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Elliptic boundary value problems with large parameter for mixed order systems

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ABSTRACT. In this paper boundary value problems are studied for systems with large parameter, elliptic in the sense of Douglis–Nirenberg. We restrict ourselves on model problems acting in the half-space. It is possible to define parameter-ellipticity for such problems, in particular we formulate Shapiro–Lopatinskii type conditions on the boundary operators. It can be shown that parameter-elliptic boundary value problems are uniquely solvable and that their solutions satisfy uniform two-sided a priori estimates in parameter-dependent norms. We essentially use ideas from Newton's polygon method and of Vishik–Lyusternik boundary layer theory.

1. Introduction

The paper is devoted to the study of boundary value problems with large parameter:

$$A(x,D)u(x) - \lambda u(x) = f(x), \quad x \in M, \tag{1.1}$$

$$B(x',D)u(x') = g(x'), \quad x' \in \partial M, \tag{1.2}$$

where $A(x,D)$ is a matrix partial differential operator elliptic in the sense of Douglis–Nirenberg (mixed order systems), and its symbol for each fixed x satisfies the parameter-ellipticity condition of [**11**] (see [**4**] and [**14**] for equivalent conditions). In the case where M is a manifold without boundary, equation (1.1) was studied in [**4**] and [**14**] where the Newton polygon of the symbol $P(x,\xi,\lambda) := \det(A(x,\xi) - \lambda I)$ played an essential role. Boundary value problems of general type for a class of scalar polynomial pencils including the pencil corresponding to $P(x,\xi,\lambda)$ were studied in [**7**]. Here the analog of the Shapiro–Lopatinskii (or, more accurately, the Agmon–Agranovich–Vishik) condition was formulated. This analog in some sense was suggested by the deep connection of mixed order problems with large parameter to the Luysternik–Vishik theory of boundary layers, developed for elliptic problems containing a small parameter in leading derivatives.

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In this paper we restrict ourselves to the case where M is the half-space and the operator matrices A and B have constant coefficients and no lower-order terms. We define Shapiro–Lopatinskii type conditions for the problem (1.1)–(1.2), investigate in detail the ODE problem obtained after Fourier transform in the tangential direction and prove the basic a priori estimate for the problem (1.1)–(1.2). Since these estimates are two-sided the standard localization and small perturbation technique permits to extend them to the case of variable coefficients and problems on a manifold with boundary. Moreover, using the results of Section 5 it is not difficult to construct the right parametrix of (1.1)–(1.2) and prove unique solvability of this problem for large $|\lambda|$.

It should be mentioned that Kozhevnikov considered in [**12**] the case where A is elliptic in the sense of Petrovskii (and satisfies the parameter-ellipticity condition of [**11**]). The matrix B of boundary operators is supposed to be upper triangular. In this case the $L_2 \to L_2$ realization of the problem (1.1) under zero boundary conditions (1.2) is investigated. Supposing unique solvability of some auxiliary boundary value problems similar to the original problem (but containing a smaller number of unknown functions), the author established for sufficiently large λ the existence of a bounded $L_2 \to L_2$ inverse. Explicit Shapiro–Lopantinskii type conditions on boundary operators are not formulated.

2. Parameter-elliptic boundary value problems

2.1. Notation. Let $A(D) = \big(a_{ij}(D)\big)_{i,j=1,\dots,N}$ be an $N \times N$-matrix of partial differential operators with constant coefficients. We assume that A is a system of mixed order, i.e. there exist integers s_j and t_j such that

$$\operatorname{ord} a_{ij}(D) \le s_i + t_j \quad (i,j = 1,\dots,N)\,. \tag{2.1}$$

Here $a_{ij}(D) = \sum_{|\alpha|\le s_i+t_j} a_{ij\alpha} D^\alpha$ where we used the standard multi-index notation $D^\alpha := (-i)^{|\alpha|}(\partial/\partial x_1)^{\alpha_1}\dots(\partial/\partial x_n)^{\alpha_n}$. For simplicity, we assume that $a_{ij}(D)$ coincides with its principal part $a_{ij}^{(0)}(D) := \sum_{|\alpha|=s_i+t_j} a_{ij\alpha} D^\alpha$ (note that $a_{ij}^{(0)}(D) = 0$ if $\operatorname{ord} a_{ij}(D) < s_i + t_j$). We impose the additional condition that the numbers $s_1, \dots, t_N$ can be chosen *nonnegative*.

We set $r_i := s_i + t_i$ and $R_i := (r_1 + \dots + r_i)/2$ (under the assumptions formulated below, the numbers R_i will be integer). We also set $R := R_N$ and $R_0 := 0$. We index the lines and columns of A such that the sequence r_i is nonincreasing. To simplify the notation, we will also assume that

$$r_1 > \dots > r_N > 0\,. \tag{2.2}$$

The operator $A(D)$ will act in the half-space $\mathbb{R}^n_+ := \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$ and will be supplemented by a matrix $B(D) := \big(b_{ij}(D)\big)_{\substack{i=1,\dots,R\\ j=1,\dots,N}}$ of boundary operators of general form. Denoting $m_i := \max_j(\operatorname{ord} B_{ij} - t_j)$, we have

$$\operatorname{ord} b_{ij}(D) \le m_i + t_j \quad (i = 1,\dots,R;\; j = 1,\dots,N). \tag{2.3}$$

We suppose that either B_{ij} is a homogeneous operator of order $m_i + t_j$, or it is identically zero. We index the boundary conditions such that the sequence $m_1, \dots, m_R$ is nondecreasing. In addition, we suppose that following conditions are satisfied:

$$m_{R_\ell} < m_{R_\ell + 1}, \quad \ell = 1, \dots, N-1\,. \tag{2.4}$$

2.2. Parameter-ellipticity condition for the inner symbol. In standard theory of elliptic systems the inner symbol is the principal part with coefficients freezed at an inner point of the manifold. In the standard case of systems with large parameter some weight is assigned to the parameter, and after this the parameter is included into the principal part. In the case where (2.2) holds this procedure cannot be realized and should be replaced by a more general procedure where all possible quasi-homogeneous principal parts obtained by assigning various weights $r > 0$ to the parameter are introduced.

For $\kappa = 1, \ldots, N$ we introduce submatrices $A_\kappa(\xi) - \lambda E_\kappa$ where

$$A_\kappa(D) := \Big(a_{ij}(D)\Big)_{i,j=1,\ldots,\kappa}$$

and E_κ is the $\kappa \times \kappa$-matrix which differs from the zero matrix only in the element at position (κ, κ) which equals 1. Adjusting the weight r_κ to the parameter λ we obtain a matrix which determinant is quasi-homogeneous. Now under an inner symbol we will understand the set of matrices $A_\kappa(\xi) - \lambda E_\kappa$.

Definition 2.1. (see [**11**] and [**4**], [**14**]) Let $\mathcal{L}$ be a closed sector in the complex plane with vertex at the origin. The matrix-symbol $A(D) - \lambda I$ is called parameter-elliptic with parameter in $\mathcal{L}$ if the following condition holds:
(i) For all $\kappa = 1, \ldots, N$, all $\xi \in \mathbb{R}^n \setminus \{0\}$ and all $\lambda \in \mathcal{L}$ we have

$$\det\big(A_\kappa(\xi) - \lambda E_\kappa\big) \neq 0\,. \tag{2.5}$$

This definition was introduced by Kozhevnikov [**10**], [**11**] and elaborated further in [**4**] and in [**14**] where several equivalent conditions for parameter-ellipticity of $A(D) - \lambda$ were discussed. In particular, the condition of parameter-ellipticity is equivalent to the estimates of elements of

$$G(\xi, \lambda) := (G_{ij}(\xi, \lambda))_{i,j=1,\ldots,N} := \big(A(\xi, \lambda) - \lambda I\big)^{-1}.$$

They are of the form

$$G_{ij}(\xi, \lambda) \leq \text{const}(|\xi| + |\lambda|^{1/r_i})^{-t_i}(|\xi| + |\lambda|^{1/r_j})^{-s_j}. \tag{2.6}$$

Let us make some comments on the above definition. Setting $\lambda = 0$ in (2.5) we obtain that all submatrices A_κ are elliptic in the sense of Douglis–Nirenberg. Their determinants are homogeneous elliptic polynomials in ξ of order $2R_\kappa$. From the ellipticity of these polynomials it follows that the algebraic equation

$$\det A_\kappa(\xi', z) = 0 \tag{2.7}$$

has no real roots for $\xi' \neq 0$ and the number $m_\kappa^\pm$ of roots in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $\mathbb{C}_- := -\mathbb{C}_+$ is independent of ξ'. It is customary to call the polynomial properly elliptic if $m_\kappa^+ = m_\kappa^-$. From this follows that $m_\kappa^\pm = R_\kappa$ and R_κ is integer. Note that for $n > 2$ the proper ellipticity condition is automatically satisfied.

The matrix $A(\xi) - \lambda I$ satisfying Definition 2.1 is called *properly parameter-elliptic*, if for $\kappa = 1, \ldots, N$ the polynomials $\det A_\kappa(\xi)$ are properly elliptic.

In what follows we will need

Lemma 2.2. *Assume that $A(D) - \lambda$ is properly parameter-elliptic in $\mathcal{L}$. Then for all $\lambda \in \mathcal{L} \setminus \{0\}$ the polynomial* $\det(A_\kappa(0, \cdot) - \lambda E_\kappa)$ *has exactly $r_\kappa/2$ roots with positive imaginary part, $\kappa = 1, \ldots, N$.*

Proof. Due to the Douglis–Nirenberg structure, the matrix $A_\kappa(0,\tau)$ has the form $(c_{ij}\tau^{s_i+t_j})_{i,j}$ with complex coefficients c_{ij}. Therefore

$$\begin{aligned}\det(A_\kappa(0,\tau)-\lambda E_\kappa) &= \det A_\kappa(0,\tau)-\lambda\det A_{\kappa-1}(0,\tau)\\ &= \tau^{2R_{\kappa-1}}\big(\tau^{r_\kappa}\det C_\kappa-\lambda\det C_{\kappa-1}\big),\end{aligned}$$

where we have set $C_\kappa := (c_{ij})_{i,j=1,\dots,\kappa}$. Due to condition (2.5) we have $\det C_\kappa \neq 0$ and $\det(A_\kappa(0,\tau)-\lambda E_\kappa)\neq 0$ for all $\tau\in\mathbb{R}\setminus\{0\}$ and $\lambda\in\mathcal{L}\setminus\{0\}$. As the number r_κ is even, the assertion follows. □

2.3. Parameter-ellipticity condition for boundary symbol. In the standard theory of elliptic systems the boundary symbol is an ODE problem on a half-line obtained after Fourier transform in tangential directions in principal parts of the system and boundary operators freezed at some point of the boundary. In the case of parameter-ellipticity the parameter is included in the boundary symbol.

In our case the role of the boundary symbol is played by two groups of ODE problems. The first group is

$$\begin{aligned}\big(A_\kappa(\xi',D_t)-\lambda E_\kappa\big)w(t)&=0\quad(t>0),\\ B_{1..\kappa}(\xi',D_t)w(0)&=g,\\ w(t)&\to 0\quad(t\to\infty).\end{aligned}\tag{2.8}$$

Here

$$B_{1..\kappa}(D) := \Big(b_{ij}(D)\Big)_{\substack{i=1,\dots,R_\kappa\\ j=1,\dots,\kappa}}.$$

If we pose $\varepsilon^{r_\kappa}=1/\lambda$ and divide the last equation in the system $\big(A_\kappa(\xi',D_t)-\lambda E_\kappa\big)w(t)=0$ by λ we obtain an ODE system with small parameter in front of the highest derivatives. We will need the study of solutions as $\varepsilon\to 0$. The Vishik–Lyusternik method suggests to consider following problems

$$\begin{aligned}\big(A_\kappa(0,D_t)-\lambda E_\kappa\big)v(t)&=0\quad(t>0),\\ B_\kappa(0,D_t)v(0)&=h,\\ v(t)&\to 0\quad(t\to\infty),\end{aligned}\tag{2.9}$$

where

$$B_\kappa(D) := \Big(b_{ij}(D)\Big)_{\substack{i=R_{\kappa-1}+1,\dots,R_\kappa\\ j=1,\dots,\kappa}}.$$

Definition 2.3. Let $\mathcal{L}$ be a closed sector in the complex plane with vertex at the origin. The problem $\big(A(D)-\lambda I, B(D)\big)$ is called parameter-elliptic with parameter in $\mathcal{L}$ if for all $\kappa=1,\dots,N$ the following conditions hold:

(i') $A(D)-\lambda I$ is properly parameter-elliptic in $\mathcal{L}$.

(ii) For all $\xi'\in\mathbb{R}^{n-1}\setminus\{0\}$, all $\lambda\in\mathcal{L}$ and all $g\in\mathbb{C}^{R_\kappa}$ the ODE problem (2.8) in $\mathbb{R}_+$ has a unique solution $w=(w_1,\dots,w_\kappa)^\top$.

(iii) For all $h\in\mathbb{C}^{r_\kappa/2}$ and all $\lambda\in\mathcal{L}\setminus\{0\}$ the ODE problem in $\mathbb{R}_+$ (2.9) has a unique solution $v=(v_1,\dots,v_\kappa)^\top$.

If we set $\lambda=0$ in condition (ii) of this definition, we obtain as a corollary condition

(iv) For each $\kappa=1,\dots,N$ the boundary value problem $\big(A_\kappa, B_{1..\kappa}\big)$ satisfies the standard Shapiro–Lopatinskii condition.

Remark 2.4. Boundary value problems of the form

$$\big\{A_\kappa(D) - \lambda E_\kappa, B_{1..\kappa}(D)\big\} \tag{2.10}$$

were studied in [**6**]. The problem was called weakly parameter-ellipticity if conditions (i)–(iv) were satisfied. In other words, the problem $\big(A(D)-\lambda, B(D)\big)$ is parameter-elliptic if for $\kappa = 1,\dots,N$ problems (2.10) are weakly parameter-elliptic in $\mathcal{L}$.

3. Main results

3.1. Functional spaces. Now we want to introduce parameter-dependent norms for the classical L_2-Sobolev spaces for which the parameter-elliptic boundary value problem $\big(A(D)-\lambda, B(D)\big)$ has a realization as a bounded operator which is invertible with bounded inverse for large λ. Here and in the following, by a bounded operator in parameter-dependent Sobolev spaces we understand a continuous operator whose norm can be estimated by a constant independent of the parameter. The definitions below are very close to Subsection 3.2 in [**7**].

For a fixed tuple $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_N) \in \mathbb{R}^N$ we set

$$\Psi_{\boldsymbol{\sigma}}(\xi,\lambda) := \prod_{j=1}^{N} (|\xi| + |\lambda|^{1/r_j})^{\sigma_j}$$

and define the parameter-dependent norm in $H^{\sigma_1+\dots+\sigma_N}(\mathbb{R}^n)$ by

$$\|v\|_{\boldsymbol{\sigma},\mathbb{R}^n} := \|F\Psi_{\boldsymbol{\sigma}}(\xi,\lambda)Fv(\xi)\|_{L_2(\mathbb{R}^n)}, \tag{3.1}$$

where F stands for the Fourier transform in $\mathbb{R}^n$. We will write $H_{\boldsymbol{\sigma}}(\mathbb{R}^n)$ for $H^{\sigma_1+\dots+\sigma_N}(\mathbb{R}^n)$ with norm (3.1). For the definition of $H_{\boldsymbol{\sigma}}(\mathbb{R}^{n-1})$ we replace $\Psi_{\boldsymbol{\sigma}}(\xi,\lambda)$ by $\Psi_{\boldsymbol{\sigma}}(\xi',\lambda) := \Psi_{\boldsymbol{\sigma}}(\xi',0,\lambda)$.

The space $H_{\boldsymbol{\sigma}}(\mathbb{R}^n_+)$ is defined as the quotient space $H_{\boldsymbol{\sigma}}(\mathbb{R}^n)/H_{\boldsymbol{\sigma}}(\mathbb{R}^n)_-$ where $H_{\boldsymbol{\sigma}}(\mathbb{R}^n)_-$ stands for the subspace of all distributions in $H_{\boldsymbol{\sigma}}(\mathbb{R}^n)$ whose support is contained in the set $\{x \in \mathbb{R}^n : x_n \le 0\}$ and is endowed with the standard quotient norm $\|\cdot\|_{\boldsymbol{\sigma},\mathbb{R}^n_+}$.

In what follows we will use norms equivalent to (3.1). To define them we need "shifted" functions $\Psi_{\boldsymbol{\sigma}}^{(-\tau)}$ which are defined for $\boldsymbol{\sigma} \in \mathbb{R}^n$ with $\sigma_i \ge 0$ for $i = 2,\dots,N$ and for $\tau > \sigma_1$ by

$$\Psi_{\boldsymbol{\sigma}}^{(-\tau)}(\xi,\lambda) := (|\xi| + |\lambda|^{1/r_K})^{\sigma_1+\dots+\sigma_K-\tau} \prod_{j=K+1}^{N} (|\xi| + |\lambda|^{1/r_j})^{\sigma_j}$$

where the index K is chosen such that

$$\sigma_1 + \dots + \sigma_{K-1} < \tau \le \sigma_1 + \dots + \sigma_K$$

(with obvious modifications for $\tau > \sigma_1 + \dots + \sigma_N$). In the case $\tau \le \sigma_1$ we pose

$$\Psi_{\boldsymbol{\sigma}}^{(-\tau)}(\xi,\lambda) := (|\xi| + |\lambda|^{1/r_1})^{\sigma_1-\tau} \prod_{j=2}^{N} (|\xi| + |\lambda|^{1/r_j})^{\sigma_j}\,.$$

Denote by $H_{\boldsymbol{\sigma}}^{(-\tau)}(\mathbb{R}^{n-1})$ the space corresponding to the shifted weight function $\Psi_{\boldsymbol{\sigma}}^{(-\tau)}(\xi,\lambda)$.

Remark 3.1. The definition of the "shifted" function has a "geometrical" interpretation (see [**5**]). Suppose Γ is a convex polygon in $\mathbb{R}^2_{(p,q)}$ with vertices (p_j, q_j), $j = 0, \dots, N$, where p_j, q_j are nonnegative. We correspond to Γ the function

$$\Xi_\Gamma(\xi, \lambda) := \sum_{j=0}^{N} |\xi|^{p_j} |\lambda|^{q_j}.$$

Suppose the components of $\boldsymbol{\sigma} \in \mathbb{R}^N$ are nonnegative and Γ is the polygon with vertices

$$(0,0), \Big(0, \frac{\sigma_1}{r_1} + \dots + \frac{\sigma_N}{r_N}\Big), \Big(\sigma_1, \frac{\sigma_2}{r_2} + \dots + \frac{\sigma_N}{r_N}\Big), \dots, (\sigma_1 + \dots + \sigma_N, 0).$$

In this case $\Xi_\Gamma(\xi, \lambda) \approx \Psi_{\boldsymbol{\sigma}}(\xi, \lambda)$. If we denote by Γ^{-r} the shift of Γ to the left parallel to the abscissa, then $\Xi_{\Gamma^{-r}}(\xi, \lambda) \approx \Psi_{\boldsymbol{\sigma}}^{-r}(\xi, \lambda)$.

If $\sigma_1 + \dots + \sigma_N \in \mathbb{N}$, the norm in $H_{\boldsymbol{\sigma}}(\mathbb{R}^n_+)$ is equivalent to the norm

$$\|v\|_{\boldsymbol{\sigma}, \mathbb{R}^n_+} := \Big(\sum_{\ell=0}^{\sigma_1+\dots+\sigma_N} \int_{\mathbb{R}^{n-1}} \big(\Psi_{\boldsymbol{\sigma}}^{(-\ell)}(\xi', \lambda)\big)^2 \|D_t^\ell (F'v)(\xi', \cdot)\|^2_{L_2(\mathbb{R}_+)} \,\mathrm{d}\,\xi' \Big)^{1/2}, \tag{3.2}$$

where F' stands for the partial Fourier transform with respect to the first $n-1$ variables (cf. [**5**]). In the following, we will only deal with norm (3.2) in the case of $\mathbb{R}^n_+$.

Now let us consider the trace operator γ_0 mapping a function v defined in $\mathbb{R}^n_+$ to $x' \mapsto v(x', 0)$. Throughout the following, the letter C stands for an unspecified constant. The following result is taken from [**5**].

Lemma 3.2. *Let* $\sigma_1 + \dots + \sigma_N > \frac{1}{2}$. *Then we have for every* $\lambda_0 > 0$

$$\|\gamma_0 v\|^{(-1/2)}_{\boldsymbol{\sigma}, \mathbb{R}^{n-1}} \le C \|v\|_{\boldsymbol{\sigma}, \mathbb{R}^n_+} \quad (v \in H_{\boldsymbol{\sigma}}(\mathbb{R}^n_+),\ |\lambda| \ge \lambda_0).$$

3.2. Realization of problem (1.1)–(1.2) **as a bounded operator.** Now we want to show that a boundary value problem $(A - \lambda, B)$ of the structure discussed in Section 2 has a realization as a bounded operator in these Sobolev spaces. Here no ellipticity is assumed. In the following, $\mathbf{e}_i$ stands for the i-th unit vector in $\mathbb{C}^N$.

Proposition 3.3. *Let the matrix operators* (A, B) *satisfy* (2.1), (2.3). *Then for every* $\boldsymbol{\sigma} \in \mathbb{R}^N$ *with* $\sigma_2, \dots, \sigma_N \ge 0$ *and* $\sigma_1 + \dots + \sigma_N > m_R + \frac{1}{2}$ *and every* $\lambda \in \mathbb{C}$ *with* $|\lambda| \ge \lambda_0 > 0$ *the operator*

$$(A - \lambda, B) \colon \prod_{i=1}^N H_{\boldsymbol{\sigma} + t_i \mathbf{e}_i}(\mathbb{R}^n_+) \to \prod_{i=1}^N H_{\boldsymbol{\sigma} - s_i \mathbf{e}_i}(\mathbb{R}^n_+) \times \prod_{j=1}^R H_{\boldsymbol{\sigma}}^{(-m_j - 1/2)}(\mathbb{R}^{n-1})$$

is continuous.

Proof. We first consider $A(D) - \lambda$ acting in $\mathbb{R}^n$. Denote $f = (A(D) - \lambda)u$. The inequality

$$\sum_{i=1}^N \|f_i\|_{\boldsymbol{\sigma} - s_i \mathbf{e}_i, \mathbb{R}^n} \le C \sum_{j=1}^N \|u_j\|_{\boldsymbol{\sigma} + t_j \mathbf{e}_j, \mathbb{R}^n}$$

is obviously equivalent to the uniform boundedness of the norm of the matrix

$$\operatorname{diag}\big(\Psi_{\boldsymbol{\sigma} - s_1 \mathbf{e}_1}, \dots, \Psi_{\boldsymbol{\sigma} - s_N \mathbf{e}_N}\big)(A(\xi) - \lambda) \operatorname{diag}\big(\Psi_{-\boldsymbol{\sigma} - t_1 \mathbf{e}_1}, \dots, \Psi_{-\boldsymbol{\sigma} - t_N \mathbf{e}_N}\big).$$

This fact follows from the inequality

$$\left|a_{ij}(\xi) - \lambda\delta_{ij}\right| \le C(|\xi| + |\lambda|^{1/r_i})^{s_i}(|\xi| + |\lambda|^{1/r_j})^{t_j}.$$

To show the continuity of $A(D) - \lambda$ acting in the half-space, we use a continuous extension operator E from $H_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\mathbb{R}^n_+)$ to $H_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\mathbb{R}^n)$. (Such an operator can be defined using the standard Hestenes construction.) Let $u \in \prod_{i=1}^N H_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\mathbb{R}^n_+)$ and $f := (A - \lambda)u$. Setting $Ef := (A - \lambda)Eu$, we obtain

$$\begin{aligned}\sum_{i=1}^N \|f_i\|_{\mathbf{r}-s_i\mathbf{e}_i,\mathbb{R}^n_+} &\le C\sum_{i=1}^N \|(Ef)_i\|_{\mathbf{r}-s_i\mathbf{e}_i,\mathbb{R}^n} \le C\sum_{i=1}^N \|(Eu)_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n} \\ &\le C\sum_{i=1}^N \|u_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+},\end{aligned}$$

which shows the continuity of $A(D) - \lambda$ acting in the half-space. In the same way, if we consider $B(D)$ as an operator acting in the half-space (without taking the trace), it is continuous from $\prod_{i=1}^N H_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\mathbb{R}^n_+)$ to $\prod_{j=1}^R H_{\boldsymbol{\sigma}-m_j\mathbf{e}_1}(\mathbb{R}^n_+)$. Indeed,

$$B_{jk}(D)u_k \in H_{\boldsymbol{\sigma}-(m_j+t_k)\mathbf{e}_1+t_k\mathbf{e}_k}(\mathbb{R}^n_+) \subset H_{\boldsymbol{\sigma}-m_j\mathbf{e}_1}(\mathbb{R}^n_+)$$

if $u_k \in H_{\boldsymbol{\sigma}+t_k\mathbf{e}_k}(\mathbb{R}^n_+)$. Taking the trace and applying Lemma 3.2, we see that

$$B(D)\colon \prod_{i=1}^N H_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\mathbb{R}^n_+) \to \prod_{j=1}^R H^{(-1/2)}_{\boldsymbol{\sigma}-m_j\mathbf{e}_1}(\mathbb{R}^{n-1})$$

is continuous.

But by definition of the "shifted" function and $\sigma_1 + \dots + \sigma_N > m_R + 1/2$ we have $\Psi^{(-1/2)}_{\boldsymbol{\sigma}-m_j\mathbf{e}_1} \equiv \Psi^{(-m_j-1/2)}_{\boldsymbol{\sigma}}$, and we obtain the continuity of the operator

$$B(D)\colon \prod_{i=1}^N H_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\mathbb{R}^n_+) \to \prod_{j=1}^R H^{(-m_j-1/2)}_{\boldsymbol{\sigma}}(\mathbb{R}^{n-1}).$$

□

3.3. The inverse of the operator related to (1.1)–(1.2) **and its estimates.** We now come to the main result of the present paper which states that for a parameter-elliptic boundary value problem the operator of Proposition 3.3 has a bounded inverse for large values of λ.

THEOREM 3.4. *Let $\mathcal{L}$ be a closed sector in the complex plane with vertex at the origin and let $\boldsymbol{\sigma} \in \mathbb{R}^N$ be fixed satisfying*

$$\begin{aligned}\sigma_1 + \dots + \sigma_k &\in [m_{R_k} + 1/2, m_{R_k+1} + 1/2] \quad (k = 1, \dots, N-1), \\ \sigma_1 + \dots + \sigma_N &> m_R + 1/2\end{aligned} \tag{3.3}$$

For simplicity, let $\sigma_1 + \dots + \sigma_N \in \mathbb{N}$.

(a) *Let the boundary value problem $(A(D) - \lambda, B(D))$ be parameter-elliptic in $\mathcal{L}$. Then there exists a $\lambda_0 > 0$ such that for every $\lambda \in \mathcal{L}$ with $|\lambda| \ge \lambda_0$ and for every*

$$f \in \prod_{i=1}^N H_{\boldsymbol{\sigma}-s_i\mathbf{e}_i}(\mathbb{R}^n_+) \text{ and } g \in \prod_{j=1}^R H^{(-m_j-1/2)}_{\boldsymbol{\sigma}}(\mathbb{R}^{n-1})$$

the boundary value problem

$$\begin{aligned} (A(D)-\lambda)u &= f \quad \text{in } \mathbb{R}^n_+, \\ B(D)u &= g \quad \text{on } \mathbb{R}^{n-1} \end{aligned} \tag{3.4}$$

has a unique solution $u \in \prod_{i=1}^N H_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\mathbb{R}^n_+)$, *and the (two-sided) a priori estimate*

$$\sum_{i=1}^N \|u_i\|_{\boldsymbol{\sigma}+t_i\mathbf{e}_i,\mathbb{R}^n_+} \le C\Big(\sum_{i=1}^N \|f_i\|_{\boldsymbol{\sigma}-s_i\mathbf{e}_i,\mathbb{R}^n_+} + \sum_{j=1}^R \|g_j\|^{(-m_j-1/2)}_{\boldsymbol{\sigma},\mathbb{R}^{n-1}}\Big) \tag{3.5}$$

holds for all $\lambda \in \mathcal{L}$, $|\lambda| \ge \lambda_0$, *with a constant* C *independent of* u *or* λ.

(b) *If the boundary value problem* (3.4) *is uniquely solvable for large* $\lambda \in \mathcal{L}$ *in the sense above and the a priori estimate* (3.5) *holds, then conditions* 2.1 (i) *and* 2.3 (ii), (iii) *are satisfied.*

The proof of part (a) is based on the main technical result of the paper.

Theorem 3.5. *Let* $(A(D)-\lambda, B(D))$ *be parameter-elliptic in* $\mathcal{L}$ *and assume that* $\boldsymbol{\sigma}$ *satisfies* (3.3). *Then there exists a* $\lambda_0 > 0$ *such that for all* $\xi' \in \mathbb{R}^n$, *all* $\lambda \in \mathcal{L}$, $|\lambda| \ge \lambda_0$ *and all* $h \in \mathbb{C}^R$ *the ODE problem*

$$(A(\xi', D_t)-\lambda)w(t) = 0 \quad (t>0), \tag{3.6}$$

$$B(\xi', D_t)w(0) = h \in \mathbb{C}^R, \tag{3.7}$$

$$|w(t)| \to 0 \quad (t \to \infty)$$

has a unique solution $w(t,\xi',\lambda) = \big(w_i(t,\xi',\lambda)\big)_{i=1,\dots,N}$ *satisfying for* $\ell = 0,1,\dots$

$$\sum_{i=1}^N \Psi^{(-\ell)}_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\xi',\lambda)\|D_t^\ell w_i(\cdot,\xi',\lambda)\|_{L_2(\mathbb{R}_+)} \le C\sum_{j=1}^R \Psi_{\boldsymbol{\sigma}}^{(-m_j-1/2)}(\xi',\lambda)\,|h_j|\,. \tag{3.8}$$

We first derive Theorem 3.4 (a) from Theorem 3.5.

Proof of Theorem 3.4 (a).

As in the proof of Proposition 3.3 we first consider the inverse of $A(D)-\lambda$ acting in $\mathbb{R}^n$. It follows from (2.6) that for sufficiently large $|\lambda|$ the norm of the matrix

$$\operatorname{diag}\big(\Psi_{\boldsymbol{\sigma}+t_1\mathbf{e}_1},\dots,\Psi_{\boldsymbol{\sigma}+t_N\mathbf{e}_N}\big)G(\xi,\lambda)\operatorname{diag}\big(\Psi_{-\boldsymbol{\sigma}+s_1\mathbf{e}_1},\dots,\Psi_{-\boldsymbol{\sigma}+t_N\mathbf{e}_N}\big)$$

is uniformly bounded. From this the a priori estimate

$$\sum_{j=1}^N \|u_j\|_{\boldsymbol{\sigma}+t_j\mathbf{e}_j,\mathbb{R}^n} \le C\sum_{i=1}^N \|f_i\|_{\boldsymbol{\sigma}-s_i\mathbf{e}_i,\mathbb{R}^n}$$

follows where $f = (A(D)-\lambda)u$.

Now we define $v := F^{-1}(A(\xi)-\lambda)^{-1}F(Ef)(\xi)$ where Ef denotes the Hestenes extension of f, and $u^{(1)} = v\big|_{\mathbb{R}^n_+}$. Then

$$\begin{aligned} \sum_{j=1}^N \|u_j^{(1)}\|_{\boldsymbol{\sigma}+t_j\mathbf{e}_j,\mathbb{R}^n_+} &\le \sum_{j=1}^N \|v_j\|_{\boldsymbol{\sigma}+t_j\mathbf{e}_j,\mathbb{R}^n} \\ &\le C\sum_{i=1}^N \|(Ef)_i\|_{\boldsymbol{\sigma}-s_i\mathbf{e}_i,\mathbb{R}^n} \le C\sum_{i=1}^N \|f_i\|_{\boldsymbol{\sigma}-s_i\mathbf{e}_i,\mathbb{R}^n_+}. \end{aligned} \tag{3.9}$$

Now we consider $u^{(2)} := u - u^{(1)}$. Taking partial Fourier transform $w(t,\xi',\lambda) := (F'u^{(2)})(\xi',t)$, we see that for almost all $\xi' \in \mathbb{R}^{n-1}$ equations (3.6)–(3.7) hold with

$$h = h(\xi',\lambda) := (F'g)(\xi') - (F'Bu^{(1)})(\xi',\lambda).$$

Due to Proposition 3.3 and (3.9), we have for $j = 1,\dots,R$

$$\Big(\int_{\mathbb{R}^{n-1}} \Big[\Psi_{\boldsymbol\sigma}^{(-m_j-1/2)}(\xi',\lambda)|h_j(\xi',\lambda)|\Big]^2 d\xi'\Big)^{1/2} \\ \leq C\,\Big(\sum_{i=1}^N \|f_i\|_{\boldsymbol\sigma - s_i\mathbf{e}_i,\mathbb{R}^n_+} + \|g_j\|^{(-m_j-1/2)}_{\boldsymbol\sigma,\mathbb{R}^{n-1}}\Big).$$

Now we apply Theorem 3.5, choosing $\lambda \in \mathcal{L}$ large enough, to obtain that (3.6)–(3.7) has a unique solution $w(t,\xi',\lambda)$, and we can define $u_2(x',t) := (F')^{-1}w(t,\xi',\lambda)$. Using the norm (3.2) and the estimate (3.8), we see

$$\begin{aligned}\|u_i^{(2)}\|_{\boldsymbol\sigma+t_i\mathbf{e}_i,\mathbb{R}^n_+} &\leq C\Big(\sum_{\ell=0}^{\sigma_1+\dots+\sigma_N}\int_{\mathbb{R}^{n-1}}\big[\Psi^{(-\ell)}_{\boldsymbol\sigma+t_i\mathbf{e}_i}(\xi',\lambda)\|D_t^\ell w(\cdot,\xi',\lambda)\|_{L_2(\mathbb{R}_+)}\big]^2\,\mathrm{d}\,\xi'\Big)^{1/2}\\ &\leq C\Big(\sum_{j=1}^R\int_{\mathbb{R}^{n-1}}\big[\Psi_{\boldsymbol\sigma}^{(-m_j-1/2)}(\xi',\lambda)|h_j(\xi',\lambda)|\big]^2\,\mathrm{d}\,\xi'\Big)^{1/2}\\ &\leq C\sum_{j=1}^R\|h_j\|^{(-m_j-1/2)}_{\boldsymbol\sigma,\mathbb{R}^{n-1}}\\ &\leq C\Big(\sum_{i=1}^N\|f_i\|_{\boldsymbol\sigma-s_i\mathbf{e}_i,\mathbb{R}^n_+}+\sum_{j=1}^R\|g_j\|^{(-m_j-1/2)}_{\boldsymbol\sigma,\mathbb{R}^{n-1}}\Big).\end{aligned}$$

From this and (3.9) we obtain the solvability of the boundary value problem (3.4) for large $\lambda \in \mathcal{L}$ in the spaces indicated in the theorem and the a priori estimate (3.5). The uniqueness of the solution follows from unique solvability of (3.6)–(3.7).

Altogether, we have shown that the assertions of (a) follow from parameter-ellipticity of $\big(A(D)-\lambda, B(D)\big)$. The proof of the necessity (part (b)) will be done in Section 6. □

3.4. Plan of further exposition. Sections 4–5 are devoted to the proof of Theorem 3.5. To prove this theorem, we construct the so-called fundamental system of solutions of the problem (3.6)–(3.7), i.e the solutions $w^{(k)}(t,\xi'\lambda) = (w_1^{(k)},\dots,w_N^{(k)})^\top$ corresponding to $h = e_k$, $\quad k = 1,\dots,R$, where e_k are unit vectors in $\mathbb{C}^R$. For the components of these solutions the main inequality (3.8) can be rewritten as

$$\|D_t^\ell w_i^{(k)}(\cdot,\xi',\lambda)\|_{L_2(\mathbb{R}_+)} \leq C\frac{\Psi_{\boldsymbol\sigma}^{(-m_k-1/2)}(\xi',\lambda)}{\Psi^{(-\ell)}_{\boldsymbol\sigma+t_i\mathbf{e}_i}(\xi',\lambda)}. \tag{3.10}$$

Solutions of the system (3.6) are expressed in terms of the roots $\tau(\xi',\lambda)$ of the algebraic equation

$$P(\xi',\tau,\lambda) := \det(A(\xi',\tau) - \lambda I_N) = 0. \tag{3.11}$$

These roots are algebraic functions of several variables, their behaviour is rather complicated and deeply connected with the Newton polygon of the polynomial P.

This question will be discussed in the next section where we also will formulate in algebraic form conditions (ii) and (iii) of Definition 2.3.

4. Some auxilary results

4.1. Remarks on boundary value problems for ODE systems in $\mathbb{R}_+$. Let $A(\tau)$ be an $N \times N$ matrix which elements are polynomials in τ of order not greater than some natural number σ. Suppose the contour $\gamma \subset \mathbb{C}_+$ does not intersect with the set of zeros of the algebraic equation

$$\det A(\tau) = 0. \tag{4.1}$$

Denote by $\mathfrak{M}_\gamma$ the subspace of solutions on the half-line of the ODE system

$$A(D_t)v(t) = 0, \quad t > 0, \tag{4.2}$$

which can be represented in the form

$$v(t) = \frac{1}{2\pi i}\int_\gamma e^{i\tau t} A^{-1}(\tau)C(\tau)d\tau \tag{4.3}$$

with a vector $C(\tau)$ whose components are polynomials in τ of order not greater than $\sigma - 1$.

It should be mentioned that in the case where γ encloses all zeros of (4.1) in $\mathbb{C}_+$, an arbitrary solution $v(t)$ of (4.2) belongs to the subspace $\mathfrak{M}_\gamma$ if and only if $|v(t)| \to 0, \quad t \to +\infty$.

Suppose γ contains R zeros of (4.1) (counted according multiplicities). Then $\dim \mathfrak{M}_\gamma = R$, so we assume that we have R boundary conditions at $t = 0$:

$$B(D_t)v(0) = g \tag{4.4}$$

Here $B(\tau)$ is an $R \times N$ polynomial matrix.

Proposition 4.1. [**13**] *For the problem* (4.2), (4.4) *the following conditions are equivalent.*

(i) *The problem* (4.2), (4.4) *has a unique solution* $v(t) \in \mathfrak{M}_\gamma$ *for arbitrary* $g \in \mathbb{C}^R$.

(ii) *The rank of the Lopatinskii matrix*

$$\Lambda := \frac{1}{2\pi i}\int_\gamma B(\tau)A^{-1}(\tau)(I_N, \tau I_N, \dots, \tau^{\sigma-1}I_N)\,\mathrm{d}\,\tau$$

is maximal (*i.e. equals* R).

(iii) *There exists an* $N \times R$ *polynomial matrix* $N(\tau)$ *such that*

$$\frac{1}{2\pi i}\int_\gamma B(\tau)A^{-1}(\tau)N(\tau)d\tau = I_R. \tag{4.5}$$

If one of these conditions holds, the unique solution is given by

$$w(t) = \Big(\frac{1}{2\pi i}\int_\gamma e^{it\tau}A^{-1}(\tau)N(\tau)\,\mathrm{d}\,\tau\Big)\, g \tag{4.6}$$

where $N(\tau)$ *is any matrix satisfying* (4.5).

Remark 4.2. Denote by Λ^* the Hermitian adjoint of Λ. Then Λ has maximal rank if and only if the product $\Lambda\Lambda^*$ is a nonsingular matrix. (Indeed, denote by $\Lambda_k, k = 1, \dots, R$, the lines in Λ treated as vectors in $\mathbb{C}^R$. Then $\Lambda\Lambda^*$ will be the

Gram matrix of this system of vectors.) In this case an explicit formula for a matrix $N(\tau)$ satisfying (4.5) is given by

$$N(\tau) = (I_N, \tau I_N, \dots, \tau^{\sigma-1} I_N)\Lambda^*(\Lambda\Lambda^*)^{-1}.$$

REMARK 4.3. Condition (ii) in Proposition 4.1 may be replaced by the equivalent and formally more general condition where Λ is replaced by

$$\Lambda' := \frac{1}{2\pi i}\int_\gamma QB(\tau)A^{-1}(\tau)(H, (\tau/q)H, \dots, (\tau/q)^{\sigma-1}H)d\tau.$$

with arbitrary number q and arbitrary nonsingular $R\times R$ matrix Q and nonsingular $N \times N$ matrix H. In fact, Λ' is the matrix corresponding (in the sense of Proposition 4.1) to the problem obtained from (4.2), (4.4) by the substitution $t' = qt$ and $u = Hv$ (note that the invertible matrix Q has no effect on the rank).

Now we apply the results of this subsection to the problems (2.8) and (2.9) where we assume $\big(A(D) - \lambda, B(D)\big)$ to be properly parameter-elliptic in $\mathcal{L}$. Due to this condition, the polynomial $\det(A_\kappa(\xi', \cdot) - \lambda E_\kappa)$ has exactly R_κ roots in $\mathbb{C}_+$ which we denote by $\tau^0_{1,\kappa}(\xi', \lambda), \dots, \tau^0_{R_\kappa,\kappa}(\xi', \lambda)$. Let $\gamma^0_\kappa(\xi', \lambda) \subset \mathbb{C}_+$ be a closed contour enclosing these zeros.

Similarly, by Lemma 2.2 the polynomial $\det(A_\kappa(0, \cdot) - \lambda E_\kappa)$ has exactly $r_\kappa/2$ zeros in $\mathbb{C}_+$ which will be denoted by $\tau^1_{R_{\kappa-1}+1}(\lambda), \dots, \tau^1_{R_\kappa}(\lambda)$. We choose a closed contour $\gamma^1_\kappa(\lambda) \subset \mathbb{C}_+$ enclosing these zeros. Now conditions (ii) and (iii) of Definition 2.1 can be formulated in the following form, where we set $\sigma := \max_{i,j} \operatorname{ord} a_{ij}$.

LEMMA 4.4. a) *For all* $\kappa = 1, \dots, N$, *all* $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ *and all* $\lambda \in \mathcal{L}$ *the* $R_\kappa \times \sigma\kappa$*-matrix*

$$M^0_\kappa(\xi', \lambda) := \frac{1}{2\pi i}\int_{\gamma^0_\kappa(\xi',\lambda)} B_{1..\kappa}(\xi', \tau)(A_\kappa(\xi', \tau) - \lambda E_\kappa)^{-1}(I_\kappa, \dots, \tau^{\sigma-1}I_\kappa)\,\mathrm{d}\,\tau$$

has rank R_κ *(i.e. maximal rank).*

b) *For all* $\kappa = 1, \dots, N$ *and all* $\lambda \in \mathcal{L} \setminus \{0\}$ *the* $r_\kappa/2 \times \sigma\kappa$*-matrix*

$$M^1_\kappa(\lambda) := \frac{1}{2\pi i}\int_{\gamma^1_\kappa(\lambda)} B_\kappa(0, \tau)(A_\kappa(0, \tau) - \lambda E_\kappa)^{-1}(I_\kappa, \dots, \tau^{\sigma-1}I_\kappa)\,\mathrm{d}\,\tau$$

has rank $r_\kappa/2$.

4.2. The roots of $\det(A(\xi', \cdot) - \lambda)$**.** Now we turn to the study of the behaviour of the roots $\tau(\xi', \lambda)$ of the algebraic equation (3.11) which belong to the upper half-plane of the complex plane. As it was shown in [**4**] the matrix $A(\xi) - \lambda I_N$ is parameter-elliptic in the sense of Definition 2.1 if and only if the polynomial in (3.11) is N-elliptic with parameter in the sense of [**4**], Definition 2.1 (see also [**7**], Definition 2.2). In particular, for large enough $|\lambda|$

$$|\det(A(\xi) - \lambda I)| \geq C \prod_{j=1}^N (|\xi|^{r_j} + |\lambda|) \tag{4.7}$$

It follows from (4.7) that the equation in $z \in \mathbb{C}$

$$\det\big(A(\xi', z) - \lambda I\big) = 0 \tag{4.8}$$

has no real roots for large enough $|\lambda|$ and the number $m^\pm$ of roots in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $\mathbb{C}_- := -\mathbb{C}_+$ is independent of (ξ', λ). Replacing (ξ', z) by $(\rho\xi', \rho z)$ and taking the limit $\rho \to \infty$, we obtain that numbers $m^\pm$ coincide with the corresponding numbers for equation (2.7) with $\kappa = N$. In the case of proper parameter-ellipticity we obtain that $m^\pm = R$.

Hence for the study of the zeros of (3.11) we can use the results of [**7**], Section 4. We only have to note that the edge principal parts P_κ and Q_κ, which play an important role in [**7**], can be calculated explicitly in terms of the matrices $A_\kappa(\xi',\tau)-\lambda E_\kappa$ (see [**4**], Section 3 for detailed exposition). As a result we have

$$P_\kappa(\xi',\tau,\lambda) = \det\big(A_\kappa(\xi',\tau) - \lambda E_\kappa\big),$$
$$Q_{\kappa,\lambda}(\tau) = \frac{\det A_\kappa(0,\tau)}{\det A_{\kappa-1}(0,\tau)} - \lambda.$$

The roots we are investigating essentially depend on the parameters $(|\xi'|,|\lambda|)$. Following [**9**] and [**7**], we indroduce a partition of the space of all ξ' and λ depending on two parameters $\rho,\ \delta > 0$ defined by

$$G(\rho) := \mathbb{R}^{n-1} \times \{\lambda \in \mathcal{L} : |\lambda| \geq \rho\} = \bigcup_{\kappa=1}^{N} G_\kappa(\rho,\delta) \cup \bigcup_{\kappa=0}^{N} \widetilde{G}_\kappa(\rho,\delta) \tag{4.9}$$

where we set

$$G_\kappa(\rho,\delta) := \{(\xi',\lambda) \in G(\rho) : \delta|\xi'|^{r_\kappa} \leq |\lambda| \leq \delta^{-1}|\xi'|^{r_\kappa} \quad (\kappa = 1,\dots,N),$$
$$\widetilde{G}_0(\rho,\delta) := \{(\xi',\lambda) \in G(\rho) : \delta^{-1}|\xi'|^{r_1} < |\lambda|\},$$
$$\widetilde{G}_\kappa(\rho,\delta) := \{(\xi',\lambda) \in G(\rho) : \delta^{-1}|\xi'|^{r_{\kappa+1}} < |\lambda| < \delta|\xi'|^{r_\kappa}\} \quad (\kappa = 1,\dots,N-1),$$
$$\widetilde{G}_N(\rho,\delta) := \{(\xi',\lambda) \in G(\rho) : |\lambda| < \delta|\xi'|^{r_N}\}.$$

For formal reasons, we set $G_0(\rho,\delta) := \emptyset$. If ρ and δ satisfy

$$\rho > \delta^{-(r_\kappa + r_{\kappa+1})/(r_\kappa - r_{\kappa+1})} \quad (\kappa = 1,\dots,N-1) \tag{4.10}$$

the sets $G_\kappa(\rho,\delta)$, $\widetilde{G}_\kappa(\rho,\delta)$ are mutually disjoint. For an interpretation of this partition in terms of the Newton polygon, we refer the reader to [**9**] and [**7**].

The zeros of $\det(A(\xi',\cdot)-\lambda)$ will be compared with the zeros of of the quasi-homogeneous polynomials P_κ and Q_κ. The following result is taken from [**7**], Section 4.

Theorem 4.5. a) *For every $\varepsilon > 0$ there exists a $\delta_0 > 0$ such that for all $\kappa = 0,\dots,N$ and all $(\xi',\lambda) \in \widetilde{G}_\kappa(1,\delta_0)$ there exist zeros $\tau_1(\xi',\lambda),\dots,\tau_R(\xi',\lambda)$ of* $\det(A(\xi',\cdot)-\lambda)$ *satisfying*

$$|\tau_k(\xi',\lambda) - \tau^0_{\kappa,k}(\xi',0)| \leq \varepsilon|\xi'| \quad (k = 1,\dots,R_\kappa),$$
$$|\tau_k(\xi',\lambda) - \tau^1_k(\lambda)| \leq \varepsilon|\lambda|^{1/r_\ell} \quad (k = R_{\ell-1}+1,\dots,R_\ell;\ \ell = \kappa+1,\dots,N).$$

b) *For every $\varepsilon > 0$ and $\delta > 0$ there exists a $\rho_0 > 0$ such that for all $\kappa = 1,\dots,N$ and all $(\xi',\lambda) \in G_\kappa(\rho_0,\delta)$ there exist zeros $\tau_1(\xi',\lambda),\dots,\tau_R(\xi',\lambda)$ of* $\det(A(\xi',\cdot)-\lambda)$ *satisfying*

$$|\tau_k(\xi',\lambda) - \tau^0_{\kappa,k}(\xi',\lambda)| \leq \varepsilon(|\xi'| + |\lambda|^{1/r_\kappa}) \quad (k = 1,\dots,R_\kappa),$$
$$|\tau_k(\xi',\lambda) - \tau^1_k(\lambda)| \leq \varepsilon\lambda^{1/r_\ell} \quad (k = R_{\ell-1}+1,\dots,R_\ell;\ \ell = \kappa+1,\dots,N).$$

4.3. Decomposition of the space of stable solutions of system (3.6)**.** According to Theorem 4.5 for $(\xi',\lambda)\in\widetilde{G}_\kappa\cup G_\kappa$ with specially chosen ρ and δ we can define a system of contours $\gamma^0_\kappa(\xi'),\gamma^1_\ell(\lambda)$, $\ell=\kappa+1,\dots,N$ with following properties:

1) these contours belong to $\mathbb{C}_+$ and the distances between them are strictly positive;

2) if the roots of (4.8) are indexed according to Theorem 4.5, then $\gamma^0_\kappa(\xi',\lambda)$ encloses $\tau_1(\xi',\lambda),\dots,\tau_\kappa(\xi',\lambda)$ and $\gamma^1_\ell(\lambda)$ encloses $\tau_{R_{\ell-1}+1}(\lambda),\dots,\tau_{R_\ell}(\lambda)$ for $\ell=\kappa+1,\dots,N$.

With the notation of Subsection 4.1, we consider the spaces $\mathfrak{M}_{\gamma^0_\kappa}(\xi',\lambda)$ and $\mathfrak{M}_{\gamma^1_\ell}(\lambda)$ of stable solutions of (5.2) which can be represented in the form (5.7) with $A(\tau)$ being replaced by $A(\xi',\tau)-\lambda$ and γ being replaced by $\gamma^0_\kappa(\xi',\lambda)$ and $\gamma^1_\ell(\lambda)$, respectively.

Note that the space $\mathfrak{M}_+(\xi,\lambda)$ of all stable solutions is, for $(\xi',\lambda)\in\widetilde{G}_\kappa\cup G_\kappa$, the direct sum of the above subspaces:

$$\mathfrak{M}_+(\xi,\lambda)=\mathfrak{M}_{\gamma^0_\kappa}(\xi',\lambda)+\mathfrak{M}_{\gamma^1_{\kappa+1}}+\dots+\mathfrak{M}_{\gamma^1_N}. \tag{4.11}$$

Note that elements of the right-hand side subspaces exponentially decrease as $t\to+\infty$. More accurately, the elements of these spaces are $O(e^{-\operatorname{const}(|\xi'|+|\lambda|^{1/r_\kappa})})$, and $O(e^{-\operatorname{const}|\lambda|^{1/r_{\kappa+1}}}),\dots,O(e^{-\operatorname{const}|\lambda|^{1/r_N}})$, respectively. As under our conditions

$$|\xi'|+|\lambda|^{1/r_\kappa}\ll|\lambda|^{1/r_\ell},\quad \ell=\kappa+1,\dots,N,$$

the elements of $\mathfrak{M}_{\gamma^1_{\kappa+1}},\dots,\mathfrak{M}_{\gamma^1_N}$ can be treated as exponential boundary layers.

In fact, in the next section we will follow in the Vishik–Lyusternik method and use conditions (i) and (ii) to construct the solution of the problem (3.6)–(3.7).

5. Proof of Theorem 3.5

5.1. Basic solutions. Theorem 3.5, in fact, contains two statements:

1) unique solvability of the problem (3.6)–(3.7) for a fixed $\xi'\in\mathbb{R}^{n-1}$ and sufficiently large $\lambda\in\mathcal{L}$ and

2) estimate of this solution in terms of the parameters (ξ',λ).

As it was stated in the preceding section, according to (4.9) the space of parameters can be covered by domains, connected with edges and vertices of the Newton polygon of P. Thus, without loss of generality, we can suppose below that

$$(\xi',\lambda)\in G_\kappa(\rho,\delta)\cup\widetilde{G}_\kappa(\rho,\delta) \tag{5.1}$$

with ρ and δ chosen below. In this case the direct decomposition (4.11) takes place. In view of Theorem 4.5, in the domain (5.1) it is natural to define the function

$$\mu_j(\xi',\lambda):=\begin{cases}|\xi'|+|\lambda|^{1/r_\kappa} & \text{if } j\le R_\kappa \text{ and } (\xi',\lambda)\in G_\kappa(\rho,\delta),\\ |\xi'| & \text{if } j\le R_\kappa \text{ and } (\xi',\lambda)\in\widetilde{G}_\kappa(\rho,\delta),\\ |\lambda|^{1/r_\ell} & \text{if } R_{\ell-1}<j\le R_\ell \text{ for some } \ell>\kappa.\end{cases}$$

(For simplicity of notation, we omit the dependence of μ_j on κ.)

According to the decomposition of the space of all stable solutions of (3.6), we will look for solutions $\widetilde{w}_j=(\widetilde{w}_{1j},\dots,\widetilde{w}_{Nj})^\top=\widetilde{w}_j(t,\xi',\lambda)$ satisfying the system

$$(A(\xi',D_t)-\lambda)\widetilde{w}_j(t,\xi',\lambda)=0\quad(t>0), \tag{5.2}$$

and belonging to the following $\kappa+1$ groups:

$$\widetilde{w}_1,\dots,\widetilde{w}_{R_\kappa}\in\mathfrak{M}_{\gamma_\kappa^0}(\xi',\lambda) \tag{5.3}$$

$$\widetilde{w}_{R_{\ell-1}+1},\dots,\widetilde{w}_{R_\ell}\in\mathfrak{M}_{\gamma_\ell^1}(\lambda),\ \ell=\kappa+1,\dots,N \tag{5.4}$$

For the solutions from group (5.3) we pose R_κ boundary conditions

$$\sum_{k=1}^N b_{ik}(\xi',D_t)\widetilde{w}_{kj}(0,\xi',\lambda)=\delta_{ij}(\mu_j(\xi',\lambda))^{m_j},\quad i,j=1,\dots,R_\kappa \tag{5.5}$$

and for the solutions from group (5.4) we pose boundary conditions

$$\sum_{k=1}^N b_{ik}(\xi',D_t)\widetilde{w}_{kj}(0,\xi',\lambda)=\delta_{ij}(\mu_j(\xi',\lambda))^{m_j},\quad i,j=R_{\ell-1}+1,,\dots,R_\ell. \tag{5.6}$$

These solutions will be called basic solutions of system (5.2). We introduce the matrix

$$H(\xi',\lambda)=(h_{ij}(\xi',\lambda))_{i,j=1,\dots,R}:=\Big(B(\xi',D_t)\widetilde{w}_1(0),\dots,B(\xi',D_t)\widetilde{w}_R(0)\Big).$$

We shall prove the existence of basic solutions and the invertibility of H. These facts immediately imply linear independence of the basic solutions $\widetilde{w}_1,\dots,\widetilde{w}_R$ and unique solvability of the ODE problem (3.6)–(3.7) with $h:=e_k\in\mathbb{C}^R$ with solution $w^{(k)}(\xi',\lambda,t)$.

As it was mentioned at the end of the last section, basic solutions (5.4) of (5.2), (5.6) can be treated as boundary layer solutions. In the case $N=2$ we have one group of boundary layers and our approach resembles the one of Frank [**8**] to general elliptic problems with small parameter. In the case $N>2$ we come to an hierarchy of boundary layers and dealing with them we will use some results from [**7**] where the similar situation was treated in the case of scalar operators.

The main step in proving Theorem 3.5 is

Proposition 5.1. *Let $(A(D)-\lambda,B(D))$ be parameter-elliptic in $\mathcal{L}$ and suppose* (5.1) *holds. Let $\delta>0$ be sufficiently small and $\rho=\rho(\delta)>0$ be sufficiently large. Then for $j=1,\dots,N$ there exists a unique solution $\widetilde{w}_j=(\widetilde{w}_{1j},\dots,\widetilde{w}_{Nj})^\top$ of the system* (5.2)*, belonging to one of the groups* (5.3), (5.4) *and satisfying boundary conditions* (5.5) (*respectively* (5.6)). *For these solutions following estimates hold*

$$\|D_t^r\widetilde{w}_{ij}(\cdot,\xi',\lambda)\|_{L_2(\mathbb{R}_+)}\le C(\mu_j(\xi',\lambda)+|\xi'|+|\lambda|^{1/r_i})^{-t_i}\ (\mu_j(\xi',\lambda))^{r-1/2} \tag{5.7}$$
$$(i=1,\dots,N;\ j=1,\dots,R;\ r=0,1,2,\dots)$$

(5.8)
$$\Big|\sum_{k=1}^N b_{ik}(\xi',D_t)\widetilde{w}_{kj}(0,\xi',\lambda)\Big|\le C(\mu_j(\xi',\lambda))^{m_i}\quad(i,j=1,\dots,R).$$

Corollary 5.2. *Under the conditions of Proposition* 5.1 *the basic solutions are linearly independent.*

Proof. Due to the boundary conditions (5.5) the groups of solutions $\{\widetilde{w}_1,\dots,\widetilde{w}_{R_\kappa}\}$ and $\{\widetilde{w}_{R_{\ell-1}+1},\dots,\widetilde{w}_{R_\ell}\}$, $\ell=\kappa+1,\dots,N$, respectively, are linearly independent. As the direct sum of subspaces $\mathfrak{M}_{\gamma_\kappa^0}(\xi',\lambda)$ and $\mathfrak{M}_{\gamma_\ell^1}(\lambda)$, $\ell=\kappa+1,\dots,N$, is $\mathfrak{M}_+$ we obtain the statement. □

Corollary 5.3. *Under the conditions of Proposition* 5.1 *elements* $h_{ij}(\xi', \lambda)$ *of matrix* H *satisfy*

$$
\begin{aligned}
&|h_{ij}(\xi', \lambda)| \leq C\ (\mu_j(\xi', \lambda))^{m_i} \quad (i, j = 1, \dots, R)\\
&h_{ij}(\xi', \lambda) = \delta_{ij}(\mu_j(\xi', \lambda))^{m_j}\\
&\qquad \textit{if } 1 \leq i, j \leq R_\kappa \textit{ or if } R_{\ell-1} < i, j \leq R_\ell \textit{ for some } \ell > \kappa.
\end{aligned}
\tag{5.9}
$$

The proof of Proposition 5.1 is rather long and cumbersome. In the proof, we will separately consider the cases

$$ j = 1, \dots, R_\kappa, \quad (\xi', \lambda) \in \widetilde{G}_\kappa(\rho, \delta) \tag{5.10} $$

$$ j = 1, \dots, R_\kappa, \quad (\xi', \lambda) \in G_\kappa(\rho, \delta) \tag{5.11} $$

and

$$ j = R_\kappa + 1, \dots, R_N, \quad (\xi', \lambda) \in \widetilde{G}_\kappa(\rho, \delta) \cup G_\kappa(\rho, \delta). \tag{5.12} $$

But before we will deal with the proof Proposition 5.1, we deduce the proof of the main Theorem 3.5.

5.2. Proof of Theorem 3.5. Still assuming parameter-ellipticity for the boundary value problem $(A(D) - \lambda, B(D))$, we now consider the fundamental system $\{w^{(1)}, \dots, w^{(R)}\}$ of solutions of (3.6) and want to prove (3.10). Throughout this section, we fix $\boldsymbol{\sigma} \in \mathbb{R}^N$ satisfying (3.3).

The proof of Theorem 3.5 uses the following result from [**7**], Section 5.

Lemma 5.4. *Suppose* (5.1) *holds for some* $\kappa \in \{0, \dots, N\}$. *Let* $H(\xi', \lambda)$ *be an* $R \times R$ *matrix whose coefficients* h_{ij} *satisfy estimates* (5.9).

Then $H(\xi', \lambda)$ *is invertible for large* λ, *and for the coefficients of the inverse matrix* $H^{-1}(\xi', \lambda) =: \big(\widetilde{h}_{ij}(\xi', \lambda)\big)_{i,j=1,\dots,R}$ *the estimate*

$$ |\widetilde{h}_{ij}(\xi', \lambda)| \leq C\mu_i^{-m_i}\ \frac{\Psi_{\boldsymbol{\sigma}}^{(-m_j-1/2)}(\xi', \lambda)}{\Psi_{\boldsymbol{\sigma}}^{(-m_i-1/2)}(\xi', \lambda)} $$

holds.

Proof of Theorem 3.5. We fix sufficiently small $\delta > 0$ and $\rho > 0$ and consider $(\xi', \lambda) \in G_\kappa(\rho, \delta) \cup \widetilde{G}_\kappa(\rho, \delta)$ for some κ. From Corollary 5.3 we know that the matrix $H(\xi', \lambda)$ satisfies the assumptions of Lemma 5.4. The invertibility of H implies unique solvability of the ODE problem (3.6)–(3.7) with $h := e_k \in \mathbb{C}^R$ with solution $w^{(k)}(\xi', \lambda, t)$. By definition of the matrix H we have $w^{(k)} = (\widetilde{w}_1, \dots, \widetilde{w}_R)H^{-1}e_k$. Thus we have to estimate

$$ \big\|D_t^\ell w_i^{(k)}(\xi', \lambda, \cdot)\big\|_{L_2(\mathbb{R}_+)} \leq \sum_{j=1}^R \big\|D_t^\ell \widetilde{w}_{ij}(\xi', \lambda, \cdot)\big\|_{L_2(\mathbb{R}_+)} \cdot |\widetilde{h}_{jk}(\xi', \lambda)|. \tag{5.13} $$

We estimate the norms on the right-hand side of (5.13) by means of (5.7) and the term $|h_{jk}(\xi', \lambda)|$ by means of Lemma 5.4. As a result, we estimate the left-hand side of (5.13) by

$$ \sum_{j=1}^R \mu_j^{\ell-m_j-1/2}(\mu_j + |\xi'| + |\lambda|^{1/r_i})^{-t_i}\,\frac{\Psi_{\boldsymbol{\sigma}}^{(-m_k-1/2)}(\xi', \lambda)}{\Psi_{\boldsymbol{\sigma}}^{(-m_j-1/2)}(\xi', \lambda)}. $$

Now inequality (3.10) is a consequence of the following technical lemma. □

LEMMA 5.5. *Suppose* (5.1) *takes place. Then*

$$\frac{\Psi^{(-\ell)}_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\xi',\lambda)}{\Psi^{(-m_j-1/2)}_{\boldsymbol{\sigma}}(\xi',\lambda)} \le \mathrm{const}\, \mu_j^{m_j+1/2-\ell}(\mu_j+|\xi'|+|\lambda|^{1/r_i})^{t_i}.$$

PROOF. We use two elementary inequalities on the weight function Ψ_σ:

$$\frac{\Psi^{(-\ell)}_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\xi',\lambda)}{\Psi^{(-m_j-1/2)}_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\xi',\lambda)} \le \mu_j^{m_j+1/2-\ell}, \tag{5.14}$$

$$\frac{\Psi^{(-m_j-1/2)}_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\xi',\lambda)}{\Psi^{(-m_j-1/2)}_{\boldsymbol{\sigma}}(\xi',\lambda)} \le (\mu_j+|\xi'|+|\lambda|^{1/r_i})^{t_i}. \tag{5.15}$$

The estimate (5.14) can be found as inequality (5-21) in [**7**], and (5.15) can be seen directly from the definition of the shifted weight function $\Psi_{\boldsymbol{\sigma}}^{(-m_j-1/2)}$. Now we only have to multiply the left-hand sides of (5.14) and (5.15) to obtain the desired inequality. □

5.3. Proof of Proposition 5.1 in the case (5.10)**.** According to Proposition 4.1, unique solvability of (5.2), (5.3) and (5.5) will follow from

PROPOSITION 5.6. *Let conditions* (i') *and* (ii) *of Definition* 2.3 *be satisfied and let* (5.10) *hold. Then we can choose* ρ *and* δ *that there exists a contour* $\gamma^0_\kappa(\xi',\lambda)$ *in* $\mathbb{C}_+$ *enveloping the first* R_κ *roots of* (4.8) *(see Theorem* 4.5*) such that the rectangular matrix*

$$\frac{1}{2\pi i}\int_{\gamma^0_\kappa(\xi',\lambda)} B_{1..\kappa,N}(\xi',\tau)(A(\xi',\tau)-\lambda I)^{-1}(I,(\tau/q)I,\dots,(\tau/q)^{\sigma-1}I)d\tau$$

has maximal rank R_κ. *Here we used the notation* $B_{1..\kappa,N} := \big(b_{ij}\big)_{\substack{i=1,\dots,R_\kappa\\ j=1,\dots,N}}$.

PROOF. By Remark 4.3, we can equivalently prove that the rectangular matrix

$$\frac{1}{2\pi i}\int_{\gamma^0_\kappa(\xi',\lambda)} QB_{1..\kappa,N}(\xi',\tau)(A(\xi',\tau)-\lambda I)^{-1}(H,(\tau/q)H,\dots,(\tau/q)^{\sigma-1}H)d\tau \tag{5.16}$$

has maximal rank R_κ under suitable choice of q, Q, H.

In the following, we will fix a real number $r_\kappa > r > r_{\kappa+1}$ and $\mathbf{s}',\mathbf{t}' \in \mathbb{R}^N$ satisfying

$$\begin{aligned} &s'_j = s_j,\ t'_j = t_j \quad (j=1,\dots,\kappa)\\ &s'_j > s_j,\ t'_j > t_j,\ s'_j+t'_j = r \quad (j=\kappa+1,\dots,N). \end{aligned} \tag{5.17}$$

We also set

$$\varepsilon := \min\{s'_j - s_j, t'_j - t_j : j=\kappa+1,\dots,N\}$$

and introduce for $z>0$ and $\mathbf{a}\in\mathbb{R}^N$ the diagonal matrix

$$\Delta_{\mathbf{a}}(z) := \mathrm{diag}(z^{a_1},\dots,z^{a_N}).$$

Note that $\Delta_{\mathbf{a}}^{-1}(z) = \Delta_{-\mathbf{a}}(z)$.

Substituting $\tau\mapsto|\xi'|\tau$ in (5.16), we obtain an integral over a bounded contour $\widetilde{\gamma}^0_\kappa(\xi',\lambda)$ which can be deformed into the contour $\widetilde{\gamma}^0_\kappa$ independent of (ξ',λ). (Here we used Theorem 4.5.) In (5.16) we set

$$q := |\xi'|,\ \xi' = q\omega',\ Q := 2\pi i q^{-1}\Delta_{-\mathbf{m}}(q),\ H := \Delta_{\mathbf{s}'-\mathbf{a}}(q)$$

where $\mathbf{a} := (a_1, \ldots, a_N)$ with $a_i := 0$ for $i \le \kappa$ and $a_i := r - r_{\kappa+1}$ if $i \ge \kappa + 1$. We also set $\Delta_{\mathbf{m}}(q) := \operatorname{diag}(q^{m_1}, \ldots, q^{m_{R_\kappa}})$. Using the homogeneity of B, we obtain that (5.16) equals

$$\int_{\widetilde{\gamma}^0_\kappa} B_{1..\kappa,N}(\omega', \tau)\Delta_{\mathbf{t}}(q)\big[A(q\omega', q\tau) - \lambda\big]^{-1}\Delta_{\mathbf{s}'-\mathbf{a}}(q)(I_N, \ldots, \tau^{\sigma-1}I_N)\,\mathrm{d}\,\tau\,. \tag{5.18}$$

The following lemma can be shown by straightforward calculation.

Lemma 5.7. *Let $r_\kappa > r > r_{\kappa+1}$ and suppose that $\mathbf{s}'$ and $\mathbf{t}'$ satisfy* (5.17). *Suppose $(\xi', \lambda) \in \widetilde{G}_\kappa(1, \delta)$. Then*

$$A(q\omega', q\tau) - \lambda I_N = \Delta_{\mathbf{s}'}(q)\left[\begin{pmatrix} A_\kappa(\omega', \tau) & 0 \\ 0 & \lambda q^{-r} I_{N-\kappa}\end{pmatrix} + O(q^{-\varepsilon} + \delta)\right]\Delta_{\mathbf{t}'}(q)\,.$$

Corollary 5.8. *If δ and $q^{-\varepsilon}$ are small enough, then*

$$\begin{aligned}[]&[A(q\omega', q\tau) - \lambda I_N]^{-1} \\ &\qquad = \Delta_{-\mathbf{t}'}(q)\left[\begin{pmatrix} (A_\kappa(\omega', \tau))^{-1} & 0 \\ 0 & \lambda^{-1} q^{r} I_{N-\kappa}\end{pmatrix} + O(q^{-\varepsilon} + \delta)\right]\Delta_{-\mathbf{s}'}(q).\end{aligned}$$

With these results, we can finish the proof of Proposition 5.6. In (5.18) we substitute the representation of Corollary 5.8 and obtain

$$\begin{aligned}\int_{\widetilde{\gamma}^0_\kappa} B_{1..\kappa,N}(\omega', \tau)\Delta_{\mathbf{t}-\mathbf{t}'}(q)\Bigg[&\begin{pmatrix} A_\kappa(\omega', \tau) & 0 \\ 0 & q^{r_{\kappa+1}}\lambda^{-1} I_{N-\kappa}\end{pmatrix} \\ &+ O(q^{-\varepsilon} + \delta)\Bigg](I_N, \ldots, \tau^{\sigma-1}I_N)\,\mathrm{d}\,\tau.\end{aligned} \tag{5.19}$$

According to the definition of $\widetilde{G}_\kappa(\rho, \delta)$ we have $|\lambda|^{-1}q^{r_{\kappa+1}} < \delta$. Further note that

$$B_{1..\kappa,N}(\omega', \tau)\Delta_{\mathbf{t}-\mathbf{t}'}(q) \to \begin{pmatrix} B_{1..\kappa}(\omega', \tau) & 0\end{pmatrix}$$

for $\delta \searrow 0$ where 0 denotes the $R_\kappa \times (N - \kappa)$ zero matrix. From this we see that for $\delta \searrow 0$ the matrix (5.19) tends to

$$\begin{aligned}&\int_{\widetilde{\gamma}^0_\kappa} \begin{pmatrix} B_{1..\kappa}(\omega', \tau) & 0\end{pmatrix}\begin{pmatrix} A_\kappa(\omega', \tau)^{-1} & 0 \\ 0 & 0\end{pmatrix}(I_N, \tau I_N, \ldots, \tau^{\sigma-1}I_N)\,\mathrm{d}\,\tau \\ &\qquad = \int_{\widetilde{\gamma}^0_\kappa} \begin{pmatrix} B_{1..\kappa}(\omega', \tau)A_\kappa(\omega', \tau)^{-1} & 0\end{pmatrix}(I_N, \tau I_N, \ldots, \tau^{\sigma-1}I_N)\,\mathrm{d}\,\tau.\end{aligned}$$

It is easily seen that Lemma 4.4 a) with $\lambda = 0$ implies that the rank of the last matrix is maximal. Therefore for sufficiently small $\delta > 0$ the rank of the matrix (5.19) and thus (5.16) is maximal, too. □

Proposition 5.6 permits us not only to prove unique solvability of the problem (5.2), (5.3) and (5.5), but establishes the estimates (5.7) and (5.8). Indeed, as we have seen above, the matrix

$$\int_{\widetilde{\gamma}^0_\kappa} B_{1..\kappa,N}(\omega', \tau)\Delta_{\mathbf{t}}(q)\Big[A(q\omega', q\tau) - \lambda\Big]^{-1}\Delta_{\mathbf{s}'}(q)(I_N, \ldots, \tau^{\sigma-1}I_N)\,\mathrm{d}\,\tau \tag{5.20}$$

has maximal rank R_κ. By Remark 4.3 we may replace $\Delta_{\mathbf{s}'}(q)$ by $\Delta_{\mathbf{s}}(q)$. Now we use (2.6) to see that the element at position (i,j) of the matrix $\Delta_{\mathbf{t}}(q)\big[A(q\omega',q\tau)-\lambda\big]^{-1}\Delta_{\mathbf{s}}(q)$ can be estimated by

$$Cq^{t_i+s_j}(q+q|\tau|+|\lambda|^{1/r_i})^{-t_i}(q+q|\tau|+|\lambda|^{1/r_j})^{-s_j}.$$

This means

$$\Big|\Delta_{\mathbf{t}}(q)\big[A(q\omega',q\tau)-\lambda\big]^{-1}\Delta_{\mathbf{s}}(q)\Big|\le C,$$

and therefore the matrix in (5.20) with $\Delta_{\mathbf{s}'}(q)$ replaced by $\Delta_{\mathbf{s}}(q)$ is bounded by a constant (and depends continuously on (q,ω',λ)). From Proposition 4.1 we know that there exists a matrix $N(q,\omega',\lambda,\tau)$ such that

(5.21)
$$\frac{1}{2\pi i}\int_{\widetilde{\gamma}^0_\kappa} B_{1..\kappa,N}(\xi',\tau)\Delta_{\mathbf{t}}(q)\Big[A(\xi',q\tau)-\lambda\Big]^{-1}\Delta_{\mathbf{s}}(q)N(q,\omega',\lambda,\tau)\,\mathrm{d}\,\tau=\Delta_{\mathbf{m}}(q)I_{R_\kappa}.$$

holds. Due to the explicit construction of $N(\tau)$ in Remark 4.2 we may assume that $N(q,\omega',\lambda,\tau)$ is bounded, too. We set for $j=1,\dots,R_\kappa$

$$\widetilde{w}_j(t,\xi',\lambda):=\left[\frac{1}{2\pi i}\int_{\widetilde{\gamma}^0_\kappa} e^{itq\tau}\Big[A(q\omega',q\tau)-\lambda\Big]^{-1}\Delta_{\mathbf{s}}(q)N(q,\omega',\lambda,\tau)\,\mathrm{d}\,\tau\right]e_j.$$

Then obviously $(A(\xi',D_t)-\lambda)\widetilde{w}_j(t)=0$ and, by (5.21) and homogeneity of A and B,

$$B_{1..\kappa,N}(\xi',D_t)\widetilde{w}_j(0)=\operatorname{diag}(q^{m_i})_{i=1,\dots,R_\kappa}I_{R_\kappa}e_j=q^{m_j}e_j=\mu_j^{m_j}e_j,$$

i.e. $\widetilde{w}_j$ is the unique solution of (5.2)–(5.5).

Again by homogeneity of A and B we can estimate

$$\begin{aligned}&\Big|\Delta_{-\mathbf{m}}(q)B(\xi',D_t)\widetilde{w}_j(0)\Big|\\ &=\left|\frac{1}{2\pi i}\int_{\widetilde{\gamma}^0_\kappa}\Delta_{-\mathbf{m}}(q)B_{1..\kappa,N}(q\omega',q\tau)\Big[A(q\omega',q\tau)-\lambda\Big]^{-1}\Delta_{\mathbf{s}}(q)N(q,\omega',\lambda,\tau)\,\mathrm{d}\,\tau\;e_j\right|\\ &\le C\end{aligned}$$

which shows (5.8). To prove (5.7) we again use (2.6) and obtain

$$\Big|D(q,\lambda)\big[A(q\omega',q\tau)-\lambda\big]^{-1}\Delta_{\mathbf{s}}(q)\Big|\le C \tag{5.22}$$

for all $(\xi',\lambda)\in\widetilde{G}_\kappa(\rho,\delta)$ and $\tau\in\widetilde{\gamma}^0_\kappa$ with $q:=|\xi'|$, $\omega'=\xi'/q$ and

$$D(q,\lambda):=\operatorname{diag}\Big((q+|\lambda|^{1/r_i})^{t_i}\Big)_{i=1,\dots,N}.$$

Therefore the estimate

$$\Big|D(q,\lambda)D_t^r\widetilde{w}_j(t)\Big|\le Cq^r\operatorname{length}(\widetilde{\gamma}^0_\kappa)\exp(-\operatorname{dist}(\widetilde{\gamma}^0_\kappa,\mathbb{R})qt)$$

holds. Integration with respect to t leads to

$$\Big\|D(q,\lambda)D_t^r\widetilde{w}_j\Big\|_{L_2(\mathbb{R}_+)}\le Cq^{r-1/2}$$

which is equivalent to (5.7). This finishes the proof of Proposition 5.1 in the case (5.10).

5.4. Proof of Proposition 5.1 in the cases (5.11),(5.12)**.** For $(\xi', \lambda) \in \widetilde{G}_\kappa$, the case $j > R_\kappa$ can be treated in a similar way as the case $j \le R_\kappa$, so we only indicate the necessary changes. For $\kappa \in \{0, \dots, N\}$, $\ell > \kappa$ and $j \in \{R_{\ell-1} + 1, \dots, R_\ell\}$ we now have to show that the matrix

$$\frac{1}{2\pi i} \int\limits_{\gamma_\ell^1(\lambda)} Q B_{\ell,N}(\xi', \tau) \Big[A(\xi', \tau) - \lambda\Big]^{-1} (H, (\tau/q)H, \dots, (\tau/q)^{\sigma-1} H) \,\mathrm{d}\,\tau \tag{5.23}$$

has maximal rank $r_\ell/2$, where we set $B_{\ell,N} := \big(b_{ij}\big)_{\substack{i=R_{\ell-1}+1,\dots,R_\ell \\ j=1,\dots,N}}$. We now fix r with $r_\ell > r > r_{\ell+1}$ and choose $\mathbf{s}'$ and $\mathbf{t}'$ satisfying (5.17) with κ replaced by ℓ.

In (5.23) we set

$$q := |\lambda|^{1/r_\ell},\ \xi' = q\omega',\ Q := 2\pi i q^{-1} \Delta_{-\mathbf{m}}(q),\ H := \Delta_{\mathbf{s}'}(q)\,.$$

After transformation $\tau \mapsto q\tau$ and deformation of the resulting contour into a bounded contour $\widetilde{\gamma}_\ell^1$ independent of λ, we obtain instead of (5.18) the matrix

$$\int_{\widetilde{\gamma}_\ell^1} B_{\ell,N}(\omega', \tau) \Delta_{\mathbf{t}}(q) \Big[A(q\omega', q\tau) - \lambda I_N\Big]^{-1} \Delta_{\mathbf{s}'}(q) (I_N, \dots, \tau^{\sigma-1} I_N) \,\mathrm{d}\,\tau. \tag{5.24}$$

Instead of Lemma 5.7 we now have

$$\begin{aligned} &A(q\omega', q\tau) - \lambda I_N \\ &\quad = \Delta_{\mathbf{s}'}(q) \left[\begin{pmatrix} A_\ell(\omega', \tau) - q^{-1}\lambda E_\ell & 0 \\ 0 & \lambda^{-1} q^r I_{N-\ell} \end{pmatrix} + O(q^{-\varepsilon'} + \delta^{\varepsilon''}) \right] \Delta_{\mathbf{t}'}(q) \end{aligned}$$

with positive constants $\varepsilon', \varepsilon''$.

Let us first assume $\kappa \ge 1$. By definition of $\widetilde{G}_\kappa$ we have

$$|\omega'| = \frac{|\xi'|}{q} = |\xi'|\,|\lambda|^{-1/r_\ell} \le |\xi'|\,|\lambda|^{-1/r_{\kappa+1}} < \delta^{1/r_{\kappa+1}}$$

and for $|\xi'| \ge 1$

$$|\lambda|^{-1} q^r = |\lambda|^{(r-r_\ell)/r_\ell} \le \Big(\frac{|\xi'|}{|\lambda|^{1/r_\ell}}\Big)^{r_\ell - r} \le \Big(\frac{|\xi'|}{|\lambda|^{1/r_{\kappa+1}}}\Big)^{r_\ell - r} < \delta^{(r_\ell - r)/r_{\kappa+1}}\,. \tag{5.25}$$

From this we see that for $\delta \searrow 0$ the matrix (5.24) tends to

$$\int_{\widetilde{\gamma}_\ell^1} \Big(B_\ell(0,\tau)(A_\kappa(0,\tau) - |\lambda|^{-1}\lambda E_\kappa)^{-1} \quad 0\Big) (I_N, \dots, \tau^{\sigma-1} I_N) \,\mathrm{d}\,\tau$$

which has rank $r_\ell/2$ by Lemma 4.4 b). Note that up to now we have found a small $\delta > 0$ such that the desired results hold for $(\xi', \lambda) \in \widetilde{G}_\kappa(1, \delta)$

Now let us consider the case $\kappa = 0$. Here we replace (5.25) by

$$|\lambda|^{-1} q^r = |\lambda|^{(r-r_\ell)/r_\ell} \le \rho_0^{(r-r_\ell)/r_\ell}$$

which holds for $(\xi', \lambda) \in \widetilde{G}_0(\rho_0, \delta)$ with sufficiently large ρ_0.

Finally, let us assume that for $\kappa \in \{1, \dots, N\}$ we have $(\xi', \lambda) \in G_\kappa(\rho, \delta)$. For $j \le R_\kappa$ the construction of the basic solutions follows in the same way as in above, now setting $r := r_\kappa$ and $q := |\xi'| + |\lambda|^{1/r_\kappa}$ which finally leads to the matrix

$$\begin{pmatrix} (A_\kappa(\omega', \tau) - \lambda q^{-r_\kappa})^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

instead of the matrix appearing in Corollary 5.8. In a similar way the case $j > R_\kappa$ can be treated as a small modification of the case $(\xi', \lambda) \in \widetilde{G}_\kappa(\rho, \delta)$.

6. Proof of the necessity

Now we want to show that parameter-ellipticity is necessary for unique solvability of (3.4) and the a priori-estimates (3.5). So the aim of the present section is to prove the following result.

THEOREM 6.1. *Suppose for a fixed* $\sigma \in \mathbb{R}^N$ *satisfying* (3.3) *the estimate* (3.5) *with right-hand sides* (3.4) *holds. Then conditions* (i)-(iii) *of Definition* 2.3 *are satisfied.*

The proof of the theorem is based on the same ideas as the corresponding proof in [**5**], Section 4. Necessity of (i), in fact, is already contained in [**4**]. The proof of Theorem 6.1 is derived from a priori estimates given below. To simplify their formulation, we introduce some weight functions which are closely related to the definition of $\Psi_{\boldsymbol{\sigma}}$ in Section 3. For fixed $\kappa \in \{1, \dots, N\}$ we set

$$\Phi_{\boldsymbol{\sigma}}(\xi, \lambda) := |\xi|^{\sigma_1+\cdots+\sigma_{\kappa-1}}(|\xi| + |\lambda|^{1/r_\kappa})^{\sigma_\kappa}, \tag{6.1}$$

$$\widetilde{\Phi}_{\boldsymbol{\sigma}}(\xi, \lambda) := (|\xi'| + i\xi_n)^{\sigma_1+\cdots+\sigma_{\kappa-1}}(|\xi'| + i\xi_n + |\lambda|^{1/r_\kappa})^{\sigma_\kappa}. \tag{6.2}$$

Moreover, we define $\widetilde{\Phi}_{\boldsymbol{\sigma}}(\xi', \lambda) := \widetilde{\Phi}_{\boldsymbol{\sigma}}(\xi', 0, \lambda)$ and the shifted weight function by

$$\widetilde{\Phi}_{\boldsymbol{\sigma}}^{(-a)}(\xi', \lambda) := \begin{cases} |\xi'|^{\sigma_1+\cdots+\sigma_{\kappa-1}-a}(|\xi'| + |\lambda|^{1/r_\kappa})^{\sigma_\kappa}, & a \le \sigma_1 + \cdots + \sigma_{\kappa-1}, \\ (|\xi'| + |\lambda|^{1/r_\kappa})^{\sigma_1+\cdots+\sigma_\kappa-a}, & a > \sigma_1 + \cdots + \sigma_{\kappa-1}. \end{cases} \tag{6.3}$$

Throughout this section, the vector $\boldsymbol{\sigma} \in \mathbb{R}^N$ is assumed to be fixed with $\sigma_i > 0$ for $i = 2, \dots, N$ and satisfying (3.3). We will write $\|\cdot\|$ instead of $\|\cdot\|_{L_2(\mathbb{R}^n)}$.

PROPOSITION 6.2. *Suppose the a priori estimate* (3.5) *holds and* $\kappa \in \{1, \dots, N\}$ *is fixed. Then following statements hold.*

(i) there exists a constant $C > 0$ *such that for all* $u^{(\kappa)} \in (C_0^\infty(\mathbb{R}^n))^\kappa$ *and* $\lambda \in \mathcal{L}$ *the following estimate holds.*

$$\sum_{i=1}^{\kappa} \|\Phi_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(D, \lambda)u_i^{(\kappa)}\| \le C \sum_{i=1}^{\kappa} \|\Phi_{\boldsymbol{\sigma}-s_i\mathbf{e}_i}(D, \lambda)f_i^{(\kappa)}\|. \tag{6.4}$$

here $f^{(\kappa)} := (A_\kappa - \lambda E_\kappa)u^{(\kappa)}$.

(ii) for $\boldsymbol{\sigma} \in \mathbb{R}^\kappa$ *there exists a constant* $C > 0$ *such that for all* $u^{(\kappa)} \in (C_0^\infty(\overline{\mathbb{R}^n_+}))^\kappa$ *and all* $\lambda \in \mathcal{L}$ *the following inequality holds.*

$$\begin{aligned} \sum_{i=1}^{\kappa} \|\widetilde{\Phi}_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(D, \lambda)u_i^{(\kappa)}\|_{L^2(\mathbb{R}^n_+)} \le C\Big(& \sum_{i=1}^{\kappa} \|\widetilde{\Phi}_{\boldsymbol{\sigma}-s_i\mathbf{e}_i}(D, \lambda)f_i^{(\kappa)}\|_{L^2(\mathbb{R}^n_+)} \\ & + \sum_{j=1}^{R_\kappa} \|\widetilde{\Phi}_{\boldsymbol{\sigma}}^{(-m_j-1/2)}(D', \lambda)g_j^{(\kappa)}\|_{L^2(\mathbb{R}^{n-1})}\Big). \end{aligned} \tag{6.5}$$

Here we have set $f^{(\kappa)} := (A_\kappa - \lambda E_\kappa)u^{(\kappa)}$ *and* $g^{(\kappa)} := B_{1..\kappa}u^{(\kappa)}$.

(iii) there exists a $C > 0$ such that for all $u^{(\kappa)} \in (C_0^\infty(\overline{\mathbb{R}}^n_+))^\kappa$ and all $\lambda \in \mathcal{L}$ the following estimate holds.

$$\begin{aligned}\sum_{i=1}^{\kappa} &\|\widetilde{\Phi}_{\boldsymbol{\sigma}+t_i\boldsymbol{e}_i}(0,D_n,\lambda)u_i\|_{L^2(\mathbb{R}^n_+)} \\ &\le C\Big[\sum_{i=1}^{\kappa}\|\widetilde{\Phi}_{\boldsymbol{\sigma}-s_i\boldsymbol{e}_i}(0,D_n,\lambda)f_i^{(\kappa)}\|_{L^2(\mathbb{R}^n_+)} \\ &\quad + \sum_{j=R_{\kappa-1}+1}^{R_\kappa}\|\widetilde{\Phi}_{\boldsymbol{\sigma}}^{(-m_j-1/2)}(0,\lambda)g_j^{(\kappa)}\|_{L^2(\mathbb{R}^{n-1})}\Big]\end{aligned}$$

where we have set $f^{(\kappa)} := (A_\kappa - \lambda E_\kappa)u^{(\kappa)}$ and $g^{(\kappa)} := B_\kappa u^{(\kappa)}$.

Proof of the Theorem.

Necessity of condition (i). Changing the constants, we may replace each norm in (6.10) by its square. We choose $u^{(\kappa)}(x) = \varphi(x)h$ with $\varphi \in C_0^\infty(\mathbb{R}^n)$, $h \in \mathbb{C}^\kappa$. Taking the Fourier transform in $\mathbb{R}^n$, we obtain

$$\begin{aligned}0 \le \int\Bigg[&\sum_{i=1}^{\kappa}\big(\Phi_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\xi,\lambda)\big)^2|\hat{\varphi}(\xi)|^2|h_i|^2 - \\ &- C\cdot\sum_{i=1}^{\kappa}\big(\Phi_{\boldsymbol{\sigma}-s_i\mathbf{e}_i}(\xi,\lambda)\big)^2\Big|\sum_{j=1}^{\kappa}(A_{ij}(\xi)\lambda\delta_{ij}\delta_{i\kappa})\hat{\varphi}(\xi)h_i\Big|^2\Bigg]d\xi \\ = \int_{\mathbb{R}^n}&\Phi_{\boldsymbol{\sigma}}(\xi,\lambda)\big)^2|\hat{\varphi}(\xi)|^2\Bigg[\sum_{i=1}^{\kappa}(|\xi|+\delta_{i\kappa}|\lambda|^{1/r_i})^{2t_i}|h_i|^2 \\ &- C\sum_{i=1}^{\kappa}(|\xi|+\delta_{i\kappa}|\lambda|^{1/r_i})^{-2s_i}\Big|\sum_{j=1}^{\kappa}(A_{ij}(\xi)-\delta_{ij}\delta_{i\kappa}\lambda)h_j\Big|^2\Bigg]d\xi.\end{aligned} \tag{6.6}$$

As φ is an arbitrary C_0^∞-function the expression in $[\cdots]$ must be nonpositive for all $\xi \in \mathbb{R}^n$, $\lambda \in \mathcal{L}$ and $h \in \mathbb{C}^\kappa$. From this we obtain

$$\det(A_\kappa(\xi) - \lambda E_\kappa) \ne 0 \quad (\xi \in \mathbb{R}^n \setminus \{0\}, \lambda \in \mathcal{L}).$$

Indeed, if this condition is not satisfied, then for some $\xi^0 \ne 0$, $\lambda^0 \in \mathcal{L}$ and $h^0 \in \mathbb{C}^\kappa$ there exists a nontrivial solution of the equation

$$(A_\kappa(\xi^0) - \lambda^0 E_\kappa)h^0 = 0.$$

For such ξ^0, λ^0 and h^0 we have

$$\sum_{j=1}^{\kappa}(A_{ij}(\xi)\delta_{ij}\delta_{i\kappa}\lambda)h_j^0 = 0$$

for $i = 1, \dots, \kappa$ and

$$\sum_{i=1}^{\kappa}(|\xi| + \delta_{i\kappa}|\lambda|^{1/r_i})^{2t_i}|h_i|^2 > 0,$$

and the bracket in (6.6) would be positive.

Necessity of condition (ii). As in the proof of (i), we replace each term in (6.10) by its square. Now we choose $u(x) = \varphi(x')v(x_n)$ with $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$,

$v = (v_1, \dots, v_\kappa)^\top$. Taking Fourier transform with respect to x' and using that φ is arbitrary we derive for v the following inequality on the half-line.

$$\sum_{i=1}^{\kappa} \|\widetilde{\Phi}_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\xi', D_n, \lambda) v_i\|^2_{L^2(\mathbb{R}_+)}$$
$$\leq C \Bigg[\sum_{i=1}^{\kappa} \|\widetilde{\Phi}_{\boldsymbol{\sigma}-s_i\mathbf{e}_i}(\xi', D_n, \lambda) \sum_{j=1}^{\kappa} (A_{ij}(\xi', D_n) - \delta_{ij}\delta_{i\kappa}\lambda) v_j\|^2_{L^2(\mathbb{R}_+)}$$
$$+ \sum_{j=1}^{R_\kappa} \Big| \widetilde{\Phi}_{\boldsymbol{\sigma}}^{(-m_j-1/2)}(\xi', \lambda) \sum_{i=1}^{\kappa} B_{ji}(\xi', D_n) v_i \Big|^2 \Bigg].$$

From this it follows that if $\xi' \neq 0$ and $v \in L^2(\mathbb{R}_+)$ is a solution of

$$(A_\kappa(\xi', D_n) - \lambda E_\kappa) v(x_n) = 0 \quad (x_n > 0), \qquad B_{1..\kappa}(\xi', D_n) v(x_n)|_{x_n=0} = 0 \tag{6.7}$$

then v vanishes identically.

From the uniqueness of the solution we see that the Lopatinskii matrix of the ODE system (6.7) with boundary conditions

$$B_\kappa(\xi', D_n) v = (c_1, \dots, c_{R_\kappa})^\top \tag{6.8}$$

has maximal rank and the problem (6.7)–(6.8) has a unique stable solution for arbitrary $(c_1, \dots, c_{R_\kappa}) \in C^{r_\kappa}$. Thus condition (ii) is proved.

Necessity of condition (iii). Repeating the proof of (ii) we obtain from Proposition **??** for $\lambda \in \mathcal{L}$, $|\lambda| = 1$, the inequality on the half-line

$$\sum_{i=1}^{\kappa} \|\widetilde{\Phi}_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(0, D_n, \lambda) v_i\|^2_{L^2(\mathbb{R}_+)}$$
$$\Bigg[\sum_{i=1}^{\kappa} \|\widetilde{\Phi}_{\boldsymbol{\sigma}-s_i\mathbf{e}_i}(0, D_n, \lambda) \sum_{j=1}^{\kappa} A_{ij}(0, D_n) v_j\|^2_{L^2(\mathbb{R}_+)}$$
$$+ \sum_{j=R_{\kappa-1}+1}^{R_\kappa} \Big| \sum_{k=1}^{\kappa} B_{jk}(0, D_n) v_k \Big|^2 \Bigg].$$

From this inequality it follows that if $v \in L^2(\mathbb{R}_+)$ and

$$(A_\kappa(0, D_n) - \lambda E_\kappa) v(x_n) = 0 \quad (x_n > 0), \qquad B_\kappa(0, D_n) v(x_n)|_{x_n=0} = 0$$

then

$$0 = \sum_{i=1}^{\kappa} \|\widetilde{\Phi}_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(0, D_n, \lambda) v_i\|^2_{L^2(\mathbb{R}_+)}$$
$$= \sum_{i=1}^{\kappa-1} \|D_n^{\sigma_1+\dots+\sigma_\kappa+t_i} v_i\|_{\mathbb{R}_+} + \|D_n^{\sigma_1+\dots+\sigma_\kappa}(iD_n+1)^{t_i} v_\kappa\|_{\mathbb{R}_+}.$$

From this we see that the components v_i are polynomials. As they belong to $L^2(\mathbb{R}_+)$, they are identically zero. From this follows condition (iii) which finishes the proof of the necessity. □

We still have to prove Proposition 6.2.

Proof of Proposition 6.2, Part (i).

If we take an infinitely smooth vector function $u : \mathbb{R}^n_+ \to \mathbb{R}^N$ with support in $\mathbb{R}^n_+$ and apply to it inequality (3.5) we obtain, setting $f := (A-\lambda)u$,

$$\sum_{i=1}^{N} \|\Psi_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(D,\lambda)u_i\| \le C \sum_{i=1}^{N} \|\Psi_{\boldsymbol{\sigma}-s_i\mathbf{e}_i}(D,\lambda)f_i\|\,. \tag{6.9}$$

Since this inequality is invariant under shifts in $\mathbb{R}^n$ we can suppose that u is an arbitrary vector function with components belonging to $C_0^\infty(\mathbb{R}^n)$.

Following [**4**], we replace for $\rho > 0$ in (6.9) λ by $\rho^{r_\kappa}\lambda$ and $u(x)$ by

$$u_\rho(x) = (u_{1\rho}(x), \ldots, u_{N\rho}(x)), \quad u_{j\rho}(x) = \rho^{\frac{n}{2}-a_\kappa-t_j(\kappa)}u_j(\rho x), \tag{6.10}$$

where

$$a_\kappa = \sigma_1 + \cdots + \sigma_\kappa + r_\kappa\Big(\frac{\sigma_{\kappa+1}}{r_{\kappa+1}} + \cdots + \frac{\sigma_N}{r_N}\Big)$$

and

$$t_j(\kappa) = t_j, \quad \text{for} \quad j \le \kappa, \quad t_j(\kappa) = \varepsilon + t_j\frac{r_\kappa}{r_j}, \quad \text{for} \quad j > \kappa \tag{6.11}$$

for some fixed $\varepsilon > 0$. After a natural change of variables we come to the inequality

$$\sum_{i=1}^{N} \|\rho^{-a_\kappa-t_i(\kappa)}\Psi_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\rho D,\rho^{r_\kappa}\lambda)u_i\| \le C\sum_{i=1}^{N} \|\rho^{-a_\kappa}\Psi_{\boldsymbol{\sigma}-s_i\mathbf{e}_i}(\rho D,\rho^{r_\kappa}\lambda)f_{\rho i}\|, \tag{6.12}$$

where

$$f_{\rho i} := \sum_{j=1}^{N} \rho^{-t_j(\kappa)}A_{ij}(\rho D)u_j - \rho^{r_\kappa - t_i(\kappa)}\lambda u_i\,.$$

Denote by

$$J_i(\rho) := \|\rho^{-a_\kappa-t_i(\kappa)}\Psi_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\rho D,\rho^{r_\kappa}\lambda)u_i\|$$

the typical term on the left-hand side of (6.12) and by

$$I_i(\rho) := \|\rho^{-a_\kappa}\Psi_{\boldsymbol{\sigma}-s_i\mathbf{e}_i}(\rho D,\rho^{r_\kappa}\lambda)f_{\rho i}\|$$

the typical term on the right-hand side of (6.12).

(i) We first consider $J_i(\rho)$. We write

$$\begin{aligned} &\rho^{-a_\kappa-t_i(\kappa)}\Psi_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\rho\xi,\rho^{r_\kappa}\lambda) \\ &\qquad = \frac{(\rho|\xi| + \rho^{r_\kappa/r_i}|\lambda|^{1/r_i})^{t_i}}{\rho^{t_i(\kappa)}} \cdot \prod_{j=1}^{\kappa}\Big(\frac{\rho|\xi| + \rho^{r_\kappa/r_j}|\lambda|^{1/r_j}}{\rho}\Big)^{\sigma_j} \\ &\qquad\quad \cdot \prod_{j=\kappa+1}^{N}\Big(\frac{\rho|\xi| + \rho^{r_\kappa/r_j}|\lambda|^{1/r_j}}{\rho^{r_\kappa/r_j}}\Big)^{\sigma_j}. \end{aligned} \tag{6.13}$$

Now we remark that for $\rho \to \infty$ we have

$$\begin{aligned} &\frac{\rho|\xi| + \rho^{r_\kappa/r_j}|\lambda|^{1/r_j}}{\rho} \longrightarrow \begin{cases} |\xi|, & 1 \le j \le \kappa-1, \\ |\xi| + |\lambda|^{1/r_j}, & j = \kappa, \end{cases} \\ &\frac{\rho|\xi| + \rho^{r_\kappa/r_j}|\lambda|^{1/r_j}}{\rho^{r_\kappa/r_j}} \longrightarrow |\lambda|^{1/r_j}, \quad j = \kappa+1, \ldots, N. \end{aligned} \tag{6.14}$$

For $i \le \kappa$ we have $t_i(\kappa) = t_i$. Inserting the limits above into (6.13), we get that for $\rho \to \infty$ the left-hand side of (6.13) tends to

$$|\lambda|^{\frac{\sigma_{\kappa+1}}{r_{\kappa+1}}+\cdots+\frac{\sigma_N}{r_N}} \cdot \Phi_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\xi,\lambda),$$

and thus

$$J_i(\rho) \longrightarrow |\lambda|^{\frac{\sigma_{\kappa+1}}{r_{\kappa+1}}+\cdots+\frac{\sigma_N}{r_N}} \|\Phi_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(D,\lambda)u_i\| \quad (i = 1,\ldots,\kappa). \tag{6.15}$$

For $i > \kappa$ we have $t_i(\kappa) = \varepsilon + t_i r_\kappa / r_i$. Inserting this into (6.13) and taking the limit $\rho \to \infty$, we obtain

$$\begin{aligned} &\lim_{\rho\to\infty} \rho^{-a_\kappa - t_i(\kappa)} \Psi_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(\rho\xi, \rho^{r_\kappa}\lambda) \\ &\qquad = \Phi_{\boldsymbol{\sigma}}(\xi,\lambda) \cdot |\lambda|^{\frac{\sigma_{\kappa+1}}{r_{\kappa+1}}+\cdots+\frac{\sigma_N}{r_N}} \cdot \lim_{\rho\to\infty} \frac{(\rho|\xi| + \rho^{r_\kappa/r_i}|\lambda|^{1/r_j})^{t_i}}{\rho^{\varepsilon + t_i r_\kappa / r_i}} = 0. \end{aligned}$$

Therefore,

$$J_i(\rho) \longrightarrow 0 \quad (i = \kappa+1,\ldots,N). \tag{6.16}$$

(ii) Now let us consider $I_i(\rho)$. In the same way as before, we write

$$\begin{aligned} &\rho^{-a_\kappa} \Psi_{\boldsymbol{\sigma}-s_i\mathbf{e}_i}(\rho\xi, \rho^{r_\kappa}\lambda) \Big[\sum_{j=1}^N A_{ij}(\rho\xi)\rho^{-t_j(\kappa)} - \rho^{r_\kappa - t_i(\kappa)}\lambda \Big] \\ &\quad = \Big(\frac{\rho|\xi| + \rho^{r_\kappa/r_i}|\lambda|^{1/r_i}}{\rho} \Big)^{-s_i} \cdot \prod_{j=1}^{\kappa} \Big(\frac{\rho|\xi| + \rho^{r_\kappa/r_j}|\lambda|^{1/r_j}}{\rho} \Big)^{\sigma_j} \\ &\qquad \cdot \prod_{j=\kappa+1}^{N} \Big(\frac{\rho|\xi| + \rho^{r_\kappa/r_j}|\lambda|^{1/r_j}}{\rho^{r_\kappa/r_j}} \Big)^{\sigma_j} \\ &\qquad \cdot \Big[\sum_{j=1}^N \rho^{-s_i - t_j(\kappa)} A_{ij}(\rho\xi) - \rho^{-s_i - t_i(\kappa) + r_\kappa}\lambda \Big]. \end{aligned} \tag{6.17}$$

For $i \le \kappa$ we use

$$\sum_{j=1}^N \rho^{-s_i - t_j(\kappa)} A_{ij}(\rho\xi) = \sum_{j=1}^{\kappa} A_{ij}(\xi) + \sum_{j=\kappa+1}^{N} \rho^{-\varepsilon + t_j(1 - r_\kappa/r_j)} A_{ij}(\xi),$$
$$\rho^{-s_i + r_\kappa - t_i(\kappa)}\lambda = \rho^{r_\kappa - r_i}\lambda$$

and obtain that for $\rho \to \infty$ the left-hand side of (6.17) tends to

$$|\lambda|^{\frac{\sigma_{\kappa+1}}{r_{\kappa+1}}+\cdots+\frac{\sigma_N}{r_N}} \Phi_{\boldsymbol{\sigma}-s_i\mathbf{e}_i}(\xi,\lambda) \Big(\sum_{j=1}^{\kappa} A_{ij}(\xi) - \delta_{i\kappa}\lambda \Big).$$

In the case $i > \kappa$ we have

$$\Big(\frac{\rho|\xi| + \rho^{r_\kappa/r_i}|\lambda|^{1/r_i}}{\rho} \Big)^{-s_i} \to 0 \quad (\rho \to \infty)$$

and

$$\rho^{-s_i - t_i(\kappa) + r_\kappa} = \rho^{(r_\kappa - r_i)(t_i - r_i)/r_i - \varepsilon} \to 0 \quad (\rho \to \infty),$$

and the left-hand side of (6.17) tends to zero. Therefore

$$I_i(\rho) \longrightarrow \begin{cases} |\lambda|^{\frac{\sigma_{\kappa+1}}{r_{\kappa+1}}+\dots+\frac{\sigma_N}{r_N}} \|\Phi_{\boldsymbol{\sigma}-s_i\mathbf{e}_i}(D,\lambda)g_i\|, & i \le \kappa, \\ 0, & i > \kappa. \end{cases} \tag{6.18}$$

From (6.15), (6.16) and (6.18) the desired result follows. □

Proof of Proposition 6.2, Part (ii).

In analogy to $\widetilde{\Phi}_{\boldsymbol{\sigma}}$, we define

$$\widetilde{\Psi}_{\boldsymbol{\sigma}}(\xi,\lambda) := \prod_{j=1}^{N} (|\xi'| + i\xi_n + |\lambda|^{1/r_j})^{\sigma_j}$$

and rewrite for $u \in (C_0^\infty(\overline{\mathbb{R}^n_+}))^N$ the main estimate (3.5) in the form

$$\begin{aligned} \sum_{i=1}^{N} \|\widetilde{\Psi}_{\boldsymbol{\sigma}+t_i\mathbf{e}_i}(D,\lambda)u_i\|_{L^2(\mathbb{R}^n_+)} \le C\Big(&\sum_{i=1}^{N} \|\widetilde{\Psi}_{\boldsymbol{\sigma}-s_i\mathbf{e}_i}(D,\lambda)f_i\|_{L^2(\mathbb{R}^n_+)} \\ &+ \sum_{j=1}^{R} \|\widetilde{\Psi}_{\boldsymbol{\sigma}}^{(-m_j-1/2)}(D',\lambda)g_j\|_{L^2(\mathbb{R}^{n-1})}\Big), \end{aligned} \tag{6.19}$$

where $f := (A-\lambda)u$ and $g := Bu$.

As in the proof of Proposition 6.2, we fix $\kappa \in \{1,\dots,N\}$, replace u by u_ρ defined in (6.10), replace λ by $\rho^{r_\kappa}\lambda$ and take the limit $\rho \to \infty$. Slightly modifying the arguments from the proof of Proposition 6.2, we see that the left-hand side and the first sum on the right-hand side of (6.19) tend to the corresponding terms in (6.5) multiplied by $|\lambda|^{\frac{\sigma_{\kappa+1}}{r_{\kappa+1}}+\dots+\frac{\sigma_N}{r_N}}$. Now we consider the typical term of the last sum on the right-hand side of (6.19) (with u replaced by u_ρ and after change of variables),

$$\begin{aligned} L_j(\rho) &:= \rho^{-a_\kappa+1/2} \Big\| \widetilde{\Psi}_{\boldsymbol{\sigma}}^{(-m_j-1/2)}(\rho D', \rho^{r_\kappa}\lambda) \sum_{i=1}^{N} \rho^{-t_i(\kappa)} b_{ji}(\rho D)u_i \Big\|_{L^2(\mathbb{R}^{n-1})} \\ &= \rho^{-a_\kappa+m_j+1/2} \Big\| \widetilde{\Psi}_{\boldsymbol{\sigma}}^{(-m_j-1/2)}(\rho D', \rho^{r_\kappa}\lambda) \\ &\quad \times \Big[\sum_{i=1}^{\kappa} b_{ji}(D)u_i + \sum_{i=\kappa+1}^{N} \rho^{-\varepsilon+t_i(1-r_\kappa/r_i)} b_{ji}(D)u_i \Big] \Big\|_{L^2(\mathbb{R}^{n-1})}. \end{aligned} \tag{6.20}$$

(i) Let $j \in \{R_{\ell-1}+1,\dots,R_\ell\}$ with $\ell \le \kappa$. According to the definition of the shifted weight functions and (3.3), we can write

$$\begin{aligned} &\rho^{-a_\kappa+m_j+1/2}\widetilde{\Psi}_{\boldsymbol{\sigma}}^{(-m_j-1/2)}(\rho\xi', \rho^{r_\kappa}\lambda) \\ &\quad = \Big(\frac{\rho|\xi'| + \rho^{r_\kappa/r_\ell}|\lambda|^{1/r_\ell}}{\rho}\Big)^{\sigma_1+\dots+\sigma_\ell-m_j-1/2} \cdot \prod_{i=\ell+1}^{\kappa} \Big(\frac{\rho|\xi'| + \rho^{r_\kappa/r_i}|\lambda|^{1/r_i}}{\rho}\Big)^{\sigma_i} \\ &\qquad \cdot \prod_{i=\kappa+1}^{N} \Big(\frac{\rho|\xi'| + \rho^{r_\kappa/r_i}|\lambda|^{1/r_i}}{\rho^{r_\kappa/r_i}}\Big)^{\sigma_i} \\ &\quad \longrightarrow |\lambda|^{\frac{\sigma_{\kappa+1}}{r_{\kappa+1}}+\dots+\frac{\sigma_N}{r_N}} \widetilde{\Phi}_{\boldsymbol{\sigma}}^{(-m_j-1/2)}(\xi',\lambda) \quad (\rho \to \infty). \end{aligned}$$

Here we took into account (6.14). Inserting this into (6.20), we see

$$L_j(\rho) \to \|\widetilde{\Phi}_{\boldsymbol{\sigma}}^{(-m_j-1/2)}(D',1)g_j^{(\kappa)}\|_{L^2(\mathbb{R}^{n-1})} \quad (j=1,\dots,R_\kappa). \tag{6.21}$$

(ii) Now let $j \in \{R_{\ell-1}+1,\dots,R_\ell\}$ with $\ell > \kappa$. In this case we have

$$\begin{aligned}&\rho^{-a_\kappa+m_j+1/2}\widetilde{\Psi}_{\boldsymbol{\sigma}}^{(-m_j-1/2)}(\rho\xi',\rho^{r_\kappa}\lambda)\\ &\quad=\rho^{-a_\kappa+m_j+1/2}(\rho|\xi'|+\rho^{r_\kappa/r_\ell}|\lambda|^{1/r_\ell})^{\sigma_1+\dots+\sigma_\ell-m_j-1/2}\\ &\qquad\prod_{i=\ell+1}^{\kappa}\Big(\frac{\rho|\xi'|+\rho^{r_\kappa/r_i}|\lambda|^{1/r_i}}{\rho^{r_\kappa/r_i}}\Big)^{\sigma_i}.\end{aligned}$$

In this expression the exponent of ρ equals

$$\begin{aligned}&-a_\kappa+m_j+1/2+\frac{r_\kappa}{r_\ell}(\sigma_1+\dots+\sigma_\ell-m_j-1/2)+\sum_{i=\ell+1}^{N}\frac{r_\kappa\sigma_i}{r_i}\\ &\quad=\Big(\frac{r_\kappa}{r_\ell}-1\Big)(\sigma_1+\dots+\sigma_\kappa-m_j-1/2)+\sum_{i=\kappa+1}^{\ell}r_\kappa\sigma_i\Big(\frac{1}{r_i}-\frac{1}{r_\ell}\Big)\\ &\quad<0.\end{aligned}$$

Inserting this into (6.20), we have

$$L_j(\rho)\to 0 \quad (j=R_\kappa+1,\dots,R).$$

Together with (6.21), this finishes the proof. ☐

Proof of Proposition 6.2, Part (iii).

We modify the proof of Proposition **??** by substituting in (6.19) a vector function of boundary layer type

$$u_\rho=(u_{1\rho},\dots,u_{N\rho}),\quad u_{j\rho}(x):=\rho^{\frac{n-1}{2}\theta+\frac12-a(\kappa)-t_j(\kappa)}\cdot u_j(\rho^\theta x',\rho x_n)$$

with θ satisfying the inequalities

$$\frac{r_{\kappa-1}}{r_\kappa}<\theta<1.$$

The calculations of the limits for $\rho\to\infty$ of the $\|\cdot\|_{L^2(\mathbb{R}^n_+)}$-norms follow the same lines as in the proof of Proposition **??**, and we will not dwell on them. For the boundary norms, we have to consider

$$M_j:=\rho^{-a(\kappa)+m_j+1/2}\Big\|\widetilde{\Psi}_{\boldsymbol{\sigma}}^{(-m_j-1/2)}(\rho^\varepsilon D',\rho^{r_\kappa}\lambda)\cdot\sum_{i=1}^{N}\rho^{t_i-t_i(\kappa)}B_{ji}(\rho^{\theta-1}D',D_n)\Big\|.$$

To compute $\lim_{\rho\to\infty}M_j(\rho)$, we choose $\ell\in\{1,\dots,N\}$ such that $R_{\ell-1}+1\le j\le R_\ell$. We distinguish the cases $\ell<\kappa$, $\ell=\kappa$ and $\ell>\kappa$.

For $\ell < \kappa$ we have

$$\rho^{-a(\kappa)+m_j+1/2}\widetilde{\Psi}_{\boldsymbol{\sigma}}^{(-m_j-1/2)}(\rho^\theta\xi',\rho^{r_\kappa}\lambda)$$

$$= \Big(\frac{\rho^\theta|\xi'|+\rho^{r_\kappa/r_\ell}|\lambda|^{1/r_\ell}}{\rho}\Big)^{\sigma_1+\cdots+\sigma_\ell-m_j-1/2} \cdot \prod_{i=\ell+1}^{\kappa}\Big(\frac{\rho^\theta|\xi'|+\rho^{r_\kappa/r_i}|\lambda|^{1/r_i}}{\rho}\Big)^{\sigma_i} \cdot \prod_{i=\kappa+1}^{N}\Big(\frac{\rho^\theta|\xi'|+\rho^{r_\kappa/r_i}|\lambda|^{1/r_i}}{\rho^{r_\kappa/r_i}}\Big)^{\sigma_i}. \tag{6.22}$$

As both products are bounded for $\rho\to\infty$ and the first factor tends to zero (note that $\theta<1, r_\kappa/r_\ell<1$ and $\sigma_1+\cdots+\sigma_\ell-m_j-1/2>0$), the right-hand side of (6.22) tends to zero for $\rho\to\infty$.

For $\ell=\kappa$ expression (6.19) equals

$$\Big(\frac{\rho^\theta|\xi'|+\rho|\lambda|^{1/r_\kappa}}{\rho}\Big)^{\sigma_1+\cdots+\sigma_\kappa-m_j-1/2}\cdot\prod_{i=\kappa+1}^{N}\Big(\frac{\rho^\theta|\xi'|+\rho^{r_\kappa/r_i}|\lambda|^{1/r_i}}{\rho^{r_\kappa/r_i}}\Big)^{\sigma_i}$$
$$\to |\lambda|^{(\sigma_1+\cdots+\sigma_\kappa-m_j-1/2)/r_\kappa+\sigma_{\kappa+1}/r_{\kappa+1}+\cdots+\sigma_N/r_N}$$

for $\rho\to\infty$. In the same way it is easily seen that for $\ell>\kappa$ expression (6.19) tends to zero as $\rho\to\infty$. Inserting these limits into (...) and taking into account the limits for the sums in (...), we obtain

$$M_j(\rho)\to\begin{cases}0 & \text{if } j\le R_{\kappa-1} \text{ or } j>R_\kappa,\\ |\lambda|^{(\sigma_1+\cdots+\sigma_\kappa-m_j-1/2)/r_\kappa+\sigma_{\kappa+1}/r_{\kappa+1}+\cdots+\sigma_N/r_N} \\ \quad\cdot\Big\|\sum_{i=1}^{\kappa}B_{ji}(0,D_n)u\Big\|_{L(\mathbb{R}^{n-1})} & \text{else}\end{cases}$$

for $\rho\to\infty$. □

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