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Flexible affine cones and flexible coverings

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Abstract We provide a new criterion for flexibility of affine cones over varieties covered by flexible affine varieties. We apply this criterion to prove flexibility of affine cones over secant varieties of Segre–Veronese embeddings and over certain Fano threefolds. We further prove flexibility of total coordinate spaces of Cox rings of del Pezzo surfaces.

Keywords Automorphism group · Transitivity · Flexibility · Affine cone · Cox ring · Segre–Veronese embedding · Secant variety · Del Pezzo surface

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In this article we study affine algebraic varieties with the following property: the (special) automorphism group acts infinitely transitively on the regular locus. The systematic study of

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this remarkable property and its complex analytic counterpart is presented in Arzhantsev et al. [1].

Let X be an affine variety over an algebraically closed field $\mathbf{k}$ of characteristic zero. We consider actions of the additive group $\mathbb{G}_a = (\mathbf{k}, +)$ on X. The subgroup of $\mathrm{Aut}(X)$ generated by all $\mathbb{G}_a$-actions is called the *special automorphism group* of X and will be denoted by $\mathrm{SAut}(X)$. We are interested in transitivity of the $\mathrm{SAut}(X)$-actions on the regular locus X_{reg}. Recall that an action of a group G on a set M is *m-transitive* if for every two m-tuples $(x_1, \ldots, x_m)$ and $(x'_1, \ldots, x'_m)$ of pairwise distinct elements of M there exists $g \in G$ such that $g \cdot x_i = x'_i$ for $i = 1, \ldots m$.

We have the following result.

Theorem 0.1 ([1, Theorem 0.1]) *Let X be an irreducible affine variety of dimension ≥ 2. Then the following conditions are equivalent.*

(i) *The group* $\mathrm{SAut}(X)$ *acts transitively on the regular locus* X_{reg}.

(ii) *The group* $\mathrm{SAut}(X)$ *acts m-transitively on X_{reg} for every $m > 0$.*

(iii) *The tangent space of every $x \in X_{\mathrm{reg}}$ is spanned by tangent vectors to orbits of $\mathbb{G}_a$-actions.*

We say that X is *flexible* if these conditions are fulfilled.

As examples of flexible varieties, let us mention affine cones over del Pezzo surfaces of degree ≥ 4 (see [30]), over flag varieties, and affine toric varieties without torus factors [5]. It is also possible to construct new flexible varieties from a given flexible one, e.g. via *suspensions* [5] or open subsets with complement of codimension ≥ 2 [14].

Our first main result provides a new criterion for flexibility of affine cones, see Sect. 1 for the proof. A similar, but independent, criterion using the notions of cylinders was provided in Perepechko [30, Th. 5].

Theorem 1.4 *Let Y be a normal projective variety covered by flexible affine open subsets U_i, $i \in I$, and H be a very ample divisor on Y. If each subset $Y \setminus U_i$ is the support of an effective $\mathbb{Q}$-divisor D_i linearly equivalent to H, then the affine cone $X = \mathrm{AffCone}_H Y$ is flexible.*

Recall that a *Segre–Veronese* variety is an embedding of the direct product $\mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_n}$ of projective spaces by the very ample line bundle of the form $\mathcal{O}(s_1) \boxtimes \cdots \boxtimes \mathcal{O}(s_n)$. Further, given a projective variety $X \subset \mathbb{P}^n$, the Zariski closure of the union of the secant (resp. tangent) lines to X is called a *secant* (resp. *tangential*) variety of X. As the first application of Theorem 1.4, we deduce the following result, see Sect. 2 for details.

Theorem 2.20 *Let $X = v_{s_1}(\mathbb{P}(V_1)) \times \cdots \times v_{s_n}(\mathbb{P}(V_n))$ be a Segre–Veronese variety. Then the affine cone over the secant variety of X is flexible. Further, if $s_1 = \cdots = s_n = 1$, then also the affine cone over the tangential variety of X is flexible.*

Our proof technique relies on triangular transformations of the affine charts of the ambient projective space. They are inspired by algebraic statistics, precisely by computation of cumulants [10,29,34,41,42].

Section 3 contains preliminaries on smooth rational T-varieties of complexity one. These are varieties X with an effective action of a torus T, where $\dim X = \dim T + 1$. Section 4 is devoted to flexibility of affine cones over such varieties.

In Arzhantsev et al. [6] it was shown that smooth varieties of this type admit a toric covering and for certain affine cones over these varieties we, indeed, obtain flexibility. For example, this applies to all known Fano threefolds with 2-torus action. We use below the list of Fano threefolds in Mori–Mukai's classification [26].

Theorem 4.5 *All the affine cones over the Fano threefolds Q, 2.29,* 2.30, 2.31, 2.32, 3.8, 3.18, 3.19, 3.23, 3.24, 4.4, *and certain elements of the families* 2.24, 3.10 *admitting a 2-torus action in Mori–Mukai's classification are flexible.*

The main tool to obtain these results is the combinatorial description of T-varieties developed in Altmann and Hausen [2] and Altmann et al. [3], which in the case of complexity one allows to study (torus equivariant) coverings as in Theorem 1.4.

While Theorems 2.20 and 4.5 are concerned with projective coordinate rings, in Sect. 5 of the paper we obtain related results for total coordinate rings or Cox rings.

Theorem 5.4 *The total coordinate spaces of smooth del Pezzo surfaces are flexible.*

This was known so far only for the toric del Pezzo surfaces (i.e. those of degree 9, 8, 7 and 6) and by Arzhantsev et al. [5, Thm. 0.2] for the case of degree 5, where the total coordinate space is known to be the affine cone over the Grassmannian $G(2, 5)$. On the other hand, this extends a result of Arzhantsev et al. [6], where flexibility was proved only outside a subset of codimension 2.

Theorem 5.9 *The total coordinate space of a complete smooth rational T-variety of complexity one is flexible.*

1 Flexibility of affine cones

Lemma 1.1 *Let $X \subset \mathbb{A}^{n+1}$ be the affine cone over a projective variety $Y \subset \mathbb{P}^n$ of dimension ≥ 1. Consider a subgroup $G \subset \operatorname{Aut} X$ such that*

- *the canonical $\mathbb{G}_m$-action on X by homotheties sends G-orbits to G-orbits,*
- *all G-orbits are locally closed, and*
- *there is an orbit $Gx \subset X\backslash\{0\}$, whose image Y^* under the projection $\pi : X\backslash\{0\} \to Y$ is an open subset in Y with complement of codimension ≥ 2.*

Then $Gx = \pi^{-1}(Y^*)$ and is open in X.

Proof Since the G-action is $\mathbb{G}_m$-equivariant, $X^* = \pi^{-1}(Y^*)$ is a union of G-orbits, whose projections coincide with Y^*. Hence $X^* = \bigcup_{\lambda\in\mathbb{G}_m} \lambda Gx$, where all G-orbits are closed in X^*.

Let us show that $Gx = X^*$. Assume the contrary. Then $\dim Gx = \dim Y$ and the stabilizer $S \subset \mathbb{G}_m$ of the orbit Gx is finite. Indeed, two points $v, v' \in Gx$ lie in the same $\mathbb{G}_m$-orbit if and only if $v = \lambda \cdot v'$ for some $\lambda \in \mathbb{G}_m$. The latter is equivalent to $\lambda Gx = Gx$, i.e., $\lambda \in S$. So, $Gv \cap \mathbb{G}_m v = Sv$ for any $v \in X^*$.

Denote by $X^\times$ the blow up of X at 0. This is the total space of the line bundle $\mathcal{O}_Y(-1)$ over Y. Consider the quotient morphism

$$\mu\colon X^\times \xrightarrow{/S} X' = X^\times/S$$

given by $t \mapsto t^{|S|}$ on every trivialization chart, where t is the coordinate of the fibers. Then $\mu(Gx)$ intersects each fiber at most once, so it is a meromorphic, nonvanishing section of the line bundle $X' \to Y$. Indeed, it is a graph of a rational function on trivialization charts of the line bundle. However, the subset $D \subset Y$, where our rational section of the line bundle $X' \to Y$ is not defined or vanishes, if non-empty, is a Cartier divisor.

On the other hand, under our assumptions $D \subseteq Y \backslash Y^*$ is of codimension at least 2. Therefore, D is empty and $\mu(Gx)$ is a global section of $X' \rightarrow Y$ disjoint with the zero-section. Since Y is of positive dimension $X' \rightarrow Y$ is non-trivial. This gives a contradiction.

So, the group G acts on X^* transitively. □

Corollary 1.2 *Under the setup of Lemma 1.1, if Y is smooth in codimension one and $\pi(Gx) = Y^*$ coincides with the regular locus Y_{reg}, then $Gx = X_{\text{reg}} \backslash \{0\}$.*

Lemma 1.3 *Let $Y \subset \mathbb{P}^n$ be a linearly nondegenerate projective variety, and let $H = Y \cap \{x_0 = 0\}$ be a hyperplane section of Y. Suppose that $U = Y \backslash H$ is endowed with a $\mathbb{G}_a$-action $\phi \colon \mathbb{G}_a \times U \rightarrow U$. Let $\pi : X \backslash \{x_0 = 0\} \rightarrow U$ be the natural projection, where $X = \operatorname{AffCone} Y \subset \mathbb{A}^{n+1}$ is the affine cone over Y. Then X admits a $\mathbb{G}_a$-action $\hat{\phi} \colon \mathbb{G}_a \times X \rightarrow X$ normalised by the $\mathbb{G}_m$-action of the cone, such that*

- *$\phi(\mathbb{G}_a \times \{\pi(x)\}) = \pi(\hat{\phi}(\mathbb{G}_a \times \{x\}))$ for any $x \in \pi^{-1}(U)$.*
- *$\hat{\phi}$ is trivial on $X \backslash \pi^{-1}(U)$.*

In other words, $\pi : X \backslash \{x_0 = 0\} \rightarrow U$ provides a correspondence between $\hat{\phi}$-orbits and ϕ-orbits.

Proof Let Y be defined by a homogeneous ideal $I \subset \mathbf{k}[x_0, \dots, x_n]$ which does not contain x_0, $\mathbf{k}[X] = \mathbf{k}[x_0, \dots, x_n]/I$, and $U = \{x_0 \neq 0\} \subset Y$.

There exists a natural embedding $\rho \colon U \hookrightarrow X$, $\rho(U) = \{x_0 = 1\} \subset X$. On the other hand, $\pi : X \backslash \{x_0 = 0\} \rightarrow U$ is a trivial $\mathbb{G}_m$-bundle. Therefore, we may extend the $\mathbb{G}_a$-action ϕ on U to a $\mathbb{G}_a$-action $\tilde{\phi}$ on $X \backslash \{x_0 = 0\}$ defined by a homogeneous locally nilpotent derivation $\tilde{\delta}$.

There exists $d \in \mathbb{N}$ such that $x_0^d \tilde{\delta}(x_i) \in \mathbf{k}[X]$ for $i = 1, \dots, n$. Since $x_0 \in \ker \tilde{\delta}$, a homogeneous derivation $\hat{\delta} = x_0^{d+1} \tilde{\delta}$ on X is locally nilpotent. The corresponding $\mathbb{G}_a$-action $\hat{\phi}$ is normalised by the $\mathbb{G}_m$-action, coincides with ϕ on the hypeplane section $\{x_0 = 1\} \cong U$ of X, and is trivial on $\{x_0 = 0\} \subset X$. □

Theorem 1.4 *Let Y be a normal projective variety covered by flexible affine open subsets U_i, $i \in I$, and H be a very ample divisor on Y. If each subset $Y \backslash U_i$ is the support of an effective $\mathbb{Q}$-divisor D_i linearly equivalent to H, then the affine cone $X = \operatorname{AffCone}_H Y$ is flexible.*

Proof The tangent space at 0 to X must contain all lines through 0 contained in X. As X is covered by such lines, the tangent space is equal to the linear span of X. If $X = \mathbb{A}^{n+1}$ the theorem is trivial. Otherwise, the origin is a singular point of X, which we assume from now on.

For each subset U_i there exists a finite number of $\mathbb{G}_a$-actions $\{\phi_{ij}\}$ such that the orbit of the group generated by them is the regular locus of U_i, see [1, Prop. 1.5]. Let $k \in \mathbb{N}$ be such that kD_i is a $\mathbb{Z}$-divisor for any $i \in I$. For each action ϕ_{ij} we can consider a lifted action $\tilde{\phi}_{ij}$ on $\operatorname{AffCone}_{kH} Y$ as in Lemma 1.3. Since the Veronese map $X \rightarrow \operatorname{AffCone}_{kH} Y$ is unramified outside the vertex, Theorem 1.3 of Masuda and Miyanishi [27] implies the existence of an action $\hat{\phi}_{ij}$ on X, whose orbits have the same image in Y as the orbits of $\tilde{\phi}_{ij}$.

Let a subgroup $G = \langle \hat{\phi}_{ij} \rangle \subset \operatorname{SAut} X$ be generated by the $\mathbb{G}_a$-actions on X which correspond to all the open subsets U_i. Then the image of the orbit Gx of a regular point $x \in X_{\text{reg}}$ under the projection $X \backslash \{0\} \rightarrow Y$ is equal to Y_{reg}. Thus, the statement follows from Corollary 1.2, as the variety Y is normal, in particular smooth in codimension one, and the G-orbits are locally closed by Arzhantsev et al. [1, Prop. 1.3]. □

Example 1.5 Consider $\mathbb{P}^n$ with coordinates $x_0, \dots, x_n$ and a smooth subvariety $Y = \mathbb{V}(x_0, f)$ of codimension 2, where f is an irreducible homogeneous polynomial of degree d. Let $q\colon X \to \mathbb{P}^n$ be the blowup in Y. We apply Theorem 1.4 to show that all the affine cones over X are flexible. Notice that X is a Fano variety if $d \le n$ holds.

Let $U_i := \mathbb{P}^n \setminus H_i := \mathbb{P}^n \setminus \mathbb{V}(x_i)$. The preimage $q^{-1}(U_0)$ is isomorphic to $U_0 \cong \mathbb{A}^n$, since Y does not intersect U_0. For $i \ne 0$ the preimage $q^{-1}(U_i)$ is given by

$$V\left(\frac{f}{x_i^d} \cdot u - \frac{x_0}{x_i} \cdot v\right) \quad \subset \quad U_i \times \mathbb{P}^1.$$

Hence, $q^{-1}(U_i)$ is covered by the affine charts

$$U_i^0 := q^{-1}(U_i) \setminus [v = 0] \quad \text{and} \quad U_i^\infty := q^{-1}(U_i) \setminus [u = 0], \tag{1}$$

the first one being an affine space and the second one being isomorphic to

$$V\left(\frac{f}{x_i^d} - \frac{x_0}{x_i} \cdot \frac{v}{u}\right) \subset \operatorname{Spec} k\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{v}{u}\right].$$

In the notation of Arzhantsev et al. [5] this is a suspension over $\mathbb{A}^{n-1}$ and hence flexible by Theorem 0.2 in loc. cit.

Given a divisor $D \subset \mathbb{P}^n$, we denote by $\widetilde{D}$ its strict transform on the blowup. We also denote by E the exceptional divisor at Y and by H the pullback of a general hyperplane. To see that the given covering is polar ([22, Def. 3.7]) with respect to every ample divisor, note that

$$X \setminus q^{-1}(U_0) = E \cup \widetilde{H}_0; \;\; X \setminus U_i^\infty = \widetilde{H}_i \cup \widetilde{H}_0; \;\; X \setminus U_i^0 = \widetilde{H}_i \cup \widetilde{\mathbb{V}(f)}; \text{ for } i > 0. \tag{2}$$

Now, X has Picard group $\mathbb{Z}^2$ with generators $[H]$ and $[E]$. The effective cone is generated by $[E]$ and $[H] - [E]$ and the nef-cone by $[H]$ and $d[H] - [E]$. Moreover, $[\widetilde{\mathbb{V}(f)}] = d[H] - [E]$ and $[\widetilde{H}_0] = [H] - [E]$ hold. So, for every affine charts from (1) the boundary components in (2) span a cone in the Neron-Severi space containing the whole nef cone of X. Hence, every ample class can be expressed as a positive linear combination of the complement components. In other words, for every affine charts U from (1) there is an effective divisor D with support $X \setminus U$, which lies in the chosen ample class. By Theorem 1.4 the flexibility of the corresponding affine cone follows.

Let further be $q'\colon X' \to \mathbb{P}^n$ the combined blowup of $\mathbb{P}^n$ in Y as above and additionally in the point $y = (1 : 0 : \cdots : 0)$. Similarly, using the same notation and denoting by E' the exceptional divisor of the blowup in the point y, we have the following flexible charts on X':

$$U_0^i := X \setminus (\widetilde{H}_0 \cup E \cup \widetilde{H}_i), \quad U_i^\infty := X \setminus (\widetilde{H}_i \cup \widetilde{H}_0 \cup E'), \quad U_i^0 := X \setminus (\widetilde{H}_i \cup \widetilde{\mathbb{V}(f)} \cup E').$$

The first two are affine spaces and the last one is a suspension as before. We see that the complements of the affine charts always consist of three components with classes corresponding to one of the triples

$$\begin{gathered}([H] - [E], [E], [H] - [E']), \quad ([H] - [E'], d[H] - [E], [E']), \\ ([H] - [E'], [H] - [E], [E']).\end{gathered}$$

Further, each triple spans a cone containing the nef cone of X', which is spanned by $d[H] - [E]$, $[H] - [E']$, and $[H]$. Hence, every ample class can be expressed as a positive linear combination of complement components. As above this implies the flexibility of the corresponding affine cone by Theorem 1.4.

2 Secant of Segre–Veronese variety

This section is based on Michalek et al. [29], where the toric covering of a Segre variety was constructed. Here we generalize that construction to a Segre–Veronese variety and hence prove Theorem 2.20. Throughout this section by a *parameterization* of a variety Z we mean a dominant morphism from a dense open subset of an affine space to Z.

Definition 2.1 Given for each $i = 1, \dots, n$ a finite-dimensional vector space V_i and its symmetric power $S^{s_i}(V_i)$, $s_i \in \mathbb{Z}_{>0}$, the *Segre–Veronese variety*

$$X = v_{s_1}(\mathbb{P}(V_1)) \times \cdots \times v_{s_n}(\mathbb{P}(V_n)) \subset \mathbb{P}(S^{s_1}(V_1) \otimes \cdots \otimes S^{s_n}(V_n))$$

is defined as the embedding of the product $\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_n)$ by the very ample line bundle $\mathcal{O}(s_1) \boxtimes \cdots \boxtimes \mathcal{O}(s_n)$.

We will be using an equivalent construction. Apart from (projective) Segre–Veronese varieties we will consider affine cones over them and refer to those as *Segre–Veronese cones*. They should not be confused with intersections of Segre–Veronese varieties with principal affine open subsets, which also play a crucial role.

For each V_i, $1 \le i \le n$, we denote $d_i = \dim V_i - 1$ and fix a basis $e^i_0, \dots, e^i_{d_i}$ of V_i. We also denote elements of the basis of $V_i^{\otimes s}$ by

$$e^i_{i_1,\dots,i_s} = e^i_{i_1} \otimes \cdots \otimes e^i_{i_s}.$$

Thus, the symmetric power is $S^{s_i}(V_i) =$

$$\{v \in V_i^{\otimes s} \mid (e^i_{i_1,\dots,i_s})^*(v) = (e^i_{i_{\sigma(1)},\dots,i_{\sigma(s)}})^*(v) \text{ for any permutation } \sigma \in S_s\}.$$

This allows us to embed the Veronese cone into the Segre cone and obtain the following diagram:

$$\begin{tikzcd}
V_1 \oplus \cdots \oplus V_n \arrow[r, "e"] \arrow[d, "\psi"] & V_1^{s_1} \oplus \cdots \oplus V_n^{s_n} \arrow[d, "\tilde{\psi}"] \\
S^{s_1}(V_1) \otimes \cdots \otimes S^{s_n}(V_n) \arrow[r, phantom, "\subset"] & V_1^{\otimes s_1} \otimes \cdots \otimes V_n^{\otimes s_n}, \arrow[l, "s_{V_1} \otimes \cdots \otimes s_{V_n}"]
\end{tikzcd}$$

where

- $e\colon (v_1, \dots, v_n) \mapsto (v_1, \dots, v_1, v_2, \dots, v_2, v_3, \dots, v_n)$ is the diagonal embedding,
- $\tilde{\psi}\colon (v_1^1, \dots, v_1^{s_1}, v_2^1, \dots, v_2^{s_2}, v_3^1, \dots, v_n^{s_n}) \mapsto v_1^1 \otimes \cdots \otimes v_n^{s_n}$ is the parameterization of the Segre cone, which is a nonlinear map,
- $\psi = \tilde{\psi}|_{\mathrm{im}(e)} \circ e$ is the parameterization of the Segre–Veronese cone, and
- $s_{V_i}\colon V_i^{\otimes s_i} \to S^{s_i}(V_i)$ are the natural symmetrizing projections.

Notation 2.2 (i) For a vector space $V = V_{i_1} \otimes \cdots \otimes V_{i_k}$ we denote

$$\hat{V} := \{x \in V \mid (e_0^{i_1} \otimes \cdots \otimes e_0^{i_k})^*(x) = 1\}$$

and regard it as a vector space with basis $\{e^{i_1}_{j_1} \otimes \cdots \otimes e^{i_k}_{j_k} \mid j_1 + \cdots + j_n > 0\}$.

(ii) We may, and often will, consider $\hat{V}$ as a complement to a hyperplane section $\{[x] : (e_0^{i_1} \otimes \cdots \otimes e_0^{i_k})^*(x) = 0\}$ of $\mathbb{P}(V)$. Thus $\hat{V}$ may be regarded as an affine open subset of $\mathbb{P}(V)$.

(iii) We denote $A := V_1^{\otimes s_1} \otimes \cdots \otimes V_n^{\otimes s_n}$.

(iv) We also define

$$B := \prod_{i=1}^{n} \hat{V}_i^{\times s_i} \subset V_1^{s_1} \oplus \cdots \oplus V_n^{s_n},$$

$$B' := \prod_{i=1}^{n} \hat{V}_i \subset V_1 \oplus \cdots \oplus V_n.$$

(v) Finally, we denote $\pi : A \backslash \{0\} \to \mathbb{P}(A)$ and obtain the following diagram of open subsets:

$$\begin{tikzcd}
B' \subset V_1 \oplus \cdots \oplus V_n \arrow[r, "e"] \arrow[d, "\psi"] & V_1^{s_1} \oplus \cdots \oplus V_n^{s_n} \supset B \arrow[d, "\tilde{\psi}"] \\
S^{s_1}(V_1) \otimes \cdots \otimes S^{s_n}(V_n) \subset & V_1^{\otimes s_1} \otimes \cdots \otimes V_n^{\otimes s_n} \supset \hat{A} \arrow[d, "\pi"] \\
 & \mathbb{P}(A) \supset \hat{A}.
\end{tikzcd}$$

Remark 2.3 Since $\mathbb{P}(S^{s_1}(V_1) \otimes \cdots \otimes S^{s_n}(V_n)) \subset \mathbb{P}(A)$, we can study the Segre–Veronese variety as a subvariety of $X = \overline{\pi \circ \psi(B')} \subset \mathbb{P}(A)$. Note that the image of $\tilde{\psi}|_B$ does not contain the origin.

Cumulants

In this setting we may apply the (nonlinear) coordinate systems of B, called cumulants and presented in Michalek et al. [29]. For the motivations to consider them, coming from algebraic statistics, we refer the reader to Refs. [34,41,42]. A general mathematical setting for these methods is well presented in Ciliberto et al. [10]. Further results are obtained for other varieties, e.g. Grassmannians and spinor varieties [28]. However, in other cases we do not obtain toric coverings. Still, we believe that similar methods can be applied to a larger class of secant and tangential varieties.

Notation 2.4 Basis elements of A are of the form $e^1_{c^1_1,\ldots,c^1_{s_1}} \otimes \cdots \otimes e^n_{c^n_1,\ldots,c^n_{s_n}}$ and are in natural correspondence with tuples $(c^1_1, \ldots, c^1_{s_1}, \ldots, c^n_1, \ldots, c^n_{s_n})$, where $0 \le c^i_j \le d_i$ for $1 \le i \le n$ and $1 \le j \le s_i$. Let us denote the set of these tuples by $C(A)$ and for each $c \in C(A)$ the corresponding basis element by $e(c)$. Finally, denote the dual basis elements by $x(c) = e(c)^*$. Similarly, we denote $C(\hat{A}) = C(A) \backslash \{(0, \ldots, 0)\}$.

Definition 2.5 (*degree, ordering*) Given a tuple $c \in C(\hat{A})$, the number of its nonzero entries is called the *degree* of c. Given $c_1, c_2 \in C(\hat{A})$, we say that $c_1 \le c_2$ if c_1 can be obtained from c_2 by setting some entries to zero.

Thus, we have a natural poset structure on $C(\hat{A})$, which induces a poset structure on the basis of $\hat{A}$. Simply speaking, the ordering on the basis is defined by replacing e^i_j by e^i_0 in the tensor product elements.

Example 2.6 Consider $t_2 := e^1_1 \otimes e^1_0 \otimes e^2_5$, $t_1 := e^1_0 \otimes e^1_0 \otimes e^2_5$ and $t_0 := e^1_1 \otimes e^1_3 \otimes e^2_4$. We have $t_1 < t_2$ and t_1, t_2 are not comparable with t_0.

Denote the set of indices

$$\operatorname{Ind}(\hat{A}) = \left\{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \dots, \left[\begin{smallmatrix}1\\s_1\end{smallmatrix}\right], \dots, \left[\begin{smallmatrix}n\\1\end{smallmatrix}\right], \dots, \left[\begin{smallmatrix}n\\s_n\end{smallmatrix}\right]\right\}$$

and endow it with a natural lexicographic order, namely, the order of appearance above. Given $c \in C(\hat{A})$ and a subset of indices $I \subset \operatorname{Ind}(\hat{A})$, we introduce the index tuple

$$c_I = (b^1_1, \dots, b^n_{s_n}), \quad \text{where } b^i_j = \left[\begin{array}{ll} c^i_j, & \left[\begin{smallmatrix}i\\j\end{smallmatrix}\right] \in I, \\ 0, & \left[\begin{smallmatrix}i\\j\end{smallmatrix}\right] \notin I. \end{array}\right.$$

Note that $\{b \in C(\hat{A}) \mid b \le c\} = \{c_I \mid I \subset \operatorname{Ind}(\hat{A})\}$. We will use either one-element subsets $I = \left[\begin{smallmatrix}i\\j\end{smallmatrix}\right]$ or subsets of the following form, where $i_1, i_2 \in \operatorname{Ind}(\hat{A})$:

$$I = [i_1 : i_2] := \{i \mid i_1 \le i < i_2\} \subset \operatorname{Ind}(\hat{A}).$$

Definition 2.7 A *thick interval partition* of a tuple $c \in C(\hat{A})$ of degree at least two is an increasing sequence of indices $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] = b_0 < \dots < b_k = \left[\begin{smallmatrix}n\\s_n\end{smallmatrix}\right]$ such that $\deg c_{[b_i:b_{i+1}]} \ge 2$ for each i. The set of all thick interval partitions of c will be denoted by $IP(c)$. It is always nonempty as it contains $\left\{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}n\\s_n\end{smallmatrix}\right]\right\}$.

Now we can recall the coordinate systems from [29, Sec. 2].

Notation 2.8 For each $c \in C(\hat{A})$ we denote

$$y(c) := \left[\begin{array}{ll} x(c), & \deg c = 1, \\ \sum_{(b) \le (c)} (-1)^{\deg(c) - \deg(b)} x(b) \prod_{c^{i_0}_{j_0} \ne b^{i_0}_{j_0}} x(c_{\left[\begin{smallmatrix}i_0\\j_0\end{smallmatrix}\right]}), & \deg c > 1. \end{array}\right.$$

Then for each $c \in C(\hat{A})$ we introduce a function in $\mathbf{k}[\hat{A}]$

$$z(c) := \left[\begin{array}{ll} y(c), & \deg c = 1, \\ \sum_{(b_0, \dots, b_k) \in IP(c)} (-1)^k \prod_{m=1}^k y(c_{[b_{m-1}:b_m]}), & \deg c > 1. \end{array}\right.$$

Lemma 2.9 *Each one of the sets* $\{x(c)\}_{c \in C(\hat{A})}$, $\{y(c)\}_{c \in C(\hat{A})}$, *and* $\{z(c)\}_{c \in C(\hat{A})}$ *is an algebraically independent system of functions generating* $\mathcal{O}(\hat{A})$. *In other words,* $\{z(c)\}$ *is a coordinate system on* $\hat{A}$ *as an affine space.*

Proof Since $y(c)$ is a sum of $x(c)$ and of terms of smaller degree, the endomorphism of $\mathbf{k}[\hat{A}]$ that maps $x(c)$ to $y(c)$ for each c is invertible. The same holds for $\{y(c)\}$ and $\{z(c)\}$. So, the statement follows. □

Secant

The secant variety $\operatorname{Sec} X \subset \mathbb{P}(A)$ of the Segre–Veronese variety X is parameterized by a map

$$\sec_X \colon \mathbb{A}^1 \times B' \times B' \to \hat{A}, \quad (t, v, w) \mapsto \pi(t \cdot \psi(v) + (1 - t) \cdot \psi(w)). \tag{3}$$

Hereinafter, given a tuple of degree one, we denote the index of its only non-zero entry by $\left[\begin{smallmatrix}i_\circ\\j_\circ\end{smallmatrix}\right]$. Generalizing [29, Lemma 3.1], we obtain the following result.

Lemma 2.10 *Let* $\{(e^i_j)'^* \mid 1 \leq i \leq n,\ 0 \leq j \leq d_i\}$ *be the set of coordinate functions on the first copy of* B' *in* (3) *and* $\{(e^i_j)''^*\}$ *the respective set on the second one.*[1] *Then*

$$\sec_X^*\colon z(c) \mapsto \left[\begin{array}{ll} t\left(e^{i_\circ}_{c^{i_\circ}_{j_\circ}}\right)'^* + (1-t)\left(e^{i_\circ}_{c^{i_\circ}_{j_\circ}}\right)''^*, & \deg c = 1 \ \textit{with}\ c^{i_\circ}_{j_\circ} \neq 0, \\ t(1-t)(1-2t)^{\deg(z(c))-2}\prod_{c^i_j\neq 0}\left(\left(e^i_{c^i_j}\right)'^* - \left(e^i_{c^i_j}\right)''^*\right), & \deg c > 1, \end{array}\right.$$

for each $c \in C(\hat{A})$.

Proof Let Y be the affine cone over the Segre product $\mathbb{P}(V_1)^{s_1} \times \cdots \times \mathbb{P}(V_n)^{s_n}$. Then the secant of Y is parameterized by

$$\sec_Y\colon \mathbb{A}^1 \times B \times B \to \hat{A},\ (t, v, w) \mapsto \pi(t\tilde{\psi}(v) + (1-t)\tilde{\psi}(w)).$$

Thus, $\sec_X = \sec_Y \circ(\mathrm{id} \times e \times e)$. The statement follows after applying [29, Lemma 3.1] to $\sec_Y$. □

Torus action

We can infer the following decomposition of $\sec_X$.

Notation 2.11 Let $\mathrm{rep}\colon \mathbb{A}^1 \times B' \times B' \to \mathbb{A}^1 \times B' \times B'$, be a reparameterization:

$$\mathrm{rep}\colon (t, v, w) \mapsto \left(\frac{t(1-t)}{(1-2t)^2}, tv + (1-t)w, (1-2t)(w-v)\right)$$

and let $m\colon \mathbb{A}^1 \times B' \times B' \to \hat{A}$, be a *monomial* map:

$$m^*\colon z(c) \mapsto \left[\begin{array}{ll} \left(e^{i_\circ}_{c^{i_\circ}_{j_\circ}}\right)'^*, & \deg c = 1 \text{ with } c^{i_\circ}_{j_\circ} \neq 0, \\ t\prod_{c^i_j\neq 0}\left(e^i_{c^i_j}\right)''^*, & \deg c > 1. \end{array}\right.$$

Lemma 2.12 *There is a decomposition* $\sec_X = m \circ \mathrm{rep}$. *In particular, m is a monomial parameterization of* $\mathrm{Sec}\,X$.

Proof Straightforward. □

This monomial parameterization of $\mathrm{Sec}\,X$ already provides a structure of a toric variety on $\mathrm{im}(m) = \hat{A} \cap \mathrm{Sec}\,X$, hence provides us with a toric chart of $\mathrm{Sec}\,X$. Below we describe in detail its structure.

Notation 2.13 Let us introduce the following closed subsets of $\hat{A}$:

$$\begin{aligned} \hat{A}_1 &= \{z(c) = 0 \mid \deg c > 1\}, \\ \hat{A}_2 &= \{z(c) = 0 \mid \deg c = 1\}, \\ S_2 &= \mathrm{Sec}\,X \cap A_2. \end{aligned}$$

---

[1] That is, $(e^i_j)'^*(v) = v^i_j$ and $(e^i_j)''^*(w) = w^i_j$ respectively.

Notation 2.14 (Lattice Polytope P) Consider the lattice $M = \bigoplus_{1 \leq i \leq n 1 \leq j \leq d_i} \mathbb{Z}\chi_j^i$, where $\chi_j^i = (e_j^i)^*$. Let $P \subset M \otimes \mathbb{Q}$ be the lattice polytope defined by inequalities

$$\begin{cases} \left(\chi_j^i\right)^* \geq 0, & 1 \leq i \leq n, 1 \leq j \leq d_i, \\ \sum\limits_{j=1}^{d_i} (\chi_j^i)^* \leq s_i, & 1 \leq i \leq n, \\ \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq d_i}} (\chi_j^i)^* \geq 2. & \end{cases}$$

Proposition 2.15 *In the terminology above,*

(i) $\operatorname{Sec} X \cap \hat{A} = \hat{A}_1 \times S_2$ *via natural projections along coordinates* $z(c)$*,*
(ii) $\hat{A}_1 = X \cap \hat{A} \cong \mathbb{A}^N$*, where* $N = \sum_{i=1}^n \dim \hat{V}_i$*,*
(iii) $S_2 \cong \operatorname{AffCone}(X_P)$*, where* X_P *is a projective toric variety with polarization corresponding to the polytope* P*, see, e.g.,* [11, §2.3] *for the construction.*

Proof The morphism m is a direct product of $m|_{\{0\} \times B' \times \{0\}}$ and $m|_{\mathbb{A}^1 \times \{0\} \times B'}$, which respectively parameterize $\hat{A}_1$ and S_2. This implies (i). The Segre–Veronese variety X is also parameterized by $m|_{\{0\} \times B' \times \{0\}}$, thus (ii) holds.

Let us consider the standard torus $T = \operatorname{Spec} \mathbf{k}[M] \subset B'$ and a projectivization $\pi_z \colon \hat{A} \backslash \{0\} \to \mathbb{P}(\hat{A})$ along $z(c)$-coordinates. Then $X_P = \overline{\pi_z(S_2)}$ is parameterized by a T-equivariant map $\pi_z \circ m|_{\{0\} \times \{0\} \times B'}$, which is defined by the set of monomials $\{(e^i_{c^i_j})^* \mid \deg c > 1\} \subset \mathcal{O}(B')$ corresponding exactly to lattice points $P \cap M$. This implies (iii). □

Proposition 2.16 *Let* $\operatorname{Tan} X \subset \mathbb{P}(A)$ *be the tangential variety of* X*. Then*

$$\operatorname{Tan} X \cap \hat{A} = \hat{A}_1 \times X'_P,$$

where $X'_P \subset \hat{A}_2$ *is a nondegenerate toric variety parameterized by* $m|_{\{-\frac{1}{4}\} \times \{0\} \times B'}$*.*

Proof We present here a sketch of a proof for $\mathbf{k} = \mathbb{C}$, see [29] and [28, Lem. 3.3] for a complete proof. Consider $(\epsilon^{-1}, v, v + \epsilon w) \in \mathbb{A}^1 \times B' \times B'$. If $\epsilon \to 0$, then $\sec_X(\epsilon^{-1}, v, v + \epsilon w)$ tends to an element of $\mathrm{T}_{\pi \circ \psi(v)} X \hookrightarrow \operatorname{Tan} X$ corresponding to w. On the other hand, by Lemma 2.10,

$$\lim_{\epsilon \to 0} z(c)(\sec_X(\epsilon^{-1}, v, v + \epsilon w)) = \left[\begin{array}{ll} v^{i_\circ}_{c^{i_\circ}_{j_\circ}} - w^{i_\circ}_{c^{i_\circ}_{j_\circ}}, & \deg c = 1 \text{ with } c^{i_\circ}_{j_\circ} \neq 0, \\ -\frac{1}{4} \prod_{c^i_j \neq 0} 2 w^i_{c^i_j}, & \deg c > 1. \end{array} \right.$$

Thus, the decomposition follows. It remains to check that X'_P is nondegenerate. Indeed, $\mathcal{O}(X'_P) \hookrightarrow \mathcal{O}(B')$ does not contain invertible elements. □

These propositions imply the following relationship of the tangential and secant varieties, which, in turn, implies Zak's theorem [40] for Segre–Veronese varieties.

Corollary 2.17 *The following conditions are equivalent:*

(i) P *is not contained in the hyperplane* $\sum_{i,j} (\chi_j^i)^* = 2$*,*
(ii) $\dim \operatorname{Sec} X = 2 \dim X + 1$*,*
(iii) $\operatorname{Sec} X \neq \operatorname{Tan} X$*,*
(iv) $\dim \operatorname{Tan} X = \dim \operatorname{Sec} X - 1$*.*

Then Sec X *is called non-degenerate.*

Proof Let $d = \dim X$. As a toric variety, X is represented by a d-dimensional polytope S, which is a product of simplices. Then P is the intersection of S with the halfspace $\sum_{i,j}(\chi_j^i)^* \geq 2$.

$(i) \Rightarrow (ii)$ The assumption implies $\dim P = d$. Hence, $\dim S_2 = d + 1$ and $\dim \operatorname{Sec} X = \dim \hat{A}_1 + \dim S_2 = 2d + 1$.

$(i) \Rightarrow (iv)$ As before, $\dim \operatorname{Cone}(P) = d$. Hence, $\dim \operatorname{Tan} X = \dim \hat{A}_1 + d = 2d$. The implications $(ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (iii)$ are obvious.

$(iii) \Rightarrow (i)$ If P is contained in the hyperplane, then all monomials corresponding to lattice points in P are of the same degree. In particular, $X'_P \cong \operatorname{AffCone}(X_P)$. □

Example 2.18 1. Consider the Veronese surface $Y = \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. It is a projective toric variety corresponding to the simplex $S = \operatorname{conv}(0, 2\chi_1, 2\chi_2) \subset \langle \chi_1, \chi_2 \rangle \cong \mathbb{Z}^2$, i.e. parameterized by characters of a two-dimensional torus that correspond to the lattice points of S. By Proposition 2.15, the variety X_P that defines the secant is parameterized by $P = S \cap \{\chi_1^* + \chi_2^* \geq 2\}$, i.e., by the lattice points $2\chi_1, \chi_1 + \chi_2, 2\chi_2$. Hence, both factors of $\operatorname{Sec} X \cap \hat{A} = \hat{A}_1 \times S_2$ are of dimension two, so $\dim \operatorname{Sec} X = 4$. Thus, the secant variety is degenerate and defined by $z(1,2)^2 = z(1,1)z(2,2)$.

2. Consider the Segre product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The representing polytope is a cube $[0,1]^3$. The secant variety is represented by a polytope with vertices $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)$. We see that the affine cone over it is the whole affine space; indeed, in this case the secant variety is non-degenerate and fills the whole ambient space. The tangential variety is a hypersurface defined by the equation $z(1,1,0)z(1,0,1)z(0,1,1) = z(1,1,1)^2$.

Theorem 2.19 *The tangential and secant varieties of a Segre–Veronese variety are covered by complements of hyperplane sections. Each complement is an affine toric variety without torus factors. In case of the secant variety, these are always normal toric varieties. In case of the tangential variety they are normal if the underlying variety is the Segre product.*

Proof By [39, Theorem 2.2] we know that $\operatorname{Sec} X$ is normal. By [29, Proposition 8.5] we know that $\operatorname{Tan} X$ is normal, when X is the Segre product. The open subsets $\operatorname{Sec} X \cap \hat{A}$ and $\operatorname{Tan} X \cap \hat{A}$ are toric varieties by Propositions 2.15 and 2.16. Moreover, they do not contain torus factors, since they are products of an affine space $\hat{A}_1$ with either an affine cone over a projective toric variety X_P or a non-degenerate toric variety X'_P. By taking such subsets for various choices of basis vectors e_0^i, $i = 1, \ldots, n$, we obtain the statement. □

Theorem 2.20 *Let* $X = v_{s_1}(\mathbb{P}(V_1)) \times \cdots \times v_{s_n}(\mathbb{P}(V_n))$ *be a Segre–Veronese variety. Then the affine cone over the secant variety of* X *is flexible. Further, if* $s_1 = \cdots = s_n = 1$, *then also the affine cone over the tangential variety of* X *is flexible.*

Proof By [5, Theorem 0.2] we know that all the affine charts from Theorem 2.19 are flexible. Now, by Theorem 1.4 we obtain flexibility of the affine cone. □

Example 2.21 Consider the third Veronese embedding $X = v_3(\mathbb{P}^1)$. It is represented by characters in the interval $S = \operatorname{conv}(0, 3)$. Then $\operatorname{Tan} X \cap \hat{A} \cong \mathbb{A}^1 \times X'_P$, where the monoid of characters associated to the toric variety X'_P is generated by $\{2, 3\}$. Namely, X'_P is the curve with a cusp singularity at the origin. Thus, $\operatorname{Tan} X$ is a surface, whose singular locus is the curve X. This example can be generalized to tangential varieties of other Segre–Veronese varieties provided that at least one of the Veronese factors is of degree at least 3.

3 The combinatorial description of T-varieties

We consider a normal variety X with an effective action of an algebraic torus $T \cong (\mathbb{G}_m)^r$. Then X is called a T-variety of complexity $(\dim X - \dim T)$. Here, the case of a complexity-one torus is the most widely studied one, with contributions by many different authors [2, 16,21,24,35,37]. In the following we restrict ourselves to the case of rational T-varieties of complexity one. Following [4] and generalising the classification of toric varieties by their fans, we introduce some combinatorial language to classify rational T-varieties of complexity one.

Let us denote the character lattice of the torus T by M and the dual lattice by N. For the associated $\mathbb{Q}$-vector spaces we write $M_{\mathbb{Q}}$ and $N_{\mathbb{Q}}$.

For a polyhedron $\Delta \subset N_{\mathbb{Q}}$ we define its tail cone as follows

$$\operatorname{tail}(\Delta) := \{v \in N_{\mathbb{Q}} \mid \Delta + \mathbb{Q}_{\geq 0} \cdot v = \Delta\}.$$

Now, we consider polyhedral complexes Ξ in $N_{\mathbb{Q}}$. Here, by polyhedral complex we mean a set of convex polyhedra, which is closed under the face relation and every pair of polyhedra intersect in a common face. Moreover, we assume that the set of tail cones has the structure of a polyhedral complex itself, which is called the tail fan of Ξ and will be denoted by $\operatorname{tail}(\Xi)$. Consider a pair $\mathcal{S} = (\sum_{P \in \mathbb{P}^1} \mathcal{S}_P \otimes P, \deg \mathcal{S})$ where $\mathcal{S}_P$ are polyhedral complexes in $N_{\mathbb{Q}}$ with some common tail fan Σ and $\deg \mathcal{S} \subset |\Sigma|$. Here, $\sum_P \mathcal{S}_P \otimes P$ is just a formal sum. The complexes $\mathcal{S}_P$ are called *slices* of $\mathcal{S}$. We assume that there are only finitely many slices that differ from the tail fan $\operatorname{tail}(\mathcal{S}) := \Sigma$. The set of the points $P \in \mathbb{P}^1$ such that $\mathcal{S}_P \neq \Sigma$ is called the support of $\mathcal{S}$ and will be denoted by $\operatorname{supp} \mathcal{S}$. Note that for every full-dimensional $\sigma \in \Sigma$ there is a unique polyhedron Δ_P^σ in $\mathcal{S}_P$ with $\operatorname{tail}(\Delta_P^\sigma) = \sigma$.

Definition 3.1 (*f-divisor*) A pair $\mathcal{S}$ as above is called an *f-divisor* if for any full-dimensional $\sigma \in \operatorname{tail}(\mathcal{S})$ we have either $\deg \mathcal{S} \cap \sigma = \emptyset$ or

$$\sum_P \Delta_P^\sigma = \deg \mathcal{S} \cap \sigma \subsetneq \sigma.$$

An f-divisor as above corresponds to a rational T-variety of complexity one, see [19, Section 1]. Moreover, this correspondence is even functorial. In particular, invariant open subvarieties correspond to f-divisors $\mathcal{S}'$, such that $\mathcal{S}'_P \subset \mathcal{S}_P$ as sets of polyhedra and $\deg \mathcal{S}' = |\operatorname{tail}(\mathcal{S}')| \cap \deg \mathcal{S}$. For simplicity we write $\mathcal{S}' \subset \mathcal{S}$ in this situation. As a consequence of Proposition 1.6 in [19], f-divisors $\mathcal{S}_1, \ldots, \mathcal{S}_\ell \subset \mathcal{S}$ give rise to an open covering if and only if their slices cover the slices of $\mathcal{S}$, i.e.

$$|\mathcal{S}| = \bigcup_i |\mathcal{S}_i|.$$

Remark 3.2 Affine charts correspond to f-divisors $\mathcal{S}$ such that $\mathcal{S}_P$ consists of a single polyhedron (and its faces) and $\deg \mathcal{S} = \sum_{P \in \mathbb{P}^1} \mathcal{S}_P$. These objects are called p-divisors in [4,19].

Example 3.3 In Fig. 1 we sketched the non-trivial slices of an f-divisor as well as its degree. It describes the blowup of the quadric threefold in one point, see [33].

Lemma 3.4 [20, Remark 1.8.] *An f-divisor describes a subtorus action on a toric variety if and only if $\mathcal{S}_P$ equals a lattice translate of the tail fan for all but at most two $P \in \mathbb{P}^1$.*

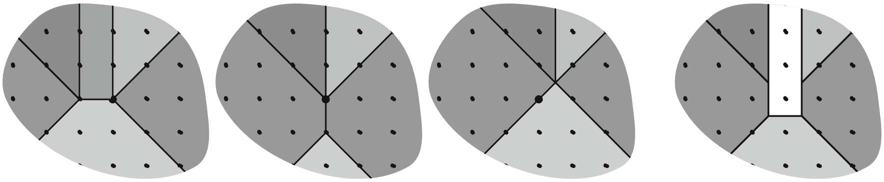

Fig. 1 An f-divisor

In the language of f-divisors we also may describe torus invariant Cartier divisors by *support functions*. A support function h on a polyhedral subdivision Ξ is a continuous function that is affine linear on every polyhedra in Ξ. We denote by $\operatorname{lin} h$ the linear part of h. This is a piecewise linear function on the tail fan defined as follows:

$$(\operatorname{lin} h)(v) := h(w+v) - h(w)$$

for some $w \in \Delta \in \Xi$ with $v \in \operatorname{tail}(\Delta)$.

Definition 3.5 (*Support function on $\mathcal{S}$*) A *support function h* on an f-divisor $\mathcal{S}$ is a collection $\{h_P\}_{P\in\mathbb{P}^1}$ of support functions on $\mathcal{S}_P$ such that

(i) all h_P have the same linear part, which will be denoted by $\operatorname{lin} h$,
(ii) only finitely many of them differ from $\operatorname{lin} h$.

We have two kinds of torus invariant prime divisors on $X(\mathcal{S})$. *Horizontal* prime divisors correspond to rays $\rho \in \operatorname{tail}(\mathcal{S})^{(1)}$ that do not intersect $\deg \mathcal{S}$ and are denoted by D_ρ. *Vertical* prime divisors correspond to vertices v in the subdivisions $\mathcal{S}_P$ and are denoted by $D_{P,v}$. Now, the divisor corresponding to the support function h is given by

$$D_h = -\sum_\rho (\operatorname{lin} h)(\rho)\cdot D_\rho - \sum_{P,v} \mu(v)\cdot h_P(v) D_{P,v}, \tag{4}$$

where we identify the ray with the ray generator and $\mu(v)$ denotes the minimal positive integer such that $\mu(v)\cdot v$ is a lattice element. In particular,

(i) if $h_P \leq 0$ for all $P \in \mathbb{P}^1$, then D_h is effective.
(ii) in this case $X(\mathcal{S})\setminus \operatorname{supp} D_h$ is given by the f-divisor

$$[h=0] := \left(\sum_P [h_P = 0]\otimes P, \quad \deg \mathcal{S} \cap [\operatorname{lin} h = 0]\right), \tag{5}$$

where $[h_P = 0]$ denotes the polyhedral subcomplex of $\mathcal{S}_P$ consisting of those polyhedra on which h_P vanishes.

By [36, Section 4] (or [31]), every invariant Cartier divisor arises in this way. We have $D_h \sim D_{h'}$ if and only if $h_P - h'_P$ is affine linear for every P, i.e. $h_P - h'_P = \langle u, \cdot\rangle + a_P$, and $\sum_P a_P = 0$. Moreover, we have a criterion for ampleness expressed in the following notation. We denote by $\Box_h$ the polytope given by

$$\Box_h = \{u \in M_\mathbb{Q} \mid \langle u, \cdot\rangle \geq (\operatorname{lin} h)\} \tag{6}$$

and consider concave piecewise affine function h_P^* on $\Box_h$ as "dual" of h_P:

$$h_P^*(u) := \inf_v(\langle u, v\rangle - h_P(v)).$$

The definition implies that $h_P^*(u)$ is finite for $u \in \Box_h$.

Theorem 3.6 *If D_h is ample, then h_P is strongly concave for every $P \in \mathbb{P}^1$ and $h_P^*(u) \geq 0$ for every $u \in \Box_h$.*

Proof By Petersen and Süß [31, Theorem 3.28], h_P has to be strongly concave and by [19, Prop. 3.1(i)] we get that $h_P^*(u) \geq 0$. □

Remark 3.7 On a T-variety every divisor is linearly equivalent to some torus invariant divisor. This follows for example from Fulton et al. [15, Theorem 1].

4 Affine cones over projective T-varieties

From now on we assume that the T-varieties which we consider are proper over the base, i.e. the corresponding f-divisors $\mathcal{S}$ satisfy the condition that all its slices $\mathcal{S}_P$ are *subdivisions* of $N_{\mathbb{Q}}$.

Definition 4.1 (*Equivariant covering by toric charts*) A T-variety is called *equivariantly covered by toric charts*, if there is an open covering by toric varieties U_i such that the torus T acts as a subtorus of the embedded torus of U_i.

Lemma 4.2 *The T-variety $X(\mathcal{S})$ is equivariantly covered by toric charts if and only if for every maximal polyhedron Δ in $\mathcal{S}_P$, $P \in \mathbb{P}^1$, all but at most two slices contain a lattice translate of* $\operatorname{tail}(\Delta)$. *In particular, either $X(\mathcal{S})$ itself is toric or there is at most one $P \in \mathbb{P}^1$ such that $\mathcal{S}_P$ does not contain a lattice vertex.*

Proof The first part is a corollary of Lemma 3.4. To prove the last statement, we consider two points P, Q such that $\mathcal{S}_P, \mathcal{S}_Q$ contain only non-lattice vertices. Now, consider a third point R and a maximal polyhedron $\Delta \subset \mathcal{S}_R$. Since there is no lattice translate of $\operatorname{tail}(\Delta)$ in $\mathcal{S}_P$ and $\mathcal{S}_Q$, Δ itself must be a translated cone. Notice that all maximal cones in $\mathcal{S}_R$ must be translated by the same lattice point. Indeed, otherwise they would not cover $N_{\mathbb{Q}}$ and there would exist a different maximal dimensional polyhedron that could not be a lattice translate of its tail cone. Hence, for any $R \notin \{P, Q\}$ the slice $\mathcal{S}_R$ is just a translated tail fan. By Lemma 3.4, $X(\mathcal{S})$ is a toric variety. □

Remark 4.3 By [6, Appendix] this criterion is fulfilled for all smooth complete rational T-varieties of complexity one. Hence, they are covered by affine charts isomorphic to affine spaces.

To get flexibility for every affine cone we need to strengthen the condition in Lemma 4.2.

Theorem 4.4 *Let $X = \mathfrak{X}(\mathcal{S})$ be a T-variety such that for any maximal polyhedron $\Delta \in \mathcal{S}_y$, $y \in \mathbb{P}^1$, at most two slices contain a polyhedron with the same tail cone* $\operatorname{tail}(\Delta)$ *that is not a lattice translate of* $\operatorname{tail}(\Delta)$. *Then for every very ample divisor H the corresponding affine cone is flexible.*

Proof By Theorem 1.4 it is enough to show that there exists a T-invariant H-polar covering by toric charts. Let us first rephrase this condition in terms of f-divisors. Remember that being H-polar means that the complement of every chart is the support of an effective divisor linearly equivalent to H. Having T-invariant charts means that we have to choose the effective divisors above to be T-invariant. Therefore, we are looking for a collection of support functions h, which via (4) give rise to divisors $D_h \sim H$, where $\sim$ denotes linear equivalence. Moreover, the open subsets $X \setminus \operatorname{supp} D_h$ have to cover X. By (5) the latter is equivalent to the fact that for every maximal polyhedron $\Delta_P \in \mathcal{S}_P$ there exists a strictly concave non-positive support function h on $\mathcal{S}$ corresponding to an effective divisor $D_h \sim H$, such that $[h_P = 0] = \Delta_P$ (i.e. $h_P|_{\Delta_P} \equiv 0$ and negative elsewhere). For being a toric covering additionally we have to impose that $[h = 0]$ has only two slices that are not lattice translates of the tail fan.

We now construct such a covering for some very ample divisor H. By Remark 3.7 every divisor is linearly equivalent to a torus invariant one. Hence, using [36, Section 4] we can assume that $H \sim D_h$ for some support function h. Fix a maximal polyhedron $\Delta \subset \mathcal{S}_Q$. Then $h_Q|_\Delta$ is affine linear, i.e. $h_Q(v) = \langle u, v\rangle + a$. By concavity this implies $u \in \Box_h$, with $\Box_h$ defined as in (6). We now consider $h' := h - u$ with $h'_P(v) := h_P(v) - \langle u, v\rangle$. Now, h'_P is again strongly concave and achieves its maximum at a polyhedron Δ_P with tail cone $\operatorname{tail}(\Delta_P) = \operatorname{tail}(\Delta)$. Moreover, by construction we have $0 \in \Box_{h'}$.

By our precondition, we may assume without loss of generality that for every point $R \in \mathbb{P} \setminus \{0, \infty\}$ the polyhedron Δ_R is a lattice translate of $\operatorname{tail}(\Delta)$. Assume further $Q \neq \infty$ and introduce h^∞ by

$$\begin{cases} h^\infty_P(v) := h'_P(v) - \max \operatorname{Im} h'_P & \text{for } P \neq \infty, \\ h^\infty_\infty(v) := h'_\infty(v) + \sum_{P \neq \infty} \max \operatorname{Im} h'_P. \end{cases}$$

It remains to check that $h'_\infty(v) + \sum_{P\neq\infty} \max \operatorname{Im} h'_P \leq 0$ to see that D_{h^∞} is indeed effective. Recall that we have $0 \in \Box_{h'}$. The claim follows from the ampleness of $D_{h'}$ and Theorem 3.6.

Now, by construction we have $D_{h^\infty} \sim H$ and $[h^\infty_Q = 0] = \Delta$. Moreover, $[h^\infty_P = 0]$ is a lattice translate of $\operatorname{tail}(\Delta)$ for each $P \notin \{0, \infty\}$. Then it describes a toric chart.

Taking these toric charts for every maximal polyhedron provides us with an H-polar covering. Now, our result follows by Theorem 1.4. ☐

Theorem 4.5 *All the affine cones over the Fano threefolds Q, 2.29,* 2.30*,* 2.31*,* 2.32*,* 3.8*,* 3.18*,* 3.19*,* 3.23*,* 3.24*,* 4.4*, and certain elements of the families* 2.24*,* 3.10 *admitting a 2-torus action in Mori–Mukai's classification are flexible.*

Proof For all Fano threefolds from Theorem 4.5 the corresponding f-divisors are listed in Süß [33]. One can easily check that the precondition of Theorem 4.4 is fulfilled in every case. ☐

Example 4.6 Let us illustrate the difference of assumptions in Lemma 4.2 and Theorem 4.4. In the lemma we are allowed to *choose* the polyhedron with the given tail cone. Hence, if we consider the variety given by the slice $(-\infty, -1], [-1, 1], [1, \infty)$ taken three times, then it does satisfy the assumptions. Indeed, if we take the maximal polytope $[-1, 1]$ in one slice, in other two slices we can take just the vertex $\{1\}$, which is a lattice shift of the tail cone $\{0\}$. On the other hand, in the theorem we ask for *all* polyhedra with the given tail cone. Here, we get three times $[-1, 1]$ which is not a lattice translate of $\{0\}$. Such a difference is only possible for cones that are not full-dimensional.

Example 4.7 We are coming back to the blowup of the quadric threefold from Example 3.3. We may check that the corresponding f-divisor in Fig. 1 fulfills the condition of Theorem 4.4. Hence, all the affine cones over the blowup of the quadric threefold are flexible.

Example 4.8 The hypersurface $\mathbb{V}(x_0 y_0^2 + x_1 y_1^2 + x_2 y_2^2) \subset \mathbb{P}^2 \times \mathbb{P}^2$ is 2.24 from our list in Theorem 4.5. Hence, every affine cone over this variety is flexible. In particular, this is true for the affine cone over the Segre embedding.

5 Total coordinate spaces

We recall the definition of Cox rings.

Definition 5.1 (*Cox sheaf, Cox ring, universal torsor, total coordinate space*) Let X be a complete normal variety, whose class group is a free abelian group. Assume that the classes of divisors $D_1, \dots, D_r$ form a basis of this class group. The *Cox sheaf* of X is defined by

$$\mathcal{R} = \bigoplus_{\mathbf{a} \in \mathbb{Z}^r_{\geq 0}} \mathcal{O}\left(\sum_{i=1}^{r} a_i D_i\right).$$

It becomes a sheaf of $\mathcal{O}_X$-algebras via the usual multiplication of sections. The algebra $\mathcal{R}(X)$ of global sections of $\mathcal{R}$ is called the *Cox ring* of X.

The relative spectrum $\hat{X} = \operatorname{Spec}_X(\mathcal{R})$ is called the *universal torsor* of X. It is an open subset of the absolute spectrum $\overline{X} = \operatorname{Spec}(\mathcal{R}(X))$, which is called the *total coordinate space* of X. By construction, the Cox ring is graded by the class group of X inducing an action of the torus $\operatorname{Spec} k[\operatorname{Cl}(X)]$ on the total coordinate space.

In the following we are studying flexibility of total coordinate spaces for several classes of varieties.

5.1 Del Pezzo surfaces

Since the smooth del Pezzo surfaces of degrees 6, 7, 8, and 9 are toric, their total coordinate spaces are just affine spaces and hence flexible. The remaining del Pezzo surfaces are blowups X_r of $\mathbb{P}^2$ in r points of general position, where $4 \leq r \leq 8$. Their Cox rings are described for example in Batyrev and Popov [8], Derenthal [13], and Testa et al. [38].

An exceptional curve on X is a curve of self-intersection -1 and anti-canonical degree 1. On every del Pezzo surface there are only finitely many of them, we denote their number by $N(r)$. Seen as an effective divisor every such curve C corresponds to a section in the degree-$[C]$ part of the Cox ring. This section is uniquely determined up to scaling by a non-zero constant.

We will use the following facts from Batyrev and Popov [8].

Theorem 5.2 [8, Thm 3.2 and Prop. 3.4] *Let $N(r)$ be the number of exceptional curves on a del Pezzo surface X_r. Denote by $e_1 \dots, e_{N(r)}$ the sections corresponding to the exceptional curves and by I the ideal of their relations. Then*

(i) $\mathcal{R}(X_r) = \mathbf{k}[e_1, \dots, e_{N(r)}]/I$ *for* $4 \leq r \leq 7$*;*
(ii) $\mathcal{R}(X_8) = \mathbf{k}[e_1, \dots, e_{N(r)}]/I \oplus \langle f_1, f_2 \rangle_{\mathbf{k}}$ *as a vector space, where* $f_1, f_2 \in H^0(X_8, \mathcal{O}(-K_{X_8}))$ *are elements of degree one with respect to the* $\mathbb{Z}$*-grading by the anti-canonical degree of a divisor class.*

Theorem 5.2 shows that the Cox ring $\mathcal{R}(X)$ is generated by elements of degree 1 and $Y_r := \operatorname{Proj}(\mathcal{R}(X_r))$ comes with an embedding into $\mathbb{P}^{N-1}$, where N (resp. $N-2$) is the number of exceptional curves in the case $4 \leq r \leq 7$ (resp. $r = 8$).

In this situation the total coordinate space $\overline{X}_r$ is the affine cone over this embedding.

Proposition 5.3 *Let e be a section corresponding to an exceptional curve. Then the principal open subset $(Y_r)_e$ is isomorphic to X_{r-1}.*

Proof This can be found for example in the proof of Proposition 3.4 in Batyrev and Popov [8]. ◻

Theorem 5.4 *The total coordinate spaces of smooth del Pezzo surfaces are flexible.*

Proof As said above, it is enough to check the statement for X_r with $4 \leq r \leq 8$. We will go by induction. The del Pezzo surface X_3 is toric. Therefore, it has a flexible total coordinate space. Now, consider X_r with $4 \leq r \leq 8$. Then we have seen that $\overline{X}_r$ is the affine cone over Y_r. Moreover, the principal open subsets corresponding to sections of exceptional curves are isomorphic to $\overline{X}_{r-1}$ and hence flexible by induction hypothesis. It remains to check that these principal open subsets cover Y_r to conclude flexibility of $\overline{X}_r$ from Theorem 1.4. For $4 \leq r \leq 7$ this follows directly from Theorem 5.2(i). For the case $r = 8$ we have to take care for the remaining generators. By Theorem 5.2(ii) their squares are contained in the ideal $(e_1 \ldots, e_N)$ generated by the sections corresponding to exceptional curves, but then the common vanishing of $e_1 \ldots, e_N$ implies the vanishing of the remaining generators and hence $Y_r = \bigcup_{i=1}^{N} (Y_r)_{e_i}$. □

5.2 Smooth complexity-one T-varieties

In Hausen and Süß [18] the Cox rings of T-varieties are studied. For the case of a complexity-one action they have a very particular form.

Proposition 5.5 [18, Corollary 4.9] *Let $\mathcal{S}$ be an f-divisor and let us denote by $\mathcal{S}^\times$ the subset of rays in* $\operatorname{tail}(\mathcal{S})^{(1)}$ *that do not intersect* $\deg \mathcal{S}$. *Then the Cox ring of $X(\mathcal{S})$ is given by*

$$\frac{\mathbf{k}[S_\rho, T_{P,v} \mid \rho \in \mathcal{S}^\times, P \in \operatorname{supp}\mathcal{S}, v \in \mathcal{S}_P^{(0)}]}{\langle z \cdot T^{\mu(0)} + T^{\mu(\infty)} + T^{\mu(z)} \mid z \in \operatorname{supp}\mathcal{S} \cap \mathbf{k}^* \rangle}$$

where $T^{\mu(P)} := \prod_{v \in \mathcal{S}_P^{(0)}} T_{P,v}^{\mu(v)}$ and $\mu(v)$ denotes the minimal positive integer such that $\mu(v) \cdot v$ is a lattice element.

If we impose the additional condition that the T-variety is equivariantly covered by toric charts (which is fulfilled in the smooth case), then we can conclude the following.

Proposition 5.6 *The Cox ring of a complexity-one T-variety equivariantly covered by toric charts is isomorphic to*

$$\mathbf{k}[S_1, \ldots, S_{n_S}; T_{\ell,j} \mid 0 \leq \ell \leq m, 1 \leq j \leq n_\ell]/\langle z_\ell \cdot A_0 + A_1 + A_\ell \mid 2 \leq \ell \leq m \rangle,$$

where

(i) $N, m, n_0, \ldots, n_m \in \mathbb{Z}_{>0}$;
(ii) $z_2, \ldots, z_m$ *are distinct elements of* $\mathbf{k}^*$;
(iii) *for* $\ell = 0, \ldots, m$, A_ℓ *is a monomial in* $\mathbf{k}[T_{\ell,1}, \ldots, T_{\ell,n_\ell}]$;
(iv) *for* $\ell = 1, \ldots, m$ *the monomial* A_ℓ *is linear in at least one variable.*

Moreover, if X is Fano, then we may assume that A_ℓ for $\ell \geq 3$ is linear in each variable.

Proof The first statements follow directly from Proposition 5.5 and Lemma 4.2. For the Fano case note that the Cox ring of a Fano variety is log-terminal by Brown [9] and Gongyo et al. [17] and factorial by [7]. Hence, by Remark 6.4 in [25] we obtain the last statement. □

Proposition 5.7 *Let X be a complexity-one T-variety equivariantly covered by toric charts and $\overline{X}$ be the total coordinate space of X. Then $\overline{X}$ is flexible.*

Proof Let $\mathbf{k}[\overline{X}]$ be as in Proposition 5.6. We assume that $\overline{X}$ is naturally embedded into an affine space $\mathbb{A}^N = \operatorname{Spec} \mathbf{k}[S_1, \dots, S_N; T_{\ell,j} \mid 0 \leq \ell \leq m, 1 \leq j \leq n_\ell]$.

The images of monomials $A_0, \dots, A_m$ in $\mathbf{k}[\overline{X}]$ span a two-dimensional subspace, and no two of them are collinear. Therefore, we may permute $A_0, \dots, A_m$ along with a proper change of their coefficients, indices of variables, and numbers z_i. □

Lemma 5.8 *The point $x \in \overline{X}$ is singular if and only if there are at least three monomials A_i such that all their partial derivatives are vanishing at x.*

Proof Let x be singular and denote $L_\ell = z_\ell \cdot A_0 + A_1 + A_\ell$ for $\ell = 2, \dots, m$. Then there is a non-trivial linear combination $L \in \langle L_2, \dots, L_m \rangle_{\mathbf{k}}$, whose partial derivatives vanish at x. Since L is a sum of at least three monomials A_i, whose partial derivatives also vanish, the statement follows.

Conversely, given three monomials with partial derivatives vanishing at x, we assume that they are A_0, A_1, A_2. Then we take $L = L_2$. □

So, for a smooth point $x \in X$, up to permutation of monomials and variables we assume that each monomial A_i, $i = 2, \dots, m$, has a non-zero partial derivative, say, $\frac{\partial A_i}{\partial T_{i,1}}(x) \neq 0$. Moreover, we may choose $T_{i,1}$ to be linear in A_i. Indeed, if $T_{i,1}$ is not linear, then $A_i(x) \neq 0$, and we take $T_{i,1}$ to be a linear variable by 5.6 (iv). We denote $B_i = \frac{\partial A_i}{\partial T_{i,1}}$, which is non-zero at x.

Given a set of arbitrary numbers $c_{0,1}, \dots, c_{0,n_0}, c_{1,1}, \dots, c_{1,n_1} \in \mathbf{k}$, we construct a $\mathbb{G}_a$-action ϕ on $\mathbb{A}^N$ in two steps. First, denoting a parameter of $\mathbb{G}_a$ by t, we let

$$\begin{aligned} \phi^* \colon \mathbf{k}[\mathbb{A}^N] &\to \mathbf{k}[\mathbb{A}^N] \otimes \mathbf{k}[t], \\ S_i &\mapsto S_i, \quad \text{for } i = 1, \dots, n_S, \\ T_{0,j} &\mapsto T_{0,j} + t c_{0,j} \prod_{k=2}^{m} B_k, \quad \text{for } j = 1, \dots, n_0, \\ T_{1,j} &\mapsto T_{1,j} + t c_{1,j} \prod_{k=2}^{m} B_k, \quad \text{for } j = 1, \dots, n_1. \end{aligned}$$

Then, for some $H_0, H_1 \in \mathbf{k}[\mathbb{A}^N] \otimes \mathbf{k}[\mathbb{G}_a]$,

$$\begin{aligned} \phi^*(A_0) =& A_0 + H_0 \prod_{k=2}^{m} B_k, \\ \phi^*(A_1) =& A_1 + H_1 \prod_{k=2}^{m} B_k. \end{aligned}$$

Now, for each $\ell = 2, \dots, m$ we let

$$\begin{aligned} T_{\ell,1} &\mapsto T_{\ell,1} - (z_\ell \cdot H_0 + H_1) \prod_{\substack{2 \leq k \leq m \\ k \neq \ell}} B_k, \\ T_{\ell,j} &\mapsto T_{\ell,j}, \quad \text{for } j = 2, \dots, n_\ell. \end{aligned}$$

Then the trinomial $z_\ell \cdot A_0 + A_1 + A_\ell$ is fixed by ϕ^*, so ϕ preserves $\overline{X} \subset \mathbb{A}^N$. Thus, we have constructed a $\mathbb{G}_a$-action on the total coordinate space $\overline{X}$, which we also denote by ϕ.

As said before, for a chosen smooth point x we have $B_i(x) \neq 0$. Let us take another smooth point $y \in \overline{X}$ with non-zero coordinates and move x to y by $\mathbb{G}_a$-actions, denoting images of x by same letter. By translations along coordinates $S_1, \dots, S_{n_S}$ we 'equalize' them, i.e., obtain $S_i(x) = S_i(y)$, $i = 1, \dots, n_S$.

Since $c_{0,1}, \dots, c_{0,n_0}, c_{1,1}, \dots, c_{1,n_1} \in \mathbf{k}$ are arbitrary, with ϕ we also equalize coordinates $T_{0,j}$, $j = 1, \dots, n_0$, and $T_{1,j}$, $j = 1, \dots, n_1$, at x and y. Now, let $T_{1,1}$ be a linear variable in A_1, then for each $\ell = 2, \dots, m$ we construct a $\mathbb{G}_a$-action ϕ_ℓ by permuting monomials A_1 and A_ℓ and applying the procedure above. Since $\frac{\partial A_1}{\partial T_{1,1}}(x) = \frac{\partial A_1}{\partial T_{1,1}}(y) \neq 0$, with ϕ_ℓ we may equalize coordinates $T_{\ell,2}, \dots, T_{\ell,m}$, but break the equality of coordinate $T_{1,1}$, which we restore with ϕ.

Proceeding in this way for each $\ell = 2, \dots, m$, we equalize all coordinates except $T_{2,1}, \dots, T_{m,1}$. But in this case the equation $z_\ell \cdot A_0 + A_1 + A_\ell = 0$ with condition $B_\ell(x), B_\ell(y) \neq 0$ implies $T_{\ell,1}(x) = T_{\ell,1}(y)$ for each ℓ. So, we may send any smooth point to any point with non-zero coordinates. Hence the action of $\operatorname{SAut} \overline{X}$ is transitive on the regular locus of $\overline{X}$. □

Theorem 5.9 *The total coordinate space of a complete smooth rational T-variety of complexity one is flexible.*

Proof By [6, Thm A.1], every rational smooth complete rational T-variety of complexity one is covered by affine spaces, so the statement follows from Proposition 5.7. □

5.3 Flexibility of total coordinate spaces vs. flexible coverings

In [6] it was proved that for a variety with an open covering by affine spaces one obtains flexibility of the universal torsor. However, it is not clear whether the flexibility property extends to the total coordinate space. This motivates the following even more general question.

Question 5.10 Provided a variety admits an open covering by flexible affine subsets, does this imply flexibility of the total coordinate space?

It is also tempting to try to connect flexibility of the total coordinate space of the Cox ring of X with that of affine cones over X. The following example shows that flexibility of the total coordinate space does not imply flexibility of all the affine cones.

Example 5.11 (Del Pezzo surfaces) We have seen in Sect. 5.1 that all total coordinate spaces of del Pezzo surfaces are flexible. On the other hand, del Pezzo surfaces are covered by affine spaces, which are flexible. Concerning flexibilty of affine cones it was shown in Perepechko [30] and Park and Won [32] that for degree 4 and 5 all the affine cones are flexible, but by Cheltsov et al. [23] and Kishimoto et al. [12] the anti-canonical affine cones over del Pezzo surfaces of degree 3, 2, and 1 are not flexible.

One may still ask if flexibility of all the affine cones implies flexibility of the total coordinate space or for a more subtle relation, e.g. involving the grading of the Cox ring.

Question 5.12 Is there a relation between flexibility of the total coordinate space of X and the fact that *all* affine cones over X are flexible?

Let us give some illustrating examples for these questions.

Example 5.13 (Toric varieties) The Cox ring of a complete toric variety is a polynomial ring. Hence, the total coordinate space is flexible. On the other hand, the torus invariant affine charts and also the affine cones of a toric variety are again toric and hence flexible by Arzhantsev et al. [5, Theorem 0.2].

Example 5.14 (Blowups of a projective space in cubic hypersurfaces inside hyperplanes) The blowup constructions from Example 1.5 give varieties for which all the affine cones are flexible, as we have seen. On the other hand, the total coordinate space is flexible, as we see in the following.

We can consider the k^*-action on $\mathbb{P}^n$ given by multiplication with the coordinate x_0. It comes with a natural quotient map to $\mathbb{P}^{n-1}$ being defined outside the isolated fixed point $(1:0:\ldots:0)$. Then the centers of our blowups are fixed points of the action and we obtain induced actions on X and X' with natural quotient maps given by composition of the original quotient map with the blowup. Now, we may use Theorem 1.2 in Hausen and Süß [18] to calculate the Cox rings

$$\mathcal{R}(X) \cong \mathbf{k}[T_0, \ldots, T_n, T_n', S_1]/(T_n T_n' - f(T_0, \ldots, T_{n-1}, 0))$$

and

$$\mathcal{R}(X') \cong \mathbf{k}[T_0, \ldots, T_n, T_n', S_1, S_2]/(T_n T_n' - f(T_0, \ldots, T_{n-1}, 0))).$$

We see that they are suspensions over an affine space and hence flexible by Theorem 0.2 in Arzhantsev et al. [5].

Example 5.15 (T-varieties of complexity one) The proof of Theorem 5.9 implies that for T-varieties of complexity one the condition of being covered by toric (and hence flexible) charts is enough to deduce the flexibility of the total coordinate space. On the other hand, to conclude flexibility of all the affine cones we had to impose the stronger (technical) condition of Theorem 4.4.

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