Mild Parametrization in O-minimal Structures

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Zusammenfassung

Eine milde Parametrisierung ist die Parametrisierung von beschränkten Mengen mit Hilfe von glatten Funktionen, deren Ableitungen spezifische Wachstumsbedingungen erfüllen, und ist ein wichtiges Werkzeug zur Abschätzung der Anzahl rationaler Punkte von beschränkten Höhe dieser Mengen. In dieser Arbeit untersuchen wir o-minimale Expansionen des reellen Körpers, wobei wir uns darauf konzentrieren, ob ihre beschränkte definierbaren Mengen eine milde Parametrisierung haben.

Der Satz von Pila-Wilkie gibt eine Grenze für die Dichte rationaler Punkte von beschränkten Mengen in o-minimalen Expansionen des reellen Körpers an. Es ist bekannt, dass die Schranke im Satz von Pila und Wilkie für $\mathbb{R}_{\text{an}}$ nicht verbessert werden kann, aber eine bessere Schranke wird von Wilkie für $\mathbb{R}_{\text{exp}}$ vermutet. Eine milde Parametrisierung war ein wichtiges Instrument, das für diese Vermutung verwendet wurde. Pila zeigte, dass, wenn $\mathbb{R}_{\text{exp}}$ eine gleichmäßige Version der milden Parametrisierung zulässt, man Wilkies Vermutung beweisen kann. Diese Arbeit leistet einen Beitrag zur Verbesserung von Wilkies Vermutung.

In Kapitel 1 geben wir einen Überblick über den Inhalt dieser Arbeit und motivieren unsere Arbeit. Wir geben eine kurze Zusammenfassung des Hintergrunds über o-minimalen Strukturen und das Zählen rationaler Punkte in Kapitel 2, die wir in dieser Arbeit verwenden werden. Kapitel 3 besteht aus Definitionen von milden Funktionen und einer milden Parametrisierung sowie einer Sammlung von Ergebnissen über ihre Eigenschaften. In diesem Kapitel wird die milde Parametrisierung und ihre Rolle als Werkzeug für Ergebnisse im Zusammenhang mit der Dichte rationaler Punkte analysiert. In Kapitel 4 stellen wir bekannte Ergebnisse zur milden Parametrisierung in o-minimalen Strukturen vor.

Die Kapitel 5, 6, 7 und 8 enthalten unsere Ergebnisse zur milden Parametrisierung verschiedener o-minimaler Expansionen des reellen Körpers. Die Hauptfrage, die uns interessiert, ist, ob diese Strukturen eine milde Parametrisierung zulassen oder nicht.

In Kapitel 5 zeigen wir, dass bei einer Algebra von reellen Funktionen $\mathcal{A}$, die

bestimmte Eigenschaften erfüllt (z.b. Quasianalytizität, abgeschlossen unter Komposition und impliziten Funktionen) die Expansion $\mathbb{R}_{\mathcal{A}}$ des reellen Körpers durch $\mathcal{A}$, eine milde Parametrisierung zulässt, wenn die beschränkten Funktionen in dieser Algebra mild sind. Dann verwenden wir dieses Ergebnis, um zu verifizieren, dass $\mathbb{R}_{\mathcal{G}}$, die Expansion des reellen Körpers um eine Klasse von Gevrey-Funktionen, eine milde Parametrisierung zulässt. Außerdem betrachten wir ein spezifisches Beispiel einer in $\mathbb{R}_{\mathcal{G}}$ definierbaren Funktion und zeigen Wilkies Vermutung für eine Fläche, die von dieser Funktion definiert wird.

In Kapitel 6 untersuchen wir spezielle Klassen von C^{∞}-Funktionen, die quasianalytische Denjoy-Carleman Klassen bezeichnet werden. Dieses Kapitel enthält unsere Ergebnisse zu milden Parametrisierungseigenschaften für diese Klassen und den Expansionen des reellen Körpers durch diesen Algebren. Wir beweisen, dass diese Klassen, wenn sie milde Funktionen enthalten, unter Differenzierung abgeschlossen sind. Wir beweisen auch durch die explizite Konstruktion eines Beispiels, dass die Umkehrung dieser Aussage nicht wahr ist. Wir betrachten auch die Expansion des reellen Körpers durch ein spezifisches Beispiel der Denjoy-Carleman-Klassen und zeigen, dass diese Struktur eine milde Parametrisierung zulässt. Wir zeigen auch, dass diese Struktur eine strenge Expansion von $\mathbb{R}_{\text{an}}$ ist.

In Kapitel 7 konzentrieren wir uns auf die Frage, ob es Expansionen des reellen Körpers gibt, die keine definierbare milde Parametrisierung zulassen. Das Hauptergebnis dieses Kapitels ist: Jede polynomiell begrenzte o-minimale Expansion des reellen Körpers, in dem eine irrationale Potenzfunktion definierbar ist, lässt keine definierbare milde Parametrisierung zu. Darüber hinaus erlaubt diese Struktur keine milde Parametrisierung mittels Funktionen, die in irgendeinen polynomiellen begrenzten Struktur definierbar sind.

Wir betrachten die Expansion von $\mathbb{R}_{\text{an}}$ durch Potenzfunktionen in Kapitel 8. Wir beweisen in diesem Kapitel, dass Kurven in $(0,1)^2$, die in dem Expansion des reellen Körpers durch Potenzfunktionen definierbar sind, eine milde Parametrisierung aufweisen. Obwohl wir in Kapitel 7 zeigen, dass diese Struktur keine *definierbare* milde Parametrisierung hat, falls in dieser Struktur eine irrationale Potenzfunktionen definierbar ist, zeigen unsere Ergebnisse in diesem Kapitel, dass eine milde

Parametrisierung für die Struktur immer noch möglich ist.

Schließlich stellen wir in Kapitel 9 weitere mögliche Forschungspfade vor, die sich aus dieser Arbeit ergeben.

Abstract

Mild parametrization is parametrization of bounded sets by means of smooth functions that have a specific growth condition on their derivatives and it is a significant tool for estimating the number of rational points of bounded height of these sets. In this thesis we study o-minimal expansions of the real field focusing on whether their bounded definable sets have mild parametrization or not. If this is true for all such sets of a given structure, then we say that this structure admits mild parametrization.

The theorem of Pila and Wilkie gives a bound on the density of rational points of bounded sets in o-minimal expansions of the real field. It is known that the bound in the Pila-Wilkie Theorem cannot be improved for $\mathbb{R}_{\mathrm{an}}$ but a better bound is conjectured for $\mathbb{R}_{\exp}$ by Wilkie. Mild parametrization has been an important tool used towards this conjecture. Pila showed that if $\mathbb{R}_{\exp}$ admits a uniform version of mild parametrization then one can prove Wilkie's conjecture. We aim to contribute to improvements toward Wilkie's conjecture with the work in this thesis.

In Chapter 1 we give an overview of the content of this thesis and provide motivation for our work. We give a brief summary of the background about o-minimal structures and counting rational points in Chapter 2 which we will be using in this thesis. Chapter 3 consists of definitions of mild functions and mild parametrization and a collection of results about their properties. This chapter analyses mild parametrization and its role as a tool for results related to the density of rational points. In Chapter 4, we present known results about mild parametrization in o-minimal structures.

The Chapters 5, 6, 7 and 8 contain our results about mild parametrization of various o-minimal expansions of the real field. The main question we are interested is whether these structures admit mild parametrization or not.

In Chapter 5 we show that, given an algebra of real functions $\mathcal{A}$ satisfying certain properties (e.g. quasianalyticity, being closed under composition and taking implicit

functions), $\mathbb{R}_{\mathcal{A}}$, the expansion of the real field by $\mathcal{A}$, admits mild parametrization if the bounded functions in this algebra are mild. Then we use this result to verify that $\mathbb{R}_{\mathcal{G}}$, the expansion of the real field by a class of Gevrey functions, admits mild parametrization. Moreover we consider a specific example of a function definable in $\mathbb{R}_{\mathcal{G}}$ and show Wilkie's conjecture is true for a surface defined using this function.

We examine special classes of C^∞ functions called quasianalytic Denjoy-Carleman classes in Chapter 6. This chapter includes our results about mild parametrization properties for these classes and the expansions of the real field by these algebras. We prove that if each function $f : (0,1)^n \to (0,1)$ in these classes is mild then they are closed under differentiation. We also prove by explicitly constructing an example that the converse of this statement is not true. We also consider the expansion of the real ordered field by a specific example of a Denjoy-Carleman class and show that this structure admits mild parametrization. We also show that this structure is a strict expansion of $\mathbb{R}_{\text{an}}$.

In Chapter 7 we focus on the question of whether or not there exist expansions of the real field which do not admit definable mild parametrization. The main result of this chapter is: any polynomially bounded o-minimal expansion of the real field in which an irrational power function is definable does not admit definable mild parametrization. Moreover, this structure does not admit mild parametrization by means of functions definable in any polynomially bounded structure.

We consider the expansion of $\mathbb{R}_{\text{an}}$ by any collection of power functions in Chapter 8. We prove in this chapter that curves in $(0,1)^2$ which are definable in this structure have mild parametrization. Although we show in Chapter 7 that such a structure does not have *definable* mild parametrization whenever it defines an irrational power, our result in this chapter shows that mild parametrization for this structure is still possible.

Finally in Chapter 9 we present further possible research paths arising from the work in this thesis.

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Contents

Chapter 1

Introduction

In this thesis we focus on the theory of o-minimal structures and its application to questions in number theory and real analytic geometry. The interaction of o-minimality with number theory began with the Pila-Wilkie Theorem ([52]) on counting rational points, a result of diophantine geometry which makes essential use of o-minimality. It attracted great interest amongst number theorists and has been used to prove ground-breaking results, including a new proof of the Manin-Mumford conjecture ([61]) and some cases of the André-Oort conjecture ([58]). Wilkie has conjectured an important improvement to the Pila-Wilkie Theorem which again may have applications to transcendental number theory. The research we describe in this thesis aims to contribute towards this conjecture by analysing, in the context of o-minimal structures, the main geometric tool used to date in approaches to the conjecture, namely mild parametrization. We will present our work aiming to address how our results contribute to this area.

The Pila-Wilkie Theorem gives an upper bound for the number of rational points of bounded height of the transcendental part of a set definable in an o-minimal expansion of the real field. It was conjectured by Wilkie that this bound can be improved for sets definable in the structure $\mathbb{R}_{\exp}$. Mild parametrization has played a central role in important steps already made towards this conjecture. The main ingredient of the proof of the Pila-Wilkie Theorem is a reparametrization theorem which states that each subset of $(0,1)^n$ definable in an o-minimal structure has parametrization using C^r functions whose derivatives are bounded by 1 in absolute value. Pila and Wilkie proved an upper bound for the number of zero sets of polynomials with bounded degree containing the rational points of the images of the parametrizing functions. The approach of using mild parametrization to address Wilkie's conjecture is to improve the bounds for the number of these zero sets containing the rational points when these rational points are lying in the images of C^∞ functions with specific bounds on their derivatives (mild functions). This approach was followed in many works. In [56], Pila showed that Wilkie's conjecture is true for graphs of Pfaffian functions, solutions of triangular systems of first order polynomial differential equations (including the function exp), if they have mild parametrization. It was also proven by Pila in [59] that mild parametrization with sufficient uniformity in parameters would be sufficient to establish Wilkie's conjecture. In [33], Jones, D. Miller and Thomas proved that $\mathbb{R}_{\text{respfaff}}$, the expansion of the real field by all restricted Pfaffian functions, admits definable mild parametrization and affirmed Wilkie's conjectures for curves definable in this structure. There are also some explicit examples of surfaces shown to verify the conjecture by Pila [59] and Butler [10], using mild parametrization. Using Pila's methods, Jones and Thomas ([34]) later proved Wilkie's conjecture for definable surfaces in $\mathbb{R}_{\exp}$ that have mild parametrization. Strategies to prove Wilkie's conjecture that are not involving mild-parametrization have been followed with some success, such as in [5], where Wilkie's conjecture is proven for sets definable in the expansion of the real field by the restricted exponential and sine functions.

In this thesis we study mild parametrization properties of o-minimal expansions of the real field. For various expansions of the real field, we ask whether the structure admits mild parametrization or not. Our main motivation for this research is to understand the interactions between mild parametrization and o-minimality and also obtain results about the density of rational points of bounded sets using that these sets have mild parametrization.

Before presenting our research, in Chapter 2, we will give the necessary background on which the work in this thesis will be based. Chapter 2 will contain information about o-minimal expansions of the real field, our general setting; Pfaffian functions, a special type of analytic functions for which a uniform bound on the number of zeros in terms of the degree and the order of the function (see Definition

2.2.2) is well understood; and brief information about the Pila-Wilkie Theorem and Wilkie's conjecture, the conjecture that motivates this work.

Since our aim is to explore mild parametrization in o-minimal structures, we devote Chapter 3 to obtaining a better understanding of mild functions and functions that have mild parametrization. We prove some closure properties for these functions. We also give examples of mild functions and functions that have mild parametrization both to perceive the behaviour of these functions and to provide tools for us to use in the later chapters. Chapter 3 contains a detailed presentation of Pila's proof of the fact that Wilkie's conjecture is true for any Pfaffian curve lying in $(0,1)^2$ that admits mild parametrization. We provide a concrete case for the reader to show how mild parametrization is used for obtaining the bound conjectured by Wilkie.

We present known results in our research area in Chapter 4. Within these results the work in [33] has a special importance related to our work. In this paper Jones, D. Miller and Thomas showed that reducts of $\mathbb{R}_{\text{an}}$ admit mild parametrization. Because of the relevance of this result with our work we explain this result in detail in Chapter 4.

We prove our main result of Chapter 5 using the parametrization theorem of Rolin and Servi ([64]). The authors in [64] develop a setting that generalizes the proof of o-minimality for many polynomially bounded expansions of the real field. They specify certain properties for algebras of real functions, such as quasianalyticity, being closed under several operations like composition, monomial division, extracting the implicit functions, and they prove that expansions of the real field by algebras satisfying these properties are polynomially bounded, model complete and o-minimal. We give a detailed overview of the paper [64] and the parametrization theorem in this thesis. We prove the following proposition.

Proposition 5.2.3. *Let $\mathcal{A}$ be an algebra satisfying properties given in [64] (pages 1211,1212). If all the functions $f : (0,1)^n \to (0,1)$ in the class $\mathcal{A}$ are mild, for all $n \in \mathbb{N}$, then the structure $\mathbb{R}_{\mathcal{A}}$ admits definable mild parametrization.*

We apply our result to the expansion of the real field by a special class of Gevrey functions, $\mathbb{R}_{\mathcal{G}}$, and verify that this structure admits mild parametrization. Then we consider a specific surface definable in $\mathbb{R}_{\mathcal{G}}$ and affirm Wilkie's conjecture for this surface.

We study special classes of C^{∞} functions called quasianalytic Denjoy-Carleman classes in Chapter 6. The content of these classes rely on a given so-called Denjoy Carleman sequence (an increasing sequence of positive real numbers satisfying specific conditions). That is the classes consist of functions whose derivatives are bounded where the bounds depend on the terms of the given sequence. In the paper [65] Rolin, Speissegger and Wilkie proved that expansions of the real field by these classes are o-minimal. In this chapter we search for an answer to the question of whether each function $f : (0,1)^n \to (0,1)$ in a given quasianalytic Denjoy-Carleman class is mild or not. We give a negative answer to this question by constructing a specific quasianalytic Denjoy-Carleman class and giving a function in this class which is not mild. We also prove the following proposition.

Proposition 6.6.4. *Let $C(M)$ be a Denjoy-Carleman class. If each function $f : (0,1)^n \to (0,1)$ is mild, then $C(M)$ is closed under differentiation.*

Our example proves also that the converse of this result is not true since the example we construct of a class containing a nonmild function is a class that is moreover closed under differentiation. We also consider a specific quasianalytic Denjoy-Carleman class associated to the sequence $L = (L_n)_{n\in\mathbb{N}}$ where $L_0 = L_1 = 1$ and $L_n = (\log n)^n$ for all $n \geq 2$. We show that $\mathbb{R}_{C(L)}$, the expansion of the real field by the class $C(L)$, is a strict expansion of $\mathbb{R}_{\text{an}}$. We also prove the following proposition.

Proposition 6.6.6. *The structure $\mathbb{R}_{C(L)}$ admits mild parametrization.*

In order to understand under which conditions an expansion of the real field admits mild parametrization, as well as investigating structures that admit mild parametrization, we also try to understand when mild parametrization is not possible. There is no example in literature of a structure that is known not to admit mild parametrization. But, in [73], Thomas constructed an o-minimal expansion of the real field that does not admit definable mild parametrization (parametrizing functions being definable in the same structure). In Chapter 7, as well as discussing the work of Thomas, we prove the following theorem.

Theorem 7.2.8. *Let $\mathcal{R}$ be a polynomially bounded o-minimal expansion of the real field where the field of exponents of $\mathcal{R}$ contains an irrational number. Then $\mathcal{R}$ does not admit definable mild parametrization.*

To prove this result we show that any mild functions used to parametrize the graph of the function $x^\alpha : (0,1) \to (0,1)$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, cannot be definable in any polynomially bounded o-minimal structure. Hence our result can be generalized as: expansions of the real field in which an irrational power function is definable do not have mild parametrization using functions definable in polynomially bounded structure.

Our result in Chapter 7 applies to the structures $\mathbb{R}_{\text{an}}^S$, the expansion of $\mathbb{R}_{\text{an}}$ by all power functions x^α where $\alpha \in S$ and S is a subfield of $\mathbb{R}$. Hence $\mathbb{R}_{\text{an}}^S$ does not admit definable mild parametrization if $S \neq \mathbb{Q}$. On the other hand this structure may still admit mild parametrization. In Chapter 8 we prove the following theorem.

Theorem 8.1.6. *Let $f : (0,1) \to (0,1)$ be a function definable in $\mathbb{R}_{an}^S$ where S is a subfield of $\mathbb{R}$. Then f has mild parametrization.*

Following our results, we will present further research paths in the area of mild parametrization in o-minimal structures in Chapter 9.

Chapter 2

General background

In this chapter we present basic information about the topics that we will be encountering in this thesis. The main setting of our work is an important type of o-minimal structures, namely o-minimal expansions of the real field. In Section 2.1, we provide information about o-minimal structures concentrating on o-minimal expansions of the real field. Instead of giving a model theoretical definition of o-minimal structures we give a geometric definition which would be enough to define o-minimal expansions of the real field. Section 2.2 is about Pfaffian functions which are considered in many works in our research area like [59], [10], [56], [33], [34] and more. The theorem of Pila and Wilkie states a bound on the density of rational points of sets definable in o-minimal expansions of the real field. It is known that it is not possible to improve the bound given in this theorem generally for all o-minimal expansions of the real field. But a better bound was conjectured by Wilkie for $\mathbb{R}_{\exp}$. Since our work is basically motivated by this conjecture, in Section 2.3, we will talk about the Pila-Wilkie Theorem and Wilkie's conjecture.

2.1 O-minimal structures

In this section we define o-minimal expansions of the real field, present properties of these structures, and give examples that we will be considering in the later chapters. For further information about o-minimal structures the reader can see [19], [42], and also [69] which is particularly about o-minimal expansions of the real field.

Definition 2.1.1. *A subset X of $\mathbb{R}^n$ is said to be* semialgebraic *if it is a finite union of sets of the form*

$$\{\bar{x} \in \mathbb{R}^n : P(\bar{x}) = 0, Q_1(\bar{x}) > 0, \ldots, Q_k(\bar{x}) > 0\}$$

where P, Q_1, ..., Q_k are polynomials with coefficients in $\mathbb{R}$.

Definition 2.1.2. *An* expansion of the real field *(or* structure *for short) is a sequence $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ where $\mathcal{S}_n$ is a collection of subsets of $\mathbb{R}^n$ that satisfy the following properties for all $n \in \mathbb{N}$.*

(S1) $\mathcal{S}_n$ contains all semialgebraic subsets of $\mathbb{R}^n$.

(S2) If A is in $\mathcal{S}_n$ then the set $\{(x_1, \dots, x_n) \in \mathbb{R}^n : (x_{\sigma(1)}, \dots, x_{\sigma(n)}) \in A\}$ is in $\mathcal{S}_n$ for any permutation σ of $\{1, \dots, n\}$.

(S3) If A and B are in $\mathcal{S}_n$ then $A \cup B$ and $A \cap B$ are in $\mathcal{S}_n$.

(S4) For all $A \in \mathcal{S}_n$ the products $A \times \mathbb{R}$ and $\mathbb{R} \times A$ are in $\mathcal{S}_{n+1}$.

(S5) If A is in $\mathcal{S}_n$ then its complement $\mathbb{R}^n \setminus A$ is in $\mathcal{S}_n$.

(S6) For all $A \in \mathcal{S}_{n+1}$ the projection of A on its first n coordinates, $\Pi_n(A)$, is in $\mathcal{S}_n$.

We say that a set A is definable in the structure $\mathcal{S}$ if $A \in \mathcal{S}_n$ for some $n \in \mathbb{N}$. We say that a function f is definable in the structure $\mathcal{S}$ if the graph of f is definable in $\mathcal{S}$.

Definition 2.1.3. *A structure $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ is called* o-minimal *if each $A \in \mathcal{S}_1$ is a finite union of intervals and points.*

Example 2.1.4 (Semialgebraic sets)**.** If for each $n \in \mathbb{N}$ the collection $\mathcal{S}_n$ is the collection of all semialgebraic subsets of $\mathbb{R}^n$ then the sequence $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ forms a structure. Indeed it is easy to see that it satisfies properties (S1) to (S5) of the Definition 2.1.2. The property (S6) is the statement of the Tarski-Seidenberg Theorem (see for example [15], Theorem 2.3). Since any semialgebraic subset of $\mathbb{R}$ is a finite union of points and intervals the structure $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ is o-minimal.

An important consequence of o-minimality is the fact that two definable functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ don't oscillate one around the other: the set of x's for which $f(x) < g(x)$ (respectively $f(x) > g(x)$, respectively $f(x) = g(x)$) is a finite union of points and intervals; in particular for all $a \in \mathbb{R}$ there is $\varepsilon > 0$ such that for all $x \in (a, a+\varepsilon)$ one either has $f(x) < g(x)$ or $f(x) > g(x)$ or $f(x) = g(x)$.

Remark 2.1.5. Using the equivalences

- $t \geq 0 \Leftrightarrow \exists s \in \mathbb{R}, t = s^2$
- $(t_1 = 0$ and $t_2 = 0) \Leftrightarrow t_1^2 + t_2^2 = 0$
- $(t_1 = 0$ or $t_2 = 0) \Leftrightarrow t_1 t_2 = 0$

one can write any semialgebraic subset of $\mathbb{R}^n$ in the form

$$\{\bar{x} \in \mathbb{R}^n : \exists \bar{y} \in \mathbb{R}^m, P(\bar{x}, \bar{y}) = 0\}$$

for some polynomial P with coefficients in $\mathbb{R}$.

Remark 2.1.6. Combining the previous remark with the fact that for any real number y, there is $t \in [-1, 1]$ such that $(1-t^2)y = t$, one can write any semialgebraic subset of $\mathbb{R}^n$ in the form

$$\{\bar{x} \in \mathbb{R}^n : \exists \bar{t} \in [-1, 1]^m, P(\bar{x}, \bar{t}) = 0\}$$

for some polynomial P with coefficients in $\mathbb{R}$.

The main way to construct (o-minimal) structures that is used throughout this thesis is the following (it is not the only way but it will suffice for our purposes):

Definition 2.1.7. *Let $\mathcal{L}$ be a collection of functions $f : \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$. The* expansion of the real ordered field by $\mathcal{L}$ *is the smallest structure $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ such that each $f \in \mathcal{L}$ is a definable function in $\mathcal{S}$. This structure is denoted $\mathbb{R}_{\mathcal{L}}$ or $(\mathbb{R}, +, -, \cdot, 0, 1, <, \mathcal{L})$; it is the closure of the collection of semialgebraic sets and graphs of functions in $\mathcal{L}$ by the operations implicit in (S1) to (S6) in Definition 2.1.2.*

Remark 2.1.8. With the notation of Definition 2.1.7, a set $X \subseteq \mathbb{R}^n$ is definable in $\mathbb{R}_{\mathcal{L}}$ if and only if there is $p \in \mathbb{N}$ and a first-order logic formula $\phi(x_1, \dots, x_n, y_1, \dots, y_p)$ in the language $(+, -, \cdot, 0, 1, <, \mathcal{L})$ with free variables $x_1, \dots, x_n, y_1, \dots, y_p$ and $(a_1, \dots, a_p) \in \mathbb{R}^p$ such that

$$X = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \phi(x_1, \dots, x_n, a_1, \dots, a_p)\}.$$

For a precise definition of a first-order logic formula the reader can see for example [45, Chapter 1].

Definition 2.1.9. *Let $\mathcal{L}$ be a collection of functions $f : \mathbb{R}^n \to \mathbb{R}$ for poossibly different $n \in \mathbb{N}$. The expansion $\mathbb{R}_{\mathcal{L}}$ of the real ordered field by $\mathcal{L}$ is* model complete *in the language $(+, -, \cdot, 0, 1, <, \mathcal{L})$ if one does not need the complement operation to define the structure $\mathbb{R}_{\mathcal{L}}$; that is, the closure of the collection of semialgebraic sets together with the graphs of functions in $\mathcal{L}$ by the operations implicit in (S1) to (S6) is $\mathbb{R}_{\mathcal{L}}$.*

In terms of formulas, the structure is model complete if and only if any definable set X can be described by an existential formula with parameters in the language $(+, -, \cdot, 0, 1, <, \mathcal{L})$.

When encountering a structure of the form $\mathbb{R}_{\mathcal{L}}$ for some specified set of functions $\mathcal{L}$ we will say that the structure is model complete if it is model complete in the language $(+, -, \cdot, 0, 1, <, \mathcal{L})$.

Remark 2.1.10. Using the equivalences from Remark 2.1.5 and untangling compositions with the equivalence

$$z = f(g_1(\bar{x}), \ldots, g_l(\bar{x})) \Leftrightarrow \exists y_1, \ldots, y_l, y_1 = g_1(\bar{x}), \ldots, y_l = g_l(\bar{x}), z = f(y_1, \ldots, y_l)$$

one gets that any set definable in a model complete structure $(\mathbb{R}; +, \cdot, \mathcal{L})$ is of the form

$$\{\bar{x} \in \mathbb{R}^n : \exists \bar{y} \in \mathbb{R}^p, P(\bar{x}, \bar{y}, f_1(\pi_1(\bar{x}, \bar{y})), \ldots, f_l(\pi_l(\bar{x}, \bar{y}))) = 0\}$$

for some polynomial P, functions $f_1, \ldots, f_l$ in $\mathcal{L}$ and maps $\pi_1 : \mathbb{R}^{n+p} \to \mathbb{R}^{n_1}, \ldots, \pi_l : \mathbb{R}^{n+p} \to \mathbb{R}^{n_l}$ whose components are of the form $(z_1, \ldots, z_{n+p}) \to z_i$ for some $i \in \{1, \ldots, n+p\}$ (since the sets $\{0\}$, $\{1\}$, $\{(x, y) \in \mathbb{R}^2 : x \leq y\}$ and $\{(x, y) \in \mathbb{R}^2 : y = x\}$ are existentially definable sets in the structure $(\mathbb{R}; +, \cdot, \mathcal{L})$), the model completeness of the structure $(\mathbb{R}; +, \cdot, \mathcal{L})$ is equivalent to the model completeness of the structure $(\mathbb{R}, +, -, \cdot, 0, 1, <, \mathcal{L})$).

2.1.1 Properties of o-minimal structures

Lemma 2.1.11. (Monotonicity Theorem, [19, Chapter 3, Theorem 1.2]) *Let $a \in \{-\infty\} \cup \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$ and $f : (a, b) \to \mathbb{R}$ be a function definable in an o-minimal structure. Then there are real numbers $a = c_0 < c_1 < \ldots < c_k < c_{k+1} = b$ such that for each $i \in \{0, \ldots, k\}$ the restriction of f to the interval (c_i, c_{i+1}) is continuous and either increasing, decreasing or constant (depending on i).*

Definition 2.1.12. *In an o-minimal structure $\mathcal{S}$ a nonempty set $D \subseteq \mathbb{R}^n$ is called a* cell *if it satisfies the following inductive property on n.*

1. *If $n = 1$, then D is either an interval (a, b) with $a \in \{-\infty\} \cup \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$ with $a < b$, or a point $\{c\}$ for some $c \in \mathbb{R}$.*
2. *If $n > 1$, then the projection C of D on the first $n - 1$ coordinates is a cell in $\mathbb{R}^{n-1}$ and D is of the form*
   - *(a) $\{(\bar{x}, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : \bar{x} \in C, y = f(\bar{x})\}$ where $f : C \to \mathbb{R}$ is a continuous definable function or,*
   - *(b) $\{(\bar{x}, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : \bar{x} \in C, g(\bar{x}) < y < h(\bar{x})\}$ where $g : C \to \mathbb{R}$ is either a continuous definable function or constantly $-\infty$ and $h : C \to \mathbb{R}$ is either a continuous definable function or constantly $+\infty$ with $g(\bar{x}) < h(\bar{x})$ for all $\bar{x} \in C$.*

Theorem 2.1.13. (Cell Decomposition Theorem, [19, Chapter 3, Theorem 2.11]) *Let $\mathcal{S}$ be an o-minimal structure and $A_1, \ldots, A_k$ be definable subsets of $\mathbb{R}^n$. Then there is a partition $\mathcal{P}$ of $\mathbb{R}^n$ into cells $C_1, \ldots, C_l$ such that for each $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, k\}$ either $C_i \subseteq A_j$ or $C_i \cap A_j = \emptyset$.*

We say that $\mathcal{P}$ is a cell decomposition of $\mathbb{R}^n$ adapted to $A_1, \ldots, A_k$.

One may easily prove by induction on n that a cell in $\mathbb{R}^n$ is a connected set; as a consequence of the Cell Decomposition Theorem, any set $X \subseteq \mathbb{R}^n$ definable in an o-minimal structure has finitely many connected components, each of which is a definable set.

Since we are working with o-minimal expansions of a field, Theorem 2.1.13 can be improved so that the functions f, g and h involved in the description of the cells are of class $\mathcal{C}^k$. To be precise what we mean by a function of class $\mathcal{C}^k$ on a possibly nonopen set, we give the following definition.

Definition 2.1.14. ([23, Definition 8.4.]) *Let A be a subset of $\mathbb{R}^n$. A map $f : A \to \mathbb{R}$ is said to be of class $\mathcal{C}^k$ with $k \in \mathbb{N} \cup \{+\infty\}$ if there is an open neighbourhood U of A in $\mathbb{R}^n$ and a map $F : U \to \mathbb{R}$ such that F is of class $\mathcal{C}^k$ and the restriction of F to A is f.*

The definition of being analytic is the same definition as Definition 2.1.14 replacing "of class C^k" with "analytic".

Definition 2.1.15. *The structure $\mathcal{S}$ is said to have $\mathcal{C}^k$ (respectively* analytic) cell decomposition *if the functions involved in the definition of the cells in the Cell Decomposition Theorem can be chosen to be of class $\mathcal{C}^k$ (respectively analytic).*

Any o-minimal structure has $\mathcal{C}^k$ cell decomposition for any $k \in \mathbb{N}$ (see [19] Chapter 7, Theorem 3.2 and Exercise 3.3); most structures encountered in this thesis have analytic cell decomposition.

2.1.2 O-minimal expansions of the real ordered field

The example we presented in Example 2.1.1, the structure $\bar{\mathbb{R}} = (\mathbb{R}, +, -, \times, 0, 1, <)$, the ordered field of reals, is the first example of an o-minimal structure, and its definable sets are the semialgebraic sets [70], which are the subject of study in real algebraic geometry. One can create expansions of $\bar{\mathbb{R}}$ by adding new function symbols to the language; in this thesis we will only be interested in the ones that still stay o-minimal.

We now present the main classical examples of o-minimal structures that will be encountered in this thesis. More examples will be introduced in their respective chapters.

Definition 2.1.16. *For each $n \in \mathbb{N}$, let $\mathcal{F}_n$ be the set of all functions that are real analytic in an open neighbourhood of $[-1, 1]^n$. For every $f \in \mathcal{F}_n$, let*

$$\tilde{f}(x) = \begin{cases} f(x) & x \in [-1,1]^n \\ 0 & \textit{otherwise.} \end{cases}$$

For all $n \in \mathbb{N}$ *and* $f \in \mathcal{F}_n$ *we will call* $\tilde{f}$ *a* restricted analytic function. *Let* $\mathcal{F} := \bigcup_{n \in N} \mathcal{F}_n$.

The expansion of the real ordered field by the restricted analytic functions is the structure

$$\mathbb{R}_{an} := \left(\bar{\mathbb{R}}, \{\tilde{f}\}_{f \in \mathcal{F}}\right).$$

The structure $\mathbb{R}_{\text{an}}$ is model complete and o-minimal (Gabrielov [26], Denef and van den Dries [16]).

Definition 2.1.17. *The expansion of the real ordered field by the exponential function is the structure*

$$\mathbb{R}_{\exp} := (\bar{\mathbb{R}}, \{\exp\}),$$

where $\exp : \mathbb{R} \to \mathbb{R}$ *is given by* $\exp(x) = e^x$.

Note that the function exp is not a restricted analytic function; we will discuss later in this section that the structure $\mathbb{R}_{\exp}$ is not a reduct of $\mathbb{R}_{\text{an}}$ (this is related to the phenomenon of "polynomially boundedness"). However the structure $\mathbb{R}_{\exp}$ is model complete and o-minimal (Wilkie [76]).

It is possible to merge the two previous examples while keeping o-minimality:

Definition 2.1.18. *With the notations of Definition 2.1.16 and 2.1.17, the expansion of the real ordered field by the restricted analytic functions and the exponential function is the structure*

$$\mathbb{R}_{\text{an,exp}} := (\bar{\mathbb{R}}, \{\tilde{f}\}_{f \in \mathcal{F}} \cup \{\exp\}).$$

The structure $\mathbb{R}_{\text{an,exp}}$ is a strict expansion of both $\mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\exp}$, which is model complete and o-minimal (Macintyre, Marker and van den Dries [20]).

Definition 2.1.19. *A* generalized power series *in the variables* $X_1, \dots, X_m$ *(with coefficients in* $\mathbb{R}$*) is a formal infinite sum*

$$f(X_1, \dots, X_m) = \sum_{(\mu_1, \dots, \mu_m) \in [0,\infty)^m} a_{\mu_1, \dots, \mu_m} X_1^{\mu_1} \dots X_m^{\mu_m}$$

with each $a_{\mu_1, \dots, \mu_m} \in \mathbb{R}$ *and such that there are well ordered subsets* $M_1, \dots, M_m$ *of* $[0, \infty)$ *such that the* support of f

$$\text{Supp}(f) := \{(\alpha_1, \dots, \alpha_m) \in [0, \infty)^m : a_{\alpha_1, \dots, \alpha_m} \neq 0\}$$

is a subset of $M_1 \times \dots \times M_m$.

With these notations, the constant term of f is the coefficient $a_{0,\ldots,0}$.

Let $m \in \mathbb{N}$ and $X = (X_1, \ldots, X_m)$. We denote by $\mathbb{R}[\![X^\star]\!]$ the set of generalized power series in X. One can check that a natural addition and multiplication are defined on $\mathbb{R}[\![X^*]\!]$ providing it with a unitary ring structure.

Note that since $\mathbb{N}$ is a well ordered subset of $[0, \infty)$, any power series is a special case of generalized power series. More generally, if $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_n)$ for $n \in \mathbb{N}$, we let $\mathbb{R}[\![X^*, Y]\!]$ be the set of generalized power series in the variables $(X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ whose support is included in $[0, \infty)^m \times \mathbb{N}^n$.

If $X = (X_1, \ldots, X_m)$, $Y = (Y_1, \ldots, Y_n)$, $f \in \mathbb{R}[\![X^*, Y]\!]$ and $g_1, \ldots, g_k$ are elements of $\mathbb{R}[\![X^*]\!]$ with constant term 0, then one can define the composition $f(X, g_1(X), \ldots, g_k(X))$ by replacing Y_i by $g_i(X)$ in the generalized power series $f(X, Y)$ (this operation is well defined thanks to the fact that supports are well ordered).

Given $m \in \mathbb{N}^+$ and $r = (r_1, \ldots, r_m) \in \mathbb{R}^m$ with $r_i > 1$ for all i, we let $\mathbb{R}[\![X^\star]\!]_r$ be the set of all $f(X_1, \ldots, X_m) = \sum_{(\mu_1,\ldots,\mu_m)\in[0,\infty)^m} a_{\mu_1,\ldots,\mu_m} X_1^{\mu_1} \ldots X_m^{\mu_m} \in \mathbb{R}[\![X^\star]\!]$ such that $\sum_{(\mu_1,\ldots,\mu_m)\in[0,\infty)^m} |a_{\mu_1,\ldots,\mu_m}| r_1^{\mu_1} \ldots r_m^{\mu_m}$ converges. Each such formal series

$$f(X_1, \ldots, X_m) = \sum_{(\mu_1,\ldots,\mu_m)\in[0,\infty)^m} a_{\mu_1,\ldots,\mu_m} X_1^{\mu_1} \ldots X_m^{\mu_m} \in \mathbb{R}[\![X^\star]\!]_r$$

gives a function defined on $[-r_1, r_1] \times \ldots \times [-r_m, r_m]$ by

$$f(x_1, \ldots, x_m) = \sum_{(\mu_1,\ldots,\mu_m)\in[0,\infty)^m} a_{\mu_1,\ldots,\mu_m} x_1^{\mu_1} \ldots x_m^{\mu_m},$$

which we call a convergent generalized power series on $[-r_1, r_1] \times \ldots \times [-r_m, r_m]$.

Definition 2.1.20. *Let $\mathcal{P}_n$ be the set of all functions given as a converging generalized power series on some $[-r_1, r_1] \times \ldots \times [-r_n, r_n]$ with $r_i > 1$ for all i.*

For every $f \in \mathcal{P}_n$, let

$$\tilde{f}(x) = \begin{cases} f(x) & x \in [-1, 1]^n \\ 0 & \textit{otherwise.} \end{cases}$$

Let $\mathcal{P} := \bigcup_{n \in N} \mathcal{P}_n$.

The expansion of the real ordered field by the restricted convergent generalized power series *is the structure*

$$\mathbb{R}_{\mathrm{an}^*} := (\bar{\mathbb{R}}, \{\tilde{f}\}_{f\in\mathcal{P}}).$$

The structure $\mathbb{R}_{an^*}$ is a strict expansion of $\mathbb{R}_{an}$ which is model complete and o-minimal but it does not define the exponential function (Speissegger and van den Dries [24]).

Definition 2.1.21. *For each $\alpha \in \mathbb{R}$, the power function x^α is defined as follows*

$$x^\alpha = \begin{cases} x^\alpha & x > 0 \\ 0 & \textit{otherwise.} \end{cases}$$

The expansion of the real ordered field by all the power functions is the structure

$$\mathbb{R}^{pow} := \left(\bar{\mathbb{R}}, \{x^\alpha\}_{\alpha\in\mathbb{R}}\right).$$

The structure $\mathbb{R}^{pow}$ is model complete and o-minimal (Chris Miller [47]).

Definition 2.1.22. *Let $\mathcal{R}$ be an o-minimal expansion of the real field. The set of all $\alpha \in \mathbb{R}$ such that the function $x \mapsto x^\alpha : (0, \infty) \to \mathbb{R}$ is definable in $\mathcal{R}$ forms a subfield of $\mathbb{R}$ and it is called the* field of exponents of $\mathcal{R}$.

Definition 2.1.23. *An o-minimal expansion of the real field is said to be* polynomially bounded *if for every one variable definable function f there is an $N \in \mathbb{N}$, and $M > 0$ such that*

$$|f(x)| \leq x^N$$

for all $x > M$.

There is a strong relationship between polynomially boundedness and definability of the exp function. The following dichotomy is due to C. Miller ([48], [49]).

Theorem 2.1.24 ([48],[49])**.** *If $\mathcal{R}$ is an o-minimal expansion of the real field, then*

- *either $\mathcal{R}$ is polynomially bounded,*
- *or the exponential function is definable in $\mathcal{R}$.*

The structures $\mathbb{R}_{exp}$ and $\mathbb{R}_{an,exp}$ are clearly *not* polynomially bounded. The structures $\mathbb{R}_{an}$ and $\mathbb{R}_{an^*}$ are polynomially bounded and their field of exponents are respectively $\mathbb{Q}$ (Łojasiewicz [41]) and $\mathbb{R}$ (van den Dries and Speissegger [24]).

All the structures $\mathbb{R}_{an}$, $\mathbb{R}_{exp}$, $\mathbb{R}_{an,exp}$ and $\mathbb{R}_{an^*}$ have analytic cell decomposition (respectively [16], [76], [20], [24]).

2.2 Pfaffian functions

Pfaffian functions are analytic functions that satisfy a triangular system of polynomial first order differential equations. They were first introduced by Khovanskiĭ [37]. Pfaffian functions can be defined on $\mathbb{R}$ or $\mathbb{C}$; here we will give the definition on $\mathbb{R}$ and always consider this case.

Definition 2.2.1. *Let U be an open subset of $\mathbb{R}^n$ and let $r \geq 0$. A sequence $(f_1, \ldots, f_r)$ of analytic functions from U to $\mathbb{R}$ satisfying the equations*

$$df_j(\bar{x}) = \sum_{1 \leq i \leq n} P_{ij}\left(\bar{x}, f_1(\bar{x}), \ldots, f_j(\bar{x})\right) dx_i$$

for all $j = 1, \ldots, r$ and $\bar{x} \in U$, where $P_{ij}\left(\bar{x}, y_1, \ldots, y_j\right) \in \mathbb{R}[\bar{x}, y_1, \ldots, y_j]$ are polynomials of degree at most α, is called a Pfaffian chain of order r and degree $\alpha \geq 1$.

Definition 2.2.2. *Let $r \geq 0$ and let $(f_1, \ldots, f_r)$ be a Pfaffian chain of degree α on the open set $U \subseteq \mathbb{R}^n$. A function $f : U \to \mathbb{R}$ is called a* Pfaffian function of order r and degree (α, β) *if there exists a polynomial $P\left(\bar{x}, y_1, \ldots, y_r\right) \in \mathbb{R}[\bar{x}, y_1, \ldots, y_r]$ with* $\deg(P) \leq \beta$ *such that*

$$f(\bar{x}) = P\left(\bar{x}, f_1(\bar{x}), \ldots, f_r(\bar{x})\right)$$

for all $\bar{x} \in U$. A curve given by a Pfaffian function is called a Pfaffian curve.

Remark 2.2.3. If $(f_1, \ldots, f_r)$ is a Pfaffian chain of degree α on the open set $U \subseteq \mathbb{R}^n$, then each function f_i is a Pfaffian function of order i and degree less than or equal to $(\alpha, 1)$.

The subsets of $\mathbb{R}^n$ that are defined like semialgebraic sets (Definition 2.1.1) but replacing polynomials with Pfaffian functions are called *semipfaffian sets*. One of the key features of systems of Pfaffian functions (proved in [37]) is that any n-tuple of Pfaffian functions from $(-1, 1)^n$ to $\mathbb{R}$ has only finitely many nondegenerate zeros and that the number of these zeros can be bounded uniformly in the order and degree of the Pfaffian functions involved. This uniformity result is an important tool to obtain strong bounds on the number of rational points of a given height in the examples of sets studied in [34], [56], [57], [59].

We now give some examples of one variable Pfaffian functions, some of which we will use later in the thesis.

Examples 2.2.4.

1. Any polynomial $P \in \mathbb{R}[X]$ in one variable with degree β is a Pfaffian function of order 0 and degree $(1, \beta)$ (if $\beta \geq 1$ it can also be seen as a Pfaffian function of order 1 and degree $(\beta - 1, 1)$).

2. Let $f_1 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the function given by $f_1(x) = 1/x$. The function f_1 is analytic on $\mathbb{R} \setminus \{0\}$ and
$$f_1'(x) = \frac{-1}{x^2} = P_1\left(x, f_1(x)\right)$$
for all $x \in \mathbb{R} \setminus \{0\}$ where $P_1(x, y) = -y^2$. So (f_1) is a Pfaffian chain of order 1 and degree 2, hence, by Remark 2.2.3 the function $x \mapsto 1/x$ is Pfaffian of order 1 and degree $(2, 1)$ on the domain $\mathbb{R} \setminus \{0\}$.

3. Similar to the first example the function $\exp : \mathbb{R} \to \mathbb{R}; x \mapsto e^x$ itself forms a Pfaffian chain of order 1 and degree 1 since $(\exp)'(x) = e^x = P_1(x, \exp(x))$ where $P_1(x, y) = y$; by Remark 2.2.3, the function exp is Pfaffian of order 1 and degree $(1, 1)$ on $\mathbb{R}$.

4. Let f_1, f_2, f_3 be functions defined on $(0, \infty)$ by $f_1(x) = 1/x$, $f_2(x) = \log(x)$, and $f_3(x) = x^x$. Note that these functions are analytic on $(0, \infty)$ and
$$\begin{aligned}&(1/x)' = -1/x^2 = P_1(x, 1/x) \text{ where } P_1(x, y) = -y^2\\&(\log(x))' = 1/x = P_2(x, 1/x, \log(x)) \text{ where } P_2(x, y, z) = y\\&(x^x)' = x^x(\log(x) + 1) = P_3(x, 1/x, \log(x), x^x) \text{ where } P_3(x, y, z, t) = zt + t.\end{aligned}$$
So (f_1, f_2, f_3) is a Pfaffian chain of order 3 and degree 2 on the domain $(0, \infty)$. By Remark 2.2.3, the function $x \mapsto x^x$ is Pfaffian of order 3 and degree $(2, 1)$ on $(0, \infty)$.

5. Let f_1, f_2, f_3 be the functions defined on $\mathbb{R} \setminus 2\pi\mathbb{Z}$ by $f_1(x) = \cot\frac{x}{2}$, $f_2(x) = \sin^2\frac{x}{2}$ and $f_3(x) = \sin x$, which are analytic on $\mathbb{R} \setminus 2\pi\mathbb{Z}$.
We have
$$f_1'(x) = -\frac{1}{2}\left(\cot^2\frac{x}{2} + 1\right) = P_1\left(x, \cot\frac{x}{2}\right)$$
where $P_1(x, y) = \dfrac{-1}{2}(y^2 + 1)$,
$$f_2'(x) = \sin\frac{x}{2}\cos\frac{x}{2} = \sin^2\frac{x}{2}\cot\frac{x}{2} = P_2\left(x, \cot\frac{x}{2}, \sin^2\frac{x}{2}\right)$$
where $P_2(x, y, z) = yz$ and
$$f_3'(x) = \cos x = 1 - 2\sin^2\frac{x}{2} = P_3\left(x, \cot\frac{x}{2}, \sin^2\frac{x}{2}\right)$$
where $P_3(x, y, z) = 1 - 2z$.
So, (f_1, f_2, f_3) is a Pfaffian chain of order 3 and degree 2 on $\mathbb{R} \setminus 2\pi\mathbb{Z}$ and, by Remark 2.2.3, the function $x \mapsto \sin x$ is Pfaffian of order 3 and degree $(2, 1)$ on the domain $\mathbb{R} \setminus 2\pi\mathbb{Z}$.

2.3 Pila-Wilkie Theorem and Wilkie's conjecture

Diophantine geometry studies integer and rational solutions of systems of algebraic equations. It is natural to extend this study to "geometrically well behaved" systems of equations. One of the first questions that can be studied is whether or not a system of equations has "many" integer/rational solutions. An early question studied in this line of research concerns the intersection points of curves with lattices. In [32], Jarník proved a bound for such points considering strictly convex arcs. It is possible to reprove Jarník's result using a different method using ideas of Dörge from [18] (see [3]). The idea of Dörge was to consider a bound on the number of lines that covers the set of integer points in the curve. This idea was developed by Bombieri and Pila in [6] by considering algebraic curves instead of straight lines covering the rational points of a graph of real analytic function f which is transcendental (a function that does not satisfy a polynomial equation). They proved a bound on the number of algebraic curves of fixed degree d that cover the rational points (with a fixed denominator) of the graph of f. Then they used the compactness of the space of algebraic curves of degree d to prove the following theorem.

Theorem 2.3.1. ([6, Theorem 1]) *Let f be a real analytic transcendental function on a closed bounded interval and let X be the graph of f. Then for all $\epsilon > 0$ there is a constant $c = c(f, \epsilon)$ such that*

$$|tX \cap \mathbb{Z}^2| \leq ct^{\epsilon}$$

for all $t \geq 1$.

The method Bombieri and Pila developed in [6], inspired by Dörge ([18]), is known today as the determinant method. We will illustrate this method in Section 3.3 via presenting in detail a proof of Pila in [56] on the density of rational points of Pfaff curves (graphs of one variable Pfaffian functions on connected domains, see [56, Definition 1.1]).

In this section we will discuss the theorem of Pila and Wilkie which generalizes the above mentioned program of research begun by Bombieri and Pila ([6]), on counting rational points of "transcendental" sets, to the o-minimal setting. Before giving the Pila-Wilkie Theorem we define the kinds of sets considered in counting rational points and the notion of height used to define the density of the rational points of sets.

A height function is a real valued function on some number field and measures the arithmetic complexity of a point of this number field. One fundamental property of height functions is that any set contains only finitely points with height less than a given bound (see [7]). Because of this property of heights, studying the growth of the number of points in a set of bounded height as the height tends to infinity can be thought of as a measure of the density of rational points in this set, even when the set has infinitely many rational points. In this thesis we will be always using the naive height function on the field of rational numbers. Other height functions on other number fields have been used for the purpose of exploring the density of rational points of given sets in research towards Wilkie's conjecture, for example in [59].

Definition 2.3.2. *Let* $q = \frac{a}{b} \in \mathbb{Q}$ *with* $\gcd(a, b) = 1$ *and* $b \neq 0$*. The* height of q *is defined by* $H(q) := \max(|a|, |b|)$.
For $\bar{q} = (q_1, \ldots, q_n) \in \mathbb{Q}^n$, $H(\bar{q}) := \max((H(q_1), \ldots, H(q_n))$.
For a subset X *of* $\mathbb{R}^n$ *and a given* $H \in \mathbb{Z}$ *the* set of rational points of X with height bounded by H *is denoted by*

$$X(\mathbb{Q}, H) := \{q \in X \cap \mathbb{Q} : H(q) \leq H\}.$$

And the density function of X *is*

$$N(X, H) := \#X(\mathbb{Q}, H).$$

In [53] Pila proved a version of Theorem 2.3.1 using the height notion. He considered the real analytic and transcendental function $f : [0, 1] \to \mathbb{R}$ and proved that for all $\epsilon > 0$ there is a constant $c > 0$ such that $N(\Gamma(f), H) \leq cH^\epsilon$. (Here and in what follows $\Gamma(f)$ denotes the graph of f.) This bound clearly fails for semialgebraic sets, even for curves; for instance the diagonal $\{(x, y) \in [0, 1]^2 : y = x\}$ has at least $H+1$ many rational points of height at most H. In higher dimensions, it may be the case that a nonsemialgebraic set contains a semialgebraic subset which may have more points of a given height than the bound in Theorem 2.3.1 allows. For example, the surface $z = x^y$, where $1 \leq x, y \leq 2$, contains the curve $\{(x, y, z) : y = 1, z = x\}$ and this curve has at least $H + 1$ many points of height at most H. Getting strong bounds on the number of rational points of a definable set requires removing from it all the connected semialgebraic sets of dimension at least 1 it may contain. For that reason we have the following definition.

Definition 2.3.3. *Let $X \subset \mathbb{R}^n$. The union of all connected semialgebraic subsets of X of positive dimension is called the* algebraic part of X *and is denoted by X^{alg}. The set $X \setminus X^{alg}$ is called the* transcendental part of X *and is denoted by X^{tr}.*

Parametric description of a set can give important information about the geometric properties of the set especially when the parametrization functions satisfy specific "nice" properties.

Definition 2.3.4. *Let $X \subseteq \mathbb{R}^n$ be a set definable in an o-minimal structure with $\dim X = d$. A finite set of functions $\phi_1, \ldots, \phi_l : (0,1)^d \to X$ is called a* parametrization of X *if*

$$\bigcup_{i=1}^{l} \mathrm{Im}(\phi_i) = X.$$

We are interested in *smooth parametrization* which have some control over the derivatives of the functions involved in the parametrization. For example being able to bound the absolute values of the derivatives of the parametrization functions to a certain order leads to important results in dynamics, analysis, diophantine and computational geometry (see for example [79]).

Pila and Wilkie proved a reparametrization theorem ([52, Theorem 2.3]) which makes it possible to see the set of rational points considered as images of C^r functions with bounds on their derivatives up to order r. Following the strategy of Bombieri and Pila from [6], Pila and Wilkie in [52] proved a bound on the number of hypersurfaces of bounded degree that covers the set of rational points of bounded height lying on the images of these parametrizing functions. The reparametrization theorem of Pila and Wilkie is obtained by improving Gromov-Yomdin's algebraic reparametrization lemma ([27], [77]) for the o-minimal setting.

Definition 2.3.5. ([52, Definition 2.2]) *Let $\mathcal{R}$ be an o-minimal expansion of the real field and let $X \subset \mathbb{R}^n$ be definable in $\mathcal{R}$. An* r-parametrization *of X is a parametrization consisting of functions $\phi_1, \ldots, \phi_l$ of class C^r such that*

$$|D^\alpha \phi_i(\bar{x})| \leq 1$$

for all $i = 1, \ldots, l$, $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq r$ and for all $\bar{x} \in (0,1)^{\dim X}$ where D^α denotes the differential operator (see introduction to Chapter 3).

We state below a special version of Theorem 2.3 in [52] which is more convenient for our setting.

Theorem 2.3.6. *Let $\mathcal{R}$ be an o-minimal expansion of the real field and let $X \subset \mathbb{R}^n$ be a bounded set definable in $\mathcal{R}$. Then for any $r \in \mathbb{N}$ there is an r-parametrization of X.*

Proposition 2.3.7. ([52, Proposition 6.1]) *Let $k, n \in \mathbb{N}$ with $k < n$. Then there are, for each $d \in \mathbb{N}$, $d \geq 1$ a nonnegative integer r and positive constants ϵ and C all depending on k, n and d with the following property. Let $\phi : (0,1)^k \to \mathbb{R}^n$ be a function of class C^r with $|D^\alpha \phi(\bar{x})| \leq 1$ for all $\alpha \in \mathbb{N}^k$ with $|\alpha| \leq r$ and all $\bar{x} \in (0,1)^k$. Let X be the image of the function ϕ, let $H \geq 1$. Then $X(\mathbb{Q}, H)$ is contained in the union of at most CH^ϵ hypersurfaces of degree less than or equal to d. Moreover $\epsilon \to 0$ as $d \to \infty$.*

Combining Theorem 2.3.6 and Proposition 2.3.7 and using o-minimality Pila and Wilkie then obtained the following theorem.

Theorem 2.3.8. [Pila-Wilkie] ([52, Theorem 1.8]) *Let $\mathcal{R}$ be an o-minimal expansion of the real field, let $X \subset \mathbb{R}^n$ and let $\epsilon > 0$. Then there is a constant $c(X, \epsilon) > 0$ such that*

$$N(X^{tr}, H) \leq cH^\epsilon.$$

There have been many applications of Pila-Wilkie Theorem to number theory. Most significant of these applications are a new proof of the Manin-Mumford conjecture in [61] and a proof for instances of André -Oort conjecture in [60]. For more information about the applications of Pila-Wilkie Theorem reader can see [34] and [35].

It is remarked in [52] that this bound is optimal in general in the sense that there is an analytic (hence definable in $\mathbb{R}_{\text{an}}$) function $f : [0,1] \to [0,1]$ which is not semialgebraic, a sequence of integers $(H_k)_{k \in \mathbb{N}}$ tending to ∞ and a sequence of positive real numbers $(\epsilon_k)_{k \in \mathbb{N}}$ tending to 0 such that, for all $k \in \mathbb{N}$, the graph of f contains at least $H_k^{\epsilon_k}$ many rational points of the form (p, q) with $H(p, q) \leq H$ ([55, Example 7.5]). However Wilkie proposed an improvement for the bound from Theorem 2.3.8 for $\mathbb{R}_{\exp}$, a conjecture which is still open in full generality.

Conjecture 2.3.9 (Wilkie's conjecture)**.** ([52, Conjecture 1.11]) *Let $X \subseteq \mathbb{R}^n$ be a definable set in $\mathbb{R}_{\exp}$. Then there are constants $c_1, c_2 > 0$ such that, for all $H \geq e$,*

$$N(X^{tr}, H) \leq c_1 (logH)^{c_2}.$$

The work in this thesis is motivated by the techniques mentioned in this section, of employing parametrizations to obtain bounds on the density of rational points of sets to approach Wilkie's conjecture. The basic idea is to use mild parametrization analogous to r-parametrization to improve the bound in the Pila-Wilkie Theorem, towards Wilkie's conjecture. We will discuss mild parametrization in more detail and provide explanation about its role in improving the bound on the density of rational points of sets in the next chapter.

Chapter 3

Mild functions and mild parametrization

The main subject of this thesis is the study of o-minimal expansions of the real field in terms of their definable sets: we explore if it is possible to have a special kind of smooth parametrization of these sets. We remind the reader here that a smooth parametrization is a parametrization obtained using functions which have specific bounds on their higher derivatives. In Section 2.3.8, we have presented the Pila-Wilkie Theorem and explained that it is obtained by improving a smooth parametrization result. We remind the reader here that this theorem gives an upper bound for the density of rational points of bounded height of the transcendental part of a set definable in an o-minimal expansion of the real field, and following this result Wilkie conjectured that this bound can be improved for definable sets of the structure $\mathbb{R}_{\exp}$. The role of the parametrization result of Pila and Wilkie in the proof of Pila-Wilkie Theorem was to provide a bound on the number of the hypersurfaces (of bounded degree) covering the set of rational points considered. The bounds on the derivatives of the parametrizing functions play a role in this bound. The motivation for considering another kind of smooth parametrization with different bounds on the derivatives of the parametrizing functions arises from the hope to improve this bound on the hypersurfaces mentioned above, towards Wilkie's conjecture.

The smooth parametrization we are interested in uses mild functions as parametrizing functions and is called mild parametrization. It has been proved by Pila in [56] that the desired improvement for the bound on the number of hypersurfaces containing the rational points of bounded height can be obtained for the images of mild functions (see Section 3.3). So it yields that mild parametrization is a promising tool to consider towards Wilkie's conjecture.

There are also other results obtained using mild parametrization. Some of these are: Pila in [59] and Butler in [10] proved Wilkie's conjecture for special surfaces using mild parametrization; in [33], Jones, D. Miller and Thomas proved that $\mathbb{R}_{\text{respfaff}}$, the expansion of the real field by all restricted Pfaffian functions, admits definable mild parametrization and proved Wilkie's conjecture for definable curves in this structure; in [34], Jones and Thomas proved Wilkie's conjecture for surfaces, definable in $\mathbb{R}_{\exp}$ which have mild parametrization and also for all surfaces definable in $\mathbb{R}_{\text{respfaff}}$. It was also proven by Pila in [59] that mild parametrization with sufficient uniformity in parameters would be sufficient to establish Wilkie's conjecture. We will discuss some of these results in detail in the rest of the thesis.

In this chapter we will focus on understanding the definition and properties of mild functions and those functions whose graph can be parametrized by some mild functions. Also we will present known results about mild parametrization and motivate our interest in mild parametrization by explaining its applications to number theory. This chapter can be considered as a self-contained one that does not rely on any knowledge and ideas from model theory but rather examines mildness from the point of view of analysis and number theory.

We will give the definition of mild parametrization and discuss its properties in Section 3.1. Section 3.2 consists of examples and nonexamples of mild and mild parametrizable functions. In particular, the mildness of the power functions will be discussed. We will discuss the fact that noninteger power functions are not mild and show however that the graph in $(0,1)^2$ of any positive power function has mild parametrization.

In Section 3.3, in order to provide more detailed explanation of how mild parametrization is beneficial, we will explain the work of Pila from [56] where he first introduced first the definition of mild parametrization and proved results on the density of the rational points of Pfaff curves lying in $[-1,1]^2$. We will also discuss the argument of Pila ([59]), about possible uniformity conditions on the mild parametrization of the definable sets of $\mathbb{R}_{\exp}$, that would imply Wilkie's conjecture. This section will provide key motivation for the usage of mild parametrization to obtain number theoretic results, particularly towards Wilkie's conjecture.

We fix some multi-index notation that we will use throughout the rest of the text.

Notation. For $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $r = (r_1, \ldots, r_d) \in \mathbb{R}^d$, we set

$$|n| = \sum_{i=1}^{d} n_i,$$

with the convention $0! = 1$, $0^0 = 1$ we set

$$n! = \prod_{i=1}^{d} n_i!, \quad r^n = \prod_{i=1}^{d} r_i^{n_i}$$

and the differential operator

$$D^n = \frac{\partial^{|n|}}{\partial x_1^{n_1} \ldots \partial x_d^{n_d}}.$$

For $n \in \mathbb{N}^d$ and a map $G : (0,1)^d \to (0,1)^m$ with coordinate functions $g_1, \ldots, g_m$, we will set

$$D^n G = (D^n g_1, \ldots, D^n g_m).$$

For $m \in \mathbb{N}^d$ and $n \in \mathbb{N}^d$ with $m = (m_1 \ldots, m_d)$ and $n = (n_1 \ldots, n_d)$, $m \leq n$ means $m_i \leq n_i$ for all $i = 1, \ldots, d$, and, given $m, n \in \mathbb{N}^d$ with $m \leq n$,

$$\binom{n}{m} = \frac{n!}{(n-m)!m!} = \prod_{i=1}^{d} \binom{n_i}{m_i}.$$

For $p, k_1, \ldots, k_d \in \mathbb{N}$ with $k_1 + \ldots + k_d = p$,

$$\binom{p}{k_1, \ldots, k_d} = \frac{p!}{k_1! k_2! \ldots k_d!}.$$

3.1 Mild functions

In this section we will investigate mild functions and their properties. We will explore them by examining examples and nonexamples in the next section. We start by giving the definition of a mild function.

Definition 3.1.1. *Let $f : (0,1)^d \to (0,1)$ be a C^∞ function and let B, C be real numbers with $B > 0$, $C \geq 0$. The function f is said to be (B,C)-*mild *if*

$$|D^n f(\bar{x})| \leq n!(B|n|^C)^{|n|}$$

for all $n \in \mathbb{N}^d$ and $\bar{x} \in (0,1)^d$ (with the convention $0! = 1$ and $0^0 = 1$). A map $F : (0,1)^d \to (0,1)^m$ is said to be (B,C)-mild if its coordinate functions are (B,C)-mild. We say that f is mild if there exist $B, C \in \mathbb{R}$ with $B > 0$, $C \geq 0$ such that f is (B,C)-mild.

Definition 3.1.2. *Let $X \subseteq (0,1)^n$. We say that X has* mild parametrization *if there exist a finite number of mild functions $\phi_1, \dots \phi_l : (0,1)^{\dim(X)} \to (0,1)^n$ such that*

$$X = \bigcup_{i=1}^{l} \operatorname{Im}(\phi_i).$$

We say that a function has mild parametrization *if its graph has.*

Remark 3.1.3. Our interest in possible mild parametrizations of sets arises from the goal of obtaining bounds on the density of rational points of bounded height of these sets. Since the function $x \mapsto 1/x$ preserves height it is enough to consider the density of rational points of sets lying in $(0,1)^n$. For that reason we only consider mild maps with domain $(0,1)^n$ and range $(0,1)^m$, $n, m \in \mathbb{N}$ and the mild parametrization of sets lying in the cartesian products of $(0,1)$.

The following lemma will enable us to deal with the cases when, in order to parametrize a set, we want to use a function satisfying the mildness bounds on the derivatives but the domain is not $(0,1)$, rather an interval in $(0,1)$.

Lemma 3.1.4. *Let $f : (0,1) \to (0,1)$ be a mild function, and let $I = (a,b) \subseteq (0,1)$. Then the function $\bar{f} : (0,1) \to (0,1)$ defined by $f(t) = f((b-a)t + a)$ is mild and $\operatorname{Im}(f\restriction_I) = \operatorname{Im}(\bar{f})$.*

Proof. The fact that $\operatorname{Im}(f\restriction_I) = \operatorname{Im}(\bar{f})$ is obvious. Let f be (B,C)-mild. Then

$$\left|\bar{f}^{(n)}(t)\right| = (b-a)^n \left|f^{(n)}((b-a)t + a)\right| \leq (b-a)^n n! B^n n^{Cn}$$

for all $t \in (0,1)$ and $n \in \mathbb{N}$. Hence $\bar{f}$ is (B,C)-mild since $(b-a) \leq 1$. □

We first consider how mild functions behave under arithmetic operations.

Proposition 3.1.5. *Let $d \in \mathbb{N}$ and let $f, g : (0,1)^d \to (0,1)$ be two mild functions. Then the functions $\dfrac{f+g}{2} : (0,1)^d \to (0,1)$ and $f \cdot g : (0,1)^d \to (0,1)$ are mild.*

Proof. Let $B, B' > 0$ and $C, C' \geq 0$ be such that f is (B,C)-mild and g is (B',C')-mild. The sum $\frac{f+g}{2}$ is $(B+B', \max\{C,C'\})$-mild:

$$\begin{aligned}\left|D^n\left(\frac{f+g}{2}\right)(\bar{x})\right| &\leq \frac{|D^n f(\bar{x})| + |D^n g(\bar{x})|}{2} \\ &\leq \frac{n!(B|n|^C)^{|n|} + n!(B'|n|^{C'})^{|n|}}{2} \\ &\leq \frac{n!(B+B')^{|n|}(|n|^{C|n|} + |n|^{C'|n|})}{2} \\ &\leq n!(B+B')^{|n|}|n|^{\max\{C,C'\}|n|}\end{aligned}$$

for all $\bar{x} \in (0,1)^d$, $n \in \mathbb{N}^d$.

Using the general Leibniz rule for multivariable functions we have

$$\begin{aligned}|D^n(f\cdot g)(\bar{x})| &= \left|\sum_{\{m:m\leq n\}} \binom{n}{m}((D^m f)(D^{n-m}g))(\bar{x})\right| \\ &\leq \sum_{\{m:m\leq n\}} \binom{n}{m}|((D^m f)(D^{n-m}g))(\bar{x})| \\ &\leq \sum_{\{m:m\leq n\}} \binom{n}{m} m!B^{|m|}|m|^{C|m|}(n-m)!B'^{|n-m|}|n-m|^{C'|n-m|} \\ &= \sum_{\{m:m\leq n\}} n!B^{|m|}B'^{|n-m|}|m|^{C|m|}|n-m|^{C'|n-m|} \\ &\leq \sum_{\{m:m\leq n\}} n!B^{|m|}B'^{|n-m|}|n|^{C|m|}|n|^{C'|n-m|}\end{aligned}$$

for all $\bar{x} \in (0,1)^d$, $n \in \mathbb{N}^d$.
Assuming without loss of generality that $B' \leq B$ and $C' \leq C$ and using the fact that $|n-m| + |m| = |n|$ we obtain the following inequality

$$|D^n(f\cdot g)(\bar{x})| \leq \sum_{\{m:m\leq n\}} n!B^{|n|}|n|^{C|n|}$$

for all $\bar{x} \in (0,1)^d$, $n \in \mathbb{N}^d$.
Note that the terms in the sum on the right hand side of the inequality do not depend on m. For $n = (n_1, \ldots, n_d)$ there are $\prod_{i=1}^d (n_i + 1)$ possible m's with $m \leq n$. We assume for now that $|n| \geq 1$. Then

$$\prod_{i=1}^d (n_i+1) \leq \prod_{i=1}^d (2|n|) = 2^d|n|^d \leq 2^d|n|^{d|n|}$$

and therefore

$$|D^n(f\cdot g)(\bar{x})| \leq n!B^{|n|}|n|^{C|n|}2^d|n|^{d|n|}.$$

Let $\beta > 0$ be such that $2^d B^{|n|} \leq \beta^{|n|}$ for all $n \in \mathbb{N}^d$ with $|n| \geq 1$ and put $\gamma = C + d$. Then

$$|D^n(f \cdot g)(\bar{x})| \leq n! \beta^{|n|} |n|^{\gamma |n|}$$

for all $\bar{x} \in (0,1)^d$. Note that if $|n| = 0$ this inequality still holds since $\mathrm{Im}(f \cdot g) \subseteq (0,1)$. Hence we can conclude that the product of the functions f and g is (β, γ)-mild. $\square$

We will next prove that the composition of two mild maps is mild as well. First we will give the following combinatorial lemmas which we will use in that proof.

Lemma 3.1.6. *Let $a, b \in \mathbb{N}^+$. Then* $\binom{a+b}{b} \leq (a+1)^b$.

Proof. Since $\frac{a+i}{i} \geq \frac{a+j}{j}$ for all $j \geq i$ we have

$$\binom{a+b}{b} = \frac{(a+1)\dots(a+b)}{b!} \leq (a+1)^b.$$

$\square$

Lemma 3.1.7. *There exist* $\binom{n-1+k}{k}$ *many different $\bar{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ such that $a_1 + \dots + a_n = k$.*

Proof. We associate each element $\bar{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ with a binary string (a string that consists of 0s and 1s) in such a way that each coordinate a_i of $\bar{a}$ is represented by a_i-many consecutive 1s in this string and between the representation of two coordinates there is a 0 to separate them. For example $(2,0,1,3)$ is associated with 110010111. So in order to calculate how many $\bar{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ exist with $a_1 + \dots + a_n = k$ we will calculate the number of different binary strings that contain k-many 1s (so that the sum of the coordinates is k) and $n-1$ many 0s to separate the n coordinates. The length of the string is $n-1+k$ and we choose k places in the string to place the 1s. This choosing can be done in

$$\binom{n-1+k}{k}$$

many different ways. $\square$

Lemma 3.1.8. *Let $k \in \mathbb{N}^+$. Then the number of different $\bar{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ such that $1 \leq a_1 + \dots + a_n \leq k$, is less than $k(kn)^k$.*

Proof. The number of different $\bar{a} \in \mathbb{N}^n$ such that $1 \leq a_1 + \dots + a_n \leq k$ is $S := \sum_{i=1}^{k} S_i$ where $S_i := \#\{\bar{a} : a_1 + \dots + a_n = i\}$ for $i = 1, \dots, k$. By Lemma 3.1.7, $S_i = \binom{n-1+i}{i}$. So by Lemma 3.1.6, $S_i \leq (n-1+i)^i$. Therefore,

$$S \leq \sum_{i=1}^{k} (n-1+i)^i \leq k(n-1+k)^k.$$

Since $k \geq 1$, for $c \in \mathbb{N}$, $c + k \leq k(c+1)$. So we have $S \leq k(kn)^k$. $\square$

We then study the compatibility of mildness with composition.

Proposition 3.1.9. *Let $f:(0,1)^m \to (0,1)$ be a mild function and let $G:(0,1)^d \to (0,1)^m$ be a mild map with coordinate functions $g_1,\dots,g_m$. Then the function $f\circ G : (0,1)^d \to (0,1)$ is mild.*

Proof. Let $B, B' > 0$ and $C, C' \geq 0$ be such that $g_1,\dots,g_m$ are (B,C)-mild and f is (B',C')-mild. We will show that there exist $\beta > 0$ and $\gamma \geq 0$ such that $f\circ G$ is (β,γ)-mild. We first recall the multivariable Faà di Bruno formula (see [14]): for any $n \in \mathbb{N}^d$

$$D^n(f\circ G) = \sum_{1\leq|\lambda|\leq|n|} D^\lambda f \sum_{s=1}^{|n|} \sum_{P_s(\lambda,n)} n! \prod_{i=1}^{s} \frac{(D^{l_i}G)^{k_i}}{k_i!(l_i!)^{|k_i|}}$$

where, for each $\lambda \in \mathbb{N}^m$ and $s \in \mathbb{N}$,

$$\begin{aligned} P_s(\lambda,n) := \{(k_1,\dots,k_s,l_1,\dots,l_s) : & k_i \in \mathbb{N}^m, l_i \in \mathbb{N}^d, \\ & |k_i| > 0, \\ & 0 \prec l_1 \prec \cdots \prec l_s, \\ & \sum_{i=1}^{s} k_i = \lambda, \\ & \sum_{i=1}^{s} |k_i| l_i = n\}, \end{aligned}$$

and $\prec$ denotes the lexicographic order on $\mathbb{N}^d$.

Note that $\sum_{i=1}^{s} |k_i| l_i = n$ implies that $\sum_{i=1}^{s} |k_i||l_i| = |n|$ and since $|k_i| > 0$, $|l_i| \leq |n|$ for all $i \in \{1,\dots,s\}$. Also $\sum_{i=1}^{s} k_i = \lambda$ implies that $\sum_{i=1}^{s} |k_i| = |\lambda|$, so $|k_i| \leq |\lambda| \leq |n|$ for all $i \in \{1,\dots,s\}$ since $0 \prec l_i$ for all $i \in \{1,\dots,s\}$.

For some fixed $s \in \mathbb{N}$ and $i \in \{1,\dots,s\}$, write $k_i = (k_{i,1}, k_{i,2}\dots,k_{i,m})$. Then

$$\left|(D^{l_i}G)^{k_i}(\bar{x})\right| = \left|\prod_{j=1}^{m} (D^{l_i}g_j)^{k_{i,j}}(\bar{x})\right| \leq \prod_{j=1}^{m} \left(l_i! B^{|l_i|} |l_i|^{C|l_i|}\right)^{k_{i,j}} = (l_i! B^{|l_i|} |l_i|^{C|l_i|})^{|k_i|}$$

for all $\bar{x} \in (0,1)^d$. So

$$\begin{aligned} \left|\prod_{i=1}^{s} \frac{(D^{l_i}G)^{k_i}(\bar{x})}{k_i!(l_i!)^{|k_i|}}\right| &\leq \prod_{i=1}^{s} \frac{(l_i! B^{|l_i|} |l_i|^{C|l_i|})^{|k_i|}}{k_i!(l_i!)^{|k_i|}} \\ &\leq \frac{B^{\sum_{i=1}^{s}|l_i||k_i|} |n|^{C\sum_{i=1}^{s}|l_i||k_i|}}{\prod_{i=1}^{s} k_i!} \\ &= \frac{B^{|n|}|n|^{C|n|}}{\prod_{i=1}^{s} k_i!} \end{aligned}$$

for all $\bar{x} \in (0,1)^d$. Using the fact that f is (B', C')-mild together with the inequality above we have

$$
\begin{aligned}
|D^n(f\circ G)(\bar{x})| &\leq \sum_{1\leq|\lambda|\leq|n|} |D^\lambda f(\bar{x})| \sum_{s=1}^{|n|} \sum_{P_s(\lambda,n)} n! \prod_{i=1}^{s} \left| \frac{(D^{l_i}G)^{k_i}(\bar{x})}{k_i!(l_i!)^{|k_i|}} \right| \\
&\leq n! B'^{|n|} |n|^{C'|n|} B^{|n|} |n|^{C|n|} \sum_{1\leq|\lambda|\leq|n|} \sum_{s=1}^{|n|} \sum_{P_s(\lambda,n)} \frac{\lambda!}{\prod_{i=1}^{s} k_i!} \qquad (3.1)
\end{aligned}
$$

for all $\bar{x} \in (0,1)^d$. We will first find an upper bound for

$$\sum_{P_s(\lambda,n)} \frac{\lambda!}{\prod_{i=1}^{s} k_i!}.$$

Let $p \in \mathbb{N}$. The multinomial Theorem,

$$(x_1 + \ldots + x_m)^p = \sum_{p_1+\ldots+p_m=p} \binom{p}{p_1,\ldots,p_m} \prod_{i=1}^{m} x_i^{p_i}$$

with $x_1 = \ldots = x_m = 1$ gives

$$m^p = \sum_{p_1+\ldots+p_m=p} \binom{p}{p_1,\ldots,p_m} = \sum_{p_1+\ldots+p_m=p} \frac{p!}{\prod_{i=1}^{m} p_i!}. \qquad (3.2)$$

Let $\lambda = (\lambda_1,\ldots,\lambda_m)$ and $s \in \mathbb{N}$ with $s \leq |n|$. For any element $(k_1,\ldots,k_s,l_1,\ldots,l_s)$ of $P_s(\lambda,n)$ with $k_i = (k_{i,1},\ldots,k_{i,m})$ for $i \in \{1,\ldots,s\}$, by the definition of $P_s(\lambda,n)$, $\sum_{i=1}^{s} k_i = \lambda$ so $\lambda_j = k_{1,j} + \ldots + k_{s,j}$ for every $j \in \{1,\ldots,m\}$. Then using the equation (3.2) we get

$$
\begin{aligned}
\sum_{P_s(\lambda,n)} \frac{\lambda!}{\prod_{i=1}^{s} k_i!} &= \sum_{P_s(\lambda,n)} \frac{\lambda_1!\ldots\lambda_m!}{\prod_{i=1}^{s} k_{i,1}!\ldots k_{i,m}!} \\
&= \sum_{P_s(\lambda,n)} \prod_{j=1}^{m} \frac{\lambda_j!}{\prod_{i=1}^{s} k_{i,j}!} \\
&\leq \prod_{j=1}^{m} \sum_{P_s(\lambda,n)} \frac{\lambda_j!}{\prod_{i=1}^{s} k_{i,j}!} \\
&= \prod_{j=1}^{m} \sum_{\bar{l}\in(\mathbb{N}^d)^s} \sum_{\substack{\bar{k}\in(\mathbb{N}^m)^s \\ (\bar{k},\bar{l})\in P_s(\lambda,n)}} \frac{\lambda_j!}{\prod_{i=1}^{s} k_{i,j}!} \\
&= \prod_{j=1}^{m} \sum_{\substack{\bar{l}\in(\mathbb{N}^d)^s \\ \exists \bar{k}\in(\mathbb{N}^m)^s: \\ (\bar{k},\bar{l})\in P_s(\lambda,n)}} \sum_{k_{1,j}+\ldots+k_{s,j}=\lambda_j} \frac{\lambda_j!}{\prod_{i=1}^{s} k_{i,j}!} \\
&\leq |K| \prod_{j=1}^{m} s^{\lambda_j} \\
&= |K| s^{|\lambda|}
\end{aligned}
$$

where $K := \{(l_1, \ldots, l_s) : \exists k_1 \ldots, k_s \in \mathbb{N}^m$ such that $(k_1, \ldots, k_s, l_1, \ldots, l_s) \in P_s(\lambda, n)\}$. The set K is a subset of the set $\{(l_1, \ldots, l_s) : l_1 \prec n, \ldots, l_s \prec n\}$ since $\sum_{i=1}^{s} |k_i| l_i = n$, for any $(k_1, \ldots, k_s, l_1, \ldots, l_s) \in P_s(\lambda, n)$. The number of different l_i with $l_i \prec n$ is bounded above by $|n|^d$. Therefore $|K| \leq |n|^{ds}$.

Then,

$$\sum_{s=1}^{|n|} \sum_{P_s(\lambda,n)} \frac{\lambda!}{\prod_{i=1}^{s} k_i!} \leq \sum_{s=1}^{|n|} |n|^{ds} s^{|\lambda|} \leq \sum_{s=1}^{|n|} |n|^{d|n|} |n|^{|n|} \leq |n|^{(d+1)|n|+1}$$

so

$$\begin{aligned} \sum_{1 \leq |\lambda| \leq |n|} \sum_{s=1}^{|n|} \sum_{P_s(\lambda,n)} \frac{\lambda!}{\prod_{i=1}^{s} k_i!} &\leq \sum_{1 \leq |\lambda| \leq |n|} |n|^{(d+1)|n|+1} \\ &\leq L \cdot |n|^{(d+1)|n|+1} \end{aligned} \tag{3.3}$$

where $L := \#\{\lambda \in \mathbb{N}^m : 1 \leq |\lambda| \leq |n|\}$. By Lemma 3.1.8, $L \leq |n|(|n|m)^{|n|}$. This bound on L and the inequalities (3.1) and (3.3) imply that

$$\begin{aligned} |D^n(f \circ G)(\bar{x})| &\leq n!(BB')^{|n|} |n|^{(C+C')|n|} |n|(|n|m)^{|n|} |n|^{(d+1)|n|+1} \\ &\leq n!(BB'm)^{|n|} |n|^{(C+C'+d+4)|n|}, \end{aligned}$$

for all $\bar{x} \in (0,1)^d$ and $n \in \mathbb{N}^d$ with $|n| \geq 1$. Note that this equation still holds when $|n| = 0$ since $(f \circ G)(\bar{x}) < 1$ for all $x \in (0,1)^d$. Therefore $f \circ G$ is (β, γ)-mild where $\beta = BB'm$ and $\gamma = C + C' + d + 4$. □

After proving that the composition of mild functions and maps preserves mildness, we can reprove Proposition 3.1.5. For mild functions f and g, $\frac{f+g}{2}$ can be regarded as the composition of the mild maps $(x,y) \mapsto \frac{x+y}{2}$ and $(x,y) \mapsto (f(x), g(x))$, and similarly $f \cdot g$ can be regarded as the composition of $(x,y) \mapsto x \cdot y$ and $(x,y) \mapsto (f(x), g(x))$. Remembering our main concern, namely the mildness and the mild parametrization of the functions definable in o-minimal structures, it is crucial to know if mildness and having mild parametrization are preserved under taking compositions, since the set of definable functions of an o-minimal structure is. The case that one of the functions f and g is mild and the other has mild parametrization is inspected in the following proposition.

Proposition 3.1.10. *Let $f : (0,1)^d \to (0,1)$ be a mild function, and let $g : (0,1)^d \to (0,1)$, $G : (0,1)^k \to (0,1)^d$ have mild parametrization. Then the functions*

(i) $\frac{f+g}{2} : (0,1)^d \to (0,1)$

(ii) $f \cdot g : (0,1)^d \to (0,1)$

(iii) $f \circ G : (0,1)^k \to (0,1)$
have mild parametrization.

Proof. Suppose that g has mild parametrization by the maps

$$\begin{array}{rccc} \Phi_i : & (0,1)^d & \to & (0,1)^{d+1} \\ & \bar{t} & \mapsto & (\phi_{i,1}(\bar{t}), \dots, \phi_{i,d+1}(\bar{t})) \end{array}$$

for $i = 1, \dots, l_g$ and that G has mild parametrization by the maps

$$\begin{array}{rccc} \Psi_j : & (0,1)^k & \to & (0,1)^{k+d} \\ & \bar{u} & \mapsto & (\psi_{j,1}(\bar{u}), \dots, \psi_{j,k+d}(\bar{u})) \end{array}$$

for $j = 1, \dots, l_G$. Then the maps

(i) $$\begin{array}{rccc} \bar{\Phi}_i : & (0,1)^d & \to & (0,1)^{d+1} \\ & \bar{t} & \mapsto & \left(\phi_{i,1}(\bar{t}), \dots, \phi_{i,d}(\bar{t}), \left(\frac{1}{2}(f \circ (\phi_{i,1}, \dots, \phi_{i,d}) + \phi_{i,d+1})\right)(\bar{t})\right) \end{array}$$
for $i = 1, \dots, l_g$,

(ii) $$\begin{array}{rccc} \tilde{\Phi}_i : & (0,1)^d & \to & (0,1)^{d+1} \\ & \bar{t} & \mapsto & (\phi_{i,1}(\bar{t}), \dots, \phi_{i,d}(\bar{t}), (f(\phi_{i,1}, \dots, \phi_{i,d})\phi_{i,d+1})(\bar{t})) \end{array}$$
for $i = 1, \dots, l_g$,

(iii) $$\begin{array}{rccc} \bar{\Psi}_j : & (0,1)^k & \to & (0,1)^{k+1} \\ & \bar{u} & \mapsto & (\psi_{j,1}(\bar{u}), \dots, \psi_{j,k}(\bar{u}), f(\psi_{j,k+1}, \dots, \psi_{j,k+d})(\bar{u})) \end{array}$$
for $j = 1, \dots, l_G$,

are respectively parametrizing maps of the graphs of the functions in the proposition and they are all mild maps by Proposition 3.1.5 and Proposition 3.1.9. □

The case of $g \circ F$ having mild parametrization when $F : (0,1)^k \to (0,1)^d$ is a mild map and $g : (0,1)^d \to (0,1)$ has mild parametrization is still open. We conjecture that there exists also mild parametrization for this case.

Conjecture 3.1.11. *Let* $F : (0,1)^d \to (0,1)^k$ *be a mild map, and let* $g : (0,1)^k \to (0,1)$ *have mild parametrization. Then* $g \circ F : (0,1)^d \to (0,1)$ *has mild parametrization.*

We want to emphasize here that a function being mild and having mild parametrization are not the same. It is obvious that a graph of a mild function $f : (0,1)^d \to (0,1)$ has mild parametrization by means of itself, namely by the mild map $t \mapsto (t, f(t))$ with domain $(0,1)^d$. But the converse is not true: there exist functions that are not mild but have mild parametrization. We will present examples of this kind in the next section.

Proposition 3.1.12. *Let $f : (0,1) \to (0,1)$ and $g : (0,1) \to (0,1)$ be functions that have mild parametrization by maps definable in an o-minimal structure. If we assume that Conjecture 3.1.11 is true, then the function $g \circ f$ has mild parametrization.*

Proof. In order to simplify the notation we assume that f and g have mild parametrization by single maps

$$\begin{array}{cccc} \Phi : & (0,1) & \to & (0,1)^2 \\ & t & \mapsto & (\phi_1(t), \phi_2(t)) \end{array}$$

and

$$\begin{array}{cccc} \Psi : & (0,1) & \to & (0,1)^2 \\ & s & \mapsto & (\psi_1(s), \psi_2(s)) \end{array}$$

respectively which are definable in an o-minimal structure.

By the Monotonicity Theorem (see Theorem 2.1.11) there exist $a_1, \ldots, a_k \in (0,1)$ with $a_1 < \ldots < a_k$ such that ψ_1 is either strictly increasing or strictly decreasing or constant on $I_i := (a_i, a_{i+1})$ for all $i = 0, \ldots, k$ where $a_0 = 0$ and $a_{k+1} = 1$.

Let $C := \{i \in \{0, \ldots, k\} : \psi_1 \text{ is constant on } I_i\}$ and $D := \{0, \ldots, k\} \setminus C$.

Let $i \in D$. Since ψ_1 restricted to I_i is continuous and monotone, it is a bijection between I_i and the open interval $\psi_1(I_i)$. By continuity of ϕ_2 and o-minimality, the inverse image $\phi_2^{-1}(\psi_1(I_i))$ is a finite union of open intervals. We denote these open intervals by $J_{i,j}$ for $j \in \{1, \ldots, l_i\}$.

Fix $j \in \{1, \ldots, l_i\}$. By definition of the intervals $J_{i,j}$, for all $t \in J_{i,j}$, the point $\phi_2(t)$ belongs to $\psi_1(I_i)$. Furthermore, since the restriction of ψ_1 to I_i realizes a bijection between I_i and $\psi_1(I_i)$, we can define its inverse $(\psi_1\restriction_{I_i})^{-1}$. So the function $(\psi_1\restriction_{I_i})^{-1} \circ (\phi_2\restriction_{J_{i,j}})$ is well defined.

Since the function ψ_1 is mild, the function $(\psi_1\restriction_{I_i})^{-1}$ has mild parametrization. Since we assume that Conjecture 3.1.11 is true, the function $(\psi_1\restriction_{I_i})^{-1} \circ (\phi_2\restriction_{J_{i,j}})$ has mild parametrization and, by Lemma 3.1.4, the function $\zeta_{i,j} := \psi_2 \circ (\psi_1\restriction_{I_i})^{-1} \circ (\phi_2\restriction_{J_{i,j}})$ has mild parametrization.

Let $u \mapsto (\alpha_{i,j,k}(u), \beta_{i,j,k}(u))$ for $k \in \{1, \ldots, p_{i,j}\}$ be a family of mild maps that parametrizes the graph of $\zeta_{i,j}$. Then for each k the map

$$\begin{array}{ccccccc} \Theta_{i,j,k} : & (0,1) & \to & (0,1) & \times & (0,1) \\ & u & \mapsto & ((\phi_1 \circ \alpha_{i,j,k}(u) & , & \beta_{i,j,k}(u)) \end{array}$$

is mild and therefore the family $\{\Theta_{i,j,k}\}_{k \in \{1,\ldots,p_{i,j}\}}$ is a mild parametrization of the graph of $(g \circ f)\restriction_{\phi_1(J_{i,j})}$.

We therefore have a parametrization of the graph of $(g \circ f)\restriction_{\bigcup_{i \in D, j \in \{1,\ldots,l_i\}} \phi_1(J_{i,j})}$; it remains to see how to parametrize the restriction of $g \circ f$ to the rest of its domain.

Since the range of ψ_1 is $(0,1)$, the union $\bigcup_{i=0}^{k} \psi_1(I_i) \cup \bigcup_{i=1}^{k} \psi_1(\{a_i\})$ is equal to $(0,1)$ and the set

$$(0,1) \setminus \bigcup_{i \in D} \psi_1(I_i) = \bigcup_{i \in C} \psi_1(I_i) \ \cup \ \bigcup_{i=1}^{k} \psi_1(\{a_i\})$$

is finite by definition of C. We denote this set $\{t_1, \ldots, t_m\}$.

Then

$$\begin{aligned}(0,1)=\phi_2^{-1}((0,1))&=\phi_2^{-1}\left(\bigcup_{i\in D}\psi_1(I_i)\cup\{t_1,\dots,t_m\}\right)\\&=\bigcup_{i\in D}\phi_2^{-1}(\psi_1(I_i))\cup\bigcup_{n=1}^{m}\phi_2^{-1}(\{t_n\})\\&=\bigcup_{i\in D,1\le j\le l_i}J_{i,j}\cup\bigcup_{n=1}^{m}\phi_2^{-1}(\{t_n\}),\end{aligned}$$

and

$$\operatorname{dom}(f)=\phi_1((0,1))=\bigcup_{i\in D,1\le j\le l_i}\phi_1(J_{i,j})\cup\bigcup_{n=1}^{m}\phi_1(\phi_2^{-1}(\{t_n\})).$$

Consider $n\in\{1,\dots,m\}$. Then the function $(g\circ f)\restriction_{\phi_1(\phi_2^{-1}(\{t_n\}))}$ is with value $g(t_n)$ and its domain is a finite union of points and intervals. It is clear that a constant function defined on a finite union of points and intervals has mild parametrization.

Since each $(g\circ f)\restriction_{\phi_1(J_{i,j})}$ and each $(g\circ f)\restriction_{\phi_1(\phi_2^{-1}(\{t_n\}))}$ has mild parametrization and $\operatorname{dom}(g\circ f)=\bigcup_{i\in D,1\le j\le l_i}\phi_1(J_{i,j})\ \cup\ \bigcup_{n=1}^{m}\phi_1(\phi_2^{-1}(\{t_n\}))$, the function $g\circ f$ has mild parametrization.

If the mild parametrizing sets of the graphs of f and g contain several mild functions then it is possible to create new parametrizing sets $\{\Phi_1,\Phi_2,\dots,\Phi_k\}$ for the graph of g with $\Phi_i=(\phi_{i1},\phi_{i2}):(0,1)\to(0,1)^2$ for $i=1,\dots,k$ and $\{\Psi_1,\Psi_2,\dots,\Psi_k\}$ for the graph of f with $\Psi_i=(\psi_{i1},\psi_{i2}):(0,1)\to(0,1)^2$ for $i=1,\dots,k$ such that the image of ψ_{i2} and ϕ_{i1} are the same for all $i=1,\dots,k$. In that case the argument above for single parametrizing functions would enable us to get a mild parametrization of $g\circ f$ restricted to an interval, say ω_i for $i=1,\dots,k$ where $\bigcup_{i=1}^{k}\omega_i=(0,1)$. Hence we get a mild parametrization of $g\circ f$. □

3.2 Examples and nonexamples

In this section we will present some examples and nonexamples of mild functions and of functions that have mild parametrization which will be useful and important for our purposes; some of them will be frequently used in the later chapters.

Considering the definition of a mild function, the most obvious and trivial examples that one thinks of are polynomials. Let $P(x)$ be a one variable polynomial in $\mathbb{R}[x]$ of degree d with $|P(x)|\le 1$ for all $x\in(0,1)$ and let T_n be the maximum value of $|D^nP(x)|$, for $x\in(0,1)$, for each $n\in\mathbb{N}$. Obviously for all $n\ge d+1$, $T_n=0$. It is also possible to find $B>0$ such that $T_n\le n!B^n$ for all $n\in\{1,\dots,d\}$. Therefore the polynomial $P(x)$ is $(B,0)$-mild. One can see that the same argument works also for any multivariable polynomial.

We remind the reader that a function is called restricted analytic if it is analytic on a neighbourhood of the box $[-1,1]^n$ for some $n \in \mathbb{N}$. We gave the full definition of the structure $\mathbb{R}_{\text{an}}$ in 2.1.16, the structure obtained by adding the function

$$\tilde{f}(x) = \begin{cases} f(x) & x \in [-1,1]^n \\ 0 & x \in \mathbb{R}^n \setminus [-1,1]^n \end{cases}$$

for each f analytic on a neighbourhood of $[-1,1]^n$ and every $n \in \mathbb{N}$. These functions and their mildness will be important for us when examining the o-minimal structure $\mathbb{R}_{\text{an}}$ and its expansions. The following fact gives us an equivalent condition for a function to be real analytic on an open set; for the proof, the reader can check 2.2.10 in [39].

Fact 3.2.1. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ be a C^∞ function. The function f is analytic on U if and only if for each $u \in U$ there exists an open neighbourhood V of u contained in U and constants $T > 0$ and $K > 0$ such that

$$|D^\mu f(\bar{x})| \leq T \frac{\mu!}{K^{|\mu|}}$$

for all $\mu \in \mathbb{N}^n$ and $\bar{x} \in V$.

Proposition 3.2.2. *Let $f \colon \mathbb{R}^d \to \mathbb{R}$ be a restricted analytic function with $f((0,1)^d) \subset (0,1)$. Then $f\restriction_{(0,1)^d}$ is mild.*

Proof. Since f is a restricted analytic function it has a C^∞ extension F on a neighbourhood U of $[0,1]^d$. By Fact 3.2.1, for all $x \in [0,1]^d$ there is a neighbourhood V_x of x with $V_x \subset U$ and constants $T_x > 0$, $K_x > 0$ such that $|D^\mu f(x)| \leq T_x \frac{\mu!}{K_x^{|\mu|}}$ for all $\mu \in \mathbb{N}^n$ and $x \in V_x$. The union $\bigcup_{x \in [0,1]^d} V_x$ covers the compact set $[0,1]^d$. By compactness, there are finitely many $x \in [0,1]^d$, say $x_1, \ldots, x_l$, such that $\bigcup_{i=1}^{l} V_{x_i}$ covers $[0,1]^d$. Putting

$$T := \max\{T_{x_i} : i = 1 \ldots, l\}$$

and

$$K := \min\{K_{x_i} : i = 1 \ldots, l\}$$

we have

$$|D^\mu f(x)| \leq T \frac{\mu!}{K^{|\mu|}}$$

for all $\mu \in \mathbb{N}^n$ and $x \in (0,1)^d$. Moreover, choosing B big enough so that $\frac{T}{K^{|\mu|}} \leq B^{|\mu|}$ for all $\mu \in \mathbb{N}^n$, we can conclude that $f\restriction_{(0,1)^d}$ is $(B,0)$-mild. $\square$

There are also examples of mild functions that are not restrictions to $(0,1)^n$ of restricted analytic functions. The flat function

$$\begin{array}{rccc} e_1: & (0,1) & \to & (0,1) \\ & x & \mapsto & e^{-1/x} \end{array}$$

is an important one of those. We will be using this mild function when parametrizing the graph of an irrational power function. It was proven by Pila in [56] that the functions

$$\begin{array}{rccc} e_m: & (0,1) & \to & (0,1) \\ & x & \mapsto & e^{-1/x^m} \end{array}$$

for $m \in \mathbb{N}$ are mild. We will give here the proof that the function e_1 is mild. Our proof uses essentially the same idea as his proof but it is given in more detail. Afterwards, we will conclude by using 3.1.9 that all the functions e_m, for $m \in \mathbb{N}$, are mild functions.

Proposition 3.2.3. *The function* $e_1 : (0,1) \to (0,1)$ *defined by* $e_1(x) = e^{-1/x}$ *is* $(8,2)$*-mild.*

Proof. We first claim that for all $n \in \mathbb{N}^+$

$$e_1{}^{(n)}(x) = e^{-1/x} \sum_{k=n+1}^{2n} a_{n,k} x^{-k} \tag{3.4}$$

where the $a_{n,k}$'s are real numbers.
We prove this claim by induction on the order n of the derivative. The initial case is obvious as

$$e_1{}^{(1)}(x) = e^{-1/x} x^{-2}$$

and it is in the desired form where $a_{1,2} = 1$. Now for the inductive part we assume (3.4) for some fixed n and take the derivative of both sides of the equation to get the (n+1)st derivative. So we have

$$\begin{aligned} e_1{}^{(n+1)}(x) &= e^{-1/x} \sum_{k=n+1}^{2n} (-k) a_{n,k} x^{-k-1} + e^{-1/x} x^{-2} \sum_{k=n+1}^{2n} a_{n,k} x^{-k} \\ &= e^{-1/x} \left(\sum_{k=n+1}^{2n} a_{n.k} (-k) x^{-k-1} + \sum_{k=n+1}^{2n} a_{n,k} x^{-k-2} \right). \end{aligned}$$

Shifting the index of each sum we get

$$\begin{aligned}
{e_1}^{(n+1)}(x) &= e^{-1/x}\left(\sum_{k=n+2}^{2n+1} -(k-1)a_{n,k-1}x^{-k} + \sum_{k=n+3}^{2n+2} a_{n,k-2}x^{-k}\right) \\
&= e^{-1/x}\Bigg(-(n+1)a_{n,n+1}x^{-(n+2)} + \\
&\quad \sum_{k=n+3}^{2n+1}(a_{n,k-2}-(k-1)a_{n,k-1})x^{-k} + a_{n,2n}x^{-(2n+2)}\Bigg) \\
&= e^{-1/x}\sum_{k=n+2}^{2n+2} a_{n+1,k}x^{-k}
\end{aligned}$$

where $a_{n+1,n+2} = -(n+1)a_{n,n+1}$, $a_{n+1,k} = a_{n,k-2} - (k-1)a_{n,k-1}$ for all $k = n+3,\ldots,2n+1$ and $a_{n+1,2n+2} = a_{n,2n}$. Hence this proves our claim.

Put $A_n = \sum_{k=n+1}^{2n} |a_{n,k}|$ for $n \in \mathbb{N}^+$. Then

$$\begin{aligned}
A_{n+1} &= |-(n+1)a_{n+1,n}| + \sum_{k=n+3}^{2n+1} |a_{n,k-2} - (k-1)a_{n,k-1}| + |a_{n,2n}| \\
&\le (n+1)|a_{n,n+1}| + (n+2)|a_{n,n+2}| + \ldots + (2n)|a_{n,2n}| + \\
&\quad |a_{n,n+1}| + \ldots + |a_{n,2n-1}| + |a_{n,2n}| \\
&\le 2nA_n + A_n \\
&= (2n+1)A_n.
\end{aligned}$$

Since $A_1 = a_{1,2} = 1$, $A_n \neq 0$ for all $n \in \mathbb{N}^+$. We have $\dfrac{A_{n+1}}{A_n} \le 2n+1$ for all $n \in \mathbb{N}^+$. Moreover

$$A_n = \frac{A_2}{A_1}\frac{A_3}{A_2}\cdots\frac{A_n}{A_{n-1}} \le \prod_{k=2}^{n} 2k = 2^{n-1}n!$$

for all $n \in \mathbb{N}^+$.

We use the notation $E_k(x)$, for $e^{-1/x}x^{-k}$. Each term of the sum

$${e_1}^{(n)}(x) = e^{-1/x}\sum_{k=n+1}^{2n} a_{n,k}x^{-k}$$

is of the form $a_{n,k}E_k(x)$ for all $n \in \mathbb{N}^+$. The first derivative of $E_k(x)$,

$$E_k(x)^{(1)} = e^{-1/x}(x^{-k-2} - kx^{-k-1})$$

vanishes when $x = 1/k$, so the maximum value of $E_k(x)$ is $(k/e)^k$ for $x \in (0,1)$. Considering that k is bounded by $2n$ we have $E_k(x) \le (2n/e)^{2n}$ for $k = n+1,\ldots,2n$. As a result, for all $x \in (0,1)$,

$$
\begin{aligned}
\left|(e_1)^{(n)}(x)\right| &= \left|\sum_{k=n+1}^{2n} a_{n,k} e^{-1/x} x^{-k}\right| \\
&\leq (2n/e)^{2n} \sum_{k=n+1}^{2n} |a_{n,k}| \\
&= (2n/e)^{2n} A_n \\
&\leq (2n/e)^{2n} 2^{n-1} n! \\
&\leq n! 8^n n^{2n}
\end{aligned}
$$

for all $n \in \mathbb{N}^+$. Note that $\left|e^{-1/x}\right| < 1$ for all $x \in (0,1)$. Hence $\left|(e_1)^{(n)}(x)\right| \leq n!8^n n^{2n}$ for all $n \in \mathbb{N}$ and $x \in (0,1)$. Therefore e_1 is $(8,2)$-mild. □

Corollary 3.2.4. *The function $e_m : (0,1) \to (0,1)$ with $e_m(x) = e^{-1/x^m}$ is mild for all $m \in \mathbb{N}$.*

Proof. The polynomial function $x \mapsto x^m$ on domain $(0,1)$ is mild. Therefore the composition $e_1 \circ x^m = e_m(x)$ is mild as well by Proposition 3.2.3 and Proposition 3.1.9. □

In [59], Pila used the mildness of the functions $e_m(x) = e^{-1/x^m}$, $m \in \mathbb{N}$ to show that the surface

$$\mathcal{X} = \{(x,y,z) \in (0,1)^3 : \log x \log y = -\log z\}$$

has mild parametrization. Then he used this result for diophantine purposes, that is, he gave an upper estimate for the number of algebraic points (over a number field) up to a given height on the surface $X = \{(x,y,z) \in (0,\infty)^3 : \log x \log y = \log z\}$ and affirmed Wilkie's conjecture for X^{tr}. Later, Butler in [10] proved the same result for the surface

$$Y = \{(x,y,z) \in (0,\infty)^3 : (\log x)^a (\log y)^b (\log z)^c = 1\}$$

where $a, b, c \in \mathbb{Q}$ after proving that the surface

$$\mathcal{Y} = \{(x,y,z) \in (0,1)^3 : |\log x|^a |\log y|^b |\log z|^c = 1\}$$

where $a, b, c \in \mathbb{Q}$ has mild parametrization.

Remark 3.2.5. In Corollary 3.2.4, we noted that the function $e_m : x \mapsto e^{-1/x^m}$ is mild. However, for a mild function $f : (0,1) \to (0,1)$ the same argument implies that the function $e^{-1/f(x)}$ is mild, which is a more general statement.

The most important examples for our purposes of functions that are not mild are power functions with positive irrational powers. We will discuss these in Subsection 3.2.1 as well as other power functions.

3.2.1 Mild parametrization of power functions

In this subsection, we will consider the power functions $x^\alpha : (0,1) \to (0,1); x \mapsto x^\alpha$ for $\alpha \in \mathbb{R}^+$. If α is a negative real number, then the image of $x^\alpha\restriction_{(0,1)}$ does not lie in $(0,1)$ and so its graph does not lie in $(0,1)^2$. For that reason we do not even consider mildness or mild parametrization of power functions x^α where α is negative. We will examine the mildness properties of power functions depending on whether α is rational or irrational.

We have already discussed the case of $\alpha \in \mathbb{N}$ in the beginning of the section as these are just polynomials, which are themselves mild functions. Except for the case when $\alpha \in \mathbb{N}$, the power function $x^\alpha\restriction_{(0,1)}$ is not mild; on the other hand we will show that $x^\alpha\restriction_{(0,1)}$ for $\alpha \in \mathbb{R}^+$ always has mild parametrization.

For a given $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$, let $k = \lceil \alpha \rceil$. Then, for all $n \geq k$, $\lim_{x\to 0} \left| \frac{d^n(x^\alpha)}{dx^n} \right| = \infty$ and so $\left| \frac{d^n(x^\alpha)}{dx^n} \right|$ is unbounded for all $n \geq k$, and therefore the power function is not mild in this case.

On the other hand, x^α for $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$ has mild parametrization. Power functions with positive rational powers can be easily parametrized with polynomials. Let $p/q \in \mathbb{Q}^+ \setminus \mathbb{N}$. Consider the power function $x^{p/q}$, whose graph is the set

$$\{(x, x^{p/q}) : x \in (0,1)\}$$

which can be rewritten as

$$\{(t^q, t^p) : t \in (0,1)\}.$$

Hence its graph can be parametrized by the map

$$\begin{array}{rccc} P_{p,q}: & (0,1) & \to & (0,1)^2 \\ & t & \mapsto & (t^q, t^p), \end{array}$$

which is a mild map because its coordinate functions are polynomials.

The above trick does not work for positive irrational powers but we can still parametrize the graph of these functions using the flat function $e_1 = e^{-1/t}$ which we showed to be mild (following Pila [56]) in Proposition 3.2.3.

Proposition 3.2.6. *Let α be a positive irrational number. Then the function $x^\alpha : (0,1) \to (0,1)$ has mild parametrization.*

Proof. We recall that the function $e_1 : (0,1) \to (0,1)$ defined by $t \mapsto e^{-1/t}$ is $(8,2)$-mild (see Proposition 3.2.3). For all $a \in \mathbb{R}^+$, define the function

$$\begin{array}{rccl} e_{1,a}: & (0,1) & \to & (0,1) \\ & t & \mapsto & e^{1/a} \cdot e^{-1/at} \end{array},$$

Then,

$$|e_{1,a}^{(n)}(t)| = |e^{1/a} a^n (e_1)^{(n)}(at)| \leq e^{1/a} a^n n! 8^n n^{2n}$$

for all $t \in (0,1)$, $a \in \mathbb{R}^+$ and $n \in \mathbb{N}$.

Let $b > 0$ such that $e^{1/a} a^n \leq b^n$. Then,

$$|e_{1,a}^{(n)}(t)| \leq n!(8b)^n n^{2n}$$

for all $t \in (0,1)$ and $n \in \mathbb{N}$. So $e_{1,a}$ is $(8b,2)$- mild for some $b \in \mathbb{R}^+$. In other words, the functions $e_{1,a}$ are mild for each $a \in \mathbb{R}^+$. Therefore, the map

$$\begin{array}{rccl} E_\alpha: & (0,1) & \to & (0,1) \\ & t & \mapsto & (e_{1,\alpha}(t), e_{1,1}(t)) \end{array}$$

is mild. The range of the function $e_{1,\alpha}$ is $(0,1)$ and $((e_{1,\alpha})(t))^\alpha = e_{1,1}(t)$ so the image of the map E_α is the graph of the function x^α restricted to $(0,1)$. Therefore the function x^α has mild parametrization by the map E_α. □

The graphs of power functions restricted to $(0,1)$ with rational powers have mild parametrization since they are semialgebraic sets. We are mostly interested in studying strict expansions of the real field by functions which are mild or have mild parametrization. For this purpose the power functions with irrational powers are more significant to investigate. Mild parametrization in expansions of the real field in which irrational power functions are definable will be considered in Chapters 7 and 8.

3.3 Mild parametrization and the density of rational points

In this section we will present the work of Pila in his paper [56] where he introduces mild parametrization and obtains results about the density of the rational points on the graphs of nonalgebraic Pfaffian functions on a connected domain (see Definition 2.2.2) which lie in $[-1,1]^2$. The graphs of nonalgebraic Pfaffian functions on a connected domain are called Pfaff curves. We have given general information on the density of rational points on sets in Section 2.3. Here the height function $H : \mathbb{Q}^n \to$

$\mathbb{R}$ is the naive height function (see Definition 2.3.2). Recall that for a set $X \subseteq \mathbb{R}^n$ and $H \in \mathbb{R}$,

$$X(\mathbb{Q}, H) := \{\bar{q} \in X \cap \mathbb{Q}^n : H(\bar{q}) \leq H\}$$

and the density function is given by

$$N(X, H) := \#X(\mathbb{Q}, H).$$

(see Section 2.3).

First, we state the main result of Pila that we mentioned above.

Theorem 3.3.1. ([56, Theorem 1.5]) *Let $X \subseteq [-1,1]^2$ be a Pfaff curve that has mild parametrization. Then there are constants $c_1, c_2 > 0$ such that (for $H > e$)*

$$N(X, H) \leq c_1(\log H)^{c_2}.$$

Pila proves the theorem above by combining two main results (which we state as Proposition 3.3.8 and Proposition 3.3.9). Proposition 3.3.9 is a geometric result of Khovanskiĭ ([37]) which we will only state. We will explain how Pila obtains Proposition 3.3.8 using a version of the determinant method which uses mild parametrization.

The determinant method is a tool in diophantine geometry. It was first developed by Bombieri and Pila in their paper [6] which is about integral points (points in $\mathbb{R}^n$ with integer coordinates) of bounded height on affine algebraic and transcendental curves. Now, we give an overview of this method.

The Determinant Method: Let $d \in \mathbb{N}$. We denote the set of two variable monomials of degree less than or equal to d by M_d, i.e.,

$$M_d := \{x^a y^b : a + b \leq d\},$$

and set

$$D := \#M_d = \frac{(d+1)(d+2)}{2}.$$

Given a subset $A = \{(a_i, b_i) : i = 1, \ldots, D\}$ of $\mathbb{R}^2$ with D many elements, we form a $D \times D$ matrix as follows, where the monomials of M_d are indexed as $m_1, \ldots, m_D$:

$$\mu_{A,d} := \begin{bmatrix} m_1(a_1, b_1) & \ldots & m_D(a_1, b_1) \\ \ldots & & \\ m_1(a_D, b_D) & \ldots & m_D(a_D, b_D) \end{bmatrix}.$$

We will obtain knowledge about the points in A using the determinant of this matrix $\mu_{A,d}$, which we will denote by $\Delta_{A,d} := \det(\mu_{A,d})$.

Lemma 3.3.2. *Assume that there is no curve of degree less than or equal to d such that all the points of A lie on that curve. Then $\Delta_{A,d} \neq 0$.*

Instead of giving a proof of this lemma we will just illustrate it for the case $d = 2$, where $M_2 = \{1, x, y, x^2, y^2, xy\}$, $D = 6$ and $A = \{(a_1, b_1), \ldots, (a_6, b_6)\}$. We obtain the 6×6 matrix

$$\mu_{A,2} := \begin{bmatrix} 1 & a_1 & b_1 & a_1^2 & b_1^2 & a_1b_1 \\ 1 & a_2 & b_2 & a_2^2 & b_2^2 & a_2b_2 \\ \vdots & & & & & \\ 1 & a_6 & b_6 & a_6^2 & b_6^2 & a_6b_6 \end{bmatrix}.$$

Consider the homogeneous system of equations with the coefficient matrix $\mu_{A,d}$,

$$\mu_{A,d} \begin{bmatrix} t_1 \\ \vdots \\ t_6 \end{bmatrix} = 0.$$

If this system of equations has a nonzero solution, which is the case when $\Delta_{A,d} = 0$, then this means that there exists an algebraic curve of degree less than or equal to 2 such that the points in A lie on it. More precisely, if the solution is $(w_1, w_2, \ldots, w_6)$, then all the points in A would satisfy the equation $w_1 + w_2x + w_3y + w_4x^2 + w_5y^2 + w_6xy = 0$. As a result, if we know that there exists no such algebraic curve, then $\Delta_{A,d} \neq 0$.

Lemma 3.3.3. *Assume all the points $(a_i, b_i) \in A$ have height less than or equal to H and there exists no algebraic curve of degree less then or equal to d that contain them. Then*

$$|\Delta_{A,d}| \geq \frac{1}{H^{2dD}}.$$

Proof. Set $a_i = \dfrac{p_i}{q_i}$ and $b_i = \dfrac{r_i}{s_i}$ with $p_i, q_i, r_i, s_i \in \mathbb{Z}$, $q_i \neq 0$, $s_i \neq 0$, $(p_i, q_i) = 1$ and $(r_i, s_i) = 1$. The ijth entry of the matrix $\mu_{A,d}$ is of the form

$$m_j\left(\frac{p_i}{q_i}, \frac{r_i}{s_i}\right) = \left(\frac{p_i}{q_i}\right)^{\alpha_j} \left(\frac{r_i}{s_i}\right)^{\beta_j}$$

where $\alpha_j + \beta_j \leq d$. Let $\mathcal{Z}$ be the matrix obtained by multiplying each ith row of $\mu_{A,d}$ by $(q_i)^d(s_i)^d$ for $i = 1, \ldots, D$. Then,

$$|\Delta_{A,d}| = \prod_{i=1}^{D} \left| \frac{1}{(q_i)^d(s_i)^d} \right| |\det(\mathcal{Z})|.$$

Note that, $\mathcal{Z}$ is a matrix with integer entries so $\det(\mathcal{Z}) \in \mathbb{Z}$. By Lemma 3.3.2 $\Delta_{A,d} \neq 0$, then $|\det(\mathcal{Z})| \neq 0$ and hence $|\det(\mathcal{Z})| \geq 1$. Then

$$|\Delta_{A,d}| \geq \prod_{i=1}^{D} \left| \frac{1}{(q_i)^d(s_i)^d} \right|.$$

Since $\frac{p_i}{q_i}$ and $\frac{r_i}{s_i}$ have height less than or equal to H,

$$|(q_i)^d(s_i)^d| \leq H^{2d},$$

therefore

$$\prod_{i=1}^{D} \left| \frac{1}{(q_i)^d(s_i)^d} \right| | \geq \frac{1}{H^{2dD}}.$$

Hence

$$|\Delta_{A,d}| \geq \left(\frac{1}{H^{2d}} \right)^{D}. \qquad \square$$

For the last lemma we will consider the points in the set A as the images of mild functions. The bound on the derivatives of these functions will play a role here.

Lemma 3.3.4. *Assume φ and ψ are (B, C)-mild functions. Let I be an interval and suppose that $A = \{(\varphi(t_i), \psi(t_i)) : t_1, \ldots, t_D \in I\}$. Then*

$$|\Delta_{A,d}| \leq D!D^{\frac{2dD}{3}} (BD^C)^{\frac{D(D-1)}{2}} |I|^{\frac{D(D-1)}{2}}.$$

Before presenting the proof of Lemma 3.3.4 we need to recall the definition of the Vandermonde determinant and some of its properties.

Definition 3.3.5. *Let $t_1, \ldots, t_D \in \mathbb{R}$. The determinant*

$$V = \begin{vmatrix} 1 & t_1 & t_1^2 & \ldots & t_1^{D-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_D & t_D^2 & \ldots & t_D^{D-1} \end{vmatrix}$$

is called the Vandermonde determinant for $t_1, \ldots, t_D$.

We will use the following property of the Vandermonde determinant together with the Generalized Schwarz Mean Value Theorem that we state afterwards. For the proofs one can check [54].

Fact 3.3.6. Let $I \subset \mathbb{R}$ be an interval, $t_1, \ldots, t_D \in I$. Let V denote the *Vandermonde determinant* for $t_1, \ldots, t_D$. Then we can write V as

$$V = \prod_{1 \leq i < j \leq D} (t_j - t_i).$$

So, we have the inequality

$$V \leq |I|^{\frac{D(D-1)}{2}}.$$

Theorem 3.3.7 (Schwarz Mean Value Theorem)**.** *Let $t_1, \ldots, t_D \in I \subset \mathbb{R}$ and let $f_1, \ldots, f_D \in C^D(I)$. Let V denote the Vandermonde determinant for $t_1, \ldots, t_D$. For all $1 \leq i, j \leq D$ there are $\xi_{ij} \in I$ such that*

$$\det[f_j(t_i)]_{ij} = \det \left[\frac{f_j^{(i-1)}(\xi_{ij})}{(i-1)!} \right] V.$$

For practical reasons we will denote $\det \left[\frac{f_j^{(i-1)}(\xi_{ij})}{(i-1)!} \right]$ by Λ.

We now give a proof of Lemma 3.3.4.

Proof of Lemma 3.3.4. Let I be an interval and let $T := \{t_1, \ldots, t_D\}$ be our usual set of D many points. A consists of the images of $t_1, \ldots, t_D$ under the map (φ, ψ) with bounded derivatives. So the matrix $\mu_{A,d}$ in this case will have entries of the form

$$(\varphi(t_i))^\alpha \cdot (\psi(t_i))^\beta,$$

for some $\alpha, \beta \in \mathbb{N}$ with $\alpha + \beta \leq d$. To see these entries as the functions f_j evaluated at $t_1, \ldots, t_D$, we re-index the elements of the set of functions $\{(\varphi(t_i))^\alpha \cdot (\psi(t_i))^\beta : \alpha, \beta \in \mathbb{N}, \alpha + \beta \leq d\}$ as $\{f_1, \ldots, f_D\}$. Then

$$\Delta_{A,d} = \Lambda \cdot V \leq \Lambda \cdot |I|^{\frac{D(D-1)}{2}}$$

where $\Lambda = \det \left[\frac{f_j^{(i-1)}(\xi_{ij})}{(i-1)!} \right]$ for some $\xi_{ij} \in I$, for $1 \leq i, j \leq D$ by Fact3.3.6 and Theorem 3.3.7. □

Upper Bound for Λ: The bound Λ:

$$\Lambda \leq D! D^{\frac{2dD}{3}} (BD^C)^{\frac{D(D-1)}{2}}$$

is obtained by means of the bounds on the derivatives of φ, ψ which are (B, C)-mild (each f_j is a product of powers of φ and ψ). For more detailed explanation where BD^C comes from, the reader can see Proposition 2.2 in [56]. In this bound $D^{\frac{2dD}{3}}$ is obtained by calculating the sum of degrees of all monomials in M (the set of two variable monomials with degree less than or equal to d). There is an isomorphism between the M and the set $\bar{M} := \{x^i y^j z^k : i+j+k = d\}$ by $x^i y^j \mapsto x^i y^j z^{d-(i+j)}$. The sum

$$\sum_{x^i y^j z^k \in \bar{M}} i+j+k = \sum_{x^i y^j z^k \in \bar{M}} d = dD$$

and

$$2 \sum_{x^i y^j z^k \in \bar{M}} (i+j+k) = \sum_{x^i y^j z^k \in \bar{M}} (i+j) + \sum_{x^i y^j z^k \in \bar{M}} (j+k) + \sum_{x^i y^j z^k \in \bar{M}} (i+k).$$

Hence by symmetry we have

$$\sum_{x^i y^j \in M} (i+j) = \frac{2}{3} \sum_{x^i y^j z^k \in \bar{M}} (i+j+k) = \frac{2dD}{3}.$$

Combining Lemmas 3.3.2, 3.3.3 and 3.3.4: Assume $X \subseteq [-1,1]^2$ is an image of the map $(\varphi, \psi) : (0,1) \to [-1,1]^2$ where φ, ψ are (B,C)-mild. For $I \subseteq (0,1)$, we denote the image of the map $x \mapsto (\varphi(x), \psi(x))$ restricted to I by X_I. Let $d \in \mathbb{N}$. Assume that $X_I(\mathbb{Q}, H)$ is not contained in an algebraic curve of degree less than or equal to d. We can choose D many points in I (for example by Lagrange interpolation), say $t_1, \ldots t_D \in I$, such that

$$(\varphi(t_i), \psi(t_i)) \in X_I(\mathbb{Q}, H).$$

These $(\varphi(t_i), \psi(t_i))$ for $i = 1, \ldots, D$ will be our set A with D many points and we will consider $\Delta_{A,d} = \det(\mu_{A,d})$ for this case. By Lemma 3.3.2, $\Delta_{A,d} \neq 0$. We have a lower bound for $|\Delta_{A,d}|$ by Lemma 3.3.3 and an upper bound for $|\Delta_{A,d}|$ by Lemma 3.3.4. Hence we get the following inequality:

$$\frac{1}{H^{2dD}} \leq |\Delta_{A,d}| \leq D! D^{\frac{2dD}{3}} (BD^C)^{\frac{D(D-1)}{2}} |I|^{\frac{D(D-1)}{2}}.$$

Then, removing $|\Delta_{A,d}|$ and making necessary computations we have

$$|I| \geq \left(H^{2dD} D! D^{\frac{2dD}{3}} (BD^C)^{\frac{D(D-1)}{2}} \right)^{\frac{-2}{D(D-1)}}.$$

Let us name the bound on the right hand side of the inequality above by $\mathcal{B} := \mathcal{B}(H, d, B, C)$. At this point we have that if $X_I(\mathbb{Q}, H)$ is not contained in an algebraic curve of degree less than or equal to d then the length of the interval I is greater than or equal to $\mathcal{B}$. This means if $|I| < \mathcal{B}$ then $X_I(\mathbb{Q}, H)$ is contained in fact in an algebraic curve of degree less than or equal to d. We can cover the interval $(0,1)$ by $1+\left\lfloor\frac{1}{\mathcal{B}}\right\rfloor$ many intervals of length less than $\mathcal{B}$. Hence $X_I(\mathbb{Q},H)$ is contained in the union of $1+\left\lfloor\frac{1}{\mathcal{B}}\right\rfloor$ algebraic curves of degree d.

Simplification of $1+\frac{1}{\mathcal{B}}$: Note that

$$1+\frac{1}{\mathcal{B}} = 1+\left(H^{2dD}D!D^{\frac{2dD}{3}}\right)^{\frac{2}{D(D-1)}}BD^C.$$

First we have

$$\frac{d}{D-1} = \frac{d}{\frac{(d+1)(d+2)}{2}-1} = \frac{2d}{d^2+3d+2-2} = \frac{2}{d+3}. \tag{3.5}$$

Then using equation 3.5 we have

$$\left(H^{2dD}\right)^{\frac{2}{D(D-1)}} = H^{\frac{4d}{D-1}} = H^{\frac{8}{d+3}}.$$

Again using equation 3.5 we have $\left(D^{\frac{2dD}{3}}\right)^{\frac{2}{D(D-1)}} = D^{\frac{4d}{3(D-1)}} = D^{\frac{8}{3(d+3)}}$. The function $d\mapsto D^{\frac{8}{3(d+3)}}$ is a decreasing function for $d\geq 5$. So

$$\left(D^{\frac{2dD}{3}}\right)^{\frac{2}{D(D-1)}} \leq 21^{1/3} \leq 3$$

for $d\geq 5$.

Finally, using Striling inequality and keeping our choice of $d\geq 5$ we have

$$(D!)^{\frac{2}{D(D-1)}} \leq e^{0.3611}.$$

Hence we have

$$\begin{aligned}1+\frac{1}{\mathcal{B}} &\leq 1+3e^{0.3611}H^{\frac{8}{d+3}}BD^C\\ &\leq 6BD^CH^{\frac{8}{d+3}}.\end{aligned}$$

Now we state the proposition we proved.

Proposition 3.3.8. *Let $d\geq 5$ and let X be the image of the map*

$$\begin{array}{rccc}\theta: & (0,1) & \to & [-1,1]^2\\ & t & \mapsto & (\varphi(t),\psi(t))\end{array}$$

where φ and ψ are (B,C)-mild. Let $H\geq 1$. Then $X(\mathbb{Q},H)$ is contained in the union of at most

$$6BD^CH^{\frac{8}{d+3}}$$

algebraic curves of degree at most d.

The second part of the proof uses a theorem of Khovanskiĭ on the intersection points of a nonalgebraic Pfaff curve (graph of a Pfaffian function on a connected domain) and an algebraic curve of degree d. We will use this result to bound the number of rational points in each of the algebraic curves that together cover X.

Proposition 3.3.9. ([56, Proposition 3.1]) *Let f be a nonalgebraic Pfaffian function of order $r \geq 1$ and degree (α, β). Assume $P(x, y) \in \mathbb{R}[x, y]$ has degree d. Then the equation*

$$P(x, f(x)) = 0$$

has at most

$$2^{\frac{r(r-1)}{2}+1} d\beta(\alpha + d\beta)^r$$

real solutions (hence rational solutions).

Now we will combine these two results to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. Assume $X \subseteq [-1, 1]^2$ is a Pfaff curve of order r and degree (α, β) that has mild parametrization. In other words, there are finitely many mild maps such that the union of their images is X. Assume there are l-many such mild maps. Then, by Proposition 3.3.8, $X(\mathbb{Q}, H)$ is contained in the union of at most

$$6BD^C H^{\frac{8}{d+3}} l \tag{3.6}$$

algebraic curves of degree $d \geq 5$. And by Proposition 3.3.9, each curve can contain at most

$$2^{\frac{r(r-1)}{2}+1} d\beta(\alpha + d\beta)^r \tag{3.7}$$

rational points. Multiplying 3.6 and 3.7 we obtain

$$N(X, H) \leq \left(6BD^C H^{\frac{8}{d+3}} l\right) \left(2^{\frac{r(r-1)}{2}+1} d\beta(\alpha + d\beta)^r\right). \tag{3.8}$$

Let H with $d = \lfloor \log H \rfloor \geq 5$. Then $d \leq \log H$ and $1 \leq \log H \leq (\log H)^2$. So

$$\begin{aligned} D = \frac{(d+1)(d+2)}{2} &\leq \frac{(\log H + 1)(\log H + 2)}{2} \\ &= \frac{(\log H)^2 + 3\log H + 1}{2} \\ &\leq \frac{(\log H)^2 + 3(\log H)^2 + (\log H)^2}{2} \\ &\leq 3(\log H)^2. \end{aligned}$$

Since $\log H \geq 1$, we have $\dfrac{\log H}{\lfloor \log H \rfloor + 3} \leq 1$, so

$$H^{\frac{8}{d+3}} = H^{\frac{8}{\lfloor \log H \rfloor + 3}} = \left(e^{\log H}\right)^{\frac{8}{\lfloor \log H \rfloor + 3}} \leq e^8.$$

Again using the assumption that $\log H \geq 1$, we obtain

$$(\alpha + d\beta)^r \leq (\alpha + \log H\beta)^r \leq ((\alpha + \beta)\log H)^r.$$

Since $d = \lfloor \log H \rfloor \geq 5$, we use the bounds we obtained above for D, $H^{\frac{8}{d+3}}$ and $(\alpha + d\beta)^r$ and rewrite inequality (3.8) as

$$\begin{aligned} N(X,H) &\leq \left(6B\big(3(\log H)^2\big)^C e^8 l\right)\left(2^{\frac{r(r-1)}{2}+1}(\log H)\beta\left((\alpha+\beta)\log H\right)^r\right) \\ &= c_1(\log H)^{c_2} \end{aligned}$$

where $c_1 = 6B3^C e^8 l 2^{\frac{r(r-1)}{2}+1}\beta(\alpha+\beta)^r$ and $c_2 = 2C + r + 1$. □

In [59] Pila proved a more general and stronger version of Proposition 3.3.8 for subsets of $(0,1)^n$ that have mild parametrization. We state this result below which is a version of Corollary 3.3 in [59].

Theorem 3.3.10. *Let $X \subset (0,1)^n$ with $\dim(X) = k$, and let X have (B, C)- mild parametrization. Then there is a constant L depending on B and a constant K depending on C such that $X(\mathbb{Q}, H)$ is contained in at most $L(\log H)^K$ intersections of X with hypersurfaces of degree $d = \lfloor(\log H)^r\rfloor$ for a constant r depending on X.*

In Section 3.2 we presented the set

$$\mathcal{X} = \{(x,y,z) \in (0,1)^3 : \log x \log y = -\log z\}$$

as an example of a set that has mild parametrization. Pila proved in [59] that $\mathcal{X}$ has mild parametrization and used Theorem 3.3.10 to prove Wilkie's conjecture for $\mathcal{X}$. Further, in [59] he conjectured that $\mathbb{R}_{\exp}$ admits a uniform version of mild parametrization (Conjecture 3.3.11) and showed that assuming this conjecture one can prove Wilkie's conjecture. Before stating Pila's conjecture we explain why he conjectured a uniform version of mild parametrization.

Assuming that $\mathbb{R}_{\exp}$ admits mild parametrization and using the result of Jones and Thomas in [34] (Wilkie's conjecture is true for the surfaces of dimension 2, definable in $\mathbb{R}_{\exp}$ which have mild parametrization) as the initial case, one would hope to prove Wilkie's conjecture by an inductive argument on the dimension of the definable sets. However it turns out that just assuming mild parametrization for definable sets is not sufficient; one would need a uniformity condition on the mildness constants.

Let $X \subset (0,1)^n$ be a definable set in $\mathbb{R}_{\exp}$ with $\dim(X) = k$. We assume Wilkie's conjecture for any definable set of dimension less than k. Fix $H \geq e$. By Theorem 3.3.10, $X(\mathbb{Q}, H)$ is contained in the intersection of X with at most

$L(\log H)^K$ hypersurfaces of degree at most $(\log H)^r$. Let γ be one of these algebraic curves. By an argument (we do not want to go into the details here), using the fact that X^{tr} does not contain any semialgebraic set of positive dimension, we only need to consider those γ for which $\dim(X \cap \gamma) < k$. Then Wilkie's conjecture is true for $(X \cap \gamma)^{tr}$ by inductive hypothesis. Let c_1 and c_2 be the constants such that

$$(X \cap \gamma)^{tr}(\mathbb{Q}, H) \leq c_1(\log H)^{c_2},$$

and let $X \cap \gamma$ have (B, C)-mild parametrization. Since the degree of γ is $(\log H)^r$, and B and C depend on γ, the constants B and C may also depend on H. If B and C depend on H then the constants c_1 and c_2 would also depend on H (since c_1, c_2 depend on B and C). Considering all the intersections of X with all possible such curves, in order to be able to obtain a bound on $X^{tr}(\mathbb{Q}, H)$ of the form $a(\log H)^b$, the dependence of B should be at most polynomial in $\log H$ and C should not even depend on H. The constants c_1 and c_2 also depend on the number of parametrizing functions so this dependence should be also at most polynomial in H. Hence Pila conjectured a uniform version of mild parametrization to avoid problems caused by possible uncontrolled dependence of B, C and the number of parametrizing functions on H.

Conjecture 3.3.11. ([59, Conjecture 3.4]) *Let $X \subset (0,1)^n$ be definable in $\mathbb{R}_{\exp}$. There exist real constants A, B, C, D, E depending only on X with the following property. Let $\mathcal{F}$ be an algebraic family of closed algebraic sets in $\mathbb{R}^n$ of degree d and let $\gamma \in \mathcal{F}$. Then $X \cap \gamma$ has (Ad^B, C)-mild parametrization using Dd^E mild maps.*

As outlined above, we have the following.

Proposition 3.3.12. *Conjecture 3.3.11 implies Wilkie's conjecture.*

Chapter 4

Definable mild parametrization

In the previous chapter we explained how mild parametrization can be useful to estimate the density of rational points of bounded sets. Following this motivation, the main subject of this thesis is mild parametrization of definable sets of o-minimal structures which lie in $(0,1)^n$ for $n \in \mathbb{N}$. We will present our results contributing to this area in the later chapters. This chapter is based on the known results about mild parametrization in o-minimal structures.

In order to interpret the interaction between mild parametrization and o-minimal structures we want to know for which o-minimal structures mild parametrization is a characteristic property for the structure. That is, for which o-minimal structures it is possible to have mild parametrization for all their definable sets lying in $(0,1)^n$ for all $n \in \mathbb{N}$.

Definition 4.0.1. *Let $\mathcal{R}$ be an o-minimal expansion of the real field. The structure $\mathcal{R}$ is said to* admit mild parametrization *if each definable subset X of $\mathcal{R}$ with $X \subseteq (0,1)^n$, for $n \in \mathbb{N}$, has mild parametrization.*

To benefit from mild parametrization to attain number theoretical applications, when considering known methods related to those outlined in the previous chapter, it is not crucial that the mild parametrizing functions are definable in the same o-minimal structure as the definable set we are considering. But it could provide us further information about the structure or help us to prove mild parametrization if we specify the definability of the parametrizing functions. We have the following definition concerning this property of structures that admit mild parametrization.

Definition 4.0.2. *Let $\mathcal{R}$ be an o-minimal expansion of the real field and X be definable in $\mathcal{R}$ such that $X \subseteq (0,1)^n$ for $n \in \mathbb{N}$. We say that X has* definable mild parametrization *if X has mild parametrization with parametrizing functions definable in $\mathcal{R}$.*

Definition 4.0.3. *Let $\mathcal{R}$ be an o-minimal expansion of the real field. The structure $\mathcal{R}$ is said to* admit definable mild parametrization *if each definable subset X of $\mathcal{R}$ with $X \subseteq (0,1)^n$, for $n \in \mathbb{N}$, has definable mild parametrization.*

Section 4.1 contains a brief summary of known results about mild parametrization in expansions of the real ordered field.

Within the known results about mild parametrization the work of Jones, D. Miller, and Thomas ([33]) is significant in terms of providing examples of o-minimal expansions of the real ordered field that admit mild parametrization. One of the main ingredients of their results is the work of Rolin, Speissegger, and Wilkie in [65]. The authors in [65] consider expansions of the real field by algebras of functions $\mathcal{C}$ that have certain properties which allow them to apply a normalization theorem to the quantifier-free definable sets (called $\mathcal{C}$-sets). This ensures that such an expansion of the real ordered field is polynomially bounded, o-minimal, and model complete. The axiomatic aspect of this result provides a setting which gives a generalization of the proof of o-minimality, polynomially boundedness and model completeness for all historically known o-minimal structures of this kind.

In [33], Jones, D. Miller and Thomas observed that $\mathcal{C}$-sets play a role in parametrizing definable subsets of $(0,1)^n$ by functions from algebras satisfying the axioms in [65] and they proved a parametrization result using this observation. They applied this parametrization result to prove that all the reducts of $\mathbb{R}_{\mathrm{an}}$ admit mild parametrization.

In Section 4.2, we will give the definition of $\mathcal{C}$-sets and present Rolin, Speissegger and Wilkie's results about these sets, which were employed in [33] by Jones, D. Miller and Thomas. We will illustrate the results of [33] in Section 4.3. Considering a special reduct of $\mathbb{R}_{\mathrm{an}}$, the real ordered field expanded by restricted Pfaffian functions, they also prove a result on the density of rational points of curves definable in this structure.

4.1 Summary of known mild parametrization results

The structures known to date that admit mild parametrization are the reducts of $\mathbb{R}_{\mathrm{an}}$ expanding the real ordered field. As noted this result is due to Jones, D. Miller, and Thomas ([33]) and will be discussed in Section 4.3. We want to point out here that these structures are polynomially bounded.

For the structure $\mathbb{R}_{\exp}$, which is not polynomially bounded, there are also some partial results that we mentioned in Section 3.2. In [59], Pila proved that the surface

$$\mathcal{X} = \{(x, y, z) \in (0, 1)^3 : \log x \log y = -\log z\}$$

which is definable in $\mathbb{R}_{\exp}$, has mild parametrization. Employing this result he obtained Wilkie's conjecture for this set. Butler in [10] obtained the same results for a more general version of the set $\mathcal{X}$. The sets he considered are

$$\mathcal{Y} = \{(x, y, z) \in (0, 1)^3 : |\log x|^a |\log y|^b |\log z|^c = 1\}$$

for $a, b, c \in \mathbb{Q}$, which is also definable in $\mathbb{R}_{\exp}$.

Another interesting parametrization theorem appears in [13] called 0-mild quasi-parametrization. As noted before it is proven in [33] that any reduct of $\mathbb{R}_{\mathrm{an}}$ admits 0-mild parametrization (mild parametrization by 0-mild functions that is $(B, 0)$-mild for some positive $B < 1$) . Similar to Conjecture 3.3.11 and Proposition 3.3.12 of Pila which we explained at the end of Section 3.3, in [52] Cluckers, Pila, and Wilkie, trying to build up on the techniques in the proof of Pila-Wilkie Theorem towards a proof of Wilkie's conjecture explain that one may wish to have a version of this result which is uniform in families in the sense that if we consider a definable family of sets, the number of 0-mild functions required to parametrize each of these sets is bounded uniformly. However Yomdin provides an example in [78] that shows that such a result cannot be obtained. The strategy followed in [13] is to relax the condition on parametrization and use quasiparametrization: instead of having functions that parametrize the set they have multivalued functions that cover the set (i.e. the union of their images is of the same dimension as that of the set but can be larger than the set itself) while keeping some "mildness" property. In the case where a multivalued function involved in such quasiparametrization happens to be single valued, this function is then 0-mild, which shows that 0-mild quasi-parametrization generalizes 0-mild parametrization. This results leads to a uniform bound as conjectured by Wilkie (Conjecture 2.3.9) on the density of rational points in the transcendental part of the members in any family of Pfaffian surfaces of fixed complexity in $[0,1]^3$ definable in $\mathbb{R}_{\text{an}}$ or even in $\mathbb{R}_{\text{an}}^{\mathbb{R}}$ (see Chapter 8 for definition).

As well as knowing which structures admit mild parametrization, understanding for which conditions mild parametrization is not possible is also important to comprehend mild parametrization in o-minimal structures. In [73] Thomas proved that there exists an o-minimal structure that does not admit definable mild parametrization. The subject of not admitting (definable) mild parametrization will be discussed in Chapter 7.

4.2 $\mathcal{C}$-sets

In [65], Rolin, Speissegger and Wilkie define an $\mathbb{R}$-algebra of functions that satisfy several conditions and so-called $\mathcal{C}$-sets obtained by means of those functions. We will give definitions and results from the paper [65] that we need to explain the parametrization result in [33] in the next section.

A set of the form $B = [a_1, b_1] \times \ldots \times [a_n, b_n]$ with $a_i < b_i$ for $i = 1, \ldots, n$ is called a compact box. For each compact box B, we associate an $\mathbb{R}$-algebra $\mathcal{C}_B$ of functions $f : B \to \mathbb{R}$, such that the collection $\mathcal{C} := \{\mathcal{C}_B : B \subseteq \mathbb{R}^n \quad \text{compact box}, n \in \mathbb{N}\}$ satisfies the following properties.

(C1) For each compact box B, the projection function $(x_1, \ldots, x_n) \mapsto x_i$ for all $i = 1, \ldots, n$ restricted to B is in $\mathcal{C}_B$, and for every $f \in \mathcal{C}_B$ the restriction of f to the interior of B is C^∞.

(C2) For each pair of compact boxes $B \subseteq \mathbb{R}^n$ and $B' \subseteq \mathbb{R}^m$ and for all $g : B' \to B$ with coordinate functions $g_1, \ldots, g_n$ in $\mathcal{C}_{B'}$ and $f \in \mathcal{C}_B$, the function $f(g_1(y), \ldots, g_n(y)) : B' \to \mathbb{R}$ belongs to $\mathcal{C}_{B'}$.

(C3) For each pair of compact boxes $B, B' \subseteq \mathbb{R}^n$ with $B' \subseteq B$ and for every $f \in \mathcal{C}_B$, the restriction of f to B' is in $\mathcal{C}_{B'}$. For every $f \in \mathcal{C}_B$ there is a compact box

$B'' \subseteq \mathbb{R}^n$ with interior of B'' containing B, and $g \in \mathcal{C}_{B''}$ such that g restricted to B is f.

For each compact box $B \subseteq \mathbb{R}^n$, by (C1) and (C3) any $f \in \mathcal{C}_B$ has a C^∞ extension $\bar{f}$ to an open neighbourhood of B.

(C4) For every $f \in \mathcal{C}_B$ and $i = 1, \dots, n$, the partial derivative $\dfrac{\partial \bar{f}}{\partial x_i}$ restricted to B is in $\mathcal{C}_B$.

For all $n \in \mathbb{N}^+$ we denote by $\mathcal{C}_n$ the collection of germs at the origin of functions f for which there is a compact box $B \subseteq \mathbb{R}^n$ containing the origin such that $f \in \mathcal{C}_B$. For a compact box $B \subset \mathbb{R}^n$ containing the origin, we will denote a function in $\mathcal{C}_B$ and its germ in $\mathcal{C}_n$ with the same letter for simplicity.

The collection $\mathcal{C}_n$ is an algebra with respect to addition and multiplication. The Taylor map $\mathcal{T} \colon \mathcal{C}_n \to \mathbb{R}[\![X]\!]$ sends each $f \in \mathcal{C}_n$ to its Taylor expansion at the origin.

Besides the properties (C1)-(C4) that $\mathcal{C}_B$ should satisfy we want $\mathcal{C}_n$ to satisfy the following properties for all $n \in \mathbb{N}^+$.

(C5) The Taylor map $\mathcal{T} \colon \mathcal{C}_n \to \mathbb{R}[\![X]\!]$ is injective.

When the algebras $\mathcal{C}_n$ for all $n \in \mathbb{N}$ satisfy property (C5), we will say that $\mathcal{C}$ forms a *quasianalytic class*.

(C6) For $n > 1$ and $x = (x_1, \dots, x_n)$ we write $x' = (x_1, \dots, x_{n-1})$. For all $f \in \mathcal{C}_n$ with $f(0) = 0$ and $\dfrac{\partial f}{\partial x_n}(0) \neq 0$, there exists $\varphi \in \mathcal{C}_{n-1}$ with $\varphi(0) = 0$ such that $f(x', \varphi(x')) = 0$. This means $\mathcal{C}_n$ is closed under extraction of implicit functions.

(C7) For $f \in \mathcal{C}_n$, if $\mathcal{T}(f) = X_i G(X)$ for some $i \leq n$ and $G \in R[\![X]\!]$, then there exists a germ $g \in \mathcal{C}_n$ with $G = \mathcal{T}(g)$ such that $f = x_i g$. This means $\mathcal{C}_n$ is closed under monomial division.

For $n \in \mathbb{N}$ and $f \in \mathcal{C}_{[-1,1]^n}$, we define the function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & x \in [-1,1]^n \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.2.1. *Given a collection $\mathcal{C} = \{\mathcal{C}_B : B \subseteq \mathbb{R}^n \text{compact box}, n \in \mathbb{N}\}$ where $\mathcal{C}_B$ is an $\mathbb{R}$-algebra of functions on the compact box B, we denote by $\mathbb{R}_\mathcal{C}$ the expansion of the real ordered field by the functions $\tilde{f}$ for all $f \in \mathcal{C}_{[-1,1]^n}$ and $n \in \mathbb{N}$.*

Theorem 4.2.2. ([65, Theorem 5.2]) *Suppose that $\mathcal{C} = \{\mathcal{C}_B : B \subseteq \mathbb{R}^n \quad \text{compact box}, n \in \mathbb{N}\}$ satisfies* (C1)-(C7)*, where, for each compact box $B \subseteq \mathbb{R}^n$, and $n \in \mathbb{N}$, $\mathcal{C}_B$ is an $\mathbb{R}$-algebra of functions on B. Then the structure $\mathbb{R}_\mathcal{C}$ is o-minimal, model complete, and polynomially bounded.*

Lemma 4.2.3. *Let $\mathcal{R}$ be a polynomially bounded o-minimal expansion of the real ordered field. For each $n \in \mathbb{N}$ and each compact box $B \subseteq \mathbb{R}^n$, let $\mathcal{C}_B(\mathcal{R})$ be the collection of all functions $f : B \to \mathbb{R}$ which are definable in $\mathcal{R}$ and have a definable C^∞ extension on some open neighbourhood of B. Then the collection $\mathcal{C}(\mathcal{R}) := \{\mathcal{C}_B(\mathcal{R}) : B \subseteq \mathbb{R}^n \text{ compact box}, n \in \mathbb{N}\}$ satisfies* (C1)-(C7).

Proof. The properties (C1)-(C4) and (C6)-(C7) are immediate consequences of basic properties of o-minimal expansions of the real closed field and by definition of the algebras $\mathcal{C}_B(\mathcal{R})$.

For (C5), assume that the map $\mathcal{T} : \mathcal{C}_n(\mathcal{R}) \to \mathbb{R}[X]$ is not injective. Then there exist $f, g \in \mathcal{C}_B(\mathcal{R})$ such that $f \neq g$ and $\mathcal{T}(f) = \mathcal{T}(g)$ so $\mathcal{T}(f - g) = 0$. Therefore, $\lim_{x\to 0} \dfrac{(f-g)(x)}{x^n} = 0$ for all $n \in \mathbb{N}$. We can furthermore assume that $f(x) > g(x)$ for sufficiently small x and by change of variables we have

$$\lim_{t\to\infty} \frac{1/(f-g)(1/t)}{(t^n)} = \infty$$

for all $n \in \mathbb{N}$. This gives a contradiction because $x \mapsto \dfrac{1}{(f-g)(1/x)}$ is definable in $\mathcal{R}$ and $\mathcal{R}$ is polynomially bounded.

□

We need further definitions from [65] in the next section where we discuss the work in [33]. For $r \in (0, \infty)^n$, we put $I_r = \prod_{i=1}^n (-r_i, r_i)$ and $\hat{I}_r$ is the topological closure of I_r.

For the rest of this section we fix $\mathcal{C} = \{\mathcal{C}_B : B \subseteq \mathbb{R}^n \text{ compact box}, n \in \mathbb{N}\}$ satisfying (C1)-(C7) , where, for each compact box $B \subseteq \mathbb{R}^n$, and $n \in \mathbb{N}$, $\mathcal{C}_B$ is an $\mathbb{R}$-algebra of analytic functions on B.

Definition 4.2.4. *Let $r \in (0, \infty)^n$ and let $f, g_1, \ldots, g_k \in \mathcal{C}_{\hat{I}_r}$. A subset of $\mathbb{R}^n$ of the form*

$$\{\bar{x} \in \hat{I}_r : f(\bar{x}) = 0, g_1(\bar{x}) > 0, \ldots, g_k(\bar{x}) > 0\}$$

is called a basic $\mathcal{C}$-set. *A set is called a* $\mathcal{C}$-set *if it is a finite union of basic $\mathcal{C}$-sets. A set $S \subseteq R^n$ is called* $\mathcal{C}$-semianalytic *if for all $\bar{a} \in \mathbb{R}^n$ there exists $r \in (0, \infty)^n$ such that $(S - \bar{a}) \cap I_r$ is a $\mathcal{C}$-set.*

Let $m \leq n$ and let $\lambda : \{1, \ldots, m\} \to \{1, \ldots, n\}$ be an injective map. We define

$$\begin{array}{rccc} \pi_\lambda : & \mathbb{R}^n & \to & \mathbb{R}^m \\ & \bar{x} & \mapsto & \left(x_{\lambda(1)}, \ldots, x_{\lambda(m)}\right). \end{array}$$

Definition 4.2.5. *Let $r \in (0, \infty)^n$. A set $M \subseteq I_r$ is called $\mathcal{C}$-*trivial *if either*
$(M1)$ it is of the form

$$M = \{\bar{x} \in I_r : x_i \star_i 0, i = 1, \ldots, n\}$$

with $\star_i \in \{<, >, =\}$ for $i \in \{1, \ldots, n\}$ or
$(M2)$ there exists a permutation σ of $\{1, \ldots, n\}$, a $\mathcal{C}$-trivial $N \subset I_s$ and $g \in \mathcal{C}_{I_s}$ where $s = \left(r_{\sigma(1)}, \ldots, r_{\sigma(n-1)}\right)$ such that $g(I_s) \subseteq \left(-r_{\sigma(n)}, r_{\sigma(n)}\right)$ and $\pi_\sigma(M)$ is the graph of g restricted to N.

Definition 4.2.6. *A $\mathcal{C}$-semianalytic manifold $M \subseteq \mathbb{R}^n$ is called* trivial *if there exists $\bar{a} \in \mathbb{R}^n$ and a $\mathcal{C}$-trivial manifold $N \subseteq R^n$ such that $M = N + \bar{a}$.*

We denote the projection map on the first k coordinates by Π_k.

Lemma 4.2.7. ([65, Proposition 4.7]) *Let S be a bounded $\mathcal{C}$-semianalytic subset of $\mathbb{R}^n$ and let $k \leq n$. Then there exist trivial $\mathcal{C}$-semianalytic manifolds $N_i \subseteq \mathbb{R}^{n_i}$ with $n_i \geq n$ for $i = 1, \ldots, p$ such that*

$$\Pi_k(S) = \bigcup_{i=1}^{p} \Pi_k(N_i)$$

where Π_k restricted to N_i is an immersion for all $i = 1, \ldots, p$.

4.3 Mild parametrization in reducts of $\mathbb{R}_{\mathbf{an}}$

We now present the parametrization result that the authors in [33] deduce from the results in [65] which we presented in the previous section.

We fix $\mathcal{C} = \{\mathcal{C}_B : B \subseteq \mathbb{R}^n \text{ compact box}, n \in \mathbb{N}\}$ satisfying (C1)-(C7) , where, for each compact box $B \subseteq \mathbb{R}^n$, and $n \in \mathbb{N}$, $\mathcal{C}_B$ is an $\mathbb{R}$-algebra of analytic functions on B and we follow the notations of the previous section.

The following lemma is stated in [33] after explaining the main idea with an example and stating that the proof of the lemma follows from an induction argument. We give here a detailed proof of this lemma.

Lemma 4.3.1. ([33, Theorem 2.7]) *Let $M \subseteq \mathbb{R}^n$ be a $\mathcal{C}$-trivial manifold with $\dim(M) = d$. Then there exists $\varphi \in \mathcal{C}_{[0,1]^d}$ such that $\varphi\left((0, 1)^d\right) = M$.*

Proof. We will prove the lemma for two cases, depending on whether $(M1)$ or $(M2)$ in Definition 4.2.5 is true for M. Let $r \in (0,\infty)^n$ and $M \subseteq I_r$.
Case 1. Assume that $(M1)$ is true. Let $M = \{\bar{x} \in I_r : x_i \star_i 0, i = 1,\dots,n\}$ where $r \in (0,\infty)^n$. Set P, N and Z to be the disjoint subsets of $\{1,\dots,n\}$ with $P \cup N \cup Z = \{1,\dots,n\}$ such that

$$\star_i = \begin{cases} > & i \in P \\ < & i \in N \\ = & i \in Z. \end{cases}$$

We define the function $\varphi : [0,1]^d \to \mathbb{R}^n$ by $\bar{x} \mapsto (\varphi_1(\bar{x}),\dots,\varphi_n(\bar{x}))$ where the coordinate functions are defined as follows:

$$\varphi_i(\bar{x}) = \begin{cases} -r_i x_i & i \in P \\ r_i x_i & i \in N \\ 0 & i \in Z. \end{cases}$$

Then the map φ is in $\mathcal{C}_{[0,1]^d}$ and $\varphi\left((0,1)^d\right) = M$.
Case 2. Assume that $(M2)$ is true. We will prove this case by induction on n. For $n = 1$, the result is clear. Let $n \geq 2$ and we assume the result for $n-1$. Let $r \in (0,\infty)^n$, let σ be the permutation of $\{1,\dots,n\}$, let $N \subset I_s$ be $\mathcal{C}$-trivial, and let $g \in \mathcal{C}_{I_s}$ be such that $s = (r_{\sigma(1)},\dots,r_{\sigma(n-1)})$, $g(I_s) \subseteq \left(-r_{\sigma(n)}, r_{\sigma(n)}\right)$ and $\pi_\sigma(M)$ is the graph of g restricted to N. Note that since $\Gamma\left(g\restriction_N\right) = \pi_\sigma(M)$, we have $\dim(N) = \dim(M) = d$. Since $N \subseteq \mathbb{R}^{n-1}$, by induction hypothesis there exists $\varphi_N : [0,1]^d \to \mathbb{R}^{n-1}$ in $\mathcal{C}_{[0,1]^d}$ such that $\varphi_N\left((0,1)^d\right) = N$. We define the function $\varphi_M : [0,1]^d \to \mathbb{R}^n$ by

$$\varphi_M(\bar{x}) = (\varphi_N(\bar{x}), g(\varphi_N(\bar{x})))$$

which is in $\mathcal{C}_{[0,1]^d}$ by property (C2), and $\varphi_M\left((0,1)^d\right) = \pi_\sigma(M)$.

The map $\tilde{\varphi}_M = \pi_{\sigma^{-1}} \circ \varphi_M$ is therefore a map in $\mathcal{C}_{[0,1]^d}$ such that $\varphi'_M\left((0,1)^d\right)$ is equal to M. □

We present now the $\mathcal{C}$-parametrization result from [33] which was proved using Lemma 4.2.7.

Proposition 4.3.2. ([33, Proposition 2.8]) *Let $X \subseteq (0,1)^n$ be a definable set in $\mathbb{R}_{\mathcal{C}}$ and let $\dim(X) = d$. Then there exist $l \in \mathbb{N}$ and maps $\phi_1,\dots,\phi_l$ whose coordinate functions are in $\mathcal{C}_{[0,1]^d}$ such that*

$$\bigcup_{i=1}^{l} \operatorname{Im}\left(\phi((0,1)^d)\right) = X.$$

Proposition 4.3.3. ([33, Proposition 1.5]) *The structure $\mathbb{R}_{an}$ admits definable mild parametrization.*

Proof. Let $\mathcal{C}^{\text{an}}$ be the collection of all functions $f : B \to \mathbb{R}$ for every compact box $B \subseteq \mathbb{R}^n$ and every $n \in \mathbb{N}$, which are definable in $\mathbb{R}_{\text{an}}$ and have definable C^∞

extension on some open neighbourhood of B. Then $\mathcal{C}^{\text{an}}$ satisfies properties (C1)-(C7) by Lemma 4.2.3 since $\mathbb{R}_{\text{an}}$ is polynomially bounded. Note that every set definable in $\mathbb{R}_{\mathcal{C}^{\text{an}}}$ is also definable in $\mathbb{R}_{\text{an}}$.

Conversely we will prove that every set definable in $\mathbb{R}_{\text{an}}$ is also definable in $\mathbb{R}_{\mathcal{C}^{\text{an}}}$. For this, it suffices to show that if $f : \mathbb{R}^n \to \mathbb{R}$ is any restricted analytic function then $f\restriction_{[-1,1]^n}$ has a C^∞ extension that is definable in $\mathbb{R}_{\text{an}}$. Consider a restricted analytic function $f : \mathbb{R}^n \to \mathbb{R}$. By definition, there is real analytic function F defined on an open neighbourhood U of $[-1,1]^n$ such that $F\restriction_{[-1,1]^n} = f\restriction_{[-1,1]^n}$.

We can find a compact box B such that the interior of B contains $[-1,1]^n$ and $B \subseteq U$. Let φ be the linear function that sends $[-1,1]^n$ to B. Then the function $g : \mathbb{R}^n \to \mathbb{R}$ given by

$$g(\bar{x}) = \begin{cases} F \circ \varphi(\bar{x}) & \bar{x} \in [-1,1]^n \\ 0 & \text{otherwise} \end{cases}$$

is a restricted analytic function. Since φ^{-1} is definable in $\mathbb{R}_{\text{an}}$, $g \circ \varphi^{-1}$ is definable in $\mathbb{R}_{\text{an}}$, but $(g \circ \varphi^{-1})\restriction_{[-1,1]^n} = f\restriction_{[-1,1]^n}$ and $g \circ \varphi^{-1}$ is analytic (hence C^∞) on the interior of B, which is an open neighbourhood of $[-1,1]^n$ so $f\restriction_{[-1,1]^n}$ has indeed a C^∞ extension that is definable in $\mathbb{R}_{\text{an}}$. Therefore $\mathbb{R}_{\mathcal{C}^{\text{an}}} = \mathbb{R}_{\text{an}}$. So for $n \in \mathbb{N}$ and any definable set $X \subseteq (0,1)^n$ with $\dim X = d$ there exist $l \in \mathbb{N}$ and maps $\phi_1, \ldots, \phi_l$ whose coordinate functions are in $\mathcal{C}^{\text{an}}$ such that $\bigcup_{i=1}^{l} \operatorname{Im}\left(\phi((0,1)^d)\right) = X$ by Proposition 4.3.2. The maps $\phi_1, \ldots, \phi_l$ are mild by Proposition 3.2.2 and they are definable in $\mathbb{R}_{\text{an}}$. Hence $\mathbb{R}_{\text{an}}$ admits definable mild parametrization. □

Corollary 4.3.4. *Let $\mathcal{R}$ be a reduct of $\mathbb{R}_{an}$ expanding the real ordered field. Then $\mathcal{R}$ admits mild parametrization.*

Remark 4.3.5. Let $\mathcal{R}$ be a reduct of $\mathbb{R}_{\text{an}}$ expanding the real ordered field and let $\mathcal{C}(\mathcal{R})$ be the collection of all functions $f : [-1,1]^n \to \mathbb{R}$ for all $n \in \mathbb{N}$ which are definable in $\mathcal{R}$ and have a definable C^∞ extension on some open neighbourhood of $[-1,1]^n$. Clearly every set definable in $\mathbb{R}_{\mathcal{C}(\mathcal{R})}$ is also definable in $\mathcal{R}$. However it seems still open that $\mathcal{R} = \mathbb{R}_{\mathcal{C}(\mathcal{R})}$, indeed given a restricted analytic function $f : \mathbb{R} \to \mathbb{R}$, it does not follow directly from the definition that $f\restriction_{[-1,1]}$ has a C^∞ extension to a neighbourhood of $[-1,1]$ that is *definable in the o-minimal structure* $(\mathbb{R}, +, -, \cdot, 0, 1, <, f)$; if f is strongly transcendental in the sense of Le Gal [40], it is expected that such a definable extension does not exist. Note that even if, as we expect, there is indeed a reduct $\mathcal{R}$ of $\mathbb{R}_{\text{an}}$ expanding the real ordered field such that $\mathcal{R} \neq \mathbb{R}_{\mathcal{C}(\mathcal{R})}$, it could still be that every reduct of $\mathbb{R}_{\text{an}}$ has definable mild parametrization; however different techniques from those used in [33] should be used to establish such a **definable** mild parametrization result.

In [33], Jones, D.Miller and Thomas considered a specific reduct of $\mathbb{R}_{\text{an}}$ which is obtained in the same way as $\mathbb{R}_{\text{an}}$ replacing analytic functions with Pfaffian functions (for the definition of Pfaffian functions see Definition 2.2.2). This structure is denoted by $\mathbb{R}_{\text{respfaff}}$.

Note that $\mathbb{R}_{\text{respfaff}}$ is a reduct of $\mathbb{R}_{\text{an}}$ so it admits mild parametrization by Corollary 4.3.4. Moreover the argument in Proposition 4.3.3 that says that a restricted analytic function has a definable C^∞ extension in $\mathbb{R}_{\text{an}}$ also applies to a restricted Pfaffian function and the structure $\mathbb{R}_{\text{respfaff}}$, so the structure $\mathbb{R}_{\text{respfaff}}$ even admits definable mild parametrization. In [33] the authors also proved Wilkie's conjecture for curves definable in $\mathbb{R}_{\text{respfaff}}$. They combined the mild parametrization of $\mathbb{R}_{\text{respfaff}}$ that they proved with Khovanskiĭ's theorem (see Proposition 3.3.9) to obtain this result. Earlier in [57] Pila had proved Wilkie's conjecture for the graph of any one variable Pfaffian function. The result of Jones, D. Miller and Thomas is more general than the result of Pila since it covers not only one variable Pfaffian functions but also all one variable functions obtained implicitly from Pfaffian functions, that is all one variable functions definable in $\mathbb{R}_{\text{respfaff}}$. In [33] Wilkie's conjecture is proven for surfaces (dimension 2) definable in $\mathbb{R}_{\text{respfaff}}$ using the mild parametrization of $\mathbb{R}_{\text{respfaff}}$ and the results presented in Section 5.3.

Chapter 5

Mild parametrization in RS-structures

In the article [64], Rolin and Servi extract a unifying setting that generalizes the proof of o-minimality of historically important examples of expansions of the real ordered field. These examples include the expansions by converging generalized power series which we define in Definition 2.1.19 ([24]), by Gevrey functions ([25]), and by quasianalytic Denjoy-Carleman classes ([65]) which we define in the coming chapters.

The idea of the proof of Rolin and Servi is the following: consider an algebra $\mathcal{A}$ of real valued functions; assume that this algebra $\mathcal{A}$ satisfies certain properties (stated precisely in Section 5.1) among which there is the quasianalyticity property: the germ of each function is determined by its expansion as a generalized power series; then the structure $\mathbb{R}_{\mathcal{A}}$ is o-minimal, model complete and polynomially bounded. An expansion of the real field by an algebra $\mathcal{A}$ satisfying the properties given by Rolin and Servi in [64] will be called an RS-structure.

The key point of the proof is getting the model completeness. This is achieved by a process of resolution of singularities. The resolution is possible "formally" for the generalized power series, and the hypothesis of quasianalyticity ensures that this "formal" resolution of singularities corresponds to an actual resolution of singularities for the corresponding germs of functions from the algebra $\mathcal{A}$. The process of resolution of singularities actually gives a result stronger than model completeness: it gives a parametrization theorem by functions in $\mathcal{A}$ generalizing Hironaka's Rectilinearization Theorem ([30]). That is, it is possible to parametrize all bounded definable sets of the structure $\mathbb{R}_{\mathcal{A}}$, using functions that belong to the algebra $\mathcal{A}$. We will describe these algebras and state the parametrization theorem of Rolin and Servi (Theorem 5.1.5) in Section 5.1.

The properties (Properties 5.1.1) stated for any algebras $\mathcal{A}$ to satisfy are similar to but less restrictive than the properties C1-C7 that the algebras $\mathcal{C}$ were requested to satisfy in Chapter 4. Roughly speaking, the functions in $\mathcal{A}$ may exhibit some extra singularities at points on some sides of the boxes on which they are defined, but these singularities stay "controlled" by a property of quasianalyticity of the development in generalized power series. An $\mathbb{R}_{\mathcal{C}}$ structure from Chapter 4 can be seen as an $\mathbb{R}_{\mathcal{A}}$ structure as described in Section 5.1.

If all the functions $f : (0,1)^n \to (0,1)$ in $\mathcal{A}$ happen to be mild, then a theorem of definable mild parametrization follows from Theorem 5.1.5. To show this we will use the ideas in the proof of model completeness from [64] to give a proof that for every definable set A in $\mathbb{R}_{\mathcal{A}}$ that lies in $[-1,1]^n$, there is a quantifier-free definable set $S \in [-1,1]^{n+m}$ with $m \in \mathbb{N}$ such that the projection of S on its first n coordinates gives A. Then using this and Theorem 5.1.5 we prove Proposition 5.2.3 that states that if all the functions $f : (0,1)^n \to (0,1)$ in the algebra $\mathcal{A}$ are mild for all $n \in \mathbb{N}$ then the structure $\mathbb{R}_{\mathcal{A}}$ admits definable mild parametrization. These results will be the subject of Section 5.2.

The real field expanded by certain Gevrey functions, $\mathbb{R}_{\mathcal{G}}$ is defined in [25]. In Section 5.3, we will examine $\mathbb{R}_{\mathcal{G}}$ and prove that this structure admits mild parametrization as a consequence of Proposition 5.2.3. Furthermore we will consider a specific example of a function definable in $\mathbb{R}_{\mathcal{G}}$ (which was given in [25]) and apply the mild parametrization result to prove a new result on the density of rational points of a surface defined using this function (Proposition 5.3.8).

For a potential further application of the parametrization result in [64] we consider relaxing the mildness condition on $\mathcal{A}$ and ask if $\mathbb{R}_{\mathcal{A}}$ admits definable mild parametrization if all the functions $f : (0,1)^n \to (0,1)$ in $\mathcal{A}$ instead have mild parametrization. We give a negative answer to this question in Chapter 7 by proving that if an irrational power function is definable in a polynomially bounded expansion of the real field then this structure does not admit definable mild parametrization

(Theorem 7.2.8).

5.1 Parametrization Theorem of Rolin and Servi

As we mentioned in the introduction of the chapter, in [64], Rolin and Servi gave a list of certain properties for algebras of smooth functions and showed that if an algebra $\mathcal{A}$ satisfies these properties then the structure $\mathbb{R}_{\mathcal{A}}$ obtained by expanding the real ordered field by the functions in $\mathcal{A}$ is polynomially bounded, model complete, and o-minimal. The model completeness of the structure $\mathbb{R}_{\mathcal{A}}$ gives a parametrization of the bounded definable sets in $\mathbb{R}_{\mathcal{A}}$ which will be useful for our purposes. Before presenting this parametrization result we will first explain here the properties highlighted by Rolin and Servi for an algebra $\mathcal{A}$ to satisfy which ensure that the structure $\mathbb{R}_{\mathcal{A}}$ is o-minimal. We will keep the notation from [64] for the rest of this section.

Notations and Conventions.

- We let m, n range over $\mathbb{N}$.
- $\bar{x} = (x_1, \ldots, x_n)$, $\bar{y} = (y_1, \ldots, y_m)$ and $\bar{z} = (z_1, \ldots, z_{n+m})$ denote tuples of variables.
- $\bar{1}$ and $\bar{0}$ will denote tuples with all coordinates 1 and 0 respectively; the number of coordinates will not be mentioned when it is clear from the context.
- $r = (s_1, \ldots, s_n, t_1, \ldots, t_m)$, $r' = (s'_1, \ldots, s'_n, t'_1, \ldots, t'_m) \in (0, \infty)^{n+m}$ and we say that $r \leq r'$ if $s_i \leq s'_i$ for all $i = 1, \ldots, n$ and $t_j \leq t'_j$ for all $j = 1, \ldots, m$.
- Given r as above, $I_{n,m,r} := (0, s_1) \times \ldots \times (0, s_n) \times (-t_1, t_1) \times \ldots \times (-t_m, t_m)$ $\hat{I}_{n,m,r} := [0, s_1) \times \ldots \times [0, s_n) \times (-t_1, t_1) \times \ldots \times (-t_m, t_m)$ and $I_{n,m,\infty} : [0, \infty)^n \times \mathbb{R}^m$.
- σ_n denotes a permutation of $\{1, \ldots, n\}$.
- $\mathcal{A}_{n,m,r}$ is an algebra of functions on $\hat{I}_{n,m,r}$ which are continuous on $\hat{I}_{n,m,r}$ and C^1 on $I_{n,m,r}$.

- $\mathcal{A}_{n,m}$ is the algebra of germs at the origin of the functions in $\mathcal{A}_{n,m,r}$ for $r \in (0,\infty)^{n+m}$. A germ of a function $f \in \mathcal{A}_{n,m,r}$ will be denoted by the same letter f for simplicity.

- $\mathcal{A}$ is the collection of all the functions in $\mathcal{A}_{n,m,r}$ for all $n, m \in \mathbb{N}$ and $r \in (0,\infty)^{n+m}$.

We now give a list of properties of the algebras $\mathcal{A}_{n,m,r}$.

Properties 5.1.1. 1. For all $i = 1, \ldots, n+m$ the functions

$$\begin{array}{rccc} f_i: & \hat{I}_{n,m,r} & \to & \mathbb{R} \\ & \bar{z} & \mapsto & z_i \end{array}$$

are in $\mathcal{A}_{n,m,r}$.

2. The algebra $\mathcal{A}_{n,m,r}$ is included in the algebra $\mathcal{A}_{n+m,0,r}$ identifying each $f(\bar{x},\bar{y})$ with $f(\bar{z}) \in \mathcal{A}_{n+m,0,r}$ where $\bar{z} = (\bar{x},\bar{y})$.

3. If $r' \leq r$ the restriction of each $f \in \mathcal{A}_{n,m,r}$ to the domain $\hat{I}_{n,m,r'}$ is in $\mathcal{A}_{n,m,r'}$.

4. For all $f \in \mathcal{A}_{n,m,r}$, there exists $r' > r$ and $g \in \mathcal{A}_{n,m,r'}$ such that the restriction of g to the domain $\hat{I}_{n,m,r}$ is the function f.

5. For every $f \in \mathcal{A}_{n,m,r}$ and $r^+ = (s_1, \ldots, s_n, s, t_1, \ldots, t_m) \in (0,\infty)^{n+m+1}$ the function

$$\begin{array}{rccc} F: & \hat{I}_{n+1,m,r^+} & \to & \mathbb{R} \\ & (\bar{x}, z, \bar{y}) & \mapsto & f(\bar{x},\bar{y}) \end{array}$$

is in $\mathcal{A}_{n+1,m,r^+}$.

6. For all $f \in \mathcal{A}_{n,m,r}$ and all permutations σ_n, the function

$$\begin{array}{rccc} f_{\sigma_n}: & \hat{I}_{n,m,r} & \to & \mathbb{R} \\ & (\bar{x},\bar{y}) & \mapsto & f\left(x_{\sigma_n(1)}, \ldots, x_{\sigma_n(n)}, \bar{y}\right) \end{array}$$

is in $\mathcal{A}_{n,m,\sigma_n(r)}$ where $\sigma_n(r) := \left(s_{\sigma_n(1)}, \ldots, s_{\sigma_n(n)}, t_1, \ldots, t_m\right)$.

7. For all $f \in \mathcal{A}_{n,m,r}$, the function

$$g : (x_1, \ldots, x_{n-1}, \bar{y}) \mapsto f(x_1, \ldots, x_{n-1}, 0, \bar{y})$$

is in $\mathcal{A}_{n-1,m,r}$.

8. For all $f \in \mathcal{A}_{n,m,r}$, the function

$$\begin{array}{rccc} f_q: & \hat{I}_{n,m,r} & \to & \mathbb{R} \\ & (\bar{x},\bar{y}) & \mapsto & f\left(\dfrac{x_1}{s_1}, \ldots, \dfrac{x_n}{s_n}, \dfrac{y_1}{t_1}, \ldots, \dfrac{y_m}{t_m}\right) \end{array}$$

is in $\mathcal{A}_{n,m,\bar{1}}$.

The next property is called $\mathcal{A}$-analyticity ([AN]) because it states that the algebra $\mathcal{A}_{n,m}$ satisfies a similar property to the real analytic functions being analytic in a whole neighbourhood of the origin.

9. [AN] For all $f \in \mathcal{A}_{n,m,r}$ and $a \in \hat{I}_{n,m,r}$ let $n' = \#\{i : 1 \leq i \leq n, a_i = 0\}$. There exists a germ $g_a \in \mathcal{A}_{n',n+m-n'}$ which satisfies

$$g_a(\bar{x}, \bar{y}) = f(x_{\sigma_n(1)} + a_{\sigma_n(1)}, \ldots, x_{\sigma_n(n)} + a_{\sigma_n(n)}, y_1 + a_{n+1}, \ldots, y_m + a_{n+m})$$

where σ_n is a permutation of $\{1, \ldots, n\}$ such that $a_{\sigma(i)} = 0$ if and only if $i \leq n'$.

The next property [QA] states that a germ in the collection $\{\mathcal{A}_{n,m} : n, m \in \mathbb{N}\}$ is uniquely determined by its generalized Taylor expansion. It is called the *quasianalyticity* property.

10. [QA] For all $n, m \in \mathbb{N}$ there exists an injective $\mathbb{R}$-algebra morphism

$$\mathcal{T}_{n,m} : \mathcal{A}_{n,m} \to \mathbb{R}[\![X^*, Y]\!].$$

Moreover for all $n' \geq n$ if either $m' \geq m$ or $n + m = n' + m'$ then we require that the morphism $\mathcal{T}_{n',m'}$ extend $\mathcal{T}_{n,m}$.

A number $\alpha \in [0, \infty)$ is called an *admissible exponent* if there are $n, m \in \mathbb{N}$, $f \in \mathcal{A}_{n,m}$, and $\beta \in Supp(\mathcal{T}(f)) \subseteq \mathbb{R}^n \times \mathbb{N}^m$ such that α is a component of β. Also we denote the specific ring and field generated using these exponents as
$\mathbb{A} :=$ semiring generated by all admissible exponents,
$\mathbb{K} :=$ the field of fractions of the ring generated by $\mathbb{A}$.

11. For every nonnegative element α in $\mathbb{K}$ the germ $g_1 : x_1 \mapsto x_1^\alpha$ is in $\mathcal{A}_1$ and $\mathcal{T}(x_1^\alpha) = X_1^\alpha$. Moreover if $f \in \mathcal{A}_{n,m}$ then $g(x, y) := f(x_1^\alpha, x_2 \ldots, x_n, \bar{y}) \in \mathcal{A}_{n,m}$ and $\mathcal{T}(g) = \mathcal{T}(f)(X_1^\alpha, X_2, \ldots, X_n, \bar{Y})$.

12. Let $\alpha \in \mathbb{K}$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_{n,m}$ and $G \in \mathbb{R}[X^*, Y]$ such that $\mathcal{T}(f)(\bar{X}, \bar{Y}) = X_1^\alpha Y_1^m G(\bar{X}, \bar{Y})$. Then there exists $g \in \mathcal{A}_{n,m}$ such that $f(\bar{x}, \bar{y}) = x_1^\alpha y_1^m g(\bar{x}, \bar{y})$.

13. For each $f \in \mathcal{A}_{n,m,r}$ and each permutation σ_n, $\mathcal{T}(f_{\sigma_n}) = \mathcal{T}(f)_{\sigma_n}$ where the function f_{σ_n} is as defined in property 6.

14. Let $f \in \mathcal{A}_{n,m}$. Then $\mathcal{T}(f(x_1, \ldots, x_{n-1}, 0, \bar{y})) = \mathcal{T}(f)(X_1, \ldots, X_{n-1}, 0, \bar{Y})$.

15. Let $g_1, \ldots, g_m \in \mathcal{A}_{n',m'}$ with $g_i(0) = 0$ and let $f \in \mathcal{A}_{n,m}$. Then $h := f(\bar{x}, g_1, \ldots, g_m) \in \mathcal{A}_{n+n',m'}$ and

$$(\mathcal{T}h) = \mathcal{T}(f)(\bar{X}, \mathcal{T}(g_1), \ldots, \mathcal{T}(g_m)).$$

16. Let $f \in \mathcal{A}_{n,m}$. If $\dfrac{\partial f}{\partial y_m}(0)$ exists and is nonzero there exists $g \in \mathcal{A}_{n,m-1}$ such that

$$f(\bar{x}, y_1, \ldots, y_{m-1}, g(\bar{x}, y_1, \ldots, y_{m-1})) = 0.$$

In the setting of [64] blow-up charts are a special type of polynomial changes of variables

$$\begin{array}{rccc} \pi : & \hat{I}_{n',m',r'} & \to & \hat{I}_{n,m,r} \\ & (\bar{x}', \bar{y}') & \mapsto & (\bar{x}, \bar{y}) \end{array}$$

where $n' \in \{n-1, n, n+1\}$ and $n' + m' = n + m$ (their definition can be found in [64, Definition 1.13]).

17. Let $f \in \mathcal{A}_{n,m}$, and let $\pi := \hat{I}_{n',m',\infty} \to \hat{I}_{n,m,\infty}$ be a blow up chart. Then $f \circ \pi \in \mathcal{A}_{n',m'}$ and $\mathcal{T}(f \circ \pi) = \mathcal{T}(f) \circ \pi$.

Definition 5.1.2. *Let $\mathcal{A}$ be a collection of algebras $\mathcal{A}_{n,m,r}$ for all $n, m \in \mathbb{R}$ and $r \in (0,\infty)^{n+m}$. We say that $\mathcal{A}$ is an RS-class if it satisfies the Properties 5.1.1.*

For $f \in \mathcal{A}_{n,m,\bar{1}}$ we define the function

$$\tilde{f}(\bar{x},\bar{y}) = \begin{cases} f(\bar{x},\bar{y}) & \text{if} \quad (\bar{x},\bar{y}) \in I_{n,m,\bar{1}} \\ 0 & \text{otherwise} \end{cases}$$

and the real field generated by $\mathcal{A}$ is

$$\mathbb{R}_{\mathcal{A}} := (\mathbb{R}, +, -, \cdot, 0, 1, <, (\tilde{f})_{f \in \mathcal{A}_{n,m,\bar{1}}}).$$

We call any structure $\mathbb{R}_{\mathcal{A}}$ an *RS-structure* if $\mathcal{A}$ is an RS-class.

Theorem 5.1.3. ([64, Theorem A]) *Let $\mathbb{R}_{\mathcal{A}}$ be an RS-structure. Then $\mathbb{R}_{\mathcal{A}}$ is polynomially bounded, model complete and o-minimal.*

We need further definitions from [64] before stating the parametrization theorem of Rolin and Servi.

Definition 5.1.4.
- *A subset of $\hat{I}_{n,m,r}$ is called an $\mathcal{A}_{n,m}$- **basic set** if it is a finite union of sets of the form*

$$\{(\bar{x},\bar{y}) \in \hat{I}_{n,m,r} : g_0(\bar{x},\bar{y}) = 0, g_1(\bar{x},\bar{y}) > 0, \ldots, g_k(\bar{x},\bar{y}) > 0\}$$

  for $g_0, \ldots, g_k \in \mathcal{A}_{n,m,r}$ and $k \in \mathbb{N}$.

- *A subset of $\mathbb{R}^{n+m}$ is called an $\mathcal{A}_{n,m}$-**semianalytic set** if for every $a \in \mathbb{R}^{n+m}$ there exists $r_a \in (0,\infty)^{n+m}$ such that, for every choice of signs $\upsilon = (\upsilon_1, \ldots, \upsilon_n) \in \{-1,1\}^n$, there exists an $\mathcal{A}_{n,m}$- basic set $A_{a,\upsilon} \subset \hat{I}_{n,m,r_a}$ with*

$$A \cap (h_{a,\upsilon}(\hat{I}_{n,m,r_a})) = h_{a,\upsilon}(A_{a,\upsilon})$$

  where $h_{a,\upsilon}(x,y) = (\upsilon_1 x_1 + a_1, \ldots, \upsilon_n x_n + a_n, y_1 + a_{n+1}, \ldots, y_m + a_{n+m})$.

- *A set $Q \subseteq \hat{I}_{n,m,r}$ is called a **subquadrant** if it is of the form $B_1 \times \ldots \times B_n$ where B_i is either $\{0\}$ or $(-r_i, 0)$ or $(0, r_i)$ where $r = (r_1, \ldots, r_n, s_1, \ldots, s_m)$. The dimension of the subquadrant Q is denoted by $\dim(Q)$ and it is the cardinality of the set $\{i : B_i \neq \{0\}\}$.*

For a set $S \in \mathbb{R}^{n+m}$ and $k \leq n+m$, the notation $\Pi_k S$ denotes the projection of S on its first k variables and the notation $\Pi^k S$ denotes the projection of S on its last k variables. We now state the parametrization theorem in [64].

Theorem 5.1.5. ([64, Theorem 3.13]) *Let $S \subseteq \mathbb{R}^{n+m}$ be a bounded $\mathcal{A}_{n,m}$-semianalytic set and let $k \leq n+m$. Then there exists $N \in \mathbb{N}$ and, for all $i = 1, \ldots, N$, there exist $n_i', m_i' \in \mathbb{N}$ with $n_i' + m_i' = n+m$, $r_i \in [0,\infty)^{n_i'+m_i'}$, a subquadrant $Q_i \subseteq \hat{I}_{n_i',m_i',r_i}$ and a map $H_i : \hat{I}_{n_i',m_i',r_i} \to \mathbb{R}^k$, whose components are in $\mathcal{A}_{n_i',m_i',r_i}$, such that*

$$H_i\restriction_{Q_i} : Q_i \to H_i(Q_i)$$

is a diffeomorphism and

$$\Pi^k(S) = \bigcup_{i=1}^{N} H_i(Q_i).$$

5.2 Mild parametrization in $\mathbb{R}_{\mathcal{A}}$

The authors in [64] pointed out that Theorem 5.1.5 applies to any $\mathbb{R}_{\mathcal{A}}$-definable set $A \subseteq [-1,1]^n$ because there exists an $\mathcal{A}$-semianalytic set $S \subseteq [-1,1]^{n+m}$ such that the projection of S onto its first n coordinates is A ([64, Remark 3.19]). We state the existence of such an $\mathcal{A}$-semianalytic set S and give a proof by adapting the arguments of Rolin and Servi in the proof of model completeness, in Subsection 3.1 of [64].

Lemma 5.2.1. *Let $\mathbb{R}_{\mathcal{A}}$ be an RS-structure. For every $n \in \mathbb{N}$ and every $\mathbb{R}_{\mathcal{A}}$-definable set $A \subseteq [-1,1]^n$, there exists $m \in \mathbb{N}$ and an $\mathcal{A}$-semianalytic set $S \subseteq [-1,1]^{n+m}$ such that $\Pi_n S = A$.*

Proof. There exist $m \in \mathbb{N}$ and a quantifier-free definable set $T \subseteq \mathbb{R}^{n+m}$ in $\mathbb{R}_{\mathcal{A}}$ such that $\Pi_n T = A$ by model completeness of $\mathbb{R}_{\mathcal{A}}$.

By Remark 2.1.10, we can assume without loss of generality that T is of the form

$$T = \{(\bar{x},\bar{y}) \in [-1,1]^n \times \mathbb{R}^m : P\left(\bar{x},\bar{y},\bar{f}_1(\bar{x},\bar{y}),\ldots,\bar{f}_k(\bar{x},\bar{y})\right) = 0\},$$

for some polynomial P in $n+m$ variables, $k \in \mathbb{N}$, and $f_1, \ldots, f_k \in \mathcal{A}$ where

$$\bar{f}_i(\bar{x},\bar{y}) = \begin{cases} f_i(\bar{x},\bar{y}) & (\bar{x},\bar{y}) \in I_{n,m,\bar{1}} \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, k$. (Since $\mathcal{A}$ is closed under compositions and contains the coordinate functions the projections used in the Remark 2.1.10 are not necessary here.)

Let $\mathcal{P}_m$ denote the collection of subsets of the set $\{1,\ldots,m\}$. We separate T as $T = \bigcup_{I\in\mathcal{P}_m} T_I$ where

$$T_\emptyset := \{(\bar{x},\bar{y}) \in [-1,1]^{n+m} : P\left(\bar{x},\bar{y},\bar{f}_1(\bar{x},\bar{y}),\ldots,\bar{f}_k(\bar{x},\bar{y})\right) = 0\}$$

and

$$\begin{aligned} T_I := \{(\bar{x},\bar{y}) \in [-1,1]^n \times \mathbb{R}^m \colon & \left(\textstyle\bigwedge_{j\in I} y_j \in \mathbb{R}\setminus[-1,1]\right) \wedge \left(\textstyle\bigwedge_{j\in\{1,\ldots,m\}\setminus I} y_j \in [-1,1]\right), \\ & P\left(\bar{x},\bar{y},\bar{f}_1(\bar{x},\bar{y}),\ldots,\bar{f}_k(\bar{x},\bar{y})\right) = 0\} \end{aligned}$$

for all nonempty subsets I of $\mathcal{P}_m$.
Fix $J \in \mathcal{P}_m \setminus \{\emptyset\}$. Without loss of generality we assume that $l \in \{1, \ldots, k\}$ is such that the functions $f_1, \ldots, f_l$ do not depend on y_j for $j \in J$ and $f_{l+1}, \ldots, f_k$ depend on some y_j with $j \in J$. By the definition of $\bar{f}_j$ for $j = l+1, \ldots, k$, $\bar{f}_j(\bar{x}, \bar{y}) = 0$ for $(\bar{x}, \bar{y}) \notin [-1,1]^{n+m}$ so T_J is of the form

$$\{(\bar{x}, \bar{y}) \in [-1,1]^n \times \mathbb{R}^m : \left(\bigwedge_{j\in J} y_j \in \mathbb{R} \setminus [-1,1]\right) \wedge \left(\bigwedge_{j\in\{1,\ldots,m\}\setminus J} y_j \in [-1,1]\right), \\ P\left(\bar{x}, \bar{y}, \bar{f}_1(\bar{x}, \bar{y}), \ldots, \bar{f}_l(\bar{x}, \bar{y}), 0, \ldots, 0\right) = 0\}.$$

We name the elements of the sets: $J = \{j_1, \ldots, j_p\}$ and $\{1, \ldots, m\} \setminus J = \{n_1, \ldots, n_{m-p}\}$. We use the notation $\hat{y}$ for an m-tuple of variables with $y_j = 0$ for all $j \in J$, and we let $\bar{z} = (z_1, \ldots, z_l)$ denote an l-tuple of variables. Consider the set

$$V_J = \{(\bar{x}, \bar{y}, \bar{z}) \in [-1,1]^n \times \mathbb{R}^m \times \mathbb{R}^l : \\ \left(\bigwedge_{j\in J} y_j \in \mathbb{R} \setminus [-1,1]\right) \wedge \left(\bigwedge_{j\in\{1,\ldots,m\}\setminus J} y_j \in [-1,1]\right) \wedge P(\bar{x}, \bar{y}, \bar{z}, 0 \ldots, 0) = 0\}.$$

Then

$$U_J := \{(\bar{x}, y_{n_1}, \ldots, y_{n_{m-p}}, \bar{z}) \in [-1,1]^n \times \mathbb{R}^{m-p} \times \mathbb{R}^l : \\ \exists y_{j_1}, \ldots, \exists y_{j_p} \bigwedge_{j=1}^m (j \notin I \iff y_j \in [-1,1]), \\ P(\bar{x}, \bar{y}, \bar{z}, 0, \ldots, 0) = 0\}$$

is the projection of V_J on the coordinates $\bar{x}, y_{n_1}, \ldots, y_{n_{m-p}}, \bar{z}$.

Since U_J is a semialgebraic set, by Remark 2.1.6 there exists a polynomial Q in $n + m - p + l + q$ variables such that

$$U_J = \{(\bar{x}, y_{n_1}, \ldots, y_{n_{m-p}}, \bar{z}) \in [-1,1]^n \times \mathbb{R}^{m-p} \times \mathbb{R}^l : \\ \exists \bar{w} \in [-1,1]^q, Q(\bar{x}, y_{n_1}, \ldots, y_{n_{m-p}}, \bar{z}, \bar{w}) = 0\}.$$

The set

$$S_J = \{(\bar{x}, y_{n_1}, \ldots, y_{n_{m-p}}) \in [-1,1]^n \times \mathbb{R}^{m-p} : \\ \exists \bar{w} \in [-1,1]^q, Q\left(\bar{x}, y_{n_1}, \ldots, y_{n_{m-p}}, \bar{f}_1(\bar{x}, \hat{y}), \ldots, \bar{f}_l(\bar{x}, \hat{y}), \bar{w}\right) = 0\}$$

is the projection of T_J on the coordinates $\bar{x}, y_{n_1}, \ldots, y_{n_{m-p}}$. The variables $y_{n_1}, \ldots, y_{n_{m-p}} \in [-1,1]$ so the set

$$R_J = \{(\bar{x}, y_{n_1}, \ldots, y_{n_{m-p}}, \bar{w}) \in [-1,1]^n \times \mathbb{R}^{m-p} \times [-1,1]^q : \\ Q\left(\bar{x}, y_{n_1}, \ldots, y_{n_{m-p}}, \bar{f}_1(\bar{x}, \hat{y}), \ldots, \bar{f}_l(\bar{x}, \hat{y}), \bar{w}\right) = 0\}$$

is a subset of $[-1,1]^{n+m-p+q}$ and

$$\Pi_n T_J = \Pi_n R_J.$$

Adding dummy variables if needed, we can assume that q does not depend on J. Let $\bar{R}_J := R_J \times \{0\}^p \subseteq [-1,1]^{n+m+q}$ for $J \neq \emptyset$, $\bar{R}_\emptyset = T_\emptyset \times \{0\}^q$ and $S = \bigcup_{I\in\mathcal{P}} \bar{R}_I$. Then $S \subseteq [-1,1]^{n+m+q}$ and $\Pi_n S = A$. □

Remark 5.2.2. Let $A \subseteq \mathbb{R}^n$ be a bounded set, definable in $\mathbb{R}_{\mathcal{A}}$ for some $\mathcal{A}$ as above.

We can find $(a_1, \ldots, a_n) \in \mathbb{R}^n$ and $(b_1, \ldots, b_n) \in (\mathbb{R}^+)^n$ such that the image $\phi(A)$ of A by the map $\phi : (x_1, \ldots, x_n) \mapsto (a_1 + b_1x_1, \ldots, a_n + b_nx_n)$ is a subset of $(0, 1)^n$. By Lemma 5.2.1 there exists a bounded semianalytic set $S \subseteq \mathbb{R}^{n+m}$ such that $\Pi_n S = \phi(A)$; then $A = \Pi_n\phi^{-1}(S)$. Since the set $\phi^{-1}(S)$ is bounded and $\mathcal{A}$-semianalytic, Theorem 5.1.5 ensures that A can be parametrized by means of functions in $\mathcal{A}$: in other words every bounded definable set in $\mathbb{R}_{\mathcal{A}}$ can be parametrized by means of functions in $\mathcal{A}$.

Going back to the main question of the thesis, we would like to identify o-minimal expansions of the real field that admit mild parametrization (definable or not). Since most known polynomially bounded examples of expansions of the real field can be considered as RS-structures and Theorem 5.1.5 gives us information about how the bounded definable sets of a given RS-structure $\mathbb{R}_{\mathcal{A}}$ are parametrized by maps whose coordinates are elements of $\mathcal{A}$, we hope to get general knowledge about the polynomially bounded structures that admit mild parametrization by exploring necessary and sufficient conditions on $\mathcal{A}$.

We first consider the case where all the functions $f : (0, 1)^n \to (0, 1)$ in $\mathcal{A}$ are mild and prove that $\mathbb{R}_{\mathcal{A}}$ admits mild parametrization for this case.

Proposition 5.2.3. *Let $\mathcal{A}$ be an RS-class. If all the functions $f : (0, 1)^n \to (0, 1)$ in the class $\mathcal{A}$ are mild, for all $n \in \mathbb{N}$, then the structure $\mathbb{R}_{\mathcal{A}}$ admits definable mild parametrization.*

Proof. Let $A \subseteq (0, 1)^n$ for some $n \in \mathbb{N}$ be a bounded definable set in $\mathbb{R}_{\mathcal{A}}$ with $\dim(A) = d$. We want to show that there exist $l \in \mathbb{N}$ and mild maps $\psi_1, \ldots, \psi_l : (0, 1)^d \to (0, 1)^n$ definable in $\mathbb{R}_{\mathcal{A}}$, such that $\bigcup_{i=1}^{l} \mathrm{Im}(\psi_i) = A$.

By Lemma 5.2.1 there exists an $\mathcal{A}$-semianalytic set $S \subseteq [-1, 1]^{n+m}$ for some $m \in \mathbb{N}$ such that

$$\Pi_n(S) = A.$$

Let σ be a permutation $\{1, \ldots, n+m\}$ that exchanges $\{1, \ldots, n\}$ and $\{1+m, \ldots, n+m\}$, keeping the order; see σ as acting on the set of coordinates of $\mathbb{R}^{n+m}$ and let $T := \sigma(S)$. The set T is still semianalytic and $\Pi^n(T) = A$. By Theorem 5.1.5 there exists $N \in \mathbb{N}$ and, for all $i = 1, \ldots, N$, there exist $n_i, m_i \in \mathbb{N}$ with $n_i + m_i = n + m$, $r_i \in [0, 1)^{n+m}$, a subquadrant $Q_i \subseteq \hat{I}_{n_i,m_i,r_i}$, and a map $H_i : \hat{I}_{n_i,m_i,r_i} \to \mathbb{R}^{n+m}$ whose components are in $\mathcal{A}_{n_i,m_i,r_i}$ such that $H_i\restriction_{Q_i} : Q_i \to H_i(Q_i)$ is a diffeomorphism and

$$A = \Pi^n(T) = \bigcup_{i=N}^{l} H_i(Q_i).$$

Write

$$Q_i = B_{i,1} \times \ldots \times B_{i,n+m},$$

$r_i = (r_{i,1} \dots r_{i,n+m})$, and $d_i = \dim(Q_i)$. Then $\max\{d_i : i = 1 \dots N\} = \dim A = d$, since each $H_i\restriction_{Q_i}$ is a diffeomorphism.
Let

$$P_i = \{l \in \{1, \dots, n_i + m_i\} : B_{i,l} = (0, r_{i,l})\},$$
$$N_i = \{l \in \{1, \dots, n_i + m_i\} : B_{i,l} = (-r_{i,l}, 0)\},$$
$$Z_i = \{l \in \{1, \dots, n_i + m_i\} : B_{i,l} = \{0\}\}.$$

Note that $d_i = |P_i| + |N_i|$. Let σ_i be the permutation of $\{1, \dots, n + m\}$ elements that takes the elements of P_i to the set $\{1, 2, \dots, |P_i|\}$, the elements of N_i to the set $\{|P_i| + 1, \dots, d_i\}$ and the elements of Z_i to the set $\{d_i + 1, \dots, n_i + m_i\}$, preserving the order within each of these sets.
Define the function $\phi_i : (0, 1)^d \to Q_i$ such that each coordinate function ϕ_{ij} of ϕ_i for $j = 1, \dots, n + m$ is defined as

$$\phi_{ij} : t \mapsto \begin{cases} r_{ij} t_{\sigma_i(j)} & j \in P_i \\ -r_{ij} t_{\sigma_i(j)} & j \in R_i \\ 0 & j \in Z_i \end{cases}$$

Consider the composition map $H_i \circ \phi_i : (0, 1)^d \to H_i(Q_i)$. For each $i = 1, \dots, N$ the component functions of H_i are in $\mathcal{A}$. Since $\mathcal{A}$ consists of mild functions and ϕ_is are linear functions, the component functions of $H_i \circ \phi_i$ are also mild by Proposition 3.1.9. In other words $H_i \circ \phi_i$ is mild. Also we have

$$A = \Pi^n(T) = \bigcup_{i=1}^{N} H_i(Q_i) = \bigcup_{i=1}^{N} \mathrm{Im}(H_i \circ \phi_i).$$

Therefore A has mild parametrization. □

5.3 Mild parametrization in $\mathbb{R}_{\mathcal{G}}$

Gevrey functions have a special importance in branches of partial and ordinary differential equations. In [25] van den Dries and Speissegger consider Tougeron's work in [74], defining a special class $\mathcal{G}$ of Gevrey functions and proving several properties of this class. Then they define the expansion $\mathbb{R}_{\mathcal{G}}$ of the real ordered field by $\mathcal{G}$ and prove that it is polynomially bounded and o-minimal.

In [33], the authors asked whether the structure $\mathbb{R}_{\mathcal{G}}$ admits mild parametrization or not. In this section we will recall the formal definition of $\mathbb{R}_{\mathcal{G}}$ and give a positive answer to this question using the results of the previous section.

In [25], Gevrey functions are defined as complex valued functions on sectors. The ones that send real values to real values are considered to form the Gevrey class $\mathcal{G}$. For simplicity we give first the definition of one variable Gevrey functions. We use the notations from [25]. For $R > 0$, $0 < \phi < \pi$ and $0 < \kappa \leq 1$, we define the open sector:

$$S = S(R, \phi, \kappa) := \{z \in \mathbb{C} : 0 < |z| < R, |\arg z| < \kappa\phi\}.$$

Definition 5.3.1. *Let $R > 0$, $0 < \phi < \pi$ and $0 < \kappa \leq 1$. The class $\mathcal{G}(R, \phi, \kappa)$ is the set of all holomorphic functions $f : S \to \mathbb{C}$ on S satisfying:*

1. *(Gevrey condition) there exist constants $A, B > 0$ (depending on f) such that*

$$\left|f^{(n)}(z)\right| \leq (n!)^{1+\kappa} A B^n$$

   for all $n \in \mathbb{N}$ and $z \in S$;

2. *$\lim_{z\to 0} f^{(n)}(z)$ exists in $\mathbb{C}$ for each $n \in \mathbb{N}$.*

The class $\mathcal{G}(R)$ is the collection of all real valued functions $f : [0, R] \to \mathbb{R}$ for which there exist $\tilde{R} > R$, $\phi \in (\pi/2, \pi)$, $\kappa_1, \ldots, \kappa_n \in (0, 1]$, and real valued functions $f_i \in \mathcal{G}(\tilde{R}, \phi, \kappa_i)$ for $i = 1, \ldots n$ on $[0, \tilde{R})$ such that

$$f(x) = f_1(x) + \ldots + f_n(x)$$

for all $x \in [0, R]$.

Proposition 5.3.2. *The restriction to $(0, 1)$ of any function $g : [0, 1] \to (0, 1)$ in $\mathcal{G}(1)$ is mild.*

Proof. By definition of $\mathcal{G}(1)$, we can find a natural number l, some $R > 1$, some $\phi \in (\pi/2, \pi)$ and for each $i \in \{1, \ldots, l\}$ some $f_i \in \mathcal{G}(R, \phi, \kappa_i)$ with $\kappa_i \in (0, 1]$, such that $(f_1 + \ldots + f_l)\restriction_{[0,1]} = g$.

By Definition 5.3.1, for each $i = 1, \ldots, l$, there are positive constant A_i and B_i such that

$$\left|f_i^{(n)}(z)\right| \leq (n!)^{1+\kappa_i} A_i B_i^n$$

for all $n \in \mathbb{N}$ and $z \in S(R, \phi, \kappa_i)$.

Let $\mathcal{B} > 0$ with $\sum_{i=1}^{l} A_i B_i^n \leq \mathcal{B}^n$ for all $n \in \mathbb{N}^+$ and let $\kappa = \max\{\kappa_1, \ldots, \kappa_l\}$.

Then

$$\left|g^{(n)}(x)\right| \leq \sum_{i=1}^{n} (n!)^{1+\kappa_i} A_i B_i^n \leq (n!)^{1+\kappa} \sum_{i=1}^{n} A_i B_i^n \leq n! \mathcal{B}^n n^{\kappa n}$$

for all $x \in (0, 1)$ and $n \in \mathbb{N}^+$. For $n = 0$, the same inequality holds since g has range in $(0, 1)$.

Hence $g\restriction_{(0,1)}$ is $(\mathcal{B}, \kappa)$-mild. □

In [25] van den Dries and Speissegger used an indirect generalization of $\mathcal{G}(R)$ to several variables due to Tougeron ([74]). We now recall the definition of several variable Gevrey functions given by van den Dries and Speissegger.

Let $\mathcal{F}_m$ be the collection of tuples $\tau = (K, R, r, \phi)$, where K is a finite nonempty subset of $(\{0\} \cup [1, \infty))^m$, $R = (R_1, \dots, R_m) \in (0, \infty)^m$, $r \in (1, \infty)$, and $\phi \in (\pi/2, \pi)$.

Given $\tau \in \mathcal{F}_m$, we define the polydisc:

$$D(R) := \{z \in \mathbb{C}^m : |z_i| < R_i \quad \text{for} \quad i = 1, \dots m\},$$

the generalised sector:

$$S(\tau) := \bigcap_{k \in K} \{z \in D(R) : k|\arg z| < \phi\},$$

and for each $p \in \mathbb{N}$

$$S_p(\tau) := \bigcap_{k \in K} \left(S(\tau) \cup \{z \in D(R) : |z^k| < \frac{R^k}{p+1}\} \right).$$

For each $p \in \mathbb{N}$ let $f_p : S_p(\tau) \to \mathbb{C}$ be a bounded holomorphic function. The equality $f =_\tau \sum f_p$ means $\sum f_p$ converges uniformly to a bounded and continuous function $f : S(\tau) \to \mathbb{C}$.

For each $\tau \in \mathcal{F}_m$, we define the set

$$\mathcal{G}_\tau := \{f : S(\tau) \to \mathbb{C} : f =_\tau \sum f_p \quad \text{for some sequence } (f_p : S_p(\tau) \to \mathbb{C})_{p \in \mathbb{N}} \text{ of bounded holomorphic functions}\}.$$

Let $[0, R] := [0, R_1] \times \dots [0, R_m]$. Then $\mathcal{G}(R)$ is defined to be the set of functions $f : [0, R] \to \mathbb{R}$ where there exists $\tilde{\tau} = (\tilde{K}, \tilde{R}, \tilde{r}, \tilde{\phi}) \in \mathcal{F}_m$ with $\tilde{R} > R$, and $\tilde{f} \in \mathcal{G}_{\tilde{\tau}}$ such that $f(x) = \tilde{f}(x)$ for all $x \in [0, R]$.

The class $\mathcal{G}$ consists of all the functions $f : \mathbb{R}^m \to \mathbb{R}$, for all $m \in \mathbb{N}$, such that $f\restriction_{[0,1]^m} \in \mathcal{G}(1, \dots, 1)$ and $f(x) = 0$ for all $x \notin [0, 1]^m$. And we define the expansion of the real field by the class $\mathcal{G}$ of Gevrey functions

$$\mathbb{R}_{\mathcal{G}} := (\mathbb{R}, +, -, \cdot, <, (f)_{f \in \mathcal{G}}).$$

Theorem 5.3.3. ([25, Theorem A]) *The structure $\mathbb{R}_{\mathcal{G}}$ is model complete, polynomially bounded and o-minimal.*

The collection of sets definable in $\mathbb{R}_{\mathcal{G}}$ contains the collection of definable sets in $\mathbb{R}_{\text{an}}$. However this inclusion is strict: the function ϕ on $(1, \infty)$ given by

$$\log \Gamma(x) = (x - \frac{1}{2}) \log x - x + \frac{1}{2} \log(2\pi) + \phi(x)$$

is noted in [25] as an example of a function definable in $\mathbb{R}_{\mathcal{G}}$ (following Nielsen [50]) but not in $\mathbb{R}_{\text{an}}$.

The class $\mathcal{G}$ as mentioned in [64] is an RS-class. The quasianaliticity of the class is proved ([25], Proposition 2.18) using the theory of multisummability which is an extension of the quasianalyticity property by Martinet, Ramis and Ecalle ([1],[46]). We just want to point out here that the choice of $\phi \in (\pi/2, \pi)$ is crucial for quasianalyticity. Other properties that the class $\mathcal{G}$ should satisfy (Properties 5.1.1) to conclude that it is an RS-class are established in [25] in Sections 4 and 5.

We now present Lemma 2.6 from [25], from which it follows that the restriction to $(0,1)^m$ of any function $f : \mathbb{R}^m \to (0,1)$ in $\mathcal{G}$ is mild.

Let $f \in \mathcal{G}_\tau$. We define

$$\|f\|_\tau := \inf\{\sum \|f_p\|_{S_p} \cdot r^p\}$$

where the infimum is taken over all the sequences (f_p) such that $f =_\tau \sum f_p$. Note that for all $f \in \mathcal{G}_\tau$, $\|f\|_\tau \in \mathbb{R}$.

Lemma 5.3.4. ([25, Lemma 2.6]) *Let $\tau = (K, R, r, \phi), \tilde{\tau} = (\tilde{K}, \tilde{R}, \tilde{r}, \tilde{\phi}) \in \mathcal{F}_m$ with $\tilde{R} < R$, $1 < \tilde{r} < r$, and $0 < \tilde{\phi} < \phi$. Let $f \in \mathcal{G}_\tau$. Then $f\restriction_{S(\tilde{\tau})}$ is C^∞ and there are constants $A, B > 0$ independent of f (depending on τ) such that*

$$|f^{(\alpha)}(z)| \leq \alpha! AB^{|\alpha|}|\alpha|^{\kappa|\alpha|}\|f\|_\tau$$

for all $\alpha \in \mathbb{N}^m$ and $z \in S(\tilde{\tau})$, where κ is $\max\{\frac{1}{|k|} : k \in K, k \neq 0\}$ if $K \neq \{0\}$ and 1 otherwise.

For any $m \in \mathbb{N}$ and for all $f : (0,1)^m \to (0,1)$ in $\mathcal{G}_\tau$, we can find $\mathcal{B} > 0$ such that $AB^{|\alpha|}\|f\|_\tau \leq \mathcal{B}^{|\alpha|}$ for all $\alpha \in \mathbb{N}^m$. Hence, by the above lemma, f is $(\mathcal{B}, \kappa)$-mild.

Theorem 5.3.5. *The structure $\mathbb{R}_{\mathcal{G}}$ admits mild parametrization.*

Proof. The theorem follows from Proposition 5.2.3 since $\mathbb{R}_{\mathcal{G}}$ is an RS-structure and all the functions $f : (0,1)^n \to (0,1)$ in $\mathcal{G}$ are mild functions by Lemma 5.3.4. □

An application. We will use a specific example of a function in $\mathcal{G}$ given in [25] and apply Theorem 5.3.5 to prove that Wilkie's conjecture holds for a specific surface defined by means of this function.

Let $f : \mathbb{C} \setminus (-\infty, 0) \to \mathbb{C}$ be defined by

$$f : z \mapsto \int_0^\infty \frac{e^{-t}}{1+zt} dt := \lim_{T \to \infty} \int_0^T \frac{e^{-t}}{1+zt} dt.$$

The authors in [25] state that the function f is in $\mathcal{G}(R, \varphi, 1)$ for any $R > 0$ and $\varphi \in (0, \pi)$. This statement can be also found in [1][Section 3.4, Exercise 3]. For more information about this function reader can check [29], Section 2.4. Note that f is definable in $\mathbb{R}_{\mathcal{G}}$ but not in $\mathbb{R}_{\text{an}}$.

Proposition 5.3.6. *The function*

$$\begin{array}{rccc} g: & (0,\infty) & \to & \mathbb{R} \\ & x & \mapsto & \displaystyle\int_0^\infty \frac{e^{-t}}{1+xt}dt \end{array}$$

is Pfaffian of order 2 *and degree* $(3, 1)$ *on* $(0, \infty)$.

Proof. By the change of variables $v = \dfrac{x}{1+xt}$, we have

$$g(x) = \int_0^\infty \frac{e^{-t}}{1+xt}dt = \frac{e^{1/x}}{x}\int_0^x \frac{e^{-1/v}}{v}dv.$$

Since $v \mapsto \dfrac{e^{-1/v}}{v}$ is analytic on $(0, \infty)$ the antiderivative $\displaystyle\int_0^x \frac{e^{-1/v}}{v}dv$ is also analytic on $(0, \infty)$. Hence g is analytic on $(0, \infty)$. Consider the functions $f(x) = 1/x$ and g on $(0, \infty)$. Then

$$f'(x) = (1/x)' = -1/x^2 = P_1(x, f(x))$$

where $P_1(x, y) = -y^2$ and

$$\begin{aligned} g'(x) = \left(\frac{e^{1/x}}{x}\int_0^x \frac{e^{-1/v}}{v}dv\right)' &= \left(\frac{e^{1/x}}{x}\int_0^x \frac{e^{-1/v}}{v}dv\right)\cdot\left(-\frac{1}{x^2}-\frac{1}{x}\right)+\frac{1}{x^2} \\ &= P_2(x, f(x), g(x)) \end{aligned}$$

where $P_2(x, y, z) = -zy^2 - zy + y^2$. So (f, g) is a Pfaffian chain of order 2 and degree 3. We have $g(x) = P(x, f(x), g(x))$ where $P(x, y, z) = z$ so g is a Pfaffian function of order 2 and degree $(3, 1)$. □

Using the function g defined in Proposition 5.3.6, we define the function

$$\begin{array}{rccc} \psi: & (0,1) & \to & (0,1) \\ & x & \mapsto & \dfrac{g(x)-1}{g(1)-1} \end{array}$$

and the surface

$$\mathcal{X} = \{(x, y, z) \in (0, 1)^3 : z = \psi(x)\psi(y)\}.$$

Note that since ψ is not definable in $\mathbb{R}_{\text{an}}$ by Theorem 11 in [21], $\mathcal{X}$ is not definable in $\mathbb{R}_{\text{an}}$ either, so we cannot apply the results of Jones, D. Miller and Thomas ([33]) to $\mathcal{X}$. On the other hand, we affirm Wilkie's conjecture for the surface $\mathcal{X}$ in Proposition 5.3.8 using a result that can be extracted from the proof of a theorem of Jones and Thomas in [34]. We first state this theorem:

Theorem 5.3.7. ([34, Proof of Theorem 5.4]) *Let $\mathcal{R} = (\overline{\mathbb{R}}, f_1, \ldots, f_l)$ be an expansion of the real field by a Pfaffian chain $(f_1, \ldots, f_l)$ on $\mathbb{R}^n$ and let $X \subseteq (0,1)^3$ be the graph of an analytic function $F : (0,1)^2 \to (0,1)$ that is existentially definable in $\mathcal{R}$. If X has mild parametrization then there exist positive constants c_1 and c_2 (both depending on X) such that*

$$N(X^{tr}, H) \leq c_1 (\log H)^{c_2}.$$

This theorem applies to the surface $\mathcal{X}$.

Proposition 5.3.8. *There exist positive constants c_1 and c_2 such that $N(\mathcal{X}^{tr}, H) \leq c_1 (\log H)^{c_2}$.*

Proof. The function g is analytic on $(0, \infty)$, so $\mathcal{X}$ is the graph of an analytic function with domain $(0,1)^2$. Furthermore $\mathcal{X}$ is quantifier-free definable, hence existentially definable, in the structure $\mathcal{R} = (\overline{\mathbb{R}}, f, g)$ where (f, g) is the Pfaffian chain introduced in the proof of Proposition 5.3.6 . Also $\mathcal{X}$ is definable in $\mathbb{R}_{\mathcal{G}}$ so has definable mild parametrization by Theorem 5.3.5. These are exactly the hypotheses needed to apply Theorem 5.3.7, so there exist positive constants c_1 and c_2 such that $N(\mathcal{X}^{tr}, H) \leq c_1(\log H)^{c_2}$. □

Chapter 6

Quasianalytic Denjoy-Carleman classes

One way to define classes of infinitely differentiable functions is to impose growth conditions on the derivatives of the functions in terms of specific sequences. In this chapter we focus on a special type of these classes called quasianalytic Denjoy-Carleman classes which is also a special case of the $\mathbb{R}_{\mathcal{C}}$-structures we examined in Chapter 4. Our interest in quasianalytic Denjoy-Carleman classes originates from the paper of Rolin, Spiessegger and Wilkie [65]. The authors of [65] proved that expansions of the real ordered field by quasianalytic Denjoy-Carleman classes are polynomially bounded and o-minimal.

We will give general information and historical notes about classes of infinitely differentiable functions associated to sequences in Section 6.1. In Section 6.2 we define Denjoy-Carleman sequences, which we use to define quasianalytic Denjoy-Carleman classes, and we examine some properties of these sequences. Then we focus on quasianalytic Denjoy-Carleman classes and examine properties of these classes in Section 6.3. In Section 6.4 we will present and discuss the work in [65] about the expansions of the real ordered field by quasianalytic Denjoy-Carleman classes.

The work in this chapter is motivated by the following question.

Question 6.0.1. *Do expansions of the real ordered field by quasianalytic Denjoy-Carleman classes admit mild parametrization?*

Since expansions of the real field by Denjoy-Carleman classes are special case of the $\mathbb{R}_{\mathcal{C}}$-structures a positive answer to Question 6.0.1 would follow from Proposition

4.3.2 if each function $f : (0,1)^n \to (0,1)$ in any quasianalytic Denjoy-Carleman class is mild. For that reason, in Section 6.5, we investigate quasianalytic Denjoy-Carleman classes in order to understand the mildness properties of the functions in these classes. In Section 6.5, we give a condition on quasianalytic Denjoy-Carleman classes for which each $f : (0,1)^n \to (0,1)$ in the class is mild. In Section 6.6 we focus on the property of classes being closed under differentiation as well as being mild. We show that if each $f : (0,1)^n \to (0,1)$ in a quasianalytic Denjoy-Carleman class is mild then this class is closed under differentiation. The converse is not true: we also prove this by constructing an example. Then we consider a specific quasianalytic Denjoy-Carleman class studied in classical analysis and, applying our results, we show that the expansion of the real field by this class is a strict expansion of $\mathbb{R}_{\text{an}}$ and admits mild parametrization.

6.1 Classes of infinitely differentiable functions

The class of real analytic functions (C^ω) is a subclass of infinitely differentiable functions (C^∞) of great historical and mathematical importance. Considering the well known properties of these classes such as

1. any closed set is the zero set of a C^∞ function (Whitney Decomposition Theorem, see [75]) but a real analytic function is identically zero if it vanishes on a set with an accumulation point;

2. the power series expansion of a real analytic function is locally convergent but it is not true generically for C^∞ functions;

3. in the class C^ω every function is determined uniquely by its Taylor expansion whereas there are many distinct C^∞ functions that have the same Taylor expansion;

one can see that there is a big gap between these two classes. Since these classes both play an important role in PDE theory it is natural to try to understand if there are interesting classes that lie in this gap and to study if they are sharing some of the properties of the class C^ω or of the class C^∞. One way to define such classes is to specify some bounds on the derivatives of all orders of the functions in the class.

Definition 6.1.1. *Let* $M = (M_n)_{n\in\mathbb{N}}$ *be a sequence of positive real numbers. Let* B *be a compact subset of* $\mathbb{R}^k$*. We define* $C_B(M)$ *to be the class of* C^∞ *functions* $f : B \to \mathbb{R}$ *such that there is an open neighbourhood* U *of* B*, a* C^∞ *function* $g : U \to \mathbb{R}$ *with* $g\restriction_B = f$ *and a constant* $A > 0$ *such that*

$$|D^\alpha g(\bar{x})| \leq A^{|\alpha|+1}\alpha! M_{|\alpha|}$$

for all $\bar{x} \in U$ *and* $\alpha \in \mathbb{N}^k$*.*
We say $C_B(M)$ *is the class associated to* M *on* B*. We will be mostly interested in* $B = [-1,1]^k$ *for* $k \in \mathbb{N}^+$ *so we use the notation* $C_k(M)$ *for the class* $C_B(M)$ *with* $B = [-1,1]^k$*. We define the class associated to* M *as the union of all* $C_k(M)$ *for all* $k \in \mathbb{N}^+$*:*

$$C(M) = \bigcup_{k\in\mathbb{N}^+} C_k(M).$$

Example 6.1.2. Let $M := (1)_{n\in\mathbb{N}}$. Then the class $C(M)$ is the class of functions $f : [-1,1]^k \to \mathbb{R}$ that have an analytic extension on a neighbourhood of $[-1,1]^k$ by Fact 3.2.1.

We will present other examples of these classes in Section 6.3.

6.2 Denjoy-Carleman sequences

Some properties of a class $C(M)$, such as quasianalyticity or being closed under differentiation, follow from properties of the associated sequence M. In this section we will study some properties of sequences M which are of relevance for the corresponding classes of functions $C(M)$ purely at the level of the sequences themselves. This will provide tools to study the influence of these properties on the corresponding classes of functions $C(M)$ and provide examples that will be the subject of later sections of this chapter.

The characterization and the properties of the sequences we give in this section can be found in classical literature on this subject. For more information reader can see [2], [38], [39], [62], [63].

Definition 6.2.1. *Let* $M = (M_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers. The sequence* M *is called* convex *if*

$$M_n \leq \frac{M_{n-1} + M_{n+1}}{2}$$

for all $n \in \mathbb{N}^+$*. For the sequence* $(M_n)_{n\in\mathbb{N}}$ *with positive terms if* $(\log(M_n))_{n\in\mathbb{N}}$ *is convex then the sequence* M *is called* logarithmically convex*. We will use the abbreviation* log-convex *for logarithmically convex.*

Definition 6.2.2. *A nondecreasing and log-convex sequence of positive real numbers* $(M_n)_{n\in\mathbb{N}}$ *with* $M_0 \geq 1$ *is called a* Denjoy-Carleman sequence*.*

Definition 6.2.3. *Let* $M = (M_n)_{n\in\mathbb{N}}$ *be a sequence of positive real numbers. The sequence* $r^M = (r_n^M)_{n\in\mathbb{N}}$ *where* $r_n^M = \dfrac{M_{n+1}}{M_n}$ *for all* $n \in \mathbb{N}$ *is called the* growth sequence *of* M*.*

Lemma 6.2.4. *Let* $M = (M_n)_{n\in\mathbb{N}}$ *be a sequence of positive real numbers. Then* M *is log-convex if and only if* r^M *is nondecreasing.*

Proof. For a sequence M with positive real number terms,

$$\log(M_n) \leq \frac{\log(M_{n-1}) + \log(M_{n+1})}{2} \iff 2\log(M_n) \leq \log(M_{n-1}M_{n+1})$$

$$\iff (M_n)^2 \leq M_{n-1}M_{n+1} \iff \frac{M_n}{M_{n-1}} \leq \frac{M_{n+1}}{M_n},$$

hence the sequence r^M is indeed nondecreasing. □

Definition 6.2.5. *Let* $M = (M_n)_{n\in\mathbb{N}}$ *be a sequence of positive real numbers. The sequence* $\sqrt{M} = (\sqrt{M}_n)_{n\in\mathbb{N}}$ *where we define* $\sqrt{M}_0 = 1$ *and, for all* $n \in \mathbb{N}^+$*,* $\sqrt{M}_n = (M_n)^{\frac{1}{n}}$*, is called the* root sequence of M*.*

Lemma 6.2.6. *Let* $M = (M_n)_{n\in\mathbb{N}}$ *be a Denjoy-Carleman sequence with* $M_0 = 1$*. Then* $\sqrt{M}$*, the root sequence of* M*, is nondecreasing.*

Proof. For any $n \in \mathbb{N}$ the $(n+1)$st term of the sequence M can be expressed as the product below since $M_0 = 1$:

$$M_{n+1} = \frac{M_{n+1}}{M_n}\frac{M_n}{M_{n-1}}\cdots\frac{M_1}{M_0}. \tag{6.1}$$

The sequence M is log-convex so by Lemma 6.2.4 the growth sequence of M is nondecreasing. That is $\dfrac{M_{i+1}}{M_i} \leq \dfrac{M_{n+1}}{M_n}$ for all $0 \leq i \leq n$. Hence if we replace each fraction on the right hand side of equation (6.1) by $\dfrac{M_{n+1}}{M_n}$ we obtain the following inequality:

$$M_{n+1} \leq \left(\frac{M_{n+1}}{M_n}\right)^{n+1}.$$

Then we have $(M_n)^{\frac{1}{n}} \leq (M_{n+1})^{\frac{1}{n+1}}$ for an arbitrary $n \geq 1$. For $n = 0$, we have $\sqrt{M}_0 = 1 = M_0 \leq M_1 = \sqrt{M}_1$ (the first equality coming from the convention $\sqrt{M}_0 = 1$ and the inequality from the fact that the sequence M is nondecreasing).

Hence the sequence $\sqrt{M}$ is a nondecreasing sequence. □

Lemma 6.2.7. *Let* $M = (M_n)_{n\in\mathbb{N}}$ *be a Denjoy-Carleman sequence. Then* $M_jM_k \leq M_{j+k}$ *for all* $(j,k) \in \mathbb{N}^2$ *if and only if* $M_0 = 1$*.*

Proof. (⇐): Let $(j,k) \in \mathbb{N}^2$. The case where j or k is 0 is obvious. We assume that $j,k>0$. The inequality $M_jM_k \leq M_{j+k}$ we want to prove is symmetric for k and j so without loss of generality we assume that $j \leq k$. Since $\sqrt{M}$ is a nondecreasing sequence by Lemma 6.2.6, we have

$$(M_k)^{\frac{1}{k}} \leq (M_{j+k})^{\frac{1}{j+k}},$$

that is

$$(M_k)^{j+k} \leq (M_{j+k})^k,$$

so

$$(M_k)^j(M_k)^k \leq (M_{j+k})^k. \tag{6.2}$$

The fact that $\sqrt{M}$ is a nondecreasing sequence together with the assumption that $j \leq k$ also implies that

$$M_j^{\frac{1}{j}} \leq M_k^{\frac{1}{k}},$$

so

$$(M_j)^k \leq (M_k)^j. \tag{6.3}$$

Combining inequalities (6.2) and (6.3) we have

$$(M_j)^k(M_k)^k \leq (M_{j+k})^k.$$

Taking the kth roots of the above equation we get

$$M_jM_k \leq M_{j+k}.$$

(⇒): Take $j=k=0$. Then $(M_0)^2 \leq M_0$. Since M is a Denjoy-Carleman sequence $M_0 \geq 1$, so the inequality can be only true if $M_0 = 1$. □

Lemma 6.2.8. *Let $M = (M_n)_{n\in\mathbb{N}}$ be a log-convex sequence. Then $\left(r_k^M\right)^{j-k} \leq \dfrac{M_j}{M_k}$ for any $(j,k) \in \mathbb{N}^2$.*

Proof. We will prove this lemma by examining the cases $k<j$ and $k \geq j$ separately.
Case 1. We first assume $k<j$. Since M is log-convex, we have that r^M is nondecreasing by Lemma 6.2.4, and hence that:

$$\frac{M_j}{M_k} = \frac{M_j}{M_{j-1}}\frac{M_{j-1}}{M_{j-2}}\cdots\frac{M_{k+1}}{M_k} = r_{j-1}^M r_{j-2}^M \cdots r_k^M \geq (r_k^M)^{j-k}.$$

Case 2. We now assume that $k \geq j$. Since M is log-convex, we have that r^M is nondecreasing by Lemma 6.2.4, and hence that:

$$\frac{M_j}{M_k} = \frac{M_j}{M_{j+1}}\frac{M_{j+1}}{M_{j+2}}\cdots\frac{M_{k-1}}{M_k} = \frac{1}{r_j^M}\frac{1}{r_{j+1}^M}\cdots\frac{1}{r_{k-1}^M} \geq \left(\frac{1}{r_k^M}\right)^{k-j} = \left(r_k^M\right)^{j-k}.$$

□

Definition 6.2.9. *A sequence $(a_n)_{n\in\mathbb{N}}$ is called* almost increasing *if there exists a constant $\lambda \geq 1$ such that $a_i \leq \lambda a_j$ for all $j \geq i \geq 2$.*

The following lemma can be found in the appendix of [65] (Lemma 6.1 (1)) with a sketch of the proof. We give here a complete proof of it.

Lemma 6.2.10. *Let $M = (M_n)_{n\in\mathbb{N}}$ be a Denjoy-Carleman sequence with $M_0 = 1$. Then the sequence $m = (m_n)_{n\in\mathbb{N}}$ where $m_0 = m_1 = 1$ and $m_n = (M_n)^{\frac{1}{n-1}}$, for all $n > 1$, is almost increasing.*

Proof. We want to find $\lambda \geq 1$ such that $m_i \leq \lambda m_j$ for all $j \geq i \geq 2$. For $i = j$, $m_i \leq \lambda m_j$ for all $\lambda \geq 1$, so we assume $(j, i) \in \mathbb{N}^2$ with $j > i \geq 2$. Since M is a Denjoy-Carleman sequence, by Lemma 6.2.4, r^M is nondecreasing. So we have

$$M_{i+1} = \frac{M_{i+1}}{M_i}\frac{M_i}{M_{i-1}}\cdots\frac{M_2}{M_1}M_1 \leq \left(\frac{M_{i+1}}{M_i}\right)^i M_1$$

for all $i \geq 0$, which means that

$$(M_i)^i \leq (M_{i+1})^{i-1}M_1$$

for all $i \geq 0$. We may take the $i(i-1)$st root of each side for all $i \geq 2$, for which we get

$$(M_i)^{\frac{1}{i-1}} \leq (M_{i+1})^{\frac{1}{i}}(M_1)^{\frac{1}{i(i-1)}}. \tag{6.4}$$

Iterating the argument that gives (6.4) $j - i - 1$ further times we obtain

$$\begin{aligned}(M_i)^{\frac{1}{i-1}} \leq (M_{i+1})^{\frac{1}{i}}(M_1)^{\frac{1}{i(i-1)}} &\leq (M_{i+2})^{\frac{1}{i+1}}(M_1)^{\frac{1}{i(i-1)}+\frac{1}{(i+1)(i)}}\\ &\leq \ldots \leq (M_j)^{\frac{1}{j-1}}(M_1)^S \end{aligned} \tag{6.5}$$

for all $i \geq 2$, where $S = \sum_{n=0}^{j-i-1}\frac{1}{(i+n)(i+n-1)}$. Note that

$$\frac{1}{(i+n)(i+n-1)} = \frac{1}{i+n-1} - \frac{1}{i+n}$$

for all $i \geq 2$ and $n \in \mathbb{N}$. So

$$S = \sum_{n=0}^{j-i-1}\left(\frac{1}{i+n-1} - \frac{1}{i+n}\right) = \frac{1}{i-1} - \frac{1}{j-1} = \frac{j-i}{(i-1)(j-1)} \leq 1, \tag{6.6}$$

for all $j > i \geq 2$. Then the inequalities (6.5) and (6.6) imply that $(M_i)^{\frac{1}{i-1}} \leq (M_j)^{\frac{1}{j-1}}M_1$. Hence, taking $\lambda = M_1 \geq 1$ we have $m_i \leq \lambda m_j$ for all $j \geq i \geq 2$. □

6.3 Quasianalytic Denjoy-Carleman classes

In this section we will examine the classes $C(M)$ when the sequence M is a Denjoy-Carleman sequence.

Definition 6.3.1. *The class $C(M)$ is called the* Denjoy-Carleman class associated to *M if $M := (M_n)_{n\in\mathbb{N}}$ is a Denjoy-Carleman sequence.*

The following proposition is well known (see for example [66]). For the sake of completeness, we give here a proof of it. For any sequence $M = (M_n)_{n\in\mathbb{N}}$ and real number λ we use the notation λM to denote the sequence given by $(\lambda M)_n = \lambda M_n$ for all $n \in \mathbb{N}$.

Proposition 6.3.2. *Let M be a Denjoy-Carleman sequence and let δ be a positive real number. Then $C(M) = C(\delta M)$.*

Proof. The case when $\delta = 1$ is trivial. For $0 < \delta < 1$ the equality $C(M) = C(\delta M)$ can be expressed as $C(\frac{1}{\delta}N) = C(N)$ where $N = (N_n)_{n\in\mathbb{N}}$ with $N_0 = 1$ and $N_n = \delta M_n$ for all $n \in \mathbb{N}^+$. So we can assume that $\delta > 1$. The inclusion $C(M) \subseteq C(\delta M)$ is obvious since $M_n \leq \delta M_n$ for all $n \in \mathbb{N}$. Let $f : [-1,1]^k \to \mathbb{R}$ be in $C(\delta M)$. Then there exists an open neighbourhood U of $[-1,1]^k$, a C^∞ function $g : U \to \mathbb{R}$ with $g \restriction_{[-1,1]^k} = f$ and a constant $A > 0$ such that

$$|D^\alpha g(\bar{x})| \leq A^{|\alpha|+1}\alpha!\delta M_{|\alpha|}$$

for all $\bar{x} \in U$ and $\alpha \in \mathbb{N}^k$. Then

$$|D^\alpha g(\bar{x})| \leq (A\cdot\delta)^{|\alpha|+1}\alpha! M_{|\alpha|}$$

for all $\bar{x} \in U$ and $\alpha \in \mathbb{N}^k$. Hence $f \in C(M)$. □

Remark 6.3.3. In the rest of the text, whenever we consider a Denjoy-Carleman sequence $M = (M_n)_{n\in\mathbb{N}}$, without loss of generality we will assume $M_0 = 1$, since the classes associated to the sequences $(M_n)_{n\in\mathbb{N}}$ and $\left(\frac{M_n}{M_0}\right)_{n\in\mathbb{N}}$ are equal by Proposition 6.3.2.

The following lemma states that in each Denjoy-Carleman class there exists a function with derivatives as big as the bounds authorize. This lemma is a variant of a classical result of Cartan and Mandelbrojt ([12]) and the function we give in the proof as an example is an example of Thilliez in [72] which originates from the work of Bang in [2].

Lemma 6.3.4. *Let $M := (M_n)_{n\in\mathbb{N}}$ be a Denjoy-Carleman sequence and let $\overline{M} := (\overline{M_n})_{n\in\mathbb{N}}$ where $\overline{M_n} = n!M_n$ for all $n \in \mathbb{N}$, and let $\overline{r} = (\overline{r}_n)_{n\in\mathbb{N}}$ denote the growth sequence of $\overline{M}$. We define the function $\phi(x) = \sum_{n=0}^{\infty}\phi_n(x)$ for $x \in \mathbb{R}$ where*

$$\begin{array}{rccl} \phi_n : & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & \dfrac{\overline{M_n}}{(2\overline{r}_n)^n}(\cos(2\overline{r}_n x) + \sin(2\overline{r}_n x)). \end{array}$$

Then the function $\psi := \phi \restriction_{[-1,1]}$ is in $C(M)$ and $|\psi^{(j)}(0)| \geq j!M_j$, for all $j \in \mathbb{N}$.

Proof. For all $j \in \mathbb{N}$,

$$\phi_n^{(j)}(x) = \frac{\overline{M_n}}{(2\overline{r}_n)^{n-j}}(s_j \cos(2\overline{r}_n x) + t_j \sin(2\overline{r}_n x))$$

for some $(s_j, t_j) \in \{-1, 1\}^2$.

Note that since M is a log-convex sequence, the sequence r^M is nondecreasing by Lemma 6.2.4 hence the sequence $\overline{r}$ is also nondecreasing, which implies that $\overline{M}$ is also log-convex by Lemma 6.2.4. Since $\overline{r}$ is the growth sequence of $\overline{M}$, we also have $\overline{r_n}^{j-n}\overline{M_n} \leq \overline{M_j}$ by Lemma 6.2.8, for any $(j, n) \in \mathbb{N}^2$. So we have

$$|\phi_n^{(j)}(x)| \leq 2\frac{\overline{M_n}}{(2\overline{r}_n)^{n-j}} \leq \frac{\overline{M_j}}{2^{n-j-1}} \tag{6.7}$$

for all $(j, n) \in \mathbb{N}^2$ and all $x \in \mathbb{R}$. The formal sum $\sum_{n=0}^{\infty} \phi_n^{(j)}(x)$ is therefore absolutely convergent for all $j \in \mathbb{N}$. Hence by an easy induction argument and [67, Theorem 7.17], ϕ is C^j for all $j \in \mathbb{N}$, so it is C^∞.

Similarly the inequality (6.7) ensures that

$$|\phi^{(j)}(x)| \leq 2^{j+1}\overline{M_j}\sum_{n=0}^{\infty}\frac{1}{2^n} = 2^{j+2}\overline{M_j} \leq 4^{j+1}\overline{M_j},$$

for all $j \in \mathbb{N}$.

Hence the function $\psi := \phi \restriction_{[-1,1]}$ belongs to $C(M)$.

On the other hand if we look at the values of the derivatives of ϕ at $x = 0$, we get that

$$|\phi^{(j)}(0)| = \sum_{n=0}^{\infty}\frac{\overline{M_n}}{(2\overline{r}_n)^{n-j}}.$$

Since all the terms in the series are positive, the sum is larger than its jth term. Therefore $|\psi^{(j)}(0)| = |\phi^{(j)}(0)| \geq \overline{M_j} = j!M_j$. □

The following theorem is also a classical result in analysis which gives a necessary and sufficient condition for inclusion of two Denjoy-Carleman classes.

Theorem 6.3.5. *Let $M = (M_n)_{n\in\mathbb{N}}$ and $N = (N_n)_{n\in\mathbb{N}}$ be Denjoy-Carleman sequences. There exists a constant $\lambda > 0$ such that $M_n \leq \lambda^n N_n$ for all $n \in \mathbb{N}$ if and only if $C(M) \subseteq C(N)$.*

Proof. ($\Rightarrow$): Let $f : [-1, 1]^k \to \mathbb{R}$ be a function in $C(M)$, so there exists an open neighbourhood U of $[-1, 1]^k$, a C^∞ function $g : U \to \mathbb{R}$ with $g \restriction_{[-1,1]^k} = f$ and a constant $A > 0$ such that

$$|D^\alpha g(\bar{x})| \leq A^{|\alpha|+1}\alpha!M_{|\alpha|}$$

for all $\bar{x} \in U$ and $\alpha \in \mathbb{N}^k$. Let $\lambda > 0$ be such that $M_n \leq \lambda^n N_n$ for all $n \in \mathbb{N}$. Then,

$$|D^\alpha g(\bar{x})| \leq A^{|\alpha|+1}\alpha!\lambda^{|\alpha|}N_{|\alpha|} \leq (A \cdot (\lambda + 1))^{|\alpha|+1}\alpha!N_{|\alpha|}$$

for all $\bar{x} \in U$ and $\alpha \in \mathbb{N}^k$. So f belongs to $C(N)$, and therefore $C(M) \subseteq C(N)$.
($\Leftarrow$): Assume for a contradiction that, for all $\lambda > 0$, there is $n_\lambda \in \mathbb{N}$ such that $\lambda^{n_\lambda} N_{n_\lambda} < M_{n_\lambda}$. Since $M_0 = N_0 = 1$, any such n_λ should be nonzero.

By Lemma 6.3.4, there is a C^∞ function ψ in $C(M)$ such that $|\psi^{(j)}(0)| \geq j!M_j$ for all $j \in \mathbb{N}$. So we have

$$|\psi^{(n_\lambda)}(0)| > n_\lambda!\lambda^{n_\lambda} N_{n_\lambda}$$

for all $\lambda > 0$. Hence there exists no $A > 0$ such that $\forall n \in \mathbb{N}, |\psi^{(n)}(0)| \leq A^{n+1}n!N_n$. Since such A would have to satisfy $A > \lambda^{\frac{n_\lambda}{n_\lambda+1}}$ for all λ, and in particular for all $\lambda > 1$ (we would have $\lambda > 1$, since $n_\lambda > 0$ and $n \mapsto \frac{n}{n+1}$ is increasing), $A > \lambda^{\frac{n_\lambda}{n_\lambda+1}} \geq \lambda^{\frac{1}{1+1}} = \sqrt{\lambda}$ which is not possible. Hence ψ does not belong to the class $C(N)$. $\square$

In this thesis we are interested in Denjoy-Carleman classes which are quasianalytic. Recall that a class of C^∞ functions is called quasianalytic if each germ of a function in the class is fully determined among the germs of functions in the class by its Taylor series (see 10. in Properties 5.1.1). For further information about quasianalytic classes the reader can see [4], [71], and [72].

Historically, the notion of quasianaliticity was first introduced at the very begining of the twentieth century by Borel's example of classes that contain infinitely differentiable functions, are nowhere analytic on the real line, and have the property that any function in the class, with all derivatives vanishing at 0, is identically the zero function ([8], [9]). Ten years after, Hadamard asked if it is possible to find a growth condition on the derivatives of the functions in the class that would imply quasianaliticity ([28]). Following this question a sufficient condition for a Denjoy-Carleman class to be quasianalytic was given by Denjoy [17] and then necessary and sufficient conditions were given by Carleman [11]. Later, other characterizations by Ostrowski [51] and Mandelbrojt [43] were given as well. The famous theorem that contains all of these characterizations is usually named as the Denjoy-Carleman Theorem in the literature. We present here a partial version of this theorem which will be enough for our purposes. We will not give the proof of this result here; the reader may find various proofs in [31], [36], [39], and [68].

Theorem 6.3.6 (Denjoy-Carleman)**.** ([39, Theorem 4.1.15]) *Let $M = (M_n)_{n\in\mathbb{N}}$ be a Denjoy-Carleman sequence. Then the following are equivalent.*

1. *$C(M)$ is a quasianalytic class.*

2. $\displaystyle\sum_{n=0}^{\infty} \frac{M_n}{M_{n+1}(n+1)} = \sum_{n=0}^{\infty} \frac{1}{r_n^M(n+1)} = \infty.$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n!M_n)^{\frac{1}{n}}} = \infty.$

In the rest of the text we will use the abbreviation QADC classes for quasianalytic Denjoy-Carleman classes.

Examples 6.3.7. 1. As we mentioned in Example 6.1.2 the Denjoy-Carleman class $C(M)$ associated to the sequence $M = (1)_{n\in\mathbb{N}}$ is the class of functions on $[-1,1]^k$, for $k \in \mathbb{N}$, which have an analytic extension on a neighbourhood of $[-1,1]^k$. It is well known that this class is quasianalytic. We can also conclude that this class is quasianalytic using Theorem 6.3.6 part 2: the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ (harmonic series) diverges to ∞.

2. Other examples of QADC classes were introduced by Denjoy ([17]) while partially answering Hadamard's question. These are the classes associated to the sequences $L^\alpha = (L_n^\alpha)_{n\in\mathbb{N}}$ for $0 \leq \alpha \leq 1$, where $L_0^\alpha = L_1^\alpha = 1$ and $L_n^\alpha = (\log(n+1))^{\alpha(n+1)}$ for $n \geq 2$. One can check that the derivative of the function
$$x \mapsto \frac{(\log(x+1))^{\alpha(x+1)}}{(\log x)^{\alpha x}}$$
is positive for all $\alpha \geq 0$ and for all $x > e$ so r^{L^α}, the growth sequence of L^α, is nondecreasing for all $n \geq 2$. Note also that $r_0^{L^\alpha} = 1 < r_1^{L^\alpha} = (\log 3)^{3\alpha} < r_2^{L^\alpha} = \frac{(\log 4)^{4\alpha}}{(\log 3)^{3\alpha}}$ (since $(\log 3)^3 < 1.4$ and $\frac{(\log 4)^4}{(\log 3)^3} > 2.7$). Then, by Lemma 6.2.4, the sequence L^α is a log-convex sequence for all $\alpha \geq 0$. Moreover L^α is a Denjoy-Carleman sequence since it is nondecreasing and $L_0^\alpha = 1$. Hence $C(L^\alpha)$ is a Denjoy-Carleman class. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n!(\log(n+1))^{\alpha(n+1)})^{\frac{1}{n}}}$ is equal to the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n!)^{\frac{1}{n}}(\log(n+1))^{\alpha}(\log(n+1))^{\alpha/n}}$. Since $(n!)^{1/n} \leq n \leq n+1$ and $\lim\limits_{n\to\infty} (\log(n+1))^{1/n} = 1$ we have,
$$\frac{1}{(n!)^{\frac{1}{n}}(\log(n+1))^{\alpha}(\log(n+1))^{\alpha/n}} \geq \frac{1}{(n+1)(\log(n+1))^{\alpha}2}$$
for sufficiently large n. The Bernstein series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^\alpha}$ diverges to ∞, for $0 \leq \alpha \leq 1$ (see [67]), so by Theorem 6.3.6 Part 3, the classes $C(L^\alpha)$ are quasianalytic for $0 \leq \alpha \leq 1$.

For the last paragraph of this section, we examine the stability of Denjoy-Carleman classes $C(M)$ under taking derivatives.

Definition 6.3.8. *We say that a class of functions is closed under differentiation if any derivative of each function in this class belongs as well to this class.*

Generally a Denjoy-Carleman class $C(M)$ need not be closed under differentiation. The necessary and sufficient condition for a Denjoy-Carleman class to be closed under differentiation was given by Mandelbrojt ([44]). Before we state Mandelbrojt's Theorem we give the following lemma that we use in order to give the proof of this theorem.

Lemma 6.3.9. *Let $M = (M_n)_{n\in\mathbb{N}}$ and $N = (N_n)_{n\in\mathbb{N}}$ be Denjoy-Carleman sequences. There exists a constant $\lambda > 0$ such that $M_n \leq \lambda^n N_n$ for all $n \in \mathbb{N}$ if and only if* $\sup_{n\in\mathbb{N}^+} \left(\frac{M_n}{N_n}\right)^{\frac{1}{n}} < \infty$.

Proof. ($\Rightarrow$): Since N is a sequence of positive real numbers we have $\frac{M_n}{N_n} \leq \lambda^n$ for all $n \in \mathbb{N}$. Therefore

$$\sup_{n\in\mathbb{N}^+} \left(\frac{M_n}{N_n}\right)^{\frac{1}{n}} \leq \sup_{n\in\mathbb{N}^+} (\lambda^n)^{\frac{1}{n}} = \lambda.$$

($\Leftarrow$): Let $s \in \mathbb{R}$ be such that $\sup_{n\in\mathbb{N}^+} \left(\frac{M_n}{N_n}\right)^{\frac{1}{n}} = s$. Since M and N are nondecreasing sequences and $M_0 = N_0 = 1$, we have $M_n, N_n \geq 1$ for all $n \in \mathbb{N}$, and therefore $s > 0$. Then, for all $n \in \mathbb{N}^+$, $M_n \leq s^n N_n$; for $n = 0$, we have $M_0 = N_0 = 1$ hence $M_0 \leq N_0$. So for $\lambda = \max\{s, 1\}$, $M_n \leq \lambda^n N_n$ for all $n \in \mathbb{N}$. □

Theorem 6.3.10. ([44, Theorem 6.9.I]) *Let $M = (M_n)_{n\in\mathbb{N}}$ be a Denjoy-Carleman sequence. The class $C(M)$ is closed under differentiation if and only if*

$$\sup_{n\in\mathbb{N}^+} \left(\frac{M_{n+1}}{M_n}\right)^{\frac{1}{n}} < \infty.$$

Proof. Let $f \in C_k(M)$. Then $\frac{\partial f}{\partial x_i} \in C_k(M^{+1})$ for all $i = 1, \ldots, k$ where $M_n^{+1} := M_{n+1}$ for all $n \in \mathbb{N}$. So the class $C(M)$ is closed under differentiation if and only if $C(M^{+1}) \subseteq C(M)$. By Theorem 6.3.5 and Lemma 6.3.9 this inclusion is equivalent to $\sup_{n\in\mathbb{N}^+} \left(\frac{M_n^{+1}}{M_n}\right)^{\frac{1}{n}} = \sup_{n\in\mathbb{N}^+} \left(\frac{M_{n+1}}{M_n}\right)^{\frac{1}{n}} < \infty$. □

6.4 QADC structures

In Section 4.2 we discussed the classes of functions defined by Rolin, Speissegger and Wilkie in [65], which they used to establish a method to construct o-minimal structures. We remind the reader that they proved that the expansions of the real ordered field by these classes are polynomially bounded and o-minimal (Theorem 4.2.2). In this section we present the main ideas and results of [65]. In [65], the authors consider classes, $\mathcal{C}$ of functions obtained by taking the union of some QADC classes, and they show that such $\mathcal{C}$ satisfy the condition (C1)-(C7) that we recalled in Section 4.2. Then they concluded that the expansion of the field of the reals by the functions in $\mathcal{C}$ is model-complete, o-minimal and polynomially bounded.

One of the properties that classes defined in [65] should satisfy in order for the expansion of the field of the reals by this class to be o-minimal is being closed under differentiation. As we mention in the previous section, QADC classes are not always closed under differentiation. Therefore the authors in [65] define the following classes.

Definition 6.4.1. *Let $C(M)$ be a QADC class associated to the sequence $M = (M_n)_{n\in\mathbb{N}}$. For all $j \in \mathbb{N}$, we define new sequences $M^{+j} = (M_n^{+j})_{n\in\mathbb{N}}$ by shifting the terms of the sequence M, that is*

$$M_n^{+j} = M_{n+j}$$

for all $n \in \mathbb{N}$. We define the shifted union of $C(M)$ to be the class

$$C^+(M) := \bigcup_{j\in\mathbb{N}} C(M^{+j}).$$

Note that for all $f \in C(M)$, $f^{(j)}$ is in $C(M^{+j})$. Therefore $C^+(M)$ is closed under differentiation.

Definition 6.4.2. *Let $C(M)$ be a QADC class. For all $k \in \mathbb{N}$ and each $f : [-1,1]^k \to \mathbb{R}$ in $C^+(M)$ we define*

$$\bar{f}(\bar{x}) = \begin{cases} f(\bar{x}) & \bar{x} \in [-1,1]^k \\ 0 & \textit{otherwise.} \end{cases}$$

The structure

$$R_{C^+(M)} = \big(\mathbb{R}, +, -, \cdot, 0, 1, <, \{\bar{f}\}_{f\in C^+(M)}\big)$$

is called a QADC structure.

The authors of [65] also show (in the introduction and the appendix) that the shifted unions of QADC classes satisfy all the other properties specified for a class $\mathcal{C}$ so that the structure $\mathbb{R}_{\mathcal{C}}$ is polynomially bounded and o-minimal, and hence they obtain the following theorem.

Theorem 6.4.3. ([65, Theorem 1]) *Let $C(M)$ be a QADC class. Then the structure $\mathbb{R}_{C^+(M)}$ is model complete, o-minimal, polynomially bounded and admits C^∞ cell decomposition.*

In [65], QADC structures are used to answer the following question.

Question 6.4.4. *Does there exist a largest o-minimal expansion of the real field such that any o-minimal expansion is a reduct of it?*

First, they proved the following theorem.

Theorem 6.4.5. ([65, Theorem 2]) *Let U be an open neighbourhood of $[-1,1]^k$ and let $f : U \to \mathbb{R}$ be a C^∞ function. Then there exist QADC classes $C(M)$ and $C(N)$ and functions $f_1 \in C(M)$ and $f_2 \in C(N)$ such that $f(\bar{x}) = f_1(\bar{x}) + f_2(\bar{x})$ for all $\bar{x} \in [-1,1]^k$.*

Then they combined Theorem 6.4.5 with Theorem 6.4.3 to give a negative answer to Question 6.4.4.

Corollary 6.4.6. *There are QADC classes $C(M)$ and $C(N)$ such that the QADC structures $\mathbb{R}_{C^+(M)}$ and $\mathbb{R}_{C^+(N)}$ are not reducts of a common o-minimal expansion of the real field. So there exists no largest o-minimal expansion of the real field.*

Note that Theorem 6.4.5 can be seen as evidence that the quasianalytic Denjoy-Carleman classes are not rare. Also this theorem would provide us knowledge about the functions that have mild parametrization which we explain in the following remark.

Remark 6.4.7. If we assume that there is a C^∞ function $f : (0,1) \to (0,1)$ which does not have mild parametrization then, by Theorem 6.4.5, there are QADC classes $C(M)$ and $C(N)$ and functions $\varphi \in C(M)$ and $\psi \in C(N)$ such that $f(x) = \varphi(x) + \psi(x)$ for all $x \in (0,1)$. Defining the map $g(x) = (\varphi(x), x)$ and the function $h(y,z) = y + \psi(z)$ we can see f as the composition of h and g. If we also assume that all the functions in any QADC class admit mild parametrization then we can conclude that the compositions of functions that have mild parametrization does not always have mild parametrization. On the other hand if we assume that

1. any composition of functions that have mild parametrization also has mild parametrization,
2. all the functions in any QADC class admit mild parametrization

then the above argument implies that any C^∞ function $f : (0,1) \to (0,1)$ has mild parametrization.

6.5 Mildness in QADC classes

In this section we examine QADC classes in terms of their mildness properties.

Definition 6.5.1. *A class C of C^∞ functions is called a* mild class *if for all $k \in \mathbb{N}$ and for all $f : U \to \mathbb{R}$ in C with $(0,1)^k \subseteq U \subseteq \mathbb{R}^k$ and $f\left((0,1)^k\right) \subseteq (0,1)$, the restriction of f to $(0,1)^k$ is mild.*

For any QADC class $C(M)$ the class $C^+(M)$ fits in the framework of classes we examined in Chapter 4. So if the class $C^+(M)$ is a mild class then the structure $\mathbb{R}_{C^+(M)}$ admits mild parametrization by Theorem 4.3.2.

There is a similarity between the mildness property for a C^∞ function $f\colon (0,1)^k \to (0,1)$ and the fact of belonging to a certain $C(M)$ class: namely, f is a mild function if and only if f is in a class $C(M)$ where $M = (M_n)_{n\in\mathbb{N}}$ is given by $M_0 = 1$ and $\forall n \in \mathbb{N}^+, M_n = n^{nc}$ for some $c \geq 0$. Therefore, for such a sequence M, the class $C(M)$ is a mild class. The following lemma states that such classes are Denjoy-Carleman classes but they are not quasianalytic unless $c = 0$. Note that when $c = 0$ the class $C(M)$ is the class of analytic functions in Example 6.1.2.

Lemma 6.5.2. *Let $c \geq 0$ and let $M^c := (M^c_n)_{n\in\mathbb{N}}$ be the real sequence where $M^c_0 = 1$ and $M^c_n = n^{cn}$ for all $n \in \mathbb{N}^+$. The class $C(M^c)$ is a Denjoy-Carleman class and it is quasianalytic if and only if $c = 0$.*

Proof. For any $c \geq 0$, let r^c be the growth sequence of M^c. Then $r^c_0 = 1$ and $r^c_n = \left(\dfrac{n+1}{n}\right)^{cn}(n+1)^c$ for all $n \in \mathbb{N}^+$. This sequence r^c is nondecreasing so by Lemma 6.2.4, M^c is a log-convex sequence. M^c is also nondecreasing and $M^c_0 = 1$, therefore it is a Denjoy-Carleman sequence. This means that $C(M^c)$ is a Denjoy-Carleman class. For the quasianalyticity of the class $C(M^c)$ we consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{M^c_n}{M^c_{n+1}(n+1)}$. We have

$$\sum_{n=0}^{\infty} \frac{M^c_n}{M^c_{n+1}(n+1)} = \sum_{n=0}^{\infty} \left(\frac{n}{n+1}\right)^{cn} \frac{1}{(n+1)^{c+1}} < \sum_{n=0}^{\infty} \frac{1}{(n+1)^{c+1}} = \sum_{n=1}^{\infty} \frac{1}{n^{c+1}}.$$

The well-known series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{c+1}}$ is convergent if $c + 1 > 1$ (see for example [67], Theorem 3.28), therefore the series $\displaystyle\sum_{n=0}^{\infty} \frac{M^c_n}{M^c_{n+1}(n+1)}$ is also convergent for $c > 0$. Hence by Theorem 6.3.6 we conclude that $C(M^c)$ is not quasianalytic for $c > 0$. On the other hand if $c = 0$ the sequence is given by $M^0_n = 1$ for all $n \in \mathbb{N}$ and in this case, the Denjoy-Carleman class associated to this sequence is quasianalytic (see Examples 6.3.7, 1.). □

In the next lemma we prove that sequences $M = (M_n)_{n\in\mathbb{N}}$ with the property that there is an $N \in \mathbb{N}$ and $c > 0$ such that $M_n \geq n^{cn}$, for all $n \geq N$, also do not give rise to quasianalytic classes.

Lemma 6.5.3. *Let $C(M)$ be a QADC class. For all $c > 0$ and for all $N \in \mathbb{N}$, there exists $n > N$ such that $M_n < n^{cn}$.*

Proof. For a contradiction, we assume that there exist $c > 0$ and $N_c \in \mathbb{N}$ such that $M_n \geq n^{cn}$ for all $n \geq N_c$. Then

$$\sum_{n=N_c}^{\infty} \frac{1}{(M_n n!)^{\frac{1}{n}}} \leq \sum_{n=1}^{\infty} \frac{1}{(n^{cn} n!)^{\frac{1}{n}}}.$$

By the Stirling Approximation Inequality

$$n! \geq \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$$

for all $n \in \mathbb{N}^+$ so we have

$$\frac{1}{(n^{cn} n!)^{\frac{1}{n}}} \leq e \cdot \frac{1}{n^{1+\frac{1}{2n}+c}} \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{1}{n}}.$$

for all $n \in \mathbb{N}^+$. Since for all $n \in \mathbb{N}^+$, $\left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{1}{n}} \leq 1$ and $\frac{1}{n^{1+\frac{1}{2n}+c}} \leq \frac{1}{n^{1+c}}$ we have

$$\frac{1}{(n^{cn} n!)^{\frac{1}{n}}} \leq \frac{e}{n^{1+c}}$$

for all $n \in \mathbb{N}^+$. The p-series $\sum_{n=1}^{\infty} \frac{e}{n^{1+c}}$ is convergent so $\sum_{n=1}^{\infty} \frac{1}{(n^{cn} n!)^{\frac{1}{n}}}$ is convergent. Hence $\sum_{n=N_c}^{\infty} \frac{1}{(M_n n!)^{\frac{1}{n}}}$ is convergent as well, so by Theorem 6.3.6 the class $C(M)$ is not quasianalytic which gives a contradiction. □

Now we consider the classes associated to those sequences that are bounded above by sequences $M^c = (n^{nc})_{n\in\mathbb{N}}$ after some $N_c \in \mathbb{N}$.

Definition 6.5.4. *Let $c \geq 0$. A sequence $M = (M_n)_{n\in\mathbb{N}}$ of positive real numbers is called a* c-mild sequence *if there exists $N_c \in \mathbb{N}$ such that $M_n \leq n^{cn}$, for all $n \geq N_c$. We say that a sequence is* mild *if it is c-mild for some $c \geq 0$.*

Remark 6.5.5. If a sequence $M = (M_n)_{n\in\mathbb{N}}$ is c-mild for some $c \geq 0$, then the class $C(M)$ is a mild class, with moreover the property that all the mild functions in $C(M)$ are c-mild. We see this as follows, let $f : [-1, 1]^k \to \mathbb{R}$ be a function in $C(M)$. Then there exists $A > 0$ such that $|D^\alpha f(\bar{x})| \leq A^{|\alpha|+1} \alpha! M_{|\alpha|}$ for all $\alpha \in \mathbb{N}^k$ and all $\bar{x} \in (0, 1)^k$. If M is c-mild then there is $N_c \in \mathbb{N}$ such that $M_n \leq n^{cn}$ for all $n \geq N_c$. Then choosing a big enough constant B we have

$$|D^\alpha f(\bar{x})| \leq B^{|\alpha|} \alpha! |\alpha|^{c|\alpha|}$$

for all $\alpha \in \mathbb{N}^k$ and all $\bar{x} \in (0, 1)^k$.

If M is a c-mild sequence then the class $C(M)$ will be mild (see Remark 6.5.5). We show below, in Proposition 6.5.7, that the converse is also true. That is, we show that if $C(M)$ is a mild class, then there exists $c > 0$ such that M is c-mild. We first state the following corollary (to Lemma 6.3.4) which introduces a function ψ in a given Denjoy-Carleman class $C(M)$ with $\psi(0,1) \subseteq (0,1)$ having derivatives as big as the bounds on the derivatives authorize.

Corollary 6.5.6. *Let $C(M)$ be a Denjoy-Carleman class. We define the function*

$$\begin{array}{rccc} \Psi: & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & \dfrac{\phi(x-1/2)+5}{10} \end{array}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is the function defined in Lemma 6.3.4. Let $\psi := \Psi\restriction_{[-1,1]}$. Then ψ is in $C(M)$, $\psi(0,1) \subset (0,1)$ and $|\psi^{(j)}(1/2)| \geq \dfrac{j!M_j}{10}$, for all $j \in \mathbb{N}$.

Proof. By inequality (6.7) in the proof of Lemma 6.3.4, we have

$$|\phi_n(x)| \leq \frac{\overline{M_0}}{2^{n-1}}$$

for all $n \in \mathbb{N}$ and hence

$$|\phi(x)| \leq \sum_{n=0}^{\infty} |\phi_n(x)| \leq 2\overline{M_0} \sum_{n=0}^{\infty} \frac{1}{2^n} = 4\overline{M_0} = 4.$$

Therefore the image of ϕ is included in $[-4, 4]$. So we have $\mathrm{Im}(\Psi) \subseteq (0,1)$ by the definition of Ψ.

Since ψ is bounded on any bounded neighbourhood of $[0,1]$ and since $|\Psi^{(j)}(x)| = \dfrac{1}{10}|\phi^{(j)}(x-1/2)|$ for all $j \in \mathbb{N}^+$ and $x \in \mathbb{R}$, we have that $\psi = \Psi\restriction_{[-1,1]}$ is in $C(M)$.

By Lemma 6.3.4, $|\phi^{(j)}(0)| \geq j!M_j$ for all $j \in \mathbb{N}$, so we have

$$|\psi^{(j)}(1/2)| \geq \frac{1}{10} j!M_j$$

for all $j \in \mathbb{N}$. □

Proposition 6.5.7. *Let $C(M)$ be a mild Denjoy-Carleman class. Then M is c-mild for some $c \geq 0$.*

Proof. Consider the function ψ in $C(M)$ defined in Corollary 6.5.6. We recall that $\psi(0,1) \subset (0,1)$ and $|\psi^{(j)}(1/2)| \geq \dfrac{j!M_j}{10}$, for all $j \in \mathbb{N}$ by Corollary 6.5.6. Since $C(M)$ is mild, $\psi\restriction_{(0,1)}$ is k-mild for some $k \geq 0$. Let $b > 0$ such that

$$|\psi^{(n)}(1/2)| \leq n!b^n n^{kn}$$

for all $n \in \mathbb{N}$. But

$$|\psi^{(n)}(1/2)| \geq \frac{1}{10} n!M_n$$

for all $n \in \mathbb{N}$. Hence, $M_n \leq 10b^n n^{kn}$ for all $n \in \mathbb{N}$. Let $c \geq 0$ be such that for all $n \geq 2$, and $10b^n n^{kn} \leq n^{cn}$. Then $M_n \leq n^{cn}$ for all $n \geq 2$, therefore M is c-mild. □

6.6 Closure under differentiation

Now we will examine the interaction between the two properties of Denjoy-Carleman classes: being closed under differentiation and being mild.

Definition 6.6.1. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers. We say that $(a_n)_{n\in\mathbb{N}}$ is* polynomially bounded *if there exists $d \geq 0$ and $N_d \in \mathbb{N}$ such that, $a_n \leq n^d$ for all $n \geq N_d$. We say that $(a_n)_{n\in\mathbb{N}}$ is* exponentially bounded *if there exists $s > 0$ and $N_s \in \mathbb{N}$ such that $a_n \leq s^n$ for all $n \geq N_s$.*

Lemma 6.6.2. *Let $M = (M_n)_{n\in\mathbb{N}}$ be a mild Denjoy-Carleman sequence. Then the growth sequence r^M of M is polynomially bounded.*

Proof. By Remark 6.5.5, there exists $c \geq 0$ and $N_c \in \mathbb{N}$ such that $M_n \leq n^{cn}$ for all $n \geq N_c$. We claim that $r_n^M \leq (2n)^{2c}$, for all $n \geq N_c$. For a contradiction, assume that there exists $n' \geq N_c$ such that $r_{n'}^M > (2n')^{2c}$. Since M is a Denjoy-Carleman sequence, by Lemma 6.2.4, r^M is nondecreasing and $M_n \geq 1$ for all $n \in \mathbb{N}$, and so we have

$$M_{2n'} \geq \frac{M_{2n'}}{M_{n'}} = \frac{M_{2n'}}{M_{2n'-1}} \cdots \frac{M_{n'+1}}{M_{n'}} \geq \left(\frac{M_{n'+1}}{M_{n'}}\right)^{n'} = \left(r_{n'}^M\right)^{n'} > (2n')^{2cn'}$$

and this gives a contradiction since $2n' \geq N_c$. Hence

$$\frac{M_{n+1}}{M_n} \leq (2n)^{2c} \leq n^{4c}$$

for all $n \geq N_c$, so r^M is polynomially bounded. □

We prove in the next lemma that the converse of Lemma 6.6.2 is true if the class $C(M)$ associated to the sequence $M = (M_n)_{n\in\mathbb{N}}$ is quasianalytic.

Lemma 6.6.3. *Let $C(M)$ be a QADC class, and let $r^M = (r_n^M)_{n\in\mathbb{N}}$ be the growth sequence of M. Then M is mild if and only if r^M is polynomially bounded.*

Proof. By Lemma 6.6.2, it remains to show that if r^M is polynomially bounded then M is mild. Let $d \geq 0$ and $N_d \in \mathbb{N}$ be such that $r_n^M \leq n^d$, for all $n \geq N_d$. We want to show that there exists $c \geq 0$ and $N_c \in \mathbb{N}$ such that $M_n \leq n^{cn}$, for all $n \geq N_c$. By Lemma 6.5.3 there exists $n_d \in \mathbb{N}$ with $n_d > N_d$ such that $M_{n_d} \leq n_d^{dn_d}$. We will pick $c = d$ and $N_c = n_d$ and proceed by an inductive argument on n. For the initial case $n = n_d$. For the inductive step we assume that $M_n \leq n^{dn}$ for some $n \geq n_d$. Since $n > n_d > N_d$, we have $r_n^M = \dfrac{M_{n+1}}{M_n} \leq n^d$ and since $e^d > 1$ it follows that $\dfrac{M_{n+1}}{M_n} \leq e^d n^d$. The sequence $\left(1 + \dfrac{1}{n}\right)^{(n+1)}$ is a decreasing sequence that converges to e as n tends to infinity. So we have

$$M_{n+1} \leq e^d n^d M_n \leq \left(1 + \frac{1}{n}\right)^{d(n+1)} n^d n^{dn} = (n+1)^{d(n+1)}.$$

Hence, $M_n \leq n^{cn}$ for all $n \geq N_c$ where $c = d$ and $N_c = n_d$. □

Theorem 6.6.4. *Let $C(M)$ be a mild Denjoy-Carleman class. Then $C(M)$ is closed under differentiation.*

Proof. By Proposition 6.5.7, M is a mild sequence. Then by Lemma 6.6.2, r^M is polynomially bounded which means there exist $d \geq 0$ and $N_d \in \mathbb{N}$ such that $\frac{M_{n+1}}{M_n} \leq n^d$ for all $n \geq N_d$. There exists $N \in \mathbb{N}$ such that $n^d \leq d^n$ for all $n \geq N$. Put $N_s = \max\{N, N_d\}$. Then for all $n \geq N_s$ we have

$$\sup_{n\in\mathbb{N}^+} \left(\frac{M_{n+1}}{M_n}\right)^{\frac{1}{n}} \leq \max\left(d, \frac{M_2}{M_1}, \ldots \left(\frac{M_{N_s}}{M_{N_s-1}}\right)^{\frac{1}{N_s-1}}\right) \leq d.$$

Therefore $C(M)$ is closed under differentiation by Theorem 6.3.10. □

After proving that mild Denjoy-Carleman classes are closed under differentiation, since we are interested in QADC classes, we ask whether all QADC classes which are closed under differentiation are mild. We give a negative answer to this question in the following proposition by providing an example of a QADC class which is closed under differentiation, and containing functions which are not mild.

Proposition 6.6.5. *There exists a QADC class $C(M)$ which is closed under differentiation and not mild.*

Proof. We first make the following claim.
Claim. There is a nondecreasing sequence of positive real numbers $r = (r_n)_{n\in\mathbb{N}}$ with $r_n \geq 1$ for all $n \in \mathbb{N}$ which satisfies the following properties:

(1) r is not polynomially bounded;

(2) r is exponentially bounded;

(3) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{r_n(n+1)} = \infty.$

Assuming the claim we define the nondecreasing sequence $M = (M_n)_{n\in\mathbb{N}}$ where $M_0 = 1$ and $M_n = \prod_{i=0}^{n-1} r_i$, for all $n \in \mathbb{N}^+$. In this case the growth sequence of M will be r and r is nondecreasing, so M is a Denjoy-Carleman sequence by Lemma 6.2.4. The class $C(M)$ is quasianalytic by Theorem 6.3.6 since

$$\sum_{n=0}^{\infty} \frac{M_n}{M_{n+1}(n+1)} = \sum_{n=0}^{\infty} \frac{1}{r_n(n+1)} = \infty.$$

Since r is exponentially bounded

$$\sup_{n\in\mathbb{N}^+} \left(\frac{M_{n+1}}{M_n}\right)^{\frac{1}{n}} = \sup_{n\in\mathbb{N}^+} (r_n)^{\frac{1}{n}} < \infty,$$

so $C(M)$ is closed under differentiation by Theorem 6.3.10. The sequence r is not polynomially bounded so M is not a mild sequence by Lemma 6.6.3. Hence $C(M)$ is not a mild class by Proposition 6.5.7.

Now we prove the claim by constructing a sequence r satisfying the required properties. We construct r in such a way that for some consecutive indices it grows like 2^n, so that r is not polynomially bounded (but still stays exponentially bounded), and for some consecutive indices it grows like $\log n$, so that r satisfies the property (3). And it alternates infinitely many times between these two behaviours.

We construct the sequence r inductively by defining (also inductively) two sequences $a = (a_i)_{i\in\mathbb{N}}$ and $b = (b_i)_{i\in\mathbb{N}}$ of natural numbers with $a_i < b_i < a_{i+1}$ for all $i \in \mathbb{N}$. Let $a_0 = 0$ and $r_0 = 1$. Given $i \in \mathbb{N}$, and given a_i and r_{a_i}, we put

$$r_{a_i+k} = \log(e^{r_{a_i}} + k)$$

for $1 \leq k \leq \alpha_i$, where α_i is the smallest positive integer α that satisfies the inequality

$$\sum_{n=a_i}^{a_i+\alpha} \frac{1}{r_n(n+1)} > 1.$$

Such α_i exists since the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ is divergent. We put b_i to be $a_i + \alpha_i$.

Given $i \in \mathbb{N}$, and given b_i and r_{b_i}, we put

$$r_{b_i+k} = r_{b_i} 2^k,$$

for $1 \leq k \leq \beta_i$, where β_i is the smallest positive integer β that satisfies

$$r_{b_i+\beta} = r_{b_i} 2^\beta > (b_i + \beta)^i.$$

Such β_i exists since the left hand side of the inequality grows exponentially in β but the right hand side grows polynomially in β. Then we choose a_{i+1} to be $b_i + \beta_i$. The sequence r is not polynomially bounded because for all $i \in \mathbb{N}$, $r_{b_i+\beta_i} > (b_i+\beta_i)^i$. Now we use induction on n to prove that $r_n \leq 2^n$ for all $n \in \mathbb{N}$. For the initial step we take $n = 0$: $r_0 = 1 \leq 2^0$. Now we assume that $r_n \leq 2^n$ for all $n \leq N$. There are two cases:
Case 1: There is $i \in \mathbb{N}$ such that $a_i \leq N < b_i$. Then since $a + b \leq ab$ for $a, b \geq 2$, we have

$$r_{N+1} = \log\left(e^{r_{a_i}} + N + 1 - a_i\right) \leq r_{a_i} 2^{N+1-a_i} \leq 2^{N+1}.$$

Case 2: There is $i \in \mathbb{N}$ such that $b_i \leq N < a_{i+1}$. Then,

$$r_{N+1} = r_{b_i} 2^{N+1-b_i} \leq 2^{b_i} 2^{N+1-b_i} = 2^{N+1}.$$

Since for all $i \in \mathbb{N}$ the sum $\sum_{n=a_i}^{a_i+\alpha_i} \frac{1}{r_n(n+1)}$ is strictly greater than 1, we have $\sum_{n=0}^{\infty} \frac{1}{r_n(n+1)} = \infty.$ □

An Application: We consider the class $C(L^1)$ where $L^1_0 = L^1_1 = 1$ and $L^1_n = (\log(n+1))^{(n+1)}$ for all $n \geq 2$. Note that $C(L^1)$ is an example of the classes $C(L^\alpha)$ which we mentioned in Examples 6.3.7(2). We remind the reader that, in Examples 6.3.7(2), we showed that the classes $C(L^\alpha)$ are QADC classes for $0 \leq \alpha \leq 1$. Here we show that $\mathbb{R}_{C(L^1)}$, expansion of the real field by $C(L^1)$ admits definable mild parametrization.

Proposition 6.6.6. *The structure $\mathbb{R}_{C(L^1)}$ admits definable mild parametrization.*

Proof. For all $c > 0$ there is a $N_c \in \mathbb{N}$ such that $(\log(n+1))^{n+1} \leq n^{cn}$ for all $n \geq N_c$ so L is c-mild sequence for all $c > 0$. Hence the class $C(L^1)$ is mild by Remark 6.5.5. Moreover, by Theorem 6.6.4, the class $C(L^1)$ is closed under differentiation. Hence $C^+(L^1) = C(L^1)$ is a mild QADC class. Finally $\mathbb{R}_{C(L^1)}$ is an o-minimal expansion of the real field by Theorem 6.4.3 and admits mild parametrization by Proposition 4.3.2. □

Remark 6.6.7. Note that $1 \leq 2(\log(n+1))^{n+1} = 2L^1_n$ for all $n \geq 2$, and $L^1_0 = L^1_1 = 1$. So $C(1) \subseteq C(L^1)$ by Theorem 6.3.5. Therefore the structure $\mathbb{R}_{C(L^1)}$ is an expansion of $\mathbb{R}_{C(1)} = \mathbb{R}_{\text{an}}$. On the other hand

$$\sup_{n\in\mathbb{N}^+} \left(\frac{(\log(n+1))^{n+1}}{1}\right)^{\frac{1}{n}} = \infty$$

so the class $C(1) \neq C(L^1)$ by Lemma 6.3.9 and Theorem 6.3.5. This means that there are functions in $C(L^1)$ which are not in $\mathbb{R}_{\text{an}}$. For example, consider the function $\varphi : (0,1) \to (0,1)$ defined by
$\varphi(x) = \frac{1}{10}\left(5 + \sum_{n=1}^{\infty} n! \log(n+1) \left(\frac{\log(n+1)}{2a_n}\right)^n \left(\cos\left(2a_n\left(x - \frac{1}{2}\right)\right) + \sin\left(2a_n\left(x - \frac{1}{2}\right)\right)\right)\right)$
where,

$$a_n = \frac{(n+1)!L^1_{n+1}}{n!L^1_n} = \frac{(n+1)(\log(n+2))^{n+2}}{(\log(n+1))^{n+1}}$$

for all $n \in \mathbb{N}^+$. By Lemma 6.3.4 and Corollary 6.5.6, φ is in $C(L^1)$ so it is mild since $C(L^1)$ is a mild class (see proof of Proposition 6.6.6). Again by Lemma 6.3.4 and Corollary 6.5.6, $|\varphi^{(j)}\left(\frac{1}{2}\right)| \geq \frac{j!(\log(j+1))^{j+1}}{10}$ for all $j \in \mathbb{N}^+$, so φ is not in $\mathbb{R}_{\text{an}}$. Hence φ is an example of a mild function which is definable in $\mathbb{R}_{C(L^1)}$ but not definable in $\mathbb{R}_{\text{an}}$.

Chapter 7

O-minimal structures without definable mild parametrization

The main subject of this chapter is o-minimal expansions of the real field for which definable mild parametrization is not possible. Towards Wilkie's conjecture the question that we ask is: is it possible to obtain a reparametrization result for the o-minimal expansions of the real field similar to Theorem 2.3.6 where the parametirzation is mild parametrization and the parametrizing functions are definable in the structure. In [73] Thomas gave a negative answer to this question for polynomially bounded expansions of the real ordered field by explicitly constructing a structure that does not admit definable mild parametrization. This result will be discussed in the Section 7.1.

Our result in this chapter also provides a negative answer to the question above for polynomially bounded expansions of the real ordered field using different methods than Thomas. We give a general condition on polynomially bounded o-minimal structures that ensures that they don't admit definable mild parametrization. We prove that any polynomially bounded structure with field of exponents containing irrational numbers does not admit definable mild parametrization. The field of exponents of the structure that Thomas constructed in [73] has field of exponents $\mathbb{Q}$. So our result is not a generalization of her result: the nonexamples of structures that we provide do not include her structure.

To prove our result we consider the power function $x^\alpha : (0,\infty) \to \mathbb{R}$ defined by

$$x \mapsto \begin{cases} x^\alpha & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and show that any set of mild functions that parametrize $x^\alpha \restriction_{(0,1)}$ contains functions that are not definable in polynomially bounded expansions of the real field. Hence our result is more general: it does not only provide examples of structures that don't admit definable mild parametrization but also provides examples of structures which don't admit mild parametrization by means of functions definable in any polynomially bounded structure. We will present our results in Section 7.2.

7.1 Example of Thomas

In [73], Thomas gave an example of a polynomially bounded expansion of the real field that does not admit definable mild parametrization. The result is actually stronger for two reasons: it is possible to relax the mildness condition to what is called G-mild parametrization; given such a definition of G-mild parametrization she proved the existence of a polynomially bounded o-minimal structure $\mathcal{R}$ and of a set X definable in $\mathcal{R}$, which is the graph of a one variable function, such that the set X does not admit G-mild parametrization by functions definable in $\mathcal{R}$; furthermore it is proven that the polynomially bounded structure $\mathcal{R}$ can be chosen to have analytic cell decomposition. We now give precise definitions of G-mild functions and definable G-mild parametrization.

Definition 7.1.1. *Let $G\colon \mathbb{N}^k \to (0,\infty)$ be a function. A smooth function $\phi\colon (0,1)^k \to (0,1)$ is said to be* G-mild *if there is an $N \in \mathbb{N}$ such that*

$$|D^\mu \phi(\bar{x})| \leq G(\mu)$$

for all $\mu \in \mathbb{N}^k$ with $|\mu| \geq N$ and all $\bar{x} \in (0,1)^k$. A map $\Phi : (0,1)^k \to (0,1)^d$ is said to be G-mild if each of its coordinate functions is G-mild.

Definition 7.1.2. *Let $\mathcal{R}$ be an o-minimal expansion of the real field and let $n \in \mathbb{N}$. A definable subset $X \subseteq (0,1)^n$ in $\mathcal{R}$ with $\dim(X) = d$ is said to have a* definable G-mild parametrization *in $\mathcal{R}$ if there is a finite collection of maps $\Phi_1, \dots, \Phi_l : (0,1)^d \to (0,1)^n$ each of which is G-mild and definable in $\mathcal{R}$ such that $\bigcup_{i=1}^l \operatorname{Im}(\Phi_i) = X$.*

Now we state the main result of Thomas in [73].

Theorem 7.1.3. ([73, Theorem 1.7]) *For every function $G : \mathbb{N} \to \mathbb{N}$, there is a polynomially bounded o-minimal expansion of the real field $\mathcal{R}$ and a 1-dimensional subset X of $\mathbb{R}^2$ definable in $\mathcal{R}$ such that X does not have a definable G-mild parametrization. Furthermore the structure $\mathcal{R}$ can be chosen to have analytic cell decomposition.*

We give brief idea of the proof of Theorem 7.1.3. Thomas first shows that having sufficiently large derivatives of high order at a sequence of points converging to zero forbids a function $H\colon (0,1) \to (0,1)$ to have G-mild parametrization by maps definable in any polynomially bounded o-minimal structure. It is then showed how to explicitly produce such a function and that the function can be chosen analytic. The next step of the proof is to take a small enough perturbation of the function to ensure that the perturbed function is generic in the sense of Le Gal ([40]), hence definable in a polynomially bounded structure while preserving its analyticity and the fact that it has large derivatives. Finally it is explained how the analytic Cell Decomposition Theorem for the structure follows from the analyticity of the function. The structure produced is the structure $\mathcal{R}$ in the theorem and the set X is the graph of the function.

Remark 7.1.4. As noted in [73], G-mild parametrization can be seen as a generalization of mild parametrization. Taking the function $G(n) = n!n^{n^2}$, if a set $X \subseteq (0,1)^2$ of dimension 1 does not have a definable G-mild parametrization, then it does not have definable mild parametrization. We can see this as follows. Assume that X has a definable (A,C)-mild parametrization for some $A > 0$, $C \geq 0$. There is $N \in \mathbb{N}$ such that $n!A^n n^{Cn} \leq G(n)$ for all $n \geq N$. Then, for any coordinate function $\phi\colon (0,1) \to (0,1)$ of a map involved in the definable (A,C)-mild parametrization of X, we would have $|\phi^{(n)}(x)| \leq n!A^n n^{Cn}$ for all $n \in \mathbb{N}$ and $x \in (0,1)$, hence $|\phi^{(n)}(x)| \leq G(n)$ for all $n \geq N$, so X would have a definable G-mild parametrization.

7.2 Definable mild parametrization and an irrational power function

The focus in this section will be on the mild parametrization of polynomially bounded expansions of the real field when the field of exponents contains irrational elements. For a given o-minimal expansion of the real field $\mathcal{R}$ we will denote the field of exponents of $\mathcal{R}$ by $\mathcal{F}^{\mathcal{R}}$.

The following Proposition 7.2.1 is a well known result of C. Miller (see [48]). We will not give the proof of this proposition here. We will derive a version of it as a corollary.

Proposition 7.2.1. *Let $\mathcal{R}$ be a polynomially bounded o-minimal expansion of the real field and let $f\colon \mathbb{R} \to \mathbb{R}$ be a definable function in $\mathcal{R}$ such that $f(x)$ is nonzero* *for sufficiently large positive values of x. Then there exist $a \in \mathcal{F}^{\mathcal{R}}$ and nonzero $L \in \mathbb{R}$ such that $\lim_{x\to\infty} \frac{f(x)}{x^a} = L$.*

Corollary 7.2.2. *Let $\mathcal{R}$ be a polynomially bounded o-minimal expansion of the real field and let $f\colon (0,1) \to \mathbb{R}$ be a definable function in $\mathcal{R}$ such that there exists $\delta > 0$ with $f(x) \neq 0$ for all $x \in (0,\delta)$. Then there exist $a \in \mathcal{F}^{\mathcal{R}}$ and nonzero $L \in \mathbb{R}$ such that $\lim_{x\to 0^+} \frac{f(x)}{x^a} = L$.*

Proof. Consider the function $g\colon (1/\delta,\infty) \to \mathbb{R}$ defined by $g(x) = f(1/x)$. Since f is definable in $\mathcal{R}$, g is definable as well. The fact that $f(x) \neq 0$ for all $x \in (0,\delta)$ implies that $g(x)$ is nonzero for all $x \in (1/\delta,\infty)$. By Proposition 7.2.1 there is $b \in \mathcal{F}^{\mathcal{R}}$ and some nonzero $L \in \mathbb{R}$ such that $\lim_{x\to\infty} \frac{g(x)}{x^b} = L$. Then $\lim_{x\to 0^+} \frac{f(x)}{x^a} = L$, where $a = -b$. □

Definition 7.2.3. *Let $f : (0,1) \to (0,1)$ be a C^∞ function. We say that f is a* regular nonmild *function at $x_0 \in \{0,1\}$ if there exists $m \in \mathbb{N}$ such that*

$$\lim_{x\to\star} f^{(k)}(x) = \begin{cases} 0 & k < m \\ +\infty \text{ or } -\infty & k \geq m \end{cases}$$

where $\star$ is 0^+ if $x_0 = 0$ and is 1^- if $x_0 = 1$.

Note that any function that is regular nonmild at 0 or 1 is not a mild function.

Remark 7.2.4. Let $\alpha \in \mathbb{R}$ be a noninteger. The function $x^\alpha : (0,1) \to (0,1)$ is a regular nonmild function at 0: for $m \in \mathbb{N}$ with $m-1 < \alpha < m$ and assuming without loss of generality that m is even, we have

$$\lim_{x\to 0^+} (x^\alpha)^{(k)} = \begin{cases} 0 & k < m \\ +\infty & k \geq m, k \text{ even} \\ -\infty & k \geq m, k \text{ odd.} \end{cases}$$

Lemma 7.2.5. *Let $f, g : (0,1) \to (0,1)$ be C^∞ functions which are definable in an o-minimal expansion of the real field with $\lim_{x\to 0^+} \frac{f(x)}{g(x)} = 1$. Then f is regular nonmild at 0 if and only if g is regular nonmild at 0.*

Proof. Since the situation is symmetric, let us assume that g is regular nonmild at 0. Let $\mathcal{R}$ be the o-minimal expansion of the real field in which f and g are definable. By o-minimality $\lim_{x\to 0^+} \frac{f^{(k)}(x)}{g^{(k)}(x)} \in \mathbb{R} \cup \{\pm\infty\}$ for all $k \in \mathbb{N}$. Let $\lim_{x\to 0^+} g(x) = \triangle$ where $\triangle \in \{\pm\infty, 0\}$. Since $\lim_{x\to 0^+} \frac{f(x)}{g(x)} = 1$, we have $\lim_{x\to 0^+} f(x) = \triangle$ as well. Since $\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} \in \mathbb{R} \cup \{\pm\infty\}$, we are allowed to use L'Hospital's rule which implies that

$$\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0^+} \frac{f(x)}{g(x)} = 1.$$

We assumed that g is a regular nonmild function, so $\lim_{x\to 0^+} g^{(k)}(x) \in \{\pm\infty, 0\}$, for all $k \in \mathbb{N}$. We can repeat the same argument inductively, and can conclude that

$$\lim_{x\to 0^+} \frac{f^{(k)}(x)}{g^{(k)}(x)} = 1$$

for all $k \in \mathbb{N}$. This means $\lim_{x\to 0^+} g^{(k)}(x) = \lim_{x\to 0^+} f^{(k)}(x)$ for all $k \in \mathbb{N}$. Hence f is regular nonmild as well. □

Remark 7.2.6. It is possible to state and prove Lemma 7.2.5 analogously for the functions f and g being regular nonmild at 1 with $\lim_{x\to 1^-} \frac{f(x)}{g(x)} = 1$.

Remark 7.2.7. We state Lemma 7.2.5 for f and g being definable in the same o-minimal expansion of the real field for simplicity. However this lemma is also true for f and g being definable in different polynomially bounded o-minimal expansions of the real field. In the proof of Lemma 7.2.5, the assumption that f and g are definable in the same o-minimal expansion of the real field is used to show $\lim_{x\to 0^+} \frac{f^{(k)}(x)}{g^{(k)}(x)} \in \mathbb{R} \cup \{\pm\infty\}$ for all $k \in \mathbb{N}$, but this can be also obtained as long as the functions f and g are definable in possibly different polynomially bounded o-minimal expansions of the real field. Let f be definable in $\mathcal{R}$ and g be definable in $\mathcal{S}$ where both $\mathcal{R}$ and $\mathcal{S}$ are polynomially bounded o-minimal expansions of the real field. Then by Corollary 7.2.2 there exist $a \in \mathcal{F}^{\mathcal{R}}, b \in \mathcal{F}^{\mathcal{S}}$ and $K, L \in \mathbb{R} \setminus \{0\}$ such that

$$\lim_{x\to 0^+} \frac{f(x)}{x^a} = L \quad \text{and} \quad \lim_{x\to 0^+} \frac{g(x)}{x^b} = K.$$

So we have

$$\lim_{x\to 0^+} \frac{f(x)}{x^{a-b}g(x)} = \frac{L}{K}$$

and therefore $\lim_{x\to 0^+} \frac{f(x)}{g(x)} \in \mathbb{R} \cup \{\pm\infty\}$. The fact that $\lim_{x\to 0^+} \frac{f^{(k)}(x)}{g^{(k)}(x)} \in \mathbb{R} \cup \{\pm\infty\}$ follows by an analogous argument.

Now we state main theorem of this chapter which says that if the power function x^α with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is definable in a polynomially bounded expansion of the real ordered field then this structure does not admit definable mild parametrization. The main ingredient of our proof is the fact that any mild function that parametrizes the graph of x^α on an interval near 0 cannot be polynomially bounded so cannot be definable in a polynomially bounded structure.

Theorem 7.2.8. *Let $\mathcal{R}$ be a polynomially bounded o-minimal expansion of the real field where $\mathcal{F}^{\mathcal{R}}$ contains an irrational number. Then $\mathcal{R}$ does not admit definable mild parametrization.*

Proof. Let $\alpha \in \mathcal{F}^{\mathcal{R}}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. We assume for a contradiction that $\mathcal{R}$ admits definable mild parametrization. Then, the graph of x^α restricted to $(0,1)$ has definable mild parametrization. The image of $(0,1)$ under each of these parametrizing functions is a connected subset of $\Gamma\left(x^\alpha\restriction_{(0,1)}\right)$, that is a set of the form $\Gamma\left(x^\alpha\restriction_I\right)$ where I is an interval included in $(0,1)$. Since these images cover $\Gamma\left(x^\alpha\restriction_{(0,1)}\right)$ closure of one of these intervals must contain 0. Let

$$\begin{array}{rccl} \Phi: & (0,1) & \to & (0,1)^2 \\ & t & \mapsto & (\varphi(t), \psi(t)) \end{array}$$

be the corresponding parametrization. Since $(0,0) \in \overline{\Phi(0,1)} \setminus \Phi(0,1)$ we have

$$\lim_{t \to 0^+} \Phi(t) = (0,0) \quad \text{or} \quad \lim_{t \to 1^-} \Phi(t) = (0,0).$$

After the change of variable $t \mapsto 1-t$ if necessary we can assume that

$$\lim_{t \to 0^+} \varphi(t) = 0 \quad \text{and} \quad \lim_{t \to 0^+} \psi(t) = 0.$$

Since $\Gamma\left(x^\alpha\restriction_{(0,1)}\right) \subseteq (0,1)^2$, $\varphi(t)$ and $\psi(t)$ are nonzero for all $t \in (0,1)$. So by Corollary 7.2.2 there exist $a, b \in \mathcal{F}^{\mathcal{R}}$ and $L, K \in \mathbb{R} \setminus \{0\}$ such that

$$\lim_{x \to 0^+} \frac{\varphi(x)}{x^a} = L \quad \text{and} \quad \lim_{x \to 0^+} \frac{\psi(x)}{x^b} = K.$$

Since $(\varphi(x))^\alpha = \psi(x)$ we have

$$\lim_{x \to 0^+} \left(\frac{\varphi(x)}{x^{\frac{b}{\alpha}}}\right)^\alpha = K,$$

then

$$\lim_{x \to 0^+} \left(\frac{\varphi(x)}{x^a} \cdot \frac{1}{x^{\frac{b}{\alpha}-a}}\right)^\alpha = K$$

and so

$$\lim_{x \to 0^+} \frac{1}{x^{b-a\alpha}} = \frac{K}{L^\alpha} \in \mathbb{R} \setminus \{0\}$$

since K and L are nonzero. If $b - a\alpha > 0$ then $\lim\limits_{x \to 0^+} \frac{1}{x^{b-a\alpha}} = \infty$ and if $b - a\alpha <$ then $\lim\limits_{x \to 0^+} \frac{1}{x^{b-a\alpha}} = 0$ so $b - a\alpha = 0$. Both a and b are nonzero since $\lim\limits_{t \to 0^+} \varphi(t) = 0$ and $\lim\limits_{t \to 0^+} \psi(t) = 0$ and $K, L \neq 0$. Therefore either a or b is irrational since α is irrational.

We assume without loss of generality that a is irrational. Then x^a is regular nonmild at 0 by Remark 7.2.4. Since $\lim\limits_{x \to 0^+} \dfrac{\varphi(x)/L}{x^a} = 1$, by Lemma 7.2.5 φ/L is regular nonmild at 0 as well and this contradicts the assumption that φ is mild. Hence $\mathcal{R}$ does not admit definable mild parametrization. $\square$

Remark 7.2.9. In the proof of Theorem 7.2.8 it is not necessary to assume that the functions φ, ψ and x^α are definable in the same structure: it is enough to assume that both φ and ψ are definable in some polynomially bounded o-minimal expansions of the real field by Remark 7.2.7. So we proved a stronger statement: the function $x^\alpha : (0,1) \to (0,1)$ does not have mild parametrization by means of mild functions definable in polynomially bounded structures. Note that the function $t \mapsto e^{-1/t}$ on $(0,1)$ that we use in Proposition 3.2.6 to parametrize the graph of x^α near 0 is flat so not definable in any polynomially bounded structure.

In order to apply our theorem we consider examples of polynomially bounded o-minimal expansions of the real ordered field whose field of exponents contains irrational elements. Such examples are $\mathbb{R}^{\mathrm{pow}}$, the expansion of the real ordered field with power functions; $\mathbb{R}^{\mathrm{pow}}_{\mathrm{an}}$, the expansion of $\mathbb{R}_{\mathrm{an}}$ with power functions; and $\mathbb{R}_{\mathrm{an}^*}$, the expansion of the real ordered field with generalised power series (see Section 2.1.2). In light of these examples we state the following corollary.

Corollary 7.2.10. *The structures $\mathbb{R}^{pow}$, $\mathbb{R}^{pow}_{an}$ and $\mathbb{R}_{an^*}$ do not admit definable mild parametrization.*

Chapter 8

Mild parametrization in $\mathbb{R}^S_{\mathbf{an}}$

We recall that by a power function $x^\alpha\colon \mathbb{R} \to \mathbb{R}$ where $\alpha \in \mathbb{R}$, we mean a one variable real function defined by

$$x \mapsto \begin{cases} x^\alpha & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(see Definition 2.1.21).

In Chapter 7 we have proved that a polynomially bounded structure $\mathcal{R}$ does not admit definable mild parametrization if an irrational power function is definable in $\mathcal{R}$. But the question of whether or not polynomially bounded o-minimal expansions of the real field in which an irrational power function is definable have mild parametrization is still open. We obtain our result in Chapter 7 by proving that any mild map that parametrizes $x^\alpha\restriction_{(0,\delta)}$ for any $\delta > 0$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, cannot be definable in any polynomially bounded o-minimal structure. On the other hand, $x^\alpha\restriction_{(0,1)}$ has mild parametrization (Proposition 3.2.6) by a map defined using the mild function $e^{-1/t} : (0,1) \to (0,1)$. So we aim to get mild parametrization results for the structures in which irrational power functions are definable by employing the function $e^{-1/t}$.

Let S be a subfield of $\mathbb{R}$. The structure

$$\mathbb{R}^S_{\text{an}} = (\mathbb{R}_{\text{an}}, (x^\alpha)_{\alpha \in S})$$

is the expansion of $\mathbb{R}_{\text{an}}$ with power functions x^α where $\alpha \in S$. In [47] C. Miller proved that $\mathbb{R}^S_{\text{an}}$ is a polynomially bounded o-minimal structure. In this chapter we prove that for any subfield S of $\mathbb{R}$, definable curves in $\mathbb{R}^S_{\text{an}}$ that lie in $(0,1)^2$ have mild parametrization.

8.1 Mild parametrization of $\mathbb{R}^S_{\mathrm{an}}$-definable curves

In [47] C. Miller proved that the theory of $\mathbb{R}^S_{\mathrm{an}}$ admits quantifier elimination, is universally axiomatizable, and that, for any subfield S of $\mathbb{R}$, $\mathbb{R}^S_{\mathrm{an}}$ is model complete. The following corollary follows from these results. It was stated in [47] and says that a definable real function in $\mathbb{R}^S_{\mathrm{an}}$ is given piecewise by terms.

The language L^S_{an} is the language of the theory of $\mathbb{R}^S_{\mathrm{an}}$.

Corollary 8.1.1. ([47, Corollary 2.7]) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function definable in $\mathbb{R}^S_{an}$. Then there are n-ary L^S_{an}-terms $t_1, t_2, \ldots, t_l$ such that, for all $a \in \mathbb{R}^n$, there exists $i \in \{1, \ldots, l\}$ with $f(a) = t_i(a)$.*

Another result in [47] which is important for our purposes is the following.

Proposition 8.1.2. ([47, Proposition 4.5]) *Let t be a unary term in $\mathbb{R}^S_{an}$, and let $\epsilon > 0$ be such that $t(x) \neq 0$ for all $x \in (0, \epsilon)$. Then there exist $d \in \mathbb{N}$, $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in S^{d+1}$ with $\alpha_i > 0$ for all $i \in \{1, \ldots, d\}$, and $F : \mathbb{R}^d \to \mathbb{R}$ analytic at the origin with $F(0) \neq 0$ such that, for all sufficiently small positive x,*

$$t(x) = x^{\alpha_0} F(x^{\alpha_1}, \ldots, x^{\alpha_d}).$$

Let $d \in \mathbb{N}$. For $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{R}^{d+1}$ we put $\alpha' = (\alpha_1, \ldots, \alpha_d)$. For $\delta > 0$ and $\alpha \in \mathbb{R}^{d+1}$, we set

$$B^{\alpha'}_\delta := [0, \delta^{\alpha_1}] \times \ldots \times [0, \delta^{\alpha_d}].$$

For any $f : U \to \mathbb{R}$ with $U \subset \mathbb{R}$ and $a \in \mathbb{R}$, we define the functions $f^+_a : x \mapsto f(a+x)$ on $U^+_a = \{x \colon a + x \in U\}$ and $f^-_a : x \mapsto f(a - x)$ on $U^+_a = \{x \colon a + x \in U\}$.

Definition 8.1.3. *Let $\delta > 0$. We say that a function $f : U \to \mathbb{R}$ where $(0, \delta) \subset U \subset \mathbb{R}$, has* S-power representation on the interval $(0, \delta)$ *if there exist $d \in \mathbb{N}$, $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in S^{d+1}$ with $\alpha_0 \geq 0$, $\alpha_i > 0$ for all $i \in \{1, \ldots, d\}$, and $F : \mathbb{R}^d \to \mathbb{R}$ analytic on an open neighbourhood of $B^{\alpha'}_\delta$ with $F(0) \neq 0$ such that*

$$f(x) = x^{\alpha_0} F(x^{\alpha_1}, \ldots, x^{\alpha_d})$$

for all $x \in (0, \delta)$.

For a function $f : (0, 1) \to \mathbb{R}$ and $a \in [0, 1]$, we will say that f has S-power representation around *a for the following cases:*

- *$a \in (0, 1)$ and the functions f^+_a and f^-_a have S-power representations on some interval $(0, \delta)$ with $\delta \leq \min\{1 - a, a\}$.*
- *$a = 0$ and f has S-power representation on some interval $(0, \delta)$ with $0 < \delta \leq 1$.*
- *$a = 1$ and the function f^-_1 has S-power representation on some interval $(0, \delta)$ with $0 < \delta \leq 1$.*

The following lemma states that any function $f : (0,1) \to (0,1)$ definable in $\mathbb{R}^S_{\text{an}}$ has S-power representation around each $a \in [0,1]$. In [22] this lemma was stated (Facts 3.1(4)) for $a = 0$ and for an S-power representation where α_0 is not constrained to be nonnegative but can be any real number in S.

Lemma 8.1.4. *Let $f : (0,1) \to (0,1)$ be a function definable in $\mathbb{R}^S_{an}$. Then f has S-power representation around a, for all $a \in [0,1]$.*

Proof. Since f is definable in $\mathbb{R}^S_{an}$, there exist unary L^S_{an}-terms $t_1, \dots, t_l$ such that, for all $a \in (0,1)$, there exists $i \in \{1, \dots, l\}$ with $f(a) = t_i(a)$ by Corollary 8.1.1. For each $i \in \{1, \dots, l\}$, the sets $A_i := \{a \in (0,1) : f(a) = t_i(a)\}$ are definable in $\mathbb{R}^S_{an}$. By o-minimality of $\mathbb{R}^S_{an}$, each A_i is the union of finitely many intervals and points. We have $(0,1) = \bigcup_{1 \le i \le l} A_i$ so there exists $\epsilon > 0$ and $i \in \{1, \dots, l\}$ such that $f(x) = t_i(x)$ for all $x \in (0,\epsilon)$. The range of f is $(0,1)$, so $t_i(x) \neq 0$ for all $x \in (0,\epsilon)$. Then, by Proposition 8.1.2, there exist $\epsilon' > 0$, $d \in \mathbb{N}$, $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d) \in S^{d+1}$ with $\alpha_1, \dots, \alpha_d > 0$, and $F : \mathbb{R}^d \to \mathbb{R}$ analytic on a neighbourhood of $B^{\alpha'}_{\epsilon'}$ with $F(0) \neq 0$ such that $t_i(x) = x^{\alpha_0} F(x^{\alpha_1}, \dots, x^{\alpha_d})$. Hence, $f(x) = x^{\alpha_0} F(x^{\alpha_1}, \dots, x^{\alpha_d})$ for all $x \in (0,\delta)$, where $\delta = \min(\epsilon, \epsilon')$. We will now prove that α_0 is also nonnegative. Assume for a contradiction that $\alpha_0 < 0$. Then x^{α_0} is unbounded for sufficiently small positive x. On the other hand, F has a continuous extension on $B^{\alpha'}_{\delta}$, so it is bounded on $B^{\alpha'}_{\delta}$. Therefore $f(x) = x^{\alpha_0} F(x^{\alpha_1}, \dots, x^{\alpha_d})$ is unbounded for sufficiently small positive x and this contradicts the fact that f is bounded. □

Now we will prove that for any $f : (0,1) \to (0,1)$ definable in $\mathbb{R}^S_{\text{an}}$, we can cover $[0,1]$ with intervals I such that $f\restriction_I$ has mild parametrization for each I.

Proposition 8.1.5. *Let $f : (0,1) \to (0,1)$ be a function definable in $\mathbb{R}^S_{an}$, and let $a \in [0,1]$. Then there is a nonempty interval I_a containing a, and open in $[0,1]$ such that $f\restriction_{I_a \cap (0,1)}$ has mild parametrization.*

Proof. Let $a \in (0,1)$; the proofs for the cases $a = 0$ or $a = 1$ are similar.
Claim. There is $\delta > 0$ such that $f^+_a\restriction_{(0,\delta)}$ has mild parametrization.
Proof of the claim. By Lemma 8.1.4, f has S-power representation around a, so there is $\delta' > 0$ such that f^+_a has S-power representation on $(0,\delta')$. Let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d) \in S^{d+1}$ with $\alpha_0 \ge 0$, $\alpha_i > 0$ for all $i \in \{1, \dots, d\}$, and let $F : \mathbb{R}^d \to \mathbb{R}$ be analytic on an open neighbourhood of $B^{\alpha'}_{\delta'}$ with $F(0) \neq 0$ such that,

$$f^+_a(x) = x^{\alpha_0} F(x^{\alpha_1}, \dots, x^{\alpha_d})$$

for all $x \in (0,\delta')$.
We will first consider the case $\alpha_0 > 0$. The fact that F is analytic on an open neighbourhood of $B^{\alpha'}_{\delta'}$ implies that F is bounded on $B^{\alpha'}_{\delta'}$. So let $M > 0$ be such that $|F(x_1 \dots, x_d)| < M$ for all $(x_1, \dots, x_d) \in B^{\alpha'}_{\delta'}$. Since $\alpha_0 > 0$, there is $\delta > 0$ with $\delta < \delta'$ such that, $0 < Mx^{\alpha_0} < 1$ for all $x \in (0,\delta)$.

Define the function

$$G(y_1, \ldots, y_d) := \frac{F(\delta^{\alpha_1} y_1, \ldots, \delta^{\alpha_d} y_d)}{M}$$

for $(y_1, \ldots, y_d) \in \mathbb{R}^d$. Then

$$F(x_1, \ldots, x_d) = M \cdot G\left(\frac{x_1}{\delta^{\alpha_1}}, \ldots, \frac{x_d}{\delta^{\alpha_d}}\right),$$

so

$$f_a^+(x) = M x^{\alpha_0} G\left(\frac{x^{\alpha_1}}{\delta^{\alpha_1}}, \ldots, \frac{x^{\alpha_d}}{\delta^{\alpha_d}}\right)$$

for all $x \in (0, \delta')$.

Now we want to parametrize the set

$$\Gamma\left(f_a^+ \restriction_{(0,\delta)}\right) = \left\{\left(x, M x^{\alpha_0} G\left(\frac{x^{\alpha_1}}{\delta^{\alpha_1}}, \ldots, \frac{x^{\alpha_d}}{\delta^{\alpha_d}}\right)\right) : x \in (0, \delta)\right\}.$$

with mild functions.

Since F is analytic on an open neighbourhood of $B_{\delta'}^{\alpha'}$, G is analytic on a neighbourhood of $[0,1]^d$. Also G is bounded by 1 on $[0,1]^d$ because F is bounded by M on $B_{\delta'}^{\alpha'}$. Therefore, by Fact 3.2.1 using the same argument as in Proposition 3.2.2, $G : (0,1)^d \to (0,1)$ is a mild function.
For every $a, b \in \mathbb{R}$ with $b > 0$ and $a \geq e^{-1/b}$, consider the function

$$\begin{array}{rccl} E_{a,b} : & (0,1) & \to & \mathbb{R} \\ & t & \mapsto & \frac{1}{a} e^{-1/bt}. \end{array}$$

The function $E_{a,b}$ is continuous and increasing on $(0,1)$ and $\lim\limits_{t \to 0^+} E_{a,b}(t) = 0$. Furthermore, $\lim\limits_{t \to 1^-} E_{a,b}(t) = \dfrac{1}{a} e^{-1/b} \leq 1$ by the inequality $a \geq e^{-1/b}$. Hence the range of the function $E_{a,b}$ is a subset of the interval $(0,1)$.

Let

$$\sigma_i := \frac{-1}{\alpha_i \log(\delta)}$$

for $i = 0, 1, \ldots, d$. Note that $0 < \delta < 1$, so $\dfrac{-1}{\log(\delta)} > 0$, and therefore $\sigma_i > 0$ for all $i = 0, 1, \ldots, d$.

We also have the inequalities $1 \geq \delta = e^{\log(\delta)}$, $1/M \geq e^{-1/\sigma_0}$ and $\delta^{\alpha_i} = e^{-1/\sigma_i}$ for all $i = 1, \ldots, d$ so the functions $E_{1,\frac{-1}{\log(\delta)}}$, $E_{\frac{1}{M},\sigma_0}$ and $E_{\delta^{\alpha_i},\sigma_i}$, for $i = 1, \ldots, d$, all lie in the class of the functions $E_{a,b}$ above and hence have a range included in $(0,1)$. Moreover the range of $E_{1,\frac{-1}{\log(\delta)}}$ is $(0, \delta)$.

We consider the map

$$\begin{array}{rccl} P_E : & (0,1) & \to & (0,1)^2 \\ & t & \mapsto & \left(E_{1,\frac{-1}{\log(\delta)}}(t), E_{\frac{1}{M},\sigma_0}(t) \cdot G\left(E_{\delta^{\alpha_1},\sigma_1}(t), \ldots, E_{\delta^{\alpha_d},\sigma_d}(t)\right)\right). \end{array}$$

By Proposition 3.2.3 and 3.1.9 every $E_{a,b}$ is mild. We also showed above that G is mild. So by Propositions 3.1.9 and 3.1.5, the map P_E is mild. The image of P_E

is the graph of $f_a^+\restriction_{(0,\delta)}$. Therefore $f_a^+\restriction_{(0,\delta)}$ has mild parametrization.

For the case $\alpha_0 = 0$, let M be as defined in the case $\alpha_0 > 0$. Then,

$$f_a^+(x) = M \cdot G\left(\frac{x^{\alpha_1}}{\delta^{\alpha_1}}, \dots, \frac{x^{\alpha_d}}{\delta^{\alpha_d}}\right)$$

for all $x \in (0,\delta)$, so the proof is analogous to the proof of the case $\alpha_0 > 0$. We define the mild function

$$\begin{array}{cccl} P_E^0: & (0,1) & \to & (0,1)^2 \\ & t & \mapsto & \left(E_{1,\frac{-1}{\log(\delta)}}(t), M \cdot G\left(E_{\delta^{\alpha_1},\sigma_1}(t), \dots, E_{\delta^{\alpha_d},\sigma_d}(t)\right)\right). \end{array}$$

Then we have

$$\Gamma\left(f_a^+\restriction_{(0,\delta)}\right) = \mathrm{Im}\left(P_E^0\right)$$

hence $f_a^+\restriction_{(0,\delta)}$ has mild parametrization, thus the claim is proved.

If $\alpha_0 > 0$, let us consider the map

$$\begin{array}{cccl} Q_E: & (0,1) & \to & (0,1)^2 \\ & t & \mapsto & \left(a + E_{1,\frac{-1}{\log(\delta)}}(t), E_{\frac{1}{M},\sigma_0}(t) \cdot G\left(E_{\delta^{\alpha_1},\sigma_1}(t), \dots, E_{\delta^{\alpha_d},\sigma_d}(t)\right)\right). \end{array}$$

The maps P_E and Q_E differ by a constant, hence they have the same derivatives at every order; since the map P_E is mild, so is the map Q_E.

Furthermore, the properties $\forall x \in (0,\delta), f(a+x) = f_a^+(x)$ and $\Gamma\left(f_a^+\restriction_{(0,\delta)}\right) = \mathrm{Im}\left(P_E\right)$ ensure that

$$\Gamma\left(f\restriction_{(a,a+\delta)}\right) = \mathrm{Im}\left(Q_E\right),$$

so $f\restriction_{(a,a+\delta)}$ has mild parametrization.

If $\alpha_0 = 0$, instead consider the map

$$\begin{array}{cccl} Q_E^0: & (0,1) & \to & (0,1)^2 \\ & t & \mapsto & \left(a + E_{1,\frac{-1}{\log(\delta)}}(t), M \cdot G\left(E_{\delta^{\alpha_1},\sigma_1}(t), \dots, E_{\delta^{\alpha_d},\sigma_d}(t)\right)\right). \end{array}$$

Likewise Q_E^0 is mild and

$$\Gamma\left(f\restriction_{(a,a+\delta)}\right) = \mathrm{Im}\left(Q_E^0\right).$$

In both cases, there exists $\delta > 0$ such that $f\restriction_{(a,a+\delta)}$ has mild parametrization.

Using the same arguments one proves that there exists $\bar{\delta} > 0$ such that $f\restriction_{(a-\bar{\delta},a)}$ has mild parametrization.

Hence there exist $\delta, \bar{\delta} > 0$ such that $f\restriction_{(a,a+\delta)}$ and $f\restriction_{(a-\bar{\delta},a)}$ have mild parametrization.

We put $\lambda = \min\{\delta, \bar{\delta}\}$ and $I_a = (a-\lambda, a+\lambda)$. Then

$$\Gamma\left(f\restriction_{I_a}\right) = \Gamma\left(f\restriction_{(a,a+\delta)}\right) \cup \Gamma\left(f\restriction_{(a-\bar{\delta},a)}\right) \cup (a, f(a))$$

and a single point always has mild parametrization, so $f\restriction_{I_a}$ has mild parametrization.

□

Theorem 8.1.6. *Let $f : (0,1) \to (0,1)$ be a function definable in $\mathbb{R}^S_{an}$. Then f has mild parametrization.*

Proof. By Proposition 8.1.5, for each a in $[0,1]$ there is an interval I_a containing a and open in $[0,1]$ such that $f\restriction_{I_a\cap(0,1)}$ has mild parametrization. Since $[0,1] = \bigcup_{a\in[0.1]} I_a$ and $[0,1]$ is compact, there is $k \in \mathbb{N}$ and $a_1,\ldots,a_k \in [0,1]$ such that $[0,1] = \bigcup_{i=1}^{k} I_{a_i}$. One can see from the proof of Proposition 8.1.5 that $f\restriction_{I_a\cap(0,1)}$ has mild parametrization using only one map. Let P_i be the mild map such that

$$\Gamma\left(f\restriction_{I_a\cap(0,1)}\right) = \mathrm{Im}\left(P_i\right).$$

Then

$$\Gamma(f) = \bigcup_{i=1}^{k} \mathrm{Im}\left(P_i\right).$$

Hence f has mild parametrization. □

Chapter 9

Conclusion

With this thesis we provide a better understanding of mild parametrization in o-minimal structures. Our work gives a detailed overview of all known results about mild parametrization of definable sets in o-minimal structures and the number theoretical results obtained using mild parametrization. In light of these results, we contribute to this research by proving new results about mild parametrization of some or all definable sets of certain expansions of the real field and by obtaining a result about the density of rational points of a specific surface as an application of mild parametrization.

All the functions considered in this chapter are real valued functions whose domains are subsets of $\mathbb{R}^n$ for some $n \in \mathbb{N}^+$.

9.1 Alternative approaches

In Chapter 3, we examined mild functions and functions that have mild parametrization in detail and prove some properties of these functions. These results enhance comprehension of mild parametrization as a tool so that this tool can be used more efficiently. But there are still unresolved questions about these functions which we plan to work on. We have conjectured (Conjecture 3.1.11) that the composition $g \circ F$ of a function g that has mild parametrization and a map F which is mild, has mild parametrization. We proved that the composition of one variable functions that have mild parametrization also has mild parametrization, assuming Conjecture 3.1.11 and assuming that the parametrizing functions are definable in an o-minimal structure. But the following more general question is still open and is of interest.

Question 9.1.1. *Let f and g be maps that have mild parametrization such that their composition is well defined. Does $f \circ g$ have mild parametrization?*

One easy way to get a positive answer to Question 9.1.1 would be to get a negative answer to the following question:

Question 9.1.2. *Is there a C^∞ function that does not have mild parametrization?*

It may appear naive to hope that every C^∞ function has mild parametrization. However, we will discuss hereafter how positive answers to Question 9.1.1 and to Question 9.1.3 would give a negative answer to Question 9.1.2. Let us first motivate and state Question 9.1.3.

Our results in Chapter 6 are motivated by the question of whether or not every function in each QADC class is mild. We proved that this is not the case by constructing an example of a QADC class and a function in it which is not mild. The explicit construction that we gave provides a good understanding of the behaviour of functions in QADC classes. Further questions we plan to consider are the following.

Question 9.1.3. *Do all QADC classes contain functions that have mild parametrization?*

Question 9.1.4. *Do all QADC structures admit mild parametrization?*

Questions 9.1.1, 9.1.2 and 9.1.3 are related to each other. Giving positive answers to two of these questions would provide a negative answer for the third one; it is not possible to give a positive answer to all three questions.

The reason why such an alternative holds is that any C^∞ function can be seen as composition of two maps from QADC classes (as we explained in Remark 6.4.7). So if we know that there exists a C^∞ function without mild parametrization (i.e. a positive answer to Question 9.1.2) and that all QADC classes admit mild parametrization (i.e. a positive answer to Question 9.1.3) then we would find two maps, each having mild parametrization, whose composition does not have mild parametrization (this would be a negative answer to Question 9.1.1).

The same argument shows that a positive answer to Questions 9.1.1 and 9.1.2 immediately provides a negative answer to Question 9.1.3.

Finally a negative answer to Question 9.1.2 (i.e. if all C^∞ maps have mild parametrization) provides a positive answer to Question 9.1.3.

The situation we find most likely to be true is that Questions 9.1.1 and 9.1.2 have a positive answer, while Question 9.1.3 has a negative answer. A future project is to construct a QADC class and a function in that class that does not have mild parametrization. The functions in a QADC class have bounds on all orders of derivatives depending on the terms of the sequence to which the class is associated. We hope that it is possible to choose infinitely many of these terms big enough to obtain a QADC class and a function lying in that class which does mild parametrization using arguments similar to those of Thomas in [73].

9.2 Reducts of $\mathbb{R}_{\text{an}}$

Chapter 4 includes information and discussion about some known results concerning mild parametrization in o-minimal structures, one of which is the result of Jones, D. Miller and Thomas stating that any reduct of $\mathbb{R}_{\text{an}}$ admits definable mild parametrization ([33]). While for some reducts of $\mathbb{R}_{\text{an}}$ (we discussed the case of $\mathbb{R}_{\text{an}}$ itself and of $\mathbb{R}_{\text{respfaff}}$) the proof given in [33] indeed provides a definable mild parametrization theorem, we remarked (Remark 4.3.5) that the argument in [33] does not seem to be complete for *every* reduct of $\mathbb{R}_{\text{an}}$. So we aim to continue our research looking for the answer to the following questions.

Question 9.2.1. *Do all reducts of $\mathbb{R}_{an}$ admit definable mild parametrization?*

As noted in Remark 4.3.5, any reduct $\mathcal{R}$ of $\mathbb{R}_{\text{an}}$ that has the definable extension property for smooth functions (that is, such that all definable C^∞ functions defined on the box $[0,1]^n$ have a *definable* C^∞ extension to an open neighbourhood of $[0,1]^n$) admits definable mild parametrization.

We expect that if $f : [0,1] \to \mathbb{R}$ is analytic and strongly transcendental in the sense of Le Gal's [40], the structure $(\bar{\mathbb{R}}, f)$ refutes the following question:

Question 9.2.2. *Do all reducts of $\mathbb{R}_{an}$ have the definable extension property for smooth functions?*

New techniques would then be needed to provide a positive answer to Question 9.2.1.

9.3 More on RS-structures

In Chapter 5 we proved that if each function $f : (0,1)^n \to (0,1)$ in an RS-algebra $\mathcal{A}$ is mild then the expansion of the real field by $\mathcal{A}$ admits mild parametrization. Since most known polynomially bounded o-minimal structures can be seen as an RS-structure, it is possible to apply this result to them, that is for such a structure $\mathcal{R}$ there is an RS-algebra $\mathcal{A}$ of some functions definable in the structure such that $\mathcal{R} = \mathbb{R}_{\mathcal{A}}$. A further goal is to improve this result to provide necessary and sufficient conditions on the functions in the RS-algebra $\mathcal{A}$ so that $\mathbb{R}_{\mathcal{A}}$ admits mild parametrization.

Question 9.3.1. *What are necessary and sufficient conditions on an RS-algebra $\mathcal{A}$ so that $\mathbb{R}_{\mathcal{A}}$ admits mild parametrization?*

A first case one would consider towards answering Question 9.3.1 would be the case in which we relax the condition that the functions in $\mathcal{A}$ are mild to requiring that the functions in $\mathcal{A}$ have mild parametrization.

Question 9.3.2. *Does $\mathbb{R}_{\mathcal{A}}$ admit mild parametrization if each function $f : (0,1)^n \to (0,1)$ in $\mathcal{A}$ has mild parametrization?*

An RS-algebra that consists of functions that have mild parametrization but contains of functions which are not mild is not known. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $\langle x^\alpha \rangle$ be the smallest RS-algebra containing the function $x^\alpha : (0,1) \to (0,1)$. The algebra $\langle x^\alpha \rangle$ can be considered as a candidate for such an RS-algebra. So we plan to work on the following question.

Question 9.3.3. *Do all the functions $f : (0,1)^n \to (0,1)$, $n \in \mathbb{N}$ in $\langle x^\alpha \rangle$ have mild parametrization?*

If Question 9.3.3 has a positive answer then we can proceed and work on the following question.

Question 9.3.4. *Does the structure $\mathbb{R}^\alpha := \mathbb{R}_{\langle x^\alpha \rangle}$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ admit mild parametrization?*

With the help of the parametrization theorem of Rolin and Servi (Theorem 5.1.5), if we can show that for all $n \in \mathbb{N}$, any function $f : (0,1)^n \to (0,1)$ in the algebra $\langle x^\alpha \rangle$ has mild parametrization (i.e. give a positive answer to Question 9.3.3), then, to give a positive answer to Question 9.3.4, it remains to show that images of maps have mild parametrization whenever the coordinate functions of these maps have mild parametrization. That is, we plan to consider the following question; a positive answer to both it and Question 9.3.3 would imply a positive answer to Question 9.3.4 using Theorem 5.1.5.

Question 9.3.5. *Let $n \in \mathbb{N}$ and let $X \subseteq (0,1)^n$ have parametrization by means of maps with each of their coordinate functions having mild parametrization. Does X have mild parametrization?*

We want to point out here that Question 9.3.4 is itself interesting for us to consider regardless of the research path following Question 9.3.2. We will discuss more about our basis to address this question below when we consider further possible research arising from our results in Chapter 8.

9.4 Wilkie's conjecture for $\mathbb{R}^\alpha$

Our results in Chapter 8 are about $\mathbb{R}^S_{\text{an}}$, the expansion of $\mathbb{R}_{\text{an}}$ by all power functions $x^\alpha : (0,\infty) \to \mathbb{R}$ with $\alpha \in S$ and S a subfield of $\mathbb{R}$. Our main result states that curves definable in $\mathbb{R}^S_{\text{an}}$ which lie in $(0,1)^2$ have mild parametrization. Hence there is a strong basis to address the following question.

Question 9.4.1. *Does $\mathbb{R}^S_{an}$ admit mild parametrization?*

Our interest in the mild parametrization of the structure $\mathbb{R}^S_{\text{an}}$ arises from Question 9.3.4. A positive answer to the above question would imply that $\mathbb{R}^\alpha$ admits mild parametrization as well. Since $\mathbb{R}^S_{\text{an}}$ is an expansion of $\mathbb{R}_{\text{an}}$, Wilkie's conjecture is not true for this structure, but any definable set in the structure $\mathbb{R}^\alpha$ is also definable in $\mathbb{R}_{\text{exp}}$ so we have great interest in the following question for our further research.

Question 9.4.2. *Is Wilkie's conjecture true for $\mathbb{R}^\alpha$?*

Working towards a solution to the questions that we have presented here, our main goal is to obtain a better understanding of mild parametrization in o-minimal structures while keeping an eye on its possible contributions to resolving Wilkie's conjecture.

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