left(\sigma^{kl}w_{k1}w_{l1}\right)+c+cw_{11}+c\operatorname{tr}w^{ij}\\&-\frac{\dot\varphi}{n}\cdot w^{pa}w^{bq}w_{ab;1}w_{pq;1}\end{aligned}$$

and, inserting this in (6.25),

$$
\begin{aligned}
PW \leq \frac{1}{w_{11}+\tilde{v}} \cdot \Bigg[& \frac{2\dot{\varphi}}{n} \cdot \frac{n+1}{1+|D\varphi|^2} \cdot \left(\sigma^{kl} w_{k1} w_{l1}\right) + c + c w_{11} + c \operatorname{tr} w^{ij} \\
& - \frac{\dot{\varphi}}{n} \cdot w^{pa} w^{bq} w_{ab;1} w_{pq;1} \Bigg] \\
& + \frac{P\tilde{v}}{w_{11}+\tilde{v}} + \frac{\dot{\varphi}}{n} \cdot w^{kl} \cdot \frac{(w_{11}+\tilde{v})_{;k}(w_{11}+\tilde{v})_{;l}}{(w_{11}+\tilde{v})^2} \\
& - \varepsilon\lambda \operatorname{tr} w^{ij} + c\lambda - \frac{\lambda\dot{\varphi}}{n} \cdot \left(w_{kl}\sigma^{kl} + \left(1+|D\varphi|^2\right) w^{kl} \varphi_{,k}\varphi_{,l} - 2|D\varphi|^2\right).
\end{aligned}
\tag{6.33}
$$

We now use the following Lemma proved in [11]:

Lemma 6.3. *Let $\left(a^{ij}\right)$ and (A_{ij}) be symmetric $n \times n$-matrices. Assume that (A_{ij}) is positive semi-definite and that $\left(a^{ij}\right)$ is positive definite with inverse $(\tilde{a}_{ij})$. Then we have the inequality*

$$
-a^{ij} A_{ij} + \frac{1}{\tilde{a}_{11}} A_{11} \leq 0.
$$

Adding the first and third-last term of inequality (6.33) and using $w_{11} \gg \tilde{v}$ (this assumption is possible since $\tilde{v}$ is bounded) and the last lemma, we achieve

$$
\begin{aligned}
\frac{1}{w_{11}+\tilde{v}} \cdot \frac{\dot{\varphi}}{n} \cdot \frac{2(n+1)}{1+|D\varphi|^2} \cdot \sigma^{ij} w_{i1} w_{j1} - \frac{\lambda\dot{\varphi}}{n} \cdot w_{kl}\sigma^{kl} & \\
= \frac{\dot{\varphi}}{n} \cdot \left(\frac{1}{w_{11}+\tilde{v}} \cdot \frac{2(n+1)}{1+|D\varphi|^2} \cdot \sigma^{ij} w_{i1} w_{j1} - \lambda \cdot w_{kl}\sigma^{kl} \right) & \\
\leq \frac{\dot{\varphi}}{n} \cdot \left(\frac{1}{w_{11}+\tilde{v}} \cdot c_1 (w_{11}+c_2)^2 - \frac{1}{c_3} \cdot \lambda w_{11} \right) & \\
\leq \frac{\dot{\varphi}}{n} \cdot w_{11} \cdot \left(c_4 - \frac{\lambda}{c_3} \right) + \frac{c_5 w_{11}}{w_{11}+\tilde{v}} + \frac{c_6}{w_{11}+\tilde{v}} & \\
\leq -c \cdot \frac{\lambda}{2} \cdot w_{11} + \frac{c w_{11}}{w_{11}+\tilde{v}} + \frac{c}{w_{11}+\tilde{v}}, &
\end{aligned}
\tag{6.34}
$$

for λ large enough.

Since $\tilde{v}$ is of the form $\tilde{v} = v' + C =: -\rho^i \varphi_i + C$, with $\rho^i : \bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that doesn't depend on φ, we obtain, analogously to the case of the double normal estimates in (6.14) and (6.15),

$$\begin{aligned}
P\tilde{v} &= \dot{\tilde{v}} - \frac{\dot{\varphi}}{n}\cdot w^{ij}\cdot\tilde{v}_{;ij} - \frac{2\dot{\varphi}}{n}\left(\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pk} - w^{pk}\right)\varphi_{,p}\cdot\tilde{v}_{;k}\\
&= \rho^l\left[-\dot{\varphi}_{,l} + \frac{\dot{\varphi}}{n}\cdot w^{ij}\varphi_{;lij} + \frac{\dot{\varphi}}{n}\cdot\frac{2(n+1)}{1+|D\varphi|^2}\cdot\sigma^{pk}\varphi_{,p}\varphi_{;lk}\right] + \frac{2\dot{\varphi}}{n}\cdot\rho^l_{;i}w^{ij}\varphi_{;lj}\\
&\quad + \frac{\dot{\varphi}}{n}\cdot w^{ij}\rho^l_{;ij}\varphi_{,l} + \frac{2\dot{\varphi}}{n}\cdot\frac{n+1}{1+|D\varphi|^2}\cdot\sigma^{pk}\varphi_{,p}\rho^l_{;k}\varphi_{,l} + \frac{2\dot{\varphi}}{n}\cdot w^{pk}\varphi_{,p}\tilde{v}_{;k}\\
&= \frac{\dot{\varphi}}{n}\cdot\Bigg[\rho^l w^{ij}\sigma_{il}\varphi_{,j} - 2v'\left(1 - \frac{1}{2}\cdot w^{ij}\sigma_{ij} - w^{ij}\varphi_{,i}\varphi_{,j}\right) + 2\rho^l_{;i}w^{ij}\sigma_{jl} - 2\rho^l_{;l}\\
&\qquad + 2\rho^l_{;i}w^{ij}\varphi_{,l}\varphi_{,j} + \rho^l_{;ij}w^{ij}\varphi_{,l} + \frac{2(n+1)}{1+|D\varphi|^2}\cdot\sigma^{pk}\rho^l_{;k}\varphi_{,l}\varphi_{,p} + 2w^{pk}\varphi_{,p}\tilde{v}_{;k}\Bigg]\\
&\le c\cdot\operatorname{tr} w^{ij} + c + \frac{2\dot{\varphi}}{n}\cdot w^{pk}\varphi_{,p}\tilde{v}_{;k},
\end{aligned}$$

where we inserted the differentiated partial differential equation (6.13), and therefore

$$\frac{P\tilde{v}}{w_{11}+\tilde{v}} \le \frac{1}{w_{11}+\tilde{v}}\cdot\left(c\operatorname{tr} w^{ij} + c + \frac{2\dot{\varphi}}{n}\cdot w^{pk}\tilde{v}_{;k}\varphi_{,p}\right).$$

Combining this last estimate and (6.34) with (6.33), provides

$$\begin{aligned}
PW \le{}& \frac{1}{w_{11}+\tilde{v}}\cdot\Bigg[c + cw_{11} + c\operatorname{tr} w^{ij}\\
&\qquad - \frac{\dot{\varphi}}{n}\cdot w^{pa}w^{bq}w_{ab;1}w_{pq;1}\Bigg]\\
&+ \frac{1}{w_{11}+\tilde{v}}\left(c\operatorname{tr} w^{ij} + c + \frac{2\dot{\varphi}}{n}\cdot w^{pk}\tilde{v}_{;k}\varphi_{,p}\right) + \frac{\dot{\varphi}}{n}\cdot\frac{w^{kl}(w_{11}+\tilde{v})_{;k}(w_{11}+\tilde{v})_{;l}}{(w_{11}+\tilde{v})^2}\\
&- \varepsilon\lambda\operatorname{tr} w^{ij} + c\lambda - \frac{\lambda\dot{\varphi}}{n}\cdot\left[\left(1+|D\varphi|^2\right)w^{kl}\varphi_{,k}\varphi_{,l} - 2|D\varphi|^2\right]\\
&- c\cdot\frac{\lambda}{2}\cdot w_{11} + \frac{cw_{11}}{w_{11}+\tilde{v}} + \frac{c}{w_{11}+\tilde{v}}.
\end{aligned}\tag{6.35}$$

The last terms we have to estimate are

$$\frac{\dot{\varphi}}{n}\cdot\left(\frac{1}{V^2}w^{kl}V_{;k}V_{;l} - \frac{1}{V}w^{pa}w^{bq}w_{ab;1}w_{pq;1}\right),$$

where we set $V := w_{11}+\tilde{v}$.

A direct application of Lemma 6.3 yields

$$\begin{aligned}
&\frac{1}{V^2}w^{kl}V_{;k}V_{;l} - \frac{1}{V}w^{pa}w^{bq}w_{ab;1}w_{pq;1}\\
&\qquad\le \frac{1}{V^2}w^{kl}V_{;k}V_{;l} - \frac{1}{V}\frac{1}{w_{11}}w^{kl}w_{1k;1}w_{1l;1}\\
&\qquad= \frac{1}{V^2}w^{kl}V_{;k}V_{;l} + \frac{1}{Vw_{11}}w^{kl}V_{;k}V_{;l} - \frac{1}{Vw_{11}}w^{kl}V_{;k}V_{;l} - \frac{1}{V}\frac{1}{w_{11}}w^{kl}w_{1k;1}w_{1l;1}\\
&\qquad= \left(\frac{1}{V^2} - \frac{1}{Vw_{11}}\right)w^{kl}V_{;k}V_{;l} + \frac{1}{Vw_{11}}w^{kl}V_{;k}V_{;l} - \frac{1}{V}\frac{1}{w_{11}}w^{kl}w_{1k;1}w_{1l;1}.
\end{aligned}$$

Since $V \geq w_{11}$, it holds $V^2 \geq Vw_{11}$ and consequently the first term is nonpositive (because of $w^{kl}V_{;k}V_{;l} \geq 0$):

$$\frac{1}{V^2}w^{kl}V_{;k}V_{;l} - \frac{1}{V}w^{pa}w^{bq}w_{ab;1}w_{pq;1} \leq \frac{1}{Vw_{11}}w^{kl}V_{;k}V_{;l} - \frac{1}{V}\frac{1}{w_{11}}w^{kl}w_{1k;1}w_{1l;1}.$$

We factor out the first term on the right-hand side of the inequality to get

$$\begin{aligned} w^{kl}V_{;k}V_{;l} &= w^{kl}(w_{11}+\tilde{v})_{;k}(w_{11}+\tilde{v})_{;l} \\ &= w^{kl}(w_{11;k}w_{11;l} + \tilde{v}_{;k}w_{11;l} + \tilde{v}_{;l}w_{11;k} + \tilde{v}_{;k}\tilde{v}_{;l}) \\ &= w^{kl}(w_{11;k}w_{11;l} + 2\tilde{v}_{;k}w_{11;l} + \tilde{v}_{;k}\tilde{v}_{;l}), \end{aligned}$$

where

$$\begin{aligned} w^{kl}w_{11;k}w_{11;l} &= w^{kl}(-\varphi_{;11k} + 2\varphi_{;1k}\varphi_{,1})(-\varphi_{;11l} + 2\varphi_{;1l}\varphi_{,1}) \\ &= w^{kl}(\varphi_{;11k}\varphi_{;11l} - 2\varphi_{;11l}\varphi_{;1k}\varphi_{,1} - 2\varphi_{;11k}\varphi_{;1l}\varphi_{,1} + 4(\varphi_{,1})^2\varphi_{;1k}\varphi_{;1l}) \\ &= w^{kl}(\varphi_{;11k}\varphi_{;11l} - 4\varphi_{,1}\varphi_{;11l}\varphi_{;1k} + 4(\varphi_{,1})^2\varphi_{;1k}\varphi_{;1l}). \end{aligned} \tag{6.36}$$

We deal with the second one, using the definition of w_{ij},

$$\begin{aligned} w^{kl}w_{1k;1}w_{1l;1} =& w^{kl}(-\varphi_{;1l1} + \varphi_{;11}\varphi_{,l} + \varphi_{;l1}\varphi_{,1})(-\varphi_{;1k1} + \varphi_{;11}\varphi_{,k} + \varphi_{;k1}\varphi_{,1}) \\ =& w^{kl}(\varphi_{;1l1}\varphi_{;1k1} - \varphi_{;1l1}\varphi_{;11}\varphi_{,k} - \varphi_{;1l1}\varphi_{;k1}\varphi_{,1} \\ &- \varphi_{;1k1}\varphi_{;11}\varphi_{,l} + \varphi_{;11}\varphi_{,l}\varphi_{;11}\varphi_{,k} + \varphi_{;11}\varphi_{,l}\varphi_{;k1}\varphi_{,1} \\ &- \varphi_{;1k1}\varphi_{;l1}\varphi_{,1} + \varphi_{;l1}\varphi_{,1}\varphi_{;11}\varphi_{,k} + \varphi_{;l1}\varphi_{,1}\varphi_{;k1}\varphi_{,1}) \\ =& w^{kl}[\varphi_{;1l1}\varphi_{;1k1} - 2\varphi_{;11}\varphi_{;1l1}\varphi_{,k} - 2\varphi_{,1}\varphi_{;1l1}\varphi_{;k1} \\ &+ (\varphi_{;11})^2\varphi_{,l}\varphi_{,k} + 2\varphi_{;11}\varphi_{,1}\varphi_{,l}\varphi_{;k1} + (\varphi_{,1})^2\varphi_{;l1}\varphi_{;k1}] \end{aligned}$$

and interchanging covariant derivatives,

$$\begin{aligned} w^{kl}w_{1k;1}w_{1l;1} =& w^{kl}[(\varphi_{;11l} - R^r{}_{11l}\varphi_{,r})(\varphi_{;11k} - R^s{}_{11k}\varphi_{,s}) \\ &- 2\varphi_{;11}(\varphi_{;11l} - R^s{}_{11l}\varphi_{,s})\varphi_{,k} - 2\varphi_{,1}(\varphi_{;11l} - R^s{}_{11l}\varphi_{,s})\varphi_{;k1} \\ &+ (\varphi_{;11})^2\varphi_{,l}\varphi_{,k} + 2\varphi_{;11}\varphi_{,1}\varphi_{,l}\varphi_{;k1} + (\varphi_{,1})^2\varphi_{;l1}\varphi_{;k1}] \\ =& w^{kl}[\varphi_{;11k}\varphi_{;11l} - \varphi_{;11l}R^s{}_{11k}\varphi_{,s} - \varphi_{;11k}R^r{}_{11l}\varphi_{,r} \\ &+ (R^s{}_{11k}\varphi_{,s})(R^r{}_{11l}\varphi_{,r}) \\ &- 2\varphi_{;11}\varphi_{;11l}\varphi_{,k} + 2\varphi_{;11}\varphi_{,k}R^s{}_{11l}\varphi_{,s} \\ &- 2\varphi_{,1}\varphi_{;11l}\varphi_{;k1} + 2\varphi_{,1}\varphi_{;k1}R^s{}_{11l}\varphi_{,s} \\ &+ (\varphi_{;11})^2\varphi_{,l}\varphi_{,k} + 2\varphi_{;11}\varphi_{,1}\varphi_{,l}\varphi_{;k1} + (\varphi_{,1})^2\varphi_{;l1}\varphi_{;k1}] \\ =& w^{kl}[\varphi_{;11k}\varphi_{;11l} - 2\varphi_{;11l}R^s{}_{11k}\varphi_{,s} + (R^s{}_{11k}\varphi_{,s})(R^r{}_{11l}\varphi_{,r}) \\ &- 2\varphi_{;11}\varphi_{;11l}\varphi_{,k} + 2\varphi_{;11}\varphi_{,k}R^s{}_{11l}\varphi_{,s} \\ &- 2\varphi_{,1}\varphi_{;11l}\varphi_{;k1} + 2\varphi_{,1}\varphi_{;k1}R^s{}_{11l}\varphi_{,s} \\ &+ (\varphi_{;11})^2\varphi_{,l}\varphi_{,k} + 2\varphi_{;11}\varphi_{,1}\varphi_{,l}\varphi_{;k1} + (\varphi_{,1})^2\varphi_{;l1}\varphi_{;k1}]. \end{aligned} \tag{6.37}$$

The terms containing the product of the third derivatives of φ vanish in the difference of (6.36) and (6.37)

$$\begin{aligned} w^{kl}\left(w_{11;k}w_{11;l}-w_{1k;1}w_{1l;1}\right) = & w^{kl}[\varphi_{;11k}\varphi_{;11l}-4\varphi_{,1}\varphi_{;11l}\varphi_{;1k}+4(\varphi_{,1})^2\varphi_{;1k}\varphi_{;1l} \\ & -\varphi_{;11k}\varphi_{;11l}+2\varphi_{;11l}R^s{}_{11k}\varphi_{,s} \\ & -(R^s{}_{11k}\varphi_{,s})(R^r{}_{11l}\varphi_{,r}) \\ & +2\varphi_{;11}\varphi_{;11l}\varphi_{,k}-2\varphi_{;11}\varphi_{,k}R^s{}_{11l}\varphi_{,s} \\ & +2\varphi_{,1}\varphi_{;11l}\varphi_{;k1}-2\varphi_{,1}\varphi_{;k1}R^s{}_{11l}\varphi_{,s} \\ & -(\varphi_{;11})^2\varphi_{,l}\varphi_{,k}-2\varphi_{;11}\varphi_{,1}\varphi_{,l}\varphi_{;k1} \\ & -(\varphi_{,1})^2\varphi_{;l1}\varphi_{;k1}] \\ = & w^{kl}[-2\varphi_{,1}\varphi_{;11l}\varphi_{;1k}+3(\varphi_{,1})^2\varphi_{;1k}\varphi_{;1l} \\ & +2\varphi_{;11l}R^s{}_{11k}\varphi_{,s}-(R^s{}_{11k}\varphi_{,s})(R^r{}_{11l}\varphi_{,r}) \\ & +2\varphi_{;11}\varphi_{;11l}\varphi_{,k}-2\varphi_{;11}\varphi_{,k}R^s{}_{11l}\varphi_{,s} \\ & -2\varphi_{,1}\varphi_{;k1}R^s{}_{11l}\varphi_{,s} \\ & -(\varphi_{;11})^2\varphi_{,l}\varphi_{,k}-2\varphi_{;11}\varphi_{,1}\varphi_{,l}\varphi_{;k1}]. \end{aligned} \tag{6.38}$$

Substituting the expression (6.12) of the Riemannian curvature tensor of the sphere, given by

$$R^s{}_{11k}=\delta^s_1\sigma_{1k}-\delta^s_k\sigma_{11},$$

we obtain for a term involving third derivatives

$$\begin{aligned} 2\varphi_{;11l}R^s{}_{11k}\varphi_{,s} &= 2\varphi_{;11l}(\delta^s_1\sigma_{1k}-\delta^s_k\sigma_{11})\varphi_{,s} \\ &= 2\varphi_{;11l}\sigma_{1k}\varphi_{,1}-2\varphi_{;11l}\sigma_{11}\varphi_{,k}. \end{aligned} \tag{6.39}$$

We do the same for

$$\begin{aligned} w^{kl}(R^s{}_{11k}\varphi_{,s})(R^r{}_{11l}\varphi_{,r}) &= w^{kl}(\delta^s_1\sigma_{1k}-\delta^s_k\sigma_{11})\varphi_{,s}(\delta^r_1\sigma_{1l}-\delta^r_l\sigma_{11})\varphi_{,r} \\ &= w^{kl}(\delta^s_1\sigma_{1k}\delta^r_1\sigma_{1l}-\delta^s_1\sigma_{1k}\delta^r_l\sigma_{11} \\ &\quad -\delta^s_k\sigma_{11}\delta^r_1\sigma_{1l}+\delta^s_k\sigma_{11}\delta^r_l\sigma_{11})\varphi_{,s}\varphi_{,r} \\ &= (\varphi_{,1})^2w^{kl}\sigma_{1k}\sigma_{1l}-2\sigma_{11}\varphi_{,1}w^{kl}\sigma_{1k}\varphi_{,l}+(\sigma_{11})^2w^{kl}\varphi_{,k}\varphi_{,l}. \end{aligned} \tag{6.40}$$

In the same way for the two other terms involving the curvature tensor it holds

$$\begin{aligned} 2w^{kl}\varphi_{;11}\varphi_{,k}R^s{}_{11l}\varphi_{,s} &= 2w^{kl}\varphi_{;11}\varphi_{,k}(\delta^s_1\sigma_{1l}-\delta^s_l\sigma_{11})\varphi_{,s} \\ &= 2\varphi_{;11}\varphi_{,1}w^{kl}\varphi_{,k}\sigma_{1l}-2\sigma_{11}\varphi_{;11}w^{kl}\varphi_{,k}\varphi_{,l}. \end{aligned} \tag{6.41}$$

and

$$\begin{aligned} 2w^{kl}\varphi_{,1}\varphi_{;k1}R^s{}_{11l}\varphi_{,s} &= 2w^{kl}\varphi_{,1}\varphi_{;k1}(\delta^s_1\sigma_{1l}-\delta^s_l\sigma_{11})\varphi_{,s} \\ &= 2(\varphi_1)^2w^{kl}\varphi_{;k1}\sigma_{1l}-2\sigma_{11}\varphi_{,1}w^{kl}\varphi_{;k1}\varphi_{,l}. \end{aligned} \tag{6.42}$$

Inserting (6.39), (6.40), (6.41) and (6.42) into the expression (6.38), that we need to estimate, provides

$$\begin{aligned}
w^{kl}\left(w_{11;k}w_{11;l}-w_{1k;1}w_{1l;1}\right)=&\,w^{kl}[-2\varphi_{,1}\varphi_{;11l}\varphi_{;1k}+3(\varphi_{,1})^2\varphi_{;1k}\varphi_{;1l}\\
&+2\varphi_{;11l}R^s{}_{11k}\varphi_{,s}-(R^s{}_{11k}\varphi_{,s})(R^r{}_{11l}\varphi_{,r})\\
&+2\varphi_{;11}\varphi_{;11l}\varphi_{,k}-2\varphi_{;11}\varphi_{,k}R^s{}_{11l}\varphi_{,s}\\
&-2\varphi_{,1}\varphi_{;k1}R^s{}_{11l}\varphi_{,s}\\
&-(\varphi_{;11})^2\varphi_{,l}\varphi_{,k}-2\varphi_{;11}\varphi_{,1}\varphi_{,l}\varphi_{;k1}]\\
=&\,w^{kl}[-2\varphi_{,1}\varphi_{;11l}\varphi_{;1k}+3(\varphi_{,1})^2\varphi_{;1k}\varphi_{;1l}\\
&+2\varphi_{;11l}\sigma_{1k}\varphi_{,1}-2\sigma_{11}\varphi_{;11l}\varphi_{,k}\\
&-(\varphi_{,1})^2\sigma_{1k}\sigma_{1l}+2\sigma_{11}\varphi_{,1}\sigma_{1k}\varphi_{,l}-(\sigma_{11})^2\varphi_{,k}\varphi_{,l}\\
&+2\varphi_{;11}\varphi_{;11l}\varphi_{,k}-2\varphi_{;11}\varphi_{,1}\varphi_{,k}\sigma_{1l}\\
&+2\sigma_{11}\varphi_{;11}\varphi_{,k}\varphi_{,l}\\
&-2(\varphi_{,1})^2\varphi_{;k1}\sigma_{1l}+2\sigma_{11}\varphi_{,1}\varphi_{;k1}\varphi_{,l}\\
&-(\varphi_{;11})^2\varphi_{,l}\varphi_{,k}-2\varphi_{;11}\varphi_{,1}\varphi_{,l}\varphi_{;k1}]
\end{aligned}$$

and consequently

$$\begin{aligned}
w^{kl}\left(w_{11;k}w_{11;l}-w_{1k;1}w_{1l;1}\right)=&\,w^{kl}[2\varphi_{,1}\varphi_{;11l}(\sigma_{1k}-\varphi_{;k1})-2\varphi_{;11l}\varphi_{,k}(\sigma_{11}-\varphi_{;11})\\
&-(\varphi_{,1})^2(\sigma_{1k}\sigma_{1l}+2\varphi_{;k1}\sigma_{1l}-3\varphi_{;l1}\varphi_{;k1})\\
&+2\varphi_{,1}\varphi_{,l}(\sigma_{11}-\varphi_{;11})(\sigma_{1k}+\varphi_{;k1})\\
&-\left((\sigma_{11})^2-2\sigma_{11}\varphi_{;11}+(\varphi_{;11})^2\right)\varphi_{,k}\varphi_{,l}]. \quad (6.43)
\end{aligned}$$

From the definition of w_{ij} it follows

$$\begin{aligned}
w^{kl}(\sigma_{1k}\sigma_{1l}+2\varphi_{;k1}\sigma_{1l}-3\varphi_{;l1}\varphi_{;k1})=&\,w^{kl}(\sigma_{1k}+3\varphi_{;k1})(\sigma_{1l}-\varphi_{;l1})\\
=&\,w^{kl}(4\sigma_{1k}-3w_{1k}+3\varphi_{,1}\varphi_{,k})(w_{1l}-\varphi_{,1}\varphi_{,l})\\
=&\,w^{kl}(4\sigma_{1k}w_{1l}-4\sigma_{1k}\varphi_{,1}\varphi_{,l}-3w_{1k}w_{1l}\\
&\quad+6w_{1k}\varphi_{,1}\varphi_{,l}-3(\varphi_{,1})^2\varphi_{,k}\varphi_{,l})\\
=&\,4\sigma_{11}-4\varphi_{,1}w^{kl}\sigma_{1k}\varphi_{,l}-3w_{11}\\
&\quad+6(\varphi_{,1})^2-3(\varphi_{,1})^2w^{kl}\varphi_{,k}\varphi_{,l} \qquad (6.44)
\end{aligned}$$

and

$$(\sigma_{11})^2-2\sigma_{11}\varphi_{;11}+(\varphi_{;11})^2=(\sigma_{11}-\varphi_{;11})^2=(w_{11}-(\varphi_{,1})^2)^2 \qquad (6.45)$$

as well as

$$\begin{aligned}
w^{kl}(\sigma_{1k}+\varphi_{;1k})&=w^{kl}(2\sigma_{1k}-w_{1k}+\varphi_{,1}\varphi_{,k})=2w^{kl}\sigma_{1k}-\delta_1^l+w^{kl}\varphi_{,1}\varphi_{,k},\\
w^{kl}(\sigma_{1k}-\varphi_{;1k})&=w^{kl}(w_{1k}-\varphi_{,1}\varphi_{,k})=\delta_1^l-w^{kl}\varphi_{,1}\varphi_{,k}.
\end{aligned}$$

For the terms involving third derivatives we therefore get

$$\begin{aligned}
&w^{kl}(2\varphi_{,1}\varphi_{;11l}(\sigma_{1k}-\varphi_{;k1})-2\varphi_{;11l}\varphi_{,k}(\sigma_{11}-\varphi_{;11}))\\
&\quad=2\varphi_{,1}\varphi_{;11l}(\delta_1^l-w^{kl}\varphi_{,1}\varphi_{,k})-2w^{kl}\varphi_{;11l}\varphi_{,k}(w_{11}-(\varphi_{,1})^2)\\
&\quad=2\varphi_{,1}\varphi_{;111}-2(\varphi_{,1})^2w^{kl}\varphi_{;11l}\varphi_{,k}-2w_{11}w^{kl}\varphi_{;11l}\varphi_{,k}+2(\varphi_{,1})^2w^{kl}\varphi_{;11l}\varphi_{,k}\\
&\quad=2\varphi_{,1}\varphi_{;111}-2w_{11}w^{kl}\varphi_{;11l}\varphi_{,k}.
\end{aligned}$$

Furthermore from

$$\begin{aligned} w^{kl}\varphi_{;11l}\varphi_{,k} =& w^{kl}(-w_{11;l} + 2\varphi_{,1}\varphi_{;1l})\varphi_{,k} \\ =& - w^{kl}w_{11;l}\varphi_{,k} + 2\varphi_{,1}w^{kl}(\sigma_{1l} - w_{1l} + \varphi_{,1}\varphi_{,l})\varphi_{,k} \\ =& - w^{kl}w_{11;l}\varphi_{,k} + 2\varphi_{,1}w^{kl}\sigma_{1l}\varphi_{,k} - 2(\varphi_{,1})^2 + 2(\varphi_{,1})^2 w^{kl}\varphi_{,k}\varphi_{,l} \end{aligned}$$

and

$$2\varphi_{,1}\varphi_{;111} = 2\varphi_{,1}(-w_{11;1} + 2\varphi_{;11}\varphi_{,1}) = -2\varphi_{,1}w_{11;1} + 4(\sigma_{11} - w_{11} + (\varphi_{,1})^2)(\varphi_{,1})^2$$

it follows

$$\begin{aligned} & w^{kl}(2\varphi_{,1}\varphi_{;11l}(\sigma_{1k}-\varphi_{;k1}) - 2\varphi_{;11l}\varphi_{,k}(\sigma_{11} - \varphi_{;11})) \\ &= 2\varphi_{,1}\varphi_{;111} - 2w_{11}w^{kl}\varphi_{;11l}\varphi_{,k} \\ &= - 2\varphi_{,1}w_{11;1} + 4(\sigma_{11} - w_{11} + (\varphi_{,1})^2)(\varphi_{,1})^2 \\ &\quad + 2w_{11}w^{kl}w_{11;l}\varphi_{,k} - 4w_{11}\varphi_{,1}w^{kl}\sigma_{1l}\varphi_{,k} \\ &\quad + 4w_{11}(\varphi_{,1})^2 - 4w_{11}(\varphi_{,1})^2 w^{kl}\varphi_{,k}\varphi_{,l} \\ &= - 2\varphi_{,1}w_{11;1} + 4(\sigma_{11} + (\varphi_{,1})^2)(\varphi_{,1})^2 \\ &\quad + 2w_{11}w^{kl}w_{11;l}\varphi_{,k} - 4w_{11}\varphi_{,1}w^{kl}\sigma_{1l}\varphi_{,k} \\ &\quad - 4w_{11}(\varphi_{,1})^2 w^{kl}\varphi_{,k}\varphi_{,l}. \end{aligned} \tag{6.46}$$

Moreover we have

$$\begin{aligned} & 2w^{kl}\varphi_{,1}\varphi_{,l}(\sigma_{11} - \varphi_{;11})(\sigma_{1k} + \varphi_{;k1}) \\ &= 2\varphi_{,1}\varphi_{,l}(w_{11} - (\varphi_{,1})^2)(2w^{kl}\sigma_{1k} - \delta_1^l + w^{kl}\varphi_{,1}\varphi_{,k}) \\ &= 2\varphi_{,1}(w_{11} - (\varphi_{,1})^2)(2w^{kl}\sigma_{1k}\varphi_{,l} - \varphi_{,1} + w^{kl}\varphi_{,1}\varphi_{,l}\varphi_{,k}) \\ &= (4\varphi_{,1}w^{kl}\sigma_{1k}\varphi_{,l} - 2(\varphi_{,1})^2 + 2(\varphi_{,1})^2 w^{kl}\varphi_{,l}\varphi_{,k})(w_{11} - (\varphi_{,1})^2). \end{aligned} \tag{6.47}$$

The sum of (6.46) and (6.47) is

$$\begin{aligned} -2\varphi_{,1}w_{11;1} &+ 2w_{11}w^{kl}w_{11;l}\varphi_{,k} - 4w_{11}\varphi_{,1}w^{kl}\sigma_{1l}\varphi_{,k} \\ &+ 4(\sigma_{11} + (\varphi_{,1})^2)(\varphi_{,1})^2 - 4w_{11}(\varphi_{,1})^2 w^{kl}\varphi_{,k}\varphi_{,l} \\ &+ (4\varphi_{,1}w^{kl}\sigma_{1k}\varphi_{,l} - 2(\varphi_{,1})^2 + 2(\varphi_{,1})^2 w^{kl}\varphi_{,l}\varphi_{,k})(w_{11} - (\varphi_{,1})^2) \\ = &- 2\varphi_{,1}w_{11;1} + 2w_{11}w^{kl}w_{11;l}\varphi_{,k} \\ &+ 2(2\sigma_{11} - w_{11} + 3(\varphi_{,1})^2)(\varphi_{,1})^2 - 2(w_{11} + (\varphi_{,1})^2)(\varphi_{,1})^2 w^{kl}\varphi_{,k}\varphi_{,l} \\ &- 4(\varphi_{,1})^3 w^{kl}\sigma_{1k}\varphi_{,l}. \end{aligned}$$

Inserting (6.44), (6.45) and this last expression into (6.43) we obtain

$$\begin{aligned} w^{kl}(w_{11;k}w_{11;l} - w_{1k;1}w_{1l;1}) = &- 2\varphi_{,1}w_{11;1} + 2w_{11}w^{kl}w_{11;l}\varphi_{,k} \\ &- (\varphi_{,1})^2(4\sigma_{11} - 4\varphi_{,1}w^{kl}\sigma_{1k}\varphi_{,l} - 3w_{11} \\ &\qquad + 6(\varphi_{,1})^2 - 3(\varphi_{,1})^2 w^{kl}\varphi_{,k}\varphi_{,l}) \\ &+ 2(2\sigma_{11} - w_{11} + 3(\varphi_{,1})^2)(\varphi_{,1})^2 \\ &- 2(w_{11} + (\varphi_{,1})^2)(\varphi_{,1})^2 w^{kl}\varphi_{,k}\varphi_{,l} \\ &- 4(\varphi_{,1})^3 w^{kl}\sigma_{1k}\varphi_{,l} \\ &- \left(w_{11} - (\varphi_{,1})^2\right)^2 w^{kl}\varphi_{,k}\varphi_{,l} \end{aligned}$$

and, adding and rearranging the terms,

$$\begin{aligned} w^{kl}(w_{11;k}w_{11;l} - w_{1k;1}w_{1l;1}) = &- 2\varphi_{,1}w_{11;1} + 2w_{11}w^{kl}w_{11;l}\varphi_{,k} \\ &- (\varphi_{,1})^2(4\sigma_{11} - 3w_{11} + 6(\varphi_{,1})^2 \\ &\qquad - 2(2\sigma_{11} - w_{11} + 3(\varphi_{,1})^2)) \\ &+ \left(3(\varphi_{,1})^4 - 2(w_{11} + (\varphi_{,1})^2)(\varphi_{,1})^2\right) w^{kl}\varphi_{,k}\varphi_{,l} \\ &- (w_{11} - (\varphi_{,1})^2)^2 w^{kl}\varphi_{,k}\varphi_{,l} \\ = &- 2\varphi_{,1}w_{11;1} + 2w_{11}w^{kl}w_{11;l}\varphi_{,k} \\ &+ w_{11}(\varphi_{,1})^2 \\ &- (w_{11})^2 w^{kl}\varphi_{,k}\varphi_{,l}. \end{aligned} \tag{6.48}$$

Since the first derivatives of W in the point (x_0, t_0) corresponds to those of v in (x_0, ξ_0, t_0), where this last function attains its maximum, we have

$$0 = W_{;i} = \frac{V_{;i}}{V} + \lambda\sigma^{rs}\varphi_{;ri}\varphi_{,s} = \frac{w_{11;i} + \tilde{v}_{;i}}{V} + \lambda\sigma^{rs}\varphi_{;ri}\varphi_{,s}.$$

This implies

$$w_{11;1} = (-\lambda V\sigma^{rs}\varphi_{;r1}\varphi_{,s} - \tilde{v}_{;1})$$

and

$$\begin{aligned} w^{kl}w_{11;l} &= w^{kl}\left(-\lambda V\sigma^{rs}\varphi_{;rl}\varphi_{,s} - \tilde{v}_{;l}\right) \\ &= -\lambda V w^{kl}\varphi_{,s}\left(\sigma^{rs}\varphi_{;rl}\right) - w^{kl}\tilde{v}_{;l} \\ &= -\lambda V w^{kl}\varphi_{,s}\left(\delta^s_l - \sigma^{rs}w_{rl} + \sigma^{rs}\varphi_{,r}\varphi_{,l}\right) - w^{kl}\tilde{v}_{;l} \\ &= -\lambda V w^{kl}\varphi_{,l} + \lambda V w^{kl}\sigma^{rs}w_{rl}\varphi_{,s} - \lambda V w^{kl}|D\varphi|^2\varphi_{,l} - w^{kl}\tilde{v}_{;l} \\ &= -\lambda V w^{kl}\varphi_{,l} + \lambda V\sigma^{ks}\varphi_{,s} - \lambda V w^{kl}|D\varphi|^2\varphi_{,l} - w^{kl}\tilde{v}_{;l}. \end{aligned}$$

It follows that in (x_0, t_0) the two first terms of (6.48) become

$$\begin{aligned} -2\varphi_{,1}w_{11;1} &= 2\varphi_{,1}\left(\lambda V\sigma^{rs}\varphi_{;r1}\varphi_{,s} + \tilde{v}_{;1}\right) \\ &= 2\varphi_{,1}\lambda V\sigma^{rs}\varphi_{;r1}\varphi_{,s} + 2\varphi_{,1}\tilde{v}_{;1} \\ &= 2\varphi_{,1}\lambda V\sigma^{rs}(\sigma_{r1} - w_{r1} + \varphi_{,r}\varphi_{,1})\varphi_{,s} + 2\varphi_{,1}\tilde{v}_{;1} \\ &= 2(\varphi_{,1})^2\lambda V - 2\varphi_{,1}\lambda V\sigma^{rs}w_{r1}\varphi_{,s} + 2(\varphi_{,1})^2\lambda V|D\varphi|^2 + 2\varphi_{,1}\tilde{v}_{;1} \end{aligned}$$

respectively

$$\begin{aligned} 2w_{11}w^{kl}w_{11;l}\varphi_{,k} =& 2w_{11}\left[-\lambda V\left(w^{kl}\varphi_{,l}-\sigma^{ks}\varphi_{,s}+w^{kl}|D\varphi|^2\varphi_{,l}\right)-w^{kl}\tilde{v}_{;l}\right]\varphi_{,k}\\ =& -2\lambda V w_{11}w^{kl}\varphi_{,k}\varphi_{,l}+2\lambda V w_{11}|D\varphi|^2\\ & -2\lambda V w_{11}w^{kl}\varphi_{,k}\varphi_{,l}|D\varphi|^2-2w_{11}w^{kl}\varphi_{,k}\tilde{v}_{;l}. \end{aligned}$$

Their sum is therefore given by

$$\begin{aligned} -2\varphi_{,1}w_{11;1}+2w_{11}w^{kl}w_{11;l}\varphi_{,k} =& 2(\varphi_{,1})^2\lambda V-2\varphi_{,1}\lambda V\sigma^{rs}w_{r1}\varphi_{,s}\\ & +2(\varphi_{,1})^2\lambda V|D\varphi|^2+2\varphi_{,1}\tilde{v}_{;1}\\ & -2\lambda V w_{11}w^{kl}\varphi_{,k}\varphi_{,l}+2\lambda V w_{11}|D\varphi|^2\\ & -2\lambda V w_{11}w^{kl}\varphi_{,k}\varphi_{,l}|D\varphi|^2-2w_{11}w^{kl}\varphi_{,k}\tilde{v}_{;l}. \end{aligned}$$

Moreover we have

$$\begin{aligned} 2w^{kl}w_{11;l}\tilde{v}_{;k} =& 2\left[-\lambda V\left(w^{kl}\varphi_{,l}-\sigma^{ks}\varphi_{,s}+w^{kl}|D\varphi|^2\varphi_{,l}\right)-w^{kl}\tilde{v}_{;l}\right]\tilde{v}_{;k}\\ =& -2\lambda V w^{kl}\tilde{v}_{;k}\varphi_{,l}+2\lambda V\sigma^{ks}\tilde{v}_{;k}\varphi_{,s}\\ & -2\lambda V w^{kl}\tilde{v}_{;k}\varphi_{,l}|D\varphi|^2-2w^{kl}\tilde{v}_{;k}\tilde{v}_{;l}. \end{aligned}$$

and in (x_0,t_0) we eventually obtain the estimate

$$\begin{aligned} \frac{1}{V^2}w^{kl}V_{;k}V_{;l}-\frac{1}{V}w^{pa}w^{bq}w_{ab;1}w_{pq;1} \le& \frac{1}{Vw_{11}}w^{kl}V_{;k}V_{;l}-\frac{1}{V}\frac{1}{w_{11}}w^{kl}w_{1k;1}w_{1l;1}\\ =& \frac{1}{Vw_{11}}\Big[-2\lambda V w^{kl}\tilde{v}_{;k}\varphi_{,l}+2\lambda V\sigma^{ks}\tilde{v}_{;k}\varphi_{,s}\\ & -2\lambda V w^{kl}\tilde{v}_{;k}\varphi_{,l}|D\varphi|^2-2w^{kl}\tilde{v}_{;k}\tilde{v}_{;l}\\ & +w^{kl}\tilde{v}_{;k}\tilde{v}_{;l}\\ & +2(\varphi_{,1})^2\lambda V-2\varphi_{,1}\lambda V\sigma^{rs}w_{r1}\varphi_{,s}\\ & +2(\varphi_{,1})^2\lambda V|D\varphi|^2+2\varphi_{,1}\tilde{v}_{;1}\\ & -2\lambda V w_{11}w^{kl}\varphi_{,k}\varphi_{,l}+2\lambda V w_{11}|D\varphi|^2\\ & -2\lambda V w_{11}w^{kl}\varphi_{,k}\varphi_{,l}|D\varphi|^2\\ & -2w_{11}w^{kl}\varphi_{,k}\tilde{v}_{;l}\\ & +w_{11}(\varphi_{,1})^2-(w_{11})^2w^{kl}\varphi_{,k}\varphi_{,l}\Big]. \end{aligned}$$

Multiplying out and regrouping some terms provides

$$\begin{aligned}\frac{1}{V^2}w^{kl}V_{;k}V_{;l}-\frac{1}{V}w^{pa}w^{bq}w_{ab;1}w_{pq;1}\le&-2\lambda\cdot\frac{1}{w_{11}}\cdot\left(1+|D\varphi|^2\right)w^{kl}\tilde{v}_{;k}\varphi_{,l}\\&+2\lambda\cdot\frac{1}{w_{11}}\cdot\sigma^{ks}\tilde{v}_{;k}\varphi_{,s}\\&-\frac{1}{Vw_{11}}w^{kl}\tilde{v}_{;k}\tilde{v}_{;l}\\&+2\lambda\frac{(\varphi_{,1})^2}{w_{11}}\cdot\left(1+|D\varphi|^2\right)-2\lambda\frac{\varphi_{,1}}{w_{11}}\sigma^{rs}w_{r1}\varphi_{,s}\\&+\frac{2}{Vw_{11}}\cdot\varphi_{,1}\tilde{v}_{;1}\\&-2\lambda w^{kl}\varphi_{,k}\varphi_{,l}\cdot\left(1+|D\varphi|^2\right)+2\lambda|D\varphi|^2\\&-\frac{2}{V}\cdot w^{kl}\varphi_{,k}\tilde{v}_{;l}\\&+\frac{1}{V}\left((\varphi_{,1})^2-w_{11}w^{kl}\varphi_{,k}\varphi_{,l}\right).\end{aligned}\tag{6.49}$$

The third and the last summand are negative since w^{kl} is positive definite. Multiplying by $\frac{\dot{\varphi}}{n}$ and estimating yields

$$\frac{1}{V}\cdot\frac{\dot{\varphi}}{n}\cdot(\varphi_{,1})^2\le\frac{c}{V}=\frac{c}{w_{11}+\tilde{v}},\tag{6.50}$$

for the second-last term, as well as

$$\frac{\dot{\varphi}}{n}\cdot 2\lambda\cdot\frac{(\varphi_{,1})^2}{w_{11}}\cdot(1+|D\varphi|^2)\le\frac{c\lambda}{w_{11}}\tag{6.51}$$

and, using (6.32),

$$-\frac{\dot{\varphi}}{n}\cdot 2\lambda\cdot\frac{\varphi_{,1}}{w_{11}}\cdot\sigma^{rs}w_{r1}\varphi_{,s}\le\frac{c\lambda}{w_{11}}\cdot(w_{11}+c)=c\lambda+\frac{c\lambda}{w_{11}}.\tag{6.52}$$

We rewrite some of the last terms of (6.49), involving $\tilde{v}$, using the fact that this map is of the form $\tilde{v} = -\rho^i\varphi_i + C$, as noted before:

$$
\begin{aligned}
-w^{kl}\tilde{v}_{;k}\varphi_{,l} &\cdot \left(1+|D\varphi|^2\right) + \sigma^{kl}\tilde{v}_{;k}\varphi_{,l} \\
&= -\, w^{kl}\left(-\rho^p\varphi_{,p}\right)_{;k}\cdot\varphi_{,l}\cdot\left(1+|D\varphi|^2\right) + \sigma^{kl}\left(-\rho^p\varphi_{,p}\right)_{;k}\cdot\varphi_{,l} \\
&= \left(w^{kl}\cdot\left(1+|D\varphi|^2\right) - \sigma^{kl}\right)\left(\rho^p_{;k}\varphi_{,p} + \rho^p\varphi_{;pk}\right)\varphi_{,l} \\
&= \left(w^{kl}\cdot\left(1+|D\varphi|^2\right) - \sigma^{kl}\right)\left(\rho^p_{;k}\varphi_{,p} + \rho^p\sigma_{pk} - \rho^p w_{pk} + \rho^p\varphi_{,p}\varphi_{,k}\right)\varphi_{,l} \\
&= \left(w^{kl}\rho^p_{;k}\varphi_{,p}\varphi_{,l} + w^{kl}\rho^p\sigma_{pk}\varphi_{,l} - \rho^l\varphi_{,l} + w^{kl}\rho^p\varphi_{,p}\varphi_{,k}\varphi_{,l}\right)\cdot\left(1+|D\varphi|^2\right) \\
&\quad - \sigma^{kl}\rho^p_{;k}\varphi_{,p}\varphi_{,l} - \rho^l\varphi_{,l} + \sigma^{kl}\rho^p w_{pk}\varphi_{,l} - \sigma^{kl}\rho^p\varphi_{,p}\varphi_{,k}\varphi_{,l} \\
&= \left(w^{kl}\rho^p_{;k}\varphi_{,p}\varphi_{,l} + w^{kl}\rho^p\sigma_{pk}\varphi_{,l} + v'\left(1 - w^{kl}\varphi_{,k}\varphi_{,l}\right)\right)\cdot\left(1+|D\varphi|^2\right) \\
&\quad - \sigma^{kl}\rho^p_{;k}\varphi_{,p}\varphi_{,l} + \sigma^{kl}\rho^p w_{pk}\varphi_{,l} + v'\left(1+|D\varphi|^2\right) \\
&= \left(w^{kl}\rho^p_{;k}\varphi_{,p}\varphi_{,l} + w^{kl}\rho^p\sigma_{pk}\varphi_{,l} + v'\left(2 - w^{kl}\varphi_{,k}\varphi_{,l}\right)\right)\cdot\left(1+|D\varphi|^2\right) \\
&\quad - \sigma^{kl}\rho^p_{;k}\varphi_{,p}\varphi_{,l} + \sigma^{kl}\rho^p w_{pk}\varphi_{,l},
\end{aligned} \tag{6.53}
$$

To estimate this last expression it is convenient to consider separately: the terms on the first line

$$
\frac{2\lambda\dot\varphi}{n}\cdot\frac{1}{w_{11}}\cdot\left(w^{kl}\rho^p_{;k}\varphi_{,l}\varphi_{,p} + w^{kl}\sigma_{pk}\rho^p\varphi_{,l} - v'w^{kl}\varphi_{,k}\varphi_{,l}\right)(1+|D\varphi|^2) \le \frac{c\lambda\,\mathrm{tr}\,w^{ij}}{w_{11}}, \tag{6.54}
$$

except for

$$
\frac{2\lambda\dot\varphi}{n}\cdot\frac{1}{w_{11}}\cdot 2v'(1+|D\varphi|^2) \le \frac{c\lambda}{w_{11}}; \tag{6.55}
$$

and then the two terms on the second line of (6.53):

$$
\frac{2\lambda\dot\varphi}{n}\cdot\frac{1}{w_{11}}\cdot\left(-\sigma^{kl}\rho^p_{;k}\varphi_{,p}\varphi_{,l}\right) \le \frac{c\lambda}{w_{11}} \tag{6.56}
$$

and

$$
\frac{2\lambda\dot\varphi}{n}\cdot\frac{1}{w_{11}}\cdot\sigma^{kl}\rho^p w_{pk}\varphi_{,l} \le \frac{c\lambda}{w_{11}}\cdot(w_{11}+c) = c\lambda + \frac{c\lambda}{w_{11}}, \tag{6.57}
$$

observing that in all this four cases we already have took care of the multiplying factors that will appear in the inequality for PW.

Analogously to (6.53), we expand

$$
\begin{aligned}
\tilde{v}_{;1}\varphi_{,1} = \left(-\rho^p\varphi_{,p}\right)_{;1}\cdot\varphi_{,1} &= -\left(\rho^p_{;1}\varphi_{,p} + \rho^p\varphi_{;p1}\right)\varphi_{,1} \\
&= -\left(\rho^p_{;1}\varphi_{,p} + \rho^p\sigma_{p1} - \rho^p w_{p1} + \rho^p\varphi_{,p}\varphi_{,1}\right)\varphi_{,1} \\
&= -\varphi_{,1}\left(\rho^p_{;1}\varphi_{,p} + \rho^p\sigma_{p1} - \rho^p w_{p1} - v'\varphi_{,1}\right)
\end{aligned} \tag{6.58}
$$

and get hence the following estimates:

$$
-\frac{\dot\varphi}{n}\cdot\frac{2}{Vw_{11}}\cdot\varphi_{,1}\left(\rho^p_{;1}\varphi_{,p} + \rho^p\sigma_{p1} - v'\varphi_{,1}\right) \le \frac{c}{w_{11}(w_{11}+\tilde{v})} \tag{6.59}
$$

as well as

$$
-\frac{\dot\varphi}{n}\cdot\frac{2}{Vw_{11}}\cdot\varphi_{,1}\cdot\left(-\rho^p w_{p1}\right) \le \frac{cw_{11}+c}{w_{11}(w_{11}+\tilde{v})} = \frac{c}{w_{11}+\tilde{v}} + \frac{c}{w_{11}(w_{11}+\tilde{v})}. \tag{6.60}
$$

In view of (6.50), (6.51), (6.52), (6.54), (6.55), (6.56), (6.57), (6.58), (6.59) and (6.60) and multiplying by $\frac{\dot{\varphi}}{n}$, (6.49) becomes

$$
\begin{aligned}
\frac{\dot{\varphi}}{n}\cdot\left(\frac{1}{V^2}w^{kl}V_{;k}V_{;l}-\frac{1}{V}w^{pa}w^{bq}w_{ab;1}w_{pq;1}\right)&\\
\le \frac{c\lambda \operatorname{tr} w^{ij}}{w_{11}}+\frac{c\lambda}{w_{11}}&\\
+\frac{c\lambda}{w_{11}}+c\lambda+\frac{c\lambda}{w_{11}}&\\
-0&\\
+\frac{c\lambda}{w_{11}}+c\lambda+\frac{c\lambda}{w_{11}}&\\
+\frac{c}{w_{11}(w_{11}+\tilde{v})}+\frac{c}{w_{11}+\tilde{v}}&\\
-\frac{\dot{\varphi}}{n}\cdot\left[2\lambda w^{kl}\varphi_{,k}\varphi_{,l}\cdot\left(1+|D\varphi|^2\right)-2\lambda|D\varphi|^2\right]&\\
-\frac{\dot{\varphi}}{n}\cdot\frac{2}{V}\cdot w^{kl}\varphi_{,k}\tilde{v}_{;l}&\\
+\frac{c}{w_{11}+\tilde{v}}+\frac{c}{w_{11}(w_{11}+\tilde{v})}-0&
\end{aligned}
\tag{6.61}
$$

and it holds therefore

$$
\begin{aligned}
\frac{\dot{\varphi}}{n}\cdot\left(\frac{1}{V^2}w^{kl}V_{;k}V_{;l}-\frac{1}{V}w^{pa}w^{bq}w_{ab;1}w_{pq;1}\right)&\\
\le \frac{c\lambda \operatorname{tr} w^{ij}}{w_{11}}+\frac{c\lambda}{w_{11}}+c\lambda+\frac{c}{w_{11}(w_{11}+\tilde{v})}+\frac{c}{w_{11}+\tilde{v}}&\\
-\frac{\dot{\varphi}}{n}\cdot\left[2\lambda w^{kl}\varphi_{,k}\varphi_{,l}\cdot\left(1+|D\varphi|^2\right)-2\lambda|D\varphi|^2\right]&\\
-\frac{2}{V}\cdot\frac{\dot{\varphi}}{n}\cdot w^{kl}\varphi_{,k}\tilde{v}_{;l}.&
\end{aligned}
\tag{6.62}
$$

In the point (x_0,t_0), since v is maximal at (x_0,ξ_0,t_0) and the derivatives (up to the second order) of v and W coincide there, as we have shown at the beginning of this section, we have $\dot{W}(x_0,t_0)\ge 0$, $W_{;ij}(x_0,t_0)\le 0$ and $W_{;k}(x_0,t_0)=0$ and consequently $0\le PW(x_0,t_0)$. Using the last inequality to further estimate the right-hand side of (6.35), yields therefore in this point

$$
\begin{aligned}
0 \le PW \le \frac{1}{V} &\cdot \left(c + c w_{11} + c \operatorname{tr} w^{ij} \right) \\
&+ \frac{1}{V} \cdot \left(c \operatorname{tr} w^{ij} + c + \frac{2\dot{\varphi}}{n} \cdot w^{pk} \tilde{v}_{;k} \varphi_{,p} \right) \\
&- \varepsilon \lambda \operatorname{tr} w^{ij} + c\lambda - \frac{\lambda \dot{\varphi}}{n} \cdot \left[\left(1 + |D\varphi|^2\right) w^{kl} \varphi_{,k} \varphi_{,l} - 2|D\varphi|^2 \right] \\
&- c \cdot \frac{\lambda}{2} \cdot w_{11} + \frac{c w_{11}}{w_{11} + \tilde{v}} + \frac{c}{w_{11} + \tilde{v}} \\
&+ \frac{c\lambda \operatorname{tr} w^{ij}}{w_{11}} + \frac{c\lambda}{w_{11}} + c\lambda + \frac{c}{w_{11}(w_{11} + \tilde{v})} + \frac{c}{w_{11} + \tilde{v}} \\
&- \frac{\dot{\varphi}}{n} \cdot \left[2\lambda w^{kl} \varphi_{,k} \varphi_{,l} \cdot \left(1 + |D\varphi|^2\right) - 2\lambda |D\varphi|^2 \right] \\
&- \frac{2}{V} \cdot \frac{\dot{\varphi}}{n} \cdot w^{kl} \varphi_{,k} \tilde{v}_{;l}.
\end{aligned}
$$

We regroup some terms and observe that some others cancel, obtaining

$$
\begin{aligned}
0 \le PW \le &\frac{c}{w_{11} + \tilde{v}} + \frac{c w_{11}}{w_{11} + \tilde{v}} + \frac{c \operatorname{tr} w^{ij}}{w_{11} + \tilde{v}} - \varepsilon \lambda \operatorname{tr} w^{ij} + c\lambda \\
&- \frac{\lambda \dot{\varphi}}{n} \cdot \left[3 \left(1 + |D\varphi|^2\right) w^{kl} \varphi_{,k} \varphi_{,l} - 4|D\varphi|^2 \right] \\
&- c \cdot \frac{\lambda}{2} \cdot w_{11} \\
&+ \frac{c\lambda \operatorname{tr} w^{ij}}{w_{11}} + \frac{c\lambda}{w_{11}} + \frac{c}{w_{11}(w_{11} + \tilde{v})}.
\end{aligned} \tag{6.63}
$$

Moreover we note that the term

$$
-\frac{\lambda \dot{\varphi}}{n} \cdot 3 \left(1 + |D\varphi|^2\right) w^{pq} \varphi_{,p} \varphi_{,q}
$$

is nonpositive, and that it holds

$$
\frac{\lambda \dot{\varphi}}{n} \cdot 4|D\varphi|^2 \le c\lambda.
$$

Combining this last observations with (6.63) we now get

$$
\begin{aligned}
0 \le PW \le &\frac{c}{w_{11} + \tilde{v}} + \frac{c w_{11}}{w_{11} + \tilde{v}} + \frac{c \operatorname{tr} w^{ij}}{w_{11} + \tilde{v}} - \varepsilon \lambda \operatorname{tr} w^{ij} + c\lambda \\
&- c \cdot \frac{\lambda}{2} \cdot w_{11} + \frac{c\lambda \operatorname{tr} w^{ij}}{w_{11}} + \frac{c\lambda}{w_{11}} + \frac{c}{w_{11}(w_{11} + \tilde{v})}
\end{aligned}
$$

as well as

$$\begin{aligned}\frac{c\lambda w_{11}}{2} &\le \frac{c}{w_{11}+\tilde v} + \frac{c w_{11}}{w_{11}+\tilde v} + \frac{c\operatorname{tr} w^{ij}}{w_{11}+\tilde v} - \varepsilon\lambda \operatorname{tr} w^{ij} + c\lambda \\ &\quad + \frac{c\lambda \operatorname{tr} w^{ij}}{w_{11}} + \frac{c\lambda}{w_{11}} + \frac{c}{w_{11}(w_{11}+\tilde v)} \\ &\le \frac{c}{w_{11}+\tilde v} + c + c\lambda + \frac{c\lambda}{w_{11}} + \frac{c}{w_{11}(w_{11}+\tilde v)} \\ &\quad + \left(\frac{c}{w_{11}+\tilde v} - \lambda\left(\varepsilon - \frac{c}{w_{11}}\right)\right)\operatorname{tr} w^{ij}.\end{aligned}$$

Finally assuming w.l.o.g. $\frac{\varepsilon}{2} - \frac{c}{w_{11}} > 0$, since otherwise w_{11} would be bounded by $\frac{2c}{\varepsilon}$, and fixing λ larger if necessary provide

$$\frac{c\lambda w_{11}}{2} \le c + \frac{c}{w_{11}+\tilde v} + c\lambda + \frac{c\lambda}{w_{11}} + \frac{c}{w_{11}(w_{11}+\tilde v)}.$$

This implies that w_{11} has to be bounded, because the right hand side of the inequality is getting smaller, when w_{11} is increasing.

We have shown that w_{11} is bounded at the point (x_0, t_0) so, since the first derivatives of φ have already been estimated, v is also bounded at (x_0, ξ_0, t_0), where it is maximal. Hence v is bounded everywhere.

Last of all we have to contemplate the possibility that the maximum of v is not in the interior of Ω.

Remaining estimates. If v is maximal on the boundary, it is convenient to distinguish three cases, that is according to whether the direction ξ_0 in which this map is maximal, is tangential, normal or neither (as in [8]).

ξ_0 *normal.* The C^2-bounds follow from the double normal estimates.

ξ_0 *tangential.* We consider a fixed point $x_0 \in \partial\Omega$ and we choose a boundary coordinate chart containing x_0, so that $\partial\Omega$ is represented locally as $\operatorname{graph}\omega$ over its tangent plane at $x_0 = (\hat x_0, x_0^n)$ in order that locally $\Omega = \{(\hat x, x^n)|\ x^n > \omega(\hat x)\}$, $D\omega(\hat x_0) = 0$ and $D^2\omega(\hat x_0) > 0$.

We differentiate covariantly the boundary condition

$$\bar\nu^i(\hat x)\varphi_{,i}(\hat x, \omega(\hat x), t) = 0$$

with respect to $\hat x^j$, $1 \le j \le n-1$,

$$\bar\nu^i_{;j}\varphi_{,i} + \bar\nu^i\varphi_{;ij} + \bar\nu^i\varphi_{;in}\omega_{,j} = 0$$

and with respect to $\hat x^k$, $1 \le k \le n-1$,

$$\begin{aligned}0 &= \bar\nu^i_{;jk}\varphi_{,i} + \bar\nu^i_{;j}\varphi_{;ik} + \bar\nu^i_{;j}\varphi_{;in}\omega_{,k} + \bar\nu^i_{;k}\varphi_{;ij} + \bar\nu^i\varphi_{;ijk} + \bar\nu^i\varphi_{;ijn}\omega_{,k} \\ &\quad + \bar\nu^i_{;k}\varphi_{;in}\omega_{,j} + \bar\nu^i\varphi_{;ink}\omega_{,j} + \bar\nu^i\varphi_{;inn}\omega_{,j}\omega_{,k} + \bar\nu^i\varphi_{;in}\omega_{;jk}.\end{aligned}$$

In (x_0, t_0), in view of $D\omega(\hat x_0) = 0$, this last two equations turn into

$$\bar\nu^i_{;j}\varphi_{,i} + \bar\nu^i\varphi_{;ij} = 0 \tag{6.64}$$

and

$$\bar{\nu}^i_{;jk}\varphi_{,i} + \bar{\nu}^i_{;j}\varphi_{;ik} + \bar{\nu}^i_{;k}\varphi_{;ij} + \bar{\nu}^i\varphi_{;ijk} + \bar{\nu}^i\varphi_{;in}\omega_{;jk} = 0.$$

So we obtain in this point for a tangential vector ξ:

$$\begin{aligned} 0 &= \left(\bar{\nu}^i_{;jk}\varphi_{,i} + \bar{\nu}^i_{;j}\varphi_{;ik} + \bar{\nu}^i_{;k}\varphi_{;ij} + \bar{\nu}^i\varphi_{;ijk} + \bar{\nu}^i\varphi_{;in}\omega_{;jk}\right)\xi^j\xi^k \\ &=: \bar{\nu}^i_{;\xi\xi}\varphi_{,i} + 2\bar{\nu}^i_{;\xi}\varphi_{;i\xi} + \bar{\nu}^i\varphi_{;i\xi\xi} + \bar{\nu}^i\varphi_{;in}\omega_{;\xi\xi}. \end{aligned}$$

Remark 6.4. We put vectors as indices to indicate products as

$$\varphi_{;\bar{\nu}\xi\xi} := \bar{\nu}^i\varphi_{;ijk}\xi^j\xi^k$$

and not covariant derivatives in the corresponding direction

$$\varphi_{;i\xi\xi} \neq \left(\xi^j\varphi_{;ij}\right)_{;\xi} = \xi^k\xi^j_{;k}\varphi_{;ij} + \xi^k\xi^j\varphi_{;ijk}.$$

Analogously we will for instance write

$$\varphi_{;\xi\xi} := \varphi_{;ij}\xi^i\xi^j, \quad w_{\xi\xi} := w_{ij}\xi^i\xi^j, \quad w_{\bar{\nu}\bar{\nu}} := w_{ij}\bar{\nu}^i\bar{\nu}^j, \quad \sigma_{\xi\xi} := \sigma_{ij}\xi^i\xi^j, \quad \ldots .$$

The C^1-estimate and the double normal estimate provide $\bar{\nu}^i_{;\xi\xi}\varphi_{,i} \leq c$ respectively $\bar{\nu}^i\varphi_{;in}\omega_{;\xi\xi} \leq c$. It follows

$$\varphi_{;\bar{\nu}\xi\xi} = \bar{\nu}^i\varphi_{;i\xi\xi} \geq -2\bar{\nu}^i_{;\xi}\varphi_{;i\xi} - c. \tag{6.65}$$

The definition of w_{ij} and again the C^1-estimate lead to

$$-2\bar{\nu}^i_{;\xi}\varphi_{;i\xi} = -2\bar{\nu}^i_{;\xi}\left(\sigma_{i\xi} - w_{i\xi} + \varphi_{,\xi}\varphi_{,i}\right) \geq 2\bar{\nu}^i_{;\xi}w_{i\xi} - c.$$

We now choose normal coordinates around x_0, so that $\sigma_{ij}(x_0) = \delta_{ij}$ and ${}^{\sigma}\Gamma^k_{ij}(x_0) = 0$, whenever $1 \leq i, j, k \leq n$. As already noted, ξ_0 is an eigenvector of $(w_{ij} + v'_{ij})(x_0, t_0)$ to an eigenvalue $\hat{\lambda}$, since it corresponds to a maximal direction. Therefore it holds

$$\bar{\nu}^i_{;\xi_0}w_{i\xi_0} = \xi_0^j\bar{\nu}^i_{;j}\left(w_{ik} + v'_{ik}\right)\xi_0^k - \xi_0^j\bar{\nu}^i_{;j}v'_{ik}\xi_0^k = \hat{\lambda}\xi_0^j\bar{\nu}^i_{;j}\sigma_{ik}\xi_0^k - \xi_0^j\bar{\nu}^i_{;j}v'_{ik}\xi_0^k,$$

where we may assume that $\hat{\lambda}$ is nonnegative, because otherwise $w_{ik} + v'_{ik}$ would be negative definite and the needed estimate would follow immediately. Moreover the strict convexity of $\partial\Omega$ implies the existence of a (small) constant $c_1 > 0$ such that

$$\xi^j\bar{\nu}^i_{;j}\sigma_{ik}\xi^k \geq c_1\xi^j\delta^i_j\sigma_{ik}\xi^k = c_1\xi^i\sigma_{ik}\xi^k,$$

for all tangential vectors ξ. Considering ξ_0, inequality (6.65) becomes hence

$$\begin{aligned} \varphi_{;\bar{\nu}\xi_0\xi_0} &\geq 2\bar{\nu}^i_{;\xi_0}w_{i\xi_0} - c - c = 2\hat{\lambda}\xi_0^j\bar{\nu}^i_{;j}\sigma_{ik}\xi_0^k - 2\xi_0^j\bar{\nu}^i_{;j}v'_{ik}\xi_0^k - c \\ &\geq 2c_1\hat{\lambda}\xi_0^i\sigma_{ik}\xi_0^k - 2\xi_0^j\bar{\nu}^i_{;j}v'_{ik}\xi_0^k - c = 2c_1\xi_0^i\left(w_{ik} + v'_{ik}\right)\xi_0^k - 2\xi_0^j\bar{\nu}^i_{;j}v'_{ik}\xi_0^k - c \\ &= 2c_1w_{\xi_0\xi_0} + 2c_1\xi_0^iv'_{ik}\xi_0^k - 2\xi_0^j\bar{\nu}^i_{;j}v'_{ik}\xi_0^k - c \geq 2c_1w_{\xi_0\xi_0} - c. \end{aligned} \tag{6.66}$$

On the other hand, since v is maximal in (x_0, ξ_0, t_0) and we are differentiating in direction of the outward pointing normal vector, it holds $v_{;\bar{\nu}} \geq 0$, i.e.

$$\begin{aligned} 0 &\leq v_{;i}(x_0, \xi_0, t_0)\bar{\nu}^i(x_0) = \left(\frac{w_{\xi_0\xi_0;i} + v'_{\xi_0\xi_0;i}}{w_{\xi_0\xi_0} + v'_{\xi_0\xi_0} + C} + \lambda\sigma^{kl}\varphi_{;ki}\varphi_{,l}\right)\bar{\nu}^i \\ &= \frac{w_{\xi_0\xi_0;\bar{\nu}} + v'_{\xi_0\xi_0;\bar{\nu}}}{w_{\xi_0\xi_0} + v'_{\xi_0\xi_0} + C} + \lambda\sigma^{kl}\varphi_{;k\bar{\nu}}\varphi_{,l}, \end{aligned}$$

as in (6.17), and then

$$0 \le w_{\xi_0\xi_0;\bar\nu} + v'_{\xi_0\xi_0;\bar\nu} + \left(w_{\xi_0\xi_0} + v'_{\xi_0\xi_0} + C\right) \cdot \lambda\sigma^{kl}\varphi_{;k\bar\nu}\varphi_{,l},$$

because of $w_{\xi_0\xi_0} + v'_{\xi_0\xi_0} + C > 0$. Using the differentiated boundary condition (6.64) and once again the strict convexity of $\partial\Omega$, we obtain for the last term

$$\begin{aligned}\left(w_{\xi_0\xi_0} + v'_{\xi_0\xi_0} + C\right) \cdot \lambda\sigma^{kl}\varphi_{;ki}\bar\nu^i\varphi_{,l} &= \left(w_{\xi_0\xi_0} + v'_{\xi_0\xi_0} + C\right) \cdot \lambda\sigma^{kl}\left(-\varphi_{,i}\bar\nu^i_{;k}\right)\varphi_{,l}\\ &\le c_1\left(w_{\xi_0\xi_0} + v'_{\xi_0\xi_0} + C\right) \cdot \lambda\sigma^{kl}\left(-\varphi_{,i}\delta^i_k\right)\varphi_{,l}\\ &= -\,c_1\left(w_{\xi_0\xi_0} + v'_{\xi_0\xi_0} + C\right) \cdot \lambda\left|D\varphi\right|^2 \le 0.\end{aligned}$$

We assert that the normal derivative of $v'_{\xi_0\xi_0}$ is bounded. The definition (6.16) of $v'(x,\xi,\xi,t) = -2\left\langle\xi,\bar\nu\right\rangle_\sigma(\xi - \left\langle\xi,\bar\nu\right\rangle_\sigma\bar\nu)^l\bar\nu^k_{;l}\varphi_{,k}$ and the C^1-estimate provide indeed

$$\begin{aligned}v'_{;i}(x_0,\xi_0,\xi_0,t_0) &= \left(-\rho^k(x_0,\xi_0,\xi_0,t_0)\varphi_{,k}(x_0,t_0)\right)_{;i}\\ &= -\left(\rho^k_{;i}\varphi_{,k} + \rho^k\varphi_{;ki}\right) = -2\left(\left\langle\xi_0,\bar\nu\right\rangle_\sigma\right)_{;i}\xi_0^l\bar\nu^k_{;l}\varphi_{,k} \le c,\end{aligned}$$

as a consequence of

$$\rho^k(x_0,\xi_0,\xi_0,t_0) = 2\left\langle\xi_0,\bar\nu\right\rangle_\sigma\left(\xi_0^l - \left\langle\xi_0,\bar\nu\right\rangle_\sigma\bar\nu^l\right)\bar\nu^k_{;l} = 0$$

and

$$\begin{aligned}\rho^k_{;i}(x_0,\xi_0,\xi_0,t_0) =& \,2\left(\left\langle\xi_0,\bar\nu\right\rangle_\sigma\right)_{;i}\left(\xi_0^l - \left\langle\xi_0,\bar\nu\right\rangle_\sigma\bar\nu^l\right)\bar\nu^k_{;l}\\ &+ 2\left\langle\xi_0,\bar\nu\right\rangle_\sigma\left[\left(\xi_0^l - \left\langle\xi_0,\bar\nu\right\rangle_\sigma\bar\nu^l\right)\bar\nu^k_{;l}\right]_{;i}\\ =& \,2\left(\left\langle\xi_0,\bar\nu\right\rangle_\sigma\right)_{;i}\xi_0^l\bar\nu^k_{;l},\end{aligned}$$

since the fact of ξ_0 being tangential, implies that all the terms containing the inner product $\left\langle\xi_0,\bar\nu\right\rangle_\sigma$ vanish. This last estimates and the expression (6.12) of the Riemannian curvature tensor yield

$$\begin{aligned}0 \le& \,w_{\xi_0\xi_0;\bar\nu} + c\\ =& -\varphi_{;\xi_0\xi_0\bar\nu} + 2\varphi_{;\xi_0\bar\nu}\varphi_{,\xi_0} + c\\ =& -\varphi_{;\xi_0\bar\nu\xi_0} - R^m{}_{\xi_0\xi_0\bar\nu}\varphi_{,m} + 2\varphi_{;\xi_0\bar\nu}\varphi_{,\xi_0} + c\\ =& -\varphi_{;\xi_0\bar\nu\xi_0} - \xi_0^i\xi_0^j\bar\nu^kR^m{}_{ijk}\varphi_{,m} + 2\varphi_{;\xi_0\bar\nu}\varphi_{,\xi_0} + c\\ =& -\varphi_{;\bar\nu\xi_0\xi_0} - \xi_0^i\xi_0^j\bar\nu^k\left(\delta^m_j\sigma_{ik} - \delta^m_k\sigma_{ij}\right)\varphi_{,m} + 2\varphi_{;\xi_0\bar\nu}\varphi_{,\xi_0} + c\\ =& -\varphi_{;\bar\nu\xi_0\xi_0} - \xi_0^i\xi_0^j\bar\nu^k\left(\varphi_{,j}\sigma_{ik} - \varphi_{,k}\sigma_{ij}\right) + 2\varphi_{;\xi_0\bar\nu}\varphi_{,\xi_0} + c\\ =& -\varphi_{;\bar\nu\xi_0\xi_0} - \xi_0^i\xi_0^j\bar\nu^k\varphi_{,j}\sigma_{ik} + 2\varphi_{;\xi_0\bar\nu}\varphi_{,\xi_0} + c,\end{aligned}$$

where one of the terms in the second-last line vanishes because of the boundary condition. The second and the third term on the last line are bounded as a consequence of the bounds for the first respectively for the tangential-normal derivatives. It follows

$$\varphi_{;\bar\nu\xi_0\xi_0}(x_0,t_0) \le c$$

and, inserting this in (6.66),

$$w_{\xi_0\xi_0}(x_0,t_0) \le c,$$

which is our desired estimate.

ξ_0 *Neither tangential nor normal.* We assume, like in the previous case, that $\partial\Omega$ is represented locally as a graph of a map ω and define by

$$\tau := \frac{\xi - \langle \xi, \bar\nu\rangle_\sigma \bar\nu}{|\xi - \langle \xi, \bar\nu\rangle_\sigma \bar\nu|}$$

the (normed) tangential component of a nonnormal vector $\xi \in \mathbb{R}^n$ with $\langle \xi, \xi\rangle_\sigma = 1$. Setting

$$\delta_\tau := \langle \xi, \tau\rangle_\sigma = \frac{\langle \xi, \xi\rangle_\sigma - \langle \xi, \bar\nu\rangle_\sigma^2}{|\xi - \langle \xi, \bar\nu\rangle_\sigma \bar\nu|} = \frac{|\xi - \langle \xi, \bar\nu\rangle_\sigma \bar\nu|^2}{|\xi - \langle \xi, \bar\nu\rangle_\sigma \bar\nu|} = |\xi - \langle \xi, \bar\nu\rangle_\sigma \bar\nu|$$

and $\delta_{\bar\nu} := \langle \xi, \bar\nu\rangle_\sigma$, we obtain $\xi = \delta_\tau\tau + \delta_{\bar\nu}\bar\nu$, with $\delta_\tau^2 + \delta_{\bar\nu}^2 = 1$. If ξ is neither tangential nor normal, then $\delta_{\bar\nu}$ and δ_τ are both nonzero. It follows

$$\begin{aligned} w_{\xi\xi} &= w_{ij}\xi^i\xi^j = w_{ij}(\delta_\tau\tau^i + \delta_{\bar\nu}\bar\nu^i)(\delta_\tau\tau^j + \delta_{\bar\nu}\bar\nu^j) \\ &= w_{ij}(\delta_\tau^2\tau^i\tau^j + \delta_\tau\delta_{\bar\nu}\tau^i\bar\nu^j + \delta_{\bar\nu}\delta_\tau\bar\nu^i\tau^j + \delta_{\bar\nu}^2\bar\nu^i\bar\nu^j) \\ &=: \delta_\tau^2 w_{\tau\tau} + \delta_{\bar\nu}^2 w_{\bar\nu\bar\nu} + 2\delta_{\bar\nu}\delta_\tau w_{\tau\bar\nu}. \end{aligned}$$

Differentiating the boundary condition in a point (x_0, t_0) on the boundary, where $D\omega = 0$, in the same way as in (6.64) yields

$$\bar\nu^i_{;j}\varphi_{,i} = -\bar\nu^i\varphi_{;ij}. \tag{6.67}$$

The definition (6.16) of $v'(x, \xi_1, \xi_2, t) = -\bar\nu^l_{;k}\varphi_{,l}\left(\langle \xi_1, \bar\nu\rangle_\sigma \xi_2'^k + \langle \xi_2, \bar\nu\rangle_\sigma \xi_1'^k\right)$ implies

$$\begin{aligned} v'(x_0, \xi, \xi, t_0) &= -2\langle \xi, \bar\nu\rangle_\sigma (\xi - \langle \xi, \bar\nu\rangle_\sigma \bar\nu)^k \bar\nu^l_{;k}\varphi_{,l} \\ &= 2\delta_{\bar\nu}(\delta_\tau\tau)^k\varphi_{;kl}\bar\nu^l \qquad \text{by (6.67)} \\ &= 2\delta_\tau\delta_{\bar\nu}\varphi_{;\tau\bar\nu} \end{aligned}$$

and consequently

$$w_{\xi\xi} = \delta_\tau^2 w_{\tau\tau} + \delta_{\bar\nu}^2 w_{\bar\nu\bar\nu} - v'_{\xi\xi} + 2\delta_{\bar\nu}\delta_\tau\left(\sigma_{ij}\tau^i\bar\nu^j + \varphi_{,\bar\nu}\varphi_{,\tau}\right)$$

in (x_0, t_0). The last terms vanish because τ and $\bar\nu$ are defined to be orthogonal with respect to σ and it holds $\varphi_{,\bar\nu} = 0$ on the boundary. The identity $v'(x, \tau, \tau, t) = v'(x, \bar\nu, \bar\nu, t) = 0$ for all $(x, t) \in \bar\Omega \times [0, t^*)$ provides hence

$$w_{\xi\xi} + v'_{\xi\xi} = \delta_\tau^2(w_{\tau\tau} + v'_{\tau\tau}) + \delta_{\bar\nu}^2(w_{\bar\nu\bar\nu} + v'_{\bar\nu\bar\nu}) \tag{6.68}$$

in (x_0, t_0). We now choose Riemannian normal coordinates centered at x_0, in order that $\sigma_{ij}(x_0) = \delta_{ij}$. Since we assumed

$$v = \log\left(\frac{w_{ij}\xi^i\xi^j + v'_{ij}\xi^i\xi^j}{\sigma_{ij}\xi^i\xi^j} + C\right) + \frac{1}{2}\lambda|D\varphi|^2$$

to be maximal in direction ξ_0, in these coordinates we obtain

$$w_{\xi_0\xi_0}(x_0, t_0) + v'(x_0, \xi_0, \xi_0, t_0) \geq w_{\xi\xi}(x_0, t_0) + v'(x_0, \xi, \xi, t_0)$$

for all $\xi \in \mathrm{S}^n$ and, combining (6.68) with this inequality,

$$\begin{aligned} &w_{\xi_0\xi_0}(x_0, t_0) + v'(x_0, \xi_0, \xi_0, t_0) \\ &\quad = \delta_\tau^2(w_{\tau\tau}(x_0, t_0) + v'(x_0, \tau, \tau, t_0)) + \delta_{\bar\nu}^2(w_{\bar\nu\bar\nu}(x_0, t_0) + v'(x_0, \bar\nu, \bar\nu, t_0)) \\ &\quad \leq \delta_\tau^2\left(w_{\xi_0\xi_0}(x_0, t_0) + v'(x_0, \xi_0, \xi_0, t_0)\right) + \delta_{\bar\nu}^2(w_{\bar\nu\bar\nu}(x_0, t_0) + v'(x_0, \bar\nu, \bar\nu, t_0)). \end{aligned}$$

It follows immediately

$$(1 - \delta_\tau^2)(w_{\xi_0\xi_0}(x_0, t_0) + v'(x_0, \xi_0, \xi_0, t_0)) \leq \delta_{\bar\nu}^2(w_{\bar\nu\bar\nu}(x_0, t_0) + v'(x_0, \bar\nu, \bar\nu, t_0))$$

and, from $(1-\delta_\tau^2) = \delta_{\bar{\nu}}^2$,

$$w_{\xi_0\xi_0}(x_0,t_0) + v'(x_0,\xi_0,\xi_0,t_0) \le w_{\bar{\nu}\bar{\nu}}(x_0,t_0) + v'(x_0,\bar{\nu},\bar{\nu},t_0).$$

We finally achieve the desired result

$$\begin{aligned} w_{\xi_0\xi_0}(x_0,t_0) &\le w_{\bar{\nu}\bar{\nu}}(x_0,t_0) - v'(x_0,\xi_0,\xi_0,t_0) \\ &\le w_{\bar{\nu}\bar{\nu}}(x_0,t_0) + c, \end{aligned}$$

where $w_{\bar{\nu}\bar{\nu}}$ is bounded as shown before.

This concludes the C^2-estimates.

References

1. B. Andrews, *Gauss curvature flow: the fate of the rolling stones*, Invent. math. **138** (1999), no. 1, 151-161.
2. J.-H. Eschenburg, *Local convexity and nonnegative curvature - Gromov's proof of the sphere theorem*, Invent. math. **84** (1986), no. 3, 507-522.
3. C. Gerhardt, *Closed Weingarten hypersurfaces in Riemannian manifolds.* J. Differential Geom. **43** (1996), no. 3, 612-641.
4. C. Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J. Differential Geom. **32** (1990), no. 1, 299-314.
5. R. A. Horn and C. R. Johnson, *Matrix Analysis*, second ed., Cambridge University Press, New York, 2013.
6. G. Huisken, *Non-parametric mean curvature evolution with boundary conditions*, J. Differential Equations **77**, (1989), no. 2, 369-378.
7. B. Lambert, *The perpendicular Neumann problem for mean curvature flow with a timelike cone boundary condition*, Trans. Amer. Math. Soc. **366** (2014), no. 7, 3373-3388.
8. P.-L. Lions, N. S. Trudinger and J. I. E. Urbas, *The Neumann problem for equations of Monge-Ampère type*, Comm. Pure Appl. Math. **39** (1986), no. 4, 539-563.
9. T. Marquardt, *Inverse mean curvature flow for star-shaped hypersurfaces evolving in a cone*, J. Geom. Anal. **23** (2013), no. 3, 1303-1313.
10. O. C. Schnürer, *Surfaces expanding by the inverse Gauß curvature flow*, J. reine angew. Math. **600** (2006), 117-134.
11. O. C. Schnürer, *Translating solutions to the second boundary value problem for curvature flows*, Manuscripta Math. **108** (2002), no. 3, 319-347.
12. O. C. Schnürer and H. R. Schwetlick, *Translating solutions for Gauß curvature flows with Neumann boundary condition*, Pacific J. Math. **213** (2004), no. 1, 89-109.
13. J. I. E. Urbas, *An expansion of convex hypersurfaces*, J. Differential Geom. **33** (1991), no. 1, 91-125.
14. J. I. E. Urbas, *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, Math. Z. **205** (1990), no. 1, 355-372.
15. J. A. Viaclovsky, *Estimates and existence results for some fully nonlinear elliptic equations on Riemannian manifolds*, Comm. Anal. Geom. **10** (2002), no. 4, 815-846.

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