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# Bounding the Range of a Rational Function over a Box*

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**Abstract**

A simple method is presented by which tight bounds on the range of a multivariate rational function over a box can be computed. The approach relies on the expansion of the numerator and denominator polynomials into Bernstein polynomials.

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## 1 Introduction

Many problems in applied mathematics and the engineering sciences can be reduced to the problem of finding the range of a function over a certain set. Here, we consider

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the problem of computing bounds for the range of a multivariate rational function over a box. The method is based on the expansion of the numerator and denominator polynomials into Bernstein polynomials; this expansion is now a well established tool. To the best of our knowledge, its application to bounding the range of rational functions, with the exception of the recent paper [3], has not yet been considered. The presented examples indicate that the new method is superior to the previous one.

# 2 Bernstein Expansion

In this section we briefly recall the most important properties of the Bernstein expansion, which will be used in the following section. Without loss of generality we may consider the unit box $I := [0,1]^n$ since any compact non-empty box in $\mathbb{R}^n$ can be mapped thereupon by an affine transformation.

Comparisons and the arithmetic operations on multiindices $i = (i_1, \ldots, i_n)^T$ are defined componentwise. For $x \in \mathbb{R}^n$ its monomials are $x^i := x_1^{i_1} \ldots x_n^{i_n}$. Using the compact notation $\sum\limits_{i=0}^{k} := \sum\limits_{i_1=0}^{k_1} \ldots \sum\limits_{i_n=0}^{k_n}$, $\binom{k}{i} := \prod\limits_{\mu=1}^{n} \binom{k_\mu}{i_\mu}$, an $n$-variate polynomial $p$, $p(x) = \sum\limits_{i=0}^{l} a_i x^i$, can be represented as

$$p(x) = \sum_{i=0}^{k} b_i^{(k)}(p) B_i^{(k)}(x), \; x \in I = [0,1]^n, \tag{1}$$

where $B_i^{(k)}(x) = \binom{k}{i} x^i (1-x)^{k-i}$ is the $i$th Bernstein polynomial of degree $k \geq l$, and the so-called *Bernstein coefficients* $b_i^{(k)}(p)$ are given by

$$b_i^{(k)}(p) = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{k}{j}} a_j, \; 0 \leq i \leq k, \;\; \text{where} \;\; a_j := 0 \;\; \text{for} \;\; j \geq l, j \neq l. \tag{2}$$

In particular, we have the *endpoint interpolation property*

$$b_i^{(k)}(p) \; = \; p(\frac{i}{k}), \;\; \text{for all} \;\; i = 0, \ldots, k \;\; \text{with} \;\; i_\mu \in \{0, k_\mu\}\,, \;\; \mu = 1, \ldots, n. \tag{3}$$

For an efficient computation of the Bernstein coefficients, see [2]. The Bernstein coefficients of order $k+1$ can easily be computed as convex combinations of the coefficients of order $k$, e.g., [1, 2].

A fundamental property for our approach is the *convex hull property*, which states that the graph of $p$ over $I$ is contained within the convex hull of the control points derived from the Bernstein coefficients, i.e.,

$$\{(x, p(x)) \mid x \in I\} \; \subseteq \; conv \left\{ \left( \frac{i}{k}, b_i^{(k)}(p) \right) \mid 0 \leq i \leq k \right\}, \tag{4}$$

where *conv* denotes the convex hull. This implies the *interval enclosing property*

$$\min_{i=0,\ldots,k} b_i^{(k)}(p) \; \leq \; p(x) \; \leq \; \max_{i=0,\ldots,k} b_i^{(k)}(p), \;\; \text{for all} \;\; x \in I. \tag{5}$$

A disadvantage of the direct use of (5) is that the number of the Bernstein coefficients to be computed explicitly grows exponentially with the number of variables $n$. Therefore, we can use a method [8] by which the number of coefficients which are needed for the enclosure only grows approximately linearly with the number of the terms of the polynomial.

# 3 Main Result

Let $p$ and $q$ be polynomials in variables $x_1, \ldots, x_n$. We consider the rational function $f := p/q$. We may assume that both $p$ and $q$ have the same degree $l$ since otherwise we can elevate the degree of the Bernstein expansion of either polynomial by component where necessary to ensure that their Bernstein coefficients are of the same order $k \geq l$. Without loss of generality we consider only the case $k = l$ and suppress the upper index for the Bernstein coefficients.

**Theorem 3.1** *Let $p$ and $q$ be polynomials with Bernstein coefficients $b_i(p)$ and $b_i(q)$, $0 \leq i \leq l$, over a box $X$, respectively. Assume that all Bernstein coefficients $b_i(q)$ have the same sign and are non-zero (this implies that $q(x) \neq 0$, for all $x \in X$). Then for $f := p/q$,*

$$\min_{i=0,\ldots,l} \frac{b_i(p)}{b_i(q)} \;\leq\; f(x) \;\leq\; \max_{i=0,\ldots,l} \frac{b_i(p)}{b_i(q)}, \quad \textit{for all } x \in X. \tag{6}$$

*Proof:* Without loss of generality we consider only the case $b_i(q) > 0$, $0 \leq i \leq l$, and prove the statement for

$$M \;:=\; \max_{i=0,\ldots,l} \frac{b_i(p)}{b_i(q)}. \tag{7}$$

Define the polynomial $s$ by $s(x) := p(x) - Mq(x)$. Then the Bernstein coefficients of $s$ are (note that the Bernstein coefficients (2) are linear in the polynomial coefficients $a_j$)

$$b_i(s) \;=\; b_i(p) - Mb_i(q),\ 0 \leq i \leq l, \tag{8}$$

which are nonpositive by (7). It follows from (5) that $s$ is nonpositive on $X$. Therefore,

$$\frac{s(x)}{q(x)} \;=\; f(x) - M \;\leq\; 0,\ x \in X, \tag{9}$$

which completes the proof. □

## Remarks

1. By the interpolation property (3), equality holds on the left or right hand side of (6), if the minimum or maximum, respectively, is attained at an index $i$ with

$$i_\mu \in \{0, l_\mu\},\ \mu = 1, \ldots, n. \tag{10}$$

2. The convex hull property (4) does not in general carry over to rational functions and control points formed from the ratios of Bernstein coefficients, according to (6), even in the univariate case ($n = 1$). A counterexample is provided by

$$p(x) \;:=\; 4x^2,\ \ q(x) \;:=\; 4x^2 + 1. \tag{11}$$

The degree 2 Bernstein coefficients of $p$ and $q$ are $(0, 0, 4)$ and $(1, 1, 5)$, respectively. The convex hull of the control points $\{(0, 0), (0.5, 0), (1, 0.8)\}$ does not contain the graph of $f := p/q$ over $I$ because, e.g., $f(0.5) = 0.5 > 0.4$, where 0.4 is the value attained by the affine function describing the upper facet of the convex hull at $x = 0.5$. Note that these control points define a degree 2 polynomial which is not a particularly good approximation to $f$ over $I$.

3. The result in [7] on complex-valued polynomials on $I$ does not carry over, even in the affine case ($l = 1$). A counterexample is provided by
$$p(x) \;:=\; ix, \;\; q(x) \;:=\; ix + 1. \tag{12}$$
Then $p(x) \in conv\{0, i\}$, $q(x) \in conv\{1, i+1\}$, $x \in I$, but the range of $f := p/q$ is not contained in $C := conv\{0, i/(i+1)\}$ because, e.g., $f(0.5) = i/(i+2)$ is not contained in $C$.
4. The bounds (6) may be improved by subdivision, i.e., the box $X$ is subdivided and the bounding approach of Theorem 3.1 is applied to the generated sub-boxes, see, e.g., [2, 9].
5. Theorem 3.1 may be used for finding bounds on the range of the partial derivatives of $f$ in order to, e.g., prove monotonicity of $f$ over $X$. Such information often considerably speeds up the application of branch and bound algorithms. For simplicity, we consider here the univariate case. The degree $l - 1$ Bernstein coefficients of $p'$ are given by
$$b_i(p') \;=\; l\left(b_{i+1}(p) - b_i(p)\right), \; i = 0, \ldots, l-1, \tag{13}$$
see, e.g., [1], and correspondingly for $q'$. The Bernstein coefficients of the products involved in $f' = (p'q - pq')/q^2$ can be computed by formula (44) in [1]. In the multivariate case, the Bernstein coefficients of the partial derivatives of $p$ and $q$ can be computed by forming differences of the Bernstein coefficients in the respective coordinate directions, see, e.g., formula (13) in [9].
6. Theorem 3.1 carries over to the Bernstein polynomials over the standard simplex in $\mathbb{R}^n$ [2].
7. The bounds (6) are integrated into the interactive theorem prover *Prototype Verification System* (PVS) [5].

# 4 Comparison with a Previous Method

Denote the Bernstein enclosures (5) for the polynomials $p$ and $q$ over a box $X$ by $P$ and $Q$, respectively. A simple enclosure for the range of $f := p/q$ over $X$ is obtained by $P/Q$, where the usual division of interval arithmetic, see, e.g., [4], is used. This method, termed the *naive method* below, neglects the dependency between the variables of both polynomials and may therefore result in gross overestimation in the range of $f$.

Following a suggestion by Arnold Neumaier, and as given in [3], $f$ can be represented in the following form
$$f(x) \;=\; \frac{p(x)}{q(x)} \;=\; r(x) + \frac{p(x) - r(x)q(x)}{q(x)}, \tag{14}$$
where $r$ is a linear approximation to $f$, viz. the linear least squares approximation of the control points $(i/l, b_i(p)/b_i(q)) = (i/l, f(i/l))$ associated with the vertices of $X$, see (3). We call this method the *least squares method*. The advantages of the representation (14) are that the range of $r$ over $X$ can be given exactly and that the Bernstein enclosure of the range of $p - rq$ over $X$ is often tighter than the Bernstein enclosure $P$. As in the naive method, we employ the Bernstein enclosure $Q$.

The *improved least squares method* is obtained by using Theorem 3.1, instead of the naive method, to obtain an enclosure for $(p(x) - r(x)q(x))/q(x)$ in (14).

## 4.1 Examples

We consider the following two examples from [6], cf. [3]. Let $f$ be given by

$$f := \frac{a(w^2 + x^2 - y^2 - z^2) + 2b(xy - wz) + 2c(xz + wy)}{w^2 + x^2 + y^2 + z^2}, \quad (15)$$

where

$$\begin{aligned} a \in [7, 9],\ b \in [-1, 1],\ c \in [-1, 1], \\ w \in [-0.9, -0.6],\ x \in [-0.1, 0.2],\ y \in [0.3, 0.7],\ z \in [-0.2, 0.1], \end{aligned} \quad (16)$$

and $g$ (a related function) be given by

$$g := \frac{2(xz + wy)}{w^2 + x^2 + y^2 + z^2}, \quad (17)$$

where the intervals for $w$, $x$, $y$, $z$ are as in (16). The box (16) spans 16 different orthants of $\mathbb{R}^7$.

In each case we consider the naive method, the least squares method, the improved least squares method, the new method based on Theorem 3.1 alone, and the true enclosure. In the first instance we compute the Bernstein coefficients over the whole box (16). Subsequently and alternatively, we compute the range enclosures and Bernstein coefficients over each orthant separately and form the union of the single orthant ranges. For a single orthant we can make use of the implicit Bernstein form [8].

The resulting enclosures for $f$ and $g$, outwardly rounded to 4 decimal places of precision, are given in Tables (1) and (2), respectively.

Table 1: Range enclosures for the rational function $f$ (15) over the box (16).

| **Bernstein expansion over** | **whole box** | **separate orthants** |
|---|---|---|
| Naive method | $[-6.6830, 18.5610]$ | $[-5.9112, 16.8223]$ |
| Least squares method | $[-5.8760, 11.2601]$ | $[-5.4357, 10.9533]$ |
| Improved least squares | $[-4.0403, 9.6863]$ | $[-3.9753, 9.5292]$ |
| New method | $[-3.1495, 8.3484]$ | $[-2.9888, 8.0550]$ |
| True range | $[-2.9561, 8.0094]$ | $[-2.9561, 8.0094]$ |

Table 2: Range enclosures for the rational function $g$ (17) over the box (16).

| **Bernstein expansion over** | **whole box** | **separate orthants** |
|---|---|---|
| Naive method | $[-3.2683, -0.2318]$ | $[-2.9778, -0.2370]$ |
| Least squares method | $[-1.3835, -0.4074]$ | $[-1.3302, -0.4250]$ |
| Improved least squares | $[-1.2333, -0.5175]$ | $[-1.2086, -0.5175]$ |
| New method | $[-1.1416, -0.5263]$ | $[-1.0878, -0.5263]$ |
| True range | $[-1, -0.5263]$ | $[-1, -0.5263]$ |

In these examples, the new method is superior to the least squares method.

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