Formation of singularities and spectral stability of shock profiles in nonlinear hyperbolic systems

Doctoral thesis for obtaining the

academic degree

Doctor of Sciences

(Dr. rer. nat.)

submitted by

Bärlin, Johannes

at the

Universität Konstanz

Faculty of Sciences

Department of Mathematics and Statistics

Konstanz, 2023

Date of oral exam: 2nd November 2023

1st Referee: Prof. Dr. Heinrich Freistühler (Universität Konstanz)
2nd Referee: Prof. Dr. Kevin Zumbrun (Indiana University Bloomington)

Abstract

The present thesis consists of two main parts, both concerned with properties of solutions to nonlinear hyperbolic systems:
In the first part we consider first-order nonlinear hyperbolic systems with a general source and prove that for appropriately chosen smooth initial data the lifespan of the associated C^1-solution u cannot be infinite. We employ ideas of John (1974) and Hörmander (1987) to show that the derivative u_x of u becomes unbounded in finite time.
In the second part we report a proof that under natural assumptions shock profiles viewed as heteroclinic travelling wave solutions to a second-order hyperbolically regularized system of conservation laws are spectrally stable if the shock amplitude is sufficiently small. This means that an associated Evans function $\mathcal{E} : \Lambda \to \mathbb{C}$ with $\Lambda \subset \mathbb{C}$ an open superset of the closed right half plane $\mathbb{H}^+ \equiv \{\kappa \in \mathbb{C} : \operatorname{Re} \kappa \geq 0\}$, has only one zero, namely a simple zero at 0. The result is analogous to the one obtained by Freistühler and Szmolyan (2002) and Plaza and Zumbrun (2004) for parabolically regularized systems of conservation laws, and also distinctly extends findings of Plaza and Zumbrun (2004), Mascia and Zumbrun (2009) and Ueda (2009) on hyperbolic relaxation systems.

Acknowledgements

First of all, I want to thank my supervisor Heinrich Freistühler for his support and guidance in the last four years during which this thesis has come into existence. The joint work has been a true pleasure, and his confident and at the same time careful manner as well as his striking mathematical intuition and honest commitment have been, and continue to be, a constant source of inspiration to me. It is to a large part due to him that I will keep my years as a doctoral student at the University of Konstanz in warm memories.

A big thanks goes to my colleagues Valentin Pellhammer, with whom I shared an office and went to multiple conferences with, Matthias Sroczinski and Moritz Reintjes for the time spent together and many lunch and coffee breaks with interesting discussions, not all necessarily of mathematical type. I'm grateful for the positive atmosphere during work and leisure times which is a thing I do not consider to be granted. A special thanks goes to Valentin for reading large parts of this manuscript and suggesting improvements and corrections.

I'd like to thank the people of the Department of Mathematics and Statistics of the University of Konstanz. I had great teachers with inspiring lectures and experienced open and kind support by the department staff. I shared my time learning mathematics with many other students of whom some have become dear friends of mine.

I can't thank enough all my friends for the happy and lasting memories of the past years. Many of them I've met during the years in Konstanz as fellow students, house and flat mates, sharing hobbies or spending time together in some other way. They have carried me through difficult and hard times, making me feel at home where I was, and I consider myself lucky to call them close friends.

I would also like to thank Professor Kevin Zumbrun who agreed to co-referee this thesis. His work has a continuing profound influence on the field of shock wave theory, and I'm happy and excited to have such an expert reviewing this text.

Finally, let me dedicate this thesis to my family, my parents Erich and Beate, my sister Franziska and my brother Matthias. They've formed me as a person and made me feel safe no matter what. I'm happy to have shared a large part of my life with them. In particular, this dedication goes to Matthias who is not only a brother and friend but also a fellow mathematician and sportsman. I remember endless discussions on mathematics, sciences, society as well as on climbing and surfing. Owing to unfortunate circumstances he had to put his mathematical passion at rest for while and was just lately able to carefully pick it up again, and so this dedication goes especially to him.

Financial support: The research that led to the results presented in this thesis has been supported by Deutsche Forschungsgemeinschaft under Grant No. FR822/10-1.

Contents

List of Figures

List of Abbreviations

cf.	**c**on**f**er
e.g.	**e**xempli **g**ratia
GSPT	**G**eometric **S**ingular **P**erturbation **T**heory
i.e.	**i**d **e**st
ODE	**O**rdinary **D**ifferential **E**quation
PDE	**P**artial **D**ifferential **E**quation
w.r.t.	**w**ith **r**espect **t**o

List of Symbols

A^*	adjoint of a matrix $A \in \mathbb{C}^{n\times n}$	p. 3
A^t	transposed of a matrix A	p. 3
$B_r(x)$	open ball of radius $r > 0$ centered at x	p. 3
$C(x)$	characteristic through x	p. 7
$\mathbb{C}$	complex numbers	p. 3
C^K	K-times continuously differentiable functions	p. 4
C_b^K	bounded C^K-functions	p. 4
C_0^K	compactly supported C^K-functions	p. 4
$\|\| \cdot \|\|_{C_b^K}$	norm on C_b^K	p. 4
$\Box u$	d'Alembertian	p. 1
$\operatorname{diag}(x_1, \ldots, x_m)$	matrix with diagonal entries $x_1, \ldots, x_m$	p. 3
Df	differential (or Jacobi matrix) of f	p. 4
E_{jj}	elementary matrix with jj-th entry equal to 1	p. 82
$\mathcal{E}$	Evans function	p. 65
$\Gamma^m(A)$	induced vector field by $A \in \mathbb{C}^{d\times d}$ on $\mathscr{G}_m^d(\mathbb{C})$	p. 40
$\mathfrak{g}_\kappa$	resolution kernel	p. 74
$\mathfrak{G}_\kappa$	resolvent kernel	p. 73
$\mathscr{G}_m^d$	Grassmannian manifold	p. 39
H^s	L^2-based Sobolev space of order s	p. 72
$\mathbb{H}^+$	closed right half of the complex plane	p. 65
$\mathcal{H}^\pm$	Evans bundles	p. 65
id	identity	p. 74
$\mathbb{1}_n$	unit matrix of size n	p. 3
f^{-1}	inverse of or preimage under a function f	p. 8
$\ker L$	kernel of a linear operator L	p. 3
Λ	domain of consistent splitting	p. 64
L^2	Lebesgue space of square integrable functions	p. 72
$L_i f$	derivative of $f = f(t, x)$ along i-th characteristic	p. 11
$M_{d,m}$	Stiefel manifold	p. 39
$\mathbb{N}$	natural numbers	p. 1
$\partial_{x_i} f, f_{x_i}$	partial derivative of f w.r.t. i-th coordinate	p. 4
$\mathbb{R}$	real numbers	p. 1
$\rho(L)$	resolvent set of a linear operator L	p. 3
$\sigma(L)$	spectrum of a linear operator L	p. 3
$\operatorname{span}\{x_1, \ldots, x_m\}$	subspace spanned by $x_1, \ldots, x_m$	p. 3
$\operatorname{supp} u$	support of a function u	p. 4
TM	tangent bundle of a smooth manifold M	p. 4
W^s	(local) stable manifold	p. 48
W^u	(local) unstable manifold	p. 48

Introduction

Let $n \in \mathbb{N}$. In this thesis we are concerned with nonlinear hyperbolic systems of the form

$$A^0(u)u_t + A^1(u)u_x = \mathcal{B}[u] \quad (u \in \mathbb{R}^n) \tag{1}$$

with temporal variable $t \in \mathbb{R}$ and a one dimensional spacial variable $x \in \mathbb{R}$. The first-order partial differential operator on the left-hand side of equation (1) is of hyperbolic type. This will usually mean that the smooth coefficient matrices A^0, A^1 have real entries, i.e. $A^0(u), A^1(u) \in \mathbb{R}^{n\times n}$, with invertible $A^0(u)$ such that in a neighborhood of a reference state $u_* \in \mathbb{R}^n$ the eigenvalues of $A^1(u)$ with respect to $A^0(u)$ are real and simple. The operator $A^0(u)u_t + A^1(u)u_x$ is *strictly hyperbolic*.

The term $\mathcal{B}[u]$ on the right-hand side in (1) has one of the two following shapes regularizing in both cases the nonlinear hyperbolic operator on the left-hand side of (1) in a certain sense:
In the first case it is a smooth function $\mathcal{B}[u] = B(u) \in \mathbb{R}^n$ of u and in this way (1) becomes a first-order quasi-linear hyperbolic system with source term. Compared to the homogeneous system

$$A^0(u)u_t + A^1(u)u_x = 0 \tag{2}$$

the augmentation of (2) by a nontrivial source term can have interesting consequences: While it is known that classical solutions of (2) tend to develop singularities in finite time [48] a source term can suppress the formation of a singularity and solutions of (1) with small initial data exist for all positive times (e.g. [97]).
In the second case $\mathcal{B}[u]$ is a second-order hyperbolic operator of the form

$$B\Box u := B(u_{xx} - u_{tt})$$

with a symmetric positive definite matrix $B \in \mathbb{R}^{n\times n}$. System (1) is now a second-order semilinear hyperbolic partial differential equation. It consists of a first-order nonlinear hyperbolic operator and a linear second-order operator which is essentially the d'Alembertian. Each of these operators has its own characteristic speeds. If these speeds are in an appropriate relation to each other then (1) is dissipative in the sense that an associated linearization has decaying Laplace modes for all nonzero wave numbers [25].

In Chapter 1 we consider the first case where $\mathcal{B}[u] = B(u)$ is a source term. The main result of this part of the thesis, Theorem 1.0.1, shows that if there is a genuinely nonlinear eigenvalue of A^1 with respect to A^0 at some reference state $u_* \in \mathbb{R}^n$, and $B(u_*) = 0$, then one can construct a class of (large) data giving

rise to solutions of (1) whose gradients blow up in finite time, regardless of a possible smoothing effect caused by the source term B. The strategy of proof follows the works of John [48] and Hörmander [41]. Because we also provide proofs for the local existence and uniqueness theory we consider Chapter 1 to be self-contained.

In Chapter 2 the operator $\mathcal{B}[u]$ is of the form $B\Box u$ with symmetric positive definite $B \in \mathbb{R}^{n\times n}$. Furthermore, the matrix-valued functions A^0 and A^1 in (1) are assumed to be symmetric and given by the Jacobian matrices $Dg = A^0$ and $Df = A^1$ of two smooth functions $g = g(u) \in \mathbb{R}^n$ and $f = f(u) \in \mathbb{R}^n$. Note that under these assumptions the first-order part in (1) is now in conservation from given by $g(u)_t + f(u)_x$. Under a suitable sub-characteristicity condition Theorem 2.4.8 states that small amplitude shock profiles of (1) are spectrally stable if the underlying shock is a Lax shock which is associated to a genuine nonlinear eigenvalue of Df w.r.t. Dg. This main result is set within Evans function theory and is proved following the geometric approach presented in [26, 27] on spectral stability of shock profiles in viscous conservation laws. The scope of the basic material needed for Chapter 2 is quite large and at most places we will refer to the existing literature but let us point out the detailed discussion of Grassmannian flows in Section 2.1 as an exception to this rule.

In order to avoid repetitions as well as jumping back and forth we do not go into further details of the discussed problems at this point. The reader can find more information on the content, the background material and the literature of each chapter in the respective introductory sections (for Chapter 2 see also Section 2.3). The results presented in this thesis have been submitted or published in the following way: The introduction to Chapter 1 and Sections 1.2, 1.3 and 1.4 form the basis of a paper which is to appear in *Nonlinear Analysis: Real World Applications* with an online version available under [5]. Sections 2.1, 2.3, 2.4, 2.6 and 2.7 are currently submitted as a paper to a journal. Preprints of these papers can be found on the arXiv-server under the numbers arXiv:2204.12101 and arXiv:2208.12165, respectively.

Before continuing with our specific subjects let us give a selection of textbooks covering many topics in the theory of hyperbolic equations. The list is by no means complete and is just meant to serve as an orientation. Hyperbolic partial differential equations are a vital part of most introductory textbooks on partial differential equations such as [18]. A detailed presentation of first-order hyperbolic systems can be found in [7]. More specific texts are [12, 68] on hyperbolic conservation laws, [78] on geometric optics, and [44] with a focus on nonlinear hyperbolic equations. A classic in wave propagation is [95], and for the theory of shock waves the reader may consult [86].

Structure of the thesis and some notation

This thesis consists of two main chapters, Chapter 1 and Chapter 2. Each chapter is divided into sections of which only Section 2.6 and Section 2.7 are divided into further subsections. At the beginning of each chapter the reader finds an introduction to the problem at hand and some background information as well as an outline of the chapter with short summaries of the content of each

section of the respective chapter. More detailed summaries are placed at the beginning of each section.

The literature of Chapter 1 is discussed in its opening words while the bulk of the literature and references of Chapter 2 is to be found at the end of Section 2.3.

Some background material for Chapter 1 is revised in Section 1.2 together with proofs of the local existence theory. Preliminaries on Grassmannian flows and geometric singular perturbation theory that are important for the developments in Chapter 2 are contained in Sections 2.1 and 2.2. We also want to highlight the fact that for each problem an explicitly solvable example is presented in Section 1.1 and Section 2.5, respectively.

The proof of the main result on the formation of singularities in solutions to nonlinear hyperbolic systems with general sources, Theorem 1.0.1, is given in Sections 1.3 and 1.4. The second main result, Theorem 2.4.8, on spectral stability of shock profiles for hyperbolically regularized systems of conservation laws is proved in Section 2.7.

As additional information Section 1.5 provides an example for a hyperbolic relaxation system with a genuine nonlinear eigenvalue, and Section 2.6 discusses the meaning of the spectral stability result for the resolvent kernel of the linearized problem at hand.

We fix some notation used throughout this text. Let $k, n, m \in \mathbb{N}$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The real and imaginary part of a complex number a are denoted by $\operatorname{Re} a$ and $\operatorname{Im} a$, respectively. The complex conjugate of a is $\overline{a}$.

The entries of a matrix $A \in \mathbb{K}^{n\times m}$ are given by A_{ij} for $1 \leq i \leq n, 1 \leq j \leq m$, and sometimes brackets are added, $(A)_{ij}$. The transposed of A is denoted by A^t with $(A^t)_{ij} = A_{ji}$, and the adjoint of A is denoted by A^* with $(A^*)_{ij} = \overline{A_{ji}}$. The real part of a matrix A is defined as $\operatorname{Re} A = 1/2(A + A^*)$. If $B \in \mathbb{K}^{m\times k}$ then $AB \in \mathbb{K}^{n\times k}$ is the matrix product of A and B. The matrix $\operatorname{diag}(x_1, \ldots, x_n) \in \mathbb{K}^{n\times n}$ has $x_i \in \mathbb{K}$ as its i-th diagonal element, and $\mathbb{1}_n = \operatorname{diag}(1, \ldots, 1) \in \mathbb{K}^{n\times n}$ is the unit matrix of dimension n. The unit vector along the i-th coordinate in $\mathbb{K}^n$ is usually denoted by $e_i \in \mathbb{K}^{n\times 1}$.

The closure of a subset $\Omega \subset X$ of a topological space X is given by $\overline{\Omega}$, and if X is metrizable, like all manifolds in this text, then we use d to denote some metric on X that metrizes the topology of X. Let X, Y be $\mathbb{K}$-vector spaces. Given $x_1, \ldots, x_m \in X$ then $\operatorname{span}\{x_1, \ldots, x_m\}$ is the subspace of X spanned by the vectors $x_1, \ldots, x_m$. If X is endowed with a scalar product $\langle \cdot, \cdot \rangle$ then we assume that $\langle \cdot, \cdot \rangle$ is linear in the first and anti-linear in the second component. If there is a commonly used scalar product on X we will usually not specify it any further. The norm induced on X by a given scalar product is denoted by $|| \cdot ||$. In case $X = \mathbb{K}^n$ we will use $| \cdot |$ for the Euclidean norm induced by the standard scalar product on X. For a normed space $(X, || \cdot ||)$ the subset $B_r(x)$ is the open ball of radius $r > 0$ centered at $x \in X$. If we are given the norms $|| \cdot ||_X$ and $|| \cdot ||_Y$ on X and Y and a linear map $L : D(L) \subset X \to Y$ defined on some linear subspace $D(L)$ of X then $||L|| = \sup_{||x||_X=1, x\in D(L)} ||L(x)||_Y$ is the operator norm of L. The subset $\ker L \subset X$ is the kernel of L, $\rho(L) \subset \mathbb{C}$ is its resolvent set and $\sigma(L)$ its spectrum.

Let $K \in \mathbb{N}_0 \cup \{\infty\}$, and let $f : \Omega \to \mathbb{K}^m$ be a continuously differentiable function defined on an open subset $\Omega \subset \mathbb{K}^n$. Then Df denotes the differential of f given at a point $x \in \Omega$ as a linear map $Df(x) : \mathbb{K}^n \to \mathbb{K}^m$. We will use $Df(x)$ also as the matrix representation of the linear map $Df(x)$ with respect to the standard bases on $\mathbb{K}^n$ and $\mathbb{K}^m$, i.e. its Jacobian matrix. The partial, directional or distributional, derivative with respect to the i-th coordinate x_i is denoted by $\partial_{x_i} f$ or f_{x_i}. The space of K-times continuously differentiable functions on Ω with values in $\mathbb{K}^m$ is $C^K(\Omega; \mathbb{K}^m)$. We write $f \in C_0^K(\Omega; \mathbb{K}^m)$ if the support $\operatorname{supp} f$ of f is compact and $f \in C^K(\Omega; \mathbb{K}^m)$.

If $f : N \to M$ is a smooth function defined on a smooth manifold N with values in another smooth manifold M then Df is the differential of f which for each $x \in N$ is a linear map $Df(x) : T_x N \to T_{f(x)} M$ for the tangent spaces TN and TM of N and M.

Here and in general, we will use the term "smooth" in the sense that the necessary number of continuous derivatives of a particular object exist.

Let $I \subset \mathbb{R}$ be an interval with interior $(a, b), a < b$. We are often met with functions $u = u(t, x)$ defined on the Cartesian product $I \times \mathbb{R}$ where the first coordinate is denoted by t, the time, and the second coordinate is the position in one dimensional space, denoted by x. The space $C_b^K(I \times \mathbb{R}; \mathbb{K}^m)$ contains all continuous functions u whose partial derivatives of order less than or equal to K exist on $(a, b) \times \mathbb{R}$ and are given by continuous and bounded functions $\partial_{t,x}^\alpha u$ defined on all of $I \times \mathbb{R}$ where $\alpha \in \mathbb{N}_0^2, |\alpha| \leq K$. The subspace consisting of functions $u \in C_b^K(I \times \mathbb{R}; \mathbb{K}^m)$ with compact support $\operatorname{supp} u \Subset I \times \mathbb{R}$ is denoted by $C_0^K(I \times \mathbb{R}; \mathbb{K}^m)$. If $K < \infty$ and $u : I \times \mathbb{R} \to \mathbb{K}^m$ is K-times continuously differentiable then set

$$||u||_{C_b^K} = \sup_{\alpha \in \mathbb{N}_0^2, |\alpha| \leq K} \sup_{(t,x) \in I \times \mathbb{R}} |\partial_{t,x}^\alpha u(t, x)|.$$

If it is clear where a function takes its values we will at times suppress the reference to the respective space in the above notation.

Chapter 1

Formation of singularities in solutions to nonlinear hyperbolic systems with general sources

Let $n \in \mathbb{N}$, and let $\mathcal{U} \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ with $0 \in \mathcal{U}$. We consider quasi-linear systems of the form

$$u_t + a(u)u_x = B(u) \text{ on } (0,\infty) \times \mathbb{R} \tag{1.1}$$

for $u \in \mathcal{U}$ and smooth functions $a : \mathcal{U} \to \mathbb{R}^{n\times n}, B : \mathcal{U} \to\in \mathbb{R}^n$. Systems like (1.1) occur in gas dynamics, nonlinear optics, numerics of conservation laws and many more applications (cf. [12]). We make three assumptions on the coefficients in (1.1):

A1 $a(0)$ has only real and simple eigenvalues;

A2 There is $k \in \{1,\dots,n\}$ such that the k-th eigenvalue of a is genuinely nonlinear at 0;

A3 $B(0) = 0$.

Assumption A1 makes the first-order differential operator $\partial_t + a(u)\partial_x$ a hyperbolic one. Assumption A2 is Lax's genuine nonlinearity condition and means

$$\langle D\lambda_k(0), r_k(0)\rangle \neq 0$$

where $a(u)r_k(u) = \lambda_k(u)r_k(u)$ (see Section 1.2). It is this non-degeneracy condition that will drive the blow-up mechanism in (1.1). By A3 we find that 0 is an equilibrium solution of (1.1).

We will be concerned with the formation of a singularity in C^1-solutions of (1.1), i.e. the finite life span of classical solutions of (1.1) for certain smooth initial data.

If $B \equiv 0$ then it is well-known that solutions u to quasi-linear hyperbolic systems of the form

$$u_t + a(u)u_x = 0 \tag{1.2}$$

may develop singularities over time. In [48] John showed that for a large class of C^1_b-small initial data the life span of the classical solution of (1.1) is finite,

extending an earlier result by Lax [60] for $n = 1, 2$ to arbitrary system sizes $n \in \mathbb{N}$. His results were then further generalized by Liu in [65], and reproved with some elegant simplifications in [41] .

A heuristic explanation for the formation of a singularity is the following: The initial data u_0 is propagated by (1.2) along certain curves in (t, x)-space called characteristics. If after some time these curves meet at a point in space then the solution u accumulates there resulting in a loss of regularity of u. Due to this focusing a singularity has formed.

If however $B \not\equiv 0$ then the situation can be dichotomous in the sense that

D1 for small data a smooth solution to (1.1) exists for all time while

D2 for some large initial data the smooth solution of (1.1) must develop a singularity.

For systems like (1.1) there can be a competition between a destabilizing effect induced by the nonlinear hyperbolic operator $\partial_t + a(u)\partial_x$ and a dissipative effect caused by the source term B. In the words of [14]:

> The prevailing conjecture is that dissipation manages to restrain waves of small amplitude but is incapable of preventing the breaking of waves with large amplitude.

While property D1 is known to hold for quite general classes notably of relaxation systems [6, 8, 13, 56, 70, 97, 98] (cf. also, e.g., [24, 63, 81]), blow-up results D2 have been shown in particular cases by ad-hoc constructions, see e.g. [4, 45, 63].

Hence, by D1, if $B \neq 0$ then we cannot in general expect a blow-up result for a class of C^1_b-small data like in [48]. The main theorem of this part of the thesis shows that, independently of the nature of the source B and property D2, it is the genuine nonlinearity of the differential operator $\partial_t + a(u)\partial_x$ that by itself entails the breakdown of certain smooth solutions. By adapting ideas of F. John [48] and L. Hörmander [41] we will prove:

Theorem 1.0.1. *If A1, A2, A3 hold then there exist smooth compactly supported functions $u^0 : \mathbb{R} \mapsto \mathcal{U}$ such that, while remaining bounded, the unique C^1-solution u of (1.1) with data $u(0, \cdot) = u^0$ exists only for finite time.*

Theorem 1.0.1 applies to all systems considered in [6, 8, 56, 70, 97, 98] if the space dimension is equal to one and the assumptions A1-A3 are satisfied. The result is sharp in the sense that there exist significant relaxation systems with linear principal part $\partial_t + a\partial_x$ − and nonlinear equilibrium system − whose solutions remain smooth for all times, see e.g. [63], Sec. 4. (This sharpness of course does not cover the general linearly degenerate case.) Finally, compared to the blow-up results in [57] our assumptions on B, namely only assumption A3, are quite weaker than those placed on B in [57] where A3, $DB(0) = 0$, and a further condition are required to hold in order to prove that the C^1-solution to (1.1) for certain initial data develops a singularity in finite time.

Outline of the chapter:

In Section 1.1 a textbook example for (1.1) consisting of Burgers' equation with

a linear source term is discussed.
In Section 1.2 the method of characteristics is applied to (1.1) and we derive a set of partial differential equations satisfied by the gradient u_x of a solution u of (1.1) that will be central for the estimates in the following sections. A lemma by Hörmander in [41] is adapted to our situation, and the reader also finds a discussion of the local existence and uniqueness theory for (1.1).
A class of data that initiate C^1-solutions of (1.1) with finite life span is constructed in Section 1.3. Some smallness properties of solutions for a scaled version of (1.1) are stated and proved, too.
The main finding of this chapter is contained in section 1.4. We show a gradient blow-up of smooth solutions of the (scaled) version of (1.1) with the afore constructed initial data.
Finally, Section 1.5 briefly discusses a well-known model for a gas in thermal non-equilibrium by Y. Zeng which is in the form of (1.1).
A short read of the chapter would consist of Section 1.2 skipping the proofs of local existence theory, and Sections 1.3 and 1.4.

1.1 A relaxed Burgers' equation

This section discusses a textbook example for equations like (1.1) built from Burgers' equation relaxed by a linear source term. The example can be found, e.g., in the books [12, 95]. It serves to illustrate the dichotomy D1, D2 that can be present in equations of the form (1.1), and to motivate the strategy employed in the proof of Theorem 1.0.1

Let $\eta \in \mathbb{R}$ and consider the scalar balance law

$$u_t + uu_x = -\eta u, \quad u(0,x) = u_0(x), \tag{1.3}$$

which is a relaxed Burgers' equation. Note that the assumptions A1-A3 are satisfied with

$$\lambda(u) = u, \quad r(u) = -1, \quad \text{and} \quad -\eta u|_{u=0} = 0.$$

For given $T > 0$ and a solution $u \in C_b^1([0,T] \times \mathbb{R})$ the characteristic curve $C(x_0)$ through a point $x_0 \in \mathbb{R}$ is defined as the graph

$$C(x_0) := \{(t,x) \in \mathbb{R}^2 : \ t \in [0,T], x = X(t;x_0)\}$$

where $X(t;x_0)$ is the solution of

$$X'(t) = \frac{d}{dt}X(t;x_0) = \lambda(u(t,X(t;x_0)) = u(t,X(t;x_0), \quad X(0;x_0) = x_0.$$

One finds that $((t,x) \mapsto e^{\eta t}u(t,x))$ is constant along characteristics. This easily leads to a proof of the following proposition.

Proposition 1.1.1. *Let $K \in \mathbb{N}, u_0 \in C_b^K(\mathbb{R})$, and for $(t,x) \in \mathbb{R}^2$ define*

$$\chi_t(x) := \begin{cases} x + u_0(x)\eta^{-1}\left(1 - e^{-\eta t}\right) & \text{if } \eta \neq 0, \\ x + u_0(x)t & \text{if } \eta = 0. \end{cases} \tag{1.4}$$

There exist $T > 0$ and a unique solution $u \in C_b^K([0,T] \times \mathbb{R})$ of (1.3) with initial data $u(0,x) = u_0(x)$ such that χ_t is invertible for all $t \in [0,T]$ with $\chi_t(x) = X(t;x)$, and

$$u(t,x) = e^{-\eta t} u_0(\chi_t^{-1}(x)). \tag{1.5}$$

If

$$\eta > \sup_x |\partial_x u_0(x)| \tag{1.6}$$

then the above holds for any $T \in [0,\infty)$.

Hence, if $\eta > 0$ then D1 in the previously discussed dichotomy holds for (1.3). But also D2 is satisfied regardless of the sign of η:

Proposition 1.1.2. *Let $u_0 \in C_b^1(\mathbb{R})$. If for some x_0*

$$\partial_x u_0(x_0) < -|\eta| \tag{1.7}$$

then the life span of the unique C^1-solution of (1.3) with $u(0,x) = u_0(x)$ is bounded from above by

$$0 < T^* = \begin{cases} -\eta^{-1} \ln\left(1 + \eta(\partial_x u_0(x_0))^{-1}\right) & \text{if } \eta \neq 0, \\ -(\partial_x u_0(x_0))^{-1} & \text{if } \eta = 0. \end{cases} \tag{1.8}$$

Proof. Let $u_0 \in C_b^1(\mathbb{R})$. The unique C^1-solution u of (1.3) with initial data u_0 is given by (1.5) as long as χ_t is strictly monotonic. Differentiating the solution formula (1.5) yields

$$\partial_x u(t, X(t;x_0)) = \frac{e^{-\eta t}\partial_x u_0(x_0)}{\partial_x \chi_t(x_0)}.$$

Because $\partial_x \chi_{T^*}(x_0) = 0$ one finds that u cannot exist longer than T^* as a classical solution of (1.3) since otherwise the gradient $\partial_x u$ would be unbounded on $[0,T^*)$. □

A sharp estimate for the lifespan of u in the above propositions is obtained by setting

$$0 < T_{max} := \inf\{t \in (0,\infty) : \ \exists x \in \mathbb{R} : \partial_x \chi_t(x) = 0\}.$$

We see from (1.6) that if the source term is strong enough compared to the derivative of the initial data (and has the correct sign) then no singularities form. If however the blow-up driven by genuine nonlinearity dominates the

effect of the source term, as quantified in (1.7), then the classical solution can only exist for finite time.

The proof of Proposition 1.1.2 hinges on the exact solution formula (1.5) which is something that is usually not available for more general systems of the form (1.1). But we find the heuristics of meeting characteristics that we mentioned before explicitly present in the above calculations: By (1.4) and in the easiest case $\eta = 0$ the characteristics are straight lines with possibly different slopes. If at some point $x_0 \in \mathbb{R}$ the gradient of the initial data $\partial_x u_0(x_0)$ is negative then some of these lines meet in finite time implying the end of the classical solution.

There is an alternative route to the proof of Proposition 1.1.2 which can be extended to general systems (1.1) quantifying the above focusing of characteristics. Suppose we are given a C^2-solution u of (1.3) with initial data $u(0,\cdot) = u_0$. Set $w := -u_x$. Differentiation of (1.3) with respect to x gives

$$\partial_t w + u\partial_x w = w^2 - \eta w.$$

Along the characteristic $C(x_0)$ we find the Bernoulli equation

$$\frac{d}{dt} w(t, X(t;x_0)) = \partial_t w + u\partial_x w = w^2 - \eta w \tag{1.9}$$

since $X'(t;x_0) = u(t, X(t;x_0))$. The explicit solution formula for (1.9) with $w(0,x_0) = -\partial_x u_0(x_0)$ is

$$w(t, X(t;x_0)) = -\frac{e^{-\eta t}\partial_x u_0(x_0)}{\partial_x \chi_t(x_0)}$$

which again proves the previous proposition for C^2-data.

The strategy for general systems (1.1) will be to choose appropriate initial data, and then to mimic the above procedure for w obtaining a Bernoulli-type differential inequality which leads to a logarithmic lifespan estimate similar to (1.8). In doing so we need to deal with n waves w_i each solving a characteristic partial differential equation. The ideas in [41, 48] will be key when estimating these waves and their interactions. Furthermore, a rescaling of the independent variables t, x will help to control the contributions coming from the source term B.

1.2 The method of characteristics

In this section we deduce the characteristic partial differential equations satisfied by the gradient $w = u_x$ of a smooth solution u of (1.1) with w being decomposed along eigenvector fields of a. A geometric lemma of Hörmander for systems (1.1) with $B \equiv 0$ is extend to (1.1) with general source terms, and we prove some local existence and uniqueness results for (1.1) using the method of characteristics.

Fix a $k \in \{1, \dots, n\}$ satisfying assumption A2. By A1 there are $\delta > 0$ and smooth functions

$$\begin{aligned} \lambda_i &: B_{2\delta}(0) \subset \mathbb{R}^n \to \mathbb{R}, \\ r_i &: B_{2\delta}(0) \subset \mathbb{R}^n \to \mathbb{R}^{n\times 1}, \\ l_i &: B_{2\delta}(0) \subset \mathbb{R}^n \to \mathbb{R}^{1\times n}, \end{aligned}$$

for all $i \in \{1, \dots, n\}$ which satisfy for all $u \in B_{2\delta}(0)$ and all i, j

$$l_i(u)a(u) = \lambda_i(u)l_i(u),\ a(u)r_i(u) = \lambda_i(u)r_i(u),\ \text{and } \lambda_i(u) < \lambda_j(u) \text{ if } i < j.$$

We normalize the above eigenvectors of a such that for all i, j

$$|l_i| = 1 \text{ and } l_i r_j = \delta_{ij}, \tag{1.10}$$

and such that, by assumption A2, it holds

$$\langle D\lambda_k(0), r_k(0)\rangle < 0. \tag{1.11}$$

Set

$$c_\lambda := \min_{i\neq k} \inf_{u\in B_\delta(0)} |\lambda_i(u) - \lambda_k(u)| > 0. \tag{1.12}$$

Fix $T > 0$ and a solution

$$u \in C^2([0, T] \times \mathbb{R}) \tag{1.13}$$

of (1.1) with

$$|u(t, x)| < \delta. \tag{1.14}$$

By differentiation of (1.1) the x-derivative of u

$$w(t, x) := u_x(t, x)$$

satisfies the equation

$$w_t + a(u)w_x = DB(u)w + \gamma(u, w) \tag{1.15}$$

where γ is a quadratic form in w. Decompose w with respect to $\{r_j(u)\}_j$ to obtain

$$w(t, x) = \sum_j w_j(t, x)r_j(u(t, x)). \tag{1.16}$$

The component functions w_i satisfy certain partial differential equations along

the i-th characteristic curves. The latter are determined by the ordinary differential equations

$$\frac{d}{dt}X_i(t;x) = \lambda_i(u(t, X_i(t;x))),\ X_i(0;x) = x, \tag{1.17}$$

which may be solved under the above hypothesis on $[0,T]$ for all $x \in \mathbb{R}$. The i-th characteristic curve $C_i(x)$ starting at $(0,x)$ is given by the trace of

$$t \mapsto (t, X_i(t;x)).$$

For given i and $(t,x) \in [0,T] \times \mathbb{R}$ let $z_i = z_i(t,x)$ be the unique element of $\mathbb{R}$ such that the i-th characteristic curve starting at $(0,z_i)$ runs through (t,x), i.e.

$$(t,x) \in C_i(z_i).$$

If $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a differentiable function then let $L_i(u)f$ denote the derivative of f with respect to t along the i-th characteristic of u:

$$L_i(u)f(t,x) := f_t(t,x) + \lambda_i(u(t,x))f_x(t,x).$$

Then for all i and $x \in \mathbb{R}$

$$\frac{d}{d\xi}(f(\xi, X_i(\xi;x))) = L_i(u)f(\xi, X_i(\xi;x)). \tag{1.18}$$

We will drop the reference to u in the operator $L_i = L_i(u)$ and simply write $L_i f$ whenever it is clear which function is to be inserted into the eigenvalue functions $\lambda_i = \lambda_i(u)$.

From (1.15) one obtains a system of partial differential equations:

$$L_i w_i = \sum_{j,m} \gamma_{ijm}(u) w_j w_m + \sum_m B_{im}(u) w_m \tag{1.19}$$

where γ_{ijm} are given in [48] and below; they satisfy for all i,j

$$\gamma_{ijj}(u) = -\delta_{ij}\langle D\lambda_i(u), r_i(u)\rangle. \tag{1.20}$$

The functions B_{im} will be discussed below, too. There are constants $\overline{\gamma}, \overline{B} > 0$ such that

$$\max_i \sup_{u \in B_\delta(0)} \sum_{j,m} |\gamma_{ijm}(u)| \le \overline{\gamma}$$
$$\max_i \sup_{u \in B_\delta(0)} \sum_m |B_{im}(u)| \le \overline{B}.$$

In the above discussion the decomposition (1.16) of w along $\{r_j(u)\}_j$ led to a diagonalization of the first-order hyperbolic differential operator on the left-hand side in (1.15). The waves $\{w_j\}_j$ now solve a system of scalar equations as given in (1.19) coupled by a zero-order right-hand side. According to (1.18)

each of these characteristic PDEs can be integrated along the respective characteristic. In this way one obtains information on a given solution at positive times from its initial data and a time integral of the lower order coupling terms. In other words, instead of analyzing the PDE (1.15) one now investigates n implicit integral equations, one for each w_i, with no derivatives of w entering. This leads, amongst other things, to a-priori estimates and finite speed of propagation results that are presented below. We employ the so-called method of characteristics (see e.g. [78]).

As a first application of this method we follow [41] in proving a lemma that simplifies the arguments of [48] in an elegant geometric way. Calculate the differential of $w_i(dx - \lambda_i(u)dt)$ and obtain by (1.19)

$$d(w_i(dx - \lambda_i(u)dt)) = \left(\sum_{j,m} \Gamma_{ijm}(u) w_j w_m + \sum_m B_{im}(u) w_m\right) dt \wedge dx \tag{1.21}$$

where the Γ_{ijm} are like in [41]. In particular for all i, j one has

$$\Gamma_{ijj}(u) = 0. \tag{1.22}$$

There is a constant $\overline{\Gamma} > 0$ such that

$$\sup_i \sup_{u \in B_\delta(0)} \sum_{j,m} |\Gamma_{ijm}(u)| \leq \overline{\Gamma}.$$

Expressions and relations for the coefficient functions $c_{ijm}, \gamma_{ijm}, \Gamma_{ijm}$ and B_{im} are given at the end of this section. We use (1.21) to prove the following statement which is a straightforward extension of Lemma 1.2.2 in [41] to systems with sources.

Lemma 1.2.1. *Let $T > 0$ and let $i \in \{1, \dots n\}$. Suppose $u \in C^2([0,T] \times \mathbb{R})$ solves (1.1) with $|u(t,x)| < \delta$. Let τ be a C^1-arc in $[0,T] \times \mathbb{R}$ intersecting the i-th characteristic curves transversally, and let $A_i(\tau)$ be the open region bounded by τ, the orbits of L_i through the end points of τ and an interval τ_{i0} where $\{t = 0\}$. Then*

$$\int_\tau |w_i(dx - \lambda_i(u)dt)| \tag{1.23}$$
$$\leq \int_{\tau_{i0}} |w_i| dx + \int_{A_i(\tau)} \left| \sum_{j,m} \Gamma_{ijm}(u) w_j w_m + \sum_m B_{im}(u) w_m) \right| dx dt.$$

Proof. The statement of Lemma 1.2.1 in case of $B \equiv 0$ is Lemma 1.2.2 in [41]. The proof for general sources B remains essentially the same as in [41] using (1.21) and Stokes' formula. Let us just comment on the case when w_i does not have constant sign on τ. Then one considers the (relatively) open set

$$\tilde{\tau} := \tau \cap \{w_i \neq 0\}.$$

Use a compact exhaustion $\{\tau_j\}_{j\in\mathbb{N}}, \tau_j \subset \operatorname{int}\tau_{j+1}, \cup_j \tau_j = \tilde{\tau}$, of $\tilde{\tau}$ and note that each τ_j consists of finitely many path-wise connected components. Then apply the considerations holding when w_i has constant sign on τ to each of these components. □

This closes our considerations on a solution (1.13), (1.14). Next the method of characteristics is used to show uniqueness and local existence of solutions to (1.1) for small initial data. Since A3 is not used in the following we allow for general smooth source terms B for the rest of this section.

Lemma 1.2.2. *Consider* (1.1) *satisfying A1 and A2 with smooth source term* B. *Let* $T > 0$ *and* $c, d \in \mathbb{R}$ *with* $c < d$. *Suppose* $u, v \in C^1([0,T]\times\mathbb{R})$ *solve* (1.1) *with*

$$u(0,x) = v(0,x) \text{ for } x \in I := [c,d].$$

Furthermore, suppose $v \in C_b^1$ *and* $|v(t,x)| < \delta$ *for all* $(t,x) \in [0,T]\times\mathbb{R}$. *Define*

$$D_{v,I} := \{(t,x) \in [0,T]\times\mathbb{R} : \ x \in [X_n(t;c), X_1(t,d)]\}$$

where X_i *are characteristics of* v. *Then*

$$u|_{D_{v,I}} = v|_{D_{v,I}}.$$

Proof. Consider the difference $V := v - u$. Then subtracting the equations for v and u from each other we obtain

$$V_t + a(v)V_x = \tilde{H}(u,v,u_x)V$$

for some smooth $\tilde{H} = \tilde{H}(u,v,w) \in \mathbb{R}^{n\times n}$. Decompose V along $\{r_j(v)\}_j$ to obtain

$$V(t,x) = \sum_j V_j(t,x) r_j(v(t,x)).$$

The components V_i solve

$$L_i(v)V_i = \sum_j H_{ij}(u,v,u_x,v_x)V_j \tag{1.24}$$

with appropriate $H_{ij} = H_{ij}(u,v,u_x,v_x)$ satisfying

$$\sup_{i;(t,x)\in D_{v,I}} \sum_j |H_{ij}(u,v,u_x,v_x)(t,x)| \le \overline{H} < \infty.$$

Define for $t \in [0,T]$ with $X_n(t;c) \le X_1(t;d)$ the quantity

$$M(t) := \sup_{j;x\in[X_n(t;c),X_1(t;d)]} |V_j(t,x)|.$$

For $i \in \{1, \dots, n\}$ and $(t,x) \in D$ let $x_0 = z_i(t,x)$, and note $(s, X_i(s;x_0)) \in D_{v,I}$ for all $s \in [0,t]$. Integrating (1.24) along the characteristic $C_i(x_0)$ yields

$$|V_i(t,x)| \leq |V_i(0,x_0)| + \overline{H} \int_0^t M(s)ds = \overline{H} \int_0^t M(s)ds$$

since $x_0 \in I$ and $V_i(0,x_0) = 0$. One finds

$$M(t) \leq \overline{H} \int_0^t M(s)ds,$$

and Gronwall's Lemma implies $M(t) = 0$ for all t under consideration. This shows

$$V|_{D_{v,I}} = 0.$$

□

As a corollary of Lemma 1.2.2 we find that C^1-solutions to (1.1) are uniquely determined by their initial data at $t = 0$, and that supports get propagated by (1.1) if A3 holds.

Corollary 1.2.3. *Consider* (1.1) *satisfying A1 and A2 with smooth source term* B. *Let* $T > 0$. *Suppose* $u, v \in C^1([0,T] \times \mathbb{R})$ *are solutions of* (1.1), *and* $v \in C_b^1$ *with* $|v(t,x)| < \delta$.

(i) If $u(0,x) = v(0,x)$ *for all* $x \in \mathbb{R}$ *then* $u = v$.

(ii) If B *satisfies A3 and for some* $\tilde{c} < \tilde{d}$ *we have*

$$\operatorname{supp} u(0,\cdot) \subset [\tilde{c}, \tilde{d}]$$

then for all $t \in [0,T]$ *it holds*

$$\operatorname{supp} u(t,\cdot) \subset [\tilde{c} + \lambda_1(0)t, \tilde{d} + \lambda_N(0)t].$$

Proof. (i): For $(t,x) \in [0,T] \times \mathbb{R}$ apply Lemma 1.2.2 with $I = [c,d]$ such that $(t,x) \in D_{v,I}$.
(ii): Apply Lemma 1.2.2 with $v \equiv 0$ and $I = [c, \tilde{c}]$ resp. $I = [\tilde{d}, d]$ for arbitrary $c < \tilde{c}$ resp. $d > \tilde{d}$. Note that assumption A3 is needed for 0 to be a solution of (1.1). □

A local existence result for (1.1) is stated in the next lemma. For later use note that there is a constant $c_1 = c_1(\delta) \geq 1$ such that for all $u \in \mathbb{R}^n, v \in B_\delta(0)$ and $u_j \in \mathbb{R}$ with $u = \sum_j u_j r_j(v)$ it holds

$$c_1^{-1}|u| \leq \sup_j |u_j| \leq c_1|u|.$$

Lemma 1.2.4. *Consider* (1.1) *satisfying A1 and A2 with smooth source term* B. *Let* $C > 0, \delta_1 \in (0, c_1^{-2}\delta)$ *and* $K \in \mathbb{N} \cup \{\infty\}$. *Then there is a time* $T = T(\delta, \delta_1, C)$ *such that for all* $u_0 \in C_b^K(\mathbb{R})$ *with* $||u_0||_{C_b^0} \leq \delta_1$ *and* $||\partial_x u_0||_{C_b^0} \leq C$ *the system* (1.1) *has a unique solution* $u \in C_b^K([0,T] \times \mathbb{R})$ *with initial data* $u(0,\cdot) = u_0$ *and* $||u||_{C_b^0} < \delta$.

Proof. We give a detailed proof adapting the strategy sketched in [41] for the case $B \equiv 0$. Also note [61] and [78] on how to deal with strictly hyperbolic systems in one space dimension.

Let $K \in \mathbb{N}$ and fix $\delta_1 \in (0, c_1^{-2}\delta)$ and $C > 0$. The proof has three steps: First one considers data in $C_b^\infty(\mathbb{R})$ proving Lemma 1.2.4 in case of $K = \infty$. This is followed by an approximation procedure using the result of the first step to deal with data $u_0 \in C_0^K(\mathbb{R})$. Then one concludes by exploiting the finite speed of propagation property given in Lemma 1.2.2.

Step 1: Let $u_0 \in C_b^\infty(\mathbb{R})$ with $|u_0(x)| \leq \delta_1$ and $|\partial_x u_0(x)| \leq C$. Set $u^0(t,x) = u_0(x)$ and $w^0(t,x) = \partial_x u_0(x)$ for $(t,x) \in \mathbb{R}^2$ and denote by u_i^0, w_i^0 the i-th component of u^0, w^0. By the linear theory for strictly hyperbolic systems with smooth and bounded coefficients (see e.g. [78]) there exists a unique solution $u^1 \in C^\infty([0,\infty) \times \mathbb{R})$ of the Cauchy problem

$$u_t^1 + a(u^0)u_x^1 = B(u^0), \quad u^1(0,\cdot) = u_0, \tag{1.25}$$

with bounded derivatives on $[0,T] \times \mathbb{R}$ for all $T > 0$. Set $w^1 = \partial_x u^1$ and decompose u^1, w^1 along $\{r_j(u^0)\}_j$ to obtain

$$u^1(t,x) = \sum_j u_j^1(t,x) r_j(u^0(t,x)) \text{ and } w^1(t,x) = \sum_j w_j^1(t,x) r_j(u^0(t,x)).$$

The waves u_i^1, w_i^1 each satisfy a characteristic PDE given by

$$L_i(u^0)u_i^1 = h_i^{10}(u^0) + \sum_{j,p} g_{ijp}^{10}(u^0) w_j^0 u_p^1, \tag{1.26}$$

$$L_i(u^0)w_i^1 = \sum_j h_{ij}^{11}(u^0) w_j^0 + \sum_{j,p} g_{ijp}^{11}(u^0) w_j^0 w_p^1. \tag{1.27}$$

with smooth coefficients $h_i^{10}, h_{ij}^{11}, g_{ijp}^{10}, g_{ijp}^{11}$ uniformly bounded on $B_\delta(0)$. For $t \in [0,\infty)$ define

$$M_0^1(t) := \sup\left\{|u_j^i(t,x)| : \ i \in \{0,1\}, j \in \{1,\dots,n\}, x \in \mathbb{R}\right\},$$
$$M_1^1(t) := \sup\left\{|w_j^i(t,x)| : \ i \in \{0,1\}, j \in \{1,\dots,n\}, x \in \mathbb{R}\right\}.$$

By integrating (1.27) along the i-th characteristics of u^0 we find

$$|w_i^1(t,x)| \leq M_1^1(0) + c_{11} \int_0^t (1 + M_1^1(s)) M_1^1(s) ds$$

for a constant $c_{11} = c_{11}(\delta) > 0$ depending only on δ. Hence it holds

$$M_1^1(t) \leq M_1^1(0) + c_{11} \int_0^t (1 + M_1^1(s)) M_1^1(s) ds, \tag{1.28}$$

and by Gronwall's Lemma we deduce

$$M_1^1(t) \leq \frac{1}{(M_1^1(0)^{-1} + 1)\, e^{-c_{11} t} - 1} \tag{1.29}$$

as long as the denominator on the right is non-zero. We have

$$M_1^1(0) \leq c_1 ||\partial_x u_0||_{C_b^0} \leq c_1 C.$$

By (1.29) there exists $T_1 = T_1(\delta, C) > 0$ such that for all $t \in [0, T_1]$

$$M_1^1(t) \leq 2 c_1 C. \tag{1.30}$$

By integration of (1.26) along the i-th characteristic of u^0 we find analogously for $t \in [0, T_1]$ that

$$|M_0^1(t)| \leq M_0^1(0) + c_{10} \left(t + \int_0^t 2 c_1 C M_0^1(s) ds \right). \tag{1.31}$$

for a constant $c_{10} = c_{10}(\delta)$. Because $c_1^2 ||u_0||_{C_b^0} \leq c_1^2 \delta_1 < \delta$ there is a possibly smaller $T_1 = T_1(\delta, \delta_1, C)$ such that for all $t \in [0, T_1]$ and $x \in \mathbb{R}$ the inequality (1.31) implies by Gronwall's Lemma

$$|u^1(t,x)| \leq c_1 M_0^1(t) \leq c_1 (c_1 ||u_0||_{C_b^0} + c_{10} T_1) e^{2 c_{10} c_1 C T_1} < \delta. \tag{1.32}$$

Let $u^2 \in C_b^\infty([0, T_1] \times \mathbb{R})$ be the solution of

$$u_t^2 + a(u^1) u_x^2 = B(u^1), \;\; u^2(0, \cdot) = u_0.$$

Decompose u^2 and $w^2 := \partial_x u^2$ along $\{r_j(u^1)\}_j$ with

$$u^2(t,x) = \sum_j u_j^2(t,x) r_j(u^1(t,x)) \text{ and } w^2(t,x) = \sum_j w_j^2(t,x) r_j(u^1(t,x)).$$

Again we find characteristic PDEs given by

$$L_i(u^1) u_i^2 = \tilde{h}_i^{10}(u^0, u^1, u^2) + \sum_{j,p} g_{ijp}^{20}(u^0, u^1) w_j^1 u_p^2, \tag{1.33}$$

$$\begin{aligned} L_i(u^1) w_i^2 = &\sum_j \tilde{h}_{ij}^{11}(u^0, u^1) w_j^1 + \sum_j h_{ij}^{21}(u^0, u^1) w_j^2 \\ &+ \sum_{j,p} g_{ijp}^{21}(u^0, u^1) w_j^1 w_p^2, \end{aligned} \tag{1.34}$$

with smooth coefficient functions uniformly bounded for $|u^0|, |u^1|, |u^2| < \delta$. Here the equation $u^1_t = B(u^0) - a(u^0)w^1$ for u^1 was used, too. Setting for all $t \in [0, T_1]$

$$M^2_0(t) := \sup\left\{|u^i_j(t,x)| : \ i \in \{0,1,2\}, j \in \{1,\dots,n\}, x \in \mathbb{R}\right\},$$
$$M^2_1(t) := \sup\left\{|w^i_j(t,x)| : \ i \in \{0,1,2\}, j \in \{1,\dots,n\}, x \in \mathbb{R}\right\},$$

and integrating (1.33) and (1.34) along characteristics of u^1 we obtain in analogy to (1.28) and (1.30)

$$M^2_1(t) \le M^2_1(0) + c_{21} \int_0^t (1 + M^2_1(s)) M^2_1(s) ds, \tag{1.35}$$

with $c_{21} = c_{21}(\delta) \ge c_{11}$, and for $T_2 = T_2(\delta, C) \in (0, T_1)$

$$M^2_1(t) \le \frac{1}{(M^2_1(0)^{-1} + 1)\, e^{-c_{21} t} - 1} \tag{1.36}$$

and

$$M^2_1(t) \le 2c_1 C. \tag{1.37}$$

Furthermore, in analogy to (1.31) and (1.32) we find for all $t \in [0, T_2]$ the relations

$$|M^2_0(t)| \le M^2_0(0) + c_{20} \left(t + \int_0^t 2c_1 C M^2_0(s) ds \right) \tag{1.38}$$

with $c_{20} = c_{20}(\delta) \ge c_{10}$, and by choosing $T_2 = T_2(\delta, \delta_1, C)$ small enough

$$|u^2(t,x)| \le c_1 M^2_0(t) \le c_1(c_1 ||u_0||_{C^0_b} + c_{20} T_2) e^{2c_{20} c_1 C T_2} < \delta \tag{1.39}$$

for all $(t,x) \in [0, T_2] \times \mathbb{R}$.

Now using (1.39) we successively define a sequence $\{u^m\}_{m \in \mathbb{N}}$ of functions $u^m \in C^\infty_b([0, T_2] \times \mathbb{R})$ solving

$$u^m_t + a(u^{m-1}) u^m_x = B(u^{m-1}), \ \ u^m(0, \cdot) = u_0.$$

For $m \ge 2$ the characteristic PDEs of u^m and $w^m := \partial_x u^m$ are given by (1.33) and (1.34) with u^0, u^1, u^2, w^1 and w^2 replaced by $u^{m-2}, u^{m-1}, u^m, w^{m-1}$ and w^m, and decompositions

$$u^m = \sum_j u^m_j r_j(u^{m-1}) \text{ and } w^m = \sum_j w^m_j r_j(u^{m-1}).$$

The quantities

$$M^m_0(t) := \sup\left\{|u^i_j(t,x)| : \ i \in \{0,\dots,m\}, j \in \{1,\dots,n\}, x \in \mathbb{R}\right\},$$
$$M^m_1(t) := \sup\left\{|w^i_j(t,x)| : \ i \in \{0,\dots,m\}, j \in \{1,\dots,n\}, x \in \mathbb{R}\right\}$$

satisfy the inequalities (1.35)-(1.39) with M_0^2 and M_0^1 replaced by M_0^m and M_1^m. In particular, $|u^m(t,x)| < \delta$ for all $m \in \mathbb{N}_0$ and $(t,x) \in [0,T_2] \times \mathbb{R}$.

Uniform estimates for higher order derivatives of u^m are obtained by differentiation of (1.34) for each iterative w^m. Successively increasing the order of derivatives one finds that for all $K' \in \mathbb{N}$ there are smooth and monotonically increasing functions $d_{K'} : \mathbb{R} \to [0,\infty)$ with $d_{K'}(0) = 0$ such that for all $m \in \mathbb{N}_0$

$$||u^m||_{C_b^0} < \delta, \tag{1.40}$$

$$||\partial_x u^m||_{C_b^{K'-1}} \leq d_{K'}(||\partial_x u_0||_{C_b^{K'-1}}). \tag{1.41}$$

For $m \geq 1$ set

$$V^m := u^{m+1} - u^m,$$

decompose V^m according to

$$V^m(t,x) = \sum_k V_j^m(t,x) r_j(u^m(t,x)),$$

and define for $t \in [0,T_2]$

$$M_V^m(t) := \sup\left\{|V_j^i(t,x)| : \ i \in \{0,\dots,m\}, j \in \{1,\dots,n\}, x \in \mathbb{R}\right\}.$$

The waves V_i^m satisfy

$$\begin{aligned} L_i(u^m)V_i^m = &\sum_j H_{ij}^1(u^m, u^{m-1}, u_x^m)V_j^{m-1} \\ &+ \sum_j H_{ij}^2(u^m, u^{m-1}, u_x^m)V_j^m \end{aligned} \tag{1.42}$$

with coefficients uniformly bounded on $B_\delta(0) \times B_\delta(0) \times B_{2c_1^2 C}(0)$. Integrating (1.42) along characteristics of u^m and $M_V^m(0) = 0$ yields for all $t \in [0,T_2]$

$$M_V^m(t) \leq c_{1V} \int_0^t M_V^{m-1}(s)ds + c_{2V} \int_0^t M_V^m(s)ds \tag{1.43}$$

with constants $c_{1V} = c_{1V}(\delta, C)$ and $c_{2V} = c_{2V}(\delta, C)$. Setting $c_V := c_{1V} e^{c_{2V} T_2}$ an application of Gronwall's Lemma to (1.43) gives for $m \geq 2$ and all $t \in [0,T_2]$

$$M_V^m(t) \leq c_V \int_0^t M_V^{m-1}(s)ds \leq 2c_1\delta_1 \frac{(c_V t)^{m-1}}{(m-1)!}$$

if for $s \in [0,T_2]$ it holds

$$M_V^{m-1}(s) \leq 2c_1\delta_1 \frac{(c_V s)^{m-2}}{(m-2)!}.$$

Since $M_V^1(t) \leq 2c_1\delta_1$ we find that $\{M_V^m(t)\}_m$ is summable uniformly for all $t \in [0, T_2]$. Hence $\{u^m\}_m$ is a Cauchy sequence in $C_b^0([0, T_2] \times \mathbb{R})$ with limit $u \in C_b^0([0, T_2] \times \mathbb{R})$. For all $\alpha \in \mathbb{N}_0^2$ there is a uniform modulus of continuity of $\{\partial_{t,x}^\alpha u^m\}_m$ because the derivatives of u^m satisfy the uniform estimates (1.40), (1.41). We find that u is a classical solution of

$$u_t + a(u)u_x = B(u), \;\; u(0, \cdot) = u_0,$$

with $u \in C_b^\infty([0, T_2] \times \mathbb{R})$ and derivatives bounded by the constants in (1.41). Lemma 1.2.4 is proved for $u_0 \in C_b^\infty(\mathbb{R})$.

Step 2: With T_2 like above consider $u_0 \in C_0^K(\mathbb{R})$ with $||u_0||_{C_b^0} \leq \delta_1$ and $||\partial u_x||_{C_b^0} \leq C$. Let $\phi \in C_0^\infty(\mathbb{R})$ with $0 \leq \phi \leq 1$ and $\int_{\mathbb{R}} \phi(x)dx = 1$. Set $\phi_m(x) := m\phi(mx)$ and mollify u_0 by defining

$$u_0^m(x) := \phi_m * u_0(x) = \int\limits_{\mathbb{R}} \phi_m(y)u(x - y)dy.$$

Then $u_0^m \in C_0^\infty(\mathbb{R})$ with

$$||\partial_x^{k'} u_0^m||_{C_b^0} = ||\partial_x^{k'} u_0||_{C_b^0}$$

for $k' \leq K$. In particular, we have $|u_0^m(x)| \leq \delta_1$ and $|\partial_x u_0^m(x)| \leq C$, and

$$||u_0^m - u_0||_{C_b^K} \to 0 \text{ for } m \to \infty. \tag{1.44}$$

By the first step of the proof we define a sequence $\{u^m\}_m$ of functions $u^m \in C_b^\infty([0, T_2] \times \mathbb{R})$ by letting u^m be the solution of

$$u_t^m + a(u^m)u_x^m = B(u^m), \;\; u^m(0, \cdot) = u_0^m$$

with

$$||u^m||_{C_b^0} < \delta \tag{1.45}$$

$$||\partial_x u^m||_{C_b^{K'-1}} \leq d_{K'}(||\partial_x u_0^m||_{C_b^{K'-1}}) = d_{K'}(||\partial_x u_0||_{C_b^{K'-1}}) \tag{1.46}$$

for all $K' \leq K$. Applying the method of characteristics and using (1.45) and (1.46) with $K' = 1$ we find that for $m_1, m_2 \in \mathbb{N}$

$$||u^{m_1} - u^{m_2}||_{C_b^0} \leq c(||u_0||_{C_b^1})||u_0^{m_1} - u_0^{m_2}||_{C_b^0}.$$

From (1.44) we deduce that $\{u^m\}_m$ is a Cauchy sequence in $C_b^0([0, T_2] \times \mathbb{R})$ with limit $u \in C_b^0([0, T_2] \times \mathbb{R})$. The uniform bounds (1.45), (1.46) imply $u \in C_b^{K-1}([0, T_2] \times \mathbb{R})$.

In order to show that u is a classical solution of

$$u_t + a(u)u_x = B(u), \;\; u(0, \cdot) = u_0 \tag{1.47}$$

in $C_b^K([0, T_2] \times \mathbb{R})$ we proceed by extending the above a-priori estimates to the modulus of continuity for the derivatives of u of order K. To this end let $m \in \mathbb{N}$.

Then $u^m \in C_b^\infty([0,T_2]\times\mathbb{R})$ solves

$$u_t^m + a(u^m)u_x^m = B(u^m),\ \ u^m(0,\cdot) = u_0^m$$

and the estimates (1.45), (1.46) hold. The derivative $w^m = \partial_x u^m$ with decomposition $w^m = \sum_j w_j^m r_j(u^m)$ satisfies by (1.19)

$$L_i(u^m)w_i^m = \sum_{j,p}\gamma_{ijp}(u^m)w_j^m w_p^m + \sum_j B_{ij}(u^m)w_j^m. \tag{1.48}$$

We concentrate on the case $K=1$. If $K>1$ then for $\alpha\in\mathbb{N}_0^2$ with $|\alpha| = K-1$ apply $\partial_{t,x}^\alpha$ to (1.48) and find

$$\partial_t w_i^{m\alpha} + \lambda_i(u^m)\partial_x w_i^{m\alpha} = \sum_j H_{ij}^{m\alpha}(t,x)w_j^{m\alpha} + G_i^{m\alpha}(t,x)$$

for $w_i^{m\alpha} = \partial^\alpha w_i^m$ and functions $H_{ij}^{m\alpha}, G_i^{m\alpha} \in C_b^1([0,T_2]\times\mathbb{R})$ depending on derivatives of u^m of order $\le K-1$ with a global Lipschitz constant which can be chosen independent of m by the uniform bounds (1.45), (1.46).

The moduli of continuity e_0 and e_1 of u_0 and $\partial_x u_0$ are well-defined due to the compact support of u_0 and are given by

$$\begin{aligned} e_0(r) &= \sup\{|u_0(x_1) - u_0(x_2)| :\ |x_1-x_2|<r\},\\ e_1(r) &= \sup\{|\partial_x u_0(x_1) - \partial_x u_0(x_2)| :\ |x_1-x_2|<r\},\end{aligned}$$

for $r>0$. Define the modulus of continuity of w^m at $t\in[0,T_2]$ for $r>0$ by

$$\epsilon(w^m,t,r) := \sup\{|w_i^m(t,x_1) - w_i^m(t,x_2)| :\ i\in\{1,\dots,n\}, |x_1-x_2|<r\}.$$

Note that for all t, r

$$\epsilon(w^m,t,r) < \infty$$

because w^m has bounded derivatives. Let $i\in\{1,\dots,n\}$ and denote by Ψ_i^m the flow of the backwards characteristics equation

$$\begin{cases} t' = -1\\ x' = -\lambda_i(u^m(t,x)).\end{cases} \tag{1.49}$$

Let $r>0$ and $x_1,x_2\in\mathbb{R}$ with $|x_1-x_2|<r$. By continuous dependence on the data of solutions to (1.49) there is $N\in\mathbb{N}$ for all m (by (1.45),(1.46)) such that

$$|\Psi_i^m(s,(t,x_1)) - \Psi_i^m(s,(t,x_2))| \le |x_1-x_2|N \le Nr$$

for all $0 \leq s \leq t \leq T_2$. Let $t \in [0, T_2]$. Integrating (1.48) along $C_i(z_i(t, x_1))$ and $C_i(z_i(t, x_2))$ and taking differences yields

$$\begin{aligned}|w_i^m(t,x_1) - w_i^m(t,x_2)| \leq \\ &\leq \epsilon(w^m, 0, Nr) + c(||u_0||_{C_b^1}) \left(Nr + \int_0^t \epsilon(w^m, s, Nr) ds \right) \\ &\leq Nc(||u_0||_{C_b^1}) \left(e_0(r) + e_1(r) + r + \int_0^t \epsilon(w^m, s, r) ds \right)\end{aligned}$$

where we have used

$$\epsilon(w^m, s, Nr) \leq N\epsilon(w^m, s, r)$$

and

$$\epsilon(w^m, 0, r) \leq c(e_0(r) + e_1(r))$$

uniformly in m. Proceeding like above for all indices i and again applying Gronwall's Lemma gives

$$\epsilon(w^m, t, r) \leq c(||u_0||_{C_b^1}) (e_0(r) + e_1(r) + r)$$

for all $t \in [0, T_2]$ (with a possibly different constant $c(||u_0||_{C_b^1})$).

Let $(t_1, x_1), (t_2, x_2) \in [0, T_2] \times \mathbb{R}$ with $|(t_1, x_1) - (t_2, x_2)| < r$ and $t_2 < t_1$ (otherwise interchange the indices). Set $(t_2, x_{i,1}) = \Psi_i^m(t_1 - t_2, (t_1, x_1))$. By integration of (1.49) from 0 to $t_1 - t_2$ with initial data (t_1, x_1) there is $M \in \mathbb{N}$ depending only on $||u_0||_{C_b^1}$ such that

$$|(t_1, x_1) - (t_2, x_{i,1})| \leq M|t_1 - t_2| \leq Mr.$$

We find

$$\begin{aligned}|w_i^m(t_1,x_1) - w_i^m(t_2,x_2)| \leq &|w_i^m(t_1,x_1) - w_i^m(t_2,x_{i,1})| \\ &+ |w_i^m(t_2,x_{i,1}) - w_i^m(t_2,x_2)| \\ \leq &c(||u_0||_{C_b^1})(|t_1 - t_2| + (M+1)\epsilon(w^m, t_2, r)) \\ \leq &c(||u_0||_{C_b^1})(r + e_0(r) + e_1(r)) =: \epsilon(r),\end{aligned}$$

i.e. for all $m \in \mathbb{N}$ the modulus of continuity of w^m is bounded from above by $\epsilon(r)$. We conclude by Arzela and Ascoli's Theorem that $u \in C_b^K([0, T_2] \times \mathbb{R})$ and that u solves (1.47).

Step 3: Finally, choose $T = T(\delta, \delta_1, 2C) > 0$ such that the conclusion of Lemma 1.2.4 holds for all initial data $u_0 \in C_0^K(\mathbb{R})$ with $||u_0||_{C_b^0} \leq \delta_1$ and $||\partial_x u_0||_{C_b^0} \leq 2C$.

Let $u_0 \in C_b^K([0, T] \times \mathbb{R})$ with $||u_0||_{C_b^0} \leq \delta_1$ and $||\partial_x u_0||_{C_b^0} \leq C$. We make use of the finite speed of propagation property as stated in Lemma 1.2.2 by considering a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1, \chi \equiv 1$ on $[-1, 1]$ and

$\operatorname{supp}\chi \in [-2,2]$. Set for $m \in \mathbb{N}$

$$\chi_m(x) := \chi(m^{-1}x) \text{ and } u_0^m = \chi_m u_0.$$

Then for all $m \geq m'$ with m' large enough we have $u_0^m \in C_0^k(\mathbb{R}), ||u_0^m||_{C_b^0} \leq \delta_1$ and $||\partial_x u_0^m||_{C_b^0} \leq 2C$. By the previous steps there is a unique solution $u^m \in C_b^K([0,T]\times\mathbb{R})$ of

$$u_t^m + a(u^m)u_x^m = B(u^m), \;\; u^m(0,\cdot) = u_0^m.$$

With the notation of Lemma 1.2.2, for $(t,x) \in [0,T]\times\mathbb{R}$ define $u = u(t,x)$ by choosing m so large that $(t,x) \in D_{u^m,[-m,m]}$ and setting

$$u(t,x) := u^m(t,x).$$

Invoking Lemma 1.2.2 we find that u is well-defined and the unique solution of

$$u_t + a(u)u_x = B(u), \;\; u(0,\cdot) = u_0,$$

with $u \in C_b^K([0,T]\times\mathbb{R})$ and $|u(t,x)| < \delta$. □

Lemma 1.2.4 finishes our discussion on local-in-time classical solutions to (1.1). In the following sections we are interested in how far one can extend these local solutions, and proving Theorem 1.0.1 will show that in fact for some initial data the classical solution to a system (1.1) satisfying A1-A3 can only be extended up to a finite time.

This section is closed with some relations and expressions concerning certain coefficient functions appearing in the above analysis. From [48], p. 380, we have

$$c_{ijm}(u) = \left(\frac{d}{ds}(l_i(u)a(u+sr_m(u))r_j(u)\right)_{s=0}$$

and ([48], p. 383)

$$\gamma_{ijm} = \gamma_{imj}, \;\; \gamma_{iii} = -c_{iii},$$
$$2\gamma_{iim} = -c_{iim} - c_{imi} + \sum_{j,j\neq i}\frac{\lambda_i - \lambda_m}{\lambda_j - \lambda_i}c_{ijm}\langle l_j, l_i\rangle \text{ if } m \neq i,$$
$$2\gamma_{ijm} = -\frac{\lambda_j - \lambda_m}{\lambda_j - \lambda_i}c_{ijm} - \frac{\lambda_m - \lambda_j}{\lambda_m - \lambda_i}c_{imj} \text{ if } j \neq i, m \neq i.$$

The coefficients Γ_{ijk} are determined by the requirement (pp. 219-220, [41])

$$\sum_{j,m}\gamma_{ijm}(u)w_j w_k + \sum_m w_i w_m\langle D\lambda_i(u), r_m(u)\rangle = \sum_{j,m}\Gamma_{ijm}(u)w_j w_m$$

for all $w \in \mathbb{R}^n$. An explicit expression for B_{im} may be obtained from the requirement that for all $w \in \mathbb{R}^n$ it holds

$$\sum_m B_{im}(u)w_m = \sum_m l_i DB(u)r_m w_m + \sum_{j,m,j\neq i} \frac{1}{\lambda_j - \lambda_i} c_{ijm}(u)(l_m B(u))(\langle l_j, l_i\rangle w_i - w_j).$$

1.3 Initial data and smallness properties

In this section a class of initial data that will entail the formation of a singularity in the associated classical solution of (1.1) is constructed. We prove some smallness properties of solutions of a scaled version of (1.1) attaining the constructed data at $t = 0$, and state a first consequence of these properties. For the rest of this chapter we consider source terms B satisfying A3.

Let us begin with the construction of the special initial data. Let $U : I \to B_\delta(0)$ be an integral curve of r_k through 0 defined on an open interval $I \subset \mathbb{R}$ with $0 \in I$:

$$U'(\xi) = r_k(U(\xi)) \ (\xi \in I), \ U(0) = 0.$$

Let $\alpha \in C_0^\infty(\mathbb{R})$ with $\alpha(\mathbb{R}) \subset I, \operatorname{supp}\alpha \subset (-1/2, 1/2)$ and $\max_x |\alpha'| = \max_x \alpha' > 0$. For $\varepsilon \in (0, 1]$ define

$$u^{0\varepsilon}(x) = U(\varepsilon\alpha(x)) \ (x \in \mathbb{R}).$$

Then $u^{0\varepsilon} \in C_0^\infty(\mathbb{R})$.

Remark 1.3.1. *For $B \equiv 0$, these data give rise to a breaking simple-wave (see [48]). The idea of using the same data for the system with non-vanishing source is that the associated blow-up of the quantity w_k persists if the source term is small.*

The main purpose of this section consists in showing the following lemma.

Lemma 1.3.2. *For any fixed $\overline{T} \in (0, \infty)$ there exist constants $c_J, c_M, c_S, c_V > 0$ and $\nu > 0$ such that if for $\varepsilon, \kappa \in (0, \nu]$ and*

$$0 < T \leq \overline{T}\varepsilon^{-1}$$

a function $u \in C^2([0, T] \times \mathbb{R})$ solves the Cauchy problem

$$\begin{cases} u_t + a(u)u_x = \varepsilon\kappa B(u), \\ u(0, x) = u^{0\varepsilon}(x). \end{cases} \tag{1.50}$$

then for all $t \in [0,T]$ the quantities

$$\begin{aligned}
J(t) &:= \sup_{s\in[0,t]} \int_{c_k(s)}^{d_k(s)} |w_k(s,x)|dx,\\
M(t) &:= \sup_{(s,x)\in[0,t]\times\mathbb{R}} |u(s,x)|,\\
S(t) &:= \sup_{s\in[0,t]} d_k(s) - c_k(s),\\
\tilde{V}(t) &:= \max_{i\neq k} \sup_{(s,x)\in[0,t]\times\mathbb{R}} |w_i(s,x)|,\\
W_k^{out}(t) &:= \sup_{\substack{(s,x)\in[0,t]\times\mathbb{R},\\ (s,x)\notin R_k(t)}} |w_k(s,x)|,\\
V(t) &:= W_k^{out}(t) + \tilde{V}(t),
\end{aligned}$$

satisfy

$$J(t) < c_J\varepsilon, \tag{1.51}$$
$$M(t) < c_M\varepsilon < \delta, \tag{1.52}$$
$$S(t) < c_S, \tag{1.53}$$
$$V(t) < c_V\varepsilon^2. \tag{1.54}$$

Here we have set

$$\begin{aligned}
c_k(t) &:= X_k(t;-1/2),\\
d_k(t) &:= X_k(t;1/2),\\
R_k(t) &:= \{(s,x)\in[0,t]\times\mathbb{R} :\ x\in[c_k(s),d_k(s)]\}.
\end{aligned}$$

Remark 1.3.3. *Since we expect the solution u to be of the same size as $u^{0\varepsilon}$ we find that J and M should be comparable to $u^{0\varepsilon}$, too, hence of order ε. The quantities $\tilde{V}$ and W_k^{out} are initially equal to zero, and, thinking of the idea of perturbing the simple wave, we hope that the interaction between $w_i, i \neq k$, and w_k is weak, and that w_k along the k-th characteristics outside R_k remains small compared to w_k inside R_k by the propagation properties of the hyperbolic differential operator in* (1.50).

Proof. Let $\overline{T} > 0$. For

$$\begin{aligned}
\overline{c} &:= \max\left\{1, \max_i \sup_{u\in B_\delta(0)} \sum_{j,m} |c_{ijm}(u)|\right\},\\
\overline{r} &:= \sup_{u\in B_\delta(0)} \sum_m |r_m(u)|,
\end{aligned}$$

set

$$\begin{aligned} c_J &:= 2\max_x \alpha'(x), \\ c_V &:= 2\overline{B}c_\lambda^{-1}c_J(1+\overline{B}\,\overline{T}), \\ c_S &:= 2\overline{c}(1+c_J\overline{T}), \\ c_M &:= 2\overline{r}(c_J + c_V(1+\overline{T}(\lambda_N(0)-\lambda_1(0)))). \end{aligned}$$

Take ν so small that

$$c_M\nu < \delta \tag{1.55}$$

holds. Let $\varepsilon, \kappa \in (0,\nu]$ with ν to be determined further later on, and let $T \in (0, \overline{T}\varepsilon^{-1}]$. Suppose we are given a solution $u \in C^2([0,T]\times\mathbb{R})$ of the Cauchy problem (1.50). By Corollary 1.2.3 we find that $u \in C_0^2([0,T]\times\mathbb{R})$. At $t=0$ it holds

$$|u(0,x)| = |u^{0\varepsilon}(x)| = \left| \int_{-\infty}^{x} \varepsilon\alpha'(y) r_k(U(\varepsilon\alpha(y))dy \right| \leq \varepsilon\overline{r}c_J \leq \frac{c_M}{2}\varepsilon < c_M\varepsilon \tag{1.56}$$

Because $u \in C_0^2$ the function M is continuous on $[0,T]$ and we find a $t_1 \in (0,T]$ such that (1.52) holds for all $t \in [0,t_1]$ since the stronger inequality (1.56) holds at $t=0$. Hence, on $[0,t_1]$ the functions $J, S, V, \tilde{V}, W_k^{out}$ are well-defined, continuous and increasing. We will show that the set

$$E_1 := \{t \in [0,T] : \ (1.51) - (1.54) \text{ hold on } [0,t]\}$$

is non-empty, open and closed, hence equal to $[0,T]$. We have by

$$u_x(0,x) = \varepsilon\alpha'(x) r_k(u(0,x)) = w_k(0,x) r_k(u(0,x))$$

that initially the waves w_j satisfy

$$w_i(0,x) = 0 \qquad \text{if } i \neq k, \tag{1.57}$$

$$w_k(0,x) = \varepsilon\alpha'(x). \tag{1.58}$$

By (1.57), (1.58) and the compact support of α in $(-1/2, 1/2)$ we find $0 \in E_1$ since

$$\begin{aligned} J(0) &\leq \varepsilon \max_x \alpha' = \varepsilon\frac{c_J}{2} \\ S(0) &= 1 \leq \frac{c_S}{2} \\ V(0) &= 0 \leq \frac{c_V}{2}. \end{aligned}$$

By continuity and monotonicity of J, M, S and V the set E_1 is open in $[0,T]$. It remains to show that E_1 is closed which is by monotonicity equivalent to:

If for $T' \in (0,T]$ (1.51)-(1.54) hold on $[0,T')$ then (1.51)-(1.54) hold at $t = T'$.

Suppose for $T' \in (0,T]$ we have $[0,T') \subset E_1$ and let $t \in [0,T')$. Note that $M(T') < \delta$ by (1.52). For $s \in [0,t]$ it holds (compare [48] p. 394)

$$\begin{aligned}\frac{d}{ds}(d_k(s) - c_k(s)) &= \int\limits_{c_k(s)}^{d_k(s)} \sum_m c_{kkm}(u(s,x))w_m(s,x)dx \\ &\leq \overline{c}(\tilde{V}(s)S(s) + J(s)).\end{aligned}$$

Integrating the last relation over $[0,t]$ we find by $t \leq \overline{T}\varepsilon^{-1}$ and $[0,t] \subset E_1$ that

$$\begin{aligned}S(t) &\leq S(0) + \overline{c}(\tilde{V}(t)S(t)t + J(t)t) \\ &\leq \overline{c}(1 + c_J\overline{T}) + \overline{c}\overline{T}c_V\varepsilon S(t).\end{aligned}$$

If ν satisfies

$$1 - \overline{c}\overline{T}c_V\nu \geq \frac{2}{3} \tag{1.59}$$

then

$$S(t) \leq \frac{3}{2}\overline{c}(1 + c_J\overline{T}) < c_S$$

which shows $S(T') < c_S$.

To estimate $J(t)$ we use Lemma 1.2.1 with $\tau = \{s\} \times [c_k(s), d_k(s)]$ for $s \in [0,t]$ to obtain

$$\int\limits_{c_k(s)}^{d_k(s)} |w_k(s,x)|dx \leq J(0) + \int\limits_{A_k(\tau)} \left| \sum_{j,m} \Gamma_{kjm}(u)w_jw_m + \varepsilon\kappa \sum_m B_{im}(u)w_m \right| dxdt.$$

By (1.22) we see that the terms involving $\Gamma_{kjm}(u)w_jw_m$ are essentially linear and we get

$$\begin{aligned}J(t) &\leq J(0) + \overline{\Gamma}[\tilde{V}(t)^2S(t)t + \tilde{V}(t)J(t)t] + \varepsilon\kappa\overline{B}[\tilde{V}(t)S(t)t + J(t)t] \\ &\leq \varepsilon\frac{c_J}{2} + \varepsilon^3\overline{\Gamma}c_V^2c_S\overline{T} + \varepsilon^2\kappa\overline{B}c_Vc_s\overline{T} + J(t)[\varepsilon\overline{\Gamma}c_V\overline{T} + \kappa\overline{B}\,\overline{T}].\end{aligned}$$

If ν satisfies

$$\nu^2(\overline{\Gamma}c_V^2c_S\overline{T} + \overline{B}c_Vc_s\overline{T}) \leq \frac{1}{8}c_J, \tag{1.60}$$

$$1 - \nu(\overline{\Gamma}c_V\overline{T} + \overline{B}\,\overline{T}) \geq \frac{5}{6} \tag{1.61}$$

then

$$J(t) \leq \frac{3}{4} c_J \varepsilon < c_J \varepsilon$$

implying $J(T') < c_J \varepsilon$.

We estimate $u(s,x)$ for all $(s,x) \in [0,t] \times \mathbb{R}$ using Corollary 1.2.3 by

$$\begin{aligned} |u(s,x)| &= \left| \int_{-\infty}^{x} u_x(s,y) dy \right| \\ &\leq \int_{-1/2+\lambda_1(0)s}^{1/2+\lambda_N(0)s} |u_x(s,x)| dy \\ &\leq \overline{r}(J(t) + V(t)(1 + t(\lambda_N(0) - \lambda_1(0)))) \\ &\leq \varepsilon \overline{r}(c_J + c_V(1 + \overline{T}(\lambda_N(0) - \lambda_1(0)))) \\ &= \varepsilon \frac{c_M}{2} \end{aligned}$$

yielding $M(T') \leq \varepsilon \frac{c_M}{2} < \varepsilon c_M < \delta$.

The estimates for $V(t)$ are the longest ones. We begin with an estimate of $W_k^{out}(t)$: Let $(s,x) \in [0,t] \times \mathbb{R}$ with $(s,x) \notin R_k(t)$. Then for $z_k = z_k(s,x)$ and all $\sigma \in [0,t]$ we find

$$(\sigma, X_k(\sigma, z_k)) \notin R_k(t) \tag{1.62}$$

because the k-th characteristic curves $C_k(y)$ may not cross for different choices of $y \in \mathbb{R}$ since they are defined through solutions of (1.17) with $i = k$. Furthermore, by (1.58) it holds

$$w_k(0, z_k) = 0. \tag{1.63}$$

Integrating $L_k w_k$ along $C_k(z_k)$ and using (1.19), (1.62) and (1.63) yields

$$\begin{aligned} |w_k(s,x)| &= \left| \int_0^s \sum_{j,m} \gamma_{kjm}(u) w_j w_m + \varepsilon\kappa \sum_m B_{im}(u) w_m d\sigma \right| \\ &\leq t\overline{\gamma}(V(t) + \varepsilon\kappa\overline{B})V(t), \end{aligned}$$

and therefore by $[0,t] \subset E_1$

$$W_k^{out}(t) \leq \overline{T}(\overline{\gamma} c_V \varepsilon + \overline{B}\kappa) V(t). \tag{1.64}$$

We turn to the estimate for $\tilde{V}(t)$. Let $i \neq k$ and $(s,x) \in [0,t] \times \mathbb{R}$. For $z_i = z_i(s,x)$ define

$$\omega_i(s,x) := \{\sigma \in [0,t] : \ (\sigma, X_i(\sigma; z_i)) \in R_k(t)\}.$$

Then $\omega_i(s,x)$ is a (possibly empty) closed interval in $[0,t]$ and the trace of

$$\omega_i(s,x) \ni \sigma \mapsto (\sigma, X_i(\sigma; z_i))$$

defines a smooth arc $\tau_i(s,x)$ in $[0,t]\times\mathbb{R}$ intersecting the k-th characteristic curves transversally by the separation of eigenvalues of a (1.12). We have $w_i(0,z_i)=0$ by (1.57) and integrating the equation (1.19) for $L_i w_i$ along $C_i(z_i)$ gives

$$|w_i(s,x)| = \left| \int_0^s \sum_{j,m} \gamma_{ijm}(u) w_j w_m + \varepsilon\kappa \sum_m B_{im}(u) w_m d\sigma \right|.$$

Using the relations (1.20) for the coefficients γ_{ijm} yields

$$|w_i(s,x)| \le (\overline{\gamma} V(t) + \varepsilon\kappa \overline{B}) \left[tV(t) + \int_0^t |w_k(\sigma, X_i(\sigma; z_i))| d\sigma. \right] \tag{1.65}$$

We need to estimate the integral of $|w_k|$ along $C_i(z_i)$. To this end we make use of Lemma 1.2.1 and (1.12):

$$\begin{aligned}
&\int_0^t |w_k(\sigma, X_i(\sigma; z))| d\sigma \qquad (1.66)\\
&\qquad \le W_k^{out}(t)t + \int_{w_i(s,x)} |w_k| \frac{\lambda_i(u)-\lambda_k(u)}{\lambda_i(u)-\lambda_k(u)} d\sigma \\
&\qquad \le V(t)t + c_\lambda^{-1} \int_{\tau_i(s,x)} |w_k(dx - \lambda_k(u)dt)| \\
&\qquad \le V(t)t + c_\lambda^{-1} J(0) \\
&\qquad\quad + c_\lambda^{-1} \int_{A_k(\tau_i)} \left| \sum_{j,m} \Gamma_{kjm}(u) w_j w_m + \varepsilon\kappa \sum_m B_{im}(u) w_m \right| dxdt \\
&\qquad \le V(t)t + c_\lambda^{-1} \left[c_J \varepsilon + t(\overline{\Gamma} V(t) + \varepsilon\kappa \overline{B})(V(t)S(t) + J(t)) \right]
\end{aligned}$$

where the relations (1.22) for Γ_{kjm} were used in the last step. Combining the estimates (1.64)-(1.66) we find a function $Q = Q(\varepsilon,\kappa)$ given by

$$\begin{aligned}
Q(\varepsilon,\kappa) =& 3\overline{T}(\overline{\gamma} c_V \varepsilon + \overline{B}\kappa) + \overline{\gamma}\varepsilon c_\lambda^{-1}[c_J + \overline{T}(\overline{\Gamma} c_V \varepsilon + \overline{B}\kappa)(c_V c_S \varepsilon + c_J)] \\
&+ \overline{B}\kappa c_\lambda^{-1} \overline{T} \varepsilon (\overline{\Gamma}(c_V c_S \varepsilon + c_J) + \overline{B} c_S \kappa),
\end{aligned}$$

and a constant $\overline{Q} := Q(1,1)$ such that for all $\varepsilon,\kappa \in (0,\nu]$ we have $Q(\varepsilon,\kappa) \le \overline{Q}\nu$ and

$$\begin{aligned}
V(t) &\le Q(\varepsilon,\kappa) V(t) + \kappa \overline{B} c_\lambda^{-1} c_J (1 + \kappa \overline{B}\,\overline{T}) \varepsilon^2 \\
&\le \overline{Q}\nu V(t) + \frac{c_V}{2}\varepsilon^2.
\end{aligned}$$

If ν satisfies

$$1 - \nu\overline{Q} \leq \frac{3}{4} \tag{1.67}$$

then

$$V(t) \leq \frac{2}{3}c_V\varepsilon^2 < c_V\varepsilon^2.$$

Hence $T' \in E_1$ follows if ν satisfies the smallness conditions (1.55), (1.59), (1.60), (1.61) and (1.67). □

As a first application of Lemma 1.3.2 we prove that any C^1-solution of (1.50) on $[0,T]$ must be a C^2-solution of (1.50) on $[0,T]$ if T is smaller than a certain ε-depended bound.

Lemma 1.3.4. *Let $\overline{T} \in (0,\infty)$. There exists $\nu > 0$ such that if for $\varepsilon, \kappa \in (0,\nu]$ and $T \in (0, \overline{T}\varepsilon^{-1}]$ a function $u \in C^1([0,T]\times\mathbb{R})$ solves (1.50) then $u \in C^2([0,T]\times\mathbb{R})$.*

Proof. Let $\nu > 0$ be so small such that the statement of Lemma 1.3.2 holds with a constant $c_M > 0$, and such that $c_1^2 c_M \nu < \delta$. For $\varepsilon, \kappa \in (0,\nu]$ and $T \in (0, \overline{T}\varepsilon^{-1}]$ suppose $u \in C^1([0,T]\times\mathbb{R})$ solves (1.50). By Corollary 1.2.3 we have $u \in C_0^1([0,T]\times\mathbb{R})$ and there is a constant $C > 0$ such that for all $(t,x) \in [0,T]\times\mathbb{R}$

$$|u_x(t,x)| \leq C.$$

Let $\Delta T = \Delta T(\delta, c_M\nu, C)$ be a local time of existence of a C^2-solution to (1.50) like in Lemma 1.2.4. Note that the replacement of the source term ($B \mapsto \varepsilon\kappa B$) allows for a ε,κ-uniform choice of ΔT if $\varepsilon\kappa \leq 1$ since only upper bounds of the source term and its derivatives play a role in the proof of Lemma 1.2.4.

We argue by continuous induction. Set

$$E_3 := \{T' \in [0,T] :\ u \in C^2([0,T']\times\mathbb{R})\}.$$

We show that E_3 is non-empty, relatively open and closed. By Lemma 1.2.4 there is a C^2-solution on $[0,\Delta T]$ of (1.50) because $u^{0\varepsilon} \in C_0^2(\mathbb{R})$ with $||u_x^{0\varepsilon}||_{C_b^0} \leq C$ and $||u^{0\varepsilon}||_{C_b^0} \leq \varepsilon c_M$ by Lemma 1.3.2. By Corollary 1.2.3 the C^2-solution has to be equal to u on $[0,\Delta T]$, i.e. $\Delta T \in E_3$.

Let $T' \in E_3$. Because $u(T',\cdot) \in C_0^2(\mathbb{R})$, $||u_x(T',\cdot)||_{C_b^0} \leq C$ and $||u(T',\cdot)||_{C_b^0} < \varepsilon c_M$, by Lemma 1.3.2, there is a solution $\tilde{u} \in C^2([T', T'+\Delta T]\times\mathbb{R})$ of

$$u_t + a(u)u_x = \varepsilon\kappa B(u) \tag{1.68}$$

with $\tilde{u}(T',x) = u(T',x)$ for $x \in \mathbb{R}$. Uniqueness again yields $u \equiv \tilde{u}$ on $[T', T'+\Delta T] \cap [T', T]$. We conclude that E_3 is open in $[0,T]$.

Let $T' \in [0,T]$. We show that if $[0,T') \subset E_3$ than $T' \in E_3$. This implies the closedness of E_3. Because $E_3 \neq \emptyset$ the case $T' = 0$ follows immediately. Therefore suppose $T' > 0$ and $[0,T') \subset E_3$. Then $u \in C^2([0,T')\times\mathbb{R})$. Set

$\tilde{T} := T' - \frac{\min\{T', \Delta T\}}{2}$. By $T' \leq T \leq \overline{T}\varepsilon^{-1}$, Lemma 1.3.2 and continuity of u on $[0,T]$, we find for all $(t,x) \in [0,T'] \times \mathbb{R}$ that

$$|u(t,x)| \leq \varepsilon c_M < \delta.$$

Because $|u_x(\tilde{T},x)| \leq C$, too, there is a C^2-solution of (1.68) on $[\tilde{T}, \tilde{T} + \Delta T]$ with initial data $u(\tilde{T},\cdot)$. By $\tilde{T} < T' < \tilde{T} + \Delta T$ we find $u \in C^2$ on $[0,T']$, i.e. $T' \in E_3$. □

1.4 The gradient blow-up

Making use of the smallness properties found in the previous section we are now ready to prove the key finding of this chapter:

Proposition 1.4.1. *Set*

$$\overline{T} := \frac{4}{\gamma_{kkk}(0) \max \alpha'}.$$

There exists $\nu \in (0,1]$ *such that if for* $\varepsilon, \kappa \in (0,\nu]$ *and* $T > 0$ *a function* $u \in C^2([0,T] \times \mathbb{R})$ *solves* (1.50) *then* T *must satisfy*

$$T < T_\varepsilon := \frac{3}{4}\overline{T}\varepsilon^{-1}. \tag{1.69}$$

Proof. Let $z \in \mathbb{R}$ such that $\alpha'(z) = \max_x \alpha'$ and let $\varepsilon, \kappa \in (0,\nu]$ for $\nu = \nu(\overline{T})$ and $c_M, c_V > 0$ like in Lemma 1.3.2. Suppose $u \in C^2([0,T_\varepsilon] \times \mathbb{R})$ solves (1.50). Then the statement of Lemma 1.3.2 applies to u since $T_\varepsilon \leq \overline{T}\varepsilon^{-1}$. For $t \in [0,T_\varepsilon]$ define

$$W(t) := w_k(t, X_k(t;z)).$$

Then $W(0) = \varepsilon\alpha'(z) > 0$ and W satisfies (1.19) with B_{im} replaced by $\varepsilon\kappa B_{im}$:

$$W' = \sum_{j,m} \gamma_{kjm}(u) w_j w_m + \varepsilon\kappa \sum_m B_{km}(u) w_m.$$

Set

$$c_W := 2\frac{c_V}{\alpha'(z)}.$$

Let ν be so small such that for all $\varepsilon \in (0,\nu]$ and for all $(t,x) \in [0,T_\varepsilon] \times \mathbb{R}$

$$\gamma_{kkk}(u(t,x)) > \frac{1}{2}\gamma_{kkk}(0) \tag{1.70}$$

which is possible by $|u(t,x)| < c_M\varepsilon$ (see (1.52)). We find

$$W' > \frac{1}{2}\gamma_{kkk}(0)W^2 - \overline{\gamma}(V^2 + V|W|) - \varepsilon\kappa\overline{B}(V + |W|).$$

We claim that for ν small enough the relations

$$W(t) > \frac{W(0)}{2} > 0 \tag{1.71}$$

$$V(t) < \varepsilon c_W W(t) \tag{1.72}$$

hold for all $t \in [0, T_\varepsilon]$. Like in the proof of Lemma 1.3.2 we proceed by continuous induction. Set

$$E_2 := \{t \in [0, T_\varepsilon] : \text{ (1.71) and (1.72) hold on } [0, t]\}.$$

Again E_2 will be non-empty, open and closed in $[0, T_\varepsilon]$. Since

$$W(0) = \varepsilon\alpha'(z) > \frac{W(0)}{2} > 0,$$
$$V(0) = 0 < \varepsilon\frac{c_W}{2}W(0) < \varepsilon c_W W(0)$$

we find by continuity of W and V a $t > 0$ with $[0, t] \subset E_2$.

Let $t \in E_2\backslash\{0\}$. We show that if ε and κ are small enough then for all $s \in [0, t]$ the stronger inequalities

$$W(s) \geq W(0) \tag{1.73}$$

$$V(s) \leq \varepsilon\frac{c_W}{2}W(s) \tag{1.74}$$

hold. By continuity this implies that E_2 is open in $[0, T_\varepsilon]$, and by the procedure in the proof of Lemma 1.3.2 it also implies its closedness in $[0, T_\varepsilon]$. For $s \in [0, t]$ (1.72) gives

$$W'(s) > \left(\frac{1}{2}\gamma_{kkk}(0) - \overline{\gamma}c_W\varepsilon(1 + c_W\varepsilon)\right)W(s)^2 - \varepsilon\kappa\overline{B}(1 + c_W\varepsilon)W(s).$$

If we choose ν so small that

$$\frac{1}{2}\gamma_{kkk}(0) - 2\overline{\gamma}c_W\nu > \frac{3}{8}\gamma_{kkk}(0), \tag{1.75}$$

$$\nu c_W + 1 < 2 \tag{1.76}$$

then

$$W'(s) > \frac{3}{8}\gamma_{kkk}(0)W(s)^2 - 2\varepsilon\kappa\overline{B}W(s). \tag{1.77}$$

If ν satisfies

$$\frac{3}{16}\gamma_{kkk}(0)\alpha'(z) - 2\nu\overline{B} > \frac{1}{8}\gamma_{kkk}(0)\alpha'(z) \tag{1.78}$$

then we obtain by (1.71)

$$\begin{aligned} W'(s) &> \left(\frac{3}{16}\gamma_{kkk}(0)\alpha'(z) - 2\kappa\overline{B}\right)\varepsilon W(s) \\ &> \frac{1}{8}\gamma_{kkk}(0)\alpha'(z)\varepsilon W(s) \\ &> 0. \end{aligned}$$

Hence W increases on $[0,t]$ and $W(s) \geq W(0)$ holds for all $s \in [0,t]$. Furthermore, by (1.54) we find for all $s \in [0,t]$

$$V(s) < c_V\varepsilon^2 = \frac{c_V}{\alpha'(z)}\varepsilon W(0) \leq \frac{c_V}{\alpha'(z)}\varepsilon W(s) = \frac{c_W}{2}\varepsilon W(s)$$

which concludes the proof of the inequalities (1.73) and (1.74) if ν satisfies the conditions (1.75), (1.76) and (1.78).

Finally let y be the solution of

$$y' = \frac{3}{8}\gamma_{kkk}(0)y^2 - 2\varepsilon\kappa\overline{B}y,\ y(0) = W(0). \tag{1.79}$$

Then by Gronwall's Lemma and (1.77) we have

$$y(s) \leq W(s)$$

as long as y and W exist. Solving (1.79) explicitly and using (1.78) we see that the lifespan of y is given by

$$\begin{aligned} T^*(\varepsilon,\kappa) &= -\frac{1}{2\varepsilon\kappa\overline{B}}\ln\left(\frac{-2\kappa\overline{B}}{\frac{3}{8}\gamma_{kkk}(0)\alpha'(z)} + 1\right) \\ &\leq \varepsilon^{-1}\left(\frac{8}{3}\frac{1}{\gamma_{kkk}(0)\alpha'(z)} + c\kappa\right) \end{aligned} \tag{1.80}$$

where the constant $c > 0$ may be chosen uniformly for all $\kappa, \varepsilon \in (0,\nu]$. If ν is so small that

$$\frac{8}{3}\frac{1}{\gamma_{kkk}(0)\alpha'(z)} + c\nu < \frac{3}{\gamma_{kkk}(0)\alpha'(z)} = \frac{3}{4}\overline{T} \tag{1.81}$$

then $T^*(\varepsilon,\kappa) < T_\varepsilon$ and W cannot be continued as a differentiable function on $[0,T_\varepsilon]$. □

Remark 1.4.2. *In* (1.80) *we find a logarithmic estimate for the lifespan of the solution u to* (1.50) *just like in Section 1.1 for solutions to Burgers' equation with linear source term* (1.3) *and focusing initial data. Furthermore, the role of κ is highlighted by the fraction in the logarithm in* (1.80)*: Taking κ small ensures that the contributions of the source term B, represented by $\overline{B}$ in* (1.80)*, is small compared to the quantities driving the blow-up, i.e. $\gamma_{kkk}(0)$ and $\alpha'(z)$ in* (1.80)*.*

The obvious scaling

$$(t,x) \mapsto ((\varepsilon\kappa)^{-1}t, (\varepsilon\kappa)^{-1}x)$$

yields

Corollary 1.4.3. *There exist ν such that for all $\varepsilon, \kappa \in (0,\nu]$ the maximal time of existence $T_{max} > 0$ for the unique solution $u \in C^1([0,T_{max}) \times \mathbb{R})$ of the Cauchy problem*

$$\begin{cases} u_t + a(u)u_x = B(u), \\ \qquad u(0,x) = u^{0\varepsilon}((\varepsilon\kappa)^{-1}x) \end{cases} \tag{1.82}$$

is finite. The solution u satisfies

$$|u(t,x)| < \delta \text{ for all } (t,x) \in [0,T_{max}) \times \mathbb{R} \tag{1.83}$$

and T_{max} is bounded by

$$T_{max} < \overline{T}\kappa.$$

Proof. Choose $\nu \in (0,1]$ so small such that the statements of Lemma 1.3.2 and Proposition 1.4.1 hold, and such that $c_1^2 c_M \nu < \delta$. By Lemma 1.2.4 and $||u^{0\varepsilon}||_\infty \leq \varepsilon c_M < \delta$ the set

$$E_4 := \{T \in [0,\overline{T}\kappa) : \exists\, u \in C^1([0,T] \times \mathbb{R}) : \ u \text{ solves (1.82)}\}$$

is non-empty, in particular there is $T' > 0$ such that $[0,T'] \subset E_4$. Let $T \in E_4 \backslash \{0\}$. Then there is $u \in C^1([0,T] \times \mathbb{R})$ solving (1.82) and $0 < T \leq \overline{T}\kappa$. We find $u \in C^2([0,T] \times \mathbb{R})$ by Lemma 1.3.4 and $|u(t,x)| < \varepsilon c_M < \delta$ for all $(t,x) \in [0,T] \times \mathbb{R}$ by Lemma 1.3.2. Applying Lemma 1.2.4 there is a $\Delta T > 0$ such that u can be extend as a C^2-solution of (1.82) onto $[0, T + \Delta T]$. We deduce that E_4 is open and of the form $E_4 = [0, T_{max})$ for some $T_{max} \in (0, \overline{T}\kappa]$. In summary: The unique C^1-solution u on $[0,T_{max})$ of (1.82) is in fact a C^2-solution on $[0,T_{max})$ and satisfies $|u(t,x)| < \delta$ for all $(t,x) \in [0,T_{max}) \times \mathbb{R}$. Finally, Proposition 1.4.1 yields $T_{max} \leq \frac{3}{4}\overline{T}\kappa < \overline{T}\kappa$. □

Proof of Theorem 1.0.1. Choose $\nu > 0$ like in Corollary 1.4.3 and let $\varepsilon, \kappa \in (0,\nu]$. Then by Corollary 1.4.3 the unique C^1-solution u of (1.1) with initial data

$$u(0,x) = u^0_{\varepsilon,\kappa}(x) := u^{0\varepsilon}((\varepsilon\kappa)^{-1}x)$$

can only exist up to a maximal time $T_{max} \in (0, \overline{T}\kappa)$. On $[0,T_{max}) \times \mathbb{R}$, u takes its values in $B_\delta(0)$. Hence, each one of the initial data $u^0_{\varepsilon,\kappa}$ with $\varepsilon, \kappa \in (0,\nu]$ provide an example that proves the statement of Theorem 1.0.1. □

Remark 1.4.4. *Note that the data in Corollary 1.4.3 are in general not small in C^1: The first derivative of these data has its supremum bounded from below*

by

$$\nu^{-1} \max_x \alpha' \min_{|u|\leq\delta} |r_k(u)| > 0.$$

1.5 Genuine nonlinearity in a model by Y. Zeng

A well-known model for gas dynamics in thermal non-equilibrium was introduced by Y. Zeng in [98]. The main objective of this section is to show the existence of a simple genuinely nonlinear eigenvalue in this model. We restrict ourselves to the necessary facts needed in the following calculations and refer the reader to [98] for more details.

The equations considered in [98] can be written as

$$w_t + f(w)_x = B(w) \tag{1.84}$$

where

$$w = \begin{pmatrix} v \\ u \\ e + \frac{u^2}{2} \\ q \end{pmatrix}, \; f(w) = \begin{pmatrix} -u \\ p \\ pu \\ 0 \end{pmatrix},$$

with a smooth nontrivial source term B. The pressure p is given as a function $p = p(v, e_1) = \overline{p}(v, s_1)$ where v is the specific volume, and e_1, s_1 and T_1 are the total specific energy, the total specific entropy and the common temperature all associated to translational and rotational modes. One assumes $\overline{p}_v < 0$, and the differentials satisfy

$$de_1 = T_1 ds_1 - pdv. \tag{1.85}$$

There are also thermodynamic quantities with subscript 2 which are associated to vibrational modes but with the exception of the total specific energy q of these modes we do not need them here.

The differential $D_w f$ of f is given by

$$D_w f = \begin{pmatrix} 0 & -1 & 0 & 0 \\ p_v & -p_{e_1} u & p_{e_1} & -p_{e_1} \\ p_v u & -p_{e_1} u^2 + p & p_{e_1} u & -p_{e_1} u \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the identity $\overline{p}_v = p_v - p p_{e_1}$ one finds that $D_w f$ has two simple eigenvalues c_f and $-c_f$ with speed of sound $c_f = \sqrt{-\overline{p}_v}$, and a semi-simple eigenvalue 0 of multiplicity 2. An eigenvector for the simple eigenvalue $\lambda = -c_f$ and the

differential of λ are given by

$$r(w) = \begin{pmatrix} 1 \\ c_f \\ uc_f - p \\ 0 \end{pmatrix} \text{ and } D_w\lambda(w) = (2c_f)^{-1} \begin{pmatrix} \overline{p}_{vv} + \overline{p}_{vs_1} s_{1v} \\ -\overline{p}_{vs_1} s_{1v} u \\ \overline{p}_{vs_1} s_{1e_1} \\ -\overline{p}_{vs_1} s_{1e_1} \end{pmatrix}.$$

We find that if $\overline{p}_{vv} \neq 0$, for instance if $\overline{p}$ is a convex pressure law, then λ is genuine nonlinear since

$$\langle D_w\lambda(w), r(w)\rangle = (2c_f)^{-1}\overline{p}_{vv} \neq 0$$

using $ps_{1e_1} = s_{1v}$ from (1.85).

Although $D_w f$ is not strictly hyperbolic some elementary calculations show that there are smooth left and right eigenvectors $\{l_i\}_i$ and $\{r_i\}_i$ satisfying the relations in (1.10). Since a straightforward extension of our result also holds for systems (1.1) with a possessing smooth left and right eigenbases satisfying (1.10) as well as a simple genuinely nonlinear eigenvalue, and with source term B satisfying A3, (cf. the Outlook) we find that Theorem 1.4.1 applies to Y. Zeng's system (1.84).

Chapter 2

Spectral stability of shock profiles for hyperbolically regularized systems of conservation laws

Let $n \in \mathbb{N}$. Partial differential equations of the form

$$g(v)_t + f(v)_x = 0 \tag{2.1}$$

with $(t, x) \in \mathbb{R}^2, v \in \mathbb{R}^n$ and $g(v), f(v) \in \mathbb{R}^n$ are called *conservation laws* for one dimensional space. They appear in many applications modelling for instance fluids and traffic flow (cf. [12]). For different purposes (2.1) is augmented by a differential operator $\mathcal{B}[v]$ yielding

$$g(v)_t + f(v)_x = \mathcal{B}[v]. \tag{2.2}$$

The operator $\mathcal{B}[v]$ can *regularize* (2.1) in the sense that the augmented system (2.2) is dissipative (e.g. [12, 25]) and the formation of singularities is suppressed and a global solution of (2.2) exists for small data (e.g. [93]). One often refers to $\mathcal{B}[v]$ as a artificial, numerical or physical viscosity (e.g. [12, 47]), depending on the context. We will consider the second-order hyperbolic operator

$$\mathcal{B}[v] = B\Box v := B(\partial_x^2 v - \partial_t^2 v)$$

with a symmetric and positive definite matrix $B \in \mathbb{R}^{n\times n}$. System (2.2) with $B\Box v$ on its right-hand side will be called a *hyperbolically regularized system of conservation laws*. In this form, we think of system (2.2) as

(a) an alternative to the more common artificial parabolic viscosity $\Delta v = \partial_x^2 v$ (cf. [35]), and

(b) a simple model for the equations of relativistic fluid dynamics with hyperbolic dissipation as introduced in [29].

Note the different sign convention in $\Box v$ compared to the usual d'Alembertian operator.

A *shock profile* of (2.2) is a smooth curve connecting two shock end states $v^-, v^+ \in \mathbb{R}^n$ with a certain shock speed $s \in \mathbb{R}$. Shock profiles define self-similar solutions of (2.2) called heteroclinic travelling waves, and it is the spectral stability of those solutions of (2.2) that we are interested in. In shock wave

theory *spectral stability* is defined using an *Evans function* (e.g. [70, 99]), an analytic function defined on an open neighborhood of the closed right half of the complex plane. Under suitable conditions on the fluxes f and g and for sufficiently small shock amplitudes, we will prove the spectral stability of shock profiles for

$$g(v)_t + f(v)_x = B\Box v \tag{2.3}$$

by showing the existence of an associated Evans function which satisfies the *Evans function condition*: The found Evans function has exactly one zero in the closed right half of the complex plane, namely at 0, and this zero is of first order. Our result is analogous to the ones obtained in [26, 77] on viscous conservation laws, and also distinctly extends findings on hyperbolic relaxation systems in [73, 77, 92]. The strategy of proof follows the work in [26, 27] and rests on the direct construction of (rescaled) Evans bundles for the shock profile at hand.

Outline of the chapter:

Section 2.1 is of preliminary type. We recall some basics on Grassmannian flows and extend results found in [26].

The following Section 2.2 is a preliminary one, too, and is concerned with the theory of normally hyperbolic manifolds containing basic definitions and a collection of important theorems.

In Section 2.3 we specify the assumptions on the coefficient functions f and g that appear in the hyperbolically regularized conservation law (2.3). The shock profiles are constructed in standard fashion and the dynamical system that will be at the center of the studies thereafter is derived. We give some more details of the general strategy of proof and briefly revise the existing literature.

The reader finds the precise statement of our main result, Theorem 2.4.8, in Section 2.4 where also an appropriate version of Evans function theory for systems like (2.3) is given.

Section 2.5 contains the discussion of an example of (2.3) based on the Burgers' equation hyperbolically regularized by $\Box v$. The stability problem for a particular standing shock profile is explicitly solved by obtaining a concrete Evans function.

In section 2.6 we show how the Evans function appears in the resolvent kernel associated to the linearization of (2.3).

Finally, our main result on spectral stability of shock profiles for hyperbolically regularized systems of conservation laws is proved in the three subsections of Section 2.7 where each subsection treats a different spectral regime.

A short read of the chapter would consist of Sections 2.3, 2.4 and 2.7.

2.1 Grassmannian flows

An important tool in the following investigations are flows on Grassmann manifolds that are induced on the latter by the flow of a linear ordinary differential equation (see [2]). In this section the definitions and results needed in the later analysis are presented including a derivation of the well-known algebraic Riccati

equation that governs the evolution of vector spaces under a linear differential equation. More background material may be found in many geometry textbooks, such as [38, 62, 94]. Furthermore, flows on Grassmannian manifolds are a popular topic in control theory, see e.g. [15], where the algebraic Riccati equation is treated, too. In [23] the reader may find a nice overview of the topic and some characterizations of smoothness of functions taking values in Grassmannian manifolds. Finally, two references on complex analysis are the textbooks [42, 80].

Let $d, m \in \mathbb{N}$ with $m \leq d$. Set

$$M_{d,m} := \{B \in \mathbb{C}^{d\times m} : \operatorname{rank} B = m\}$$

and for $B \in M_{d,m}$ denote by $\operatorname{span} B$ the m-dimensional subspace of $\mathbb{C}^d$ spanned by the columns of B. For any matrix $B \in \mathbb{C}^{d\times m}$ we will write

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

for appropriate matrices $B_1 \in \mathbb{C}^{m\times m}$ and $B_2 \in \mathbb{C}^{(d-m)\times m}$, and $B = (b_j)_j$ will mean that $b_j \in \mathbb{C}^d$ is the j-th column of B. $M_{d,m}$ is an open subset of $\mathbb{C}^{d\times m}$ and is endowed with the standard analytic structure as an open submanifold of $\mathbb{C}^{d\times m}$. The set

$$\mathscr{G}_m^d(\mathbb{C}) := \{X : X \text{ is an } m \text{ dim. subspace of } \mathbb{C}^d\}.$$

can be given an analytic structure yielding a compact complex analytic manifold of dimension $(d-m)m$ which is called a Grassmann or Grassmannian manifold. For an ordered basis $\mathcal{B} = (b_1, \ldots, b_d)$ of $\mathbb{C}^d$ canonical local coordinates of $\mathscr{G}_m^d(\mathbb{C})$ centered at $X = \operatorname{span}\{b_1, \ldots, b_m\}$ are defined by

$$\begin{aligned} \phi_{\mathcal{B}} : \mathbb{C}^{(d-m)\times m} &\to \mathscr{G}_m^d(\mathbb{C}) \\ T &\mapsto \operatorname{span} M_{\mathcal{B}} \begin{pmatrix} \mathbb{1}_m \\ T \end{pmatrix} \end{aligned}$$

where

$$M_{\mathcal{B}} = (b_1 \ldots b_d) \in \mathrm{GL}_d(\mathbb{C})$$

and $\mathrm{GL}_d(\mathbb{C})$ is the set of the invertible elements of $\mathbb{C}^{d\times d}$. Note $\phi_{\mathcal{B}}(0) = X$. The analytic manifolds $M_{d,m}$ and $\mathscr{G}_m^d(\mathbb{C})$ are connected by the fact that

$$\pi_{d,m} : M_{d,m} \to \mathscr{G}_m^d(\mathbb{C}),\ B \mapsto \operatorname{span} B$$

is an analytic quotient map.
Let $A \in \mathbb{C}^{d\times d}$ and consider the linear ordinary differential equation

$$\xi' = A\xi \quad (\xi \in \mathbb{C}^d) \tag{2.4}$$

with flow $((t,\xi) \mapsto e^{At}\xi)$. For $X \in \mathscr{G}_m^d(\mathbb{C})$ with basis $\mathcal{B}_X = (b_1, \dots, b_m)$ the Grassmannian flow on $\mathscr{G}_m^d(\mathbb{C})$ associated to (2.4) starting in X is defined by

$$X(t) = \pi_{d,m}((e^{At}b_j)_{j\le m}) = \operatorname{span}(e^{At}b_j)_{j\le m}.$$

The definition of $(t \mapsto X(t))$ is independent of the choice of basis of X and gives rise to a flow $((t,X) \mapsto \Phi(t,X))$ on $\mathscr{G}_m^d(\mathbb{C})$ by letting $(t \mapsto \Phi(t,X))$ be the Grassmannian flow on $\mathscr{G}_m^d(\mathbb{C})$ associated to (2.4) starting in X. Augment $\mathcal{B}_X$ to a basis $\mathcal{B} = (b_1, \dots, b_d)$ of $\mathbb{C}^d$. Then the map

$$\begin{aligned} \tilde{\Phi}_{\mathcal{B}} : \mathbb{R} \times \mathbb{C}^{(d-m)\times m} &\to M_{d,m} \\ (t,T) &\mapsto e^{At} M_{\mathcal{B}} \begin{pmatrix} \mathbb{1}_m \\ T \end{pmatrix} \end{aligned}$$

is smooth in t and analytic in T. For all $(t,T) \in \mathbb{R} \times \mathbb{C}^{(d-m)\times m}$ we have by definitions

$$\Phi(t, \phi_{\mathcal{B}}(T)) = \pi_{d,m}\tilde{\Phi}_{\mathcal{B}}(t,T) \tag{2.5}$$

which shows that Φ is smooth in t and analytic in X w.r.t. to the analytic structure of $\mathscr{G}_m^d(\mathbb{C})$. We give a local representation of the vector field induced by Φ on $\mathscr{G}_m^d(\mathbb{C})$ by calculating the derivative

$$\Gamma^m(A)(X) := \left.\frac{\partial}{\partial t}\right|_{t=0} \Phi(t,X) \in T\mathscr{G}_m^d(\mathbb{C}).$$

in $\phi_{\mathcal{B}}$-coordinates. First we are interested in the differential of $\psi := \phi^{-1} \circ \pi|_U$ where $\phi = \phi_{\mathcal{B}}, \pi = \pi_{d,m}$ and $U := \pi^{-1}(\phi(\mathbb{C}^{(d-m)\times m}))$ is open. For $B \in U$ there is a $T \in C^{d-m\times m}$ with $\operatorname{span} B = \phi(T)$. Hence, we may find $E \in \mathbb{C}^{m\times m}$ such that

$$M_{\mathcal{B}} \begin{pmatrix} \mathbb{1}_m \\ T \end{pmatrix} = BE \quad \Leftrightarrow \quad \begin{pmatrix} \mathbb{1}_m \\ T \end{pmatrix} = \tilde{B}E$$

with $\tilde{B} = M_{\mathcal{B}}^{-1}B$. We deduce that $\tilde{B}_1$ is invertible since $\tilde{B}_1 E = \mathbb{1}_m$, and that $T = \tilde{B}_2\tilde{B}_1^{-1}$ holds. Vice versa, if for $B \in M_{d,m}$ the matrix $\tilde{B} = M_{\mathcal{B}}^{-1}B$ has invertible submatrix $\tilde{B}_1$ then

$$\operatorname{span}\tilde{B} = \operatorname{span}\begin{pmatrix} \mathbb{1}_m \\ \tilde{B}_2\tilde{B}_1^{-1} \end{pmatrix}$$

because multiplying $\tilde{B}$ by $\tilde{B}_1^{-1}$ from the right results only in a change of basis of $\operatorname{span} B$. This implies $\operatorname{span} B = \phi(\tilde{B}_2\tilde{B}_1^{-1})$ and we have found

$$U = \{B \in M_{d,m} : \ \tilde{B}_1 \in \mathrm{GL}_m(\mathbb{C}) \text{ where } \tilde{B} = M_{\mathcal{B}}^{-1}B\}.$$

From these considerations we find that the map ψ is given by $\psi = \tilde{\psi} \circ M_{\mathcal{B}}^{-1}$ where

$$\begin{aligned} \tilde{\psi} : \tilde{U} := M_{\mathcal{B}}^{-1} U &\to \mathbb{C}^{(d-m)\times m}, \\ \tilde{B} &\mapsto \tilde{B}_2 \tilde{B}_1^{-1}. \end{aligned}$$

The differential of $\tilde{\psi}$ at $\tilde{B} \in \tilde{U}$ is calculated to be

$$D\tilde{\psi}(\tilde{B}) : \mathbb{C}^{d\times m} \to \mathbb{C}^{(d-m)\times m}, E \mapsto E_2 \tilde{B}_1^{-1} - \tilde{B}_2 \tilde{B}_1^{-1} E_1 \tilde{B}_1^{-1}. \tag{2.6}$$

If $\tilde{B}_1 = \mathbb{1}_m$ then

$$D\tilde{\psi}(\tilde{B})E = E_2 - \tilde{B}_2 E_1 = (-\tilde{B}_2, \mathbb{1}_{d-m})E.$$

We find

$$\begin{aligned} \left.\frac{\partial}{\partial t}\right|_{t=0} (\phi^{-1} \circ \Phi)(t, \phi(T)) &= \left.\frac{\partial}{\partial t}\right|_{t=0} (\psi \circ \tilde{\Phi}_{\mathcal{B}})(t, T) \\ &= D\psi(\tilde{\Phi}_{\mathcal{B}}(0, T)) \left.\frac{\partial}{\partial t}\right|_{t=0} \tilde{\Phi}_{\mathcal{B}}(t, T) \\ &= D\tilde{\psi}\left(M_{\mathcal{B}}^{-1} M_{\mathcal{B}} \begin{pmatrix} \mathbb{1}_m \\ T \end{pmatrix}\right) M_{\mathcal{B}}^{-1} A M_{\mathcal{B}} \begin{pmatrix} \mathbb{1}_m \\ T \end{pmatrix} \\ &= (-T, \mathbb{1}_{d-m}) A_{\mathcal{B}} \begin{pmatrix} \mathbb{1}_m \\ T \end{pmatrix} \end{aligned}$$

where $A_{\mathcal{B}} \in \mathbb{C}^{d\times d}$ is the representative of A with respect to $\mathcal{B}$, i.e. $A_{\mathcal{B}} = M_{\mathcal{B}}^{-1} A M_{\mathcal{B}}$. Hence, in canonical local coordinates $\phi_{\mathcal{B}}$ the Grassmannian flow consists of solutions to the Riccati-type equation

$$T' = (-T, \mathbb{1}_{d-m}) A_{\mathcal{B}} \begin{pmatrix} \mathbb{1}_m \\ T \end{pmatrix} \text{ on } \mathbb{C}^{(d-m)\times m}. \tag{2.7}$$

The global expression for the vector field associated to Φ reads

$$X' = \Gamma^m(A)(X) \text{ on } \mathscr{G}_m^d(\mathbb{C}). \tag{2.8}$$

The fact that the Grassmannian flow induced by a linear ordinary differential equations is locally of Riccati-type is of course well-known. This is summarized in the following lemma which is also a more general form of Lemma 3 (i) in [26].

Lemma 2.1.1. *Let $\mathcal{B}$ be an ordered basis of $\mathbb{C}^d$. Then in canonical local coordinates $\phi_{\mathcal{B}}$ the Grassmannian flow on $\mathscr{G}_m^d(\mathbb{C})$ associated to (2.4) consists of solutions to the Riccati-type equation*

$$T' = (-T, \mathbb{1}_{d-m}) A_{\mathcal{B}} \begin{pmatrix} \mathbb{1}_m \\ T \end{pmatrix} \text{ on } \mathbb{C}^{(d-m)\times m} \tag{2.9}$$

where $A_{\mathcal{B}} \in \mathbb{C}^{d\times d}$ is the representative of A w.r.t. the basis $\mathcal{B}$. If $A_{\mathcal{B}}$ is diagonal with

$$A_{\mathcal{B}} = \operatorname{diag}(\mu_1, \dots, \mu_d)$$

then (2.9) *reads for* $1 \le j \le d-m, 1 \le k \le m$

$$(T)'_{jk} = (\mu_{m+j} - \mu_k)(T)_{jk}.$$

On $\mathbb{C}^{(d-m)\times m}$ let $\langle\langle \cdot, \cdot \rangle\rangle$ denote the Frobenius inner product given by

$$\langle\langle E, F \rangle\rangle = \sum_{j=1}^{d-m} \sum_{k=1}^{m} (E)_{jk} \overline{(F)}_{jk} \quad (E, F \in \mathbb{C}^{(d-m)\times m})$$

and let $||\cdot||$ be the norm on $\mathbb{C}^{(d-m)\times m}$ induced by $\langle\langle \cdot, \cdot \rangle\rangle$. We give some useful consequences of Lemma 2.1.1.

Corollary 2.1.2. *Let $\mathcal{B} = (b_1, \dots, b_d)$ be an ordered basis of $\mathbb{C}^d$ such that $X := \operatorname{span}\{b_1, \dots, b_m\}$ and $Y := \operatorname{span}\{b_{m+1}, \dots, b_d\}$ are complementary invariant subspaces of A. Then*

$$A_{\mathcal{B}} = \begin{pmatrix} A^X & 0 \\ 0 & A^Y \end{pmatrix}$$

for appropriate $A^X \in \mathbb{C}^{m\times m}, A^Y \in \mathbb{C}^{(d-m)\times(d-m)}$ and the local representation of (2.8) *at X in $\phi_{\mathcal{B}}$-coordinates is given by the linear ordinary differential equation*

$$T' = -TA^X + A^Y T. \tag{2.10}$$

If there exist $c^X, c^Y \in \mathbb{R}$ with $c^X + c^Y > 0$ and for all $\eta^X \in \mathbb{C}^m, \eta^Y \in \mathbb{C}^{d-m}$

$$\begin{cases} \operatorname{Re}\langle \eta^X, A^X \eta^X \rangle \ge c^X |\eta^X|^2 \\ \operatorname{Re}\langle \eta^Y, A^Y \eta^Y \rangle \le -c^Y |\eta^Y|^2 \end{cases} \tag{2.11}$$

then X is a hyperbolic attractor for the Grassmannian flow Φ. For any $r > 0$ the set $\phi_{\mathcal{B}}(B_r(0))$ is a positively invariant neighborhood of X in $\mathscr{G}_m^d(\mathbb{C})$.

Proof. The assertion on $A_{\mathcal{B}}$ and the local form of Γ^m follow immediately by linear algebra and (2.7). Denote by

$$F(T) := -TA^X + A^Y T$$

the local representation of the vector field Γ^m as given in (2.10). X is a fixed point of Φ since $\phi_{\mathcal{B}}(0) = X$ and $F(0) = 0$. We show that $DF(0)$ is negative definite and that for any $r > 0$ the function

$$\begin{aligned} L : B_r(0) &\to \mathbb{R} \\ T &\mapsto ||T||^2 \end{aligned}$$

is a strict Lyapunov function for (2.7). The differential of F at 0 is

$$DF(0)E = -EA^X + A^Y E \ \ (E \in \mathbb{C}^{(d-m)\times m}).$$

For $E \in \mathbb{C}^{(d-m)\times m}$ denote by $\tilde{E}_i$ the i-th row of E and by E_j the j-th column of E. Then

$$\begin{aligned}
2\mathrm{Re}\,\langle\langle E, DF(0)E\rangle\rangle =& -\langle\langle E, EA^X\rangle\rangle + \langle\langle E, A^Y E\rangle\rangle \\
& -\langle\langle EA^X, E\rangle\rangle + \langle\langle A^Y E, E\rangle\rangle \\
=& -\sum_{i=1}^{d-m}\langle \tilde{E}_i, (A^X)^t\tilde{E}_i\rangle + \sum_{j=1}^{m}\langle E_j, A^Y E_j\rangle \\
& -\sum_{i=1}^{d-m}\langle (A^X)^t\tilde{E}_i, \tilde{E}_i\rangle + \sum_{j=1}^{m}\langle A^Y E_j, E_j\rangle \\
=& 2\left(-\sum_{i=1}^{d-m}\mathrm{Re}\,\langle \tilde{E}_i, (A^X)^t\tilde{E}_i\rangle + \sum_{j=1}^{m}\mathrm{Re}\,\langle E_j, A^Y E_j\rangle\right) \\
\leq& 2\left(-c^X\sum_{i=1}^{d-m}|\tilde{E}_i|^2 - c^Y\sum_{j=1}^{m}|E_j|^2\right) \\
=& -2(c^X + c^Y)||E||^2,
\end{aligned}$$

i.e. $DF(0)$ is negative definite and X a hyperbolic fixed point of Φ since $c^X + c^Y > 0$. Note that for the last inequality we have used

$$\langle \eta, A^t\eta\rangle = \langle \overline{\eta}, A\overline{\eta}\rangle.$$

for $\eta \in \mathbb{C}^m$. By exactly the same calculation we find

$$L(T)' = 2\mathrm{Re}\,\langle\langle T, T'\rangle\rangle = 2\mathrm{Re}\,\langle\langle T, F(T)\rangle\rangle \leq -2(c^X + c^Y)||T||^2$$

showing that L is a strict Lyapunov function and $B_r(0)$ is positively invariant for (2.10). □

Corollary 2.1.3. *Let $m_1, m_2 \in \mathbb{N}$ such that $m + m_1 + m_2 = d$. Let $\mathcal{B} = (b_1, \ldots, b_d)$ be an ordered basis of $\mathbb{C}^d$ such that $X := \mathrm{span}\,\{b_1, \ldots, b_m\}, Y := \mathrm{span}\,\{b_{m+1}, \ldots, b_{m+m_1}\}$ and $Z := \mathrm{span}\,\{b_{m+m_1+1}, \ldots, b_d\}$ are complementary invariant subspaces of A. Then*

$$A_{\mathcal{B}} = \begin{pmatrix} A^X & 0 & 0 \\ 0 & A^Y & 0 \\ 0 & 0 & A^Z \end{pmatrix}$$

for appropriate $A^X \in \mathbb{C}^{m\times m}, A^Y \in \mathbb{C}^{m_1\times m_1}, A^Z \in \mathbb{C}^{m_2\times m_2}$ and the local representation of $\Gamma^m(A)$ at X in $\phi_{\mathcal{B}}$-coordinates is given by the linear ordinary

differential equation

$$T' = -TA^X + \begin{pmatrix} A^Y & 0 \\ 0 & A^Z \end{pmatrix} T. \tag{2.12}$$

If $\ker A = X$ *and for some* $c > 0$ *and all* $\eta^Y \in \mathbb{C}^{m_1}, \eta^Z \in \mathbb{C}^{m_2}$

$$\begin{cases} \operatorname{Re} \langle \eta^Y, A^Y \eta^Y \rangle \leq -c|\eta^Y|^2 \\ \operatorname{Re} \langle \eta^Z, A^Z \eta^Z \rangle \geq c|\eta^Y|^2 \end{cases} \tag{2.13}$$

then $A^X = 0$ *and* X *is a hyperbolic saddle of* Φ.

Proof. The first part follows from the same arguments as used in the beginning of the proof of Lemma 2.1.2. Suppose $\ker A = X$ and (2.13) hold. Set

$$V^Y := \left\{ E \in \mathbb{C}^{(d-m)\times m} \,\middle|\, \exists E_1 \in \mathbb{C}^{m_1 \times m} : E = \begin{pmatrix} E_1 \\ 0 \end{pmatrix} \right\},$$
$$V^Z := \left\{ E \in \mathbb{C}^{(d-m)\times m} \,\middle|\, \exists E_2 \in \mathbb{C}^{m_2 \times m} : E = \begin{pmatrix} 0 \\ E_2 \end{pmatrix} \right\}.$$

Let $F(T) = \begin{pmatrix} A^Y & 0 \\ 0 & A^Z \end{pmatrix} T$. Then

$$DF(0)E = \begin{pmatrix} A^Y & 0 \\ 0 & A^Z \end{pmatrix} E$$

and V^Y resp. V^Z are complementary invariant subspace of $DF(0)$ of dimension $m_1 m$ resp. $m_2 m$. With a similar calculation as in the proof of Lemma 2.1.2 one finds

$$\operatorname{Re} \langle E, DF(0)E \rangle \leq -c||E||^2 \text{ for all } E \in V^Y,$$
$$\operatorname{Re} \langle E, DF(0)E \rangle \geq c||E||^2 \text{ for all } E \in V^Z.$$

Hence, X is a hyperbolic saddle for Φ with an $m_1 m$-dimensional stable manifold and an $m_2 m$-dimensional unstable manifold for Φ at X. □

This section is closed by a variant of Lemma 6 in [26] replacing diagonalization by a suitable block-diagonalization which will be used in the proof of spectral stability of small amplitude shock profiles of (2.3) in the outer and outmost spectral regime (see section 2.7).

Lemma 2.1.4. *For every* $d \in \mathbb{N}$ *there exists a constant* $c = c(d)$ *such that the following holds: Let* $c^>, c^< : J \to \mathbb{R}$ *be functions with* $c^>(\tau) + c^<(\tau) > 0, \tau \in J$, *and let* $A, R : J \to \mathrm{GL}_{2d}(\mathbb{C})$ *be smooth functions such that*

$$R(\tau)^{-1} A(\tau) R(\tau) = \begin{pmatrix} A^>(\tau) & 0 \\ 0 & A^<(\tau) \end{pmatrix} \in \mathbb{C}^{2d} \ (\tau \in J) \tag{2.14}$$

with $A^{\gtrless}(\tau) \in \mathbb{C}^{d\times d}$ *and*

$$\begin{cases} \operatorname{Re}\langle \eta, A^{>}(\tau)\eta\rangle \geq c^{>}(\tau)|\eta|^2, \\ \operatorname{Re}\langle \eta, A^{<}(\tau)\eta\rangle \leq -c^{<}(\tau)|\eta|^2, \end{cases} \tag{2.15}$$

for all $\eta \in \mathbb{C}^d$ *and* $\tau \in J$. *With* $\chi : \mathbb{R} \to (-1,1)$ *being the solution of*

$$\chi' = (1-\chi^2)h(\chi),\ \chi(0) = 0, \tag{2.16}$$

for some smooth function $h : J \to (0,\infty)$ *consider the equation*

$$X' = \Gamma^d(A(\chi))(X) \text{ on } \mathscr{G}_d^{2d}(\mathbb{C}) \tag{2.17}$$

which is the associated Grassmannian version for d-dimensional vector spaces of the non-autonomous linear system

$$\xi' = A(\chi)\xi \tag{2.18}$$

for $\xi \in \mathbb{C}^{2d}$. *Let* $U(\tau), S(\tau)$ *denote the invariant subspace of* $A(\tau), \tau \in J$, *spanned by the first d resp. last d columns of* $R(\tau)$ *and define* $X^{\pm} : \mathbb{R} \to \mathscr{G}_d^{2d}(\mathbb{C})$ *as the two solutions of* (2.17) *with*

$$X^{-}(-\infty) = U(-1),\ X^{+}(+\infty) = S(+1). \tag{2.19}$$

If

$$\left| R(0)^{-1}\frac{d}{d\tau}R(\tau)\right| \leq c\ (\tau \in J) \tag{2.20}$$

then

$$X^{-}(+\infty) = U(+1),\ X^{+}(-\infty) = S(-1). \tag{2.21}$$

Remark 2.1.5. *If in the above lemma the constants* $c^{>}, c^{<}$ *are larger than zero then* $U(\tau)$ *resp.* $S(\tau)$ *is the unstable resp. stable subspace of* $A(\tau)$.

Proof. We treat X^{-}. Fix a $\tau \in J$. Let $\phi_\tau : \mathbb{C}^{d\times d} \to \mathcal{S}_\tau$ be the canonical coordinates on a maximal portion $\mathcal{S}_\tau \subset \mathscr{G}_d^{2d}$ adapted to the columns of $R(\tau)$ as an ordered basis of $\mathbb{C}^{2d}$ and centered at $\operatorname{span}(R(\tau)_j)_{1\leq j\leq d}$. Then equation (2.17) reads in local coordinates

$$T' = (-T, \mathbb{1}_n)\tilde{A}(\tau)\begin{pmatrix}\mathbb{1}_n \\ T\end{pmatrix} =: F_\tau(T) \tag{2.22}$$

where $T \in \mathbb{C}^{d\times d}$ and $\tilde{A}(\tau)$ is the representative of $A(\tau)$ with respect to the basis given by the columns of $R(\tau)$, i.e.

$$\tilde{A}(\tau) = R(\tau)^{-1}A(\tau)R(\tau) = \begin{pmatrix} A^{>}(\tau) & 0 \\ 0 & A^{<}(\tau)\end{pmatrix}.$$

In particular,

$$F_\tau(T) = -TA^>(\tau) + A^<(\tau)T.$$

The point 0 corresponds in φ_τ-coordinates to the subspace $U(\tau)$. Using Corollary 2.1.2 we find that the differential $D_T F_\tau(T)$ of $F_\tau(T)$ with respect to T at $T = 0$ is negative definite. In particular, if $\tau = -1$ then the coupled system (2.16)-(2.17) has exactly one repelling direction at $(-1, U(-1))$ coming from the χ-equation. To see this note that $\{\tau = -1\}$ is an invariant set for the system (2.16)-(2.17), and in local $\mathrm{id}_1 \times \phi_{-1}$-coordinates the vector field for the coupled (τ, X)-equations has its differential given by

$$\begin{pmatrix} 2h(-1) & 0 \\ * & D_T F_{-1}(0) \end{pmatrix}$$

at $(-1, 0)$. Using the stable manifold theorem we find a unique orbit giving rise to the solution X^- of (2.17) satisfying (2.19).
Furthermore, Corollary 2.1.2 tells us that for any $\sigma > 0$ the ball $B_\sigma(0)$ of radius σ centered at 0 is a positively invariant set for the local representation

$$T' = (-T, \mathbb{1}_n)\tilde{A}(\tau)\begin{pmatrix} \mathbb{1}_n \\ T \end{pmatrix}$$

of the Grassmannian flow induced by $A(\tau)$ on $\mathscr{G}_d^{2d}(\mathbb{C})$ in ϕ_τ-coordinates. Therefore the construction of an appropriate Lyapunov function for the coupled (τ, X)-system carries over from the proof of Lemma 6 in [26], and we find

$$X^-(+\infty) = U(+1).$$

□

2.2 Geometric singular perturbation theory

In subsection 2.7.1 the general spectral stability problem for (2.3) is reduced for small frequencies to a canonical problem provided by (a variant of) the Burgers' equation. Following [26] this reduction rests on geometric singular perturbation theory (GSPT) in the sense of Fenichel [22, 50, 90] which is based on the perturbation theory for normally hyperbolic invariant manifolds [19, 40]. This section first recalls the notion of normal hyperbolicity and states a theorem on the existence and persistence of stable and unstable manifolds [19, 20, 21, 40, 96]. Then we follow the general coordinate-free development of GSPT in [22], and a local form of GSPT is presented along the lines of [50]. For the basics in differential geometry we refer the reader to, e.g., the textbooks [62, 87].

Let $r \in \mathbb{N} \cup \{\infty\}$ and let M be an n-dimensional C^r-manifold. Suppose we are given a C^r-vector field X on M, i.e. X is a C^r-section of TM. Denote by $F = F^t(m)$ the flow of X. This means that for each $m \in M$ the map

$\gamma : t \mapsto F^t(m)$ is the maximal integral curve of

$$\gamma' = \frac{d}{dt}\gamma = X\gamma \tag{2.23}$$

through $\gamma(0) = m$. The flow domain is an open subset of $\mathbb{R} \times M$ and F is a C^r-function. We also use the common short hand $(t, m) \mapsto m \cdot t = F^t(m)$ for the flow F. For a point $m \in M$ and an interval $J \subset \mathbb{R}$ such that $\{m\} \times J$ is in the domain of F set

$$m \cdot J := \{m \cdot t : \ t \in J\}.$$

A subset $V \subset M$ is called locally invariant if there exists a neighborhood $U \subset M$ of V such that for all $m \in V$ and $t \in [0, \infty)$ the inclusion $m \cdot [0, t] \subset U$ implies $m \cdot [0, t] \subset V$, and the inclusion $m \cdot [-t, 0] \subset U$ implies $m \cdot [-t, 0] \subset V$. It is called positively invariant if $m \cdot [0, \infty) \subset V$ for all $m \in V$, negatively invariant if $m \cdot (-\infty, 0] \subset V$ for all $m \in V$, and invariant if $m \cdot (-\infty, \infty) \subset V$. For any subset $U \subset M$ define

$$\begin{aligned} A^+(U) &:= \{m \in U : \ \overline{m \cdot [0, \infty)} \subset U\} \\ A^-(U) &:= \{m \in U : \ \overline{m \cdot (-\infty, 0]} \subset U\} \\ I(U) &:= \{m \in U : \ \overline{m \cdot (-\infty, \infty)} \subset U\} \end{aligned}$$

called, in the above order, the maximal positively invariant set in U, the maximal negatively invariant set in U, and the maximal invariant set in U. An element $m \in M$ is called a fixed, or equilibrium, point of X if $X(m) = 0$. If a subset $U \subset M$ consists only of fixed points then it is called a critical set of X. A critical set of X is a simple example for an invariant set of F.

Define some Riemannian metric for TM on M with associated norm $||v|| = ||v||_m$ for $m \in M, v \in T_mM$.

Definition 2.2.1. Let V be an invariant compact embedded C^r-submanifold of M. For $t \in (-\infty, \infty)$ and $m \in V$ let

$$DF^t(m) : T_mM \to T_{F^t(m)}M$$

be the differential of F^t at m. Then V is called a normally hyperbolic invariant manifold of X if there exist a continuous splitting

$$TM|_V = N^u \oplus TV \oplus N^s,$$

with $DF^t(m)N^{u/s}_m \subset N^{u/s}_{F^t(m)}$ for all $t \in (-\infty, \infty), m \in V$, and constants $\nu > \rho \geq 0$ and $C > 0$ such that for all $t \in [0, \infty), m \in V$ and $n^u \in N^u_m, v \in TV_m, n^s \in N^s_m$

the estimates

$$\begin{aligned} ||DF^{-t}(m)n^u||_{F^{-t}(m)} &\leq C||n^u||_m e^{-\nu t}, \\ ||DF^{-t}(m)v||_{F^{-t}(m)} &\leq C||v||_m e^{\rho t}, \\ ||DF^{t}(m)v||_{F^{t}(m)} &\leq C||v||_m e^{\rho t}, \\ ||DF^{t}(m)n^s)||_{F^{t}(m)} &\leq C||n^s||_m e^{-\nu t} \end{aligned}$$

hold. We refer to the constants ρ and ν as exponential rates.

Remark 2.2.2. *The above definition of normally hyperbolicity of a compact invariant manifold V does not depend on the chosen metric on M due to compactness of V.*

The estimates in Definition 2.2.1 mean that, on V, the differential of F expands N^u faster than TV while contracting N^s faster than TV. This gives normally hyperbolic invariant manifolds certain stability properties which is why they are interesting from a mathematical and from a modelling point of view (cf. [96]). We will encounter normally hyperbolic invariant manifolds only in the form of critical sets. In this case we can choose ρ arbitrarily small in Definition 2.2.1, and we will simply speak of a critical manifold. Nonetheless, in the following theorem we also allow for noncritical manifolds but with empty boundary.

Theorem 2.2.3. *Let $V \subset M$ be a normally hyperbolic invariant manifold of X with hyperbolic splitting $TM|_V = N^u \oplus TV \oplus N^s$ and exponential rates $\nu > \rho \geq 0$. Suppose V has empty boundary, and let $K \subset M$ be precompact and open with $V \subset K$. If*

$$\frac{\rho}{\nu} \leq \frac{1}{r}$$

then there exist a neighborhood U of V in M and C^r-manifolds W^u and W^s with boundary called a local unstable manifold and a local stable manifold for V, respectively, such that

(i) $W^u = A^-(U)$ and $W^s = A^+(U)$, and

(ii) W^u is tangent to $N^u \oplus TV$ along V and W^s is tangent to $TV \oplus N^s$ along V.

There are C^r-families of manifolds with boundary $\{W^u(m)\}_{m\in V}, \{W^s(m)\}_{m\in V}$ invariant in the sense that for all $(t, m) \in [0, \infty) \times V$

(iii) $F^{-t}(W^u(m)) \subset W^u(F^{-t}(m))$ and $F^t(W^s(m)) \subset W^s(F^t(m))$.

They define foliations of

(iv) $W^u = \cup_{m\in V} W^u(m)$ and $W^s = \cup_{m\in V} W^s(m)$,

with $m \in W^{s/u}(m)$ for all $m \in V$, and

(v) $W^u(m)$ is tangent to N^u_m at $m \in V$ and $W^s(m)$ is tangent to N^s_m at $m \in V$.

If Y is another C^r-vector field on M sufficiently close to X in C^1 on K then there is a unique normally hyperbolic manifold of Y, C^r-diffeomorhpic to V, and the above objects persist for Y in place of X satisfying (i)-(v) and being C^r-diffeomorphic to their X-counterparts.

Remark 2.2.4. *Let V be a locally invariant submanifold V of M. Then one calls a family of submanifolds $\{S(m) : m \in V\}$ a C^r-family of manifolds if*

$$S^* := \{(m, m') : m \in V, m' \in S(m)\}$$

is a C^r-submanifold of $M \times M$ (cf. section 8 in [22]).

Theorem 2.2.3 is similar to Theorem 4.1 in [40] and summarizes Theorems 4.5.1 and 5.6.1 of [96] in the case of an invariant manifold without boundary. The latter two theorems state, and slightly improve, results from [19, 20, 21] for normally hyperbolic manifolds. Asymptotic estimates of the flow F on the leaves $W^{u/s}(m)$ can be found in [20, 21, 96].

Turning to the case of critical sets we continue with the coordinate-free description of GSPT as found in sections 6-9 of [22] (see also Section 2 in [90]). In the following we assume that M is a C^{r+1}-manifold. Let $\phi : U \to V \subset \mathbb{R}^n$ be a coordinate chart of M and denote by

$$\begin{aligned} \Phi : TM|_U &\to V \times \mathbb{R}^n, \\ (m, v) &\mapsto (\phi(m), D\phi(m)v) \end{aligned}$$

the canonical local trivialization of TM over U associated to the chart (U, ϕ). For $z = \phi(m)$ the local representation of the differential equation (2.23) in z-coordinates reads

$$z' = X_\phi(z) := D\phi(\phi^{-1}(z))X(\phi^{-1}(z)). \tag{2.24}$$

The variational equation of (2.24) along a solution z is

$$\delta z' = DX_\phi(z)\delta z \quad (\delta z \in \mathbb{R}^n) \tag{2.25}$$

where we think of the differential $DX_\phi(z)$ of X_ϕ at z as its associated Jacobi matrix. Usually the coefficient matrix $DX_\phi(z)$ in (2.25) is coordinate dependent due to second derivatives of coordinate changes appearing when comparing the coefficient matrix in (2.25) in different charts. However, if $m_0 = \phi^{-1}(z_0)$ is an equilibrium point of X, i.e. $X(m_0) = 0$, then $DX_\phi(z_0)$ transforms by similarity and

$$\begin{aligned} LX(m_0) : T_{m_0}M &\to T_{m_0}M, \\ v &\mapsto D\phi^{-1}(\phi(m_0))DX_\phi(\phi(m_0))D\phi(m_0)v \end{aligned}$$

is a well-defined linear map on $T_{m_0}M$ which we call the linearization of X at m_0. With this construction in mind we turn to a global formulation of geometric singular perturbation theory.

Remark 2.2.5. *The above is a precise version of Fenichel's short hand in [22] and shows that on a critical set the linearization of X can be defined in a coordinate-free way.*

Let $\{X^\varepsilon\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ be a family of vector fields on M such that $(m,\varepsilon)\mapsto X^\varepsilon(m)$ is a C^r-function. Let $\mathscr{E}$ be a C^r-submanifold of M consisting entirely of equilibrium points. Denote by $\mathbf{0}$ the zero vector field on $(-\varepsilon_0,\varepsilon_0)$, and define a vector field $X^\varepsilon\times\mathbf{0}$ on $M\times(-\varepsilon_0,\varepsilon_0)$ in the canonical way. The flow of both X^ε and $X^\varepsilon\times\mathbf{0}$ is denoted by $(t,m)\mapsto m\cdot t$. Since X^0 vanishes on $\mathscr{E}$ the tangent space $T_m\mathscr{E}$ is in the kernel of the linearization $LX^0(m)$ of X^0 at all $m\in\mathscr{E}$. For all $m\in\mathscr{E}$ the vector space $T_m\mathscr{E}$ is invariant under $LX^0(m)$, and the linearization $LX^0(m)$ induces a linear map

$$QX^0(m):T_mM\backslash T_m\mathscr{E}\to T_mM\backslash T_m\mathscr{E}$$

on the quotient space $T_mM\backslash T_m\mathscr{E}$. The eigenvalues of $QX^0(m)$ are the non-trivial eigenvalues of $LX^0(m)$.

Let $\mathscr{E}_R\subset\mathscr{E}$ be the open set where all non-trivial eigenvalues of $LX^0(m)$ are non-zero, and denote by $\mathscr{E}_H\subset\mathscr{E}_R$ the open set where all these eigenvalues have non-vanishing real parts. There is a C^{r-1}-splitting $TM|_{\mathscr{E}_R}=T\mathscr{E}|_{\mathscr{E}_R}\oplus N$ where the bundle N consists of the unique complement N_m of $T\mathscr{E}_m$ that is invariant under $LX^0(m)$. The associated projection map $\pi^{\mathscr{E}}:TM|_{\mathscr{E}_R}\to T\mathscr{E}_{\mathscr{E}_R}$ is C^{r-1}. In $\mathscr{E}_R$ the reduced vector field X_R is defined by

$$X_R(m)=\pi^{\mathscr{E}}\partial/\partial\varepsilon X^\varepsilon(m)|_{\varepsilon=0}$$

and is itself a C^{r-1}-vector field on $\mathscr{E}_R$.

We need some notions from invariant manifold theory. Let $K\subset\mathscr{E}$ be a compact subset such that for $m\in K$ the quotient map $QX^0(m)$ has k^s eigenvalues with real parts smaller than zero, k^u ones with real parts larger than zero, and k^c purely imaginary eigenvalues, all counted with multiplicities. Consequently, for all $m\in K$, the linearization $L(X^\varepsilon\times\mathbf{0})(m,0)$ has a k^s dimensional stable subspace E^s_m, a $n-(k^s+k^u)+1$ dimensional center subspace E^c_m, and a k^u dimensional unstable subspace E^s_m in $T_{(m,0)}M\times(-\varepsilon_0,\varepsilon_0)$.

Definition 2.2.6. A manifold $\mathscr{C}^s$ is called a center-stable manifold for $X^\varepsilon\times 0$ near K if $K\times\{0\}\subset\mathscr{C}^s$, $\mathscr{C}^s$ is locally invariant under the flow of $X^\varepsilon\times\mathbf{0}$, and $\mathscr{C}^s$ is tangent to $E^s_m\oplus E^c_m$ at $(m,0)$ for all $m\in K$.

One obtains the definitions of an unstable-center manifold $\mathscr{C}^u$ and of a center manifold $\mathscr{C}$ for $X^\varepsilon\times\mathbf{0}$ near K by replacing $E^s_m\oplus E^c_m$ by $E^u_m\oplus E^c_m$ and E^c_m in Definition 2.2.6, respectively.

Definition 2.2.7. Let $\mathscr{C}^s$ be a center-stable manifold for $X^\varepsilon\times\mathbf{0}$ near K. A family $\{\mathscr{F}^s(p):p\in\mathscr{C}^s\}$ is called a C^{r_2}-family of C^{r_1}-stable manifolds for $\mathscr{C}^s$ near K if

(i) $\mathscr{F}^s(p)$ is a C^{r_1}-manifold for each $p\in\mathscr{C}^s$,

(ii) $p\in\mathscr{F}^s(p)$ for each $p\in\mathscr{C}^s$,

(iii) $\mathscr{F}^s(p)$ and $\mathscr{F}^s(q)$ are disjoint or identical for each p and q in $\mathscr{C}^s$,

(iv) $\mathscr{F}^s(m,0)$ is tangent to E^s_m at $(m,0)$ for each $m \in K$,

(v) $\{\mathscr{F}^s(p) : p \in \mathscr{C}^s\}$ is a positively invariant C^{r_2}-family of manifolds.

Given a center-unstable manifold $\mathscr{C}^u$ near K one defines in the same manner a C^{r_2}-family of C^{r_1}-unstable manifolds $\{\mathscr{F}^u(p) : p \in \mathscr{C}^u\}$ for $\mathscr{C}^u$ near K by replacing E^s_m by E^u_m in (iv) and positive invariance by negative invariance in (v) of Definition 2.2.7. We are now ready to state the central result of global GSPT which is Theorem 9.1 in [22].

Theorem 2.2.8. *Let M be a C^{r+1}-manifold with $1 \leq r < \infty$. Let $X^\varepsilon, \varepsilon \in (-\varepsilon_0, \varepsilon_0)$, be a family of vector fields on M such that $(m,\varepsilon) \mapsto X^\varepsilon(m)$ is a C^r-function, and let $\mathscr{E}$ be a C^r-submanifold of M consisting entirely of equilibrium points of X^0. Let k^s, k^c and k^u be fixed integers, and let $K \subset \mathscr{E}$ be a compact subset such that $QX^0(m)$ has k^s eigenvalues with real parts smaller than zero, k^u eigenvalues with real parts larger than zero and k^c purely imaginary eigenvalues for all $m \in K$. Then the following statements hold:*

(i) There are a C^r-center-stable manifold $\mathscr{C}^s$, a C^r-center-unstable manifold $\mathscr{C}^u$ and a C^r-center manifold $\mathscr{C}$ for $X^\varepsilon \times \mathbf{0}$ near K. There is a neighborhood U of $K \times \{0\}$ such that $A^-(U) \subset \mathscr{C}^u, A^+(U) \subset \mathscr{C}^s$ and $I(U) \subset \mathscr{C}$.

(ii) There is a C^{r-1}-family $\{\mathscr{F}^s(p) : p \in \mathscr{C}^s\}$ of C^r-stable manifolds for $\mathscr{C}^s$ near K. If $p \in M \times \{\varepsilon\}$ then $\mathscr{F}^s(p) \subset M \times \{\varepsilon\}$. Each manifold $\mathscr{F}^s(p)$ intersects $\mathscr{C}$ transversally in exactly one point. There is a C^{r-1}-family $\{\mathscr{F}^u(p) : p \in \mathscr{C}^u\}$ of C^r-unstable manifolds for $\mathscr{C}^u$ near K. If $p \in M \times \{\varepsilon\}$ then $\mathscr{F}^u(p) \subset M \times \{\varepsilon\}$. Each manifold $\mathscr{F}^u(p)$ intersects $\mathscr{C}$ transversally in exactly one point.

(iii) Let K^s be larger than the real parts of the eigenvalues of $QX^0(m)$ in the left half of the complex plane for all $m \in K$. Then there is a constant C_s such that if $p \in \mathscr{C}^s$ and $q \in \mathscr{F}^s(p)$ then

$$d(p \cdot t, q \cdot t) \leq C_s e^{K_s t} d(p,q)$$

for all $t \geq 0$ such that $p \cdot [0,t] \subset \mathscr{C}^s$. If $p \in A^+(\mathscr{C}^s)$ then

$$\mathscr{F}^s(p) = \{q \in U : d(p \cdot t, q \cdot t) \leq C_s e^{K_s t} d(p,q) \text{ for all } t \geq 0\}.$$

Let K^u be larger than the real parts of the eigenvalues of $QX^0(m)$ in the right half of the complex plane for all $m \in K$. Then there is a constant C_u such that if $p \in \mathscr{C}^u$ and $q \in \mathscr{F}^u(p)$ then

$$d(p \cdot t, q \cdot t) \leq C_u e^{K_u t} d(p,q)$$

for all $t \leq 0$ such that $p \cdot [t,0] \subset \mathscr{C}^u$. If $p \in A^-(\mathscr{C}^u)$ then

$$\mathscr{F}^u(p) = \{q \in U : d(p \cdot t, q \cdot t) \leq C_u e^{K_u t} d(p,q) \text{ for all } t \leq 0\}.$$

(iv) If $K \subset \mathscr{E}_H$ define for $(m, \varepsilon) \in \mathscr{C}$

$$X_{\mathscr{C}}(m,\varepsilon) := \begin{cases} \varepsilon^{-1} X^{\varepsilon}(m) \text{ if } \varepsilon \neq 0, \\ X_R(m) \text{ if } \varepsilon = 0. \end{cases}$$

Then $X_{\mathscr{C}}$ is a C^{r-1}-vector field on $\mathscr{C}$ near $K \times \{0\}$.

From among the conclusions of Theorem 2.2.8 we want to highlight statement (iv): If K is a subset of the normally hyperbolic part $\mathscr{E}_H$ of $\mathscr{E}$ then one can investigate the flow $F_{\mathscr{C}}$ of $X_{\mathscr{C}} \times \mathbf{0}$ on $\mathscr{C}$ which is defined by the maximal integral curves $\tau \mapsto (\gamma(\tau), \varepsilon)$ of the differential equation

$$\begin{aligned} \dot{\gamma} &:= \frac{d}{d\tau}\gamma = X_{\mathscr{C}}(\gamma, \varepsilon), \\ \dot{\varepsilon} &:= \frac{d}{d\tau}\varepsilon = 0, \end{aligned}$$

on $\mathscr{C}$. If $r > 1$ we may expand $X_{\mathscr{C}}$ in orders of ε and gain information on the flow $F_{\mathscr{C}}$ by analyzing the reduced vector field X_R and applying regular perturbation theory. For $\varepsilon > 0$ this information translates to the the original dynamics induced by $X^{\varepsilon} \times \mathbf{0}$ on $M \times (-\varepsilon_0, \varepsilon_0)$ by the simple scaling $\tau \mapsto \tau\varepsilon^{-1}$. Hence, two time scales are present, and the time scale given by t is called fast whereas the one given by τ is called slow. The dynamics of $X_{\mathscr{C}}$ are usually easier to cope with since the dimension of the original problem n is diminished to $n - (k^s + k^u)$ and often the reduced vector field X_R is of particularly simple form. One carries out a center manifold reduction.

Moving to the local picture and following the exposition of GSPT in [50] will make the above comment more transparent. We will apply GSPT to families of vector fields that in a suitable coordinate chart define ordinary differential equations of the form

$$\begin{cases} x' = \varepsilon f(x, y, \varepsilon), \\ y' = g(x, y, \varepsilon), \end{cases} \tag{2.26}$$

where $' = d/dt, x \in \mathbb{R}^l, y \in \mathbb{R}^k$ with $l + k = n$, and f, g are C^r-functions defined on $U \times (-\varepsilon_0, \varepsilon_0)$ with U an open subset of $\mathbb{R}^n$ and $\varepsilon_0 > 0$. Changing the time scale to $\tau = \varepsilon t$ we find that system (2.26) reads

$$\begin{cases} \dot{x} = f(x, y, \varepsilon), \\ \varepsilon\dot{y} = g(x, y, \varepsilon), \end{cases} \tag{2.27}$$

with $\dot{} = d/d\tau$. The system (2.26) is called the fast system, and (2.27) is called the slow system. One refers to x and y as the slow variable and the fast variable, respectively. Let us set $X^{\varepsilon}(x, y) := (\varepsilon f(x, y, \varepsilon), g(x, y, \varepsilon))$.

There are natural limit problems as $\varepsilon \to 0$ to each of the systems (2.26) and (2.27). Taking the limit $\varepsilon \to 0$ one obtains from (2.26) the so-called reduced

problem

$$\begin{cases} x' = 0, \\ y' = g(x, y, 0), \end{cases} \tag{2.28}$$

and from (2.27) the so-called layer problem

$$\begin{cases} \dot{x} = f(x, y, 0), \\ 0 = g(x, y, 0). \end{cases} \tag{2.29}$$

In (2.28) the slow variable x remains constant and enters as a parameter into the differential equation for y. The layer problem (2.29) can be solved under suitable non-degeneracy conditions by first solving the constraint $g(x, y, 0) = 0$ for y in terms of x, and then plugging the result into $\dot{x} = f(x, y, 0)$ which yields a differential equation for x alone. Note that the set $\{(x, y) \in U : g(x, y, 0) = 0\}$ is the set of fixed points for (2.28). A goal achieved by GSPT is to find a perturbation of the layer problem (2.29), which enables one to deduce information on the dynamics of (2.26) from those of (2.29), and to describe the flow of the fast equations (2.26) near compact (normally hyperbolic) critical sets of (2.28) by combining properties of the solutions to the limiting problems (2.28) and (2.29).

More concretely, let $K \subset \mathbb{R}^l$ be a compact contractible domain whose boundary is an $l-1$ dimensional embedded C^r-submanifold of $\mathbb{R}^l$, and let $h^0 : K \to \mathbb{R}^k$ be a C^r-function. Let k^s, k^u be integers such that $k^s + k^u = k$. Suppose that all elements (x, y) of the graph

$$M_0 := \{(x, y) : y = h^0(x)\}$$

satisfy $g(x, y, 0) = 0$ while the differential $D_y g(x, y, 0)$ is invertible possessing k^s eigenvalues with real parts smaller than zero and k^u eigenvalues with real parts greater than zero. Then M_0 is a normally hyperbolic invariant manifold of the vector field $X^0 = (0, g(x, y, 0))$ consisting only of fixed points of X^0. Thus M_0 is a critical manifold of X^0. We find a center manifold like the one in Theorem 2.2.8 which is given, for constant ε, as a graph over K.

Theorem 2.2.9. *If $\varepsilon > 0$ is sufficiently small then there is a function $y = h^\varepsilon(x)$, defined on K, such that the graph*

$$M_\varepsilon := \{(x, y) : x \in K, y = h^\varepsilon(x)\}$$

has a locally invariant interior under the flow of (2.26). Moreover, h^ε is C^r jointly in ε and x

Theorem 2.2.9 is Theorem 2 of [50]. One calls the manifold M_ε a slow manifold. The reduced vector field X_R is easily calculated using the function h^ε and defines in the slow time scaling the so-called slow equations

$$\dot{x} = f(x, h^\varepsilon(x), \varepsilon). \tag{2.30}$$

The limit $\varepsilon \to 0$ is well-defined for the slow equations and solutions to (2.30) with $\varepsilon = 0$ solve the layer problem (2.29). If r is sufficiently large then regular perturbation theory is applicable to (2.30) in order to deduce information on how these solutions of the layer problem perturb on M_ε as ε is varied away from zero.

By a change of variables we may assume $h^0 \equiv 0$. Furthermore, splitting the fast variable $y = (a, b) \in \mathbb{R}^{k^s} \times \mathbb{R}^{k^u}$ we will encounter system (2.26) in the special form

$$\begin{cases} x' = \varepsilon f(x, y, \varepsilon), \\ a' = A(x)a + G_1(x, y, \varepsilon), \\ b' = B(x)b + G_2(x, y, \varepsilon), \end{cases} \tag{2.31}$$

with matrices $A(x) \in \mathbb{R}^{k_s \times k_s}, B(x) \in \mathbb{R}^{k_u \times k_u}$ such that for all $x \in K$ and $(a, b) \in \mathbb{R}^{k_s} \times \mathbb{R}^{k_u}$ and some constants $K_s < 0 < K_u$ the inequalities

$$\begin{aligned} \langle a, A(x)a \rangle &\leq K_s |a|^2, \\ \langle b, B(x)b \rangle &\geq K_u |b|^2, \end{aligned}$$

hold, and with C^r-functions G_i such that

$$G_i(x, y, \varepsilon) = \mathcal{O}(\varepsilon)$$

uniformly for (x, y) in a compact neighborhood of $K \times \{0\} \subset \mathbb{R}^n$. We will say that system (2.26) is in Fenichel normal form if it is of the form (2.31) with the stated properties.

This closes our review of GSPT as well as our preliminary discussion on some of the basic mathematical material used in the following sections.

2.3 Hyperbolically regularized systems of conservation laws

Let us continue by specifying the assumptions that will be imposed on the coefficients of the hyperbolically regularized conservation law (2.3) in the remaining sections of this chapter. Then we construct the shock profiles and deduce the dynamical system that will be at the center of our stability studies. At the end of this section the reader finds some background information and a brief review of the literature.

Let $\mathcal{V} \subset \mathbb{R}^n$ be open with $v_* \in \mathcal{V}$, and let $k \in \{1, \dots, n\}$. We consider a hyperbolically regularized conservation law of the form

$$g(v)_t + f(v)_x = B\Box v = B(v_{xx} - v_{tt}) \ \ (v \in \mathbb{R}^n) \tag{2.32}$$

for $(t, x) \in \mathbb{R}^2$ with smooth $f, g : \mathcal{V} \to \mathbb{R}^n$ and $B \in \mathbb{R}^{n \times n}$ satisfying

B1 $\forall v \in \mathcal{V}$: B and the Jacobian matrices $Df(v)$ and $Dg(v)$ of f and g at v are symmetric with B and $Dg(v^*)$ being positive definite;

B2 $Df(v_*)$ has only real and simple eigenvalues w.r.t. $Dg(v_*)$;

B3 The k-th eigenvalue of Df w.r.t. Dg is genuinely nonlinear at v_*;

B4 Sub-characteristicity: $Dg(v_*) \pm Df(v_*)$ is positive definite.

Let us briefly discuss assumptions B1-B4:

Assumption B1 is classical: Symmetric hyperbolic conservation laws are ubiquitous in mathematical physics [30, 36]. In fact, choosing appropriate variables one may often assume that f and g possess potentials [9, 36, 82]. Thus we think of f and g as gradient fields which justifies assumption B1.

Assumption B2 means that there are n pairwise distinct real numbers $\mu_i \in \mathbb{R}$ and n vectors $r_i \in \mathbb{R}^n \backslash \{0\}$ such that

$$Df(v_*)r_i = \mu_i Dg(v_*)r_i.$$

Hence, the conservation law $g(v)_t + f(v)_x = 0$ is a hyperbolic partial differential equation (near v_*). Furthermore, B2 is equivalent to strict hyperbolicity of

$$Dg(v_*)^{-1/2} Df(v_*) Dg(v_*)^{-1/2}$$

which is a common assumption on the flux of hyperbolic conservation laws.

Assumption B3 states that the k-th eigenvalue of $Dg^{-1/2} Df Dg^{-1/2}$ is genuinely nonlinear at v_*. This genuine nonlinearity in the sense of Lax [59] is used, on the one hand, to show the existence of shock profiles [37, 69] determining in the spirit of an entropy condition the direction in which the end-states of a shock can be smoothly connected. On the other hand it also enters in various contexts in the proofs on stability of said shock profiles enabling, vaguely speaking, control of contributions that emerge along the k-th eigenvector field (see e.g. [26, 37, 77, 89]). Assumption B3 will play a similar role in the following analysis. (For results beyond genuine nonlinearity the reader may consult, e.g., [28, 31, 51, 64].)

By assumption B4 we find

$$\sigma(Dg(v_*)^{-1/2} Df(v_*) Dg(v_*)^{-1/2}) \subset (-1, 1) \tag{2.33}$$

which is a typical stability condition for hyperbolic equations with wave hierarchy: If the characteristic speeds of the lower order hyperbolic operator lie in-between those of the higher order hyperbolic operator then the compound equation is linearly stable (see [95]). In our case the higher order operator is (essentially) the d'Alembertian with symmetrically distributed characteristic speeds -1 and 1 bounding by (2.33) the respective speeds of the lower order operator in (2.32). On the linearized level B4 makes system (2.32) dissipative which in our case is, quite remarkably, equivalent to causality of (2.32) [25]. The sub-characteristicity condition B4 (cf. [88]) is inspired by the notion of dissipativity in hyperbolic-parabolic systems introduced by Shizuta and Kawashima in [85] and is thereby closely related to considerations concerning hyperbolic relaxation systems (see e.g. [56, 66, 97, 98]). Hence one expects the dispersion curves of shock profiles of system (2.32) to look similar to the ones in the case of hyperbolic relaxation systems [70] (see Section 2.4).

Without loss of generality we may assume

$$f(v_*) = g(v_*) = 0 \text{ and } v_* = 0.$$

The above discussion of assumptions B1-B4 readily leads to:

Lemma 2.3.1. *If B1-B4 hold then there exist $\delta > 0$ and functions $\mu_i \in C^\infty(B_\delta(0);\mathbb{R}), r_i \in C^\infty(B_\delta(0);\mathbb{R}^n\backslash\{0\})$ for $i = 1,\dots,n$ such that for all $v \in B_\delta(0) \subset \mathbb{R}^n$ the matrices $Dg(v)$ and $Dg(v) \pm Df(v)$ are positive definite, and for all $v, w \in B_\delta(0)$ and i, j*

$$\begin{aligned} Df(v)r_i(v) &= \mu_i(v)Dg(v)r_i(v), \\ r_i(v)^t Dg(v)r_j(v) &= \delta_{ij}, \end{aligned}$$

where δ_{ij} denotes the Kronecker delta, and

$$\langle D\mu_k(0), r_k(0)\rangle < 0. \tag{2.34}$$

Furthermore, for all $v, w \in B_\delta(0)$ and i, j we have

$$|\mu_i(v)| < 1 \tag{2.35}$$

and

$$\mu_i(v) < \mu_j(w) \text{ if } i < j. \tag{2.36}$$

Multiplying (2.32) from the left by $B^{-1/2}$ and changing variables via $\tilde{v} = B^{1/2}v$ we may assume, and do so from now on, that

$$B = \mathbb{1}_n$$

in (2.32) while keeping the assumptions B1-B4 (and the conclusions of Lemma 2.3.1).

Definition 2.3.2. A triple $(v^-, v^+, s) \in \mathcal{V} \times \mathcal{V} \times \mathbb{R}$ is called a shock of the conservation law $g(v)_t + f(v)_x = 0$ if the Rankine-Hugoniot condition

$$f(v^+) - f(v^-) = s(g(v^+) - g(v^-))$$

is satisfied. It is called a (Lax-)p-shock for $p \in \{1,\dots,d\}$ if Lax's inequalities

$$\mu_p(v^+) < s < \mu_p(v^-)$$

and

$$(\mu_j(v^-) - s)(\mu_j(v^+) - s) > 0 \text{ for } j \neq p,$$

hold.

We will move into the rest frame of a given p-shock (v^-, v^+, s) with $s \in (-1, 1)$ by the Lorentz-boost

$$\begin{pmatrix} t \\ x \end{pmatrix} \mapsto \begin{pmatrix} \gamma(s)(t - sx) \\ \gamma(s)(x - st) \end{pmatrix}$$

of speed s where for $s \in (-1, 1)$ one has

$$\gamma(s) := (1 - s^2)^{-1/2}.$$

Under such a transformation the hyperbolically regularized conservation law (2.32) keeps its form.

Lemma 2.3.3. *Let $s \in (-1, 1)$. Then a Lorentz-boost of speed s transforms (2.32) into*

$$g_s(v)_t + f_s(v)_x = B\Box v$$

where

$$\begin{aligned} g_s(v) &:= \gamma(s)(g(v) - sf(v)) \\ f_s(v) &:= \gamma(s)(f(v) - sg(v)). \end{aligned}$$

The assumptions B1-B4 hold with f and g replaced by f_s and g_s. For $\delta > 0$ like in Lemma 2.3.1 the eigenvalues of Df_s w.r.t. Dg_s on $B_\delta(0)$ are given by

$$\mu_{j;s} = \frac{\mu_j - s}{1 - \mu_j s} \quad (j \in \{1, \dots, d\}) \tag{2.37}$$

with eigenvectors r_j and satisfy (2.34), (2.35), (2.36).

Proof. The claimed transformation of equation (2.32) with the given f_s and g_s is a direct calculation. Assumption B1 for f_s and g_s follows immediately, too. Using formula (2.37) and the fact that suitable eigenvectors are still given by r_j, yields B2 and B3 for f_s and g_s as well as (2.34) and (2.36). B4 for f_s and g_s, implying (2.35), is a direct calculation. □

Since $|\mu_k(0)| < 1$ we can perform a Lorentz-boost of speed $s = \mu_k(0)$, and in view of Lemma 2.3.3 we assume from now on without loss of generality that

$$\mu_k(0) = 0$$

by making the replacements $f \to f_{\mu_k(0)}$ and $g \to g_{\mu_k(0)}$. To simplify our set-up further we prove a small lemma.

Lemma 2.3.4. *Let $F \in \mathbb{R}^{n\times n}$ be symmetric. If there are $c > 0$ and a splitting*

$$V^- \oplus \ker F \oplus V^+ = \mathbb{R}^n$$

such that for all $\eta^- \in V^-$ *and* $\eta^+ \in V^+$

$$\langle \eta^-, F\eta^- \rangle \leq -c|\eta^-|^2,$$
$$\langle \eta^+, F\eta^+ \rangle \geq c|\eta^+|^2,$$

then, counting with multiplicities, F *has exactly* $\dim V^-$ *negative and* $\dim V^+$ *positive eigenvalues.*

Proof. Let W^- resp. W^+ be the maximal F-invariant subspace of $\mathbb{R}^n$ such that $F|_{W^-}$ resp. $F|_{W^+}$ is negative resp. positive definite. Obviously,

$$W^- \oplus \ker F \oplus W^+ = \mathbb{R}^n.$$

Assume $\dim W^- > \dim V^-$. Then $W^- \cap (\ker F \oplus V^+)$ is non-trivial, so we can pick $w = v^0 + v^+ \in W^- \cap (\ker F \oplus V^+) \backslash \{0\}$ with $v^0 \in \ker F$ and $v^+ \in V^+$. Since $w \notin \ker F$ we find $v^+ \neq 0$ and

$$\langle w, Fw \rangle = \langle v^+, Fv^+ \rangle \geq c|v^+|^2 > 0,$$

a contradiction to the negative definiteness of $F|_{W^-}$. Hence, $\dim W^- \leq \dim V^-$. In the same way one shows $\dim W^+ \leq \dim V^+$. Since $\dim W^- + \dim W^+ = \dim V^- + \dim V^+$ we find $\dim W^- = \dim V^-$ and $\dim W^+ = \dim V^+$. □

Lemma 2.3.5. *Let* δ, μ_i *and* r_i *be like in Lemma 2.3.1. Then there is an orthogonal change of variables leaving* (2.32) *invariant such that*

$$Df(0) = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$$

with

$$\lambda_k = 0 \text{ and } \lambda_j < 0 < \lambda_{j'} \text{ if } j < k < j',$$

and

$$e_k = \frac{r_k(0)}{|r_k(0)|}$$

where $e_k \in \mathbb{C}^n$ *is the unit vector along the* k*-th axis. With respect to this change of basis it holds*

$$\langle D\mu_k(0), r_k(0) \rangle = |r_k(0)|^3 (\partial_k^2 f(0))_k. \tag{2.38}$$

Proof. Set $F := Df(0), G := G(0)$, and

$$V^- := \operatorname{span}\{r_j(0)\}_{1 \leq j < k} \text{ and } V^+ := \operatorname{span}\{r_j(0)\}_{k < j \leq d}.$$

The matrix F is symmetric and $\ker F = \operatorname{span}\{r_k(0)\}$. Then $V^- \oplus \ker F \oplus V^+ = \mathbb{R}^n$. On $G^{1/2}V^-$ resp. $G^{1/2}V^+$ the matrix $G^{-1/2}FG^{-1/2}$ is negative resp. positive definite. Because G is positive definite, too, this implies F being negative resp. positive definite on V^- resp. V^+. An application of Lemma 2.3.4 shows that there is an orthonormal set of eigenvectors of F with the k-th eigenvector

equal to $r_k/|r_k|$ inducing a change of basis that diagonalizes F in the claimed way.
To show (2.38) use the above change of basis and apply the differential operator $\langle r_k(0), D_v\rangle = |r_k|\partial_k$ to the identity (cf. [86])

$$r_k(v)^t(Df(v) - \mu_k(v)Dg(v))r_k(v) = 0 \text{ for } v \in B_\delta(0).$$

□

From now on fix δ, μ_i, r_i like in Lemma 2.3.1 and change basis like in Lemma 2.3.5. For small enough $\varepsilon_0 \in (0,1]$ consider a smooth family of triples

$$\{(v_\varepsilon^-, v_\varepsilon^+, s(\varepsilon))\}_{\varepsilon\in[0,\varepsilon_0]} \subset \mathcal{V}\times\mathcal{V}\times(-1,1)$$

such that:

(i) For $\varepsilon > 0$ the triple $(v_\varepsilon^-, v_\varepsilon^+, s(\varepsilon))$ is a k-shock of $g(v)_t + f(v)_x = 0$;

(ii) $s(0) = 0$ and $s'(0) = 0$;

(iii) $v_\varepsilon^\pm = \pm\varepsilon(e_k + \mathcal{O}(\varepsilon))$;

(iv) v_ε^- and v_ε^+ are the only solutions of

$$f(v) - s(\varepsilon)g(v) = f(v_\varepsilon^-) - s(\varepsilon)g(v_\varepsilon^-)$$

for $v \in \overline{B_{3\varepsilon}(0)}$.

Remark 2.3.6. *One can construct such a family of k-shocks by the usual (local) structure of the Rankine-Hugoniot locus*

$$\{v \in \mathcal{V} |\ \exists s \in \mathbb{R} : f(v) - sg(v) = f(v_\varepsilon^-) - sg(v_\varepsilon^-)\}$$

at v_ε^- (see [61], and for a proof e.g. [18]). Note $\mu_k(0) = 0$.

For simplicity we only consider shocks with end-states symmetrically distributed over the k-th direction, i.e.

(iii)′ $(v_\varepsilon^\pm)_k = \pm\varepsilon$.

Definition 2.3.7. We call $\phi \in C^\infty(\mathbb{R})$ a shock profile for a shock $(v^-, v^+, s) \in \mathcal{V}\times\mathcal{V} \in (-1,1)$ of $g(v)_t + f(v)_x = 0$ if ϕ is a solution of the boundary value problem

$$\phi' = f_s(\phi) - f_s(v^-), \quad \phi(\pm\infty) = v^\pm. \tag{2.39}$$

For $\varepsilon \in [0,\varepsilon_0]$ set

$$c_\varepsilon := f_{s(\varepsilon)}(v_\varepsilon^-) \text{ and } \gamma_\varepsilon = \gamma(s(\varepsilon)).$$

Viewing the construction of shock profiles by Majda and Pego [69] from a geometric perspective one can obtain a smooth family of shock profiles $\{\phi_\varepsilon\}_\varepsilon$ for

the family of shocks $\{(v_\varepsilon^-, v_\varepsilon^+, s(\varepsilon)\}_\varepsilon$ by a slow-fast decomposition like in [26]. To make the slow-fast structure apparent set $\varepsilon u = \phi$ in (2.39) for the shock $(v_\varepsilon^-, v_\varepsilon^+, s(\varepsilon))$. Then u must solve

$$u' = \tilde{f}_\varepsilon(u), \quad u(\pm\varepsilon) = u_\varepsilon^\pm \tag{2.40}$$

where

$$\tilde{f}_\varepsilon(u) := \varepsilon^{-1} f_{s(\varepsilon)}(\varepsilon u) - \varepsilon^{-1} c_\varepsilon \text{ for } \varepsilon \in (0, \varepsilon_0], u \in \overline{B_3(0)},$$

and

$$u_\varepsilon^\pm := \varepsilon^{-1} v_\varepsilon^\pm.$$

A Taylor expansion shows

$$\begin{aligned}\varepsilon^{-1} f_{s(\varepsilon)}(\varepsilon u) =& \operatorname{diag}(\lambda_1, \dots, \lambda_d) u + \frac{\varepsilon}{2} D^2 f(0)[u, u] \\ &+ \mathcal{O}(\varepsilon^2)\end{aligned}$$

and

$$\varepsilon^{-1} c_\varepsilon = \mathcal{O}(\varepsilon) \text{ with } (\varepsilon^{-1} c_\varepsilon)_k = \varepsilon a + \mathcal{O}(\varepsilon^2)$$

for

$$a := \frac{1}{2} (\partial_k^2 f(0))_k < 0.$$

Hence, $\tilde{f}_\varepsilon(u)$ and $u_\varepsilon^\pm$ extend smoothly onto $\varepsilon = 0$ and (2.40) is in Fenichel normal form (2.31). The fast variables are u_j for $j \neq k$, the slow variable is u_k. The set $\{u_j = 0\}_{j \neq k} \cap \overline{B_3(0)}$ is a normally hyperbolic critical manifold for (2.40) and Theorem (2.2.9) shows (see also [50, 90]):

Lemma 2.3.8. *For ε_0 small enough there exists a smooth function $u = u_\varepsilon(\tau)$ for $\varepsilon \in [0, \varepsilon_0], \tau \in J := [-1, 1]$, with $u_\varepsilon(\tau)_j = \mathcal{O}(\varepsilon)$ if $j \neq k$ and $(u_\varepsilon(\tau))_k = \tau$ defining a slow manifold $u_\varepsilon(J)$ of (2.40) with slow flow given by*

$$\tau' = \varepsilon(1 - \tau^2) h_\varepsilon(\tau) \tag{2.41}$$

for a smooth function $h = h_\varepsilon(\tau) = (-a + \mathcal{O}(\varepsilon))$.

We obtain a smooth family of shock profiles $\{\phi_\varepsilon\}_\varepsilon$ connecting v_ε^- to v_ε^+ by letting ϕ_ε be the solution of

$$\phi' = f_{s(\varepsilon)}(\phi) - c_\varepsilon, \quad \phi(0) = \varepsilon u_\varepsilon(0).$$

They correspond to solutions χ_ε of the equations

$$\chi_\varepsilon' = \varepsilon(1 - \chi_\varepsilon^2) h_\varepsilon(\chi_\varepsilon), \quad \chi_\varepsilon(0) = 0, \tag{2.42}$$

via

$$\phi_\varepsilon = \varepsilon u_\varepsilon(\chi_\varepsilon).$$

The profiles ϕ_ε define travelling wave solutions v_ε of (2.32) with

$$v_\varepsilon(t,x) = \phi_\varepsilon(\gamma_\varepsilon(x - s(\varepsilon)t)).$$

We apply a Lorentz-boost of speed $s(\varepsilon)$ to move into the rest frame of v_ε. Then (2.32) reads

$$g_{s(\varepsilon)}(v)_t + f_{s(\varepsilon)}(v)_x = \Box v. \tag{2.43}$$

Linearizing (2.43) at $v_\varepsilon(t,x) = \phi_\varepsilon(x)$ yields

$$Dg_{s(\varepsilon)}(\phi_\varepsilon)v_t + (Df_{s(\varepsilon)}(\phi_\varepsilon)v)_x = \Box v. \tag{2.44}$$

The Laplace modes $e^{\kappa t}p(x), \kappa \in \mathbb{C}$, of (2.44) are defined by solutions p of

$$l_{\kappa,\varepsilon}p := p_{xx} - (Df_{s(\varepsilon)}(\phi_\varepsilon)p)_x - (\kappa^2 \mathbb{1}_n + \kappa Dg_{s(\varepsilon)}(\phi_\varepsilon))p = 0.$$

Setting $q = p_x - Df_{s(\varepsilon)}(\phi_\varepsilon)p$ the above Laplace-mode differential equation has a first-order formulation

$$\xi' = A_{s(\varepsilon),\kappa}(\phi_\varepsilon)\xi \tag{2.45}$$

for $\xi = (p,q) \in \mathbb{C}^{2n}$ and

$$A_{s,\kappa}(v) = \begin{pmatrix} Df_s(v) & \mathbb{1}_n \\ \kappa^2\mathbb{1}_n + \kappa Dg_s(v) & 0 \end{pmatrix} \in \mathbb{C}^{2n\times 2n}.$$

The study of the non-autonomous, parameter depending and linear ordinary differential equation (2.45) is the main objective of the following sections. (The reader can find more information on ordinary differential equations and dynamical systems in, e.g., [10, 79, 91], and specifically on systems like (2.93) in [11, 39, 83].)

We will follow the work of Freistühler and Szmolyan on hyperbolic-parabolic conservation laws in [26] and [27]: Instead of treating the Laplace-mode equation (2.45) as a non-autonomous system one couples (2.45) to the reduced profile equation (2.41) and obtains an autonomous system in τ and ξ which depends on the parameters ε and κ. Then invariant manifold theory is applied: Inspired by the construction of the shock profiles ϕ_ε one uses normally hyperbolic theory [19, 40], most times in the form of geometric singular perturbation theory [22, 50, 90]. Making use of the linear structure of (2.45) the evolution of vector spaces is tracked by the associated Grassmannian flows (cf. [2], see Section 2.1 for some basics concerning flows on Grassmann manifolds). Since the resulting state space is invariant and compact, normally hyperbolic theory yields unique invariant manifolds perturbing smoothly from the initial manifolds. By considering three spectral regimes (see Section 2.7) in combination with various rescalings a detailed construction of Evans bundles for the profiles

ϕ_ε is achieved.

The use of spectral theory to answer questions on the stability of travelling waves has become popular in the last five decades starting with the fundamental work of Evans on nerve axon equations [17]. The *Evans function method* was further developed by Jones [49] and Alexander et al. [2], and since then Evans function theory has been applied successfully to reaction-diffusion equations, e.g. [2, 32, 33, 49], viscous conservation laws, e.g. [26, 34, 51, 71, 77, 99], and hyperbolic relaxation systems, e.g. [70, 72, 77], yielding stability results in many different contexts. When studying the stability of travelling wave solutions associated to shocks of, e.g., viscous conservation laws one is met with the problem of the absence of a spectral gap in the sense that the essential spectrum touches the imaginary axis. At first glance this limits the scope of results of an ansatz by Evans function theory since an analytic extension of the Evans function into the essential spectrum is necessary [32, 76]. But after the discovery that Evans bundles of shock profiles have suitable analytic extensions [34, 53], it was Zumbrun and collaborators that showed the profound importance of spectral stability: Under natural assumptions spectral stability implies linearized and nonlinear orbital stability of shock profiles for hyperbolic-parabolic conservation laws [71, 99] and hyperbolic relaxation systems [70, 72]. Whether these implications also hold for hyperbolically regularized systems of conservation laws such as (2.32) is an open research question. However the reader may find a discussion of the associated resolvent kernel in Section 2.6. We view our results as the first steps towards other stability results such as linearized and nonlinear orbital stability.

Parallel to developments within the Evans function framework non-standard energy methods are used in the stability analysis of travelling waves. Along the lines of Sattinger [84], Matsumura and Nishihara [74] and Goodman [37] this ansatz was employed to great success, e.g. [31, 55, 67, 89, 92], often solving directly the nonlinear stability problem. The proofs are tailor-made for the specific problem at hand yielding strong results in the particular situation but are often difficult to generalize.

Despite the similarities to hyperbolic relaxation systems our result is not contained in the results on (spectral) stability of shock profiles for such systems [46, 64, 70, 72, 73, 77, 92]. This is due to the fact that in general g need not be the identity, and hence it seems not possible to transform (2.32) into a relaxation equation of Jin-Xin-type [47] (cf. [58, 92]). Furthermore, in the setting of special relativity as in [29] we need to allow for $g \neq \mathrm{id}$ since by Lemma 2.3.3 the assumption $g = \mathrm{id}$ is in general not invariant under a Lorentz boost. Finally, we want to mention the work of Lattanzio et al. [58] where stability of travelling waves for the Allan-Cahn model with relaxation is discussed. The underlying model being a scalar semi-linear damped wave equation they also arrive at a quadratic nonlinear eigenvalue problem. The dispersion curves of the latter equation lie entirely in the left half of the complex plane and do not touch the imaginary axis which is a major qualitative difference to the models considered here.

2.4 Evans function theory for second-order hyperbolic operators

Some important definitions of Evans function theory for second-order hyperbolic operators like the one defining (2.32) are given in the following as well as the notion of spectral stability of shock profiles used in our setting. The section concludes with a statement of our main result, Theorem 2.4.8.

We begin with a definition and a lemma on dispersion curves for certain families of holomorphic, matrix-valued functions.

Definition 2.4.1. Let $\mathcal{A} = \{A(\kappa)\}_{\kappa\in\mathbb{C}} \subset \mathbb{C}^{2n\times 2n}$ be a smooth family of matrices and denote by $U(\kappa)$ resp. $S(\kappa)$ the unstable resp. stable subspace of $A(\kappa)$. Then a domain of constant splitting $\tilde{\Lambda}$ for $\mathcal{A}$ is an open subset of $\mathbb{C}$ with

$$\dim U(\kappa) = n = \dim S(\kappa) \text{ for all } \kappa \in \tilde{\Lambda}.$$

Lemma 2.4.2. *Let $F, G \in \mathbb{R}^{n\times n}$ satisfy the following assumptions:*

C1 F, G are symmetric, and G is positive definite;

C2 $G \pm F$ is positive definite.

For $\kappa \in \mathbb{C}$ define

$$A(\kappa) := A_{F,G}(\kappa) := \begin{pmatrix} F & \mathbb{1}_n \\ \kappa^2\mathbb{1}_n + \kappa G & 0 \end{pmatrix}$$

and set $\mathcal{A} := \{A(\kappa)\}_{\kappa\in\mathbb{C}}$. Then

$$\tilde{\Lambda} := M_\nu := \left\{ \kappa \in \mathbb{C} \,\middle|\, -\nu\frac{(\operatorname{Im}\kappa)^2}{1+(\operatorname{Im}\kappa)^2} < \operatorname{Re}\kappa \right\} \tag{2.46}$$

is a domain of constant splitting for $\mathcal{A}$. Furthermore, zero is not an element of any domain of constant splitting for $\mathcal{A}$.

Proof. We invoke the results on dissipative symmetric hyperbolic-hyperbolic systems as found in [25]. Let $\kappa \in \mathbb{C}$. Then $A(\kappa)$ is not hyperbolic iff for some $\xi \in \mathbb{R}$

$$i\xi \in \sigma(A(\kappa))$$

which is equivalent to the dispersion relation

$$\det(\kappa G + i\xi F + \kappa^2\mathbb{1}_n + \xi^2\mathbb{1}_n) = 0. \tag{2.47}$$

But (2.47) holds iff $e^{\kappa t + i\xi x}$ solves

$$Gv_t + Fv_x = \Box v.$$

By Theorem 3 and Proposition 1 in [25] there exists a constant $\nu > 0$ such that the last statement implies

$$\operatorname{Re}\kappa \leq -\nu\frac{\xi^2}{1+\xi^2}.$$

As is easy to see, there exists a constant $c > 0$ such that for all solutions $(\kappa, \xi) \in \mathbb{C} \times \mathbb{R}$ of the dispersion relation (2.47) it holds

$$|\kappa| \leq c|\xi|,$$

in particular

$$|\operatorname{Im}\kappa| \leq c|\xi|.$$

By shrinking $\nu > 0$ if necessary we find that $\kappa \in M_\nu$ implies

$$i\mathbb{R} \cap \sigma(A(\kappa)) = \emptyset.$$

Therefore, by continuity, the real parts of the eigenvalues of $A(\kappa)$ may not change signs on M_ν, and considering large $\kappa \in (0, \infty)$ one finds that the stable space $S(\kappa)$ and the unstable space $U(\kappa)$ of $A(\kappa)$ satisfy

$$\dim S(\kappa) = n = \dim U(\kappa)$$

if $\kappa \in M_\nu$. □

Definition 2.4.3. (cf. [2]) Let $\mathcal{A} = \{(A^-(\kappa), A^+(\kappa))\}_{\kappa\in\mathbb{C}} \subset (\mathbb{C}^{2n\times 2n})^2$ be a smooth family of pairs of matrices, and set $\mathcal{A}^\pm = \{A^\pm(\kappa)\}_{\kappa\in\mathbb{C}}$. Let $\tilde{\Lambda}^\pm$ be a domain of constant splitting for $\mathcal{A}^\pm$. Then a domain of consistent splitting $\Lambda \subset \mathbb{C}$ for $\mathcal{A}$ is defined as the intersection

$$\Lambda = \tilde{\Lambda}^- \cap \tilde{\Lambda}^+.$$

Suppose we are given a shock $(v^-, v^+, s) \in \mathcal{V}\times\mathcal{V}\times(-1, 1)$ of $g(v)_t + f(v)_x = 0$ with end-states $v^\pm$ connected by a smooth shock profile $\phi : \mathbb{R} \to \mathcal{V}$. Consider the system

$$\xi' = A_{s,\kappa}(\phi(x))\xi \ \text{ on } \ \mathbb{C}^{2n}. \tag{2.48}$$

Set at the end-states

$$A^\pm_\kappa := A_{s,\kappa}(v^\pm) = A_{Df^\pm_s, Dg^\pm_s}(\kappa)$$

with

$$Df^\pm_s := Df_s(v^\pm) \ \text{ and } \ Dg^\pm_s := Dg_s(v^\pm),$$

and let $\mathcal{A} := \{(A^-_\kappa, A^+_\kappa)\}_{\kappa\in\mathbb{C}}$. Then $Df^\pm_s, Dg^\pm_s$ satisfy C1, C2 of Lemma 2.4.2, and as a direct consequence we obtain:

Corollary 2.4.4. *There exists $\nu \in (0,\infty)$ such that $\Lambda := M_\nu$ is a domain of consistent splitting for $\mathcal{A}$. In particular, setting $\mathbb{H}^+ := \{\kappa \in \mathbb{C} |\ \mathrm{Re}\,\kappa \geq 0\}$ we have*

$$\mathbb{H}^+ \backslash \{0\} \subset \Lambda.$$

Note $0 \notin \Lambda$.

Let Λ be a domain of consistent splitting as found in Corollary 2.4.4. For $\kappa \in \Lambda$ there are unique n-dimensional subspaces [11]

$$\mathcal{H}^{\pm}(\kappa) \subset \mathscr{G}_n^{2n}(\mathbb{C})$$

such that any solution ξ of (2.48) with $\xi(0) \in \mathcal{H}^{\pm}(\kappa)$ decays to zero at $\pm\infty$. Here $\mathscr{G}_n^{2n}(\mathbb{C})$ denotes the complex Grassmann manifold consisting of all n - dimensional subspace of $\mathbb{C}^{2n}$ (see Section 2.1). On Λ the subspaces $\mathcal{H}^{\pm}(\kappa)$ depend analytically on κ [2, 83]:

Definition 2.4.5. The analytic bundles

$$\mathcal{H}^{\pm} : \Lambda \to \mathscr{G}_n^{2n}(\mathbb{C})$$

are called Evans bundles for the shock profile ϕ. An Evans function

$$\mathcal{E} : \Lambda \to \mathbb{C}$$

is the Wronskian determinant of an analytic choice of bases for $\mathcal{H}^-$ and $\mathcal{H}^+$ on Λ.

Later on we will restrict attention to the bundles $\mathcal{H}^{\pm}$ on $\mathbb{H}^+\backslash\{0\}$.

Remark 2.4.6. *In the construction of Evans bundles $\mathcal{H}^{\pm}_\varepsilon$ for the shock profile ϕ_ε we will use the characterization that for $\xi_0 \in \mathbb{C}^{2n}\backslash\{0\}$ and $\kappa \in \mathbb{H}^+\backslash\{0\}$, hence for κ being an element of a domain of consistent splitting, one has $\xi_0 \in \mathcal{H}^-$ if and only if the solution ξ of* (2.48) *starting in $\xi(0) = \xi_0$ satisfies*

$$\mathrm{dist}(\mathrm{span}\,\{\xi(-x)\}, U^-_{\varepsilon,\kappa}) \to 0 \text{ as } x \to \infty,$$

and analogously with "$-$" and $U^-_{\varepsilon,\kappa}$ replaced by "$+$" and $S^+_{\varepsilon,\kappa}$ (cf. [83, 27]). Here $U^-_{\varepsilon,\kappa}$ and $S^+_{\varepsilon,\kappa}$ are the unstable subspace of $A_{s(\varepsilon),\kappa}(v^-_\varepsilon)$ and the stable subspace of $A_{s(\varepsilon),\kappa}(v^+_\varepsilon)$, respectively, and $\mathrm{dist}(\mathrm{span}\,\{\xi(-x)\}, U^-_{\varepsilon,\kappa})$ *is the distance of the point* $\mathrm{span}\,\{\xi(-x)\}$ *from the set $U^-_{\varepsilon,\kappa}$ in $\mathscr{G}_1^{2n}(\mathbb{C})$.*

Definition 2.4.7. The shock profile ϕ is called spectrally stable if the Evans bundles $\mathcal{H}^{\pm}$ possess analytic extensions on $\mathbb{H}^+$, and an Evans function $\mathcal{E}$ which is defined by taking the Wronskian determinant of an analytic choice of bases of the extensions of $\mathcal{H}^-$ and $\mathcal{H}^+$ on $\mathbb{H}^+$ satisfies the Evans function condition (see [70]):

$$(\forall \kappa \in \mathbb{H}^+ :\ \mathcal{E}(\kappa) = 0 \ \Leftrightarrow\ \kappa = 0) \ \text{ and } \ (\mathcal{E}'(0) \neq 0). \tag{2.49}$$

With the above definitions we can give a short statement of the central result of this chapter:

Theorem 2.4.8. *For sufficiently small $\varepsilon > 0$, the shock profile ϕ_ε is spectrally stable.*

Theorem 2.4.8 is proved in section 2.7 by construction of holomorphic bundles $H^\pm_\varepsilon$ on $\mathbb{H}^+$ which vary smoothly in ε and are rescaled versions $H^\pm_\varepsilon(\zeta) = \mathcal{H}^\pm_\varepsilon(\varepsilon^2\zeta)$ of the Evans bundles $\mathcal{H}^\pm_\varepsilon$ of ϕ_ε.

2.5 A hyperbolic Burgers' equation

Before we continue the discussion of spectral stability of shock profiles for system (2.32) let us consider a scalar example, the hyperbolic Burgers' equation, where spectral stability of a standing shock profile can be verified by explicit expressions for the Evans bundles and an Evans function without relying on the general developments in the following sections. One obtains concrete objects for the notions introduced in the previous section. Furthermore, the parabolic version of Burgers' equation will be important in the stability analysis for shock profiles of (2.32) as a canonical equation appearing in the singular limit of (2.45) (cf. [26]). The spectral stability of a particular shock profile to the parabolic Burgers' equation will be passed on to shock profiles of (2.32) in the small frequency regime. Following [3] we treat both a hyperbolic and a parabolic version of Burgers' equation in the same vein, with focus on the first of the two equations.

For $v \in \mathbb{R}$ consider the scalar equation

$$v_t - \frac{1}{4}(v^2)_x = \Box v = v_{xx} - v_{tt} \tag{2.50}$$

on $(t, x) \in (0, \infty) \times \mathbb{R}$ consisting of a conservation law on the left side with flux $f(v) = -(1/4)v^2$ and the d'Alembertian on the right side. We call (2.50) a hyperbolic Burgers' equation. The factor $1/4$ ensures that for $|v| \leq 1$ the conservation law on the left of (2.50) is sub-characteristic with respect to the d'Alembertian.

A standing viscous shock profile τ for (2.50) connecting the states $v^- = -1$ and $v^+ = 1$ satisfies the boundary value problem

$$\tau' = \frac{1}{4}(1 - \tau^2),\ \tau(\pm\infty) = \pm 1. \tag{2.51}$$

The solution of (2.51) with $\tau_0(0) = 0$ is given by $\tau_0(x) = \tanh(x/4)$.

Remark 2.5.1. *Note $f(-1) - f(1) = 0$ and $f'(v^-) = 1/2 > 0 > -1/2 = f'(v^+)$, in particular compare $|f'(v^\pm)| < 1$ to assumption A4.*

The linearization of (2.50) at τ_0 is

$$v_t - \frac{1}{2}(\tau_0 v)_x = v_{xx} - v_{tt}. \tag{2.52}$$

and the Laplace modes $e^{t\kappa}p(x)$ of (2.52) solve the ODE

$$l_\kappa p := p_{xx} + \frac{1}{2}(\tau_0 p)_x - (\kappa^2 + \kappa)p = 0$$

with a first order formulation

$$\xi' = A_\kappa(\tau_0)\xi \tag{2.53}$$

where $\xi = (p, q)$ and

$$A_\kappa(\tau) = \begin{pmatrix} -\frac{1}{2}\tau & 1 \\ k(\kappa) & 0 \end{pmatrix}$$

for $\kappa \in \mathbb{C}, \tau \in [-1, 1]$ and $k(\kappa) := \kappa^2 + \kappa$. Note that replacing $\Box v$ by $\partial_x^2 v$ in (2.50) gives the parabolic Burgers' equation, and the same deductions as above lead to an ODE (2.53) with $k(\kappa)$ replaced by $k_{par}(\kappa) := \kappa$.

The asymptotic values of $A_\kappa(\tau)$ are

$$A_\kappa^\pm = A_\kappa(\pm 1) = \begin{pmatrix} \mp\frac{1}{2} & 1 \\ k(\kappa) & 0 \end{pmatrix}, \tag{2.54}$$

and the eigenvalue problem for $A_\kappa^\pm$ reads

$$\begin{cases} \mu v_1 = \mp\dfrac{1}{2}v_1 + v_2, \\ \mu v_2 = k(\kappa)v_1, \end{cases} \tag{2.55}$$

for $\mu \in \mathbb{C}$ and $v = (v_1, v_2) \in \mathbb{C}^2$. From (2.55) we find the quadratic equation

$$\mu^2 \pm \frac{1}{2}\mu - \kappa - \kappa^2 = 0. \tag{2.56}$$

Purely imaginary eigenvalues $\mu = i\eta, \eta \in \mathbb{R}$, of $A_\kappa^\pm$ define the dispersion curves $\kappa^\pm(\eta)$ which lie in the set

$$D := \{\kappa \in \mathbb{C} |\ \exists \eta \in \mathbb{R} :\ i\eta \in \sigma(A_\kappa^+) \cup \sigma(A_\kappa^-)\}.$$

If $\mu = i\eta$ one gets from (2.56)

$$\kappa_1^\pm(\eta) = -\frac{1}{2} + \sqrt{\tfrac{1}{4} - \eta^2 \pm i\tfrac{1}{2}\eta},\ \kappa_2^\pm(\eta) = -\frac{1}{2} - \sqrt{\tfrac{1}{4} - \eta^2 \pm i\tfrac{1}{2}\eta}, \tag{2.57}$$

where $\sqrt{\cdot}$ is the branch of the square root with $\sqrt{1} = 1$. It is not hard to see that

$$\operatorname{Re}\kappa_{1,2}^\pm(\eta) < 0\ (\eta \neq 0),$$

i.e. the dispersion curves of $A_\kappa^\pm$ lie in the left half plane of $\mathbb{C}$ and intersect the imaginary axis only at the origin. Note that here the aforementioned sub-characteristicity is important.

Next the expansions of $\kappa_{1,2}^{\pm}(\eta)$ at $|\eta| \to 0$ and $|\eta| \to \infty$ are given. Since $\kappa_1^{\pm}(\eta) + \kappa_2^{\pm}(\eta) = -1$ the expansion at 0 is only given for $\kappa_1^{\pm}$. One gets

$$\kappa_1^{\pm}(\eta) = \pm i\frac{1}{2}\eta - \frac{3}{4}\eta^2 + \mathcal{O}(|\eta|^3) \tag{2.58}$$

which is in leading order a parabola in $\mathbb{C}$ opened towards the left. The expansions at infinity are

$$\kappa_1^{\pm}(\eta;\tau) = i\eta - \frac{2 \mp 1}{4} + \mathcal{O}(|\eta|^{-1}),\ \kappa_2^{\pm}(\eta;\tau) = -i\eta - \frac{2 \pm 1}{4} + \mathcal{O}(|\eta|^{-1})$$

showing asymptotic approximation of $\kappa_{1,2}^{\pm}$ to lines which are parallel to the imaginary axis and pass through either $-1/4$ or $-3/4$.

A domain of consistent splitting Λ is given by the open and simply connected subset of $\mathbb{C}$ lying to the right of all the dispersion curves (2.57). We have $\mathbb{H}^+\backslash\{0\} \subset \Lambda$, and by (2.58) the boundary of Λ touches the imaginary axis at 0 to second order.

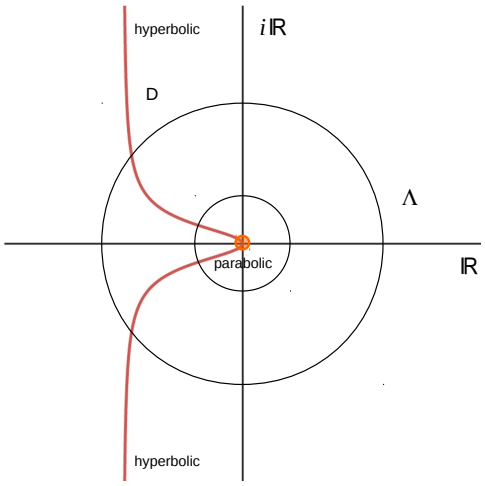


FIGURE 2.1: The essential spectrum of l_κ touches the imaginary axis at the origin the latter being in the extended point spectrum.[1]

The situation is sketched in Figure 2.1. We see a portion of the complex plane centered at the origin. The orange curves stands for the right-most dispersion curve in D touching the imaginary axis at the origin to second order. The origin itself is highlighted by an orange circle since the operator l_0 has τ_0' as an eigenfunction, i.e. $\tau_0' \in \ker l_0$. Furthermore, there are two circles splitting the complex plane into three parts: For small frequencies $|\kappa| \ll 1$ we have $k(\kappa) \approx \kappa$ and we expect solutions of (2.53) to behave w.r.t κ as if we were considering a viscous, or parabolically regularized, conservation law. For large frequencies $|\kappa| \gg 1$ the hyperbolic nature of equation (2.50) kicks in and the dispersion curves ran parallel to the imaginary axis. Remembering the domains of consistent splitting M_ν in Lemma 2.4.2 we see that Figure 2.1 captures the situation

[1]This figure was generated on www.desmos.com/calculator and edited with Libre Office 5.1.6.2.

in the general context, too (see also the discussion of the inner and the outmost regime in Section 2.7).

From the quadratic relation (2.56) we find that the eigenvalues of $A^{\pm}_{\kappa}$ are given by

$$\mu^{\pm,u}(\kappa) = \mp\frac{1}{4} + \sqrt{\frac{1}{16} + k(\kappa)},\ \mu^{\pm,s}(\kappa) = \mp\frac{1}{4} - \sqrt{\frac{1}{16} + k(\kappa)},$$

with corresponding eigenvectors

$$R^{\pm,u}(\kappa) = \begin{pmatrix} 1 \\ -\mu^{\pm,s}(\kappa) \end{pmatrix},\ R^{\pm,s}(\kappa) = \begin{pmatrix} 1 \\ -\mu^{\pm,u}(\kappa) \end{pmatrix}.$$

An expansion of $\mu^{\pm,u/s}$ for large $\kappa \in \mathbb{H}^+$ gives

$$\mu^{\pm,u}(\kappa) = \kappa + \frac{2 \mp 1}{4} + \mathcal{O}(|\kappa|^{-1}),\ \mu^{\pm,s}(\kappa) = -\kappa - \frac{2 \pm 1}{4} + \mathcal{O}(|\kappa|^{-1}).$$

With $\kappa = i\eta$ we see that the asymptotic matrices $A^{\pm}_{i\eta}$ have some uniform hyperbolicity left at the boundary of $\mathbb{H}^+$ if $\eta \gg 1$ (compare this to the treatment in Subsection 2.7.3).

To compute the Evans bundles we must find solutions to

$$\begin{cases} p' = -\frac{1}{2}\tau_0 p + q \\ q' = k(\kappa)p \end{cases} \tag{2.59}$$

with the appropriate decay properties. By [3] an appropriate choice for p is

$$p^{\pm}_{\kappa}(x) = \left(-\frac{1}{4}\tau_0(x) \mp \sqrt{k(\kappa) + \frac{1}{16}} \right) \sqrt{\tau_0'(x)} e^{\mp x\sqrt{k(\kappa)+\frac{1}{16}}}$$

where the signs indicate the decay in x to 0 at $+$ or $-$ infinity. q may be read off the first line in the ODE-system (2.59) and is given by

$$q^{\pm}_{\kappa}(x) = k(\kappa)\sqrt{\tau_0'(x)} e^{\mp x\sqrt{k(\kappa)+\frac{1}{16}}}.$$

For all $\kappa \in \Lambda$ we know that the real part $\mu^{+,u}(\kappa)$ is non-zero. For $\kappa \in (0, \infty)$ it holds $\mu^{+,u}(\kappa) > 0$. By continuity the real part of $\mu^{+,u}(\kappa)$ may not change its sign in Λ and we deduce

$$\operatorname{Re}\left(\sqrt{k(\kappa) + \frac{1}{16}} \right) > \frac{1}{4}\ (\kappa \in \Lambda).$$

This shows for $\xi^{\pm}_{\kappa} = (p^{\pm}_{\kappa}, q^{\pm}_{\kappa})$ that

$$\xi^{\pm}_{\kappa}(x) \to 0 \text{ for } x \to \pm\infty \text{ and } \kappa \in \Lambda.$$

The absolute value of the decay rate is larger than $1/4$ for $\kappa \in \Lambda$.

Furthermore, in the spirit of evolving subspaces we find that $\operatorname{span}\{\xi^{\pm}_{\kappa}(x)\}$

approaches the stable resp. unstable subspace of $A_\kappa^\pm$ at $\pm\infty$ since (read for $+$: first superscript s; for $-$: second superscript u):

$$\begin{aligned}\operatorname{span}\{\xi_\kappa^\pm(x)\} &= \operatorname{span}\left\{\begin{pmatrix} -\frac{1}{4}\tau_0(x) \mp \sqrt{k(\kappa)+\frac{1}{16}} \\ k(\kappa)\end{pmatrix}\right\} \\ &\to \operatorname{span}\left\{\begin{pmatrix} \mp\frac{1}{4} \mp \sqrt{k(\kappa)+\frac{1}{16}} \\ k(\kappa)\end{pmatrix}\right\} \text{ for } x\to\pm\infty \\ &= \operatorname{span}\left\{\begin{pmatrix} \mu^{\pm,s/u}(\kappa) \\ k(\kappa)\end{pmatrix}\right\} \\ &= \operatorname{span}\left\{R^{\pm,s/u}(\kappa)\right\}.\end{aligned}$$

The limits are understood in the topology of the Grassmannian manifold $\mathscr{G}_1^2(\mathbb{C})$.

Hence, $\xi_\kappa^\pm = (p_\kappa^\pm, q_\kappa^\pm)$ are indeed the desired solutions and the Evans bundles are given by

$$\mathcal{H}^\pm(\kappa) := \operatorname{span}\left\{\begin{pmatrix} \mp\sqrt{k(\kappa)+\frac{1}{16}} \\ k(\kappa)\end{pmatrix}\right\} \tag{2.60}$$

with Evans function

$$\mathcal{E}(\kappa) := 2k(\kappa)\sqrt{k(\kappa)+\frac{1}{16}}. \tag{2.61}$$

The Evans function $\mathcal{E}$ satisfies the Evans function condition (2.49) and τ_0 is a spectrally stable shock profile of (2.50).

We close this section by giving expressions for the Evans bundles and an Evans function for the parabolic Burgers' equation (2.50) with $\partial_x^2 v$ instead of $\Box v$. This amounts to simply replacing k by k_{par} in (2.60) and (2.61):

$$h_{p,par}^\pm(\kappa) := \mp\sqrt{\kappa+1/16} \tag{2.62}$$

$$h_{q,par}^\pm(\kappa) := \kappa \tag{2.63}$$

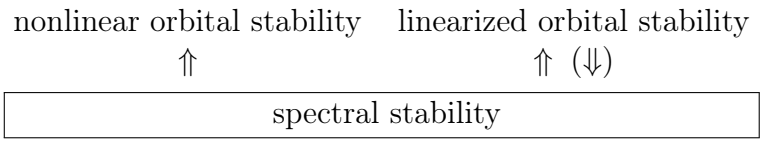
$$\mathcal{H}_{par}^\pm(\kappa) := \operatorname{span}\left\{\begin{pmatrix} h_{p,par}^\pm(\kappa) \\ h_{q,par}^\pm(\kappa)\end{pmatrix}\right\} \tag{2.64}$$

$$\mathcal{E}_{par}(\kappa) := 2\kappa\sqrt{\kappa+1/16} \tag{2.65}$$

2.6 Resolvent kernel

The use of a spectral approach towards stability of shock profiles in regularized conservation laws is at first obstructed by the fact that the essential spectrum touches the imaginary axis at the origin hindering an extension of the resolvent of the linearized problem as a meromorphic function into the left half of the complex plane. Furthermore, at the frequency $\kappa = 0$ there is an eigenfunction given by the derivative of the shock profile but 0 cannot be an isolated eigenvalue since it is embedded in the essential spectrum (see also Figure 2.1). This problem can be elegantly taken care of by considering the so-called resolvent

kernel instead. The resolvent kernel is a meromorphic function whose isolated poles coincide on a domain of consistent splitting with those of the resolvent but with a well-defined meromorphic continuation into the left half of the complex plane. It also allows for an extended notion of point spectrum (see [99]). The precise analysis of the properties of the resolvent kernel lies at the heart of the program of Zumbrun and collaborators where the verification of linearized and nonlinear orbital stability of shock profiles is reduced to proving their spectral stability (see the sketch below).



The program is complete for, e.g., Lax shocks in viscous conservation laws and hyperbolic relaxation systems [26, 70, 72, 77, 99].

This section discusses the appearance of an Evans function in the construction of the resolvent kernel for the linearized system (2.44) thereby giving a meaning to the Evans function condition in Definition 2.4.7. As discussed above, following [99] this marks the starting point of an extended spectral theory for the linear operator associated with (2.44) yielding an extension of its resolvent into the essential spectrum while characterizing the latter's regularity as well. For the definitions of the various parts of the spectrum we refer the reader to the textbooks [39, 52] (see also [54]). The relevant information on distribution theory and Sobolev spaces can be found in [1, 43], and treatments of semi-group theory are given in, e.g., [16, 75].

2.6.1 Semi-group theory

Suppose that we are given a tuple $(v^-, v^+, s) \in \mathcal{V} \times \mathcal{V} \times (-1, 1)$ and $\phi \in C^\infty(\mathbb{R})$ such that

S1 $dg(v^\pm), dg(v^-) \pm df(v^-)$ and $dg(v^+) \pm df(v^+)$ are positive definite;

S2 $df(v^\pm)$ is strictly hyperbolic w.r.t. $dg(v^\pm)$;

S3 (v^-, v^+, s) is a p-shock for $g(v)_t + f(v)_x = 0$;

S4 ϕ is a shock profile of (2.32) for the shock (v^-, v^+, s).

A Lorentz boost of speed s turns ϕ into a stationary solution of

$$g_s(v)_t + f_s(v)_x = \Box v$$

Linearizing at ϕ yields

$$Gv_t + (Fv)_x = \Box v. \tag{2.66}$$

where we have set

$$G(x) := Dg_s(\phi(x)) \text{ and } F(x) := Df_s(\phi(x)).$$

We will discuss the semi-group theory of (2.66) and its resolvent kernel. Associated to (2.66) we define the closed linear operator

$$D(L) := H^2(\mathbb{R};\mathbb{C}^n) \times H^1(\mathbb{R};\mathbb{C}^n),$$
$$LV := \begin{pmatrix} V_2 \\ \partial_x^2 V_1 - \partial_x(FV_1) - GV_2 \end{pmatrix},$$

on the Hilbert space

$$H = H^1(\mathbb{R};\mathbb{C}^n) \times L^2(\mathbb{R};\mathbb{C}^n),$$
$$\langle V, W\rangle_H := \langle V_1, W_1\rangle_{L^2} + \langle \partial_x V_1, \partial_x W_1\rangle_{L^2} + \langle V_2, W_2\rangle_{L^2}.$$

L is equal to $A + B$ where

$$D(A) := D(L),\ AV := \begin{pmatrix} V_2 \\ \partial_x^2 V_1 \end{pmatrix}$$

and

$$D(B) := H,\ BV := -\begin{pmatrix} 0 \\ \partial_x(FV_1) + GV_2. \end{pmatrix}$$

Here $H^m, m \in \mathbb{N}$, denotes the usual L^2-based Sobolev space of order m. By Theorem 4.5 in Chapter 7 of Pazy's book [75] A is the generator of a C_0-group $\{e^{At}\}_{t\in\mathbb{R}}$ on H satisfying

$$||e^{At}|| \leq e^{|t|}.$$

The operator B is a linear and bounded operator on H and therefore L is a perturbation of A by the bounded linear operator B. By Theorem 1.1 in Chapter 3 of [75] L is the generator of a C_0-group $\{e^{Lt}\}_{t\in\mathbb{R}}$ of operators on H satisfying

$$||e^{Lt}|| \leq e^{(1+||B||)|t|}.$$

Hence, we arrive at

Lemma 2.6.1. *(Corollary 7.5, Chapter 1, [75]) Let e^{At} be the group of operators generated by L in H. If $\gamma > 0$ is large enough then for all $V \in D(L^2)$ it holds*

$$e^{At}V = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\kappa t}(\kappa - L)^{-1}V d\kappa \in D(L) \tag{2.67}$$

and for every $0 < \delta < 1$, the integral converges in H uniformly in t for $|t| \in [\delta, 1/\delta]$.

Remark 2.6.2. *(i) Note $D(L^2) = \{V \in H : V \in D(L), LV \in D(L)\}$. Ellipticity of ∂_x^2 implies $D(L^2) = H^3(\mathbb{R};\mathbb{C}^n) \times H^2(\mathbb{R};\mathbb{C}^n)$.*

(ii) The integral in (2.67) *is a principal value integral in the sense that for all* $U \in D(L^2)$ *the line integral*

$$\int_{\gamma-iK}^{\gamma+iK} e^{\kappa t}(\kappa - L)^{-1} V d\kappa$$

converges for $K \to \infty$ *in* H *uniformly in* t *for* $|t| \in [\delta, 1/\delta]$.

The Green's function $\mathfrak{G}(t) = \mathfrak{G}(t, x, y)$ for $\partial_t - L$ is defined by its action on test functions $h, k \in C_0^\infty(\mathbb{R}; \mathbb{C}^n) \times C_0^\infty(\mathbb{R}; \mathbb{C}^n)$ via

$$\mathfrak{G}(t)(h, k) := (k, e^{Lt} h)$$

where for $h, k \in L^2(\mathbb{R}; \mathbb{C}^n) \times L^2(\mathbb{R}; \mathbb{C}^n) \cong L^2(\mathbb{R}; \mathbb{C}^{2n})$ we have used the pairing

$$(h, k) = \int_{\mathbb{R}} h(x)^t k(x) dx \in \mathbb{C}.$$

On the resolvent set $\rho(L)$ of L the resolvent kernel $\mathfrak{G}_\kappa = \mathfrak{G}_\kappa(x, y)$ is similarly defined by

$$\mathfrak{G}_\kappa(h, k) = (k, (L - \kappa)^{-1} h).$$

Hence, for $\gamma > 0$ large enough one finds, in the sense of distributions,

$$\mathfrak{G}(t) = -\frac{1}{2\pi i} P.V. \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\kappa t} \mathfrak{G}_\kappa d\kappa.$$

For $\kappa \in \rho(L)$ the resolvent kernel is the uniquely defined distribution satisfying

$$(L - \kappa)\mathfrak{G}_\kappa = \delta_0(x - y) \text{ and } \mathfrak{G}_\kappa(h) \in H.$$

Here δ_0 is the Dirac distribution at 0 and for $h, k \in C_0^\infty(\mathbb{R}; \mathbb{C}^{2n})$ the expression $\delta_0(x-y)$ denotes the distribution defined by inserting h, k into the pairing $(\cdot, \cdot)$, i.e.

$$\delta_0(h, k) = (h, k).$$

Remark 2.6.3. *As customary we frequently switch between interpreting* $\mathfrak{G}_\kappa$ *as a distribution acting on test functions in* $C_0^\infty(\mathbb{R}^2; \mathbb{C}^{2n})$ *and as a map taking test functions* $C_0^\infty(\mathbb{R}; \mathbb{C}^{2n})$ *to distributions acting on* $C_0^\infty(\mathbb{R}; \mathbb{C}^{2n})$. *Since* $\mathfrak{G}_\kappa$ *is constructed from Dirac distributions and measurable functions there is no difficulty in changing between these view points.*

2.6.2 Construction of the resolvent kernel

For $\kappa \in \mathbb{C}$ let

$$D(l_\kappa) := H^2(\mathbb{R}; \mathbb{C}^n) \subset L^2(\mathbb{R}; \mathbb{C}^n)$$

and define the closed linear operator

$$l_\kappa v := \partial_x^2 v - (Fv)_x - \kappa(\kappa + G)v, \quad v \in D(l_\kappa),$$

on $L^2(\mathbb{R};\mathbb{C}^n)$. For $\kappa \in \mathbb{C}$ with $0 \in \rho(l_\kappa)$ let $\mathfrak{g}_\kappa$ be the uniquely defined distribution satisfying

$$l_\kappa \mathfrak{g}_\kappa = \delta_0(x-y) \text{ and } \mathfrak{g}_\kappa(h) \in L^2,$$

where again for $h, k \in C_0^\infty(\mathbb{R};\mathbb{C}^n)$

$$\delta_0(h,k) = \int_{\mathbb{R}} h(x)^t k(x) dx.$$

We find that indeed $\mathfrak{g}_\kappa$ is well-defined and for $h \in C_0^\infty(\mathbb{R};\mathbb{C}^n)$ we have

$$\mathfrak{g}_\kappa(h) = l_\kappa^{-1} h.$$

In particular, $\mathfrak{g}_\kappa(h) \in H^2$. We refer to $\mathfrak{g}_\kappa$ as the resolution kernel of l_κ.

Lemma 2.6.4. *Let $\kappa \in \mathbb{C}$. Let $S_\kappa : H \to H$ denote the bounded multiplication operator*

$$S_\kappa(V)(x) = \begin{pmatrix} \mathbb{1}_n & 0 \\ \kappa + G(x) & \mathbb{1}_n \end{pmatrix} V(x),$$

If $0 \in \rho(l_\kappa)$ then the linear operator

$$\begin{pmatrix} 0 & l_\kappa^{-1} \\ \mathrm{id}_n & \kappa l_\kappa^{-1} \end{pmatrix} \circ S_\kappa : H \to H$$

is a bounded inverse of $L - \kappa$. If $\kappa \in \rho(L)$ then the linear operator

$$(\mathrm{id}_n, 0)(L-\kappa)^{-1} \begin{pmatrix} 0 \\ \mathrm{id}_n \end{pmatrix} : L^2(\mathbb{R};\mathbb{C}^n) \to L^2(\mathbb{R};\mathbb{C}^n)$$

is a bounded inverse of l_κ. In particular, it holds

$$0 \in \rho(l_\kappa) \Leftrightarrow \kappa \in \rho(L).$$

Proof. By elementary algebraic manipulations we have

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \ker(L-\kappa) \Leftrightarrow f_1 \in \ker l_\kappa \text{ and } \kappa f_1 = f_2.$$

In particular,

$$\ker(L-\kappa) = \{0\} \Leftrightarrow \ker l_\kappa = \{0\}.$$

Since for $0 \in \rho(l_\kappa)$ a bounded right inverse for $L - \kappa$ is given by

$$\begin{pmatrix} 0 & l_\kappa^{-1} \\ \mathrm{id}_n & \kappa l_\kappa^{-1} \end{pmatrix} \circ S_\kappa,$$

and for $\kappa \in \rho(L)$ a bounded right inverse for l_κ is given by

$$(\mathrm{id}_n, 0)(L - \kappa)^{-1} \begin{pmatrix} 0 \\ \mathrm{id}_n \end{pmatrix},$$

the claim follows. □

By the previous lemma, or by direct calculation and use of definitions, we obtain for $\kappa \in \rho(L)$ the fundamental relation

$$\mathfrak{G}_\kappa(x,y) = \begin{pmatrix} \mathfrak{g}_\kappa(x,y)(\kappa \mathbb{1}_n + G(y)) & \mathfrak{g}_\kappa(x,y) \\ \delta_0(x-y) + \kappa \mathfrak{g}_\kappa(x,y)(\kappa \mathbb{1}_n + G(y)) & \kappa \mathfrak{g}_\kappa(x,y) \end{pmatrix}. \tag{2.68}$$

Hence, a representation of $\mathfrak{g}_\kappa$ leads directly to a representation of $\mathfrak{G}_\kappa$. In the following we construct $\mathfrak{g}_\kappa$ in standard fashion from appropriately decaying solutions $(p, p')^t$ where p solves $l_\kappa p = 0$: Consider the linear non-autonomous ordinary differential equation

$$\xi' = \tilde{A}_\kappa(x)\xi \quad (\xi = (p, p')^t \in \mathbb{C}^{2n}) \tag{2.69}$$

with coefficient matrix

$$\tilde{A}_\kappa(x) = \begin{pmatrix} 0 & \mathbb{1}_n \\ \kappa(\kappa \mathbb{1}_n + G(x)) + F'(x) & F \end{pmatrix}.$$

The differential system (2.69) is the standard first-order formulation of $l_\kappa p = 0$. Set $F^\pm := Df_s(v^\pm), G^\pm := Dg_s(v^\pm)$ and $\tilde{A}_\kappa^\pm := \tilde{A}_\kappa(\pm\infty)$. Note that $F^\pm$ is invertible because (v^-, v^+, s) is a p-shock. For any $m \in \mathbb{N}_0$ there is a constant $\theta > 0$ such that

$$\lim_{x \to \pm\infty} (|(\partial_x^m (F(x) - F^\pm)| + |\partial_x^m (G(x) - G^\pm)|) e^{\theta|x|} = 0,$$

i.e. the coefficient matrix $\tilde{A}_\kappa$ in (2.69) is asymptotically constant with exponential rate. Because

$$\tilde{A}_\kappa^\pm \sim A_{F^\pm, G^\pm}(\kappa) = \begin{pmatrix} F^\pm & \mathbb{1}_n \\ \kappa(\kappa + G^\pm) & 0 \end{pmatrix}$$

and $F^\pm, G^\pm$ satisfy the assumptions of Lemma 2.4.2 there exists $\nu > 0$ such that $\Lambda := M_\nu$ is a domain of consistent splitting for $\{(\tilde{A}_\kappa^-, \tilde{A}_\kappa^+)\}_\kappa$. On Λ the spectral projections $P_\kappa^{\pm,S}$ and $P_\kappa^{\pm,U}$ onto the n-dim. stable subspace $S_\kappa^\pm$ and the n-dim. unstable subspace $U_\kappa^\pm$ of $\tilde{A}_\kappa^\pm$ are well-defined and holomorphic in κ. The next lemma shows that these spectral projections have holomorphic extensions in a small neighborhood of zero.

Lemma 2.6.5. *Let F, G satisfy the assumptions of Lemma 2.4.2. Suppose in addition that F is invertible, and let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of F counted without their multiplicities. Denote by $\nu_1, \dots, \nu_{\tilde{m}}$ the eigenvalues of $F^{-1}G (\sim G^{1/2}F^{-1}G^{1/2})$ with corresponding projections $p_1, \dots, p_{\tilde{m}}$. Set $A(\kappa) := A_{F,G}(\kappa)$ and $\mathcal{A} := \{A(\kappa)\}_\kappa$, and let $\tilde{\Lambda}$ be a domain of constant splitting for $\mathcal{A}$ as found in Lemma 2.4.2. On $\tilde{\Lambda}$ denote by $P^S = P^S(\kappa)$ resp. $P^U = P^U(\kappa)$ the holomorphic spectral projection onto the stable resp. unstable subspace of $A = A(\kappa)$. Then for some $\gamma \geq 1$ the slow eigenvalues of $A(\kappa)$ expand as*

$$\nu_{jk}(\kappa) = -\nu_j \kappa + \beta_{jk}\kappa^2 + \mathcal{O}(\kappa^{2+\frac{1}{\gamma}})$$

with total projection $P_j = P_j(\kappa)$ where β_{jk} are the eigenvalues of

$$-(1-(\nu_j)^2)p_j F^{-1} p_j|_{\mathcal{R}(p_j)} = \nu_j((\nu_j)^2 - 1)p_j G^{-1} p_j|_{\mathcal{R}(p_j)}$$

with $\operatorname{sgn}\nu = \operatorname{sgn}\beta_{jk}$, and the fast eigenvalues of $A(\kappa)$ expand as

$$\lambda_j(\kappa) = \lambda_j + \mathcal{O}(\kappa).$$

For $|\kappa| \ll 1$ denote by $P^<(\kappa)$ the total projection belonging to the λ_j-groups with $\lambda_j < 0$ and by $P^>(\kappa)$ the total projection belonging to the λ_j-groups with $\lambda_j > 0$. Then

$$\dim\left(P^>(\kappa) + \sum_{\nu_j<0} P_j(\kappa)\right) = n = \dim\left(P^<(\kappa) + \sum_{\nu_j>0} P_j(\kappa)\right),$$

and if in addition $\kappa \in \Lambda$ then

$$P^>(\kappa) + \sum_{\nu_j<0} P_j(\kappa) = P^U(\kappa),$$
$$P^<(\kappa) + \sum_{\nu_j>0} P_j(\kappa) = P^S(\kappa).$$

In particular, $P^> + \sum_{\nu_j<0} P_j$ is a holomorphic extension of P^U into a small neighborhood of 0, and likewise for $P^< + \sum_{\nu_j>0} P_j$ and P^S.

Proof. Since F is symmetric hyperbolic we may assume without loss of generality that $F = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$ where the eigenvalues λ_j appear according to the ordering $\lambda_j < \lambda_{j'}, j < j'$, counted with multiplicities. A matrix consisting of eigenvectors of $A(0)$ is given by

$$\mathcal{W} := \begin{pmatrix} \mathbb{1}_n & \mathbb{1}_n \\ 0 & -F \end{pmatrix}.$$

$\mathcal{W} \in \mathbb{R}^{2n\times 2n}$ is invertible with inverse

$$\mathcal{W}^{-1} = \begin{pmatrix} \mathbb{1}_n & F^{-1} \\ 0 & -F^{-1} \end{pmatrix}.$$

Use $\mathcal{W}^{-1}$ to change basis and write A as

$$\begin{aligned} T(\kappa) &:= \mathcal{W}^{-1} A^{\pm}(\kappa) \mathcal{W} \\ &= \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} + \kappa \begin{pmatrix} C & C \\ -C & -C \end{pmatrix} + \kappa^2 \begin{pmatrix} F^{-1} & F^{-1} \\ -F^{-1} & -F^{-1} \end{pmatrix} \\ &:= T^{(0)} + \kappa T^{(1)} + \kappa^2 T^{(2)} \end{aligned}$$

where $C := F^{-1}G$. We are interested in the 0-group of T. Notice that the structure of T is almost the same as in [88] and we refer to the results therein after making sure that the technique carries over (almost word by word). By hyperbolicity of F we may separate the n-fold eigenvalue 0 of $T(0)$ from the eigenvalues of F and define for $|\kappa|$ small enough the total projection $P_0(\kappa)$ for the 0-group of eigenvalues of $T(\kappa)$ by the usual Dunford formula. One has

$$P_0 := P_0(0) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_n \end{pmatrix}.$$

In order to apply the reduction process to expand $T(\kappa)P_0(\kappa)$ consider

$$T_0(\kappa) := \frac{1}{\kappa} T(\kappa) P_0(\kappa)$$

which is holomorphic by semi-simplicity of the eigenvalue 0. In particular it holds

$$T_0(0) = P_0 T^{(1)} P_0 = \begin{pmatrix} 0 & 0 \\ 0 & -C \end{pmatrix}.$$

Because G is positive definite and

$$C \cong G^{1/2} F^{-1} G^{1/2}$$

with the right hand side being symmetric and invertible we see that C has only the pairwise distinct semi-simple eigenvalues $\{\nu_1, \ldots, \nu_{\tilde{m}}\} \subset \mathbb{R}\backslash\{0\}$. In fact, the eigenvalues of C are exactly the reciprocal eigenvalues of F w.r.t. G, and by $G \pm F > 0$ we find $|\nu_j| > 1$. Again the total projections $P_j(\kappa)$ of the $-\nu_j$-group of T_0 are defined and holomorphic for $|\kappa|$ small enough and satisfy $P_0(\kappa) = \sum_j P_j(\kappa)$. If p_j is the eigenprojection of the eigenvalue ν_j of C one has

$$P_j := P_j(0) = \begin{pmatrix} 0 & 0 \\ 0 & p_j \end{pmatrix}.$$

Carrying on as in the proof of Proposition 2.2. and Corollary 2.3. in [88] yields the claimed expansions of the eigenvalues of $A(\kappa)$ for small κ.
For $|\kappa| \ll 1$ and $\kappa \in \tilde{\Lambda} \cap \mathbb{R}$ we deduce from the signs of the leading coefficients

in the eigenvalue expansions that

$$P_>(\kappa) + \sum_{\nu_j<0} P_j(\kappa) = P^U(\kappa),$$
$$P_<(\kappa) + \sum_{\nu_j>0} P_j(\kappa) = P^S(\kappa).$$

By analytic extension and the fact that the real parts of the eigenvalues of $A(\kappa)$ may not change sign in $\tilde{\Lambda}$, the above equalities of projections hold true for all small enough $\kappa \in \tilde{\Lambda}$. Because nearby projections have images of the same dimensions by Lemma 4.10 in[54] and $\dim P^U(\kappa) = n = \dim P^S(\kappa)$, we find for $|\kappa|$ small

$$\dim \left(P_>(\kappa) + \sum_{\nu_j<0} P_j(\kappa) \right) = n = \dim \left(P_<(\kappa) + \sum_{\nu_j>0} P_j(\kappa) \right).$$

For fixed j the coefficients β_{jk} have as eigenvalues of

$$\begin{aligned} -(1-\nu_j^2)p_j F^{-1} p_j &= -(1-\nu_j^2)p_j C G^{-1} p_j \\ &= \nu_j(\nu_j^2 - 1)p_j G^{-1} p_j, \end{aligned}$$

restricted to the range $\mathcal{R}(p_j)$ of p_j the same signs as the eigenvalue ν_j of $C = F^{-1}G$ since $|\nu_j| > 1$ and G^{-1} is positive definite. □

Remark 2.6.6. *In analogy to Remark 2.2. in [99] we call*

$$B_{eff}^{\pm} := \sum_j p_j (G^{\pm})^{-1} p_j$$

an "effective artificial viscosity matrix".

Using Lemma 2.6.5 with $\tilde{A}_\kappa^{\pm} \sim A_{F^{\pm},G^{\pm}}(\kappa)$ standard techniques in asymptotic ODE-theory (e.g. [34, 70, 99]) imply:

Lemma 2.6.7. *Let $r > 0$ so small that the conclusions of Lemma 2.6.5 hold. Let $\kappa_0 \in \Lambda \cup B_r(0)$. Then there exists locally in $\Lambda \cup B_r(0)$ analytic solutions $\phi_j^+(x;\kappa), \phi_{n+j}^-(x;\kappa), j = 1, \dots, n$, of (2.69) such that*

(i) $\operatorname{span}\{\phi_{n+j}^-(x;\kappa)\}_j \to U_\kappa^-$ *for* $x \to -\infty$, *and* $\operatorname{span}\{\phi_j^+(x;\kappa)\}_j \to S_\kappa^+$ *for* $x \to +\infty$;

(ii) If $\kappa_0 \in \Lambda$ then there exists $\theta_0 > 0$ such that for all $|\kappa - \kappa_0| \ll 1$ the solutions ϕ_j^+, ϕ_{n+j}^- decay according to $|\phi_j^+(x;\kappa)| = \mathcal{O}(e^{-\theta_0|x|}), x \geq 0$, and $|\phi_{n+j}^-(x;\kappa)| = \mathcal{O}(e^{-\theta_0|x|}), x \leq 0$.

Following [99] $\mathfrak{g}_\kappa$ has to satisfy the jump condition

$$\begin{bmatrix} \mathfrak{g}_\kappa & \partial_x \mathfrak{g}_\kappa \\ \partial_y \mathfrak{g}_\kappa & \partial_x \partial_y \mathfrak{g}_\kappa \end{bmatrix}_{x=y} = J(y) := \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & F(y) \end{pmatrix}.$$

Moreover, setting

$$\Phi^+ = (\phi_j^+)_{1\le j\le n} \text{ and } \Phi^- = (\phi_{n+j}^-)_{1\le j\le n}$$

one finds for $x > y$

$$\begin{pmatrix} \mathfrak{g}_\kappa & \partial_x \mathfrak{g}_\kappa \\ \partial_y \mathfrak{g}_\kappa & \partial_x \partial_y \mathfrak{g}_\kappa \end{pmatrix} = (\Phi^+(x;\kappa), 0)(\Phi^+(y;\kappa), \Phi^-(y;\kappa))^{-1} J(y)$$

and for $x < y$

$$\begin{pmatrix} \mathfrak{g}_\kappa & \partial_x \mathfrak{g}_\kappa \\ \partial_y \mathfrak{g}_\kappa & \partial_x \partial_y \mathfrak{g}_\kappa \end{pmatrix} = -(0, \Phi^-(x;\kappa))(\Phi^+(y;\kappa), \Phi^-(y;\kappa))^{-1} J(y).$$

Because the solution propagator of (2.69) is holomorphic in κ with values in $\mathrm{GL}_{2n}(\mathbb{C})$, the regularity of $\mathfrak{g}_\kappa$ in κ and the position and order of its poles are determined by the local Evans function

$$\mathscr{E}(\kappa) = \det(\Phi^+(0;\kappa), \Phi^-(0;\kappa))$$

where the zeros of $\mathscr{E}$ correspond to poles of $\mathfrak{g}_\kappa$. Summarizing we obtain the analogue of Proposition 4.7 [99]:

Proposition 2.6.8. *For $\theta > 0$ small enough the distribution $\mathfrak{g}_\kappa(x,y)$ is meromorphic on $\mathbb{H}^+ - \theta = \{\kappa \in \mathbb{C} |\ \operatorname{Re}\kappa \ge -\theta\}$ with isolated poles of finite multiplicity corresponding in position and multiplicity to the zeros of a local Evans function of (2.69). On $\Lambda\backslash\sigma(L)$ the distribution $\mathfrak{g}_\kappa$ agrees with the resolution kernel of l_κ.*

Using the relation (2.68) we obtain as a corollary of the previous proposition:

Corollary 2.6.9. *For $\theta > 0$ small enough the distribution $\mathfrak{G}_\kappa(x,y)$ defined through $\mathfrak{g}_\kappa$ by (2.68) is meromorphic on $\mathbb{H}^+ - \theta$ with isolated poles of finite multiplicity corresponding in position and multiplicity to the zeros of a local Evans function of (2.69). On $\Lambda\backslash\sigma(L)$ the distribution $\mathfrak{G}_\kappa$ agrees with the resolvent kernel of L.*

2.7 Spectral stability: Proof of main theorem

Let us turn to the proof of Theorem 2.4.8, our main result on spectral stability of shock profiles. Set

$$A_{\varepsilon,\kappa}(\tau) := A_{s(\varepsilon),\kappa}(\varepsilon u_\varepsilon(\tau)).$$

Instead of the non-autonomous ordinary differential equation (2.45) it will be convenient to consider the autonomous system

$$\begin{cases} \tau' = \varepsilon(1-\tau^2) h_\varepsilon(\tau), \\ \xi' = A_{\varepsilon,\kappa}(\tau)\xi, \end{cases} \tag{2.70}$$

on $J \times \mathbb{C}^{2n}$ with the obvious correspondence between solutions of (2.70) and (2.45). Making use of the linear structure of the ξ-equations in (2.70) Grassmannian versions

$$X' = \Gamma^m(A_{\varepsilon,\kappa}(\tau))(X) \text{ on } \mathscr{G}_m^{2n}(\mathbb{C}) \tag{2.71}$$

of the ξ-equations in (2.70), still coupled to the dominant τ-equation, are investigated for various $m \in \{1, \dots, 2n\}$. Here the Grassmannian flow is again defined by taking the span of the ξ-part of appropriately many solutions to the system (2.70) with linearly independent ξ-initial values. With this point of view the Evans bundles $\mathcal{H}_\varepsilon^\pm(\kappa)$ are constructed in a way that, besides analyticity in κ, implies the smooth dependence on ε, too. In fact, certain rescalings of κ and ε will lead to rescaled bundles revealing different properties of the Evans bundles in each of the following three spectral regimes (cf. [26, 27]):
Inner regime: For all $r_0 > 0$ and small $\varepsilon_0 = \varepsilon_0(r_0) > 0$ consider all $\varepsilon \in (0, \varepsilon_0]$ and $\kappa \in \mathbb{H}^+$ with

$$0 \leq |\kappa| \leq r_0 \varepsilon^2;$$

Outer regime: For some $r_0, r_1 > 0$ and $\varepsilon_0 > 0$ consider all $\varepsilon \in (0, \varepsilon_0]$ and $\kappa \in \mathbb{H}^+$ with

$$r_0 \varepsilon^2 \leq |\kappa| \leq r_1;$$

Outmost regime: For all $r_1 > 0$ and small $\varepsilon_0 = \varepsilon_0(r_1) > 0$ consider all $\varepsilon \in (0, \varepsilon_0]$ and $\kappa \in \mathbb{H}^+$ with

$$r_1 \leq |\kappa|.$$

The different regimes are treated in the following subsections. Before we enter into the considerations regarding the spectral regimes let us give a useful block-diagonalization of

$$(\varepsilon, u) \mapsto Df_{s(\varepsilon)}(\varepsilon u)$$

where $u \in B_r(0)$ for some $r > 0$ and $\varepsilon \leq \varepsilon_0 = \varepsilon_0(r)$. Here and in the following subsections we make use of regular perturbation theory as presented in [54].

Lemma 2.7.1. *Let $r > 0$. Then there exist $\varepsilon_0 = \varepsilon_0(r)$ and smooth functions*

$$\begin{aligned} W &: [0, \varepsilon_0] \times B_r(0) \to \mathrm{SO}_n(\mathbb{R}), \\ M^< &: [0, \varepsilon_0] \times B_r(0) \to \mathbb{R}^{(k-1)\times(k-1)}, \\ M^0 &: [0, \varepsilon_0] \times B_r(0) \to \mathbb{R}, \\ M^> &: [0, \varepsilon_0] \times B_r(0) \to \mathbb{R}^{(n-k)\times(n-k)}, \end{aligned}$$

such that for all $u \in B_r(0)$

$$\begin{aligned} W(0,u) &= \mathbb{1}_n, \\ M^<(0,u) &= \operatorname{diag}(\lambda_1, \dots, \lambda_{k-1}), \\ M^0(0,u) &= 0, \\ \partial_\varepsilon M^0(0,u) &= e_k^t \partial_{v_k} D_v f(0) u, \\ M^>(0,u) &= \operatorname{diag}(\lambda_{k+1}, \dots, \lambda_n), \end{aligned}$$

and all $(\varepsilon, u) = [0, \varepsilon_0] \times B_r(0)$

$$W(\varepsilon,u)^t D f_{s(\varepsilon)}(\varepsilon u) W(\varepsilon,u) = \begin{pmatrix} M^<(\varepsilon,u) & 0 & 0 \\ 0 & M^0(\varepsilon,u) & 0 \\ 0 & 0 & M^>(\varepsilon,u) \end{pmatrix}.$$

Proof. Let $r > 0$ and consider all $\varepsilon \geq 0$ such that $B_{\varepsilon r}(0) \subset \mathcal{V}$. Define the smooth function $F(\varepsilon, u) := D f_{s(\varepsilon)}(\varepsilon u)$. Then $F(0,u) = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ for all u with

$$\lambda_j < 0 = \lambda_k = 0 < \lambda_{j'} \text{ for } j < k < j'.$$

Hence, for ε small enough there are total spectral projections

$$\begin{aligned} P^<(\varepsilon,u) &= \frac{1}{2\pi i} \int_{\Gamma^<} (z - F(\varepsilon,u))^{-1} dz, \\ P^0(\varepsilon,u) &= \frac{1}{2\pi i} \int_{\Gamma^0} (z - F(\varepsilon,u))^{-1} dz, \\ P^>(\varepsilon,u) &= \frac{1}{2\pi i} \int_{\Gamma^>} (z - F(\varepsilon,u))^{-1} dz, \end{aligned}$$

onto the algebraic eigenspaces of the groups of eigenvalues of $F(\varepsilon, u)$ with real parts smaller, close-to and larger than zero. $\Gamma^{\gtrless}, \Gamma^0$ are suitably chosen closed curves enclosing the respective group of eigenvalues. The above projections are smooth in (ε, u), form a disjoint partition of unity in the sense that

$$\begin{aligned} P^< + P^0 + P^> &= \mathbb{1}_n, \\ P^< P^0 = P^0 P^< &= 0, \\ P^> P^0 = P^0 P^> &= 0, \\ P^< P^> = P^> P^< &= 0, \end{aligned}$$

and for all u it holds

$$\begin{aligned} P^<(0,u) = \Pi^< &:= \begin{pmatrix} \mathbb{1}_{k-1} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{n \times n}, \\ P^0(0,u) &= E_{kk} \in \mathbb{C}^{n \times n}, \\ P^>(0,u) = \Pi^> &:= \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{n-k} \end{pmatrix} \in \mathbb{C}^{n \times n}, \end{aligned}$$

where for $j = 1, \dots, n$ we have set

$$E_{jj} := e_j e_j^t.$$

Because $F(\varepsilon, u)$ is real symmetric, we find

$$P^{\gtrless}, P^0 \in \mathrm{Sym}_n(\mathbb{R}).$$

Following Chapter 2.4. in [54] on the construction of transformation functions adapted to several (disjoint) projections set

$$Q = (\partial_\varepsilon P^<)P^< + (\partial_\varepsilon P^0)P^0 + (\partial_\varepsilon P^>)P^>.$$

Then Q is again smooth in (ε, u) and real skew-symmetric. For $u \in B_r(0)$ let $W = W(\varepsilon, u)$ solve

$$\partial_\varepsilon W(\varepsilon, u) = Q(\varepsilon, u)W(\varepsilon, u),\ W(0, u) = \mathbb{1}_n. \tag{2.72}$$

Because solutions of linear ordinary differential equations depend smoothly on parameters, W is smooth in (ε, u), and $\det W > 0$. Since Q is real skew-symmetric we find

$$W^{-1} = W^t,$$

i.e. $W \in \mathrm{SO}_n(\mathbb{R})$. Furthermore, for all (ε, u) the relations

$$\begin{aligned} W(\varepsilon, u)P^{\gtrless}(0, u)W(\varepsilon, u)^t &= P^{\gtrless}(\varepsilon, u), \\ W(\varepsilon, u)P^0(0, u)W(\varepsilon, u)^t &= P^0(\varepsilon, u), \end{aligned}$$

hold. We find

$$\begin{aligned} M(\varepsilon, u) :=& W(\varepsilon, u)^t F(\varepsilon, u)W(\varepsilon, u) \\ =& \Pi^< M(\varepsilon, u)\Pi^< + E_{kk}M(\varepsilon, u)E_{kk} + \Pi^> M(\varepsilon, u)\Pi^> \\ =& \begin{pmatrix} M^<(\varepsilon, u) & 0 & 0 \\ 0 & M^0(\varepsilon, u) & 0 \\ 0 & 0 & M^>(\varepsilon, u) \end{pmatrix}. \end{aligned}$$

The functions $M^{\gtrless}, M^0$ are smooth in (ε, u) with

$$M^<(0, u) = \mathrm{diag}(\lambda_1, \dots, \lambda_{k-1}), M^0(0, u) = 0, M^>(0, u) = \mathrm{diag}(\lambda_{k+1}, \dots, \lambda_n)$$

for all u. A direct calculation shows

$$\partial_\varepsilon M^0(0, u) = (\partial_\varepsilon F(0, u))_{kk} = e_k^t \partial_{v_k} D_v f(0)u.$$

□

Set

$$F_\varepsilon(\tau) := Df_{s(\varepsilon)}(\varepsilon u_\varepsilon(\tau)), G_\varepsilon(\tau) := Dg_{s(\varepsilon)}(\varepsilon u_\varepsilon(\tau)),$$

and $F_0 := F_0(0), G_0 := G_0(0)$. For appropriate $\nu_\varepsilon > 0$ let

$$\Lambda_\varepsilon := M_{\nu_\varepsilon}$$

be a domain of consistent splitting for $\{(A_{\varepsilon,\kappa}(-1), A_{\varepsilon,\kappa}(+1))\}_\kappa$ as guaranteed to exist by Corollary 2.4.4.

2.7.1 Inner regime

Let $r_0 > 0$ and set

$$D_{r_0} := \{\kappa \in \mathbb{H}^+ : \ |\kappa| \leq r_0\}.$$

The scalings

$$\varepsilon^2\zeta = \kappa \ \text{ and } \ \varepsilon\tilde{q} = q$$

yield

$$A^I_{\varepsilon,\zeta}(\tau) = \begin{pmatrix} F_\varepsilon(\tau) & \varepsilon\mathbb{1}_d \\ \varepsilon^3\zeta^2\mathbb{1}_d + \varepsilon\zeta G_\varepsilon(\tau) & 0 \end{pmatrix}$$

with

$$A^I_{\varepsilon,\zeta}(\tau) \sim A_{s(\varepsilon),\varepsilon^2\zeta}(\tau) \ \text{ if } \ \varepsilon > 0.$$

Hence, for $\varepsilon > 0$ we can consider

$$\begin{cases} \tau' = \varepsilon(1-\tau^2)h_\varepsilon(\tau), \\ \xi' = A^I_{\varepsilon,\zeta}(\tau)\xi, \end{cases} \tag{2.73}$$

on $J\times\mathbb{C}^{2n}$ instead of (2.70). First we are interested in the splitting of eigenvalues and eigenspaces of $A^I_{\varepsilon,\zeta}(\tau)$ at the end points $\tau = \pm 1$ for $\zeta \in D_{r_0}$ and small $\varepsilon \geq 0$. Fix some notation by setting

$$\begin{aligned} F^\pm_\varepsilon &:= F_\varepsilon(\pm 1), \\ G^\pm_\varepsilon &:= G_\varepsilon(\pm 1), \\ \underline{G}_0 &:= (G_0)^{-1}_{kk} G_0, \\ A^I_0 &:= A^I_{0,0}(0), \\ A^{I,\pm}_{\varepsilon,\zeta} &:= A^I_{\varepsilon,\zeta}(\pm 1). \end{aligned}$$

Lemma 2.7.2. *(i) For any $r_0 > 0$ there exists $\varepsilon_0 = \varepsilon_0(r_0)$ such that for $\varepsilon \in [0, \varepsilon_0]$ there are holomorphic mappings*

$$\begin{aligned} &S^{\pm,f}_\varepsilon : D_{r_0} \to \mathscr{G}^{2n}_{k-1}(\mathbb{C}), \quad S^{\pm,sf}_\varepsilon : D_{r_0} \to \mathscr{G}^{2n}_1(\mathbb{C}), \quad S^{\pm,ss}_\varepsilon : D_{r_0} \to \mathscr{G}^{2n}_{n-k}(\mathbb{C}), \\ &U^{\pm,f}_\varepsilon : D_{r_0} \to \mathscr{G}^{2n}_{n-k}(\mathbb{C}), \quad U^{\pm,sf}_\varepsilon : D_{r_0} \to \mathscr{G}^{2n}_1(\mathbb{C}), \quad U^{\pm,ss}_\varepsilon : D_{r_0} \to \mathscr{G}^{2n}_{k-1}(\mathbb{C}), \end{aligned}$$

and

$$N_\varepsilon^\pm : D_{r_0} \to \mathscr{G}_{n+1}^{2n}(\mathbb{C})$$

depending smoothly on ε such that for all $\zeta \in D_{r_0}$ their images define a splitting of $\mathbb{C}^{2n}$ consisting of invariant subspaces of $A_{\varepsilon,\zeta}^{I,\pm}$:
For all $\varepsilon \in [0, \varepsilon_0]$ and $\zeta \in D_{r_0}$ it holds

$$\mathbb{C}^{2n} = S_{\varepsilon,\zeta}^{\pm,f} \oplus N_{\varepsilon,\zeta}^{\pm} \oplus U_{\varepsilon,\zeta}^{\pm,f}$$

and

$$N_{\varepsilon,\zeta}^{\pm} = S_{\varepsilon,\zeta}^{\pm,sf} \oplus S_{\varepsilon,\zeta}^{\pm,ss} \oplus U_{\varepsilon,\zeta}^{\pm,sf} \oplus U_{\varepsilon,\zeta}^{\pm,ss}.$$

Furthermore, for all $\zeta \in D_{r_0}$

$$\begin{aligned} S_{0,\zeta}^{\pm,f} &= S_0^f := \mathbb{C}^{k-1} \times \{0\}^{n-k+1} \times \{0\}^n, \\ U_{0,\zeta}^{\pm,f} &= U_0^f := \{0\}^k \times \mathbb{C}^{n-k} \times \{0\}^n, \end{aligned}$$

and

$$N_{0,\zeta}^{\pm} = N_0 := \{0\}^{k-1} \times \mathbb{C} \times \{0\}^{n-k} \times \mathbb{C}^n.$$

The super-slow invariant subspaces are constant on $\{\varepsilon = 0\}$, i.e.

$$\begin{aligned} S_{0,\zeta}^{\pm,ss} &= S_0^{ss} := S_{0,0}^{+,ss}, \\ U_{0,\zeta}^{\pm,ss} &= U_0^{ss} := U_{0,0}^{+,ss}. \end{aligned}$$

(ii) The eigenvalues

$$\{\mu_{j;\varepsilon,\zeta}^{\pm,S,f}\}_{j<k}, \{\mu_{j;\varepsilon,\zeta}^{\pm,U,f}\}_{j>k}, \mu_{\varepsilon,\zeta}^{\pm,S,sf}, \mu_{\varepsilon,\zeta}^{\pm,U,sf}, \{\mu_{j;\varepsilon,\zeta}^{\pm,S,ss}\}_{j>k}, \{\mu_{j;\varepsilon,\zeta}^{\pm,U,ss}\}_{j<k}$$

of $A_{\varepsilon,\zeta}^{I,\pm}$ expand as

$$\left\{\begin{array}{ll} \lambda_j + \mathcal{O}(\varepsilon) & \text{for } j < k \\ \lambda_j + \mathcal{O}(\varepsilon) & \text{for } j > k \\ \varepsilon(\tilde{\mu}_\zeta^{S,sf} + \mathcal{O}(\varepsilon)) & \\ -\varepsilon\zeta(G_0)_{kk}(\tilde{\mu}_\zeta^{S,sf-1} + \mathcal{O}(\varepsilon)) & \\ \varepsilon\zeta(G_0)_{kk}(\tilde{\mu}_\zeta^{S,sf-1} + \mathcal{O}(\varepsilon)) & \\ -\varepsilon(\tilde{\mu}_\zeta^{S,sf} + \mathcal{O}(\varepsilon)) & \\ -\varepsilon^2\zeta(\mu_j^{-1}/2 + \mathcal{O}(\varepsilon)) & \text{for } j > k \\ -\varepsilon^2\zeta(\mu_j^{-1}/2 + \mathcal{O}(\varepsilon)) & \text{for } j < k \end{array}\right\} \text{on} \left\{\begin{array}{c} S_{\varepsilon,\zeta}^{\pm,f} \\ U_{\varepsilon,\zeta}^{\pm,f} \\ S_{\varepsilon,\zeta}^{+,sf} \\ U_{\varepsilon,\zeta}^{+,sf} \\ S_{\varepsilon,\zeta}^{-,sf} \\ U_{\varepsilon,\zeta}^{-,sf} \\ S_{\varepsilon,\zeta}^{\pm,ss} \\ U_{\varepsilon,\zeta}^{\pm,ss} \end{array}\right\}$$

where

$$\tilde{\mu}_\zeta^{U/S,sf} := a \pm \sqrt{a^2 + \zeta(G_0)_{kk}}$$

satisfy

$$-\zeta(G_0)_{kk} = \tilde{\mu}_\zeta^{S,sf}\tilde{\mu}_\zeta^{U,sf}, \operatorname{Re}\tilde{\mu}_\zeta^{U,sf} > 0 \text{ and } \operatorname{Re}\tilde{\mu}_\zeta^{S,sf} \le a < 0 \text{ for } \zeta \in \mathbb{H}^+\backslash\{0\}.$$

(iii) There is a smooth choice of bases

$$\{R^{\pm,S,f}_{j;\varepsilon,\zeta}\}_{j<k}, \{R^{\pm,U,f}_{j;\varepsilon,\zeta}\}_{j<k}, R^{\pm,S,sf}_{\varepsilon,\zeta}, R^{\pm,U,sf}_{\varepsilon,\zeta}, \{R^{\pm,S,ss}_{j;\varepsilon,\zeta}\}_{j>k}, \{R^{\pm,U,ss}_{j;\varepsilon,\zeta}\}_{j<k}$$

for the above invariant subspaces of $A^{I,\pm}_{\varepsilon,\zeta}$ such that for $\varepsilon = 0$ the fast subspaces are spanned by

$$R^{\pm,S,f}_{j;0,\zeta} = \tilde{e}_j := \begin{pmatrix} e_j \\ 0 \end{pmatrix} \text{ for } j < k,$$
$$R^{\pm,U,f}_{j;0,\zeta} = \tilde{e}_j := \begin{pmatrix} e_j \\ 0 \end{pmatrix} \text{ for } j > k,$$

the slow-fast subspaces by

$$R^{+,S,sf}_{0,\zeta} = \begin{pmatrix} e_k \\ -\tilde{\mu}^{U,sf}_\zeta \underline{G}_0 e_k \end{pmatrix},$$
$$R^{+,U,sf}_{0,\zeta} = \begin{pmatrix} e_k \\ -\tilde{\mu}^{S,sf}_\zeta \underline{G}_0 e_k \end{pmatrix},$$
$$R^{-,S,sf}_{0,\zeta} = \begin{pmatrix} e_k \\ \tilde{\mu}^{S,sf} \underline{G}_0 e_k \end{pmatrix},$$
$$R^{-,U,sf}_{0,\zeta} = \begin{pmatrix} e_k \\ \tilde{\mu}^{U,sf} \underline{G}_0 e_k \end{pmatrix},$$

and the super-slow subspaces by

$$R^{\pm,S,ss}_{j;0,\zeta} = \begin{pmatrix} 0 \\ \underline{G}_0 r_j \end{pmatrix} \text{ for } j > k,$$
$$R^{\pm,U,ss}_{j;0,\zeta} = \begin{pmatrix} 0 \\ \underline{G}_0 r_j \end{pmatrix} \text{ for } j < k.$$

Proof. Let $r_0 > 0$. We only consider the right end $\tau = +1$, i.e. only the $+$-superscript, since the arguments in the case $\tau = -1$ are the same modulo obvious modifications. Let $\varepsilon_0 > 0$ be so small that Lemma 2.7.1 holds, and in its notation we may assume

$$F_\varepsilon = \begin{pmatrix} M^<_\varepsilon(+1) & 0 & 0 \\ 0 & M^0_\varepsilon(+1) & 0 \\ 0 & 0 & M^>_\varepsilon(+1) \end{pmatrix}$$

for

$$M_\varepsilon^<(\tau) := M^<(\varepsilon, u_\varepsilon(\tau)),$$
$$M_\varepsilon^0(\tau) := M^0(\varepsilon, u_\varepsilon(\tau)),$$
$$M_\varepsilon^>(\tau) := M^>(\varepsilon, u_\varepsilon(\tau)).$$

For $\varepsilon = 0$ we find that

$$A_{0,\zeta}^I = A_0^I = \begin{pmatrix} \operatorname{diag}(\lambda_1, \dots, \lambda_n) & 0 \\ 0 & 0 \end{pmatrix}$$

is independent of ζ and of diagonal form. Hence, for small ε and $\zeta \in D_{r_0}$ the total projections

$$P_{\varepsilon,\zeta}^{S,f}, P_{\varepsilon,\zeta}^s \text{ and } P_{\varepsilon,\zeta}^{U,f}$$

onto the eigenvalues with real part (strongly) smaller, close-to and (strongly) larger than zero are well-defined and smooth in their parameters. For $\varepsilon = 0$ they have constant values

$$P_0^{S,f} := P_{0,\zeta}^{S,f} = \begin{pmatrix} \Pi^< & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.74}$$
$$P_0^s := P_{0,\zeta}^s = \begin{pmatrix} E_{kk} & 0 \\ 0 & \mathbb{1}_n \end{pmatrix},$$
$$P_0^{U,f} := P_{0,\zeta}^{U,f} = \begin{pmatrix} \Pi^> & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.75}$$

We apply Kato's reduction process [54] to the 0-group. A Taylor expansion shows that

$$A_{\varepsilon,\zeta}^{I,s} := \frac{1}{\varepsilon} A_{\varepsilon,\zeta}^I P_{\varepsilon,\zeta}^s$$

defines a smooth function with

$$A_{0,\zeta}^{I,s} = P_0^s \partial_\varepsilon A_{0,\zeta}^I P_0^s.$$

From Lemma 2.7.1 and $u_\varepsilon(\tau) = \varepsilon\tau e_k + \mathcal{O}(\varepsilon^2)$ one finds $\partial_\varepsilon M_0^0 = 2a$, and hence

$$A_{0,\zeta}^{I,s} = \begin{pmatrix} 2aE_{kk} & E_{kk} \\ \zeta G_0 E_{kk} & 0 \end{pmatrix}.$$

For $\zeta \in D_{r_0}$ the matrix $A_{0,\zeta}^{I,s}$ has two slow-fast eigenvalues given by

$$\tilde{\mu}_\zeta^{U/S,sf} = a \pm \sqrt{a^2 + \zeta (G_0)_{kk}} \tag{2.76}$$

and an $2(n-1)$-fold zero eigenvalue. It holds for all $\zeta \in D_{r_0}$ that

$$\operatorname{Re} \tilde{\mu}_\zeta^{S,sf} < a < 0 \leq \operatorname{Re} \tilde{\mu}_\zeta^{S,sf}$$

implying that for ε small and all $\zeta \in D_{r_0}$ there is a spectral gap of size $a/2$ separating a stable simple eigenvalue $\tilde{\mu}^{S,sf}_{\varepsilon,\zeta}$ from all other eigenvalues of $A^{I,s}_{\varepsilon,\zeta}$. We find a stable slow-fast eigenvalue $\mu^{S,sf}_{\varepsilon,\zeta}$ of $A^{I}_{\varepsilon,\zeta}$ with

$$\mu^{S,sf}_{\varepsilon,\zeta} = \varepsilon(\tilde{\mu}^{S,sf}_{\zeta} + \mathcal{O}(\varepsilon))$$

and one dimensional eigenprojection $P^{S,sf}_{\varepsilon,\zeta}$ which is smooth in ε and holomorphic in ζ.
To get a hold of the slow-fast eigenvalue associated to $\tilde{\mu}^{U,sf}_{\zeta}$ let us consider

$$A^{I,U,sf}_{\varepsilon,\zeta} = A^{I,s}_{\varepsilon,\zeta}(\mathbb{1}_{2n} - P^{S,sf}_{\varepsilon,\zeta}).$$

Using the Dunford formula we find at $\zeta = 0$

$$P^{s}_{\varepsilon,0} = \begin{pmatrix} E_{kk} & -\varepsilon\tilde{F}^{-1}_{\varepsilon} \\ 0 & \mathbb{1}_n \end{pmatrix}$$

where

$$\tilde{F}^{-1}_{\varepsilon} = \begin{pmatrix} M^{<}_{\varepsilon}(+1)^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M^{>}_{\varepsilon}(+1)^{-1} \end{pmatrix}$$

which leads to

$$A^{I,s}_{\varepsilon,0} = \varepsilon^{-1}A^{I}_{\varepsilon,0}P^{s}_{\varepsilon,0} = \begin{pmatrix} \frac{M^{0}_{\varepsilon}(+1)}{\varepsilon}E_{kk} & E_{kk} \\ 0 & 0 \end{pmatrix}$$

and

$$P^{S,sf}_{\varepsilon,0} = \begin{pmatrix} E_{kk} & \frac{\varepsilon}{M^{0}_{\varepsilon}(+1)}E_{kk} \\ 0 & 0 \end{pmatrix}.$$

We obtain for all small ε

$$A^{I,U,sf}_{\varepsilon,0} = A^{I,s}_{\varepsilon,0}(\mathbb{1}_{2n} - P^{S,sf}_{\varepsilon,0}) = 0. \tag{2.77}$$

Set $\Delta^{sf}_{\zeta} := \tilde{\mu}^{S,sf}_{\zeta} - \tilde{\mu}^{U,sf}_{\zeta}$ and

$$\tilde{R} := \big((r_j)_{j<k}, 0, (r_j)_{j>k}\big) \in \mathbb{R}^{n\times n}$$

with r_j from Lemma 2.3.1. By the just quoted lemma and Lemma 2.3.5 the columns of

$$\tilde{R} + (G_0)^{-1}_{kk}E_{kk}$$

are eigenvectors of F_0 w.r.t. G_0, and

$$(\tilde{R}^t + E_{kk})G_0(\tilde{R} + (G_0)^{-1}_{kk}E_{kk}) = \mathbb{1}_n.$$

Diagonalize $A^{I,s}_{0,\zeta}$ via

$$R_\zeta := \begin{pmatrix} \mathbb{1}_n & E_{kk} \\ -\frac{\tilde\mu_\zeta^{S,sf}}{(G_0)_{kk}} G_0 E_{kk} & G_0\left(\tilde R - \frac{\tilde\mu_\zeta^{U,sf}}{(G_0)_{kk}} E_{kk}\right) \end{pmatrix}, \tag{2.78}$$

$$R_\zeta^{-1} := \begin{pmatrix} \mathbb{1}_n - \frac{\tilde\mu_\zeta^{S,sf}}{\Delta_\zeta^{sf}} E_{kk} & -\frac{1}{\Delta_\zeta^{sf}} E_{kk} \\ \frac{\tilde\mu_\zeta^{S,sf}}{\Delta_\zeta^{sf}} E_{kk} & \tilde R^t + \frac{1}{\Delta_\zeta^{sf}} E_{kk} \end{pmatrix},$$

to find

$$R_\zeta^{-1} A^{I,s}_{0,\zeta} R_\zeta = \begin{pmatrix} \tilde\mu_\zeta^{U,sf} E_{kk} & 0 \\ 0 & \tilde\mu_\zeta^{S,sf} E_{kk} \end{pmatrix}.$$

Hence, at $\varepsilon = 0$ the slow-fast stable projection is given by

$$P^{S,sf}_{0,\zeta} = R_\zeta \begin{pmatrix} 0 & 0 \\ 0 & E_{kk} \end{pmatrix} R_\zeta^{-1}. \tag{2.79}$$

A calculation shows

$$A^{I,U,sf}_{0,\zeta} = \tilde\mu_\zeta^{U,sf} P_\zeta^{U,sf} = -\zeta \frac{(G_0)_{kk}}{\tilde\mu_\zeta^{S,sf}} P_\zeta^{U,sf} \tag{2.80}$$

where

$$P_\zeta^{U,sf} = R_\zeta \begin{pmatrix} E_{kk} & 0 \\ 0 & 0 \end{pmatrix} R_\zeta^{-1}. \tag{2.81}$$

By combining (2.77) and (2.80) there exists a ε-smooth and ζ-holomorphic function $H = H(\varepsilon, \zeta)$ such that for ε small and $\zeta \in D_{r_0}$

$$A^{I,U,sf}_{\varepsilon,\zeta} = \tilde\mu_\zeta^{U,sf} P_\zeta^{U,sf} + \varepsilon\zeta H(\varepsilon, \zeta).$$

We find an unstable slow-fast eigenvalue $\mu^{U,sf}_{\varepsilon,\zeta}$ of $A^I_{\varepsilon,\zeta}$ with ε-smooth and ζ-holomorphic one dimensional eigenprojection $P^{U,sf}_{\varepsilon,\zeta}$ and

$$\mu^{U,sf}_{\varepsilon,\zeta} = -\varepsilon\zeta\left(\frac{(G_0)_{kk}}{\tilde\mu_\zeta^{S,sf}} + \mathcal{O}(\varepsilon)\right),$$
$$P^{U,sf}_{0,\zeta} = P^{U,sf}_\zeta.$$

For the last reduction step consider

$$A^{I,ss}_{\varepsilon,\zeta} := \frac{1}{\varepsilon\zeta} A^{I,U,sf}_{\varepsilon,\zeta}(\mathbb{1}_{2n} - P^{U,sf}_{\varepsilon,\zeta}) = \frac{1}{\varepsilon\zeta} A^{I,s}_{\varepsilon,\zeta}(\mathbb{1}_{2n} - P^{sf}_{\varepsilon,\zeta})$$

where $P^{sf} = P^{S,sf} + P^{U,sf}$. Noting

$$\tilde{F}_0^{-1} = \begin{pmatrix} (M_0^<(+1))^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (M_0^>(+1))^{-1} \end{pmatrix}$$

and

$$\begin{aligned} \partial_\varepsilon A^I_{0,\zeta} &= \begin{pmatrix} \partial_\varepsilon F_0(1) & \mathbb{1}_n \\ \zeta G_0 & 0 \end{pmatrix}, \\ \partial^2_\varepsilon A^I_{0,\zeta} &= \begin{pmatrix} \partial^2_\varepsilon F_0(1) & 0 \\ \zeta \partial_\varepsilon G_0(1) & 0 \end{pmatrix}, \\ \partial_\varepsilon P_{0;0,\zeta} &= \begin{pmatrix} 0 & -\tilde{F}_0^{-1} \\ -\zeta G_0 \tilde{F}_0^{-1} & 0 \end{pmatrix}, \\ P^{sf}_{0,\zeta} &= \begin{pmatrix} E_{kk} & 0 \\ 0 & \underline{G}_0 E_{kk} \end{pmatrix}, \end{aligned}$$

one finds with $p^{ss} := \mathbb{1}_n - \underline{G}_0 E_{kk}$ that

$$A^{I,ss}_{0,\zeta} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & -p^{ss} G_0 \tilde{F}_0^{-1} p^{ss} \end{pmatrix}.$$

Because

$$\begin{aligned} (\tilde{R}^t + E_{kk}) p^{ss} G_0 \tilde{F}_0^{-1} p^{ss} G_0 (\tilde{R} + (G_0)^{-1}_{kk} E_{kk}) \\ = \operatorname{diag}(\mu_1^{-1}, \ldots, \mu_{k-1}^{-1}, 0, \mu_{k+1}^{-1}, \ldots, \mu_n^{-1}) \end{aligned}$$

we find one dimensional eigenprojections $P^{S,ss}_{j;\varepsilon,\zeta}, j > k$, and $P^{U,ss}_{j;\varepsilon,\zeta}, j < k$, associated to $n-k$ stable eigenvalues $\mu^{S,ss}_{j;\varepsilon,\zeta}, j > k$, and $k-1$ unstable eigenvalues $\mu^{U,ss}_{j;\varepsilon,\zeta}, j < k$, of $A^I_{\varepsilon,\zeta}$ satisfying

$$\begin{aligned} \mu^{S/U,ss}_{j;\varepsilon,\zeta} &= -\varepsilon^2 \zeta \left(\frac{\mu_j^{-1}}{2} + \mathcal{O}(\varepsilon) \right), \\ P^{S/U,ss}_{j;0,\zeta} &= \begin{pmatrix} 0 & 0 \\ 0 & G_0 \tilde{R} E_{jj} \tilde{R}^t \end{pmatrix}. \end{aligned} \tag{2.82}$$

For ε small enough and all $\zeta \in D_{r_0}$ the vectors

$$\begin{aligned}
R^{S,f}_{j;\varepsilon,\zeta} &= P^{S,f}_{\varepsilon,\zeta}\begin{pmatrix} e_j \\ 0 \end{pmatrix} && \text{for } \ j<k,\\
R^{U,f}_{j;\varepsilon,\zeta} &= P^{U,f}_{\varepsilon,\zeta}\begin{pmatrix} e_j \\ 0 \end{pmatrix} && \text{for } \ j>k,\\
R^{S,sf}_{\varepsilon,\zeta} &= P^{S,sf}_{\varepsilon,\zeta}\begin{pmatrix} e_k \\ -\tilde{\mu}^{U,sf}_{\zeta}\underline{G}_0 e_k \end{pmatrix}, &&\\
R^{U,sf}_{\varepsilon,\zeta} &= P^{U,sf}_{\varepsilon,\zeta}\begin{pmatrix} e_k \\ -\tilde{\mu}^{S,sf}_{\zeta}\underline{G}_0 e_k \end{pmatrix}, &&\\
R^{S,ss}_{j;\varepsilon,\zeta} &= P^{S,ss}_{j;\varepsilon,\zeta}\begin{pmatrix} 0 \\ \underline{G}_0 r_j \end{pmatrix} && \text{for } \ j>k,\\
R^{U,ss}_{j;\varepsilon,\zeta} &= P^{U,ss}_{j;\varepsilon,\zeta}\begin{pmatrix} 0 \\ \underline{G}_0 r_j \end{pmatrix} && \text{for } \ j<k.
\end{aligned}$$

define ε-smooth and ζ-holomorphic choices of bases for the invariant subspaces of $A^I_{\varepsilon,\zeta}$ which are given by the images of the respective (total) projections. Using the explicit expressions (2.74), (2.75), (2.79), (2.81), (2.82) at $\varepsilon = 0$ finishes the proof of the lemma. □

Remark 2.7.3. *By Corollary 2.4.4 we have*

$$\mathbb{H}^+\backslash\{0\} \subset \Lambda_\varepsilon.$$

Hence, in the situation of Lemma 2.7.2 the real parts of $A^{I,\pm}_{\varepsilon,\zeta}$ may not change signs as (ε,ζ) is varied in $(0,\varepsilon_0]\times(D_{r_0}\backslash\{0\})$, and the eigenvalues with superscript S resp. U indeed have real parts smaller resp. greater than 0 if $\varepsilon,\zeta \neq 0$.

With the help of Lemma 2.7.2 and singular perturbation theory we give a detailed construction of the (rescaled) Evans bundles for ϕ_ε following [26]. Define the one dimensional submanifolds

$$\begin{aligned}
\mathcal{C}^{S,f}_0 &= J\times\{S^f_0\} \subset J\times\mathscr{G}^{2n}_{k-1}(\mathbb{C}),\\
\mathcal{C}^{U,f}_0 &= J\times\{U^f_0\} \subset J\times\mathscr{G}^{2n}_{n-k}(\mathbb{C}),\\
\mathcal{N}_0 &= J\times\{N_0\} \subset J\times\mathscr{G}^{2n}_{n+1}(\mathbb{C}).
\end{aligned}$$

The statement for the fast part reads:

Lemma 2.7.4. *For all $r_0>0$ there exists $\varepsilon_0=\varepsilon_0(r_0)$ such that for all $\varepsilon\in[0,\varepsilon_0]$ there are holomorphic mappings*

$$\begin{aligned}
H^{+,f}_\varepsilon &: D_{r_0} \to \mathscr{G}^{2n}_{k-1}(\mathbb{C}),\\
H^{-,f}_\varepsilon &: D_{r_0} \to \mathscr{G}^{2n}_{n-k}(\mathbb{C}),
\end{aligned}$$

depending smoothly on ε and the solutions X^+ and X^- of

$$\begin{aligned}
X^{+\prime} &= \Gamma^{k-1}(A^I_{\varepsilon,\zeta}(\chi_\varepsilon))(X^+), \quad X^+(0) = H^{+,f}_\varepsilon(\zeta), \ \text{ on } \mathscr{G}^{2n}_{k-1}(\mathbb{C}),\\
X^{-\prime} &= \Gamma^{n-k}(A^I_{\varepsilon,\zeta}(\chi_\varepsilon))(X^-), \quad X^-(0) = H^{-,f}_\varepsilon(\zeta), \ \text{ on } \mathscr{G}^{2n}_{n-k}(\mathbb{C}),
\end{aligned}$$

converge on the left and right ends:

$$X^+(\pm\infty) = S^{\pm,f}_{\varepsilon,\zeta},$$
$$X^-(\pm\infty) = U^{\pm,f}_{\varepsilon,\zeta}.$$

Proof. Let $r_0 > 0$ and choose ε_0 so small such that the conclusions of Lemma 2.7.2 hold. Since

$$A^I_{0,\zeta}(\tau) = A^I_0 := \begin{pmatrix} \operatorname{diag}(\lambda_1, \ldots, \lambda_n) & 0 \\ 0 & 0 \end{pmatrix}$$

we find by Corollary 2.1.2 that $\mathcal{C}^{S,f}_0$ is a repelling normally hyperbolic critical manifold for the system

$$\begin{cases} \tau' = \varepsilon(1-\tau^2)h_\varepsilon(\tau), \\ X' = \Gamma^{k-1}(A^I_{\varepsilon,\zeta}(\tau))(X), \end{cases} \quad \text{on } \ J \times \mathscr{G}^{2n}_{k-1}(\mathbb{C}). \tag{2.83}$$

By Fenichel Theory [22], i.e. Theorem 2.2.8, and compactness of $J \times \mathscr{G}^{2n}_{k-1}(\mathbb{C})$ it perturbs smoothly to a unique repelling slow manifold $\mathcal{C}^{S,f}_{\varepsilon,\zeta}$. Because $\mathcal{C}^{S,f}_0$ is the graph of

$$C^{S,f}_0(\tau) := S^f_0$$

we deduce from Theorem 2.2.9 that for ε small enough there exists an ε-smooth and ζ-holomorphic function

$$C^{S,f}_{\varepsilon,\zeta} : J \to \mathscr{G}^{2n}_{k-1}(\mathbb{C})$$

such that $C^{S,f}_{0,\zeta} = C^{S,f}_0$ and $\mathcal{C}^{S,f}_{\varepsilon,\zeta}$ is a graph of $C^{S,f}_{\varepsilon,\zeta}$ for all $\zeta \in D_{r_0}$. Since $(\pm 1, S^{\pm,f}_{\varepsilon,\zeta})$ are fixed points of (2.83) and $S^{\pm,f}_{\varepsilon,\zeta}$ is ε-close to S^f_0 we find

$$(\pm 1, S^{\pm,f}_{\varepsilon,\zeta}) \in \mathcal{C}^{S,f}_{\varepsilon,\zeta} \ \text{ and } \ C^{S,f}_{\varepsilon,\zeta}(\pm 1) = S^{\pm,f}_{\varepsilon,\zeta}.$$

In a similar fashion one finds a unique attracting slow manifold $\mathcal{C}^{U,f}_{\varepsilon,\zeta}$ of

$$\begin{cases} \tau' = \varepsilon(1-\tau^2)h_\varepsilon(\tau), \\ X' = \Gamma^{n-k}(A^I_{\varepsilon,\zeta}(\tau))(X), \end{cases} \quad \text{on } \ J \times \mathscr{G}^{2n}_{n-k}(\mathbb{C}) \tag{2.84}$$

which perturbs from $\mathcal{C}^{U,f}_0$ and is given as the graph of a smooth function $C^{U,f}_{\varepsilon,\zeta}$ with $C^{U,f}_{0,\zeta}(\tau) = U^f_0$ and

$$C^{U,f}_{\varepsilon,\zeta}(\pm 1) = U^{\pm,f}_{\varepsilon,\zeta}.$$

With an appropriately small choice of $\varepsilon_0 > 0$ define for all $(\varepsilon, \zeta) \in [0, \varepsilon_0] \times D_{r_0}$ the fast bundles by setting

$$H^{-,f}_\varepsilon(\zeta) := C^{U,f}_{\varepsilon,\zeta}(0) \ \text{ and } \ H^{+,f}_\varepsilon(\zeta) := C^{S,f}_{\varepsilon,\zeta}(0).$$

From their definitions the bundles $H^{-,f}_{\varepsilon}$ and $H^{+,f}_{\varepsilon}$ inherit the desired smoothness properties form $C^{U,f}_{\varepsilon,\cdot}$ and $C^{S,f}_{\varepsilon,\cdot}$. Because $H^{-,f}_{\varepsilon}(\zeta)$ resp. $H^{+,f}_{\varepsilon}(\zeta)$ are ε-close to U^f_0 resp. S^f_0, transversality of the fast bundles follows from transversality of S^f_0 and U^f_0. □

Let $r_0 > 0$ and choose $\varepsilon_0 > 0$ so small that the conclusions of Lemma 2.7.2 hold. We turn to the slow part of the Evans bundles. By Corollary 2.1.3, N_0 is a hyperbolic saddle for the constant coefficient equation

$$X' = \Gamma^{n+1}(A^I_0)(X) \text{ on } \mathscr{G}^{2n}_{n+1}(\mathbb{C}).$$

Hence, the manifold

$$\mathcal{N}_0 = J \times \{N_0\} \subset J \times \mathscr{G}^{2n}_{n+1}(\mathbb{C})$$

is a normally hyperbolic critical manifold of the system

$$\begin{cases} \tau' = \varepsilon(1-\tau^2)h_\varepsilon(\tau), \\ X' = \Gamma^{n+1}(A^I_{\varepsilon,\zeta}(\tau))(X), \end{cases} \quad \text{on } J \times \mathscr{G}^{2n}_{n+1}(\mathbb{C}). \tag{2.85}$$

Again by Fenichel theory and compactness of $J \times \mathscr{G}^{2n}_{n+1}(\mathbb{C})$ the critical manifold $\mathcal{N}_0$ perturbs to a unique slow manifold $\mathcal{N}_{\varepsilon,\zeta}$ depending smoothly on ε and holomorphically on ζ. It is the graph of a smooth function

$$N_{\varepsilon,\zeta} : J \to \mathscr{G}^{2n}_{n+1}(\mathbb{C}).$$

Set $\hat{p} := (p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_n)^t \in \mathbb{C}^{n-1}$ for $p \in \mathbb{C}^n$. By choosing an appropriate canonical chart for $\mathscr{G}^{2n}_{n+1}(\mathbb{C})$ centered at N_0 there is a function

$$\tilde{N}_{\varepsilon,\zeta} : J \to \mathbb{C}^{(n-1)\times(n+1)}$$

depending smoothly on ε and holomorphically on ζ such that

$$N_{\varepsilon,\zeta}(\tau) = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{C}^{2n} \,\middle|\, \hat{p} = \tilde{N}_{\varepsilon,\zeta}(\tau) \begin{pmatrix} p_k \\ q \end{pmatrix} \right\}.$$

In linear (p_k, q)-coordinates the slow equations associated to (2.85) on $\mathcal{N}_{\varepsilon,\zeta}$ read

$$\begin{aligned} \dot{\tau} &= (1-\tau^2)h_\varepsilon(\tau), \\ \dot{p}_k &= 2a\tau p_k + q_k + \mathcal{O}(\varepsilon) \begin{pmatrix} p_k \\ q \end{pmatrix}, \\ \dot{q}_j &= \zeta (G_0)_{jk} p_k + \mathcal{O}(\varepsilon) \begin{pmatrix} p_k \\ q \end{pmatrix}, \end{aligned}$$

with $p_j, j \neq k$, being the fast variables and $p_k, q_1, \dots, q_n$ being the slow variables. By the eigenvalue expansions in Lemma 2.7.2 we obtain the analogue of Lemma 4 (ii) and Corollary 3 in [26]:

Lemma 2.7.5. *For all $r_0 > 0$ there exists $\varepsilon_0 = \varepsilon_0(r_0)$ such that the following two statements hold:*

(i) There are unique holomorphic bundles

$$H_\varepsilon^{+,s} : D_{r_0} \to \mathscr{G}^{2n}_{n-k+1}(\mathbb{C}),$$
$$H_\varepsilon^{-,s} : D_{r_0} \to \mathscr{G}^{2n}_{k}(\mathbb{C}),$$

depending smoothly on ε such that for all $\varepsilon > 0$ and $\zeta \in D_{r_0}$ the solutions Y^+ and Y^- of

$$Y^{+\prime} = \Gamma^{n-k+1}(A^I_{\varepsilon,\zeta}(\chi_\varepsilon))(Y^+), \quad Y^+(0) = H_\varepsilon^{+,s}(\zeta), \text{ on } \mathscr{G}^{2n}_{n-k+1}(\mathbb{C}),$$
$$Y^{-\prime} = \Gamma^{k}(A^I_{\varepsilon,\zeta}(\chi_\varepsilon))(Y^-), \quad Y^-(0) = H_\varepsilon^{-,s}(\zeta), \text{ on } \mathscr{G}^{2n}_{k}(\mathbb{C}),$$

converge on the right resp. left end:

$$Y^+(+\infty) = S^{+,s}_{\varepsilon,\zeta},$$
$$Y^-(-\infty) = U^{-,s}_{\varepsilon,\zeta}.$$

(ii) Restrict (2.73) *on the slow manifold $\mathcal{N}_{\varepsilon,\zeta}$, i.e.*

$$\dot\tau = (1-\tau^2)h_\varepsilon(\tau), \quad \dot\xi = \varepsilon^{-1}\tilde A_{\varepsilon,\zeta}(\tau)\xi,$$

on $\mathcal{N}_{\varepsilon,\zeta}$, and consider the 1-dimensional Grassmannian version of the restricted flow:

$$\dot Z = \varepsilon^{-1}\Gamma^1(A^I_{\varepsilon,\zeta}(\chi_\varepsilon))(Z), \quad Z(0) \in \{V \in \mathscr{G}^{2n}_1(\mathbb{C}) : \ V \subset N_{\varepsilon,\zeta}(0)\}.$$

Then there are unique holomorphic bundles

$$H_\varepsilon^{\pm,sf} : D_{r_0} \to \mathscr{G}^{2n}_1(\mathbb{C})$$

depending smoothly on $\varepsilon \in [0,\varepsilon_0]$ such that the solutions $Z^\pm$ of

$$\dot Z^\pm = \varepsilon^{-1}\Gamma^1(A^I_{\varepsilon,\zeta}(\chi_\varepsilon))(Z^\pm), \quad Z^\pm(0) = H^{\pm,sf}_{\varepsilon,\zeta},$$

satisfy

$$Z^+(+\infty) = S^{+,sf}_{\varepsilon,\zeta} \text{ and } Z^-(-\infty) = U^{-,sf}_{\varepsilon,\zeta}.$$

Furthermore, at $\varepsilon = 0$ we have

$$H_0^{+,s}(\zeta) = H_0^{+,sf}(\zeta) \oplus S^{ss} \text{ and } H_0^{-,s}(\zeta) = H_0^{-,sf}(\zeta) \oplus U^{ss}.$$

Proof. Let $r_0 > 0$, and choose $\varepsilon_0 > 0$ so small such that the conclusions of Lemma 2.7.2 hold, and such that the slow manifold $\mathcal{N}_{\varepsilon,\zeta}$ exists for all $(\varepsilon,\zeta) \in [0,\varepsilon_0] \times D_{r_0}$ depending smoothly on its parameters. We shall only treat the right end $\tau = +1$, the statement on the bundles with asymptotics at $\tau = -1$ is proved in a similar fashion with obvious modifications.
It is convenient to change basis such that $N^+_{\varepsilon,\zeta}$ is constant and spanned by the

vectors

$$\begin{pmatrix} e_k \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ e_n \end{pmatrix} \in \mathbb{C}^{n+1},$$

and such that the (linear) slow (p_k, q)-flow is given by a diagonal matrix at $\tau = +1$. To this end let $\mathscr{S}^+_{\varepsilon,\zeta}$ be the ε-smooth and ζ-holomorphic solution of

$$\partial_\varepsilon \mathscr{S}^+_{\varepsilon,\zeta} = Q^+_{\varepsilon,\zeta} \mathscr{S}^+_{\varepsilon,\zeta}, \quad \mathscr{S}^+_{0,\zeta} = \mathbb{1}_{2n}$$

where

$$\begin{aligned} Q^+_{\varepsilon,\zeta} :=& (\partial_\varepsilon P^{+,S,f}_{\varepsilon,\zeta}) P^{+,S,f}_{\varepsilon,\zeta} + (\partial_\varepsilon P^{+,U,f}_{\varepsilon,\zeta}) P^{+,U,f}_{\varepsilon,\zeta} \\ &+ (\partial_\varepsilon P^{+,S,sf}_{\varepsilon,\zeta}) P^{+,S,sf}_{\varepsilon,\zeta} + (\partial_\varepsilon P^{+,U,sf}_{\varepsilon,\zeta}) P^{+,U,sf}_{\varepsilon,\zeta} \\ &+ \sum_{j>k} (\partial_\varepsilon P^{+,S,ss}_{j;\varepsilon,\zeta}) P^{+,S,ss}_{\varepsilon,\zeta} + \sum_{j<k} (\partial_\varepsilon P^{+,U,ss}_{j;\varepsilon,\zeta}) P^{+,U,ss}_{\varepsilon,\zeta} \end{aligned}$$

and the above projections have been constructed in the proof of Lemma 2.7.2. Then setting

$$\mathcal{W}^+_{\varepsilon,\zeta} = \mathscr{S}^+_{\varepsilon,\zeta} R^+_\zeta$$

with $R^+_\zeta = R_\zeta$ from (2.78) the change of basis

$$\mathcal{W}^+_{\varepsilon,\zeta} \xi^+ = \xi$$

gives the desired equality

$$N^+_{\varepsilon,\zeta} = \operatorname{span} \left\{ \begin{pmatrix} e_k \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ e_n \end{pmatrix} \right\}$$

for all ε and ζ. In fact we have fixed the space

$$\begin{aligned} S^{+,sf}_{\varepsilon,\zeta} &= \{0\}^{k-1} \times \{0\} \times \{0\}^{n-k} \times \{0\}^{k-1} \times \mathbb{C} \times \{0\}^{n-k}, \\ U^{+,sf}_{\varepsilon,\zeta} &= \{0\}^{k-1} \times \mathbb{C} \times \{0\}^{n-k} \times \{0\}^{k-1} \times \{0\} \times \{0\}^{n-k}, \\ U^{+,ss}_{\varepsilon,\zeta} &= \{0\}^{k-1} \times \{0\} \times \{0\}^{n-k} \times \mathbb{C}^{k-1} \times \{0\} \times \{0\}^{n-k}, \\ S^{+,ss}_{\varepsilon,\zeta} &= \{0\}^{k-1} \times \{0\} \times \{0\}^{n-k} \times \{0\}^{k-1} \times \{0\} \times \mathbb{C}^{n-k}, \end{aligned}$$

at constant values, too. The system (2.73) reads

$$\begin{cases} \tau' = \varepsilon(1-\tau^2) h_\varepsilon(\tau), \\ \xi^{+\prime} = A^{I,+}_{\varepsilon,\zeta}(\tau) \xi^+, \end{cases} \tag{2.86}$$

with

$$A^{I,+}_{\varepsilon,\zeta}(\tau) = (\mathcal{W}^+_{\varepsilon,\zeta})^{-1} A^I_{\varepsilon,\zeta}(\tau) \mathcal{W}^+_{\varepsilon,\zeta}.$$

At $\varepsilon = 0$ we still have

$$A^{I,+}_{0,\zeta}(\tau) = A^I_0.$$

Hence, with $\xi^+ = (p^+, q^+)$ the fast variables are again $p^+_j, j \neq k$, and the slow variables are $p^+_k, q^+_1, \dots, q^+_n$. The slow equations for the system (2.86) on $\mathcal{N}_{\varepsilon,\zeta}$ in (p^+_k, q^+)-coordinates read

$$\begin{cases} \dot\tau = \varepsilon(1-\tau^2)h_\varepsilon(\tau), \\ \begin{pmatrix} \dot p^+_k \\ q^+ \end{pmatrix} = \tilde A^+_{\varepsilon,\zeta}(\tau) \begin{pmatrix} p^+_k \\ q^+ \end{pmatrix}, \end{cases} \tag{2.87}$$

with a matrix-valued function $\tilde A^+_{\varepsilon,\zeta}(\tau) \in \mathbb{C}^{(n+1)\times(n+1)}$ depending smoothly on ε and τ, and holomorphically on ζ. At $\tau = +1$ we have

$$N_{\varepsilon,\zeta}(+1) = \left\{(p,q) \in \mathbb{C}^{2n} \mid \hat p = 0\right\},$$

and

$$A^{I,+}_{\varepsilon,\zeta}(+1) = \operatorname{diag}(A^{I,+,S}_{\varepsilon,\zeta}, \mu^{+,U,sf}_{\varepsilon,\zeta}, A^{I,+,U}_{\varepsilon,\zeta}, (\mu^{+,U,ss}_{j;\varepsilon,\zeta})_{j<k}, \mu^{+,S,sf}_{\varepsilon,\zeta}, (\mu^{+,S,ss}_{j;\varepsilon,\zeta})_{j>k})$$

for appropriate matrices

$$A^{I,+,S}_{\varepsilon,\zeta} \in \mathbb{C}^{(k-1)\times(k-1)} \text{ and } A^{I,+,U}_{\varepsilon,\zeta} \in \mathbb{C}^{(n-k)\times(n-k)}.$$

Hence, we find for ε and ζ that at the right end the matrix describing the linear slow (p^+_k, q^+)-flow in (2.87) is given by the diagonal matrix

$$\tilde A^+_{\varepsilon,\zeta}(+1) = \varepsilon^{-1}\operatorname{diag}(\mu^{+,U,sf}_{\varepsilon,\zeta}, (\mu^{+,U,ss}_{j;\varepsilon,\zeta})_{j<k}, \mu^{+,S,sf}_{\varepsilon,\zeta}, (\mu^{+,S,ss}_{j;\varepsilon,\zeta})_{j>k}). \tag{2.88}$$

In (p^+_k, q^+)-coordinates the (constant) spaces $S^{+,sf}_{\varepsilon,\zeta}, U^{+,sf}_{\varepsilon,\zeta}, S^{+,ss}_{\varepsilon,\zeta}$ and $U^{+,ss}_{\varepsilon,\zeta}$ have unique constant counter parts give by

$$\begin{aligned} \tilde S^{+,sf}_{\varepsilon,\zeta} &= \tilde S^{+,sf}_0 := \{0\} \times \{0\}^{k-1} \times \mathbb{C} \times \{0\}^{n-k}, \\ \tilde U^{+,sf}_{\varepsilon,\zeta} &= \tilde U^{+,sf}_0 := \mathbb{C} \times \{0\}^{k-1} \times \{0\} \times \{0\}^{n-k}, \\ \tilde S^{+,ss}_{\varepsilon,\zeta} &= \tilde S^{+,sf}_0 := \{0\} \times \{0\}^{k-1} \times \{0\} \times \mathbb{C}^{n-k}, \\ \tilde U^{+,ss}_{\varepsilon,\zeta} &= \tilde U^{+,ss}_0 := \{0\} \times \mathbb{C}^{k-1} \times \{0\} \times \{0\}^{n-k}. \end{aligned}$$

Consider the Grassmannian flow

$$\begin{cases} \dot\tau = (1-\tau^2)h_\varepsilon(\tau,) \\ \tilde Y' = \Gamma^{n-k+1}(\tilde A^{I,+}_{\varepsilon,\zeta}(\tau))(\tilde Y), \end{cases} \text{ on } J \times \mathscr{G}^{n+1}_{n-k+1}(\mathbb{C}) \tag{2.89}$$

associated to (2.87). Like in [26] we need to distinguish between the two cases $\mathcal{O}(\varepsilon + |\zeta|)$ and $\mathcal{O}(\varepsilon)$ with $\zeta \in D_{r_0}\backslash\{0\}$ uniformly bounded away from zero.

To treat the first case define

$$\mathcal{C}^{+,s} = \{+1\} \times \{\tilde{Y} \in \mathscr{G}^{n+1}_{n-k+1}(\mathbb{C}) |\ \tilde{S}_0^{+,sf} \subset \tilde{Y}\}$$

which is an invariant manifold of (2.89) for all ε and ζ by (2.88). For $\varepsilon = 0$ and $\zeta = 0$ one has

$$\tilde{A}^{I,+}_{0,0}(+1) = \mathrm{diag}(0, \ldots, 0, 2a, 0, \ldots, 0) \tag{2.90}$$

with $a < 0$. Invoking Lemma 2.1.1 we find that $\mathcal{C}^{+,s}$ is a critical manifold without boundary for (2.89) with the τ-direction being attracting and $\mathcal{C}^{+,s}$ being repelling in $\{\tau = 1\}$. Applying the theory on normally hyperbolic manifolds in the form of Theorem 2.2.3 shows that if ε and $|\zeta|$ are small enough then there is a stable manifold $W^S_{\varepsilon,\zeta}(\mathcal{C}^{+,s})$ possessing an invariant fibration with one dimensional leaves $W^S_{\varepsilon,\zeta}((1,\tilde{Y}))$ each based at some point $(1,\tilde{Y}) \in \mathcal{C}^{+,s}$. If the base point of the leaf is a fixed point of (2.89) then the leaf is itself invariant under the flow of (2.89). The intersection of $\{\tau = 0\}$ with the leaf $W^S_{\varepsilon,\zeta}((1,\tilde{S}^{+,s}))$ based at the fixed point

$$\tilde{S}_0^{+,s} := \tilde{S}_0^{+,sf} \oplus \tilde{S}_0^{+,ss}$$

varies smoothly in ε and holomorphically in ζ. Hence, by setting

$$\tilde{H}^{+,s}_\varepsilon(\zeta) := W^S_{\varepsilon,\zeta}((1,\tilde{S}_0^{+,s})) \cap \{\tau = 0\}$$

we obtain a holomorphic bundle depending smoothly on ε, and the solution $(\tau, \tilde{Y}^{+,s})$ of (2.89) starting at $(0, \tilde{H}^{+,s}_\varepsilon(\zeta))$ satisfies

$$(\tau, \tilde{Y}^{+,s})(+\infty) = (1, \tilde{S}_0^{+,s}).$$

To treat the second case, $\mathcal{O}(\varepsilon)$ and $\zeta \neq 0$, consider arbitrary $r_1 \in (0, r_0)$. Instead of (2.90) we have

$$\tilde{A}^{I,+}_{0,\zeta}(+1) = \mathrm{diag}(\tilde{\mu}_\zeta^{+,U,sf}, 0, \ldots, 0, \tilde{\mu}_\zeta^{+,S,sf}, 0, \ldots, 0),$$

with $\tilde{\mu}_\zeta^{+,S/U,sf} = \tilde{\mu}_\zeta^{S/U,sf}$ from (2.76). There exists a constant $c > 0$ such that for all $\zeta \in \mathbb{H}^+, r_1 \leq |\zeta| \leq r_0$, the inequalities

$$\mathrm{Re}\,\tilde{\mu}_\zeta^{+,U,sf} \geq c > 0 > a > \mathrm{Re}\,\tilde{\mu}_\zeta^{+,S,sf}$$

and

$$\mathrm{Re}\,(\tilde{\mu}_\zeta^{+,U,sf} - \tilde{\mu}_\zeta^{+,S,sf}) = 2\mathrm{Re}\,\sqrt{a^2 + (G_0)_{kk}\zeta} \geq c$$

hold. Replacing $\mathcal{C}^{+,s}$ by

$$\tilde{\mathcal{C}}^{+,s} = \{+1\} \times \{\tilde{Y} \in \mathscr{G}^{n+1}_{n-k+1}(\mathbb{C}) |\ \tilde{S}_0^{+,sf} \subset \tilde{Y} \subset \{0\} \times \mathbb{C}^n\}$$

one finds that $\tilde{\mathcal{C}}^{+,s}$ is a critical manifold for (2.89) for $\varepsilon = 0$ and all $\zeta \in \mathbb{H}^+, r_1 \leq |\zeta| \leq r_0$. One argues like above with the difference that the direction associated

to the eigenvalue $\tilde{\mu}_\zeta^{U,sf}$ belongs to the normally hyperbolic repelling part, too.

Noting that with our choice of basis we have

$$S_{\varepsilon,\zeta}^{+,s} = \{0\}^{n-1} \times \tilde{S}_0^{+,s}$$

and returning to the full system (2.86) we find the desired slow Evans bundle

$$H_\varepsilon^{+,s}(\zeta) := \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{C}^{2n} \,\middle|\, \begin{pmatrix} p_k \\ q \end{pmatrix} \in \tilde{H}_\varepsilon^{+,s}(\zeta), \hat{p} = N_{\varepsilon,\zeta}(0) \begin{pmatrix} p_k \\ q \end{pmatrix} \right\}.$$

This proves part (i) of the lemma. We continue with the resolution of the slow bundle $H_\varepsilon^{+,s}(\zeta)$ at $\varepsilon = 0$. Using the definition of the reduced vector field associated to (2.86) on the slow manifold $\mathcal{N}_{\varepsilon,\zeta}$ one finds after a short calculation that at $\varepsilon = 0$ the slow flow is given by

$$\tilde{A}_{0,\zeta}^{I,+}(\tau) = \begin{pmatrix} * & *e_k^t \\ *e_k & *E_{kk} \end{pmatrix} \tag{2.91}$$

where each $*$ stands for some (different) holomorphic scalar-valued function. From (2.91) we see that the spaces

$$\begin{aligned} V &:= \mathbb{C} \times \{0\}^{k-1} \times \mathbb{C} \times \{0\}^{n-k} \\ \tilde{U}_0^{ss} &:= \{0\} \times \mathbb{C}^{k-1} \times \{0\} \times \{0\}^{n-k} \\ \tilde{S}_0^{ss} &:= \{0\} \times \{0\} \times \{0\} \times \mathbb{C}^{n-k} \end{aligned}$$

are a splitting for $\mathbb{C}^{n+1}$ and define disjoint invariant sets

$$J \times V, J \times \tilde{U}_0^{ss}, J \times \tilde{S}_0^{ss}$$

of the slow equations (2.87). By the asymptotics

$$\tilde{Y}^{+,s}(+\infty) = \tilde{S}_0^{+,s} = \tilde{S}_0^{+,sf} \oplus \tilde{S}_0^{+,ss}$$

the holomorphic bundle $\tilde{H}_0^{+,s}(\zeta) \cap V$ must be one dimensional, and

$$\tilde{S}_0^{ss} \subset \tilde{H}_0^{+,s}(\zeta).$$

Finally, let us consider the Grassmannian flow

$$\begin{cases} \dot{\tau} = (1-\tau^2) h_\varepsilon(\tau), \\ \tilde{Y}' = \Gamma^1(\tilde{A}_{\varepsilon,\zeta}^{I,+}(\tau))(\tilde{Y}), \end{cases} \quad \text{on } J \times \mathscr{G}_1^{n+1}(\mathbb{C}). \tag{2.92}$$

associated to (2.87). Since the matrix $\tilde{A}_{0,\zeta}^{I,+}(\tau)$ at $\tau = 1$ is of diagonal form given in (2.90) and there is a positive spectral gap between the eigenvalue $\tilde{\mu}_\zeta^{+,S,sf}$ and the other n eigenvalues of $\tilde{A}_{0,\zeta}^{I,+}(1)$, we find by Lemma 2.1.1 that $\{1\} \times \tilde{S}_0^{+,sf}$ is a hyperbolic fixed point of (2.92) for all ζ. At $\{1\} \times \tilde{S}_0^{+,sf}$ the τ-direction is attracting and in $\{1\} \times \mathscr{G}_1^{n+1}(\mathbb{C})$ the point $\{1\} \times \tilde{S}_0^{+,sf}$ is repelling. Applying

the stable manifold theorem yields a unique holomorphic bundle

$$\tilde{H}_\varepsilon^{+,sf} : D_{r_0} \to \mathscr{G}_1^{n+1}(\mathbb{C})$$

depending smoothly on ε such that the solution $(\tau, \tilde{Z}^{+,sf})$ of (2.92) starting at $(0, \tilde{H}_\varepsilon^{+,sf}(\zeta))$ satisfies

$$\tilde{Z}^{+,sf}(+\infty) = \tilde{S}_0^{+,sf}.$$

In particular, by sharp asymptotics we find

$$\tilde{H}_0^{+,sf}(\zeta) = \tilde{H}_0^{+,s}(\zeta) \cap V.$$

Embedding $\tilde{H}_\varepsilon^{+,sf}$ into the full phase space like $\tilde{H}_\varepsilon^{+,s}$ before yields the slow-fast bundle $H_\varepsilon^{+,sf}$ and finishes the proof of the lemma. □

From the slow equations and the decay properties in Lemma 2.7.5 we can resolve the structure of the bundles $H_\varepsilon^{\pm,s}(\zeta)$ at $\varepsilon = 0$ by determining the slow-fast bundle $H_0^{\pm,sf}(\zeta)$:

Lemma 2.7.6. *Setting $\tilde{\zeta}(\zeta) = (4a)^{-2}(G_0)_{kk}\zeta$ for $\zeta \in D_{r_0}$ it holds*

$$H_0^{\pm,sf}(\zeta) = \operatorname{span}\left\{ \begin{pmatrix} h_{p,par}^{\pm}(\tilde{\zeta}(\zeta))e_k \\ 4|a| h_{q,par}^{\pm}(\tilde{\zeta}(\zeta))\underline{G}_0 e_k \end{pmatrix} \right\}$$

with $h_{p/q,par}^{\pm}$ from (2.62), (2.63).

Proof. We use the results on the standing shock profile τ_0 for the parabolic Burgers' equation from B. Barker's master thesis [3] presented in Section 2.5.

For $\varepsilon = 0$ the slow equations are

$$\begin{aligned} \dot{\tau} &= (1-\tau^2)|a|, \\ \dot{p}_k &= 2a\tau p_k + q_k, \\ \dot{q}_j &= \zeta (G_0)_{jk} p_k. \end{aligned}$$

Rescaling the independent variable with $4|a|$, setting $\tilde{\zeta} = (4a)^{-2}(G_0)_{kk}\zeta$ and changing basis via

$$\begin{pmatrix} p_k \\ q \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 4|a|\underline{G}_0 \end{pmatrix}^{-1} \begin{pmatrix} p_k \\ q \end{pmatrix}$$

yields the system

$$\begin{aligned} \dot{\tau} &= \frac{1}{4}(1-\tau^2), \\ \dot{p}_k &= -\frac{1}{2}\tau p_k + e_k^T \underline{G}_0 q, \\ \dot{q}_j &= \delta_{jk}\tilde{\zeta} p_k. \end{aligned}$$

The subsystem for (τ, p_k, q_k)-variables is exactly the system (2.59) for the standing shock profile τ_0 of the parabolic Burgers' equation discussed in Section 2.5.

With

$$\begin{pmatrix} h^{\pm}_{p,par}(\tilde{\zeta}) \\ h^{\pm}_{q,par}(\tilde{\zeta}) \end{pmatrix} \in \mathbb{C}^2$$

from (2.62), (2.63) spanning the Evans bundles for τ_0 we find by changing back to original variables that the dynamical parts of the slow bundles are

$$H_0^{\pm,sf}(\zeta) = \text{span}\left\{ \begin{pmatrix} h^{\pm}_{p,par}(\tilde{\zeta}(\zeta))e_k \\ 4|a|h^{\pm}_{q,par}(\tilde{\zeta}(\zeta))\underline{G}_0 e_k \end{pmatrix} \right\}$$

with $\tilde{\zeta}(\zeta) = (4a)^{-2}(G_0)_{kk}\zeta$. □

The previous considerations yield:

Corollary 2.7.7. *Let $r_0 > 0$ and choose $\varepsilon_0 = \varepsilon_0(r_0)$ such that the statements of Lemma 2.7.4 and Lemma 2.7.5 hold. For $\varepsilon \in [0, \varepsilon_0]$ define holomorphic bundles*

$$H^{\pm}_{\varepsilon} := H^{\pm,f}_{\varepsilon} \oplus H^{\pm,s}_{\varepsilon} \text{ on } D_{r_0}.$$

If $\varepsilon > 0$ then the bundles $H^{\pm}_{\varepsilon}$ are rescaled versions of the Evans bundles $\mathcal{H}^{\pm}_{\varepsilon}$ of ϕ_{ε} via

$$\mathcal{H}^{\pm}_{\varepsilon}(\varepsilon^2\zeta) = H^{\pm}_{\varepsilon}(\zeta)$$

with corresponding Evans functions

$$\det(\mathcal{H}^{-}_{\varepsilon}(\kappa), \mathcal{H}^{+}_{\varepsilon}(\kappa)) =: \mathcal{E}_{\varepsilon}(\kappa) = E_{\varepsilon}(\zeta) := \det(H^{-}_{\varepsilon}(\zeta), H^{+}_{\varepsilon}(\zeta))$$

for $\kappa = \varepsilon^2\zeta$. At $\varepsilon = 0$ there is $c \in \mathbb{R}\backslash\{0\}$ such that

$$E_0(\zeta) = c\mathcal{E}_{par}(\tilde{\zeta}(\zeta))$$

with $\mathcal{E}_{par}$ from (2.65). If $\varepsilon \to 0$ then

$$H^{\pm}_{\varepsilon} \to H^{\pm}_0 \text{ and } E_{\varepsilon} \to E_0$$

as holomorphic functions on D_{r_0}. For ε small enough the Evans function condition on D_{r_0}

$$E_{\varepsilon}(\zeta) = 0 \text{ for } \zeta \in D_{r_0} \text{ iff } \zeta = 0, \text{ and } E'_{\varepsilon}(0) \neq 0$$

holds.

Proof. Using

$$H_0^{-,f}(\zeta) = \{0\}^k \times \mathbb{C}^{n-k} \times \{0\}^n,$$
$$H_0^{+,f}(\zeta) = \mathbb{C}^{k-1} \times \{0\}^{n-k+1} \times \{0\}^n,$$

the form of $H_0^{\pm,s}(\zeta)$ in Lemma 2.7.6 and the basis of U_0^{ss} and S_0^{ss} in Lemma 2.7.2 one finds

$$\begin{aligned} E_0(\zeta) &= \tilde{c}|a| \det \begin{pmatrix} h^-_{p,par}(\tilde{\zeta}(\zeta)) & h^+_{p,par}(\tilde{\zeta}(\zeta)) & 0 \\ h^-_{q,par}(\tilde{\zeta}(\zeta)) & h^+_{q,par}(\tilde{\zeta}(\zeta)) & 0 \\ * & * & C^{ss} \end{pmatrix} \\ &= \tilde{c}|a| \det \begin{pmatrix} h^-_{p,par}(\tilde{\zeta}(\zeta)) & h^+_{p,par}(\tilde{\zeta}(\zeta)) \\ h^-_{q,par}(\tilde{\zeta}(\zeta)) & h^+_{q,par}(\tilde{\zeta}(\zeta)) \end{pmatrix} \det C^{ss} \\ &= c\mathcal{E}_{par}(\tilde{\zeta}(\zeta)) \end{aligned}$$

where $\tilde{c} \in \{-1, 1\}$ and $C^{ss} \in \mathrm{GL}_{n-1}(\mathbb{C})$ consists of the $n-1$ non-zero rows of $(\underline{G}_0 r_j)_{j \neq k} \in \mathbb{C}^{n \times (n-1)}$, namely all but the k-th row. Because $\mathcal{E}_{par}$ satisfies the Evans function condition (2.49) so does E_0, in particular $E'(0) \neq 0$. Hence, we deduce from $E_\varepsilon \to E_0$ as $\varepsilon \to 0$ that $E'_\varepsilon(0) \neq 0$ for ε small enough.

Let $\varepsilon > 0$. It remains to show that $E_\varepsilon(0) = 0$ in order to verify that E_ε satisfies condition (2.49) on D_{r_0} for small ε. If $\zeta = 0$ then system (2.73) is given by

$$\begin{cases} \tau' = \varepsilon(1-\tau^2)h_\varepsilon(\tau), \\ \begin{pmatrix} p \\ q \end{pmatrix}' = \begin{pmatrix} F_\varepsilon(\tau) & \varepsilon \mathbb{1}_n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \end{cases} \tag{2.93}$$

The set $J \times \{q = 0\}$ is invariant for (2.93). Furthermore, since $A^I_{\varepsilon,0}(\tau)\mathbb{C}^{2n} \subset \{q = 0\}$ we find that $U^{-,sf}_{\varepsilon,0}$ resp. $S^{+,sf}_{\varepsilon,0}$ is a subset of $\{q = 0\}$ with an unique counter part $\tilde{U}^{-,sf}_{\varepsilon,0}$ resp. $\tilde{S}^{+,sf}_{\varepsilon,0}$ in $\mathbb{C}^n$. The counter part is the invariant subspace of $F_\varepsilon(-1)$ resp. $F_\varepsilon(+1)$ associated to the slow-fast eigenvalue $\mu^{-,U,sf}_{\varepsilon,0}$ resp. $\mu^{+,S,sf}_{\varepsilon,0}$. On $J \times \{q = 0\}$ (2.93) reduces to

$$\begin{cases} \tau' = \varepsilon(1-\tau^2)h_\varepsilon(\tau), \\ p' = F_\varepsilon(\tau)p. \end{cases} \tag{2.94}$$

Consider the Grassmannian flow

$$\begin{cases} \tau' = \varepsilon(1-\tau^2)h_\varepsilon(\tau), \\ X' = \Gamma^1(F_\varepsilon(\tau))(X), \end{cases} \quad \text{on } J \times \mathscr{G}^n_1(\mathbb{C}), \tag{2.95}$$

and note that by $F_0(\tau) = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and Lemma 2.1.1 the manifold

$$\mathcal{M}_0 := J \times \{\mathrm{span}\,\{e_k\}\}$$

is a one dimensional normally hyperbolic critical manifold for (2.95). It gives rise to a slow manifold $\mathcal{M}_\varepsilon$. Because

$$\tilde{U}^{-,sf}_{0,0} = \mathrm{span}\,\{e_k\} = \tilde{S}^{+,sf}_{0,0}$$

the fixed points $(-1, \tilde{U}^{-,sf}_{\varepsilon,0})$ and $(+1, \tilde{S}^{+,sf}_{\varepsilon,0})$ of (2.95) are elements of $\mathcal{M}_\varepsilon$. For $\varepsilon > 0$ there is a point $(0, \tilde{H}^{sf}_\varepsilon) \in \mathcal{M}_\varepsilon$ such that the solution (τ, X) of (2.95)

starting in $(0, \tilde{H}^{sf}_{\varepsilon})$ satisfies

$$X(-\infty) = \tilde{U}^{-,sf}_{\varepsilon,0} \text{ and } X(+\infty) = \tilde{S}^{+,sf}_{\varepsilon,0}.$$

Returning to full phase space we find by Lemma 2.7.5 (ii) and sharp asymptotics that

$$H^{-,sf}_{\varepsilon}(0) = \tilde{H}^{sf}_{\varepsilon} \times \{0\}^n = H^{+,sf}_{\varepsilon}(0),$$

i.e. $H^{-}_{\varepsilon}(0) \cap H^{+}_{\varepsilon}(0) \neq \{0\}$ and

$$E_{\varepsilon}(0) = 0.$$

□

Remarks on the application of GSPT

We give some local representations of the Grassmannian versions of the above vector fields and make a comment on analyticity of the objects obtained by GSPT.

Local representation of (2.85): Remember that $(\tilde{e}_1, \ldots, \tilde{e}_{2n})$ form a basis of $\mathbb{C}^{2n}$ consisting of the unit vectors of $\mathbb{C}^{2n}$ pointing into the respective coordinate directions. Consider the ordered basis

$$\mathcal{B} = (\tilde{e}_k, \tilde{e}_{n+1}, \ldots, \tilde{e}_{2n}, \tilde{e}_1, \ldots, \tilde{e}_{k-1}, \tilde{e}_{k+1}, \ldots, \tilde{e}_n)$$

of $\mathbb{C}^{2n}$. Then (2.85) reads in the canonical chart of $\mathscr{G}^{2n}_{n+1}(\mathbb{C})$ adapted to $\mathcal{B}$ and centered at N_0 as

$$\begin{cases} \tau' = \varepsilon(1-\tau^2)h_{\varepsilon}(\tau), \\ T' = (-T, \mathbb{1}_{n-1})(A^I_{\varepsilon,\zeta}(\tau))_{\mathcal{B}} \begin{pmatrix} \mathbb{1}_{n+1} \\ T \end{pmatrix}, \end{cases} \tag{2.96}$$

on $J \times \mathbb{C}^{(n-1)\times(n+1)}$. At $\varepsilon = 0$ we find

$$\begin{cases} \tau' = 0, \\ T' = \operatorname{diag}(\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_n)T. \end{cases}$$

The local representation of $\mathcal{N}_0$ given by $J \times \{0\} \subset J \times \mathbb{C}^{(n-1)\times(n+1)}$ is obviously a critical manifold for the vector field (2.96). The directions $\{T_{ij}\}_{1\leq i\leq k-1}$ are normally attracting, and the directions $\{T_{ij}\}_{k\leq i\leq n-1}$ are normally repelling. The local form of GSPT as given in Theorem 2.2.9 yields the existence of a smooth function $\tilde{N}_{\varepsilon,\zeta} : J \to \mathbb{C}^{(n-1)\times(n+1)}$ depending smoothly on ε and holomorphically on ζ with $\tilde{N}_{0,\zeta} \equiv 0$. The graph of $\tilde{N}_{\varepsilon,\zeta}$ is a compact invariant slow manifold for (2.96) near $J \times \{0\}$ with dimension equal to 1. Returning to the compact phase space $J \times \mathscr{G}^{2n}_{n+1}(\mathbb{C})$ and noting that any slow manifold of (2.85) near $\mathcal{N}_0$ is of dimension 1 and must contain the maximal invariant subset of a neighborhood of $\mathcal{N}_0$ (see (i) in Theorem 2.2.8) we find that any such slow manifold must be equal to the image of the aforementioned graph under the coordinate map $\phi_{\mathcal{B}}$.

Local representation of (2.89)**:** For $i \in \{1, \dots, n+1\}$ denote by $\hat{e}_i \in \mathbb{C}^{n+1}$ the unit vector in the i-th coordinate direction. Consider an ordered basis

$$\mathcal{B} = (\hat{e}_{k+1}, \hat{e}_{i_1}, \dots, \hat{e}_{i_n})$$

of $\mathbb{C}^{n+1}$ where $(i_1, \dots, i_n)$ is a rearrangement of $(1, \dots, k, k+2, \dots, n+1)$. In canonical local coordinates of $\mathscr{G}^{n+1}_{n-k+1}$ adapted to $\mathcal{B}$ and centered at

$$\operatorname{span}\{\hat{e}_{k+1}, \hat{e}_{i_1}, \dots, \hat{e}_{i_{n-k}}\}$$

the system (2.89) reads

$$\begin{cases} \tau' = \varepsilon(1-\tau^2)h_\varepsilon(\tau), \\ T' = (-T, \mathbb{1}_k)(\tilde{A}^{I,+}_{\varepsilon,\zeta}(\tau))_{\mathcal{B}} \begin{pmatrix} \mathbb{1}_{n-k+1} \\ T \end{pmatrix}, \end{cases} \tag{2.97}$$

on $J \times \mathbb{C}^{k\times(n-k+1)}$. The portion of $\mathcal{C}^{+,s}$ in the chosen coordinates of $\mathscr{G}^{n+1}_{n-k+1}(\mathbb{C})$ is given by the set

$$\mathcal{C}^{+,s}_{\mathcal{B}} = \{T \in \mathbb{C}^{k\times(n-k+1)} | T_{i1} = 0, 1 \le i \le k\}.$$

If $\varepsilon = 0$ and $\zeta = 0$ we find that the T-equation in (2.97) is given by

$$T' = -T\operatorname{diag}(2a, 0, \dots, 0)$$

and $\mathcal{C}^{+,s}_{\mathcal{B}} \cap \overline{B}_R(0)$ is a critical manifold of (2.97) for any $R > 0$ with repelling normal directions $\{T_{i1}\}_i$ and attracting τ-direction. As exponential rate in the τ-direction one may choose a positive number smaller than $2|a|$. Since $\mathcal{C}^{+,s}$ can be covered by finitely many balls $B_R(0)$ in charts of $\mathscr{G}^{n+1}_{n-k+1}$ like the one above, with $R > 0$ large enough, we find that Theorem 2.2.3 is applicable. In this way one may also show that $\mathcal{C}^{+,s}$ is indeed an embedded submanifold of $\mathscr{G}^{n+1}_{n-k+1}$ by invoking the slice chart criterion (see [62]). This covers the case $\mathcal{O}(\varepsilon + |\zeta|)$. Local representation for the case $\mathcal{O}(\varepsilon)$ with ζ uniformly bounded away from 0 can be obtained by considering an ordered basis of the form

$$\mathcal{B} = (\hat{e}_{k+1}, \hat{e}_{i_1}, \dots, \hat{e}_{i_{n-k}}, \hat{e}_1, \hat{e}_{i_{n-k+1}}, \dots, \hat{e}_{i_{n-1}})$$

of $\mathbb{C}^{n+1}$ where $(i_1, \dots, i_{n-1})$ is a rearrangement of $(2, \dots, k, k+2, \dots, n+1)$. Like above we obtain a local representation of the vector field in (2.89) with T-equation given by

$$T' = -T\operatorname{diag}(\mu^{+,S,sf}_\zeta, 0, \dots, 0) + \operatorname{diag}(\mu^{+,U,sf}_\zeta, 0, \dots, 0)T.$$

if $\varepsilon = 0$. The portion of $\tilde{\mathcal{C}}^{+,s}$ in the chosen coordinates of $\mathscr{G}^{n+1}_{n-k+1}(\mathbb{C})$ is given by the set

$$\tilde{\mathcal{C}}^{+,s}_{\mathcal{B}} = \{T \in \mathbb{C}^{k\times(n-k+1)} | T_{i1} = 0, 1 \le i \le k \ \&\ T_{1j} = 0, 1 \le j \le n-k+1\}.$$

We see that $\tilde{\mathcal{C}}_{\mathcal{B}}^{+,s}$ is a critical manifold for (2.97) with repelling normal directions $\{T_{i1}\}_i$ and $\{T_{1j}\}_j$, and attracting τ-direction. We conclude similar to the case of $\mathcal{O}(\varepsilon + |\zeta|)$ above.

A comment on analyticity: The theorems in Section 2.2 do not address complex manifolds, or analytic dependence of the objects obtained by geometric singular perturbation theory. When applying GSPT to systems like (2.73) with complex linear ξ-equation or its Grassmannian versions like (2.83) whose local representation is complex and algebraic in T (see Lemma 2.1.1) we first think of the real versions via $\mathbb{C} \cong \mathbb{R}^2$ of the aforementioned equations. Complexifying again yields the desired objects such as slow manifolds for the originally considered equations (see [2] for a similar comment).

Addressing the issue of holomorphic dependence first note that in (2.73) and (2.83) there is a hierarchy present: The τ-equation is independent of the complex variables ζ and ξ and dominates in this way the ξ-equation and its Grassmannian version. One finds that the flows of (2.73) and (the local representation) of (2.83) conserve analyticity on ζ since they depend in an algebraic, hence analytic, fashion on ξ and T, respectively. The (locally) invariant manifolds in the Theorems 2.2.3 and 2.2.8 are constructed locally as graphs via contraction mappings. These mappings themselves are graph transforms built from the flows of (2.73) and (2.83) which is why the contractions conserve analyticity as well. This allows for sequences of iterates depending holomorphically on ζ (and smoothly on ε) converging uniformly on compacta. Therefore the unique limiting objects of these sequences inherit their smoothness properties from the iterates.

2.7.2 Outer regime

Set for $\kappa \in \mathbb{C}$ and $\varepsilon \in [0, \varepsilon_0]$ (cf. [27])

$$\beta^2 = |\kappa|, \varepsilon = \alpha\beta, \text{ and } \tilde{\kappa} = \frac{\kappa}{|\kappa|} \text{ if } \kappa \neq 0.$$

Then finding appropriate $r_0, r_1 > 0$ and $\varepsilon_0 > 0$ for the outer regime $r_0\varepsilon^2 \leq |\kappa| \leq r_1$ translates to finding r_1, r_0^{-1} small enough:

$$\begin{aligned} \beta &\leq r_1^{1/2}, \\ \alpha &\leq r_0^{-1/2}. \end{aligned}$$

We consider

$$A^{II}_{\alpha,\beta,\tilde{\kappa}}(\tau) := \begin{pmatrix} F_{\alpha\beta}(\tau) & \beta\mathbb{1}_n \\ \beta^3\tilde{\kappa}^2\mathbb{1}_n + \beta\tilde{\kappa}G_{\alpha\beta}(\tau) & 0 \end{pmatrix}$$

for $\alpha, \beta \geq 0$ and

$$\tilde{\kappa} \in \partial D_1^+ := \{\kappa \in \mathbb{H}^+ : \ |\kappa| = 1\}.$$

If $\beta > 0$ then

$$A^{II}_{\alpha,\beta,\tilde{\kappa}}(\tau) \sim A_{\alpha\beta,\beta^2\tilde{\kappa}}(\tau).$$

The task is to find $\alpha_0, \beta_0 > 0$ small enough such that for all

$$(\alpha, \beta, \tilde{\kappa}) \in [0, \alpha_0] \times (0, \beta_0] \times \partial D_1^+$$

the non-autonomous system

$$\xi' = A^{II}_{\alpha,\beta,\tilde{\kappa}}(\chi_{\alpha\beta})\xi \ \ (\xi \in \mathbb{C}^{2n}) \tag{2.98}$$

has Evans bundles with trivial intersection. First we make an observation on the spectrum of $A^{II}_{\alpha,\beta,\tilde{\kappa}}(\tau)$.

Lemma 2.7.8. *There are $\alpha_0, \beta_0 > 0$ such that for all*

$$(\alpha, \beta, \tilde{\kappa}, \tau) \in [0, \alpha_0] \times (0, \beta_0] \times \partial D_1^+ \times J$$

the stable subspace $S_{\alpha,\beta,\tilde{\kappa}}(\tau)$ and the unstable subspace $U_{\alpha,\beta,\tilde{\kappa}}(\tau)$ of $A^{II}_{\alpha,\beta,\tilde{\kappa}}(\tau)$ satisfy

$$\dim S_{\alpha,\beta,\tilde{\kappa}}(\tau) = n = \dim U_{\alpha,\beta,\tilde{\kappa}}(\tau).$$

In particular, $i\mathbb{R} \cap \sigma(A^{II}_{\alpha,\beta,\tilde{\kappa}}(\tau)) = \emptyset$.

Proof. Let $(\alpha, \beta, \tilde{\kappa}, \tau) \in [0, \alpha_0] \times (0, \beta_0] \times \partial D_1^+ \times J$ with $\alpha_0\beta_0 (= \varepsilon_0)$ small enough. Because $F_{\alpha\beta}(\tau), G_{\alpha\beta}(\tau)$ satisfy the assumptions of Lemma 2.4.2 there exists a $\nu > 0$ such that $\tilde{\Lambda} := M_\nu$ is a domain of constant splitting for $\{A_{F_{\alpha\beta}(\tau),G_{\alpha\beta}(\tau)}(\kappa)\}_\kappa$ with

$$\mathbb{H}^+ \backslash \{0\} \subset \tilde{\Lambda}. \tag{2.99}$$

Because $\beta > 0$ we have $A^{II}_{\alpha,\beta,\tilde{\kappa}}(\tau) \sim A_{\alpha\beta,\beta^2\tilde{\kappa}}(\tau) = A_{F_{\alpha\beta}(\tau),G_{\alpha\beta}(\tau)}(\beta^2\tilde{\kappa})$ and $\beta^2\tilde{\kappa} \in \mathbb{H}^+\backslash\{0\}$, thus the claim follows from (2.99). □

With the above lemma in mind we can hope to apply Lemma 2.1.4. To this end we need slowly varying block-diagonalizers for $A^{II}_{\alpha,\beta,\tilde{\kappa}}(\tau)$ splitting the latter into a contracting and an expanding part of dimension n.

Lemma 2.7.9. *There exist $\alpha_0, \beta_0 > 0$ and smooth functions*

$$\begin{aligned} R &: [0, \alpha_0] \times [0, \beta_0] \times \partial D_1^+ \times J \to \mathrm{GL}_{2n}(\mathbb{C}), \\ A^> &: [0, \alpha_0] \times [0, \beta_0] \times \partial D_1^+ \times J \to \mathbb{C}^n \\ A^< &: [0, \alpha_0] \times [0, \beta_0] \times \partial D_1^+ \times J \to \mathbb{C}^n, \end{aligned}$$

such that for all $(\alpha, \beta, \tilde{\kappa}, \tau) \in [0, \alpha_0] \times [0, \beta_0] \times \partial D_1^+ \times J$ it holds

$$R^{-1}_{\alpha,\beta,\tilde{\kappa}}(\tau) A^{II}_{\alpha,\beta,\tilde{\kappa}}(\tau) R_{\alpha,\beta,\tilde{\kappa}}(\tau) = \begin{pmatrix} A^>_{\alpha,\beta,\tilde{\kappa}}(\tau) & 0 \\ 0 & A^<_{\alpha,\beta,\tilde{\kappa}}(\tau) \end{pmatrix},$$

and if $\beta > 0$ then

$$\operatorname{Re} A^{<}_{\alpha,\beta,\tilde{\kappa}}(\tau) < 0 < \operatorname{Re} A^{>}_{\alpha,\beta,\tilde{\kappa}}(\tau).$$

Furthermore, there is a constant such that

$$\begin{aligned} |R_{\alpha,\beta,\tilde{\kappa}}(0)^{-1}| &\leq c, \\ |\partial_\tau R_{\alpha,\beta,\tilde{\kappa}}(\tau)| &\leq c(\alpha+\beta). \end{aligned}$$

Proof. We proceed in a similar fashion as in the proof of Lemma 2.7.2. Let $r > 0$ such that $u_\varepsilon(\tau) \in B_r(0)$ for all $\varepsilon \in [0, \varepsilon_0]$, and, if necessary, shrink ε_0 such that the statement of Lemma 2.7.1 holds. Then we may assume that with the notation of the latter lemma

$$F_{\alpha\beta}(\tau) = \begin{pmatrix} M^{<}_{\alpha\beta}(\tau) & 0 & 0 \\ 0 & M^{0}_{\alpha\beta}(\tau) & 0 \\ 0 & 0 & M^{>}_{\alpha\beta}(\tau) \end{pmatrix}$$

for

$$\begin{aligned} M^{<}_{\alpha\beta}(\tau) &= M^{<}(\alpha\beta, u_{\alpha\beta}(\tau)), \\ M^{0}_{\alpha\beta}(\tau) &= M^{0}(\alpha\beta, u_{\alpha\beta}(\tau)), \\ M^{>}_{\alpha\beta}(\tau) &= M^{>}(\alpha\beta, u_{\alpha\beta}(\tau)). \end{aligned}$$

For $\beta = 0$ we find that

$$A^{II}_{\alpha,0,\tilde{\kappa}}(\tau) = \begin{pmatrix} \operatorname{diag}(\lambda_1, \ldots, \lambda_n) & 0 \\ 0 & 0 \end{pmatrix}$$

is independent of $(\alpha, \tilde{\kappa}, \tau)$ and of diagonal form. Hence for small β the total projections

$$P^{S,f}_{\alpha,\beta,\tilde{\kappa}}(\tau), P^{s}_{\alpha,\beta,\tilde{\kappa}}(\tau) \text{ and } P^{U,f}_{\alpha,\beta,\tilde{\kappa}}(\tau)$$

onto the eigenvalues with real part (strongly) smaller, close-to and (strongly) larger than zero are well-defined and smooth in their parameters. For $\beta = 0$ they have constant values

$$\begin{aligned} P^{S,f}_0 &:= P^{S,f}_{\alpha,0,\tilde{\kappa}}(\tau) = \begin{pmatrix} \Pi^{<} & 0 \\ 0 & 0 \end{pmatrix}, \\ P^{s}_0 &:= P^{s}_{\alpha,0,\tilde{\kappa}}(\tau) = \begin{pmatrix} E_{kk} & 0 \\ 0 & \mathbb{1}_n \end{pmatrix}, \\ P^{U,f}_0 &:= P^{U,f}_{\alpha,0,\tilde{\kappa}}(\tau) = \begin{pmatrix} \Pi^{>} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

We apply Kato's reduction process [54] to the 0-group. A Taylor expansion shows that

$$A^{II,s}_{\alpha,\beta,\tilde\kappa}(\tau) := \frac{1}{\beta} A^{II}_{\alpha,\beta,\tilde\kappa}(\tau) P^s_{\alpha,\beta,\tilde\kappa}(\tau)$$

defines a smooth function with

$$A^{II,s}_{\alpha,0,\tilde\kappa}(\tau) = P^s_0 \partial_\beta A^{II}_{\alpha,0,\tilde\kappa}(\tau) P^s_0.$$

From Lemma 2.7.1 and $u_\varepsilon(\tau) = \varepsilon\tau e_k + \mathcal{O}(\varepsilon^2)$ one finds $\partial_\beta M^0_0(\tau) = 2\alpha a\tau$ and hence

$$A^{II,s}_{\alpha,0,\tilde\kappa}(\tau) = \begin{pmatrix} 2\alpha a\tau E_{kk} & E_{kk} \\ \tilde\kappa G_0 E_{kk} & 0 \end{pmatrix}.$$

For small $\alpha \geq 0$ and all $(\tilde\kappa, \tau)$ the matrix $A^{II,s}_{\alpha,0,\tilde\kappa}(\tau)$ has two non-zero simple eigenvalues

$$\tilde\mu^{U/S,sf}_{\alpha,\tilde\kappa}(\tau) = \alpha a\tau \pm \sqrt{(\alpha a\tau)^2 + \tilde\kappa (G_0)_{kk}}$$

Hence, for small $\alpha \geq 0, \beta > 0$ there is a simple unstable and a simple stable slow-fast eigenvalues $\mu^{U/S,sf}_{\alpha,\beta,\tilde\kappa}(\tau)$ of $A^{II}_{\alpha,\beta,\tilde\kappa}(\tau)$ with

$$\mu^{U/S,sf}_{\alpha,\beta,\tilde\kappa}(\tau) = \beta \left(\tilde\mu^{U/S,sf}_{\alpha,\tilde\kappa}(\tau) + \mathcal{O}(\beta) \right)$$

and smooth one dimensional eigenprojections $P^{U/S,sf}_{\alpha,\beta,\tilde\kappa}(\tau)$. For $\alpha \geq 0, \beta > 0$ small enough we have for all $(\tilde\kappa, \tau)$

$$\operatorname{Re} \mu^{U/S,sf}_{\alpha,\beta,\tilde\kappa}(\tau) = \beta \left(\pm \operatorname{Re} \sqrt{\tilde\kappa (G_0)_{kk}} + \mathcal{O}(\alpha + \beta) \right) \gtrless 0.$$

Let $P^{ss}_{\alpha,\beta,\tilde\kappa}(\tau)$ be the total projection onto the group of eigenvalues of $A^{II,s}_{\alpha,\beta,\tilde\kappa}(\tau)$ associated to the $2(n-1)$-fold semi-simple zero eigenvalue of $A^{II,s}_{\alpha,0,\tilde\kappa}(\tau)$. Set $\Delta^{sf}_{\alpha,\tilde\kappa}(\tau) := \tilde\mu^{S,sf}_{\alpha,\tilde\kappa}(\tau) - \tilde\mu^{U,sf}_{\alpha,\tilde\kappa}(\tau)$ and remember

$$\tilde R = \left((r_j)_{j<k}, 0, (r_j)_{j>k} \right) \in \mathbb{R}^{n\times n}.$$

The matrices

$$R_{\alpha,\tilde\kappa}(\tau) = \begin{pmatrix} \mathbb{1}_n & E_{kk} \\ -\frac{\tilde\mu^{S,sf}}{(G_0)_{kk}} G_0 E_{kk} & G_0 \left(\tilde R - \frac{\tilde\mu^{U,sf}}{(G_0)_{kk}} E_{kk} \right) \end{pmatrix},$$

$$R_{\alpha,\tilde\kappa}(\tau)^{-1} = \begin{pmatrix} \mathbb{1}_n - \frac{\tilde\mu^{S,sf}}{\Delta^{sf}} E_{kk} & -\frac{1}{\Delta^{sf}} E_{kk} \\ \frac{\tilde\mu^{S,sf}}{\Delta^{sf}} E_{kk} & \tilde R^t + \frac{1}{\Delta^{sf}} E_{kk} \end{pmatrix},$$

diagonalize $A^{II,s}_{\alpha,0,\tilde{\kappa}}(\tau)$ to

$$R_{\alpha,\tilde{\kappa}}(\tau)^{-1}A^{II,s}_{\alpha,0,\tilde{\kappa}}(\tau)R_{\alpha,\tilde{\kappa}}(\tau) = \begin{pmatrix} \tilde{\mu}^{U,sf}_{\alpha,\tilde{\kappa}}(\tau)E_{kk} & 0 \\ 0 & \tilde{\mu}^{S,sf}_{\alpha,\tilde{\kappa}}(\tau)E_{kk} \end{pmatrix}.$$

With the help of the above diagonalizers we find

$$\begin{aligned} P^{U,sf}_{\alpha,\tilde{\kappa}}(\tau) :=& P^{U,sf}_{\alpha,0,\tilde{\kappa}}(\tau) \\ =& R_{\alpha,\tilde{\kappa}}(\tau)\begin{pmatrix} E_{kk} & 0 \\ 0 & 0 \end{pmatrix} R_{\alpha,\tilde{\kappa}}(\tau)^{-1}, \\ P^{S,sf}_{\alpha,\tilde{\kappa}}(\tau) :=& P^{S,sf}_{\alpha,0,\tilde{\kappa}}(\tau) \\ =& R_{\alpha,\tilde{\kappa}}(\tau)\begin{pmatrix} 0 & 0 \\ 0 & E_{kk} \end{pmatrix} R_{\alpha,\tilde{\kappa}}(\tau)^{-1}, \\ P^{ss}_{0} :=& P^{ss}_{\alpha,0,\tilde{\kappa}}(\tau) \\ =& R_{\alpha,\tilde{\kappa}}(\tau)\begin{pmatrix} \mathbb{1}_n - E_{kk} & 0 \\ 0 & \mathbb{1}_n - E_{kk} \end{pmatrix} R_{\alpha,\tilde{\kappa}}(\tau)^{-1} \\ =& \begin{pmatrix} \mathbb{1}_n - E_{kk} & 0 \\ 0 & p^{ss} \end{pmatrix}, \end{aligned}$$

where $p^{ss} = \mathbb{1}_n - \underline{G}_0 E_{kk}$. Carrying on with the reduction process define the smooth function

$$A^{II,ss}_{\alpha,\beta,\tilde{\kappa}}(\tau) = \frac{1}{\beta}P^{ss}_{\alpha,\beta,\tilde{\kappa}}(\tau)A^{II,s}_{\alpha,\beta,\tilde{\kappa}}(\tau)P^{ss}_{\alpha,\beta,\tilde{\kappa}}(\tau).$$

With

$$\tilde{F}_0^{-1} = \begin{pmatrix} (M_0^<(0))^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (M_0^<(0))^{-1} \end{pmatrix}$$

it holds

$$A^{II,ss}_{\alpha,0,\tilde{\kappa}}(\tau) = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & -\tilde{\kappa}p^{ss}G_0\tilde{F}_0^{-1}p^{ss} \end{pmatrix}$$

where we have used

$$\partial_\beta P^{s}_{\alpha,0,\tilde{\kappa}}(\tau) = \begin{pmatrix} 0 & -\tilde{F}_0^{-1} \\ -\tilde{\kappa}G_0 & 0 \end{pmatrix}.$$

Like in the previous treatment of the inner regime we have

$$\begin{aligned} (\tilde{R}^t + E_{kk})p^{ss}G_0\tilde{F}_0^{-1}p^{ss}G_0(\tilde{R}+&(G_0)^{-1}_{kk}E_{kk}) \\ &= \mathrm{diag}(\mu_1^{-1},\dots,\mu_{k-1}^{-1},0,\mu_{k+1}^{-1},\dots,\mu_n^{-1}), \end{aligned}$$

and the non-zero and simple eigenvalues of $A^{II,ss}_{\alpha,0,\tilde{\kappa}}(\tau)$ are

$$\{-\tilde{\kappa}(2\mu_j)^{-1}\}_{j\neq k}.$$

Hence, there are $n-1$ simple eigenvalues of A^{II} expanding as

$$\begin{aligned}\mu^{U,ss}_{j;\alpha,\beta,\tilde{\kappa}}(\tau) &= \beta^2(-\tilde{\kappa}(2\mu_j)^{-1}+\mathcal{O}(\beta)) \ \ (j<k)\\ \mu^{S,ss}_{j;\alpha,\beta,\tilde{\kappa}}(\tau) &= \beta^2(-\tilde{\kappa}(2\mu_j)^{-1}+\mathcal{O}(\beta)) \ \ (j>k)\end{aligned}$$

with smooth eigenprojections $P^{S/U;ss}_{j;\alpha,\beta,\tilde{\kappa}}(\tau)$. For small $\beta>0$ the inequalities

$$\operatorname{Re}\mu^{S,ss}_{j;\alpha,\beta,1}(\tau)<0<\operatorname{Re}\mu^{U,ss}_{j;\alpha,\beta,1}(\tau)$$

hold. Furthermore, the super-slow projections are given for $\beta=0$ by

$$P^{S/U;ss}_j := P^{S/U;ss}_{j;\alpha,0,\tilde{\kappa}}(\tau) = R_{\alpha,\tilde{\kappa}}(\tau)\begin{pmatrix}0&0\\0&E_{jj}\end{pmatrix}R_{\alpha,\tilde{\kappa}}(\tau)^{-1} = \begin{pmatrix}0&0\\0&G_0\tilde{R}E_{jj}\tilde{R}^t\end{pmatrix}.$$

Set

$$\begin{aligned}Q =&\partial_\beta P^{S,f}P^{S,f}+\partial_\beta P^{U,f}P^{U,f}+\partial_\beta P^{S,sf}P^{S,sf}+\partial_\beta P^{U,sf}P^{U,sf}\\ &+\sum_{j>k}\partial_\beta P^{S,ss}_j P^{S,ss}_j+\sum_{j<k}\partial_\beta P^{U,ss}_j P^{U,ss}_j,\end{aligned}$$

and for $(\alpha,\tilde{\kappa},\tau)$ let $\mathscr{S}_{\alpha,\beta,\tilde{\kappa}}(\tau)$ solve

$$\partial_\beta\mathscr{S}_{\alpha,\beta,\tilde{\kappa}}(\tau)=Q_{\alpha,\beta,\tilde{\kappa}}(\tau)\mathscr{S}_{\alpha,\beta,\tilde{\kappa}}(\tau),\ \ \mathscr{S}_{\alpha,0,\tilde{\kappa}}(\tau)=\mathbb{1}_{2n}.$$

Define

$$R_{\alpha,\beta,\tilde{\kappa}}(\tau) := \mathscr{S}_{\alpha,\beta,\tilde{\kappa}}(\tau)R_{\alpha,\tilde{\kappa}}(\tau)$$

and note that for all $(\tilde{\kappa},\tau)$

$$R_{0,0,\tilde{\kappa}}(\tau)=R_{0,\tilde{\kappa}}(\tau)=R_{\tilde{\kappa}}:=R_{0,\tilde{\kappa}}(0),$$

which implies the claimed properties of R by smoothness of R, the compactness of the parameter domain and the identity

$$R_{\alpha,\beta,\tilde{\kappa}}(\tau)=R_{\tilde{\kappa}}+\int_0^1\nabla_{\alpha,\beta}R_{y\alpha,y\beta,\tilde{\kappa}}(\tau)dy\begin{pmatrix}\alpha\\\beta\end{pmatrix}.$$

After taking care of rearrangements by multiplication of R with an elementary matrix, and naming the resulting product R again, we find that

$$R^{-1}A^{II}R=\begin{pmatrix}A^{>}&0\\0&A^{<}\end{pmatrix}$$

where

$$A^{>} = \begin{pmatrix} A^{>,f} & 0 \\ 0 & \operatorname{diag}(\mu_1^{U,ss}, \dots, \mu_{k-1}^{U,ss}, \mu^{U,sf}) \end{pmatrix},$$
$$A^{<} = \begin{pmatrix} A^{<,f} & 0 \\ 0 & \operatorname{diag}(\mu^{S,sf}, \mu_{k+1}^{S,ss}, \dots, \mu_n^{S,ss}) \end{pmatrix}$$

with $A^{>,f} = A^{>,f}_{\alpha,\beta,\tilde{\kappa}}(\tau) \in \mathbb{C}^{(n-k)\times(n-k)}$ and $A^{<,f} = A^{<,f}_{\alpha,\beta,\tilde{\kappa}}(\tau) \in \mathbb{C}^{(k-1)\times(k-1)}$ satisfying

$$A^{>,f}_{\alpha,\beta,\tilde{\kappa}}(\tau) = \operatorname{diag}(\lambda_{k+1}, \dots, \lambda_n) + \mathcal{O}(\beta),$$
$$A^{<,f}_{\alpha,\beta,\tilde{\kappa}}(\tau) = \operatorname{diag}(\lambda_1, \dots, \lambda_{k-1}) + \mathcal{O}(\beta).$$

Furthermore, for $\alpha \geq 0, \beta > 0$ small enough and all $(\tilde{\kappa}, \tau)$ the inequalities

$$\operatorname{Re}\mu^{S,sf}, \operatorname{Re}\mu_j^{S,ss} < 0 < \operatorname{Re}\mu^{U,sf}, \operatorname{Re}\mu_j^{U,ss}$$

hold because these inequalities are true for $\tilde{\kappa} = 1$ and $\beta > 0$ small, and, by continuity, real parts of eigenvalues of $A^{II}_{\alpha,\beta,\tilde{\kappa}}(\tau)$ may not change signs as $\tilde{\kappa}$ is varied in ∂D_1^+. The latter argument applies by Lemma 2.7.8 which states $i\mathbb{R} \cap \sigma(A^{II}_{\alpha,\beta,\tilde{\kappa}}(\tau)) = \emptyset$ if $\beta > 0$. Hence we find for $\beta > 0$ small enough the desired inequalities

$$\operatorname{Re} A^{<}_{\alpha,\beta,\tilde{\kappa}}(\tau) < 0 < \operatorname{Re} A^{>}_{\alpha,\beta,\tilde{\kappa}}(\tau). \tag{2.100}$$

□

Combing Lemma 2.7.9 and Lemma 2.1.4 the parametrization $\varepsilon = \alpha\beta$ and $\beta^2 = |\kappa|$ in the outer regime yields:

Corollary 2.7.10. *There exist $r_0, r_1, \varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$ and $\zeta \in \mathbb{H}^+$ with $\varepsilon^2 r_0 \leq \varepsilon^2|\zeta| \leq r_1$ the rescaled Evans bundles*

$$H_\varepsilon^{\pm}(\zeta) = \mathcal{H}_\varepsilon^{\pm}(\varepsilon^2\zeta)$$

are transversal:

$$H_\varepsilon^{-}(\zeta) \cap H_\varepsilon^{+}(\zeta) = \{0\}.$$

Remark 2.7.11. *It is in the outer regime where assumption B2, strict hyperbolicity of Df w.r.t. Dg, enters in an essential way yielding an eigenbasis for the super-slow eigenvalues and ruling out the existence of Jordan blocks that might hinder the important estimates in* (2.100)*. We think that small amplitude shock profiles of* (2.32) *continue to be spectrally stable if B2 is replaced by a weaker assumption, such as simplicity of the genuinely nonlinear eigenvalue μ_k, but the necessary modifications in the outer regime are not obvious.*

2.7.3 Outmost regime

The main purpose of this subsection is to show that Lemma 2.1.4 is applicable to the system

$$\xi' = A_{\varepsilon,\kappa}(\chi_\varepsilon)\xi \tag{2.101}$$

in a large portion of the outmost regime: $r \leq |\kappa|, \operatorname{Re}\kappa \geq 0$ for $r > 0$ large enough. The connection to the entire outmost regime $r_1 \leq |\kappa|$ with boundary defined by arbitrary $r_1 > 0$ is then an easy consequence of uniform hyperbolicity of $A_{\varepsilon,\kappa}$ on the compact set $r_1 \leq |\kappa| \leq r$ [11, 39, 83].
In order to apply Lemma 2.1.4 we need to find appropriate block-diagonalizers $R_{\varepsilon,\kappa}(\tau)$ for all $\varepsilon \in [0,\varepsilon_0], \tau \in J$ and all κ with $|\kappa|$ large enough. First note that by changing basis via

$$\begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & \kappa \end{pmatrix}$$

for $\kappa \neq 0$ and then via

$$\begin{pmatrix} \mathbb{1}_n & \mathbb{1}_n \\ \mathbb{1}_n & -\mathbb{1}_n \end{pmatrix}$$

one arrives at the representation

$$A_{\varepsilon,\kappa}(\tau) \sim A^{III}_{\varepsilon,\kappa}(\tau) = \kappa \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix} + \begin{pmatrix} B^{>}_\varepsilon(\tau) & B^{>}_\varepsilon(\tau) \\ B^{<}_\varepsilon(\tau) & B^{<}_\varepsilon(\tau) \end{pmatrix}$$

where

$$B^{\gtrless}_\varepsilon(\tau) := \frac{1}{2}\left(F_\varepsilon(\tau) \pm G_\varepsilon(\tau)\right).$$

By assumption A4, if ε_0 is small enough then there are constants $c^{\gtrless} > 0$ such that for all $\varepsilon \in [0,\varepsilon_0], \tau \in J$ and $\eta \in \mathbb{C}^n$ it holds

$$\operatorname{Re}\langle \eta, B^{>}_\varepsilon(\tau)\eta\rangle \geq c^{>}|\eta|^2, \tag{2.102}$$
$$\operatorname{Re}\langle \eta, B^{<}_\varepsilon(\tau)\eta\rangle \leq -c^{<}|\eta|^2. \tag{2.103}$$

For $r > 0$ set

$$\mathbb{H}^+_r := \{\kappa \in \mathbb{H}^+ |\ |\kappa| \geq r\}.$$

Lemma 2.7.12. *There exist $r, \varepsilon_0 > 0$ and smooth functions*

$$\begin{aligned} R &: [0,\varepsilon_0] \times \mathbb{H}^+_r \times J \to \mathrm{GL}_{2n}(\mathbb{C}), \\ A^{<} &: [0,\varepsilon_0] \times \mathbb{H}^+_r \times J \to \mathbb{C}^{n\times n}, \\ A^{>} &: [0,\varepsilon_0] \times \mathbb{H}^+_r \times J \to \mathbb{C}^{n\times n}, \end{aligned}$$

such that for all $(\varepsilon, \kappa, \tau) \in [0, \varepsilon_0] \times \mathbb{H}_r^+ \times J$ *it holds*

$$R_{\varepsilon,\kappa}(\tau)^{-1} A^{III}_{\varepsilon,\kappa}(\tau) R_{\varepsilon,\kappa}(\tau) = \begin{pmatrix} A^<_{\varepsilon,\kappa}(\tau) & 0 \\ 0 & A^>_{\varepsilon,\kappa}(\tau) \end{pmatrix}$$

and

$$\operatorname{Re} A^<_{\varepsilon,\kappa}(\tau) < 0 < \operatorname{Re} A^>_{\varepsilon,\kappa}(\tau).$$

Furthermore, there is a constant $c > 0$ *such that*

$$\begin{aligned} |R_{\varepsilon,\kappa}(0)^{-1}| &\leq c, \\ |\partial_\tau R_{\varepsilon,\kappa}(\tau)| &\leq c\varepsilon. \end{aligned}$$

Proof. We will carry out the matrix bifurcation of

$$B_{\varepsilon,\lambda}(\tau) = \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix} + \lambda \begin{pmatrix} B^>_\varepsilon(\tau) & B^>_\varepsilon(\tau) \\ B^<_\varepsilon(\tau) & B^<_\varepsilon(\tau) \end{pmatrix}$$

uniformly in ε and τ for $|\lambda| \leq r^{-1}$. Note that for $\kappa \neq 0$ the identity

$$A^{III}_{\varepsilon,\kappa}(\tau) = \kappa B_{\varepsilon,\kappa^{-1}}(\tau)$$

holds. For $\nu > 0$ small and $r > 0$ large enough the total projections

$$P^{U/S}_{\varepsilon,\lambda}(\tau) := \frac{1}{2\pi i} \int\limits_{\partial B_\nu(\pm 1)} (z\mathbb{1}_{2n} - B_{\varepsilon,\lambda}(\tau))^{-1} dz$$

are well-defined, smooth in ε and τ and holomorphic in λ. Then

$$\begin{aligned} P_{\varepsilon,0}(\tau) = P^U_0 &:= \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & 0 \end{pmatrix}, \\ P_{\varepsilon,0}(\tau) = P^S_0 &:= \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_n \end{pmatrix}, \end{aligned}$$

and a short calculation shows

$$\frac{\partial}{\partial\lambda} P^{U/S}_{\varepsilon,0}(\tau) = \pm\frac{1}{2} \begin{pmatrix} 0 & B^>_\varepsilon(\tau) \\ B^<_\varepsilon(\tau) & 0 \end{pmatrix}.$$

Following again Kato's reduction process [54] let

$$Q_{\varepsilon,\lambda}(\tau) := (\partial_\lambda P^U_{\varepsilon,\lambda}(\tau)) P^U_{\varepsilon,\lambda}(\tau) + (\partial_\lambda P^S_{\varepsilon,\lambda}(\tau)) P^S_{\varepsilon,\lambda}(\tau)$$

and let $\mathscr{S}_{\varepsilon,\lambda}(\tau)$ be the solution of

$$\partial_\lambda \mathscr{S}_{\varepsilon,\lambda}(\tau) = Q_{\varepsilon,\lambda}(\tau) \mathscr{S}_{\varepsilon,\lambda}(\tau), \ \mathscr{S}_{\varepsilon,0}(\tau) = \mathbb{1}_{2n}.$$

Then

$$P_0^{U/S} = \mathscr{S}_{\varepsilon,\lambda}(\tau)^{-1} P_{\varepsilon,\lambda}^{U/S}(\tau) \mathscr{S}_{\varepsilon,\lambda}(\tau),$$

and $B_{\varepsilon,\lambda}(\tau)$ is given with respect to the transformation $\mathscr{S}_{\varepsilon,\lambda}(\tau)$ by

$$\begin{aligned} M_{\varepsilon,\lambda}(\tau) &= \mathscr{S}_{\varepsilon,\lambda}(\tau)^{-1} B_{\varepsilon,\lambda}(\tau) \mathscr{S}_{\varepsilon,\lambda}(\tau) \\ &= P_0^U M_{\varepsilon,\lambda}(\tau) P_0^U + P_0^S M_{\varepsilon,\lambda}(\tau) P_0^S \end{aligned}$$

which is of the form

$$\begin{pmatrix} M_{\varepsilon,\lambda}^{>}(\tau) & 0 \\ 0 & M_{\varepsilon,\lambda}^{<}(\tau) \end{pmatrix}.$$

We expand $M_{\varepsilon,\lambda}(\tau)$:

$$\begin{aligned} M_{\varepsilon,\lambda}(\tau) &= M_{\varepsilon,0}(\tau) + \lambda \frac{\partial}{\partial \lambda} M_{\varepsilon,0}(\tau) + \mathcal{O}(\lambda^2) \\ &= \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix} + \lambda \begin{pmatrix} B_\varepsilon^{>}(\tau) & 0 \\ 0 & B_\varepsilon^{<}(\tau) \end{pmatrix} + \mathcal{O}(\lambda^2), \end{aligned}$$

hence

$$M_{\varepsilon,\lambda}^{\gtrless}(\tau) = \pm \mathbb{1}_n + \lambda B_\varepsilon^{\gtrless}(\tau) + \mathcal{O}(\lambda^2).$$

By choosing $r > 0$ large enough we find by setting

$$R_{\varepsilon,\kappa}(\tau) := \mathscr{S}_{\varepsilon,\kappa^{-1}}(\tau)$$

and

$$A_{\varepsilon,\kappa}^{\gtrless}(\tau) := \kappa M_{\varepsilon,\kappa^{-1}}^{\gtrless}(\tau) = \pm \kappa \mathbb{1}_n + B_\varepsilon^{\gtrless}(\tau) + \mathcal{O}(|\kappa|^{-1})$$

that for all $\varepsilon \in [0, \varepsilon_0], \tau \in J$ and $|\kappa| \geq r$

$$R_{\varepsilon,\kappa}(\tau)^{-1} A_{\varepsilon,\kappa}^{III}(\tau) R_{\varepsilon,\kappa}(\tau) = \begin{pmatrix} A_{\varepsilon,\kappa}^{>}(\tau) & 0 \\ 0 & A_{\varepsilon,\kappa}^{<}(\tau) \end{pmatrix}.$$

We have for $r > 0$ large enough and $|\kappa| \geq r, \operatorname{Re} \kappa \geq 0$, that for all $\eta \in \mathbb{C}^n$

$$\begin{aligned} \operatorname{Re} \langle \eta, A_{\varepsilon,\kappa}^{>}(\tau)\eta \rangle &= \operatorname{Re} \kappa |\eta|^2 + \operatorname{Re} \langle \eta, B_\varepsilon^{>}(\tau)\eta \rangle + \operatorname{Re} \langle \eta, \mathcal{O}(\kappa^{-1})\eta \rangle \\ &\geq (\operatorname{Re} \kappa + c^{>} + \mathcal{O}(|\kappa|^{-1}))|\eta|^2 \\ &\geq \frac{1}{2} c^{>} |\eta|^2, \\ \operatorname{Re} \langle \eta, A_{\varepsilon,\kappa}^{<}(\tau)\eta \rangle &= -\operatorname{Re} \kappa |\eta|^2 + \operatorname{Re} \langle \eta, B_\varepsilon^{<}(\tau)\eta \rangle + \operatorname{Re} \langle \eta, \mathcal{O}(\kappa^{-1})\eta \rangle \\ &\leq (-\operatorname{Re} \kappa - c^{<} + \mathcal{O}(|\kappa|^{-1}))|\eta|^2 \\ &\leq -\frac{1}{2} c^{<} |\eta|^2. \end{aligned}$$

Since $\mathscr{S}_{\varepsilon,\kappa^{-1}}(0)^{-1}$ is a continuous function and ε and κ^{-1} range over a compact set there is a constant $c_1 > 0$ such that

$$|R_{\varepsilon,\kappa}(0)^{-1}| \leq c_1.$$

Furthermore, since $\mathscr{S}_{\varepsilon,\tau}(\kappa^{-1})$ is continuously differentiable in τ and ε (and its arguments range, again, over a compact set) there is a constant $c_2 > 0$ such that

$$\left|\frac{\partial^2}{\partial\varepsilon\partial\tau}R_{\varepsilon,\kappa}(\tau)\right| \leq c_2.$$

By $R_{0,\kappa}(\tau) = R_{0,\kappa}(0)$ (for $\varepsilon = 0$ there is no τ dependence of $A^{III}_{\varepsilon,\kappa}(\tau)$) we deduce from

$$R_{\varepsilon,\kappa}(\tau) = R_{0,\kappa}(0) + \varepsilon\int_0^1 \frac{\partial}{\partial\varepsilon}R_{s\varepsilon,\kappa}(\tau)ds$$

that

$$\left|\frac{\partial}{\partial\tau}R_{\varepsilon,\kappa}(\tau)\right| \leq \varepsilon c_2.$$

□

Combining the estimate on $R_{\varepsilon,\kappa}$ in Lemma 2.7.12 gives

$$\left|R_{\varepsilon,\kappa}(0)^{-1}\frac{\partial}{\partial\tau}R_{\varepsilon,\kappa}(\tau)\right| \leq c_1c_2\varepsilon,$$

and Lemma 2.1.4 is applicable if ε is small enough. Finally, note that Lemma 2.1.4 makes a statement on the evolution of vector spaces which is independent of the choice of basis, hence its conclusions hold for the equations (2.42) and (2.101), i.e. for

$$\begin{aligned}\chi_\varepsilon' &= \varepsilon(1-\chi_\varepsilon^2)h_\varepsilon(\chi_\varepsilon),\\ \xi' &= A_{\varepsilon,\kappa}(\chi_\varepsilon)\xi.\end{aligned}$$

This proves:

Corollary 2.7.13. *For any $r_1 > 0$ there exists $\varepsilon_0 = \varepsilon_0(r_1)$ such that for all $\varepsilon \in (0,\varepsilon_0]$ and all $\zeta \in \mathbb{H}^+$ with $r_1 \leq \varepsilon^2\zeta$ the rescaled Evans bundles are transversal:*

$$H^-_\varepsilon(\zeta) \cap H^+_\varepsilon(\zeta) = \{0\}.$$

Proof of Theorem 2.4.8. Choose $r_0 > 0$ in Corollary 2.7.7 large enough and $r_1 > 0$ in Corollary 2.7.13 small enough such that for small $\varepsilon_0 > 0$ the statement of Corollary 2.7.10 holds for the constants r_0, r_1, ε_0. With this choice of constants the three spectral regimes addressed in the Corollaries 2.7.7, 2.7.10 and 2.7.13 connect to cover all of $\mathbb{H}^+$, and we find that if ε is small enough then ϕ_ε is spectrally stable in the sense of Definition 2.4.7. □

Remark 2.7.14. *We close with the side remark that for $\varepsilon > 0$ small enough there exists a $d_\varepsilon > 0$ such that the Evans bundles $\mathcal{H}^\pm_\varepsilon$ and an Evans function $\mathscr{E}_\varepsilon$ are holomorphically defined on*

$$\mathbb{H}^+ - d_\varepsilon = \{\kappa \in \mathbb{C} |\ \operatorname{Re}\kappa \geq -d_\varepsilon\}$$

with the Evans function condition holding on the above superset of $\mathbb{H}^+$, too.

Outlook

A natural extension of the result in Chapter 1 on the formation of singularities in solutions to nonlinear hyperbolic systems is to prove the statement of Theorem 1.0.1 for systems (1.1) with assumptions A1 and A2 replaced by

A1' There exist $\delta > 0$ and functions

$$\lambda_i \in C^\infty(B_{2\delta}(0);\mathbb{R}), l_i \in C^\infty(B_{2\delta}(0);\mathbb{R}^{n\times 1}) \text{ and } r_i \in C^\infty(B_{2\delta}(0);\mathbb{R}^{1\times n})$$

which satisfy for all $u \in B_{2\delta}(0)$ and all i, j

$$l_i(u)a(u) = \lambda_i(u)l_i(u),\ a(u)r_i(u) = \lambda_i(u)r_i(u), \text{ and } \lambda_i(u) \leq \lambda_j(u) \text{ if } i \leq j.$$

Furthermore, the normalization

$$|l_i| = 1 \text{ and } l_i r_j = \delta_{ij}$$

holds.

A2' There is $k \in \{1, \dots, n\}$ such that the k-th eigenvalue of a is simple and genuinely nonlinear at 0.

In fact it suffices to show that the relations (1.20) and (1.22) for the coefficiens γ_{ijm} and Γ_{ijm} continue to hold under the assumptions A1', A2'. In the case $B \equiv 0$ this is easily done using simple waves like in [41] which then translates to arbitrary source terms B since the quadratic coefficients in (1.19) and (1.21) do not depend on B. Hence, Theorem 1.0.1 holds with A1 and A2 replaced by A1' and A2'.

Concerning the results on spectral stability of shock profiles in hyperbolically regularized conservation laws there are numerous directions for further research that fall into the line of the findings presented in Chapter 2:
A first extension would be to prove Theorem 2.4.8 for planar small amplitude shock profiles in systems like (2.32) with multiple space dimensions comparable to the results in [27] on multi-dimensional viscous conservation laws. Presumably only profiles of extreme shocks will be spectrally stable (see [27]).
It is also desirable to consider more general hyperbolic operators $\mathcal{B}[v]$ then the d'Alembertian $\Box v$, for instance suitable quasi-linear second-order hyperbolic operators like in [29].
Last but not least, in the case of hyperbolically regularized systems of conservation laws the entire program whether spectral stability implies linearized and nonlinear orbital stability of shock profiles is an interesting open research question.

Deutsche Zusammenfassung (German summary)

Sei $n \in \mathbb{N}$. Die vorgelegte Arbeit setzt sich aus zwei Hauptbestandteilen zusammen:
Sei $\mathcal{U} \subset \mathbb{R}^n$ offen und $0 \in \mathcal{U}$. Im ersten Teil, Kapitel 1, werden quasi-lineare hyperbolische Gleichungen erster Ordnung mit allgemeinen Quelltermen in der Form

$$u_t + a(u)u_x = B(u) \ \ (u \in \mathcal{U}) \tag{1'}$$

mit $(t, x) \in \mathbb{R}^2$ und glatten Funktionen $a : \mathcal{U} \to \mathbb{R}^{n\times n}, B : \mathcal{U} \to \mathbb{R}^n$ betrachtet. Unter den Voraussetzungen

A1 Die Eigenwerte von $a(0)$ sind reell und einfach;

A2 Für ein $k \in \{1, \dots, n\}$ ist der k-te Eigenwert von a echt nicht-linear, „genuine nonlinear“, bei 0;

A3 $B(0) = 0$;

zeigen wir mit Theorem 1.0.1 die Existenz von glatten Funktionen $u^0 : \mathbb{R} \to \mathcal{U}$ mit kompakten Träger, sodass die eindeutige C^1-Lösung u von (1') mit Anfangswerten $u(0, \cdot) = u^0$, obwohl sie beschränkt bleibt, nur für endliche Zeiten existieren kann. Die Methode der Charakteristiken findet Anwendung und die Beweisstrategie folgt den Ideen von John [48] und Hörmander [41], die Singularitätenbildung in C^1-Lösungen von (1') mit $B \equiv 0$ behandeln. Außerdem werden Beweise zur lokalen Existenztheorie klassischer Lösungen ausgeführt und beispielhaft wird die echte Nichtlinearität eines einfachen Eigenwerts in einem bekannten hyperbolischen Relaxationssystem [98] gezeigt.

Sei $k \in \{1, \dots, n\}$, und sei $\mathcal{V} \subset \mathbb{R}^n$ offen mit $0 \in \mathcal{V}$. Im zweiten Teil, Kapitel 2, untersuchen wir hyperbolisch regularisierte Systeme von Erhaltungsgleichungen in der Form

$$g(v)_t + f(v)_x = B\Box v = B(\partial_x^2 v - \partial_t^2 v) \ \ (v \in \mathcal{V}) \tag{2'}$$

mit $(t, x) \in \mathbb{R}^2$, glatten Funktionen $f, g : \mathcal{V} \to \mathbb{R}^n$ und einer Matrix $B \in \mathbb{R}^{n\times n}$. Wir fordern, dass die Koeffizienten in (2') die folgenden Voraussetzungen erfüllen:

B1 $\forall v \in \mathcal{V} : B$ und die Jacobi-Matrizen $Df(v)$ und $Dg(v)$ von f und g bei v sind symmetrisch mit positiv definiten B und $Dg(0)$;

B2 Die Eigenwerten von $Df(0)$ bezüglich $Dg(0)$ sind reell und einfach;

B3 Der k-te Eigenwert μ_k von Df bezüglich Dg ist echt nicht-linear bei 0;

B4 Sub-Charakterizität: $Dg(0) \pm Df(0)$ ist positiv definit.

Ohne Einschränkung der Allgemeinheit können wir $f(0) = g(0) = 0, B = \mathbb{1}_n$ und $\mu_k(0) = 0$ annehmen. Nach einer geeigneten Wahl einer Basis von $\mathbb{R}^n$ betrachten wir eine glatte Familie $\{(v_\varepsilon^-, v_\varepsilon^+, s(\varepsilon))\}_{\varepsilon\in[0,\varepsilon_0]}, \varepsilon_0 > 0$, von Lax-$k$-Schocks zur Erhaltungsgleichung $g(v_t) + f(v)_x = 0$ mit dazugehöriger glatter Familie von Schockprofilen $\{\phi_\varepsilon\}_{\varepsilon\in[0,\varepsilon_0]}$ von (2'). Wir nehmen

$$s(0) = s'(0) = 0 \text{ und } v_\varepsilon^\pm = \pm\varepsilon(e_k + \mathcal{O}(\varepsilon)) \text{ mit } (v_\varepsilon^\pm)_k = \pm\varepsilon$$

an. Mit den Methoden aus [26, 27] wird gezeigt, dass für genügend kleines ε das Schockprofil ϕ_ε, aufgefasst als selbst-ähnliche Lösung von (2'), spektral stabil im Sinne der Theorie der Evans-Funktionen ist, d.h. es gibt ein zu ϕ_ε gehörende Evans-Funktion $\mathcal{E}_\varepsilon : \Lambda \to \mathbb{C}$, die analytisch auf einer Umgebung $\Lambda \subset \mathbb{C}$ der abgeschlossenen rechten komplexen Halbebene $\mathbb{H}^+$ ist und die Evans-Funktion-Bedingung [70, 99] erfüllt:

$$(\forall\kappa \in \Lambda :\ \mathcal{E}_\varepsilon(\kappa) = 0 \ \Leftrightarrow\ \kappa = 0) \ \text{ und } \ (\mathcal{E}_\varepsilon'(0) \neq 0).$$

Das Hauptresultat, Theorem 2.4.8, ist analog zu den Resultaten in [26, 77] für parabolisch regularisierte Erhaltungsgleichungen, und erweitert in gewisser Weise die Ergebnisse in [73, 77, 92] über Schockprofile in hyperbolischen Relaxationssystemen. Wir führen die Begrifflichkeiten der Evans-Funktionen Theorie für (2') ein, und ziehen die Verbindung zwischen spektraler Stabilität der Schockwelle ϕ_ε und dem Resolventkern der Linearisierung von (2') bei ϕ_ε. Außerdem wird in die Theorie der geometrischen singulären Störungsrechnung eingeführt, und grundlegendes zu Flüsse auf Grassmann-Mannigfaltigkeiten bewiesen, wobei einige Ergebnisse aus [26] verallgemeinert werden.

In beiden Kapitel findet sich ein explizit lösbares Beispiel, welches die jeweilige Problematik veranschaulicht.

Bibliography

[1] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*. 2nd ed. Vol. 140. Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, 2003, pp. xiv+305.

[2] J. Alexander, R. Gardner, and C. Jones. "A topological invariant arising in the stability analysis of travelling waves". *J. Reine Angew. Math.* 410 (1990), pp. 167–212.

[3] B. H. Barker. "Evans Function Computations". MA thesis. Brigham Young University, 2009.

[4] J. Bärlin. *Blow-up of solutions to relaxed compressible Navier-Stokes equations in divergence form*. arXiv:2202.05634. 2022.

[5] J. Bärlin. "Formation of singularities in solutions to nonlinear hyperbolic systems with general sources". *Nonlinear Anal. Real World Appl.* 73 (2023), Paper No. 103901.

[6] K. Beauchard and E. Zuazua. "Large time asymptotics for partially dissipative hyperbolic systems". *Arch. Ration. Mech. Anal.* 199.1 (2011), pp. 177–227.

[7] S. Benzoni-Gavage and D. Serre. *Multidimensional hyperbolic partial differential equations*. Oxford Mathematical Monographs. First-order systems and applications. The Clarendon Press, Oxford University Press, Oxford, 2007, pp. xxvi+508.

[8] S. Bianchini, B. Hanouzet, and R. Natalini. "Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy". *Comm. Pure Appl. Math.* 60.11 (2007), pp. 1559–1622.

[9] G. Boillat. "Sur l'existence et la recherche d'équations de conservation supplémentaires pour les systèmes hyperboliques". *C. R. Acad. Sci. Paris Sér. A* 278 (1974), pp. 909–912.

[10] C. Chicone. *Ordinary differential equations with applications*. 2nd ed. Vol. 34. Texts in Appl. Math. Springer, New York, 2006, pp. xx+636.

[11] W. A. Coppel. *Stability and asymptotic behavior of differential equations*. D. C. Heath and Company, Boston, Mass., 1965, pp. viii+166.

[12] C. M. Dafermos. *Hyperbolic Conservation Laws in Continuum Physics*. Vol. 325. Grundlehren der math. Wissenschaften. Springer, Berlin, 2016.

[13] C. M. Dafermos and L. Hsiao. "Hyperbolic systems and balance laws with inhomogeneity and dissipation". *Indiana Univ. Math. J.* 31.4 (1982), pp. 471–491.

[14] C. M. Dafermos and L. Hsiao. "Development of singularities in solutions of the equations of nonlinear thermoelasticity". *Quart. Appl. Math.* 44.3 (1986), pp. 463–474.

[15] B. F. Doolin and C. F. Martin. *Introduction to differential geometry for engineers*. Vol. 136. Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1990, pp. xii+163.

[16] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*. Vol. 194. Graduate Texts in Mathematics. Springer-Verlag, New York, 2000, pp. xxii+586.

[17] J. W. Evans. "Nerve axon equations. IV. The stable and the unstable impulse". *Indiana Univ. Math. J.* 24.12 (1974/75), pp. 1169–1190.

[18] L. C. Evans. *Partial differential equations*. 2nd ed. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010, pp. xxii+749.

[19] N. Fenichel. "Persistence and smoothness of invariant manifolds for flows". *Indiana Univ. Math. J.* 21 (1971/72), pp. 193–226.

[20] N. Fenichel. "Asymptotic stability with rate conditions". *Indiana Univ. Math. J.* 23 (1973/74), pp. 1109–1137.

[21] N. Fenichel. "Asymptotic stability with rate conditions. II". *Indiana Univ. Math. J.* 26.1 (1977), pp. 81–93.

[22] N. Fenichel. "Geometric singular perturbation theory for ordinary differential equations". *J. Differential Equations* 31.1 (1979), pp. 53–98.

[23] J. Ferrer, M. I. García, and F. Puerta. "Differentiable families of subspaces". *Linear Algebra Appl.* 199 (1994), pp. 229–252.

[24] H. Freistühler. "Time-asymptotic stability for first-order symmetric hyperbolic systems of balance laws in dissipative compressible fluid dynamics". *Quart. Appl. Math.* 80.3 (2022), pp. 597–606.

[25] H. Freistühler and M. Sroczinski. "A class of uniformly dissipative symmetric hyperbolic-hyperbolic systems". *J. Diff. Equ.* 288 (2021), pp. 40–61.

[26] H. Freistühler and P. Szmolyan. "Spectral stability of small shock waves". *Arch. Ration. Mech. Anal.* 164.4 (2002), pp. 287–309.

[27] H. Freistühler and P. Szmolyan. "Spectral stability of small-amplitude viscous shock waves in several space dimensions". *Arch. Ration. Mech. Anal.* 195.2 (2010), pp. 353–373.

[28] H. Freistühler, P. Szmolyan, and J. Wächtler. "Spectral stability of shock waves associated with not genuinely nonlinear modes". *J. Differential Equations* 257.1 (2014), pp. 185–206.

[29] H. Freistühler and B. Temple. "Causal dissipation and shock profiles in the relativistic fluid dynamics of pure radiation". *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 470.2166 (2014), pp. 20140055, 17.

[30] K. O. Friedrichs and P. D. Lax. "Systems of conservation equations with a convex extension". *Proc. Nat. Acad. Sci. U.S.A.* 68 (1971), pp. 1686–1688.

[31] C. Fries. "Stability of viscous shock waves associated with non-convex modes". *Arch. Ration. Mech. Anal.* 152.2 (2000), pp. 141–186.

[32] R. Gardner and C. K. R. T. Jones. "Stability of travelling wave solutions of diffusive predator-prey systems". *Trans. Amer. Math. Soc.* 327.2 (1991), pp. 465–524.

[33] R. A. Gardner and C. K. R. T. Jones. *Stability of one-dimensional waves in weak and singular limits*. SIAM, Philadelphia, PA, 1991, pp. 32–48.

[34] R. A. Gardner and K. Zumbrun. "The gap lemma and geometric criteria for instability of viscous shock profiles". *Comm. Pure Appl. Math.* 51.7 (1998), pp. 797–855.

[35] I. M. Gelfand. "Some problems in the theory of quasi-linear equations". *Uspehi Mat. Nauk* 14.2 (86) (1959), pp. 87–158.

[36] S. K. Godunov. "An interesting class of quasi-linear systems". *Dokl. Akad. Nauk SSSR* 139 (1961), pp. 521–523.

[37] J. Goodman. "Nonlinear asymptotic stability of viscous shock profiles for conservation laws". *Archive for Rational Mechanics and Analysis* 95.4 (1986), pp. 325–344.

[38] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978, pp. xii+813.

[39] D. Henry. *Geometric theory of semilinear parabolic equations*. Vol. 840. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1981, pp. iv+348.

[40] M. W. Hirsch, C. C. Pugh, and M. Shub. *Invariant manifolds*. Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977, pp. ii+149.

[41] L. Hörmander. "The lifespan of classical solutions of nonlinear hyperbolic equations". *Pseudodifferential operators (Oberwolfach, 1986)*. Vol. 1256. Lecture Notes in Math. Springer, Berlin, 1987, pp. 214–280.

[42] L. Hörmander. *An introduction to complex analysis in several variables*. 3rd ed. Vol. 7. North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam, 1990, pp. xii+254.

[43] L. Hörmander. *The analysis of linear partial differential operators. I*. 2nd ed. Classics in Mathematics. Distribution theory and Fourier analysis. Springer-Verlag, Berlin, 1990, pp. xii+440.

[44] L. Hörmander. *Lectures on nonlinear hyperbolic differential equations*. Vol. 26. Mathématiques & Applications (Berlin) [Mathematics & Applications]. Springer-Verlag, Berlin, 1997, pp. viii+289.

[45] Y. Hu and N. Wang. "Global existence versus blow-up results for one dimensional compressible Navier-Stokes equations with Maxwell's law". *Math. Nachr.* 292.4 (2019), pp. 826–840.

[46] J. Humpherys. "Stability of Jin-Xin relaxation shocks". *Quarterly of Applied Mathematics* 61.2 (2003), pp. 251–263.

[47] S. Jin and Z. P. Xin. "The relaxation schemes for systems of conservation laws in arbitrary space dimensions". *Comm. Pure Appl. Math.* 48.3 (1995), pp. 235–276.

[48] F. John. "Formation of singularities in one-dimensional nonlinear wave propagation". *Comm. Pure Appl. Math.* 27 (1974), pp. 377–405.

[49] C. K. R. T. Jones. "Stability of the travelling wave solution of the Fitz-Hugh-Nagumo system". *Trans. Amer. Math. Soc.* 286.2 (1984), pp. 431–469.

[50] C. K. R. T. Jones. "Geometric singular perturbation theory". Lecture Notes in Math. 1609 (1995), pp. 44–118.

[51] C. K. R. T. Jones, R. Gardner, and T. Kapitula. "Stability of travelling waves for nonconvex scalar viscous conservation laws". *Comm. Pure Appl. Math.* 46.4 (1993), pp. 505–526.

[52] T. Kapitula and K. Promislow. *Spectral and dynamical stability of nonlinear waves*. Vol. 185. Applied Mathematical Sciences. Springer, New York, 2013, pp. xiv+361.

[53] T. Kapitula and B. Sandstede. "Stability of bright solitary-wave solutions to perturbed nonlinear Schrödinger equations". *Phys. D* 124.1-3 (1998), pp. 58–103.

[54] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995, pp. xxii+619.

[55] S. Kawashima, A. Matsumura, and K. Nishihara. "Asymptotic behavior of solutions for the equations of a viscous heat-conductive gas". *Proc. Japan Acad. Ser. A Math. Sci.* 62 (1986).

[56] S. Kawashima and W.-A. Yong. "Dissipative structure and entropy for hyperbolic systems of balance laws". *Arch. Ration. Mech. Anal.* 174.3 (2004), pp. 345–364.

[57] D.-X. Kong. *Cauchy problem for quasilinear hyperbolic systems*. Vol. 6. MSJ Memoirs. Mathematical Society of Japan, Tokyo, 2000, pp. viii+213.

[58] C. Lattanzio et al. "Analytical and numerical investigation of traveling waves for the Allen-Cahn model with relaxation". *Math. Models Methods Appl. Sci.* 26.5 (2016), pp. 931–985.

[59] P. D. Lax. "Hyperbolic systems of conservation laws. II". *Comm. Pure Appl. Math.* 10 (1957), pp. 537–566.

[60] P. D. Lax. "Development of singularities of solutions of nonlinear hyperbolic partial differential equations". *J. Mathematical Phys.* 5 (1964), pp. 611–613.

[61] P. D. Lax. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973, pp. v+48.

[62] John M. Lee. *Introduction to smooth manifolds*. 2nd ed. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708.

[63] T. Li and H. Liu. "Critical thresholds in hyperbolic relaxation systems". *J. Differential Equations* 247.1 (2009), pp. 33–48.

[64] H. Liu. "Asymptotic stability of relaxation shock profiles for hyperbolic conservation laws". *J. Differential Equations* 192.2 (2003), pp. 285–307.

[65] T.-P. Liu. "Development of singularities in the nonlinear waves for quasi-linear hyperbolic partial differential equations". *J. Differential Equations* 33.1 (1979), pp. 92–111.

[66] T.-P. Liu. "Hyperbolic conservation laws with relaxation". *Comm. Math. Phys.* 108.1 (1987), pp. 153–175.

[67] T.-P. Liu. "Pointwise convergence to shock waves for viscous conservation laws". *Comm. Pure Appl. Math.* 50.11 (1997), pp. 1113–1182.

[68] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*. Vol. 53. Applied Mathematical Sciences. Springer-Verlag, New York, 1984, pp. viii+159.

[69] A. Majda and R. L. Pego. "Stable viscosity matrices for systems of conservation laws". *Journal of Differential Equations* 56.2 (1985), pp. 229 – 262.

[70] C. Mascia and K. Zumbrun. "Pointwise Green's function bounds and stability of relaxation shocks". *Indiana Univ. Math. J.* 51.4 (2002), pp. 773–904.

[71] C. Mascia and K. Zumbrun. "Stability of large-amplitude viscous shock profiles of hyperbolic-parabolic systems". *Arch. Ration. Mech. Anal.* 172.1 (2004), pp. 93–131.

[72] C. Mascia and K. Zumbrun. "Stability of large-amplitude shock profiles of general relaxation systems". *SIAM J. Math. Anal.* 37.3 (2005), pp. 889–913.

[73] C. Mascia and K. Zumbrun. "Spectral stability of weak relaxation shock profiles". *Comm. Partial Differential Equations* 34.1-3 (2009), pp. 119–136.

[74] A. Matsumura and K. Nishihara. "On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas". *Japan J. Appl. Math.* 2.1 (1985), pp. 17–25.

[75] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Vol. 44. Applied Mathematical Sciences. Springer-Verlag, New York, 1983, pp. viii+279.

[76] R. L. Pego and M. I. Weinstein. "Eigenvalues, and instabilities of solitary waves". *Philos. Trans. Roy. Soc. London Ser. A* 340.1656 (1992), pp. 47–94.

[77] R. Plaza and K. Zumbrun. "An Evans function approach to spectral stability of small-amplitude shock profiles". *Discrete Contin. Dyn. Syst.* 10.4 (2004), pp. 885–924.

[78] J. Rauch. *Hyperbolic partial differential equations and geometric optics*. Vol. 133. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2012, pp. xx+363.

[79] C. Robinson. *Dynamical systems*. Studies in Advanced Mathematics. Stability, symbolic dynamics, and chaos. CRC Press, Boca Raton, FL, 1995, pp. xii+468.

[80] W. Rudin. *Real and complex analysis*. 3rd ed. McGraw-Hill Book Co., New York, 1987, pp. xiv+416.

[81] T. Ruggeri. "Entropy principle and relativistic extended thermodynamics: global existence of smooth solutions and stability of equilibrium state". *Nuovo Cimento Soc. Ital. Fis. B* 119.7-9 (2004), pp. 809–821.

[82] T. Ruggeri and A. Strumia. "Main field and convex covariant density for quasilinear hyperbolic systems. Relativistic fluid dynamics". *Ann. Inst. H. Poincaré Sect. A (N.S.)* 34.1 (1981), pp. 65–84.

[83] B. Sandstede. "Stability of travelling waves". *Handbook of dynamical systems*. Vol. 2. North-Holland, Amsterdam, 2002, pp. 983–1055.

[84] D. H. Sattinger. "On the stability of waves of nonlinear parabolic systems". *Advances in Math.* 22.3 (1976), pp. 312–355.

[85] Y. Shizuta and S. Kawashima. "Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation". *Hokk. Math. J.* 14.2 (1985), pp. 249–275.

[86] J. Smoller. *Shock waves and reaction-diffusion equations*. 2nd ed. Vol. 258. Grundlehren der math. Wissenschaften. Springer-Verlag, New York, 1994, pp. xxiv+632.

[87] M. Spivak. *A comprehensive introduction to differential geometry. Vol. I*. Brandeis Univ., Waltham, Mass., 1970, pp. iii+656.

[88] M. Sroczinski. "Asymptotic stability in a second-order symmetric hyperbolic system modeling the relativistic dynamics of viscous heat-conductive fluids with diffusion". *J. Differential Equations* 268.2 (2020), pp. 825–851.

[89] A. Szepessy and Z. P. Xin. "Nonlinear stability of viscous shock waves". *Arch. Rational Mech. Anal.* 122.1 (1993), pp. 53–103.

[90] P. Szmolyan. "Transversal heteroclinic and homoclinic orbits in singular perturbation problems". *J. Differential Equations* 92 (1991).

[91] G. Teschl. *Ordinary differential equations and dynamical systems*. Vol. 140. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2012, pp. xii+356.

[92] Y. Ueda. "Stability of travelling wave solutions to a semilinear hyperbolic system with relaxation". *Math. Methods Appl. Sci.* 32.4 (2009), pp. 419–434.

[93] Y. Ueda and S. Kawashima. "Large time behaviour of solutions to a semilinear hyperbolic system with relaxation". *J. Hyperbolic Diff. Equ.* 4.1 (2007), pp. 147–179.

[94] R. O. Wells Jr. *Differential analysis on complex manifolds*. 2nd ed. Vol. 65. Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1980, pp. x+260.

[95] G. B. Whitham. *Linear and nonlinear waves*. Pure and Applied Mathematics (New York). Reprint of the 1974 original, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999, pp. xviii+636.

[96] S. Wiggins. *Normally hyperbolic invariant manifolds in dynamical systems*. Vol. 105. Applied Mathematical Sciences. Springer-Verlag, New York, 1994, pp. x+193.

[97] W.-A. Yong. "Entropy and global existence for hyperbolic balance laws". *Arch. Ration. Mech. Anal.* 172.2 (2004), pp. 247–266.

[98] Y. Zeng. "Gas Dynamics in Thermal Nonequilibrium and General Hyperbolic Systems with Relaxation". *Arch. Ration. Mech. Anal.* 150.3 (1999), pp. 225–279.

[99] K. Zumbrun and P. Howard. "Pointwise semigroup methods and stability of viscous shock waves". *Indiana Univ. Math. J.* 47.3 (1998), pp. 741–871.