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EULER-LIKE METHOD FOR THE SIMULTANEOUS INCLUSION OF POLYNOMIAL ZEROS WITH WEIERSTRASS' CORRECTION

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Abstract An improved iterative method of Euler's type for the simultaneous inclusion of polynomial zeros is considered. To accelerate the convergence of the basic method of fourth order, Carstensen-Petković's approach [7] using Weierstrass' correction is applied. It is proved that the R-order of convergence of the improved Euler-like method is (asymptotically) $2 + \sqrt{7} \approx 4.646$ or 5, depending of the type of applied inversion of a disk. The proposed algorithm possesses great computational efficiency since the increase of the convergence rate is obtained without additional calculations. Initial conditions which provide the guaranteed convergence of the considered method are also studied. These conditions are computationally verifiable, which is of practical importance.

1. INTRODUCTION

Iterative methods for the simultaneous inclusion of polynomial zeros, realized in complex circular interval arithmetic, produce resulting disks that contain the zeros of a given polynomial. The main advantage of circular arithmetic methods lies in automatic computation of rigorous error bounds (given by the radii of resulting inclusion disks) on approximate solutions. For more details about inclusion methods for polynomial zeros see the books [1], [6] and [8] and references cited therein.

Recently, Carstensen and Petković [2], [7] have proposed a procedure for accelerating the convergence order of total-step iterative methods for the simultaneous inclusion of polynomial zeros. The idea consists of the use of a suitable correction term which accelerates the convergence of the midpoints of inclusion circular approximations to the zeros. Since the convergence of the midpoints and the convergence of the radii are coupled, the increased convergence of the midpoints improves the convergence of the radii.

The third order method of Borsch-Supan's type (known also as Petković's interval method) is considered in [7]. In this case, Weierstrass' correction term was used to increase the convergence properties of the basic method, while Schroeder's correction was applied to the Gargantini simultaneous inclusion method for polynomial zeros in [2].

The purpose of this paper is to present the Euler-like method with Weierstrass' correction which improves the convergence of the basic method. The improved method is constructed using Carstensen-Petković's approach and realized in circular complex arithmetic. The new method can be regarded as a modification of the fourth order method of Euler's type presented in [9]. The main subject of the paper is the convergence analysis of the proposed method, including computationally verifiable initial conditions for the convergence.

Convergence analysis of the proposed Euler-like method needs the basic properties and operations of the so-called circular complex arithmetic. Disk Z with the center c and the radius r is denoted by $Z = \{c; r\}$. We deal with two types of the inversion of a disk,

$$Z^{-1} = \{c; r\}^{-1} = \frac{\{\bar{c}; r\}}{|c|^2 - r^2} \quad (0 \notin Z) \quad \textit{(exact inversion)}, \tag{1}$$

where the bar denotes the complex conjugate, and

$$Z^I = \{c; r\}^I := \left\{\frac{1}{c}; \frac{r}{|c|(|c| - r)}\right\} \quad (0 \notin Z) \quad \textit{(centered inversion)}. \tag{2}$$

The square root of a disk $\{c; r\}$ in the centered form, where $c = |c|e^{i\varphi}$ and $|c| > r$, is defined as the union of two disks (see [3]):

$$\{c; r\}^{1/2} := \left\{\sqrt{|c|}e^{i\frac{\varphi}{2}}; \sqrt{|c|} - \sqrt{|c| - r}\right\} \bigcup \left\{-\sqrt{|c|}e^{i\frac{\varphi}{2}}; \sqrt{|c|} - \sqrt{|c| - r}\right\}.$$

A review of circular arithmetic operations can be found in the books [1, Ch. 5] and [8, Ch. 1].

2. EULER-LIKE INCLUSION METHODS

Let P be a monic polynomial with simple real or complex zeros $\zeta_1,\dots,\zeta_n$. Assume that we have found disjoint disks $Z_1,\dots,Z_n$ containing the zeros, that is, $\zeta_i \in Z_i$ for each $i \in I_n := \{1,\dots,n\}$. Let $z_i = \operatorname{mid} Z_i$ $(i \in I_n)$ be the center of the inclusion disk Z_i. We introduce the following abbreviations:

$$W_i = \frac{P(z_i)}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}(z_i - z_j)} \qquad \text{(Weierstrass' correction)},$$

$$G_i = \sum_{j\neq i}\frac{W_j}{z_i - z_j}, \qquad T_i(Z) = \frac{W_i}{(1+G_i)^2}\sum_{j\neq i}\frac{W_j\,\mathrm{INV}(Z - z_j)}{z_i - z_j},$$

where Z is a disk and $\mathrm{INV} \in \{()^{-1}, ()^I\}$ denotes one of the inversions of a disk defined by (1) and (2).

The following inclusion method has been stated in [9]:

Algorithm 1: *Let* $(Z_1,\dots,Z_n) =: (Z_1^{(0)},\dots,Z_n^{(0)})$ *be initial disks such that* $\zeta_i \in Z_i$ $(i = 1,\dots,n)$. *Writing* $z_i := \operatorname{mid} Z_i$ *and* $r_i := \operatorname{rad} Z_i$ *for the center and the radius of the disk* Z_i, *one step of the new Euler-like inclusion algorithm reads* $(Z_1,\dots,Z_n) \mapsto (\widehat{Z}_1,\dots,\widehat{Z}_n)$ *with*

$$\widehat{Z}_i := z_i - \frac{2W_i}{1+G_i}\left(1 + \sqrt{1 + 4T_i(Z_i)}\right)^{-1} \qquad (i \in I_n). \tag{3}$$

As shown in [10], the convergence order of Euler-like method (3) is four. The convergence of the inclusion method (3) can be increased without additional calculations taking, under suitable conditions, the removed disk $Z_i - W_i$ instead of Z_i in (3). In this way, the following algorithm can be constructed:

Algorithm 2: *Let* $(Z_1,\dots,Z_n) =: (Z_1^{(0)},\dots,Z_n^{(0)})$ *be initial disks such that* $\zeta_i \in Z_i$ $(i = 1,\dots,n)$. *One step of the modified Euler-like inclusion algorithm reads* $(Z_1,\dots,Z_n) \mapsto (\widehat{Z}_1,\dots,\widehat{Z}_n)$ *with*

$$\widehat{Z}_i := z_i - \frac{2W_i}{1+G_i}\left(1 + \sqrt{1 + 4T_i(Z_i - W_i)}\right)^{-1} \qquad (i \in I_n). \tag{4}$$

For simplicity, when we consider Algorithm 2 we will write T_i instead of $T_i(Z_i - W_i)$.

Remark. In both iterative formulas (3) and (4) we assume the principal branch of the square root of a disk.

3. CONVERGENCE ANALYSIS

In this section we determine the R-order of convergence of the improved method (4). Using the concept of the R-order of convergence introduced by Ortega and Rheinboldt [5], it can be proved that the R-order of the radii of inclusion disks produced by the Euler-like method (4) is at least $2+\sqrt{7} \cong 4.646$ if $\mathrm{INV} = ()^{-1}$ and 5 if $\mathrm{INV} = ()^{I}$.

Let $Z_1 = \{z_1; r_1\}, \ldots, Z_n = \{z_n; r_n\}$ be disks containing the zeros $\zeta_1, \ldots, \zeta_n$ of a polynomial P. Let us introduce the notation

$$r = \max_{1\le i\le n} r_i, \quad \rho = \min_{\substack{i,j\\ i\ne j}} \{|z_i - z_j| - r_j\}, \quad \epsilon_i = z_i - \zeta_i, \quad \epsilon = \max_{1\le i\le n} |\epsilon_i|.$$

In what follows we will always assume that $n \ge 3$. Furthermore, let

$$v_{ij} := z_i - W_i - z_j, \quad T_i = \{t_i; \eta_i\}, \quad H_i = 1 + \sqrt{1+4T_i} =: \{u_i; d_i\},$$

and let us assume that the following condition is satisfied:

$$\rho > 4(n-1)r. \tag{5}$$

It is easy to derive the following upper bounds

$$|\mathrm{mid\ INV}(Z)| \le \frac{|z|}{|z|^2 - r^2} \quad \text{and} \quad \mathrm{rad\ INV}(Z) \le \frac{r}{|z|(|z|-r)}, \tag{6}$$

where $\mathrm{INV} \in \{()^{-1}, ()^{I}\}$ and $Z = \{z\ ; r\}$.

First we prove two lemmas.

Lemma 1 *If* (5) *holds, then the implication*

$$\zeta_i \in Z_i \implies \zeta_i \in Z_i - W_i \tag{7}$$

is valid for each $i \in I_n$.

Proof. Since $z \in \{c; r\} \iff |z-c| \le r$, it is sufficient to show the implication

$$|z_i - \zeta_i| = |\epsilon_i| \le r_i \implies |z_i - W_i - \zeta_i| \le r_i.$$

Let

$$b_j^{(i)} := \frac{\epsilon_j}{z_i - z_j}, \quad (j = 1, \ldots, n;\ j \ne i)$$

and let

$$M_\mu := \sum_{j_1<j_2<\cdots<j_\mu} b_{j_1}^{(i)} b_{j_2}^{(i)} \cdots b_{j_\mu}^{(i)}, \quad M_0 = 1$$

be the symmetric function of $b_j^{(i)}$. Evidently,

$$|M_\mu| \le \sum_{j_1<j_2<\cdots<j_\mu} |b_{j_1}^{(i)}| \cdot |b_{j_2}^{(i)}| \cdots |b_{j_\mu}^{(i)}| \le \binom{n-1}{\mu} \left(\frac{r}{\rho}\right)^{\mu}.$$

Using (5) and the fact that

$$\Big(1+\frac{1}{4(n-1)}\Big)^{n-1} =: \alpha_n < \lim_{n\to+\infty} \alpha_n = e^{1/4} \approx 1.284, \tag{8}$$

we obtain

$$\begin{aligned}
|z_i - W_i - \zeta_i| &= \left|\epsilon_i - \epsilon_i \cdot \prod_{j\neq i}\left(1+\frac{\epsilon_j}{z_i - z_j}\right)\right| = |\epsilon_i|\left|1-\sum_{\mu=0}^{n-1} M_\mu\right| \\
&\le |\epsilon_i| \sum_{\mu=1}^{n-1} |M_\mu| \le |\epsilon_i| \sum_{\mu=1}^{n-1} \binom{n-1}{\mu}\Big(\frac{r}{\rho}\Big)^{\mu} \\
&= |\epsilon_i|\left[\Big(1+\frac{r}{\rho}\Big)^{n-1} - 1\right] \le |\epsilon_i|(\alpha_n - 1) \\
&< |\epsilon_i|\left(e^{1/4}-1\right) < \frac{1}{3}|\epsilon_i| < r_i,
\end{aligned}$$

which proves (7). ■

Lemma 2 *If the condition* (5) *is satisfied, and if* $\zeta_i \in Z_i$ *for each* $i \in I_n$, *then the inversion in* (4) *exists, that is* $0 \notin Z_i - W_i - z_j$ *and* $0 \notin H_i$ $(i \in I_n)$.

Proof. Using the introduced notation we have $Z_i - W_i - z_j = \{v_{ij}; r_i\}$. According to (5) and (8), we get

$$|v_{ij}| = |z_i - W_i - z_j| \ge |z_i - z_j| - |W_i| > \rho - \alpha_n r > (4n-6)r > r_i. \tag{9}$$

Hence $0 \notin Z_i - W_i - z_j$.

To prove that $0 \notin H_i$ we estimate $|t_i| = |\text{mid}\, T_i|$ and $\eta_i = \text{rad}\, T_i$. Applying (6), (9), the proof of Lemma 1 and the condition (5), we find

$$\begin{aligned}
|t_i| = |\text{mid}\, T_i| &\le \frac{|W_i|}{|1+G_i|^2} \sum_{j\neq i} \frac{|W_j||\text{mid INV}(Z_i - W_i - z_j)|}{|z_i - z_j|} \\
&< \frac{\alpha_n^2 \epsilon^2}{(1-\alpha_n/4)^2} \frac{n-1}{\rho} \frac{|v_{ij}|}{|v_{ij}|^2 - r^2} < \frac{\alpha_n^2\epsilon^2}{(1-\alpha_n/4)^2} \frac{(n-1)(\rho-2r)}{\rho(\rho-r)(\rho-3r)} \\
&< \Big(\frac{\alpha_n}{1-\alpha_n/4}\Big)^2 \frac{4n-6}{4(4n-5)(4n-7)} < \frac{5}{32}.
\end{aligned}$$

According to this it follows $t_i \in \{0; 5/32\}$, wherefrom, using circular arithmetic operations,

$$1 + 4t_i \in \left\{1; \frac{5}{8}\right\} \implies |1 + 4t_i| > 1 - \frac{5}{8} = \frac{3}{8}$$

and

$$1 + \sqrt{1 + 4t_i} \in \left\{2; \frac{5/8}{1 + \sqrt{1 - 5/8}}\right\} \subset \left\{2; \frac{5}{8}\right\}.$$

Hence we obtain the lower bound of $|u_i|$,

$$|u_i| = |1 + \sqrt{1 + 4t_i}| > 2 - \frac{5}{8} = \frac{11}{8}.$$

Using the same argumentation we find

$$\eta_i = \operatorname{rad} T_i \le \frac{|W_i|}{|1 + G_i|^2} \sum_{j \ne i} \frac{|W_j| \operatorname{rad}\ \mathrm{INV}(Z_i - W_i - z_j)}{|z_i - z_j|}$$

$$< \frac{\alpha_n^2 \epsilon^2}{(1 - \alpha_n/4)^2} \frac{(n-1)r}{\rho |v_{ij}|(|v_{ij}| - r)} < \frac{\alpha_n^2 \epsilon^2}{(1 - \alpha_n/4)^2} \frac{(n-1)r}{\rho(\rho - 2r)(\rho - 3r)}$$

$$< \left(\frac{\alpha_n}{1 - \alpha_n/4}\right)^2 \frac{1}{4(4n-6)(4n-7)} < \frac{1}{32}.$$

Applying the last results, the initial condition (5) and the definition of the square root of a disk, we obtain

$$d_i = \operatorname{rad}\left(\sqrt{1 + 4T_i}\right) < \operatorname{rad}\left(\sqrt{\{1 + 4t_i;\ 4\eta_i\}}\right)$$

$$= \sqrt{|1 + 4t_i|} - \sqrt{|1 + 4t_i| - 4\eta_i} = \frac{4\eta_i}{\sqrt{|1 + 4t_i|} + \sqrt{|1 + 4t_i| - 4\eta_i}}$$

$$< \frac{\dfrac{4\alpha_n^2 \epsilon^2}{(1 - \alpha_n/4)^2} \dfrac{(n-1)r}{\rho(\rho - 2r)(\rho - 3r)}}{\sqrt{3/8} + \sqrt{2/8}} < \frac{1}{8}.$$

According to the previous we have

$$|u_i| - d_i > \frac{5}{4},$$

which means that $0 \notin \{u_i; d_i\} = H_i$. ■

Now, we are able to state the convergence theorem for the improved Euler-like method (4).

Theorem 1 *Let* $(Z_1,\dots,Z_n) =: (Z_1^{(0)},\dots,Z_n^{(0)})$ *be initial disks such that* $\zeta_i \in Z_i$ $(i \in I_n)$ *and let* $\{Z_i^{(m)}\}$ *denote the sequences of the disks obtained by the iterative formula* (4)*, where* $m = 0, 1, \dots$ *is the iteration index. If*

$$\rho^{(0)} > 4(n-1)r^{(0)} \tag{10}$$

holds, where

$$r^{(m)} := \max_{1\le i\le n} \operatorname{rad} Z_i^{(m)}$$

and

$$\rho^{(m)} := \min_{\substack{1\le i,j\le n\\ i\ne j}} \left\{ \left| \operatorname{mid} Z_i^{(m)} - \operatorname{mid} Z_j^{(m)} \right| - \operatorname{rad} Z_j^{(m)} \right\},$$

then $\zeta_i \in Z_i^{(m)}$ *for each* $i \in I_n$ *and* $m = 0, 1, \dots$ *and the sequences of radii* $\{\operatorname{rad} Z_i^{(m)}\}$ $(i \in I_n;\ m = 0, 1, \dots)$ *monotonically tend to 0.*

Proof. We will prove Theorem 1 by induction and, first, we consider typical step for $m = 0$. From the iterative formula (4) we obtain

$$\hat r_i = \operatorname{rad} \widehat{Z}_i \le \frac{2|W_i|}{|1+G_i|}\,\frac{d_i}{|u_i|(|u_i|-d_i)},$$

wherefrom, according to (8) and the estimates for d_i and u_i (see the proof of Lemma 2), we find

$$\begin{aligned} \hat r_i &< \frac{8(n-1)\alpha_n^3|\epsilon_i|\epsilon^2 r_i}{(1-\alpha_n/4)^3|u_i|(|u_i|-d_i)\rho(\rho-2r)(\rho-3r)} \\ &< \frac{256(n-1)\alpha_n^3}{55(1-\alpha_n/4)^3}\cdot\frac{|\epsilon_i|\epsilon^2 r_i}{\rho(\rho-2r)(\rho-3r)} \\ &< \frac{64}{55}\cdot\frac{\alpha_n^3}{(1-\alpha_n/4)^3}\cdot\frac{r_i}{(\rho/r-2)(\rho/r-3)}. \end{aligned}$$

Hence, by the initial condition (5), we obtain for $n \ge 3$

$$\hat r_i < \frac{8r_i}{(4n-6)(4n-7)} \le \frac{4}{15} r_i. \tag{11}$$

Furthermore, combining results of Lemma 1 and applying the inclusion property, we get

$$\zeta_i \in \widehat{Z}_i, \qquad \text{that is} \qquad |\hat z_i - \zeta_i| < \hat r_i < \frac{4}{15} r_i.$$

Since $\zeta_i \in Z_i$, that is $|z_i - \zeta_i| < r$, using the triangle inequality we obtain

$$|\hat{z}_i - z_i| < \frac{19}{15} r_i.$$

Having in mind the definition of ρ, the last inequality and (10), we obtain the following bound

$$|\hat{z}_i - \hat{z}_j| \geq |z_i - z_j| - |\hat{z}_i - z_i| - |\hat{z}_j - z_j| > \rho + r_j - \frac{19}{15} r_i - \frac{19}{15} r_j$$

$$> 4(n-1)r - \frac{23}{15} r > \left(15(n-1) - \frac{23}{4}\right) \hat{r}.$$

According to this we find for each pair of indices $i, j \in I_n$ $(i \neq j)$

$$|\hat{z}_i - \hat{z}_j| > 2\hat{r} \geq \hat{r}_i + \hat{r}_j,$$

which implies that the disks $\widehat{Z}_1, \ldots, \widehat{Z}_n$ are mutually disjoint. Also, for any pair of indices $i, j \in I_n$ $(i \neq j)$ we have

$$|\hat{z}_i - \hat{z}_j| - \hat{r}_j > \left(15(n-1) - \frac{23}{4}\right) \hat{r} - \hat{r} > 4(n-1)\hat{r}.$$

Hence it follows

$$\hat{\rho} > 4(n-1)\hat{r}.$$

Therefore we have proved that the initial condition (10) implies the inequality which is of the same form as (10), but for the index $m = 1$. Let us note that the estimate (11) of the form $r^{(1)} < 4r^{(0)}/15$ points to the contraction of the produced circular approximations $Z_1^{(1)}, \ldots, Z_n^{(1)}$.

Repeating the presented procedure for an arbitrary value of the index $m \geq 0$ we can derive similar relations for the index $m + 1$. Since these relations have already been proved for $m = 0$, according to mathematical induction it follows that, under the initial condition (10), they hold for every $m \geq 1$. In particular, we have for every $m = 1, 2, \ldots$

$$\rho^{(m)} > 4(n-1)r^{(m)}$$

and

$$r^{(m+1)} < \frac{4}{15} r^{(m)}. \tag{12}$$

Hence, all assertions of Lemmas 1 and 2 hold for every $m = 0, 1, \ldots$.

For fixed $i \in I_n$, according to Lemma 1 we have $\zeta_i \in Z_i^{(m)} - W_i^{(m)}$, and applying the inclusion property we obtain $\zeta_i \in Z_i^{(m+1)}$. Since $\zeta_i \in Z_i^{(0)}$, according to mathematical induction it follows that $\zeta_i \in Z_i^{(m)}$ for each $i \in I_n$ and $m = 0, 1, \ldots$.

Taking into account the assertions of Lemma 2 we infer that the inversions in (4) produce closed disks in each iterative step so that the simultaneous iterative method (4) is well defined. Besides, since $\rho^{(m)} > 2r^{(m)}$ it follows that the disks $Z_1^{(m)}, \ldots, Z_n^{(m)}$ are pairwise disjoint. Finally, the inequality (12) shows that the sequences of radii $\{r_i^{(m)}\}$ $(i \in I_n;\ m = 0, 1, \ldots)$ monotonically converge to 0. ∎

In order to determine the R-order of convergence of the improved inclusion method (4), we will use the Landau symbol $\mathcal{O}()$ to stress asymptotical behaviour of some quantities. For each $i \in I_n$ i $m = 0, 1, \ldots$ we define

$$\epsilon_i^{(m)} := z_i^{(m)} - \zeta_i, \quad \epsilon^{(m)} := \max_{1 \le i \le n} |\epsilon_i^{(m)}|, \quad r^{(m)} := \max_{1 \le i \le n} r_i^{(m)}.$$

Sometimes, for simplicity, we omit iteration index m in the current iterative step and use the symbol $\hat{}$ (*"hat"*) for the quantities in the $(m+1)^{st}$ iterative step. For example, we use ϵ and $\hat{\epsilon}$ instead of $\epsilon^{(m)}$ and $\epsilon^{(m+1)}$, respectively.

In the similar way as in [7] and [2], we can derive the following assertion which proof is given in [11].

Lemma 3 *Let*

$$\omega = \begin{cases} 1\ , & if\ \text{INV} = ()^{-1} \\ 0\ , & if\ \text{INV} = ()^{I} \end{cases},$$

where INV *denotes the inversion of disks appearing in* (4)*. Then, for the inclusion method* (4)*, the following relations can be stated:*

(*i*) $\hat{r} = \mathcal{O}(r\epsilon^3)$;

(*ii*) $\hat{\epsilon} = \mathcal{O}(\epsilon^5 r^2) + \mathcal{O}(\epsilon^5) + \omega\mathcal{O}(\epsilon^3 r^2)$.

In order to determine the R-order of convergence of the inclusion method (4), we will use some results from the theory of iterative processes.

Let IM be an iterative numerical method which generates k sequences $\{z_1^{(m)}\}, \ldots, \{z_k^{(m)}\}$ for the approximation of the solutions $z_1^*, \ldots, z_k^*$. To estimate the order of convergence of the iterative method IM one usually introduces the error-sequences

$$\epsilon_i^{(m)} = ||z_i^{(m)} - z_i^*|| \quad (i = 1, \ldots, k).$$

The convergence analysis of inclusion methods with corrections needs the following assertion given in [4]:

Theorem 2 *Given the error-recursion*

$$\epsilon_i^{(m+1)} \le \lambda_i \prod_{j=1}^{k} (\epsilon_j^{(m)})^{f_{ij}}, \quad (i = 1, \ldots, k;\ m \ge 0), \tag{13}$$

where $f_{ij} \ge 0,\ \lambda_i > 0,\ 1 \le i, j \le k$. *Denote the matrix of exponents appearing in* (13) *with* F, *that is* $F = [f_{ij}]_{k\times k}$. *If the non-negative matrix* F *has the spectral radius* $\sigma(F) > 1$ *and a corresponding eigenvector* $\boldsymbol{x}_\sigma > 0$, *then all sequences* $\{\epsilon_i^{(m)}\}$ *have at least the R-order* $\sigma(F)$.

In the sequel the matrix $F_k = [f_{ij}]$ will be called the *R-matrix.*

Theorem 3 *Let* $O_R(4)$ *denote the R-order of convergence of the interval iterative method* (4), *where* $\mathrm{INV} \in \{()^{-1}, ()^I\}$. *Then*

$$O_R(4) \ge \begin{cases} 2 + \sqrt{7} \approx 4.646 & if\ \ \mathrm{INV} = ()^{-1}, \\ 5 & if\ \ \mathrm{INV} = ()^I. \end{cases}$$

Proof. According to the assertion (*ii*) of Lemma 3 it follows

$$\hat{\epsilon} = \begin{cases} \mathcal{O}(\epsilon^3 r^2) & \text{if } \omega = 1, \\ \mathcal{O}(\epsilon^5) & \text{if } \omega = 0. \end{cases} \tag{14}$$

In our consideration we will distinguish these two cases.

- $\omega = 1$ (that is $\mathrm{INV} = ()^{-1}$):
  Using (14) and (*i*) of Lemma 3 we find

$$\hat{\epsilon} = \mathcal{O}(\epsilon^3 r^2), \qquad \hat{r} = \mathcal{O}(\epsilon^3 r).$$

According to these relations and Theorem 2 we form the R-matrix $F_2 = \begin{bmatrix} 3 & 2 \\ 3 & 1 \end{bmatrix}$, with the spectral radius $\sigma(F_2) = 2+\sqrt{7}$ and the corresponding eigeinvector $\boldsymbol{x}_\sigma = (1/3 + \sqrt{7}/3, 1) > 0$. Therefore, using Theorem 2, we obtain

$$O_R(4)_{\omega=1} \ge \sigma(F_2) = 2 + \sqrt{7} \cong 4.646.$$

- $\omega = 0$ (that is $\mathrm{INV} = ()^I$):
  In the similar way, from Lemma 3 (*i*) and (14) we obtain the relations

$$\hat{\epsilon} = \mathcal{O}(\epsilon^5), \qquad \hat{r} = \mathcal{O}(\epsilon^3 r)$$

which give the R-matrix $F_2 = \begin{bmatrix} 5 & 0 \\ 3 & 1 \end{bmatrix}$. Since $\sigma(F_2) = 5$ and $\boldsymbol{x}_\sigma = (4/3, 1) > 0$, in regard to Theorem 2 we conclude that

$$O_R(4)_{\omega=0} \ge \sigma(F_2) = 5.$$

This completes the proof of Theorem 3. ■

As noted in [7], the increase of the convergence rate of Algorithm 2 in comparison to Algorithm 1 is forced by the very fast convergence of the sequences $\{z_i^{(m)}\}$ of the centers of disks, which converge with the convergence rate *five*. This acceleration of convergence is attained since the better approximation mid $(Z_i - W_i) = z_i - W_i$ to the zero ζ_i is used instead of the former approximation mid $Z_i = z_i$, which accelerates the convergence of the radii of inclusion disks.

The inclusion methods of Euler's type (3) and (4) have been tested on a number of polynomial equations. Experimental results coincide very well with the theoretical results concerning the convergence speed of Euler-like methods (3) and (4), showing the advantage of the proposed method (4) with corrections. Besides, these results show that initial approximations can be chosen under a weaker condition compared to (10). Because of the page limitation, we have not presented here numerical results, but they can be found at the web site

`www.informatik.uni-leipzig.de/∼vranic/ccas/` .

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