Universität Konstanz

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Konstanzer Schriften in Mathematik und Informatik
Nr. 224, Februar 2007

ISSN 1430-3558

Exponential Stability of Wave Equations with Potential and Indefinite Damping

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Abstract

First, we consider the linear wave equation $u_{tt} - u_{xx} + a(x)u_t + b(x)u = 0$ on a bounded interval $(0, L) \subset \mathbb{R}$. The damping function a is allowed to change its sign. If $\overline{a} := \frac{1}{L}\int_0^L a(x)dx$ is positive and the spectrum of the operator $(\partial_{xx} - b)$ is negative, exponential stability is proved for small $\|\overline{a} - a\|_{L^2}$. Explicit estimates of the decay rate ω are given in terms of $\overline{a}$ and the biggest eigenvalue of $(\partial_{xx} - b)$. Second, we show the existence of a global, small, smooth solution of the corresponding nonlinear wave equation $u_{tt} - \sigma(u_x)_x + a(x)u_t + b(x)u = 0$, if, additionally, the negative part of a is small enough compared with ω. This is an extension of the results of Racke and Muñoz Rivera [17](b=0) and Benaddi and Rao [1] ($\|a\|_\infty$ small).

1 Introduction

The linear wave equation

$$u_{tt} - u_{xx} + a(x)u_t + b(x)u = 0. \tag{1}$$

is considered on the domain $\Omega := (0, L),\ L > 0$. We assume that the function $u = u(x, t)$, $(x, t) \in \Omega \times (0, \infty)$ satisfies the following initial and Dirichlet boundary conditions

$$u(x, 0) = u_0,\ u_t(x, 0) = u_1 \text{ for } x \in \Omega \quad \text{and} \quad u(0, t) = u(L, t) = 0 \text{ for } t \in (0, \infty).$$

Convention. *We will write L^p and H_0^1 instead of $L^p(\Omega)$ and $H_0^1(\Omega)$.*

We assume that the functions $a \in L^\infty$ and $b \in L^\infty$ are time independent. Our main interest is exponential stability of (1). Because there are a lot of results on decay rates if the damping is definite i.e. $a \geq 0$ (see for example [5], [6], [8], [18], [19] and [25]), we will focus our attention on indefinite damping i.e. a is allowed to change its sign. If $b = 0$ and the function a is positive definite (i.e. $a(x) \geq 0$ and $a(x) > 0$ on a subinterval of Ω), it is a well known fact that the equation (1) is exponentially stable. Thus the key problem is to discover a condition which describes the *positiveness* of the function a in the right manner. So Chen, Fulling, Narcovich and Sun formulated in [3] a conjecture concerning exponential stability for the case $b = 0$.

Conjecture 1.1. *Let $b = 0$. If there exists $\gamma > 0$ such that*

$$\forall\, n \in \mathbb{N} \int_0^L a(x)\sin^2\left(\frac{n\pi x}{L}\right)dx \ \geq \gamma, \tag{2}$$

is satisfied, then the energy $E(t) = \int_0^L u_x^2 + u_t^2 dx$ decays exponentially in time; i.e there are constants $C > 0$ and $\omega > 0$ independent of the initial data, such that $E(t) \leq\ Ce^{-\omega t}E(0)$ for all $t \in [0, \infty)$.

We will call the real number $\omega > 0$ *a (possible) decay rate*. Behind condition (2) stands the intuitive idea that it should be more effective to damp the string at the locations with high amplitudes than with low amplitudes of vibrations. But Freitas [9] outlined that the conjecture above is, in general, false. He also showed that condition (2) is not sufficient to guarantee exponential stability if $\|a\|_{L^\infty}$ is large. By replacing a by εa Freitas and Zuazua were able to show the following result (see [10]).

Proposition 1.2 (Freitas and Zuazua 1996). *Let $\tilde{a} \in BV$, $b = 0$ and (2) be satisfied. Then there exists $\varepsilon > 0$ such that the equation (1) with $a := \varepsilon\tilde{a}$ is exponentially stable.*

This result was extended to the case $b \neq 0$ by Benaddi and Rao [1] (see Proposition 1.3) and to higher space dimensions by Liu, Rao and Zhang [16]. Whereas K. Liu, Z. Liu and Rao [15] gave an abstract treatment to these results.

Proposition 1.3 (Benaddi and Rao 2000). *Let $\lambda_n,\ n \in \mathbb{N}$ be the eigenvalue of the operator $(\partial_{xx} - b)$ belonging to the eigenfunction v_n, which is normalized in L^2. Let $\tilde{a} \in BV$ and $b \in L^1$. If*

(i) $0 > \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \rightarrow -\infty \quad (n \rightarrow \infty)$

(ii) $\exists\, \gamma > 0\, .:\ \forall\, \mathbb{N} \ni n \geq 1 \quad \int_0^L \tilde{a}(x) v_n^2(x)\, dx \geq \gamma$

holds, then there exists $\varepsilon > 0$ such that the equation (1) with $a := \varepsilon\tilde{a}$ is exponentially stable.

We want to remark that in order to apply Proposition 1.2 and 1.3, the function a needs to be small in the $\|\cdot\|_{L^\infty}$ norm. Racke and Muñoz Rivera [17] were able to determine an easier condition by using the mean value $\overline{a} := \frac{1}{L}\int_0^L a(x)dx$ and the deviation $\|\overline{a} - a\|_{L^2}$ to measure the positiveness of the function a for the one dimensional case with $b = 0$. Their main result for the linear wave equation is stated as

Proposition 1.4 (Racke and Muñoz Rivera 2004). *Let $a \in L^\infty$ and $b = 0$. If $\overline{a} > 0$, then there exists $\varepsilon > 0$ such that if $\|\overline{a} - a\|_{L^2} < \varepsilon$, then the equation (1) is exponentially stable. One can chose $\omega = 2a_0$ as a decay rate, if a_0 satisfies $0 < a_0 < -\operatorname{Re}\left(-\frac{\overline{a}}{2} + \sqrt{\left(\frac{\overline{a}}{2}\right)^2 - \frac{\pi^2}{L^2}}\right)$.*

As one sees, the function a needs not to be small in the $\|\cdot\|_{L^\infty}$ norm. To illustrate this proposition in comparison to the result of Freitas and Zuazua (see Proposition 1.2), Racke and Muñoz Rivera [17] stated a simple example. Because Racke and Muñoz Rivera used an explicit determination of the spectrum and of the resolvent of the operator $(\partial_{xx} - \overline{a}\partial_t)$, the question remains open, if their condition of *positiveness* also makes sense in more general situations. For instance, if $b \neq 0$, the spectrum and resolvent of the operator $(\partial_{xx} - \overline{a}\partial_t - b)$ cannot be determined easily in general. Our first main result approves that the result of Racke and Muñoz Rivera (see Proposition 1.4) can be generalized to the case $b \neq 0$. More precisely, we prove the following proposition in section 2.

Proposition 1.5 (Linear Case). *Let $a \in L^\infty$, $b \in L^\infty$ and let $\tilde{\eta}_1$ be the greatest eigenvalue of the operator $\tilde{A}_0 : L^2 \supset H_0^1 \cap H^2 =: D(\tilde{A}_0) \rightarrow L^2$, that is defined as*

$$\tilde{A}_0 p := (\partial_{xx} - b(x))p \quad \textit{for} \quad p \in D(\tilde{A}_0). \tag{3}$$

If $\overline{a} > 0$ and $0 > \tilde{\eta}_1$ then there exists $\varepsilon > 0$ such that if $\|\overline{a} - a\|_{L^2} < \varepsilon$, then the equation (1) is exponentially stable. One can chose $\omega = 2a_0$ as a decay rate, if a_0 satisfies $0 < a_0 < -\operatorname{Re}\left(-\frac{\overline{a}}{2} + \sqrt{\left(\frac{\overline{a}}{2}\right)^2 - |\tilde{\eta}_1|}\right)$.

Thus, we also improved the result of Benaddi and Rao (see Proposition 1.3) that had the stronger assumption of smallness of $\|a\|_{L^\infty}$. We also lifted the argument of Racke and Muñoz Rivera from a concrete level to an abstract one. Only knowledge about the distribution of the spectrum of the operator

$\tilde{A}_0 = (\partial_{xx} - b)$ is needed. Instead of an explicit determination of the resolvent we will use the Hilbert-Schmitt representation. Hence, the argument is now applicable to a wider class of models.

In the second part we will prove the existence of a global, smooth, small solution of the corresponding nonlinear wave equation

$$u_{tt} - \sigma(u_x)_x + a(x)u_t + b(x)u = 0 \tag{4}$$

with initial and Dirichlet boundary conditions on the domain Ω under certain hypotheses (see Proposition 1.6). We assume that the functions $a \in C^3$ and $b \in C^3$ are time independent. The non-linear function σ is assumed to satisfy

$$\sigma \in C^3(\mathbb{R}), \quad d_0 := \sigma'(0) > 0, \text{ and } \sigma''(0) = 0. \tag{5}$$

For example, condition (5) is satisfied for the vibrating string, where $\sigma(y) = \frac{y}{\sqrt{1+y^2}}$ holds. One can rewrite (4) as

$$u_{tt} - d_0 u_{xx} + a(x)u_t + b(x)u = c(u_x)u_{xx}, \tag{6}$$

where $c(u_x)$ is defined as

$$c(u_x) := \sigma'(u_x) - d_0 = \sigma'(u_x) - \sigma'(0). \tag{7}$$

Thus the associated linear system of (4) is

$$u_{tt} - d_0 u_{xx} + a(x)u_t + b(x)u = 0. \tag{8}$$

The exponential stability of the associated linear system (8) follows directly by a transformation of coordinates from Proposition 1.5 (see for instance [17]) under analog hypotheses. As a consequence we can apply a standard technique in nearly the same way as Racke and Muñoz Rivera to get the existence of a global, smooth, small solution of (4). Additionally, one only needs the negative part of a to be small enough compared with a decay rate of (8). In section 3 we will proof our second main result, which generalizes the statement of Racke and Muñoz Rivera adequately to the case $b \neq 0$.

Proposition 1.6 (Non-Linear Case). *Let $a \in C^3$, $b \in C^3$ and let σ satisfy the condition (5). Let $\tilde{\eta}_1$ be the greatest eigenvalue of the operator $\tilde{A}_{d_0} : L^2 \supset H_0^1 \cap H^2 =: D(\tilde{A}_{d_0}) \to L^2$, that is defined as*

$$\tilde{A}_{d_0} p := (d_0 \partial_{xx} - b(x))p \quad \textit{for} \quad p \in D(\tilde{A}_{d_0}). \tag{9}$$

Let $\overline{a} > 0$ and $0 > \tilde{\eta}_1$. If the associated linear system (8) is exponentially stable with a decay rate $\omega = 2a_0$ and if the negative part of a is sufficiently small, i.e. $a_\infty^- := |\min_{x\in\Omega}(0, a(x))|_{L^\infty} < a_0$, then there exists $\delta > 0$ such that if $\|(u_0, u_1)\|_{H^4 \times H^3} < \delta$, there exists a unique global solution u of the non-linear system (4) satisfying

$$u \in \bigcap_{k=0}^{3} C^k \left([0,\infty), H^{4-k}(\Omega) \cap H_0^1(\Omega)\right) \cap C^4 \left([0,\infty), L^2(\Omega)\right).$$

Moreover let $V := (u_x, u_t)^T$ and $V_0 := (\partial_x u_0, u_1)^T$, then there are constants $c_0 = c_0(V_0) > 0$ and $c_1 > 0$ such that $\|V(t)\|_{H^2} \le c_0 e^{-a_0 t}$ and $\|V(t)\|_{H^3} \le c_1 \|V_0\|_{H^3} e^{a_\infty^- t}$, $t \ge 0$.

2 Linear Case

To derive exponential stability of (1) we will use a standard result which was obtained by Gearhart [11] and Huang [12] independently (see for example page 852 in [21]).

Theorem 2.1. *The C_0-semigroup $S(t) = e^{At}$ is exponentially stable if and only if*

$$\text{(C1)} \quad \sup\{\operatorname{Re}\lambda \,:\, \lambda \in \sigma(A)\} \,<\, 0 \quad \text{and} \quad \text{(C2)} \quad \sup\{\|R_\lambda(A)\| \,:\, \operatorname{Re}\lambda \geq 0\} \,<\, \infty$$

holds. Moreover, if also for a fixed $\delta > 0$

$$\text{(C3)} \qquad \sup\{\|R_\lambda(A)\| \,:\, \operatorname{Re}\lambda \geq -\delta + \varepsilon\} \,<\, \infty, \quad \forall\, \varepsilon > 0$$

is satisfied, then one can choose any ω as a decay rate that satisfies $0 < \omega < 2\delta$.

Motivated by the article of Racke and Muñoz Rivera [17] we verify conditions (C1) and (C3) for an associated system, where the function a is exchanged in (1) by its mean value $\overline{a}$. In a second step we transfer the conditions (C1) and (C3) to the original system by using a fixed point argument. The first step represents the crucial part of the proof of Proposition 1.5.

For the rest of this section we assume the hypothesis of Proposition 1.5 to be satisfied. We now translate (1) into the language of operator theory. Let $\mathcal{H} := H_0^1(\Omega) \times L^2(\Omega)$ be a Hilbert space endowed with the inner product

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle_{\mathcal{H}} := \langle \nabla f_1, \nabla g_1 \rangle_{L^2} + \langle g_2, g_2 \rangle_{L^2} \text{ with } \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{H}.$$

We define the operator $A : \mathcal{H} \supset D(A) \to \mathcal{H}$ as

$$A \begin{pmatrix} p \\ q \end{pmatrix} := \begin{pmatrix} O & \operatorname{Id} \\ \partial_{xx} - b(x) & -a(x) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} p \\ q \end{pmatrix} \in (H_0^1(\Omega) \cap H^2(\Omega)) =: D(A).$$

A straightforward calculation shows that the Dirichlet problem of (1) is equivalent to

$$\begin{pmatrix} p \\ q \end{pmatrix}_t = A \begin{pmatrix} p \\ q \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} u \\ u_t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p \\ q \end{pmatrix}(t=0) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u_0 \\ \partial_t u_0 \end{pmatrix}. \tag{10}$$

Since the operator A generates a C_0-semigroup we have existence and uniqueness of a strong solution of (10). One easily verifies the following statement (cp. for example [20]).

Lemma 2.2. *There exists $\omega > 0$ such that the operator $(A - \omega)$ is dissipative. Therefore the following conditions are satisfied:*

(i) $\|(\lambda + \omega)x - Ax\|_{\mathcal{H}} \geq \lambda \|x\|_{\mathcal{H}}$ *for all $x \in D(A)$ and $\lambda > 0$.*

(ii) If for some $\lambda_0 > \omega$, $R(\lambda_0 \operatorname{Id} - A) = \mathcal{H}$, then $R(\lambda \operatorname{Id} - A) = \mathcal{H}$ for all $\lambda > \omega$.

2.1 The Associated Operator $\overline{A}$ and the Reduced Associated Operator $\overline{A}_0$

The associated differential operator $\overline{A}$ is defined as the operator A, only a has to be exchanged by $\overline{a}$. We want to verify (C1) and (C3) for the operator $\overline{A}$. For this purpose we consider the eigenvalue problem of the operator $\overline{A}$. Let $\lambda \in \mathbb{C}$ be arbitrary. For every $F = (f_1, f_2)^T \in \mathcal{H}$, we want to find a unique $U = (p, q)^T \in D(\overline{A})$ which solves the equation

$$\lambda U - \overline{A}\, U = F. \tag{11}$$

The first component of (11) gives $q = \lambda p - f_1$. Substituting q in the second component leads to

$$(\lambda^2 + \lambda\, \overline{a} + b(x))\, p - \partial_{xx}\, u = f_2 + (\lambda + \overline{a}) f_1. \tag{12}$$

In (12) we add $(b_{min} - b_{min})p = 0$, where $b_{min} := -|\min_{x\in\Omega}(0, b(x))|_{L^\infty}$. Thus we obtain

$$\underbrace{(\lambda^2 + \lambda\,\overline{a} + b_{min})}_{=:\eta}\; p - \underbrace{(\partial_{xx} + b_{min} - b(x))}_{=:\overline{A}_0}\; p = \underbrace{f_2 + (\lambda + \overline{a})f_1}_{=:g}. \tag{13}$$

By defining the reduced associated operator $\overline{A}_0 : L^2 \supset H_0^1 \cap H^2 \to L^2$ as

$$\overline{A}_0 p := (\partial_{xx} + b_{min} - b(x))p \quad \text{for} \quad p \in D(\overline{A}_0) := H_0^1 \cap H^2,$$

we derive a new eigenvalue problem from (13): $\eta p - \overline{A}_0 p = g$. In summary we have shown the following proposition, which contains a strong connection between the eigenvalue problems of the operator $\overline{A}$ and the operator $\overline{A}_0$.

Proposition 2.3. *Let $F := (f_1, f_2)^T \in \mathcal{H}$, $\lambda \in \mathbb{C}$. If $\eta = \lambda^2 + \overline{a}\lambda + b_{min} \in \varrho(\overline{A}_0)$, then $R_\lambda(\overline{A})F =: (p, q)^T$ is determined as $p = R_\eta(\overline{A}_0)(f_2 + (\lambda + \overline{a})f_1)$ and $q = \lambda p - f_1$.*

Remark 2.4. *Let η_1 and $\tilde{\eta}_1$ be the biggest eigenvalue of the operator $\overline{A}_0$, respectively $\tilde{A}_0$. Then*

$$\begin{aligned} &(i) && 0 > \tilde{\eta}_1 \iff b_{min} > \tilde{\eta}_1 + b_{min} = \eta_1 \\ &(ii) && |\tilde{\eta}_1| = |\eta_1 - b_{min}| \end{aligned} \tag{14}$$

holds. Therefore, we assume that $b_{min} > \eta_1$ is satisfied.

In order to apply Proposition 2.3 it is necessary to know for which $\lambda \in \mathbb{C}$ holds $\eta := \lambda^2 + \overline{\lambda} + b_{min} \in \varrho(\overline{A}_0)$. Proposition 2.6 provides an answer to this question.

Definition 2.5. *Let $\nu := \operatorname{Re}\left(-\frac{\overline{a}}{2} + \sqrt{\left(\frac{\overline{a}}{2}\right)^2 - |\eta_1 - b_{min}|}\right)$ and let $\varepsilon^* > 0$ be sufficiently small that $\varepsilon^* + \nu < 0$. The area of parameters $\Gamma \subset \mathbb{C}$ is defined as $\Gamma := \{z \in \mathbb{C} \,:\, \operatorname{Re} z \geq \varepsilon^* + \nu\}$.*

Proposition 2.6. *If $\lambda \in \Gamma$, then $\eta := \lambda^2 + \overline{\lambda} + b_{min} \in \varrho(\overline{A}_0)$ holds.*

Proof. Let $\lambda = \mu_1 + i\,\mu_2 \in \Gamma$. It follows that $\mathbb{R} \ni \mu_1 \geq \varepsilon^* + \nu \geq \varepsilon^* - \frac{\overline{a}}{2}$. One can determine $\eta \in \mathbb{C}$ as

$$\eta = \lambda^2 + \overline{a}\lambda + b_{min} = \mu_1^2 + \overline{a}\mu_1 + b_{min} - \mu_2^2 + i\,(\overline{a} + 2\mu_1)\mu_2. \tag{15}$$

In the first case let $\operatorname{Im}\lambda = \mu_2 \neq 0$. Because

$$(\overline{a} + 2\mu_1) \geq (\overline{a} + 2\varepsilon^* - 2\frac{\overline{a}}{2}) = 2\varepsilon^* > 0 \tag{16}$$

we know that $\operatorname{Im}\eta = (\overline{a} + 2\mu_1)\mu_2 \neq 0$. Because the spectrum of the self-adjoint operator $\overline{A}_0$ is real, it follows that $\eta \in \varrho(\overline{A}_0)$.

In the second case let $\operatorname{Im}\lambda = \mu_2 = 0$. Because $2\nu + \overline{a} \geq 0$, a short calculation gives

$$\mu_1^2 + \overline{a}\mu_1 \geq (\varepsilon^* + \nu)^2 + \overline{a}(\varepsilon^* + \nu) \geq (\varepsilon^*)^2 + \nu^2 + \overline{a}\nu. \tag{17}$$

Let $\left(\frac{\overline{a}}{2}\right)^2 - |\eta_1 - b_{min}| > 0$. It follows that $\nu^2 + \overline{a}\nu = -|\eta_1 - b_{min}|$. By using (17) we can estimate η as

$$\eta = \mu_1^2 + \overline{a}\mu_1 + b_{min} \geq (\varepsilon^*)^2 + \nu^2 + \overline{a}\nu + b_{min} = (\varepsilon^*)^2 - |\eta_1 - b_{min}| + b_{min} > \eta_1. \tag{18}$$

Because η_1 is the biggest eigenvalue of the operator $\overline{A}_0$, we get $\eta \in \varrho(\overline{A}_0)$.
Finally, let $\left(\frac{\overline{a}}{2}\right)^2 - |\eta_1 - b_{min}| < 0$. Then $\nu = -\frac{\overline{a}}{2}$ holds. So we can estimate η by using (17) as

$$\eta = \mu_1^2 + \overline{a}\mu_1 + b_{min} \geq (\varepsilon^*)^2 - \frac{\overline{a}^2}{4} + b_{min} \geq (\varepsilon^*)^2 - |\eta_1 - b_{min}| + b_{min} > \eta_1. \tag{19}$$

Again we get $\eta \in \varrho(\overline{A}_0)$. □

By using the concept of lower semi bounded operators (see for example [2]) we are able to compare the eigenvalues of $\overline{A}_0$ with the eigenvalues of the operator ∂_{xx}. Let us introduce the operator $B : L^2 \supset H_0^1 \cap H^2 \to L^2$, which is defined as $B = (\partial_{xx} - |b_{min} - b|_{L^\infty})$. Applying now Theorem 4 on page 227 in [2] to the operators $-\overline{A}_0$ and $-B$ and to the operators $-\partial_{xx}$ and $-\overline{A}_0$ gives the following statement.

Lemma 2.7. *Let η_k, $k \in \mathbb{N}$ be the eigenvalues of $\overline{A}_0$ in decreasing order. Then*

$$-\frac{\pi^2}{L^2}k^2 - |b_{min} - b|_{L^\infty} \leq \eta_k \leq -\frac{\pi^2}{L^2}k^2.$$

As a direct consequence we get a lot of information about the distribution of the spectrum of $\overline{A}_0$.

Corollary 2.8. *There exists $n \in \mathbb{N}$, such that for all $\mathbb{N} \ni k \geq n$,*

$$-\frac{\pi^2}{L^2}(k+1)^2 \leq \eta_k \leq -\frac{\pi^2}{L^2}k^2. \tag{20}$$

From the Hilbert-Schmidt theorem (see for example [23]) the following representation of the resolvent of $\overline{A}_0$ can be easily deduced.

Proposition 2.9. *Let $\eta \in \varrho(\overline{A}_0)$ and $g \in L^2$. Then there is an orthonormal set $\{u_k\}_{k=1}^\infty$ of eigenfunctions and corresponding eigenvalues $0 \neq \eta_k \in \mathbb{R}$ of the operator $\overline{A}_0$, such that*

$$R_\eta(\overline{A}_0)g = \sum_{k=1}^\infty \frac{\langle g, u_k\rangle_{L^2}}{\eta - \eta_k}u_k \quad \textit{and} \quad \|R_\eta(\overline{A}_0)\|_{L^2}^2 = \sum_{k=1}^\infty \frac{1}{|\eta - \eta_k|^2}|\langle g, u_k\rangle_{L^2}|^2.$$

The first sum converges in the $\|\cdot\|_{L^2}$-norm.

2.2 Uniform Convergence of Sums

This subsection is devoted to the proof of Proposition 2.10. It contains the uniform convergence of certain sums, which will play a crucial role in the proof of Proposition 1.5 i.e. in the verification of (C1) and (C3) for the associated operator $\overline{A}$ and in the fix point argument. Notice that Lemma 2.2 also holds for the associated operator $\overline{A}$.

Proposition 2.10. *Let $\omega > 0$ be as in Lemma 2.2 and let $\lambda \in \Gamma^{\mathrm{I}} := \Gamma \cap \{z \in \mathbb{C} : \operatorname{Re} z \leq 2\omega\}$. Let $\eta := \lambda^2 + \overline{a}\lambda + b_{min}$ and let η_k be the ordered eigenvalues associated with the eigenfunctions $u_k,\ k \in \mathbb{N}$ of the operator $\overline{A}_0$. Then there is a constant $0 < C < \infty$ independent of $\lambda \in \Gamma^{\mathrm{I}}$, such that*

$$\sum_{k=1}^\infty \frac{1}{|\eta - \eta_k|} \leq C \quad \sum_{k=1}^\infty \frac{|\lambda^2|}{|\eta - \eta_k|^2} \leq C \quad \sum_{k=1}^\infty \frac{|\eta|}{|\eta - \eta_k|^2} \leq C.$$

and

$$\sum_{k=1}^\infty \frac{|\lambda^4|}{|\eta_k|}\frac{1}{|\eta - \eta_k|^2} \leq C \quad \sum_{k=1}^\infty \frac{|\lambda^2|}{|\eta_k|}\frac{|\eta|}{|\eta - \eta_k|^2} \leq C \quad \sum_{k=1}^\infty \frac{|\lambda^2|}{|\eta_k|}\frac{1}{|\eta - \eta_k|} \leq C.$$

Before proceeding to the proof of Proposition 2.10, some preparatory work is required. Let $\lambda = \mu_1 + i\ \mu_2 \in \Gamma^{\mathrm{I}}$. It follows that $\varepsilon^* + \nu \leq \mu_1 \leq 2\omega$ and $\mu_2 \in \mathbb{R}$ is arbitrary. From (15) we get $\eta = C_1 - \mu_2^2 + i\ C_2\mu_2$, where the variables $C_1 := \mu_1^2 + \overline{a}\mu_1 + b_{min}$ and $C_2 := 2\mu_1 + \overline{a}$ are independent of μ_2. From (16) we get $C_2 > 0$. From (18) and (19) it follows directly that

$$m := (2\omega)^2 + 2\overline{a}\omega + b_{min} = \max_{\lambda \in \Gamma^{\mathrm{I}}} C_1 \geq C_1 \geq (\varepsilon^*)^2 + \eta_1. \tag{21}$$

Without loss of generality, one only has to consider the case $\mu_2 \geq 0$ for reasons of symmetry. So $\eta := \eta(\mu_2)$ can be handled as a function of the variable μ_2. We will now define an auxiliary function ψ as $\psi : (-\infty, C_1) \to \mathbb{R}$ with $\psi(x) := \operatorname{Im}\lambda = \mu_2$, where μ_2 is the unique non negative real number such that $\operatorname{Re}(\eta(\mu_2)) = C_1 - \mu_2^2 = x$. A short calculation shows that the function ψ is given by

$$\psi\left(x\right) = \mu_2 = \sqrt{C_1 - x} \tag{22}$$

and is thus monotone decreasing. With the auxiliary function ψ, we get the following representation of the function η

$$\eta(\psi(x)) = x + i\, C_2 \sqrt{C_1 - x}. \tag{23}$$

With the help of (22) and (23), $|\eta|$ can be estimated in the following way (See also Figure 1).

Lemma 2.11. *Let $n \in \mathbb{N}$. If $-\frac{\pi^2}{L^2}(n+1)^2 < \operatorname{Re}(\eta) \leq -\frac{\pi^2}{L^2}n^2$, then*

$$\sqrt{\frac{\pi^4}{L^4}n^4 + C_2^2(C_1 + \frac{\pi^2}{L^2}n^2)} \leq |\eta| < \sqrt{\frac{\pi^4}{L^4}(n+1)^4 + C_2^2(C_1 + \frac{\pi^2}{L^2}(n+1)^2)}.$$

So it is also valid that $C_2\sqrt{C_1 + \frac{\pi^2}{L^2}n^2} \leq |\operatorname{Im}\eta| < C_2\sqrt{C_1 + \frac{\pi^2}{L^2}(n+1)^2}$.

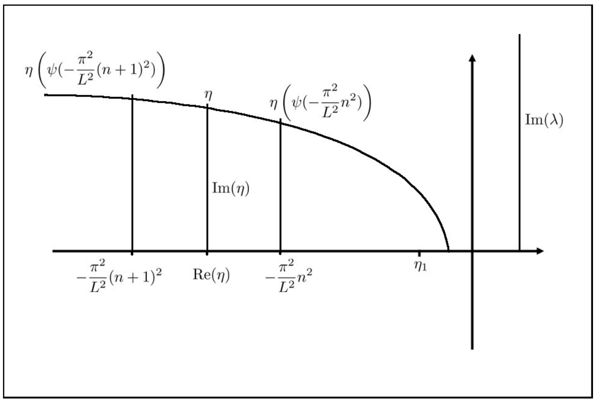

Figure 1: Illustration of Lemma 2.11

In order to prove the uniform convergence, we will split the sums into two parts

$$\sum_{k=1}^{\infty} = \sum_{k=1}^{\tilde{n}} + \sum_{k=\tilde{n}+1}^{\infty}.$$

We determine now the index $\tilde{n} \in \mathbb{N}$. Let $\tilde{n}_2 \in \mathbb{N}$ be the smallest $n \in \mathbb{N}$ for which

$$n^{-2}\eta_1 + \frac{\pi^2}{L^2} \geq \frac{\pi^2}{2L^2} \tag{24}$$

is satisfied. It follows directly with (21) that for all $\mathbb{N} \ni n \geq \tilde{n}_2$

$$C_1 + n^2\frac{\pi^2}{L^2} \geq \eta_1 + n^2\frac{\pi^2}{L^2} \geq n^2\frac{\pi^2}{2L^2} \geq \frac{\pi^2}{2L^2} \tag{25}$$

holds. The index $\tilde{n} \in \mathbb{N}$ is defined as

$$\tilde{n} = \max(\tilde{n}_1,\ \tilde{n}_2), \tag{26}$$

where $\tilde{n}_1$ is taken as in Corollary 2.8. Notice that $\tilde{n}$ only depends on the function b.

Convention. *In this section, C is designated to be a generic constant $0 < C < \infty$, which is independent of $\lambda \in \Gamma^{\mathrm{I}}$.*

We will proof now the uniform convergence of the first sum i.e.

Lemma 2.12. *Let $\lambda \in \Gamma^{\mathrm{I}}$. Then $\sum_{k=1}^{\infty} \frac{1}{|\eta-\eta_k|} \leq C < \infty$, where the constant C is independent of λ.*

Proof. As mentioned before, we divide the sum into two parts

$$\sum_{k=1}^{\infty} \frac{1}{|\eta-\eta_k|} = \sum_{k=1}^{\tilde{n}} \frac{1}{|\eta-\eta_k|} + \sum_{k=\tilde{n}+1}^{\infty} \frac{1}{|\eta-\eta_k|}.$$

By an elementary estimation one can control the first term $\sum_{k=1}^{\tilde{n}} \frac{1}{|\eta-\eta_k|}$ without any problems. Therefore we put our attention on the more interesting term $\sum_{k=\tilde{n}+1}^{\infty} \frac{1}{|\eta-\eta_k|}$. Without loss of generality we assume that $-\frac{\pi^2}{L^2}\,(n)^2 \geq \mathrm{Re}(\eta) > -\frac{\pi^2}{L^2}\,(n+1)^2$ with $\mathbb{N} \ni n \geq \tilde{n}+3$. We divide again the second term in the following way

$$\sum_{k=\tilde{n}+1}^{\infty} \frac{1}{|\eta-\eta_k|} = \underbrace{\sum_{k=\tilde{n}+1}^{n-2} \frac{1}{|\eta-\eta_k|}}_{=:\mathrm{I}} + \underbrace{\sum_{k=n-1}^{n+1} \frac{1}{|\eta-\eta_k|}}_{=:\mathrm{II}} + \underbrace{\sum_{k=n+2}^{\infty} \frac{1}{|\eta-\eta_k|}}_{\mathrm{III}}.$$

It is also helpful to compare the following calculations with Figure 2 to get an intuitive understanding of

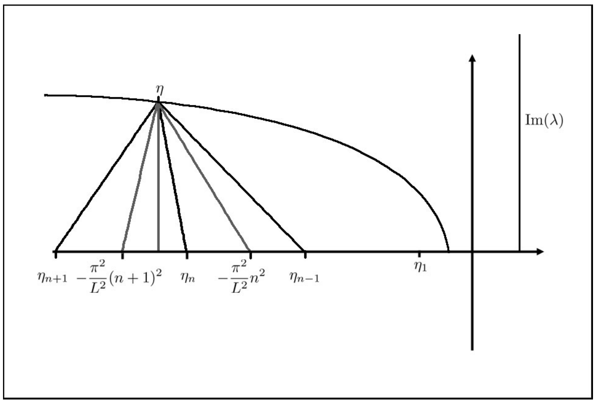

Figure 2: Estimation of $|\eta - \eta_n|$

the estimation. Let us now estimate the summand I.

$$\mathrm{I} \leq \sum_{k=\tilde{n}+1}^{n-2} \frac{1}{|\,\mathrm{Re}(\eta-\eta_k)|} \leq \sum_{k=\tilde{n}+1}^{n-2} \frac{1}{|\frac{\pi^2}{L^2}(k+1)^2 - \frac{\pi^2}{L^2}n^2|} = \frac{L^2}{\pi^2} \sum_{i=1}^{n-\tilde{n}-2} \frac{1}{|(n-i)^2 - n^2|} \leq C.$$

We will now estimate the summand II. We use $n > \tilde{n}$, (16) and (25) to obtain

$$\mathrm{II} \leq \sum_{k=n-1}^{n+1} \frac{1}{|\,\mathrm{Im}(\eta)|} \leq \sum_{k=n-1}^{n+1} \frac{1}{|C_2(C_1 + \frac{\pi^2}{L^2}n^2)^{\frac{1}{2}}|} \leq \sum_{k=n-1}^{n+1} \frac{1}{C_2|(\frac{\pi^2}{2L^2})^{\frac{1}{2}}|} \leq C.$$

We will now estimate the summand III.

$$\mathrm{III} \leq \sum_{k=n+2}^{\infty} \frac{1}{|\operatorname{Re}(\eta-\eta_k)|} \leq \sum_{k=n+2}^{\infty} \frac{1}{|\frac{\pi^2}{L^2}k^2 - \frac{\pi^2}{L^2}(n+1)^2|} \leq \frac{L^2}{\pi^2}\sum_{i=1}^{\infty}\frac{1}{i^2} \leq C.$$

Overall we have shown that

$$\sum_{k=\tilde{n}+1}^{\infty} \frac{1}{|\eta-\eta_k|} \leq C,$$

where the constant C is independent of $\lambda \in \Gamma^{\mathrm{I}}$. □

The proof of the uniform convergence of the second sum $\sum_{k=1}^{\infty} \frac{|\lambda^2|}{|\eta-\eta_k|^2}$ is roughly the same as the proof for the first sum. Hence, we will omit it. Because $|\eta| = |\lambda^2+\bar{a}\lambda+b_{min}|$, the uniform convergence of the third sum $\sum_{k=1}^{\infty} \frac{|\eta|}{|\eta-\eta_k|^2}$ can be reduced to the uniform convergence of the second sum $\sum_{k=1}^{\infty} \frac{|\lambda|^2}{|\eta-\eta_k|^2}$. We will now show the uniform convergence of the 4th sum of Proposition 2.10. It is the hardest sum to control and the estimation is sophisticated.

Lemma 2.13. *Let $\lambda \in \Gamma^{\mathrm{I}}$. Then $\sum_{k=1}^{\infty} \frac{|\lambda^4|}{|\eta_k||\eta-\eta_k|^2} \leq C < \infty$, where the constant C is independent of λ.*

Proof. Let $\lambda = \mu_1 + i\mu_2 \in \Gamma^{\mathrm{I}}$. Because $\operatorname{Re}\lambda$ is bounded it is sufficient to estimate $\Sigma := \sum_{k=1}^{\infty} \frac{\mu_2^4}{|\eta_k||\eta-\eta_k|^2}$ uniformly. We assume without loss of generality that

$$\mu_2^2 \geq 4\left(\frac{\pi^2}{L^2}(\tilde{n}+3)^2 + m\right) \geq 4\left(\frac{\pi^2}{L^2}(\tilde{n}+1)^2 + m\right) \geq \frac{\pi^2}{L^2}(\tilde{n}+1)^2 + m. \tag{27}$$

We split up Σ into $\sum_{k=1}^{\tilde{n}} + \sum_{k=\tilde{n}+1}^{\infty} =: \Sigma^{\mathrm{I}} + \Sigma^{\mathrm{II}}$, where the integer $\tilde{n}$ is defined as in (26). We put our attention on the sum Σ^{I}. From (27) it follows from straightforward calculation that

$$0 < 2\mu_2^{-2}\left(\frac{\pi^2}{L^2}(\tilde{n}+1)^2 + m\right) \leq \frac{1}{2}. \tag{28}$$

By using (27) and (20) a direct forward calculation also results in (see also Figure 3)

$$\operatorname{Re}\eta = C_1 - \mu_2^2 \leq m - \mu_2^2 \leq -\frac{\pi^2}{L^2}(\tilde{n}+1)^2 \leq \operatorname{Re}\eta_{\tilde{n}}. \tag{29}$$

So one can estimate for all $k \in \{1, \cdots, \tilde{n}\}$

$$|\eta-\eta_k|^2 \geq (\operatorname{Re}(\eta-\eta_{\tilde{n}}))^2 \geq \left(C_1 - \mu_2^2 + \frac{\pi^2}{L^2}(\tilde{n}+1)^2\right)^2 \geq \mu_2^4 - 2\mu_2^2\left(\frac{\pi^2}{L^2}(\tilde{n}+1)^2 + m\right). \tag{30}$$

From (27) follows $\mu_2^4 - 2\mu_2^2\left(\frac{\pi^2}{L^2}(\tilde{n}+1)^2 + m\right) > 0$. Finally by using (28) and (30) we obtain

$$\Sigma^{\mathrm{I}} \leq \frac{C}{\eta_1}\sum_{k=1}^{\tilde{n}} \frac{\mu_2^4}{\mu_2^4 - 2\mu_2^2\left(\frac{\pi^2}{L^2}(\tilde{n}+1)^2 + m\right)} \leq C\sum_{k=1}^{\tilde{n}} \frac{1}{1-\frac{1}{2}} \leq C.$$

We put now our attention on Σ^{II}. From (27) we also get $\operatorname{Re}\eta = C_1 - \mu_2^2 \leq m - \mu_2^2 \leq -\frac{\pi^2}{L^2}(\tilde{n}+3)^2$. Thus we can assume, without loss of generality, that there is a $\mathbb{N} \ni n \geq \tilde{n}+3$, such that

$$-\frac{\pi^2}{L^2}(n+1) < \operatorname{Re}\eta(\mu_2) \leq -\frac{\pi^2}{L^2}n.$$

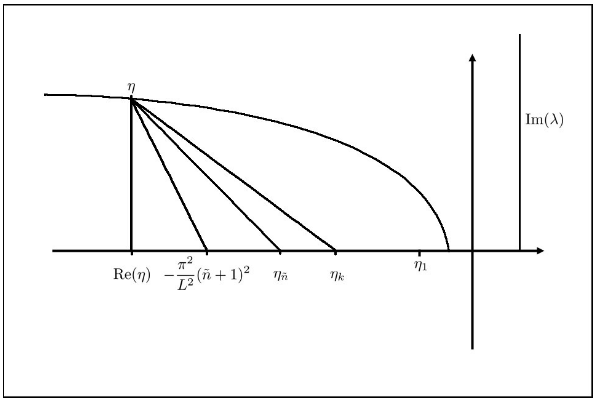

Figure 3: Illustration of the situation in (29)

By using the monotony of the auxiliary function ψ and Lemma 2.11 we get

$$C_1 + \frac{\pi^2}{L^2}\, n^2 \leq \mu_2^2 < C_1 + \frac{\pi^2}{L^2}(n+1)^2 \leq m + \frac{\pi^2}{L^2}(n+1)^2 \leq Cn^2.$$

Thus we can estimate Σ^{II} in the following manner

$$\Sigma^{\text{II}} \leq C \left[\underbrace{\sum_{k=\tilde{n}+1}^{n-2} \frac{n^4}{|\eta_k||\eta-\eta_k|^2}}_{=:\text{I}} + \underbrace{\sum_{k=n-1}^{n+1} \frac{n^4}{|\eta_k||\eta-\eta_k|^2}}_{=:\text{II}} + \underbrace{\sum_{k=n+1}^{\infty} \frac{n^4}{|\eta_k||\eta-\eta_k|^2}}_{=:\text{III}} \right].$$

We will estimate the term II by the following calculation using (16) and (25)

$$\text{II} \leq C \sum_{k=n-1}^{n+1} \frac{n^4}{(n-1)^2(\text{Im}(\eta))^2} \leq C \sum_{k=n-1}^{n+1} \frac{n^2}{(n-1)^2} \frac{n^2}{C_2^2(C_1 + \frac{\pi^2}{L^2} n^2)} \leq C.$$

In the estimation of the term I one encounters difficulties. If $n \to \infty$, a pole arises in the denominator (compare with the next calculation). To solve this problem we estimate I and III together. There is an intuitive idea behind this proceeding. The sum III converges so nice that one can use free capacities to compensate the difficulties of the term I. The term $\text{I} + \text{III}$ is estimated by the following calculation

$$\begin{aligned}
\text{I} + \text{III} &\leq C \sum_{k=\tilde{n}+1}^{n-2} \frac{n^4}{k^2(n^2-(k+1)^2)^2} + C \sum_{k=n+2}^{\infty} \frac{n^4}{k^2(k^2-(n+1)^2)^2} \\
&\leq C \sum_{j=1}^{n-\tilde{n}-2} \underbrace{\frac{n^4}{(n-j-1)^2(j^2-2nj)^2}}_{=:T_j^1} + C \sum_{j=1}^{\infty} \underbrace{\frac{n^4}{(n+j+1)^2(j^2+2jn)^2}}_{=:T_j^2} \\
&\leq C \left[\sum_{j=1}^{n-\tilde{n}-2} T_j^1 + T_j^2 - \frac{1}{j^2} \right] + C \sum_{j=1}^{n-\tilde{n}-2} \frac{1}{j^2} + C \sum_{j=n-\tilde{n}-1}^{\infty} T_j^2. \qquad (31)
\end{aligned}$$

As one easily sees, $T_j^2 \le \frac{1}{j^2}$ holds. Hence it only remains to estimate the term $\sum_{j=1}^{n-\tilde{n}-2} T_j^1 + T_j^2 - \frac{1}{j^2}$. Before proceeding we need some auxiliary results. Let

$$A := \sum_{0\le r+s\le 4} \alpha_{r,s} n^r j^s := (n-j-1)^2(2n-j)^2 \quad \text{and} \quad B := \sum_{0\le r+s\le 4} \beta_{r,s}\, n^r j^s := (n+j+1)^2(j+2n)^2.$$

By a straightforward calculation one gets

$$\alpha_{4,0} = 4, \quad \alpha_{3,1} = -12, \quad \alpha_{3,0} = -8 \quad \text{and} \quad \beta_{4,0} = 4, \quad \beta_{3,1} = 12, \quad \beta_{3,0} = 8.$$

Thus we can estimate the amount of the free capacity as

$$T_j^2 - \frac{1}{j^2} = \frac{1}{j^2}\left[\frac{n^4 - (n+j+1)^2(j+2n)^2}{(n+j+1)^2(j+2n)^2}\right] \le \frac{1}{j^2}\left[\frac{-3n^4 - 12n^3 j - 8n^3}{(n+j+1)^2(j+2n)^2}\right].$$

The next calculation shows that terms of lower order can be neglected in the estimation. Let

$$D := A\,B = (n-j-1)^2(j-2n)^2(n+j+1)^2(j+2n)^2.$$

Then $D > 0$ for $j \in \{1, \ldots, n-\tilde{n}-2\}$. If $r \le 2$ and $r+s \le 4$, then

$$\sum_{j=1}^{n-\tilde{n}-2} \frac{n^4}{j^2}\frac{n^r j^s}{D} \le \sum_{j=1}^{n-\tilde{n}-2} \frac{n^6}{D} \le \sum_{j=1}^{n-\tilde{n}-2} \frac{1}{(n-j-1)^2} = \sum_{k=\tilde{n}+1}^{n-2} \frac{1}{k^2} \le C.$$

Now the preparatory work is completed and we can return to the estimation of $\mathrm{I} + \mathrm{III}$.

$$\begin{aligned}
&\sum_{j=1}^{n-\tilde{n}-2} T_j^1 + T_j^2 - \frac{1}{j^2} \le \sum_{j=1}^{n-\tilde{n}-2} \frac{1}{j^2}\left[\frac{n^4}{A} + \frac{-3n^4 - 12n^3 j - 8n^3}{B}\right] \\
&\le \sum_{j=1}^{n-\tilde{n}-2} \frac{1}{j^2}\left[\frac{n^4(\beta_{4,0}n^4 + \beta_{3,1}n^3 j + \beta_{3,0}n^3) + (-3n^4 - 12n^3 j - 8n^3)(\alpha_{4,0}n^4 + \alpha_{3,1}n^3 j + \alpha_{3,0}n^3)}{D}\right] \\
&\quad + C \sum_{0\le r+s\le 4, r\le 2} [|\beta_{r,s}| + |\alpha_{r,s}|] \sum_{j=1}^{n-\tilde{n}-2} \frac{n^6}{D} \\
&\le \sum_{j=1}^{n-\tilde{n}-2} \frac{1}{j^2}\left[\frac{n^4(4n^4 + 12n^3 j + 8n^3) - 3n^4(4n^4 - 12n^3 j - 8n^3) - (12n^3 j + 8n^3)4n^4}{D}\right] \\
&\quad + \sum_{j=1}^{n-\tilde{n}-2} \frac{1}{j^2}\frac{(12n^3 j + 8n^3)(\alpha_{3,1}n^3 j + \alpha_{3,0}n^3)}{D} + C \\
&\le \sum_{j=1}^{n-\tilde{n}-2} \frac{1}{j^2}\left[\frac{-2n^4 4n^4 + 4n^4(12n^3 j + 8n^3) - (12n^3 j + 8n^3)4n^4}{D}\right] + C \le C.
\end{aligned}$$

Recalling (31) we get $\mathrm{I} + \mathrm{III} \le C$, which completes the proof. □

The uniform convergence of the fifth sum is easily reduced to the uniform convergence of the fourth sum. Because

$$\sum_{k=1}^{\infty} \frac{|\lambda^2|}{|\eta_k|}\frac{1}{|\eta - \eta_k|} = \sum_{k=1}^{\infty} \frac{|\lambda^2|}{|\eta_k|}\frac{|\eta - \eta_k|}{|\eta - \eta_k|^2} \le \sum_{k=1}^{\infty} \frac{|\lambda^2|}{|\eta_k|}\frac{|\eta|}{|\eta - \eta_k|^2} + \sum_{k=1}^{\infty} \frac{|\lambda^2|}{|\eta - \eta_k|^2} \le C,$$

we have reduced the uniform convergence of the sixth sum to the uniform convergence of the second and fifth sum. Therefore, the proof of Proposition 2.10 is finished.

2.3 Exponential Stability of the Associated System

This subsection is devoted to the proof of exponential stability of the associated linear wave equation i.e. we will show the following statement

Proposition 2.14. *The operator $\overline{A}$ satisfies the conditions of exponential stability*

(C1) $\quad \sup\left\{\operatorname{Re}\lambda \;:\; \lambda \in \sigma(\overline{A})\right\} \;<\; 0$

(C3) $\quad \sup\left\{\|R_\lambda(\overline{A})\| \;:\; \operatorname{Re}\lambda \geq \nu + \varepsilon^*\right\} \;<\; \infty \quad \forall\, \varepsilon^* > 0,$

where ν is as in Definition 2.5.

First, we will proof that $R\|_\lambda(\overline{A})\|$ is uniformly bounded in $\lambda \in \Gamma^{\mathrm{I}}$.

Lemma 2.15. *Let $\lambda \in \Gamma^{\mathrm{I}}$, then $\|R_\lambda(\overline{A})\| \leq C$, where the constant is independent of $\lambda \in \Gamma^{\mathrm{I}}$.*

Proof. Let $\lambda \in \Gamma^{\mathrm{I}}$ and $F = (f_1, f_2)^T \in \mathcal{H}$ be arbitrary. From Proposition 2.3 and 2.6 we know, that the equation $\lambda U - \overline{A}U = F$ has a unique solution $R_\lambda(\overline{A})F = (p, q)^T$, where p and q are given as

$$q = \lambda p - f_1 \quad \text{and} \quad p = R_\eta(\overline{A}_0)(f_2 + (\lambda + \overline{a})f_1).$$

We know that$\|R_\lambda(\overline{A})F\|^2_{\mathcal{H}} = \|\nabla p\|^2_{L^2} + \|q\|^2_{L^2} = \|\nabla p\|^2_{L^2} + \|\lambda p - f_1\|^2_{L^2}$. By applying Lemma 2.16 the statement is obtained. □

Lemma 2.16. *Let p and F be defined as in the proof of Lemma 2.15. Then there is a constant $0 < C < \infty$ independent of $\lambda \in \Gamma^{\mathrm{I}}$ such that $\|\nabla p\|^2_{L^2} \leq C\|F\|^2_{\mathcal{H}}$ and $\|\lambda p\|^2_{L^2} \leq C\|F\|^2_{\mathcal{H}}$.*

Proof. Let $g := f_2 + (\lambda + \overline{a})f_1$ and $p = R_\eta(\overline{A}_0)(g)$. Using Proposition 2.9 and Proposition 2.10 we can estimate $\|\nabla p\|_{L^2}$ as

$$\begin{aligned}
\|\nabla p\|^2_{L^2} &= \left\langle p, -\overline{A}_0 p\right\rangle_{L^2} + \langle p, (b_{min} - b)p\rangle_{L^2} \leq \left\langle p, -\overline{A}_0 p\right\rangle_{L^2} = \langle p, g\rangle_{L^2} + \langle p, -\eta p\rangle_{L^2} \\
&\leq \sum_{k=1}^{\infty} \underbrace{\left[\frac{1}{|\eta - \eta_k|} + \frac{|\eta|}{|\eta - \eta_k|^2}\right]}_{=:\chi_k} |\left\langle f_2 + (\lambda + \overline{a})f_1, u_k\right\rangle_{L^2}|^2 \\
&\leq C\|f_2\|^2_{L^2} + \underbrace{\sum_{k=1}^{\infty} |\lambda^2|\; \chi_k\; |\left\langle f_1, u_k\right\rangle_{L^2}|^2}_{=:\Sigma} + C\,\overline{a}^2\; \|f_1\|^2_{L^2}
\end{aligned} \tag{32}$$

A proposition of Hilbert and Courant (see page 288 in [4]) states that there is a constant C, such that for all $k \in \mathbb{N}$ holds $|u_k|_{L^\infty} \leq C$. Thus one can derive

$$|\left\langle f_1, (b_{min} - b)\frac{u_k}{\sqrt{\eta_k}}\right\rangle_{L^2}|^2 \leq C\|f_1\|^2_{L^2} \quad \text{and} \quad \|\nabla \frac{u_k}{\sqrt{\eta_k}}\|^2_{L^2} \leq C, \tag{33}$$

where $0 < C$ is independent of $k \in \mathbb{N}$. Hence we can estimate Σ by using Proposition 2.10 and (33) as

$$\begin{aligned}
\Sigma &\leq \sum_{k=1}^{\infty} \frac{|\lambda^2|}{|\eta_k|^2}\; \chi_k\; |\left\langle f_1, \partial_{xx} u_k\right\rangle_{L^2}|^2 + \sum_{k=1}^{\infty} \frac{|\lambda^2|}{|\eta_k|}\; \chi_k\; |\left\langle f_1, (b_{min} - b)\frac{u_k}{\sqrt{\eta_k}}\right\rangle_{L^2}|^2 \\
&\leq \sum_{k=1}^{\infty} \frac{|\lambda^2|}{|\eta_k|}\; \chi_k\; \|\nabla f_1\|^2_{L^2} \|\nabla \frac{u_k}{\sqrt{\eta_k}}\|^2_{L^2} + C\sum_{k=1}^{\infty} \frac{|\lambda^2|}{|\eta_k|}\; \chi_k\; \|f_1\|^2_{L^2} \leq C\|\nabla f_1\|^2_{L^2}.
\end{aligned} \tag{34}$$

By assembling the estimates (32) and (34) we obtain

$$\|\nabla p\|_{L^2}^2 \le C(\|\nabla f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2) = C\|F\|_{\mathcal{H}}^2,$$

where the constant $C > 0$ is independent of $\lambda \in \Gamma^{\mathrm{I}}$. The estimation of $\|\lambda p\|_{L^2}^2$ is very similar to the estimation of $\|\nabla p\|_{L^2}^2$. Thus we get (compare with (32) and (34))

$$\begin{aligned}\|\lambda p\|_{L^2}^2 &= \sum_{k=1}^{\infty} \frac{|\lambda^2|}{|\eta - \eta_k|^2} \,|\, \langle f_2 + (\lambda + \overline{a}) f_1, u_k \rangle_{L^2} \,|^2 \\ &\le C\|f_2\|_{L^2}^2 + \sum_{k=1}^{\infty} \frac{|\lambda^4|}{|\eta - \eta_k|^2} \,|\, \langle f_1, u_k \rangle_{L^2} \,|^2 + \overline{a}^2 C \|f_1\|_{L^2}^2 \\ &\le C\|f_2\|_{L^2}^2 + C\|\nabla f_1\|_{L^2}^2 \le C\|F\|_{\mathcal{H}}^2,\end{aligned}$$

where the constant $C > 0$ is again independent of $\lambda \in \Gamma^{\mathrm{I}}$. □

By analyzing the proof of Lemma 2.15 and regarding Lemma 2.2, that is also valid for the operator $\overline{A}$, one sees that (C1) is satisfied. By using a standard argument for dissipative operators (see also Lemma 2.2) it is possible to expand the statement of Lemma 2.15 to the whole area Γ. Therefore (C3) is also satisfied and the proof of Proposition 2.14 is complete.

2.4 The Fixed Point Argument

In this section we proof Proposition 1.5 by using a fixed point argument. Let $\lambda \in \Gamma$ and $F = (f_1, f_2)^T \in \mathcal{H}$ be arbitrary. We will now define the map $\Phi(\lambda, F) : \mathcal{H} \to \mathcal{H}$, which is supposed to have a fixed point. Let $V = (v_1, v_2)^T \in \mathcal{H}$. Then $\Phi(\lambda, F)(V) := U \in D(A)$, where U is the unique solution of the equation

$$\lambda U - \overline{A} U = F - \begin{pmatrix} 0 \\ (\overline{a} - a) v_2 \end{pmatrix}.$$

The next lemma is verified by a straightforward calculation and shows why a fixed point of the map $\Phi(\lambda, F)$ would be very useful.

Lemma 2.17. *Let $U \in D(A)$ be a unique fixed point of the map $\Phi(\lambda, F)$, then U is the unique solution of the equation $\lambda U - AU = F$.*

The next statement gives an answer to the question when the map $\Phi(\lambda, F)$ has a unique fixed point.

Lemma 2.18. *Let $\lambda \in \Gamma^{\mathrm{I}}$, then there exists $\varepsilon > 0$ independent of λ such that if $\|\overline{a} - a\|_{L^2} < \varepsilon$, then the map $\Phi(\lambda, F)$ is contracting. In this case, by using the Banach‘s fixed point theorem, we know that the map $\Phi(\lambda, F)$ has a unique fixed point.*

Proof. Let $V^1 = (v_1^1, v_2^1)^T \in \mathcal{H}$ and $V^2 = (v_1^2, v_2^2)^T \in \mathcal{H}$. One has to show the property of contraction for the map $\Phi(\lambda, F) : \mathcal{H} \to \mathcal{H}$. Let $U^1 = (p_1^1, p_2^1)^T := \Phi(\lambda, F)(V^1) \in D(A)$ and $U^2 := (p_1^2, p_2^2)^T := \Phi(\lambda, F)(V^2) \in D(A)$. Then $(U^1 - U^2)$ is the unique solution of the equation

$$\lambda \tilde{U} - \overline{A} \tilde{U} = \begin{pmatrix} 0 \\ (\overline{a} - a)(v_2^1 - v_2^2) \end{pmatrix}.$$

Thus, by using Proposition 2.3 and Proposition 2.6 it follows that

$$U^1 - U^2 = R_\lambda(\overline{A}) \left[\begin{pmatrix} 0 \\ (\overline{a} - a)(v_2^1 - v_2^2) \end{pmatrix} \right] = \begin{pmatrix} p \\ \lambda p \end{pmatrix}$$

holds and p is determined as $p = R_\eta(\overline{A}_0)[(\overline{a}-a)(v_2^1 - v_2^2)]$. Using Lemma 2.19 we obtain

$$\|U^1 - U^2\|_{\mathcal{H}}^2 = \|\nabla p\|_{L^2}^2 + \|\lambda p\|_{L^2}^2 \le C\|\overline{a}-a\|_{L^2}^2 \|V^1 - V^2\|_{\mathcal{H}}^2,$$

where the constant $0 < C < \infty$ is independent of $\lambda \in \Gamma^{\mathrm{I}}$. If we take $\|\overline{a}-a\|_{L^2} < \frac{1}{C} =: \varepsilon$, then the map $\Phi(\lambda, F)$ is contracting. □

Lemma 2.19. *Let p, V^1 and V^2 be defined as in the proof of Lemma 2.18. Then there is a constant $0 < C < \infty$ independent of $\lambda \in \Gamma^{\mathrm{I}}$ such that*

$$\|\nabla p\|_{L^2}^2 \le C\|\overline{a}-a\|_{L^2}^2 \|V^1 - V^2\|_{\mathcal{H}}^2 \quad \textit{and} \quad \|\lambda p\|_{L^2}^2 \le C\|\overline{a}-a\|_{L^2}^2 \|V^1 - V^2\|_{\mathcal{H}}^2.$$

Proof. The Estimation of $\|\nabla p\|_{L^2}$ and $\|\lambda p\|_{L^2}$ is performed in the same way as in the proof of Lemma 2.16. One only has to regard that the function g is now given by $g := (\overline{a}-a)(v_2^1 - v_2^2)$ and that $|\left\langle(\overline{a}-a)(v_2^1 - v_2^2), u_k\right\rangle_{L^2}|^2 \le C\|\overline{a}-a\|_{L^2}^2 \|V^1 - V^2\|_{\mathcal{H}}^2$ is satisfied by using the proposition of Courant and Hilbert mentioned before. □

We assume now that $\|\overline{a}-a\|_{L^2} < \varepsilon$, where $\varepsilon > 0$ is taken as in Lemma 2.18. By regarding Lemma 2.2, Lemma 2.17 and Lemma 2.18 one sees that condition (C1) is satisfied for the operator A. To verify condition (C3) we have to estimate the norm of the resolvent $R_\lambda(A)$ for $\lambda \in \Gamma$.

Lemma 2.20. *Let $\lambda \in \Gamma^{\mathrm{I}}$ and let $\|\overline{a}-a\|_{L^2} < \varepsilon$, where $\varepsilon > 0$ is taken as in Lemma 2.18. Then there exists a constant $0 < C < \infty$ independent of λ, such that $\|R_\lambda(A)\| \le C$.*

Proof. Let $\lambda \in \Gamma^{\mathrm{I}}$ and $F \in \mathcal{H}$ be arbitrary. By Lemma 2.18 the map $\Phi(\lambda, F)$ has a unique fixed point $U \in D(A)$. Let $\tilde{U} := \Phi(\lambda, F)(0) = R_\lambda(\overline{A})F$. Let $0 \le d < 1$ be the constant of the property of contraction of the map $\Phi(\lambda, F)$. From Lemma 2.15 we know that there is a constant $0 < C < \infty$ independent of $\lambda \in \Gamma^{\mathrm{I}}$, such that $\|R_\lambda(\overline{A})\| \le C$. Thus we obtain

$$\|U\|_{\mathcal{H}} - \|\tilde{U}\|_{\mathcal{H}} \le \|\Phi(\lambda, F)(0) - \Phi(\lambda, F)(U)\|_{\mathcal{H}} \le d\|0 - U\|_{\mathcal{H}}.$$

So it follows that

$$\|U\|_{\mathcal{H}} \le \frac{1}{1-d}\|\tilde{U}\|_{\mathcal{H}} = \frac{1}{1-d}\|R_\lambda(\overline{A})F\|_{\mathcal{H}} \le \frac{C}{1-d}\|F\|_{\mathcal{H}},$$

where the constant $0 < C < \infty$ is independent of λ. □

As in section 2.3 we can extend the statement of the last lemma to the whole area of parameters Γ with the help of Lemma 2.2. Therefore (C3) is also satisfied for the Operator A. As a consequence the linear wave equation (1) is exponentially stable. The statement about a possible decay rate follows directly from the definition of Γ. Thus the proof of Proposition 1.5 is complete.

3 Non-Linear Case

This section is devoted to the proof of Proposition 1.6. The argumentation is very similar to the one that Racke and Muñoz Rivera used in [17] for their statement in the case $b = 0$. After deducing a high energy estimate and a weighted a priori estimate the local solution of (4) is extended to a global solution by a continuation argument (compare for example [22] or [13]). The *smallness* of the global solution follows automatically from the weighted a priori estimate. We assume, for the whole section, that the hypotheses of Proposition 1.6 are satisfied.

Remark 3.1. *Let the operator* $\overline{A}_{d_0} : L^2 \supset H_0^1 \cap H^2 =: D(\overline{A}_{d_0}) \to L^2$ *be defined as* $\overline{A}_{d_0} p := (d_0 \partial_{xx} + b_{min} - b(x))p$*, where* $p \in D(\overline{A}_{d_0})$*. Let* η_1 *and* $\tilde{\eta}_1$ *be the biggest eigenvalue of the operator* $\overline{A}_{d_0}$*, respectively* $\tilde{A}_{d_0}$*. Then* $0 > \tilde{\eta}_1$ *is equivalent to* $b_{min} > \tilde{\eta}_1 + b_{min} = \eta_1$*. Therefore, we assume that* $b_{min} > \eta_1$ *is satisfied.*

We translate now the problem (4) into the language of operator theory. The operator A is defined as

$$A : \mathcal{H} \supset D(A) \to \mathcal{H} \quad \text{with} \quad A \begin{pmatrix} p \\ q \end{pmatrix} := \begin{pmatrix} O & \mathrm{Id} \\ d_0 \partial_{xx} - b(x) & -a(x) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix},$$

where $D(A) := (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ and $(p, q)^T \in D(A)$. Let $U := (u, u_t)^T$. By using (6) one can rewrite (4) as

$$U_t - AU = U_t - \begin{pmatrix} O & \mathrm{Id} \\ d_0 \partial_{xx} - b(x) & -a(x) \end{pmatrix} U = \begin{pmatrix} 0 \\ c(u_x) u_{xx} \end{pmatrix} =: F(U_x, U_{xx}), \tag{35}$$

with initial condition $U(t = 0) = (u_0, u_1)^T =: U_0$. The operator A generates a C_0-semigroup, thus for $F = 0$ the solution U of (35) is given by $U(t) = e^{tA} U_0$. The local existence of a solution of (4) and of (35) respectively is obtained as in [17], because the terms $a(x)u_t$ and $b(x)u$ are of lower order (compare also [7] or page 97 in [13]). Alternatively one can use the local existence theorem stated in the article of Shibata and Tsutsumi [24], where the conditions on the regularity can be improved, as one sees, for example, in the book of Kato [14]. Thus we have

Proposition 3.2. *There is a* $T = T(\|u_0, u_1\|_{H^4 \times H^3}) > 0$*, which depends continuously on* $\|u_0, u_1\|_{H^4 \times H^3}$*, such that (4) has a unique local solution*

$$u \in \bigcap_{k=0}^{3} C^k \left([0, T], H^{4-k} \cap H_0^1\right) \cap C^4 \left([0, T], L^2\right).$$

We assume without loss of generality that the local solution $U \in \mathcal{H}$ of (35) for arbitrary F is small enough a priori i.e. $\|U\|_{\mathcal{H}^3} < \delta < 1$ is sufficiently small such that

$$(d_0 + c(u_x) = \sigma'(u_x) \geq \frac{d_0}{2} > 0. \tag{36}$$

We are working with the following *energies*.

Definition 3.3. *Let* $U := (p, q)^T \in \mathcal{H}$*. The* canonical energy $\tilde{E}_{d_0}$ *of the equation (35) is defined as*

$$E_{d_0}[U] := d_0 \|\partial_x p\|_{L^2}^2 + \|q\|_{L^2}^2 + \langle bp, p \rangle.$$

Let $s \in 0, 1, 2, 3$ *and let* U *be the local solution of (35) from Proposition 3.2. Then the* high energy norm $\| \cdot \|_{\mathcal{H}^s}$ *and the* canonical high energy $E_{d_0}^s$ *of* U *is defined as*

$$\|U\|_{\mathcal{H}^s} := \left(\sum_{l=0}^{s} \|(\partial_x)^{l+1} p\|_{L^2}^2 + \|(\partial_x)^l q\|_{L^2}^2 \right)^{\frac{1}{2}} \quad \textit{and} \quad E_{d_0}^s[U] := \frac{1}{2} \sum_{s=0}^{3} E_{d_0}[\partial_t^s U]. \tag{37}$$

In the proof of Proposition 1.6 there is just one important difference to the argumentation outlined in [17]. It appears in the proof of the high energy estimate. By reproducing the proof of [17] step by step one can not derive the high energy estimate for the high energy norm $\mathcal{H}^3$ but for the canonical high energy $E_{d_0}^3$ i.e. one gets the statement

Lemma 3.4 (High Energy Estimate). *Let $U \in \mathcal{H}$ be the local solution of (35) defined on $[0,T]$, $T > 0$. Then there are constants $0 < C_1, C_2$, not depending on U_0 or T, such that for all $t \in [0,T]$:*

$$E^3_{d_0}[U(t)] \leq C_1 E^3_{d_0}[U_0]\; e^{2a^-_\infty t} e^{C_2 \int_0^t (\|U(r)\|_{\mathcal{H}^2} + \|U(r)\|^2_{\mathcal{H}^2} + \|U(r)\|^3_{\mathcal{H}^2}) dr}. \tag{38}$$

In Proposition 3.6 we will proof the equivalence of $\mathcal{H}^3$ and $E^3_{d_0}$. As a consequence the *original* high energy estimate as in [17] is obtained. The rest of the proof of Proposition 1.6 can be carried out in exactly the same way as Racke and Muñoz Rivera did. Therefore we only have to verify Proposition 3.6. We need the following auxiliary result

Lemma 3.5. *Let $U = (u,v)^T \in \mathcal{H}$ and $\tilde{E}_{d_0}[U] := d_0\|u_x\|^2_{L^2} + \|v\|^2_{L^2}$. If $\tilde{\eta}_1 < 0$, then there are constants $0 < C_i < \infty$, $i = 1\ldots 4$ independent of U such that*

$$C_1 \tilde{E}_{d_0}[U] \leq E_{d_0}[U] \leq C_2 \tilde{E}_{d_0}[U] \quad \textit{and} \quad C_3\|U\|^2_{\mathcal{H}} \leq E_{d_0}[U] \leq C_4\|U\|^2_{\mathcal{H}}$$

is satisfied. Moreover, we get $E_{d_0}[U] \geq 0$ for all $U \in \mathcal{H}$.

Convention. *If it is clear which argument $U \in \mathcal{H}$ is used, we will write E_{d_0} and $\tilde{E}_{d_0}$ instead of $E_{d_0}(U)$ and $\tilde{E}_{d_0}(U)$ for convenience.*

Proof. Let $U = (u,v)^T \in \mathcal{H}$. We assume without loss of generality that $b_{min} < 0$. We want to show that there exists a constant $0 < C_1 < \infty$ such that $C_1\tilde{E}_{d_0}[U] \leq E_{d_0}[U]$. Recall that η_1 is the biggest eigenvalue of the operator $\overline{A}_{d_0}$. So one can calculate

$$-d_0\|u_x\|^2_{L^2} + \langle u, (b_{min} - b)u\rangle = \langle u, (d_0\partial_{xx} + b_{min} - b)u\rangle_{L^2} \leq \eta_1 \langle u, u\rangle_{L^2}$$

By multiplying the last inequality with $0 \leq \frac{b_{min}}{\eta_1} < 1$ one can deduce that

$$b_{min}\langle u, u\rangle_{L^2} \geq -\frac{b_{min}}{\eta_1} d_0\|u_x\|^2_{L^2} - \langle u, (b - b_{min})u\rangle_{L^2}. \tag{39}$$

Transforming (39) results into

$$\langle u, bu\rangle_{L^2} \geq -\frac{b_{min}}{\eta_1} d_0\|u_x\|^2_{L^2}. \tag{40}$$

By using (40) one can finally estimate E_{d_0} as

$$E_{d_0}(t) \geq (1 - \frac{b_{min}}{\eta_1}) d_0\|u_x\|^2_{L^2} + \|v\|^2_{L^2} \geq (1 - \frac{b_{min}}{\eta_1})\tilde{E}_{d_0}(t).$$

The remaining statements are trivial or follow immediately. □

Proposition 3.6. *Let $U = (u,u_t)^T(t) \in \mathcal{H}$ be a local solution of (35) on the interval $[0,T]$. Then there are constants $0 < C_1, C_2 < \infty$ independent of T or U_0, such that*

$$C_1\|U\|^2_{\mathcal{H}^3} \leq E^3_{d_0}[U] \leq C_2\|U\|^2_{\mathcal{H}^3}. \tag{41}$$

Proof. In this proof C always stands for a constant $0 < C < \infty$, which is independent of T and U_0 and serves the purpose. We will now explain, how to prove the left handed side of the inequality (41). By using Lemma 3.5 it is sufficient to show that $\|U\|^2_{\mathcal{H}^3} \leq C\sum_{s=0}^3 \|\partial_t^s U\|^2_{\mathcal{H}}$ i.e. we have to verify that the terms $|u_{xx}|^2$, $|u_{xxx}|^2$, $|u_{txx}|^2$, $|u_{txxx}|^2$, $|u_{ttxx}|^2$ and $|u_{xxxx}|^2$ can be estimated by

$$C\sum_{k=0}^{3} |\partial_t^{k+1}u|^2 + |\partial_t^k u_x|^2.$$

But this can be done easily by using the differential equation (4) to obtain information about the derivatives of u. For the estimation of higher order terms, one has to differentiate (4) once or twice in respect to t or x and use the already calculated lower order estimations. In the calculations one also needs the a priori assumption (36). Ocurring redundant terms can also be neglected, because we obtain by using the Sobolev Embedding theorem that

$$\forall\, t \in [0,T]\ |\partial_x^s \partial_t^l u| \leq C < \infty,$$

where $s, l \in \mathbb{N}$, $l = 0, 1$ and $s + l \leq 3$. Here we see that the proof depends on the dimension of the domain of the wave equation. In order to derive the right handed side of the inequality (41) one can proceed as in the first part of the proof. By using Lemma 3.5 again, it is sufficient to show that $\sum_{s=0}^{3} \|\partial_t^s U\|_{\mathcal{H}}^2 \leq C\|U\|_{\mathcal{H}^3}^2$ i.e. we have to verify that the terms $|u_{tt}|^2$, $|u_{ttt}|^2$, $|u_{ttx}|^2$, $|u_{tttx}|^2$, $|u_{tttt}|^2$ and $|u_{ttxx}|^2$ can be estimated by

$$C \sum_{k=0}^{3} |\partial_x^{k+1} u|^2 + |\partial_x^k u_t|^2.$$

This can be done by direct forward calculations, which are the same kind as in the first part of the proof. □

Acknowledgement. *I would like to thank Professor Reinhard Racke for bringing this topic to my attention and for the fruitful discussions.*

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