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THE NEWTON POLYGON APPROACH FOR BOUNDARY VALUE PROBLEMS WITH GENERAL BOUNDARY CONDITIONS

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1. Introduction

In elliptic theory there is a lot of problems where the equation depends on a complex parameter. A wide and very important class of such problems was treated by Agmon [1] and Agranovich-Vishik [3] who introduced a very clear notion of ellipticity with parameter. This concept was further elaborated by Grubb [11] in the framework of pseudodifferential operators and Boutet de Monvel problems. In the classical theory there are two main concepts of ellipticity: ellipticity without parameter in the sense of, e.g., Agmon–Douglis–Nirenberg [2] and ellipticity with parameter in the sense of Agmon [1] and Agranovich–Vishik [3]. During the last years, however, several elliptic problems were treated which do not belong to one of these classes. For instance, the resolvent of a Douglis–Nirenberg system (mixed order system) depends on a complex parameter but in general its principal symbol is not quasi-homogeneous with respect to the co-variables and the parameter which implies that the theory of ellipticity with parameter cannot be applied. On closed manifolds such systems were considered by Kozhevnikov in [12] and by the authors in [4], where rather complete results concerning solvability and a priori estimates was obtained.

On manifolds with boundary (for instance, bounded domains in $\mathbb{R}^n$) the situation is more complicated. Here one has to impose boundary conditions, and one has to ask for the analog of the Shapiro–Lopatinskii condition (which is well-known both in the parameter-independent case and in the case of ellipticity with parameter) and for the norms on the boundary which appear in the a priori estimates.

These questions are not restricted to the resolvent of mixed order systems but appear for general parameter-dependent operator pencils (scalar or matrix) which do not satisfy the Agmon–Agranovich–Vishik condition. Let us consider scalar operators of the form

$$A(x,D,\lambda) = A_{2m}(x,D) + \lambda A_{2m-1}(x,D) + \cdots + \lambda^{2m-2\mu} A_{2\mu}(x,D) \quad (1)$$

with $m > \mu > 0$, where $A_j(x,D) = \sum_{|\alpha|\le j} a_{\alpha j}(x)D^\alpha$ $(j = 1,\dots,m)$ are partial differential operators with smooth coefficients acting on a compact manifold M with boundary ∂M. Here and in the following, we use the standard multi-index notation $D^\alpha = D_1^{\alpha_1}\dots D_n^{\alpha_n}$ with $D_j = -i\partial/\partial x_j$. The Dirichlet problem connected with (1) was treated in [5] (see also [8]). In the present paper we want to deal with general boundary conditions.

The symbol of the operator (1) is of the same structure as the determinant of the symbol of a matrix-valued operator which is of Douglis–Nirenberg type and depends on the complex parameter λ in the way

$$A(x,D,\lambda) = \begin{pmatrix} A_{11}(x,D) & A_{12}(x,D) \\ A_{21}(x,D) & A_{22}(x,D) - \lambda^{2m-2\mu} \end{pmatrix}, \tag{2}$$

where we assume (for simplicity) that $A_{ij}(x,D)$ are scalar partial differential operators with $\operatorname{ord} A_{11} = 2\mu$ and $\operatorname{ord} A_{22} = 2m - 2\mu$. Again the main question is to find Shapiro–Lopatinskii type conditions which lead to a priori estimates.

The essential tool for analyzing operators (1) and (2) is the so-called Newton polygon which can be assigned to general parameter-dependent partial differential operators. For the present text we don't want to summarize the general theory for which we refer the reader to [10], [4]–[6] and also to [8], but restrict ourselves to the application of the Newton polygon approach to the operators (1) and (2).

In the following, we will mainly deal with operators of the form (1) and only indicate the corresponding definitions and results for (2). A more detailed exposition of the results below (and additional considerations) can be found in the preprints [6] for operators of the form (1) and [7] for operators of the form (2); in the latter preprint also the case that $A_{ij}(x,D)$ itself is a mixed order matrix operator is treated.

2. Ellipticity conditions

The definition of ellipticity for the operator (1) (as well as the formulation of the a priori estimate) uses a parameter-dependent weight function $W_r(\xi,\lambda)$ defined for $\xi \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$ and $r = (r_1,r_2) \in \mathbb{R}^2$ by

$$W_r(\xi,\lambda) := (1+|\xi|)^{r_1}(|\lambda|+|\xi|)^{r_2} \tag{3}$$

and the similarly defined (ξ,λ)–homogeneous function

$$V_r(\xi,\lambda) := |\xi|^{r_1}(|\lambda|+|\xi|)^{r_2}\,.$$

For integer and positive r_1 and r_2 these functions appear as the weight function corresponding to the Newton polygon N_r. This polygon is defined as the convex hull of the set

$$\{(0,0),\ (0,r_2),\ (r_1,r_2),\ (r_1+r_2,0)\}$$

in $\mathbb{R}^2$. In the first step we have to define ellipticity for the operators (1) without boundary conditions.

For this we set $A_{ij}^{(0)}(x,\xi) := \sum_{|\alpha|=j} a_{\alpha j}(x)\xi^\alpha$ and define the principal symbol of the operator (1) by

$$A^{(0)}(x,\xi,\lambda) := A_{2m}^{(0)}(x,\xi) + \lambda A_{2m-1}^{(0)}(x,\xi) + \cdots + \lambda^{2m-2\mu} A_{2\mu}^{(0)}(x,\xi)\,. \tag{4}$$

The Newton polygon assigned to $A^{(0)}(x,\xi,\lambda)$ is given by $N_{(2\mu,2m-2\mu)}$. We define ellipticity for the pencil $A(x,D,\lambda)$ of the form (1) in accordance to the general Newton polygon approach:

Definition 1. The operator (1) is called weakly parameter-elliptic along the ray $[0,\infty)$ if the inequality $|A^{(0)}(x,\xi,\lambda)| \geq C\, V_{(2\mu,2m-2\mu)}(x,\xi)$, i.e.,

$$|A^{(0)}(x,\xi,\lambda)| \geq C|\xi|^{2\mu}(|\xi|+\lambda)^{2m-2\mu} \tag{5}$$

holds for all $x \in \overline{M}$, $\xi \in \mathbb{R}^n$ and $\lambda \in [0,\infty)$ with a constant C which does not depend on x, ξ or λ.

To define the Shapiro–Lopatinskii condition for (1), we fix a point $x^0 \in \partial M$ and local coordinates in a neighbourhood of x^0 which correspond to x^0 in the sense that in this system locally M is given by the inequality $x_n > 0$. We use in $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$ the coordinates $x = (x', x_n)$ and the dual coordinates $\xi = (\xi', \xi_n)$. From (5) it follows that $A^{(0)}(x^0,\xi,\lambda) \neq 0$ for all $\xi \in \mathbb{R}^n \backslash \{0\}$ and $\lambda \geq 0$. If $n > 2$, this implies that the polynomial $A^{(0)}(x^0,\xi',\cdot,\lambda)$ has exactly m roots with positive imaginary part; for $n = 2$ we will assume this in the following.

It is easily seen that in the case of weak parameter-ellipticity the polynomial $A^{(0)}(x^0,0,\cdot,1)$ has no real roots with the exception of zero which is a root of order 2μ. We say that $A(x,D,\lambda)$ degenerates regularly at the boundary ∂M if for every $x^0 \in \partial M$ the polynomial $A^{(0)}(x^0,0,\cdot,1)$has exactly $m-\mu$ roots with positive imaginary part. Let us remark that the condition of regular degeneration is known from the theory of singular perturbations as considered by Vishik–Lyusternik [14], Nazarov [13], Frank [9] and others. The connection of (1) to singularly perturbed problems is obvious if we set $\lambda = \varepsilon^{-1}$ in (1).

Now let us consider boundary operators $B_j(x,D) = \sum_{|\beta|\leq m_j} b_{j\beta}(x) D^\beta$ for $j = 1,\ldots,m$ where we assume that

$$m_1 \leq \cdots \leq m_\mu < m_{\mu+1} \leq \cdots \leq m_m\,. \tag{6}$$

Definition 2. The boundary value problem $\big(A(x,D,\lambda), B_1(x,D),\ldots,B_m(x,D)\big)$ is called weakly parameter-elliptic in $[0,\infty)$ if the following conditions hold:

a) $A(x,D,\lambda)$ is weakly parameter-elliptic in $[0,\infty)$ in the sense of Definition 1.
b) For every $x^0 \in \partial M$, $\xi' \in \mathbb{R}^{n-1}\backslash\{0\}$ and $\lambda \in [0,\infty)$ and every $(h_1,\dots,h_m) \in \mathbb{C}^m$ the ordinary differential equation on the half-line $\mathbb{R}_+$

$$\begin{aligned} A^{(0)}(x^0,\xi',D_t,\lambda)\,w(t) &= 0 \quad (t>0)\,, \\ B_j^{(0)}(x^0,\xi',D_t)\,w(t) &= h_j \quad (j=1,\dots,m)\,, \\ w(t) &\to 0 \quad (t\to\infty) \end{aligned}$$

is uniquely solvable. Here we have set $D_t := -i\partial/\partial t$.
c) The boundary value problem $\big(A_{2\mu}(x,D), B_1(x,D),\dots,B_\mu(x,D)\big)$ satisfies the classical Shapiro–Lopatinskii condition.
d) For every $x^0 \in \partial M$ and every $(h_{\mu+1},\dots,h_m) \in \mathbb{C}^{m-\mu}$ the ordinary differential equation on the half-line

$$\begin{aligned} A^{(0)}(x^0,0,D_t,1)\,w(t) &= 0 \quad (t>0)\,, \\ B_j^{(0)}(x^0,0,D_t)\,w(t) &= h_j \quad (j=\mu+1,\dots,m)\,, \\ w(t) &\to 0 \quad (t\to\infty) \end{aligned}$$

is uniquely solvable.

Let us make some comments on these conditions. The first and second condition seem to be quite natural and have their direct counterparts in the classical theories. However, for $\xi'=0$ condition b) is in general not satisfied even for positive λ, differently to the Agmon–Agranovich–Vishik theory. In some sense conditions c) and d) which seem to be less natural are connected with this fact. Both in c) and d) we take only some of the boundary operators; this is typical for singularly perturbed problems. We will see below that the last two conditions find their explanation in the proof of the a priori estimate; they are closely connected with the structure of the solution, in particular condition d) leads to the boundary layer nature of the solution. From the point of view of singular perturbation theory and asymptotic expansions, the last two conditions are very natural, too.

3. A priori estimates

We still consider the operator (1) with general boundary operators $B_1,\dots,B_m$. If one looks through the proofs of a priori estimates in elliptic theory, one can realize that one main step consists in the description of the zeros of the symbol, written in local coordinates and considered as a polynomial of the co-variable ξ_n. The same is true for weakly parameter-elliptic operators of the form (1), so let us first consider the behaviour of these zeros for large λ.

Throughout this section, we assume that $A(x, D, \lambda)$ degenerates regularly at the boundary and that $(A, B_1, \dots, B_m)$ is weakly parameter-elliptic in the sense of Definition 2. We fix $x^0 \in \partial M$ and assume that the boundary value problem is written in local coordinates corresponding to x^0 as described above.

The following lemma (which is taken from [6]) shows that the zeros of the polynomial $A^{(0)}(x^0, \xi', \cdot, \lambda)$ split, for large λ, into two groups, one group staying bounded and the other group being of order λ. Note that in this lemma the polynomials $A^{(0)}_{2\mu}(x^0, \xi', \cdot)$ and $A^{(0)}(x^0, 0, \cdot, 1)$ play a particular role; these polynomials also appear in conditions c) and d) of Definition 2.

Lemma 3. *There exists a $\lambda_0 > 0$ such that with a suitable numbering*

$$\tau_1(x^0, \xi', \lambda), \dots, \tau_m(x^0, \xi', \lambda)$$

of the roots of $A^{(0)}(x^0, \xi', \cdot, \lambda)$ with positive imaginary part the following assertions hold:

(i) *Let $\gamma^{(1)}(x^0, \xi')$ denote a contour in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ which encloses all zeros of $A^{(0)}_{2\mu}(x^0, \xi', \cdot)$ with positive imaginary part. Then for all $|\xi'| = 1$ and all $\lambda \geq \lambda_0$ the contour $\gamma^{(1)}$ encloses $\tau_1(x^0, \xi', \lambda), \dots, \tau_\mu(x^0, \xi', \lambda)$.*

(ii) *Let $\gamma^{(2)}(x^0)$ denote a contour in $\mathbb{C}_+$ which encloses all zeros of $A^{(0)}(x^0, 0, \cdot, 1)$ with strictly positive imaginary part. Then for all $|\xi'| = 1$ and all $\lambda \geq \lambda_0$ the contour $\gamma^{(2)}$ encloses $\tau_{\mu+1}(x^0, \xi', \lambda)/\lambda, \dots, \tau_m(x^0, \xi', \lambda)/\lambda$.*

The next result is the key for proving a priori estimates. In its formulation we use the "shifted" weight function

$$V_r^{(-a)}(\xi, \lambda) := \begin{cases} |\xi|^{r_1 - a}(|\lambda| + |\xi|)^{r_2}, & a \leq r_1, \\ (|\lambda| + |\xi|)^{r_1 + r_2 - a}, & a > r_1, \end{cases}$$

defined for $r = (r_1, r_2) \in \mathbb{R}^2$ and $a \in \mathbb{R}$. The name "shifted" comes from the fact that for positive integers r_1 and r_2 and for $0 \leq a \leq r_1 + r_2$ the function $V_r^{(-a)}$ corresponds to the Newton polygon which is constructed from N_r by a shift of length a to the left parallel to the horizontal axis. The function $W_r^{(-a)}$ is defined analogously, with $|\xi|^{r_1 - a}$ replaced by $(1 + |\xi|)^{r_1 - a}$.

For the remainder of this section, let us fix the tuple $r = (r_1, r_2)$ where we assume that r_1 and r_2 are (for simplicity) integer numbers with $r_1 + r_2 \geq m_m + 1$ and $m_\mu + 1 \leq r_1 \leq m_{\mu+1}$.

Theorem 4. *For $\xi' \in \mathbb{R}^{n-1} \backslash \{0\}$, $\lambda \in [0, \infty)$ and $j \in \{1, \dots, m\}$ let $w_j = w_j(t, \xi', \lambda)$ be the unique solution of the problem*

$$\begin{aligned} A^{(0)}(x^0, \xi', D_t, \lambda)\, w(t) &= 0 \quad (t > 0), \\ B_k^{(0)}(x^0, \xi', D_t)\, w(t) &= \delta_{kj} \quad (k = 1, \dots, m), \\ w(t) &\to 0 \quad (t \to \infty). \end{aligned}$$

Then for $l = 0, 1, 2, \dots$ *the estimate*

$$\|D_t^l w_j(\cdot, \xi', \lambda)\|_{L_2(\mathbb{R}_+)} \le C \, \frac{V_r^{(-m_j-1/2)}(\xi', \lambda)}{V_r^{(-l)}(\xi', \lambda)} \tag{7}$$

holds with a constant C *independent of* ξ' *and* λ.

Sketch of proof. The proof of the above estimate is rather envolved, so we only want to mention the main ideas. We may restrict ourselves to the case $|\xi'| = 1$ and large λ.

First of all, it is possible to show, using conditions 2 c) and d), respectively, and Lemma 3, that there exist polynomials (with respect to τ) $N_k(x^0, \xi', \tau, \lambda)$ $(k = 1, \dots, m)$ such that

$$\frac{1}{2\pi i} \int_{\gamma^{(1)}} \frac{B_l(x^0, \xi', \tau) N_k(x^0, \xi', \tau, \lambda)}{A_1(x^0, \xi', \tau, \lambda)} \, d\tau = \delta_{kl} \quad (k, l = 1, \dots, \mu) \tag{8}$$

and

$$\frac{1}{2\pi i} \int_{\gamma^{(2)}} \frac{B_l(x^0, \xi'/\lambda, \tau) N_k(x^0, \xi', \tau, \lambda)}{A_2(x^0, \xi'/\lambda, \tau, 1)} \, d\tau = \delta_{kl} \quad (k, l = \mu + 1, \dots, m). \tag{9}$$

Here $\gamma^{(1)}$ and $\gamma^{(2)}$ are the contours appearing in Lemma 3, and we have set

$$A_1(x^0, \xi', \tau, \lambda) := \prod_{j=1}^{\mu} (\tau - \tau_j(x^0, \xi', \lambda)) \,,$$
$$A_2(x^0, \xi', \tau, \lambda) := \prod_{j=\mu+1}^{m} (\tau - \tau_j(x^0, \xi', \lambda)) \,.$$

The relations (8) and (9) allow us to write w_j in the form

$$\begin{aligned} w_j(t, \xi', \lambda) = & \sum_{k=1}^{\mu} \psi_k(\xi', \lambda) \frac{1}{2\pi i} \int_{\gamma^{(1)}} \frac{N_k(x^0, \xi', \tau, \lambda)}{A_1(x^0, \xi', \tau, \lambda)} \, e^{it\tau} \, d\tau \\ & + \sum_{k=\mu+1}^{m} \psi_k(\xi', \lambda) \frac{1}{2\pi i} \int_{\gamma^{(2)}} \frac{N_k(x^0, \xi', \tau, \lambda)}{A_2(x^0, \xi'/\lambda, \tau, 1)} \, e^{it\lambda\tau} \, d\tau \end{aligned} \tag{10}$$

with unknown functions $\psi_k(\xi', \lambda)$. It can be shown that the conditions

$$B_k(\xi', D_t) w_j(t, \xi', \lambda) = \delta_{kj} \quad (k = 1, \dots, m)$$

lead to a linear equation system for $\psi_1, \dots, \psi_m$ which is uniquely solvable for sufficiently large λ; we also obtain estimates (with respect to λ for $|\xi'| = 1$) for its solution. From this and from estimates for the integrands in (10) (where we use Lemma 3 again) one can see that

$$\|D_t^l w_j(\xi', \cdot, \lambda)\|_{L_2(\mathbb{R}_+)} \le \begin{cases} O(1) + O(\lambda^{l-m_{\mu+1}-1/2}), & j \le \mu, \\ O(\lambda^{m_\mu - m_j}) + O(\lambda^{l-m_j-1/2}), & j > \mu \end{cases}$$

holds for $|\xi'| = 1$ and large λ. Now it remains to compare this with the right-hand side of (7), using the definition of the shifted weight function. □

Inequality (7) already gives an idea which norms should appear in the a priori estimate. For arbitrary $r \in \mathbb{R}^2$ we define the Sobolev space $H_r(\mathbb{R}^n)$ as the set of all distributions $u \in H^{r_1+r_2}(\mathbb{R}^n)$ depending on the parameter λ for which a constant $C(u)$ exists such that

$$\|u\|_{r,\mathbb{R}^n} := \|F^{-1} W_r(\xi, \lambda) F u\|_{L_2(\mathbb{R}^n)} \le C(u)$$

holds. Here F stands for the Fourier transform. By standard methods, it is possible to define the analog of the parameter-dependent norms $\|\cdot\|_{r,\mathbb{R}^n}$ in the half-space $\mathbb{R}^n_+$, in $\mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$ and on the manifold M and its boundary ∂M. The Sobolev space connected with the shifted weight function $W_r^{(-a)}$ will be denoted by $H_r^{(-a)}$ with norm $\|\cdot\|_r^{(-a)}$.

Theorem 5 (A priori estimate). *There exists a $\lambda_0 > 0$ and a constant C such that for all $\lambda \ge \lambda_0$ the inequality*

$$\|u\|_{r,M} \le C\Big(\|A(x, D, \lambda)u\|_{(r_1-2\mu, r_2-2(m-\mu)),M} + \sum_{j=1}^m \|B_j(x, D)u\|_{r,\partial M}^{(-m_j-1/2)} + \lambda^{r_2} \|u\|_{L_2(M)} \Big)$$

holds.

Sketch of proof. Again we restrict on mentioning some ideas of the proof. By the standard method of freezing the coefficients (localization method), it suffices to consider for fixed x^0 the operator $A^{(0)}(x^0, D, \lambda)$ as an operator acting in the whole space for $x^0 \in M$, and the same operator supplemented with the boundary operators $B_j^{(0)}(x^0, D)$ as a boundary value problem in $\mathbb{R}^n_+$ for $x^0 \in \partial M$. The case of the whole space is easily proved using the inequality of Definition 1.

For the half-space $\mathbb{R}^n_+$ we have to investigate

$$\begin{aligned} A^{(0)}(x^0, D, \lambda)\, u(x) &= f \quad \text{in } \mathbb{R}^n_+, \\ B_j^{(0)}(x^0, D)\, u(x) &= g_j \quad (j = 1, \dots, m) \text{ on } \mathbb{R}^{n-1}. \end{aligned}$$

Again by standard methods (using the results in $\mathbb{R}^n$), we may assume that $f = 0$. Now we take partial Fourier transform F' with respect to $x' \in \mathbb{R}^{n-1}$ and obtain for the function $w(t) := (F'v)(\xi', t)$ the problem on the half-line

$$\begin{aligned} A^{(0)}(x^0, \xi', D_t, \lambda)\, w(t) &= 0 \quad (t > 0), \\ B_j^{(0)}(x^0, \xi', D_t)\, w(t) &= (F'g_j)(\xi') \quad (j = 1, \dots, m)\,. \end{aligned}$$

Due to condition 2 b), for $\xi' \neq 0$ the solution is unique and given by

$$w = w(t, \xi', \lambda) = \sum_{j=1}^{m} w_j(t, \xi', \lambda)(F'g_j)(\xi')\,.$$

It is not difficult to see that $(V_r(\xi, \lambda))^2$ is equivalent to

$$\sum_{l=0}^{r_1+r_2} \Big[\xi_n^l V_r^{(-l)}(\xi', 0, \lambda)\Big]^2$$

and that therefore the norm $\|u\|^2_{r,\mathbb{R}^n_+}$ is equivalent to the norm

$$\int_{\mathbb{R}^{n-1}} \sum_{l=0}^{r_1+r_2} \Big[V_r^{(-l)}(\xi', 0, \lambda)\|D_t^l w(\xi', \cdot, \lambda)\|_{L_2(\mathbb{R}_+)}\Big]^2 d\xi' + \lambda^{2r_2}\|u\|^2_{L_2(\mathbb{R}^n_+)}\,.$$

Note that due to our assumptions on r_1 and r_2 we have $r_1 + r_2 \in \mathbb{N}$. Now the basic estimate for w_j in Theorem 4 leads to the desired a priori estimate. □

Remark 6. In [7] it is also shown that the a priori estimate of the previous theorem implies weak parameter-ellipticity in the sense of Definition 2, so these two assertions are equivalent. In particular, this shows that the conditions of Definition 2 are in some sense the "correct" conditions. Moreover, it is possible to construct a (right) parametrix to the operator $\big(A(x, D, \lambda), B_1(x, D), \dots, B_m(x, D)\big)$, see Section 5 of [7]. The definitions and results above have a direct counterpart for singularly perturbed problems.

4. Remarks on the matrix operator (2)

As already mentioned in the introduction, the above concept of weak parameter-ellipticity can be used for matrix operators of the form (2), too. In the present section we only want to indicate the main definitions and results of this approach.

For the remainder of this section, let $A(x, D, \lambda)$ be an operator given in (2). We assume that there exist nonnegative integers s_1, s_2, t_1, t_2 such that

$$A_{ij}(x, D) = \sum_{|\alpha| \le s_i + t_j} a_{ij\alpha}(x) D^\alpha$$

with $s_1+t_1=2\mu$ and $s_2+t_2=2m-2\mu$. Then we can define the principal symbol (in the sense of Douglis–Nirenberg)

$$A_{ij}^{(0)}(x,\xi) := \sum_{|\alpha|=s_i+t_j} a_{ij\alpha}(x)\xi^\alpha\,,$$
$$A^{(0)}(x,\xi,\lambda) := \begin{pmatrix} A_{11}^{(0)}(x,\xi) & A_{12}^{(0)}(x,\xi) \\ A_{21}^{(0)}(x,\xi) & A_{22}^{(0)}(x,\xi)-\lambda^{2m-2\mu} \end{pmatrix}.$$

The determinant of this symbol has the same structure as the symbol of the scalar pencil (1), so we can define:

Definition 7. The operator $A(x,D,\lambda)$ is weakly parameter-elliptic in $[0,\infty)$, if

$$|\det A^{(0)}(x,\xi,\lambda)| \ge C\,|\xi|^{2\mu}(|\xi|+\lambda)^{2m-2\mu}$$

holds for all $x\in\overline{M}$, $\xi\in\mathbb{R}^n$ and $\lambda\in[0,\infty)$.

It can be seen easily that the analog of the condition of regular degeneration at the boundary is satisfied automatically.

Now let us assume that we have a matrix

$$B(x,D) = \Big(B_{jk}(x,D)\Big)_{\substack{j=1,\dots,m\\ k=1,2}}$$

of boundary operators with $\operatorname{ord} B_{jk}\le m_j+t_k$ for some integers m_j for which (6) holds. We define the principal symbols $B_{jk}^{(0)}(x,\xi)$ and $B^{(0)}(x,\xi)$ in an obvious way.

Definition 8. The boundary value problem $\big(A(x,D,\lambda),B(x,D)\big)$ is called weakly parameter-elliptic if the following conditions hold:

a) $A(x,D,\lambda)$ is weakly parameter-elliptic in $[0,\infty)$ in the sense of Definition 7.

b) For every $x^0\in\partial M$, $\xi'\in\mathbb{R}^{n-1}\setminus\{0\}$ and $\lambda\in[0,\infty)$ and every $(h_1,\dots,h_m)\in\mathbb{C}^m$ the ordinary differential equation system on the half-line

$$A^{(0)}(x^0,\xi',D_t,\lambda)\begin{pmatrix} w_1(t)\\ w_2(t)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad (t>0)\,,$$
$$B_{k1}^{(0)}(x^0,\xi',D_t)\,w_1(t)+B_{k2}^{(0)}(x^0,\xi',D_t)\,w_2(t) = h_k \quad (k=1,\dots,m)\,,$$
$$w_j(t)\to 0 \quad (t\to\infty;\quad j=1,2)$$

is uniquely solvable.

c) The boundary value problem $\big(A_{11}(x,D),B_{11}(x,D),\dots,B_{\mu1}(x,D)\big)$ satisfies the classical Shapiro–Lopatinskii condition.

d) For every $x^0 \in \partial M$ and every $(h_{\mu+1}, \dots, h_m) \in \mathbb{C}^{m-\mu}$ the ordinary differential equation system on the half-line

$$A^{(0)}(x^0, 0, D_t, 1) \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (t > 0)\,,$$

$$B_{k1}^{(0)}(x^0, 0, D_t)\, w_1(t) + B_{k2}^{(0)}(x^0, 0, D_t)\, w_2(t) = h_k \quad (k = \mu+1, \dots, m)\,,$$

$$w_j(t) \to 0 \quad (t \to \infty; \quad j = 1, 2)$$

is uniquely solvable.

The analogy to Defintion 2 is obvious. If (A, B) is weakly parameter-elliptic, the assertions of Lemma 3 hold for the polynomial $\det A^{(0)}(x^0, \xi', \cdot, \lambda)$. Along similar steps as for scalar operators of the form (2), we can estimate the solution $w_j(t) = \begin{pmatrix} w_{j1}(t, \xi', \lambda) \\ w_{j2}(t, \xi', \lambda) \end{pmatrix}$ of the boundary value problem in Definition 8 b) with $(h_1, \dots, h_m)$ replaced by the j-th unit vector. For fixed integers r_1, r_2 with $r_1 + r_2 \geq m_m + 1$ and $m_\mu + 1 \leq r_1 \leq m_{\mu+1}$ we obtain:

Theorem 9. *For $l = 0, 1, 2, \dots$ the inequalities*

$$\|D_t^l w_{j1}(\cdot, \xi', \lambda)\| \leq C\, \frac{V_r^{(-m_j - 1/2)}(\xi', \lambda)}{V_{(r_1+t_1, r_2)}^{(-l)}(\xi', \lambda)}\,,$$

$$\|D_t^l w_{j2}(\cdot, \xi', \lambda) \leq C\, \frac{V_r^{(-m_j - 1/2)}(\xi', \lambda)}{V_{(r_1, r_2+t_2)}^{(-l)}(\xi', \lambda)}$$

hold.

Note that this estimate is slightly different from the estimate in Theorem 4 due to the Douglis–Nirenberg structure of our operators. We also want to mention that the proofs for the matrix case are technically somewhat more difficult that in the scalar case.

Now the operators A and B act in products of Sobolev spaces. It turns out that an appropriate choice of these spaces is given by

$$(A, B)\colon H_{(r_1+t_1, r_2)}(M) \times H_{(r_1, r_2+t_2)}(M) \to H_{(r_1-s_1, r_2)}(M) \times H_{(r_1, r_2-s_2)}(M) \times \prod_{j=1}^{m} H_r^{(-m_j-1/2)}(\partial M)\,.$$

For these spaces, the operator (A, B) is continuous in the sense that it is a bounded operator with norm bounded by a constant which is independent of λ. In the case of weak parameter-ellipticity the inequalities of Theorem 9 lead to the desired a priori estimate:

Theorem 10. *Let (A, B) be weakly parameter-elliptic in the sense of Definition 8. Then there exists a $\lambda_0 > 0$ and a constant C such that for all $\lambda \geq \lambda_0$ the a priori estimate*

$$\|u_1\|_{(r_1+t_1,r_2),M} + \|u_2\|_{(r_1,r_2+t_2),M} \leq C\Big(\|f_1\|_{(r_1-s_1,r_2),M} + \|f_2\|_{(r_1,r_2-s_2),M} + \sum_{j=1}^{m} \|g_j\|_{r,\partial M}^{(-m_j-1/2)} + \lambda^{r_2}\|u_1\|_{L_2(M)} + \lambda^{r_2+t_2}\|u_2\|_{L_2(M)}\Big)$$

holds for $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, where we have set $f = A(x, D, \lambda)u$ and $g = B(x, D)u$.

Remark 11. Due to the Douglis–Nirenberg structure of the boundary value problem, the numbers m_j appearing in the above a priori estimate (and, consequently, also r_1) may be negative. This is one main difference to the scalar problem and one of the reasons why we defined the shifted weight function $W_r^{(-a)}$ also for negative a, where the definition does not coincide with the geometrical interpretation.

Final remarks. The operators considered in the present text show that the concept of weak ellipticity (which is a particular case of ellipticity connected with the Newton polygon) is useful for various classes of boundary value problems. The considerations above are part of a larger program which includes the resolvent of Douglis–Nirenberg systems (already solved on closed manifolds in [4]) and general elliptic and parabolic problems. Applications of this approach can be found, e.g., for singularly perturbed problems, transmission problems and problems with free boundary.

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