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Abstract

In this paper we investigate the compressible Navier-Stokes-Cahn-Hilliard equations (the so-called NSCH model) derived by Lowengrub and Truskinowsky. This model describes the flow of a binary compressible mixture; the fluids are supposed to be macroscopically immiscible, but partial mixing is permitted leading to narrow transition layers. The internal structure and macroscopic dynamics of these layers are induced by a Cahn-Hilliard law that the mixing ratio satisfies. The PDE constitute a strongly coupled hyperbolic-parabolic system. We establish a local existence and uniqueness result for strong solutions.

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1. The model

One way to describe the flow of immiscible fluids and the motion of interfaces between these fluids is based on the assumption that Euler or Navier-Stokes equations apply to both sides of the interface and across this interface certain jump conditions are prescribed. However this model breaks down when near interfaces a molecular mixing of the immiscible fluids occurs in such a large amount that the model of sharp interfaces cannot be maintained. Another problem of this model concerns the description of merging and reconnecting interfaces. One way out is to replace the sharp interface by a narrow transition layer, that is, one allows a partial mixing in a small interfacial region.

For this purpose one firstly introduces the mass concentrations $c_i = M_i/M$ with $M = M_1 + M_2$, where M_i denotes the mass of the fluid i in the representative volume V. Notice that this implies $c_1 + c_2 = 1$ as well as $0 \leq c_i \leq 1$. A basic hypothesis is the identification of an order parameter c with a constituent concentration, e.g. $c = c_1$, or with the difference of both concentrations, $c = c_1 - c_2 \equiv 2c_1 - 1$. Choosing the latter case, c varies continuously between -1 and 1 in the interfacial region and takes the values -1 and 1 in the absolute fluids. Let u_1, u_2 denote the velocities of the corresponding fluids and $\tilde\rho_1 := \frac{M_1}{V}$, $\tilde\rho_2 := \frac{M_2}{V}$ the associated apparent densities which both fulfil the equation of mass balance. Then, introducing the total density $\rho := \tilde\rho_1 + \tilde\rho_2$ and the mass-averaged velocity $\rho u := \tilde\rho_1 u_1 + \tilde\rho_2 u_2$, we obtain the equation of mass balance for ρ and u,

$$\partial_t \rho + \nabla\cdot(\rho u) = 0, \quad (t,x) \in J \times \Omega.$$

The total energy $E_G(t)$ in a volume $G \subset \Omega$ is to be given as the sum of kinetic energy and (specific) Helmholtz free energy, that is, it is assumed that

$$E_G(t) := \int_G \frac{1}{2}\rho|u|^2 + \rho\psi(c,\rho,\nabla c)\, dx.$$

Here ψ denotes the specific Helmholtz free energy density at a given temperature, which may depend on ρ, c and ∇c. If we choose $\psi(c,\rho,\nabla c)$ as follows

$$\psi(c,\rho,\nabla c) := \overline{\psi}(c,\rho) + \tfrac{1}{2}\varepsilon(c,\rho)|\nabla c|^2,$$

also being known as the Cahn-Hilliard specific free energy density, then the convected analogue of the Cahn-Hilliard equation can be derived (using the second law of thermodynamics/local dissipation inequality etc., see [16])

$$\partial_t(\rho c) + \nabla\cdot(\rho u c) = \nabla\cdot(\gamma\nabla\mu), \quad (t,x) \in J \times \Omega.$$

The generalized chemical potential μ is given by

$$\mu = \partial_c\psi - \rho^{-1}\nabla\cdot\left(\rho\frac{\partial\psi}{\partial\nabla c}\right) \equiv \partial_c\psi - \rho^{-1}\nabla\cdot(\varepsilon\rho\nabla c), \quad \partial_c\psi = \overline{\psi}_c(c,\rho) + \tfrac{1}{2}\varepsilon_c(c,\rho)|\nabla c|^2.$$

Here the parameter $\varepsilon(c,\rho) > 0$ measures the interface thickness and $\gamma(c,\rho) > 0$ the mobility of the concentration field c. Further, it is supposed that the stress tensor $\mathcal{T}$ is given as the sum of a viscous and non-viscous contribution, that is, $\mathcal{T} := \mathcal{S}(\rho,u) + \mathcal{P}(\rho,c)$ with

$$\mathcal{S} := 2\eta(\rho)\mathcal{D}(u) + \lambda(\rho)\nabla\cdot u\,\mathcal{I}, \quad \mathcal{D}(u) := \tfrac{1}{2}(\nabla u + \nabla u^T),$$

where $\mathcal{I}$ denotes the identity, $\mathcal{S}$ the Cauchy stress tensor with viscosity coefficients $\eta(\rho)$ and $\lambda(\rho)$, and $\mathcal{P}$ the non-hydrostatic Cauchy stress tensor, which is assumed to be of the form

$$\mathcal{P} := -\rho^2\partial_\rho\psi\,\mathcal{I} - \rho\nabla c \otimes \frac{\partial\psi}{\partial\nabla c} = -\rho^2\partial_\rho\psi\,\mathcal{I} - \rho\varepsilon\nabla c\otimes\nabla c, \quad \partial_\rho\psi = \partial_\rho\overline{\psi} + \tfrac{1}{2}\varepsilon_\rho(\rho,c)|\nabla c|^2.$$

The given function $\pi := \rho^2\psi_\rho$ constitutes the pressure and the extra contribution $-\rho\nabla c\otimes\frac{\partial\psi}{\partial\nabla c}$ in the stress tensor represents capillary forces due to surface tension. Thus the Navier-Stokes equations read as

$$\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) - \nabla\cdot(\mathcal{S}(\rho,u) + \mathcal{P}(\rho,c)) = \rho f_{ext}, \quad (t,x) \in J \times \Omega,$$

where f_{ext} stands for external forces.

A complete derivation of this model can be found in [16], cf. also [9] and [2].

2. Mathematical Formulation

To become more specific, we consider a bounded domain $\Omega \subset \mathbb{R}^n$, with compact boundary $\Gamma := \partial\Omega$ of class C^4 decomposing disjointly as $\Gamma = \Gamma_d \cup \Gamma_s$, where each set may be empty. The outer unit normal of Γ at position x is denoted by $\nu(x)$. Further, let $J = [0, T]$ be a compact time interval. The two-component (binary) viscous compressible fluid is characterized by its total density (of the mixture) $\rho : J \times \overline{\Omega} \to \mathbb{R}_+$, velocity field $u : J \times \overline{\Omega} \to \mathbb{R}^n$, and mass concentration $c : J \times \overline{\Omega} \to [-1, 1]$, that is, we have chosen as order parameter $c := 2c_1 - 1$. Then the unknown functions ρ, u and c are governed by the Navier-Stokes-Cahn-Hilliard (NSCH) system reading

$$\partial_t(\rho u) + \nabla\cdot(\rho u \otimes u) - \nabla\cdot\mathcal{S} - \nabla\cdot\mathcal{P} = \rho f_{ext}, \quad (t,x) \in J \times \Omega, \tag{2.1}$$

$$\partial_t(c\rho) + \nabla\cdot(c\rho u) - \nabla\cdot(\gamma\nabla\mu) = 0, \quad (t,x) \in J \times \Omega, \tag{2.2}$$

$$\partial_t\rho + \nabla\cdot(\rho u) = 0, \quad (t,x) \in J \times \Omega, \tag{2.3}$$

with

$$\begin{aligned} &\mathcal{S} = 2\eta(\rho)\mathcal{D}(u) + \lambda(\rho)\nabla\cdot u\,\mathcal{I}, \quad \mathcal{P} = -\pi\mathcal{I} - \rho^2\varepsilon_\rho|\nabla c|^2\mathcal{I} - \rho\varepsilon(\rho,c)\nabla c \otimes \nabla c, \\ &\mu = \partial_c\psi - \rho^{-1}\nabla\cdot(\varepsilon(\rho,c)\rho\nabla c), \quad \psi = \overline{\psi}(\rho,c) + \tfrac{1}{2}\varepsilon|\nabla c|^2, \quad \pi = \rho^2\partial_\rho\overline{\psi}. \end{aligned} \tag{2.4}$$

These equations have to be complemented by initial conditions

$$u(0,x) = u_0(x), \quad c(0,x) = c_0(x), \quad \rho(0,x) = \rho_0(x), \quad x \in \Omega, \tag{2.5}$$

and boundary conditions. Two natural boundary conditions are of interest for u, namely the non-slip condition

$$u = 0, \quad (t,x) \in J \times \Gamma_d \tag{2.6}$$

and the pure slip condition

$$(u|\nu) = 0, \quad \mathcal{Q}\mathcal{S}(u)\cdot\nu \equiv 2\eta(\rho)\mathcal{Q}\mathcal{D}(u)\cdot\nu = 0, \quad (t,x) \in J \times \Gamma_s \tag{2.7}$$

with $\mathcal{Q}(x) := \mathcal{I} - \nu(x) \otimes \nu(x)$. As boundary conditions for c, we consider

$$\partial_\nu\mu(\rho,c)(t,x) = 0, \quad \partial_\nu c(t,x) = 0, \quad (t,x) \in J \times \Gamma, \tag{2.8}$$

meaning that no diffusion through the boundary occurs and the diffuse interface is orthogonal to the boundary of the domain. Finally, the viscosity coefficients may depend on t, x and ρ, the interface thickness ε as well as mobility γ may depend on t, x, ρ and c.

2.1. Function spaces and main result

To begin with, let the compact time interval J and the domain Ω be as described before. Then we are looking for solutions $w := (u, c, \rho)$ of problem (2.1)-(2.8) in the regularity class $\mathcal{Z}(J) := \mathcal{Z}_1(J) \times \mathcal{Z}_2(J) \times \mathcal{Z}_3(J)$ where the spaces $\mathcal{Z}_i(J)$ are defined by

$$\begin{aligned} \mathcal{Z}_1(J) &:= \mathrm{H}_p^{3/2}(J; \mathrm{L}_p(\Omega;\mathbb{R}^n)) \cap \mathrm{H}_p^1(J; \mathrm{H}_p^2(\Omega;\mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}_p^4(\Omega;\mathbb{R}^n)), \\ \mathcal{Z}_2(J) &:= \mathrm{H}_p^1(J; \mathrm{L}_p(\Omega)) \cap \mathrm{L}_p(J; \mathrm{H}_p^4(\Omega)), \\ \mathcal{Z}_3(J) &:= \mathrm{H}_p^{2+1/4}(J; \mathrm{L}_p(\Omega)) \cap \mathrm{B}(J; \mathrm{H}_p^3(\Omega)), \end{aligned}$$

$p \in (1,\infty)$. As usual, here and in the sequel H_p^s denote the Bessel potential spaces and W_p^s the Slobodeckij spaces ($W_p^s \equiv B_{pp}^s$ Besov spaces). The space of bounded functions $\mathrm{B}(J)$ is equipped with the norm $\|\cdot\|_\infty := \sup_{s\in J}\|\cdot\|$. Furthermore, if $\mathcal{F}(I)$ is any function space with $I \subseteq \mathbb{R}_+$ and $0 \in I$, then we set ${}_0\mathcal{F}(I) := \{v \in \mathcal{F}(I) : v_{|t=0} = 0\}$, whenever traces exist. Furthermore, we shall need the function spaces

$$\begin{aligned} Z(J) &:= Z_1(J)\times Z_2(J)\times Z_3(J),\\ Z_1(J) &:= \mathrm{H}_p^1(J;\mathrm{L}_p(\Omega;\mathbb{R}^n))\cap \mathrm{L}_p(J;\mathrm{H}_p^2(\Omega;\mathbb{R}^n)),\\ Z_2(J) &:= \mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega))\cap \mathrm{L}_p(J;\mathrm{H}_p^2(\Omega)),\\ Z_3(J) &:= \mathrm{H}_r^1(J;\mathrm{L}_p(\Omega))\cap \mathrm{B}(J;\mathrm{H}_p^1(\Omega)). \end{aligned}$$

Of course, the parameters p and r have to be restricted,

$$p\in(\hat p,\infty),\quad r\in[1,\infty),\quad \hat p := \max\{4,n\}.$$

Regarding the coefficients η ,λ, γ, ε we have to prescribe positivity, that is, these functions are subject to the condition

$$\eta(z),\ 2\eta(z)+\lambda(z)>0,\quad \forall z\in J\times\overline{\Omega}\times\mathbb{R},\quad \varepsilon(z),\ \gamma(z)>0,\quad \forall z\in J\times\overline{\Omega}\times\mathbb{R}^2; \tag{2.9}$$

with respect to their regularity,

$$\begin{aligned} \eta,\lambda &\in \mathrm{C}^\beta(J;\mathrm{C}(\overline{\Omega};\mathrm{C}^4(\mathbb{R})))\cap \mathrm{C}(J;\mathrm{C}^2(\overline{\Omega};\mathrm{C}^4(\mathbb{R}))),\quad \beta>1/2,\\ \gamma &\in \mathrm{C}(J\times\overline{\Omega};\mathrm{C}^2(\mathbb{R}^2)),\quad \varepsilon\in \mathrm{C}(J\times\overline{\Omega};\mathrm{C}^4(\mathbb{R}^2)). \end{aligned} \tag{2.10}$$

Further, the external force f_{ext} has to be in

$$\mathcal{X}_1(J;\mathbb{R}^n) := \mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega;\mathbb{R}^n))\cap \mathrm{L}_p(J;\mathrm{H}_p^2(\Omega;\mathbb{R}^n)).$$

Our main result in the homogeneous case is the following.

Theorem 2.1 *Let Ω be a bounded domain in $\mathbb{R}^n$, $n\geq 1$, with compact C^4-boundary Γ decomposing disjointly as $\Gamma=\Gamma_d\cup\Gamma_s$, $J=[0,T]$ with $T\in(0,\infty)$ and $p\in(\hat p,\infty)$. Further, let $\overline{\psi}\in \mathrm{C}^{5-}(\mathbb{R}^2)$ and assume (2.9), (2.10). Then for each $f_{ext}\in\mathcal{X}_1(J;\mathbb{R}^n)$ and initial data (u_0,c_0,ρ_0) in*

$$\mathcal{V} := \mathrm{W}_p^{4-\frac{2}{p}}(\Omega;\mathbb{R}^n)\times \mathrm{W}_p^{4-\frac{4}{p}}(\Omega)\times\{\varphi\in \mathrm{H}_p^3(\Omega;\mathbb{R}_+):\varphi(x)>0,\ \forall x\in\overline{\Omega}\}$$

satisfying the compatibility conditions

$$\begin{aligned} &u_{0|\Gamma_d}=0,\quad (u_0|\nu)_{|\Gamma_s}=0,\quad \mathcal{QS}(u)_{|t=0,\Gamma_s}\cdot\nu_{|\Gamma_s}=0,\quad \partial_\nu c_0=0,\quad \partial_\nu\mu(\rho_0,c_0)=0,\\ &-\nabla\cdot\mathcal{S}(u)_{|t=0,\Gamma_d}=\nabla\cdot\mathcal{P}_{|t=0,\Gamma_d}+(\rho f_{ext})_{|t=0,\Gamma_d}\in \mathrm{W}_p^{2-\frac{3}{p}}(\Gamma_d;\mathbb{R}^n),\\ &-(\nabla\cdot\mathcal{S}(u)_{|t=0}|\nu)_{|\Gamma_s}=(\nabla\cdot\mathcal{P}_{|t=0}-\rho_0\nabla u_0\cdot u_0+\rho_0 f_{ext|t=0}|\nu)_{|\Gamma_s}\in \mathrm{W}_p^{2-\frac{3}{p}}(\Gamma_s),\\ &-\mathcal{QS}(\nabla\cdot\mathcal{S}(u))_{|t=0,\Gamma_s}\cdot\nu_{|\Gamma_s}=\mathcal{QS}(\nabla\cdot\mathcal{P}-\rho\nabla u\cdot u+\rho f_{ext})_{|t=0,\Gamma_s}\cdot\nu_{|\Gamma_s}\in \mathrm{W}_p^{1-\frac{3}{p}}(\Gamma_s;\mathbb{R}^n), \end{aligned} \tag{2.11}$$

there is a unique solution (u,c,ρ) of (2.1)-(2.8) on a maximal time interval $[0,T^*)$, $T^*\leq T$. The solution (u,c,ρ) belongs to the class $\mathcal{Z}(J_0)$ for each interval $J_0=[0,T_0]$ with $T_0<T^*$. The maximal time interval is characterized by the property:

$$\lim_{t\to T^*} w(t)\quad \textit{does not exist in } \mathcal{V}. \tag{2.12}$$

The solution map $(u_0, c_0, \rho_0) \to (u, c, \rho)(t)$ *generates a local semiflow on the phase space* $\mathcal{V}_p := \{v \in \mathcal{V} : v$ *satisfies* (2.11) $\}$ *in the autonomous case.*

Remark 2.1 Our result is on the original Lowengrub-Truskinovsky system. A similar model has recently been treated by Abels and Feireisl [2]. These authors simplify the Lowengrub-Truskinovsky system by suppressing the factor ρ in the Helmholtz free energy and show existence of global weak solutions for the modified system; they do not show uniqueness or regularity. A similar model for incompressible fluids was studied by Boyer [4], Liu and Shen [15], Starovoitov [22], and Abels [1].

Remark 2.2 The methods of this paper apply also to the Navier-Stokes and the Navier-Stokes-Allen-Cahn system, cf. [12], [13].

Remark 2.3 *(i)* The purpose of this remark is supposed to clarify the choice of the solution class $\mathcal{Z}(J)$ and the spaces $Z_i(J)$ as well as the conditions on p and q. First of all, the central auxiliary means is the contraction mapping theorem, that is, we have to find a fixed point formulation being equivalent to the starting problem, and to establish selfmapping and contraction for this equation. Having in mind these both conditions, let us begin with the regularity class $\mathcal{Z}_3(J)$ which is of substantial interest. At first, observe that the Cahn-Hilliard equation contains a third order term of ρ, and thus we need $\rho \in \mathrm{L}_p(J; \mathrm{H}^3_p(\Omega))$ at least, when looking for strong solutions. Since ρ is governed by the hyperbolic equation (2.3), ρ only inherits the spatial regularity prescribed by the data ρ_0 and u. Hence we have to demand $\rho_0 \in \mathrm{H}^3_p(\Omega)$ and $u \in \mathrm{L}_p(J; \mathrm{H}^4_p(\Omega; \mathbb{R}^n))$. On the other hand, to obtain such a high spatial regularity for u, we are forced to study the Navier-Stokes equation in $\mathrm{L}_p(J; \mathrm{H}^2_p(\Omega; \mathbb{R}^n))$ at least, which causes a strong coupling between (2.1) and (2.2). In fact, let us suppose that ρ and all lower order terms of u are given and have sufficient regularity. Then, from the maximal L_p-regularity point of view the Cahn-Hilliard equation (2.2) can be uniquely solved in $\mathcal{Z}_2(J)$. Now, due to the mixed derivative theorem we deduce

$$\partial_{x_i} \nabla c \in \mathcal{X}_1(J; \mathbb{R}^n) = \mathrm{H}^{1/2}_p(J; \mathrm{L}_p(\Omega; \mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}^2_p(\Omega; \mathbb{R}^n)), \quad i = 1, \ldots, n$$

which are the highest order terms of c in the Navier-Stokes equation (2.1). Considering these terms as input or, in other words, taking $\mathcal{X}_1(J; \mathbb{R}^n)$ as the basic space for (2.1), we realise that $\partial_{x_i} \nabla c$ are of the same order as $\partial_t u$ and $\nabla \cdot \mathcal{S}(u)$ and thus responsible for the strong coupling. Also notice that we might expect $u \in \mathcal{Z}_1(J)$ in view of maximal L_p-regularity for the Navier-Stokes equation. Of course, we left out of consideration a precise characterization of the regularity of ρ, which is in fact essential, because several terms of ρ appear in (2.1) and (2.2). But, if we for the time being neglect this circumstance then selfmapping does work, since only first order terms of u appear in the Cahn-Hilliard equation (2.2) and this input is compatible with the basic space $\mathcal{X}_2(J) := \mathrm{L}_p(J; \mathrm{L}_p(\Omega))$ which in turn gives rise to the expected regularity $\mathcal{Z}_2(J)$ for c.

(ii) Turning to the proof of contraction with the setting above, one realises that it seems to be impossible to derive a contraction inequality for (2.3) in terms of the classes $\mathcal{Z}_1(J)$, $\mathcal{Z}_2(J)$ and $\mathcal{Z}_3(J)$, whereas in $Z_1(J)$, $Z_2(J)$ and $Z_3(J)$ the situation changes completely, see remark 3.1. Exactly on that account the second assembly of function spaces are of vital importance for approaching contraction in this manner, see [8] taking up this idea as well. Moreover, these spaces have another advantage over the classes $\mathcal{Z}_i(J)$ due to the relation $\mathcal{Z}_i(J) \subset Z_i(J)$, $i = 1, 2, 3$, which truly results in fewer estimates.

As remarked above, the contraction mapping principle is the central tool to tackle the nonlinear problem (2.1)-(2.8). For this, we introduce the closed subset $\Sigma \subset \mathcal{Z}_1(J) \times \mathcal{Z}_2(J)$,

$$\Sigma := \{(u,c) \in \mathcal{Z}_1(J) \times \mathcal{Z}_2(J) : \ (u,c)(0) = (u_0, c_0), \\ \|(u,c) - (\overline{u}, \overline{c})\|_{{}_0\mathcal{Z}_1(J) \times {}_0\mathcal{Z}_2(J)} \leq R_0\}, \quad (2.13)$$

in which solutions (u,c) of (2.3) - (2.8) are to be sought. Here the parameters R_0, T and the reference function $(\overline{u}, \overline{c})$ can be chosen appropriate to make the proof of contraction and selfmapping possible. As for the unknown ρ, we will see in Section 3.2 that there is a solution operator L depending on u and ρ_0 such that $\rho(t,x) = L[u, \rho_0](t,x)$ is the unique solution of (2.3). Inserting this solution formula into the PDE for (u,c), the starting problem is reduced to a nonlocal, fully nonlinear equation for (u,c), which is then locally solved by means of a fixed point argument. Afterwards this unique solution (u,c) gives rise to $\rho \in \mathcal{Z}_3(J)$ according to $\rho = L[u, \rho_0]$.

Picking up the idea of showing the contraction inequality with respect to the topology of $Z(J)$, one has to ensure that Σ is a closed subset in $Z(J)$, which proves to be true if $\mathcal{Z}(J) \hookrightarrow\hookrightarrow Z(J)$ or $\mathcal{Z}(J) \overset{d}{\hookrightarrow} Z(J)$ with $\mathcal{Z}(J)$ reflexive.

Lemma 2.1 *$\Sigma \subset Z(J)$ is a closed subset regarding to the topology of $Z(J)$.*

Proof. The assertion of this lemma bases on one of the following more general statements:
Auxiliary Lemma A.1 *Let X, Y be Banach spaces, such that, the identity operator I_{id} belongs to $\mathcal{K}(Y,X)$, the set of all compact operators $K : Y \to X$, $K \in \mathcal{L}(Y,X)$. Then the ball $B_r(0) := \{y \in Y : \|y\|_Y \leq r\}$ is closed regarding to the topology of X.*

Proof of the auxiliary Lemma A.1. Let $y_n \in B_r(0)$ be a sequence converging to y in X, that is, $\|y_n - y\|_X \to 0$. Then we have to show $y \in B_r(0)$. Since $B_r(0)$ is bounded in Y, there exists a subsequence y_{n_k} such that $y_{n_k} \to \tilde{y}$ weakly, $\tilde{y} \in Y$. To see $y = \tilde{y}$, we consider $y - \tilde{y}$ in X which can be estimated by

$$\|y - \tilde{y}\|_X \leq \|y_{n_k} - y\|_X + \|y_{n_k} - \tilde{y}\|_X.$$

The first norm converges to 0 as $k \to \infty$, because of the assumption. Since $I_{id} : Y \to X$ is a compact mapping, weak converging sequences are mapped to strong converging sequences, that is, from $y_{n_k} \to \tilde{y}$ weakly we may deduce $\|y_{n_k} - \tilde{y}\|_X \equiv \|I_{id}(y_{n_k} - \tilde{y})\|_X \to 0$, as $k \to \infty$. Finally, it generally holds: $\|y\|_Y \leq \liminf_{n\to\infty} \|y_n\|_Y$ and this shows $y \in B_r(0)$. □

Auxiliary Lemma A.2 *Let X, Y be Banach spaces with $Y \hookrightarrow X$ densely, Y reflexive. Then the ball $B_r(0) := \{y \in Y : \|y\|_Y \leq r\}$ is closed with respect to the topology of X.*

Proof of the auxiliary Lemma A.2. This time we reason with the difference $y - \tilde{y}$ differently. In view of the assumption $Y \overset{d}{\hookrightarrow} X$ with Y reflexive, we know by [3, Proposition 1.4.8, p. 271] that $X' \overset{d}{\hookrightarrow} Y'$ and $\langle x'|y\rangle_{X',X} = \langle x'|y\rangle_{Y',Y}$ for all $y \in Y$, $x' \in X'$. But this implies

$$\forall x' \in X' : \quad \langle x'|y - \tilde{y}\rangle_{X',X} = \langle x'|y - y_{n_k}\rangle_{X',X} + \langle x'|y_{n_k} - \tilde{y}\rangle_{Y',Y} < \varepsilon,$$

for all $\varepsilon > 0$, as $y - y_{n_k}$ converges strongly in X and $y_{n_k} - \tilde{y}$ weakly in Y. But this means $y - \tilde{y} = 0$. □

Thus, by the Auxiliary Lemma A.1, we only need to show that $\mathcal{Z}(J)$ is compactly embedded into $Z(J)$. But this follows from the mixed derivative theorem. For instance, it holds $\mathcal{Z}_2(J) \hookrightarrow \mathrm{H}^{\theta}_p(J;\mathrm{H}^{4(1-\theta)}_p(\Omega))$ for all $\theta \in (0,1)$. Now choosing $\theta = 3/4$ and $\theta = 1/4$, we get $\mathcal{Z}_2(J) \hookrightarrow \mathrm{H}^{3/4}_p(J;\mathrm{H}^1_p(\Omega)) \hookrightarrow\hookrightarrow \mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\Omega))$ and $\mathcal{Z}_2(J) \hookrightarrow \mathrm{H}^{1/4}_p(J;\mathrm{H}^3_p(\Omega)) \hookrightarrow\hookrightarrow \mathrm{L}_p(J;\mathrm{H}^2_p(\Omega))$, respectively. The space $\mathcal{Z}_1(J)$ can be treated similarly.

Considering unbounded domains, the compact embeddings used above are no longer valid, but all embeddings are still dense. Therefore and by reflexivity of L_p-spaces, $p \in (1,\infty)$, the Auxiliary Lemma A.2 gives rise to the wished result for unbounded domains as well.

□

Remark 2.4 If one aims at solving quasilinear problems strongly, it is required that all coefficients belong to multiplier spaces associated to the chosen basic spaces. This fact brings about the condition $p > \hat{p}$. Note that if we switch over to constant coefficients η, λ, γ, ε our problem stays quasilinear, because in any case ρ is present in front of $\partial_t u$ and $\partial_t c$.

Remark 2.5 At last we want to point out that an energy identity is available by means of multiplying (2.1) with u, integrating over Ω, integration by parts, and using the identity $\nabla\cdot\mathcal{P} \equiv -\rho\nabla(\psi + \rho\partial_\rho\psi) + \rho\,\mu\nabla c$. This leads to the result

$$\frac{d}{dt}E_\Omega(t) + \int_\Omega \mathcal{S}(u,\rho) : \mathcal{D}(u)\,dx + \int_\Omega \gamma(\rho,c)|\nabla\mu|^2\,dx = \int_\Omega \rho f_{ext}\cdot u\,dx, \quad \forall t > 0.$$

2.2. Formulation of the fixed point equation

In this section the nonlinear equations (2.1), (2.2) and their corresponding boundary conditions are reformulated such that the left-hand side becomes linear and the starting problem can be transferred to a fixed point equation. We point out that the linearisation is carried out to such an extent that the elliptic operator, appearing in the Cahn-Hilliard, maintains its divergence structure and can be viewed as the square of an elliptic operator. This feature will be essential to accomplish a contraction inequality for (2.2) in the space $Z_2(J)$, see section 3.3. The governing equations for u and c can be written as

$$\begin{aligned} \tilde{\rho}\partial_t u - \nabla\cdot\tilde{\mathcal{S}}(u) + \tilde{\rho}^2\tilde{\varepsilon}_\rho\nabla\tilde{c}\cdot\nabla^2 c + \tilde{\rho}\tilde{\varepsilon}\nabla\tilde{c}\cdot[\Delta c\mathcal{I} + \nabla^2 c] &= F_1(u,c,\rho), && (t,x)\in J\times\Omega,\\ j = d,s:\quad \mathcal{B}_j u &= \sigma_j(u,\rho), && (t,x)\in J\times\Gamma_j,\\ u &= u_0, && (t,x)\in\{0\}\times\Omega, \end{aligned} \tag{2.14}$$

$$\begin{aligned} \tfrac{\varepsilon_0\rho_0}{\gamma_0}\partial_t c + \nabla\cdot(\varepsilon_0\nabla\nabla\cdot(\varepsilon_0\nabla c)) &= F_2(u,c,\rho), && (t,x)\in J\times\Omega,\\ \partial_\nu c = 0,\quad \partial_\nu\nabla\cdot(\varepsilon_0\nabla c) &= \partial_\nu g(\rho,c) && (t,x)\in J\times\Gamma,\\ c &= c_0, && (t,x)\in\{0\}\times\Omega, \end{aligned} \tag{2.15}$$

where we have set $a_0 := a_{|t=0}$ for $a \in \{\gamma,\varepsilon\}$, $\tilde{a} := a(\tilde{\rho},\tilde{c})$ for $a \in \{\varepsilon,\varepsilon_\rho\}$ and $\tilde{\mathcal{S}}(u) := 2\tilde{\eta}\mathcal{D}(u) + \tilde{\lambda}\nabla\cdot u\mathcal{I}$ with $\tilde{a} := a_{|\rho=\tilde{\rho}}$ for $a \in \{\rho,\eta,\lambda\}$. Here the function $(\tilde{\rho},\tilde{c})$ belongs to $\mathcal{Z}_2(\mathbb{R}_+)\times\mathcal{Z}_3(\mathbb{R}_+)$ and fulfils the constraints $\tilde{c}_{|t=0} = c_0$, $\partial_t^k\tilde{\rho}(0) = \partial_t^k\rho(0)$ for $k = 0,1,2$. Observe that $\partial_t^k\rho(0)$ for $k = 0,1,2$ is completely known due to the possibility of taking the trace at $t = 0$ in (2.1) and (2.3). This kind of approximation[1] is needed, for instance, as the

[1]For instance, let $\tilde{\rho}$ be the solution of $\partial_t\tilde{\rho} + \nabla\cdot(\tilde{\rho}\tilde{u}) = 0$, where $\tilde{u} \in \mathcal{Z}_1(J)$ satisfies $\tilde{u}(0) = u_0$ and $\partial_t\tilde{u}(0) = -\nabla u_0\cdot u_0 + \rho_0^{-1}[\mathcal{S}_{|t=0} + \mathcal{P}_{|t=0}] + f_{|t=0}] \equiv \partial_t u(0)$. This is possible due to the 'high regularities'.

boundary equations for u have to be considered in trace classes with high time regularity, cf. condition 2. in Theorem 3.2. In the case of Cahn-Hilliard we are working in the 'usual L_p-setting', which makes possible to take ρ_0 as approximation. The boundary operators $\mathcal{B}_j$ acting on Γ_j and the data σ_j are defined according to

$$\mathcal{B}_d u := u_{|\Gamma_d}, \quad \mathcal{B}_s u := ((u|\nu)_{|\Gamma_s}, \mathcal{Q}\tilde{\mathcal{S}}(u)\cdot \nu_{|\Gamma_s}), \quad \mathcal{Q}\tilde{\mathcal{S}}(u)\cdot\nu_{|\Gamma_s} = 2\tilde{\eta}\mathcal{Q}\mathcal{D}(u)\cdot\nu_{|\Gamma_s},$$
$$\sigma_d(u,\rho) := 0, \quad \sigma_s(u,\rho) := (0, \mathcal{Q}[\tilde{\mathcal{S}}(u) - \mathcal{S}(u)]\cdot\nu_{|\Gamma_s}) = (0, 2(\tilde{\eta}-\eta)\mathcal{Q}\mathcal{D}(u)\cdot\nu_{|\Gamma_s}).$$

The nonlinearities F_1, F_2 and g, given by

$$\begin{aligned} F_1(u,c,\rho) &:= (\tilde{\rho}-\rho)\partial_t u - \rho\nabla u\cdot u + \nabla\cdot[\mathcal{S}(u)-\tilde{\mathcal{S}}(u)] - [\rho^2\varepsilon_\rho\nabla c - \tilde{\rho}^2\tilde{\varepsilon}_\rho\nabla\tilde{c}]\cdot\nabla^2 c \\ &\quad - \tfrac{1}{2}\nabla(\rho^2\varepsilon_\rho)|\nabla c|^2 - [\rho\varepsilon\nabla c - \tilde{\rho}\tilde{\varepsilon}\nabla\tilde{c}](\Delta c\mathcal{I} + \nabla^2 c) - \nabla(\rho\varepsilon)\cdot\nabla c\nabla c \\ &\quad + \rho f_{ext}, \\ F_2(u,c,\rho) &:= \tfrac{\varepsilon_0}{\gamma_0}\big\{\partial_t\big([\rho_0-\rho]c\big) - \nabla\cdot(c\rho u) + \nabla\cdot\big([\gamma_0-\gamma]\nabla(\nabla\cdot(\varepsilon_0\nabla c)+g)\big)\big\} \\ &\quad - \nabla\cdot(\varepsilon_0\nabla g) - \tfrac{\varepsilon_0^2}{\gamma_0}\nabla(\tfrac{\gamma_0}{\varepsilon_0})\cdot\nabla[\nabla\cdot(\varepsilon_0\nabla c) - g], \\ g(\rho,c) &:= \nabla\cdot([\varepsilon-\varepsilon_0]\nabla c) + \rho^{-1}\varepsilon\nabla\rho\cdot\nabla c - \partial_c\psi \end{aligned} \tag{2.16}$$

comprise all nonlinear terms of lower order as well as perturbations of quasilinear terms. In the following we want to associate (2.14) and (2.15) with the abstract equation

$$\mathcal{L}(u,c) \equiv (\mathcal{L}_1 u, \mathcal{L}_2 c) = (\mathcal{F}_1(u,c,\rho), u_0, \mathcal{F}_2(u,c,\rho), c_0) =: \mathcal{F}(u,c,\rho), \quad \rho = L[u,\rho_0], \tag{2.17}$$

i.e. $\mathcal{L}$ reflects the linear operator of the left-hand side (2.14), (2.15) splitting up to $\mathcal{L}_1$ and $\mathcal{L}_2$ due to decoupling of the associated linear problems, and $L[u,\rho_0]$ denotes the solution operator to the equation of conservation of mass, see section 3.2. Further, $\mathcal{F}_i$ comprises the nonlinearity F_i as well as the nonlinear boundary data,

$$\begin{aligned} \mathcal{F}_1(u,c,\rho) &:= (F_1(u,c,\rho), \sigma_d(u,\rho), \sigma_s(u,\rho)), \\ \mathcal{F}_1(u,c,\rho) &:= (F_2(u,c,\rho), 0, \partial_\nu g_0(\rho,c)). \end{aligned} \tag{2.18}$$

Then the equation (2.17) defines a nonlinear mapping $\mathcal{G} : \mathcal{Z}_1(J)\times\mathcal{Z}_2(J) \to \mathcal{Z}_1(J)\times\mathcal{Z}_2(J)$ according to

$$\begin{aligned} \mathcal{G} : w := (u,c) &\longrightarrow w' := (u',c'), \\ \mathcal{L}(u',c') &= \mathcal{F}(u,c,L[u,\rho_0]), \end{aligned} \tag{2.19}$$

for which we want to prove selfmapping in Σ and contraction regarding to the weaker topology of Z.

3. Preliminary results

3.1. Maximal regularity for Cahn-Hilliard and a viscous fluid

The isomorphism property of $\mathcal{L}$ corresponds to prove maximal regularity for (2.14) and (2.15) with given right-hand side. Since in this case the equations for u and c decouple, that is, one firstly solves the linear Cahn-Hilliard equation (2.15) and put this solution into

(2.14), we are in the position to study separated problems. Therefore, in the formulation below all terms in (2.14) involving c are known and plugged into the data f.

The first result concerns the linear Cahn-Hillard problem (2.15). More precisely, the linear operator reflected by these equations turns out to be an isomorphism between $\mathcal{Z}_2(J)$ and a certain basic space. The equations we have to study are

$$\begin{aligned} \tfrac{\varepsilon_0\rho_0}{\gamma_0}\partial_t c + \nabla\cdot(\varepsilon_0\nabla\nabla\cdot(\varepsilon_0\nabla c)) &= f(t,x), && (t,x)\in J\times\Omega,\\ \partial_\nu c = \sigma_1(t,x), \quad \partial_\nu\nabla\cdot(\varepsilon_0\nabla c) &= \sigma_2(t,x), && (t,x)\in J\times\Gamma,\\ c &= c_0(x), && (t,x)\in\{0\}\times\Omega, \end{aligned} \tag{3.1}$$

for which existence and uniqueness in $\mathcal{Z}_2(J)$ can be proved.

Theorem 3.1 *Let Ω be a bounded domain in $\mathbb{R}^n$, $n\geq 1$, with compact C^4-boundary Γ, $J=[0,T]$ a compact time interval, and $p>\max\{1,\frac{n}{3}\}$ with $p\neq\frac{5}{3}, 5$. Further, assume that ρ_0, γ_0, $\varepsilon_0\in \mathrm{H}^3_p(\Omega)$ and $\rho_0(x)$, $\gamma_0(x)$, $\varepsilon_0(x)>0$ for all $x\in\overline{\Omega}$. Then problem (3.1) possesses a unique solution $c\in\mathcal{Z}_2(J)$ if and only if the data f, $\sigma=(\sigma_1,\sigma_2)$, c_0 satisfy the following conditions*

1. $f\in\mathcal{X}_2(J) := \mathrm{L}_p(J;\mathrm{L}_p(\Omega))$;
2. $(\sigma_1,\sigma_2)\in\mathcal{Y}_1(J)\times\mathcal{Y}_3(J)$ *with* $\mathcal{Y}_k(J) := \mathrm{W}_p^{1-\frac{k}{4}-\frac{1}{4p}}(J;\mathrm{L}_p(\Gamma))\cap\mathrm{L}_p(J;\mathrm{W}_p^{4-k-\frac{1}{p}}(\Gamma))$;
3. $c_0\in\mathrm{W}_p^{4-\frac{4}{p}}(\Omega)$;
4. $\partial_\nu c_0 = \sigma_{1|t=0}$ *in* $\mathrm{W}_p^{3-\frac{5}{p}}(\Gamma)$ *for* $p>\frac{5}{3}$ *and* $\partial_\nu\varepsilon_0\Delta c_0 = \sigma_{2|t=0}$ *in* $\mathrm{W}_p^{1-\frac{5}{p}}(\Gamma)$ *for* $p>5$.

Proof. This result is very well-known and follows from [5], also cf. [20] and [21]. □

The remainder equations of the linearization represent linear Navier-Stokes (without density) supplemented with boundary conditions and initial data,

$$\begin{aligned} \tilde{\rho}\partial_t u - \nabla\cdot\tilde{\mathcal{S}}(u) &= f(t,x), && (t,x)\in J\times\Omega,\\ u=\sigma_d(t,x),\quad (t,x)\in J\times\Gamma_d,\quad ((u|\nu),\mathcal{Q}\tilde{\mathcal{S}}(u)\cdot\nu) &= \sigma_s(t,x) && (t,x)\in J\times\Gamma_s,\\ u &= u_0(x), && (t,x)\in\{0\}\times\Omega \end{aligned} \tag{3.2}$$

with $\tilde{\mathcal{S}}(u) = 2\tilde{\eta}\mathcal{D}(u) + \tilde{\lambda}\nabla\cdot u\mathcal{I}$ and $\sigma_s=(\sigma_1,\sigma_2)\in\mathbb{R}\times\mathbb{R}^n$, for which the following (non-standard) maximal regularity result can be proved.

Theorem 3.2 *Let Ω be a bounded domain in $\mathbb{R}^n$, $n\geq 1$, with compact C^4-boundary Γ decomposing disjointly $\Gamma=\Gamma_d\cup\Gamma_s$. Let $J=[0,T]$ and $p>\max\{\frac{4}{3},\frac{n}{3}\}$ with $p\neq\frac{3}{2}$, 3. Further, assume that $\tilde{\rho}$, $\tilde{\eta}$, $\tilde{\lambda}\in \mathrm{C}^\beta(J;\mathrm{C}(\overline{\Omega}))\cap\mathrm{C}(J;\mathrm{C}^2(\overline{\Omega}))$, $\beta>1/2$, and $\tilde{\eta}$, $\tilde{\lambda}\in \mathrm{H}_p^{1/2}(J;\mathrm{H}^2_p(\Omega))\cap\mathrm{L}_\infty(J;\mathrm{H}^3_p(\Omega))$, as well as $\tilde{\rho}(t,x)$, $\tilde{\eta}(t,x)$, $2\tilde{\eta}(t,x)+\tilde{\lambda}(t,x)>0$ for all $(t,x)\in J\times\overline{\Omega}$. Then problem (3.2) possesses a unique solution in*

$$\begin{aligned} \mathcal{Z}_{1,\mathcal{B}}(J) := \big\{v\in\mathcal{Z}_1(J):\quad & \mathcal{B}_d v\in\mathrm{W}_p^{2-\frac{1}{2p}}(J;\mathrm{L}_p(\Gamma_d;\mathbb{R}^n)),\\ & \mathcal{B}_s v\in\mathrm{W}_p^{2-\frac{1}{2p}}(J;\mathrm{L}_p(\Gamma_s))\times\mathrm{W}_p^{\frac{3}{2}-\frac{1}{2p}}(J;\mathrm{L}_p(\Gamma_s;\mathbb{R}^n))\big\}, \end{aligned}$$

if and only if the data f, σ_d, $\sigma_s=(\sigma_1,\sigma_2)$, u_0 satisfy the following conditions

1. $f\in\mathcal{X}_{1,\Gamma} := \{\varphi\in\mathcal{X}_1 : \varphi_{|t=0,\Gamma_d}\in\mathrm{W}_p^{2-3/p}(\Gamma_d),\quad (\varphi_{|t=0}|\nu)_{|\Gamma_s}\in\mathrm{W}_p^{2-3/p}(\Gamma_s),$ $\mathcal{Q}\tilde{\mathcal{S}}(\varphi)_{|t=0,\Gamma_s}\in\mathrm{W}_p^{1-3/p}(\Gamma_s;\mathbb{R}^n)\}$;

2. $(\sigma_d, \sigma_1, \sigma_2) \in \mathcal{Y}_{0,d}(J; \mathbb{R}^n) \times \mathcal{Y}_{0,s}(J) \times \mathcal{Y}_{1,s}(J; \mathbb{R}^n)$ *with* $\mathcal{Y}_{k,i}(J; E) :=$ $\mathrm{W}_p^{2-\frac{k}{2}-\frac{1}{2p}}(J; \mathrm{L}_p(\Gamma_i; E)) \cap \mathrm{L}_p(J; \mathrm{W}_p^{4-k-\frac{1}{p}}(\Gamma_i; E))$, $k = 0, 1$, $i = d, s$, $E \in \{\mathbb{R}^n, \mathbb{R}\}$;
3. $u_0 \in \mathrm{W}_p^{4-\frac{2}{p}}(\Omega; \mathbb{R}^n)$;
4. $u_{0|\Gamma_d} = \sigma_{d|t=0}$ *in* $\mathrm{W}_p^{4-\frac{3}{p}}(\Gamma_d; \mathbb{R}^n)$;
5. $(u_0|\nu)_{|\Gamma_s} = \sigma_{1|t=0}$ *in* $\mathrm{W}_p^{4-\frac{3}{p}}(\Gamma_s)$, $\mathcal{Q}\tilde{\mathcal{S}}(u)_{|t=0} \cdot \nu_{|\Gamma_s} = \sigma_{2|t=0}$ *in* $\mathrm{W}_p^{3-\frac{3}{p}}(\Gamma_s; \mathbb{R}^n)$;
6. $\tilde{\rho}_{|t=0,\Gamma_d} \partial_t \sigma_{d|t=0} - \nabla \cdot \tilde{\mathcal{S}}(u)_{|t=0,\Gamma_d} = f_{|t=0,\Gamma_d}$ *in* $\mathrm{W}_p^{2-\frac{3}{p}}(\Gamma_d; \mathbb{R}^n)$ *and* $\tilde{\rho}_{|t=0,\Gamma_s} \partial_t \sigma_{1|t=0} - (\nabla \cdot \tilde{\mathcal{S}}(u)_{|t=0}|\nu)_{|\Gamma_s} = (f_{|t=0}|\nu)_{|\Gamma_s}$ *in* $\mathrm{W}_p^{2-\frac{3}{p}}(\Gamma_s)$ *if* $p > \frac{3}{2}$;
7. $\partial_t \sigma_{2|t=0} - (\frac{\partial_t \tilde{\eta}}{\tilde{\eta}})_{|t=0,\Gamma_s} \sigma_{2|t=0} - \mathcal{Q}\tilde{\mathcal{S}}(\tilde{\rho}^{-1} \nabla \cdot \tilde{\mathcal{S}}(u))_{|t=0,\Gamma_s} \cdot \nu_{|\Gamma_s} = \mathcal{Q}\tilde{\mathcal{S}}(\tilde{\rho}^{-1} f)_{|t=0,\Gamma_s} \cdot \nu_{|\Gamma_s}$ *in* $\mathrm{W}_p^{1-\frac{3}{p}}(\Gamma_s; \mathbb{R}^n)$ *if* $p > 3$.

Proof. The crucial point is to verify the higher regularities of u. In fact, maximal L_p-regularity is very well-known, for instance, a consequence of [5].

(i) Necessity. The necessary part is only a consequence of trace theory, where one has to be attentive in respect of all possible traces in the differential equation and thus additional compatibility conditions for the data. Also note that the additional spaces in $\mathcal{Z}_{1,\mathcal{B}}$ issue from the "better" regularity of boundary data which would actually give rise to a more regular solution $u \in \mathrm{H}_p^2(J; \mathrm{L}_p(\Omega; \mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}_p^4(\Omega; \mathbb{R}^n))$.

(ii) Sufficiency. Since maximal L_p-regularity for this problem in the 'usual setting', is very well-known, our task actually consists in recalculating the regularity of u on the basis of a solution formula. For this, we have to go back to the associated half (and full) space problems with constant coefficients, because in this case an explicit solution formula is available. For the sake of brevity, we will only deal with the localised problem issuing from the boundary Γ_s. (The other case is even simpler and can be approached in the same way.) As for the localization, we follow the strategy for general parabolic problems. The starting point is localisation w.r.t. space: we choose a partition of unity $\varphi_j \in \mathrm{C}_0^\infty(\mathbb{R}^n)$, $j = 1, \ldots, N$, with $0 \leq \varphi_j \leq 1$ and $\operatorname{supp} \varphi_j =: U_j$, such that the domain is covered $\overline{\Omega} \subset \bigcup_{j=1}^N U_j$. After multiplying all equations of (3.2) by each φ_j and commuting φ_j with differential operators we obtain local problems for $(u_j, \rho_j) := (\varphi_j u, \varphi_j \rho)$, $j = 1, \ldots, N$. Considering local coordinates in $\overline{\Omega} \cap U_j$ and coordinate transformations θ_j which are C^{5-}-diffeomorphisms due to smoothness assumptions on the boundary, the original problem is reduced to a finite number of so-called full-space problems related to $U_j \subset \mathring{\Omega}$ $(U_j \cap \partial\Omega = \emptyset)$ and half-space problems for $U_j \cap \partial\Omega \neq \emptyset$. Further, the transformed differential operators enjoy the same ellipticity properties etc. as before, i.e. the principal part remains unchanged. Note that the transformation induces isomorphisms between Sobolev spaces, i.e.

$$\theta_j : \mathrm{W}_p^s(\Omega \cap U_j; E) \longrightarrow \mathrm{W}_p^s(\mathbb{R}_+^n \cap \theta_j(U_j); E), \quad E \text{ any Banach space,}$$

for each $p \in [1, \infty]$ and $0 \leq s \leq 4$. For these (full- and half-space) problems unique solutions will be available, and after summing up all local solutions we obtain a fixed point equation which can be solved first on a small time interval(!). Proceeding in this way the problem can be solved on the entire interval $[0, T]$ after finitely many steps. As to literature of localisation techniques for bounded domains, we refer to [14], [6]; a very detailed description of these techniques, with application to an example, can be found, for instance, in [25] and [11].

By means of localising and flattening the boundary, such that $\nu = (0, -1)^T$, we obtain model problems in the half space $\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times \mathbb{R}_+$ having the form

$$\begin{aligned} \partial_t u + u - \tilde{\eta}\Delta u - (\tilde{\lambda} + \tilde{\eta})\nabla\nabla\cdot u = f(t,y,x), \quad & t>0, \quad y>0, \quad x \in \mathbb{R}^{n-1}, \\ -\partial_y u^t = \theta(t,x), \quad u^n = \vartheta(t,x), \quad & t>0, \quad y=0, \quad x \in \mathbb{R}^{n-1}, \\ u = u_0(y,x), \quad & t=0, \quad y>0, \quad x \in \mathbb{R}^{n-1}, \end{aligned} \tag{3.3}$$

where we have set $u = (u^t, u^n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. The term u at the left side was inserted to make $-\Delta_x + I$ invertible, which is always possible as we localized a bounded domain. At first, we point out that maximal L_p-regularity in the 'usual setting' gives $u \in Z(J) := \mathrm{H}^1_p(J; \mathrm{L}_p(\mathbb{R}^n_+; \mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}^2_p(\mathbb{R}^n_+; \mathbb{R}^n))$. Next let us transfer the regularity assumptions and compatibility conditions to this half space problem. What is known about the data is the following

$$\begin{aligned} & f \in \mathrm{H}^{1/2}_p(J; \mathrm{L}_p(\mathbb{R}^n_+)) \cap \mathrm{L}_p(J; \mathrm{H}^2_p(\mathbb{R}^n_+)), \\ & \quad f^n_{|t=0,y=0} \in \mathrm{W}^{2-3/p}_p(\mathbb{R}^n), \quad \partial_y f^t_{|t=0,y=0} \in \mathrm{W}^{1-3/p}_p(\mathbb{R}^n; \mathbb{R}^{n-1}), \quad u_0 \in \mathrm{W}^{4-2/p}_p(\mathbb{R}^n_+; \mathbb{R}^n), \\ & \vartheta \in \mathrm{W}^{(4-1/p)\frac{1}{2}}_p(J; \mathrm{L}_p(\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J; \mathrm{W}^{4-1/p}_p(\mathbb{R}^{n-1})), \quad \partial^\beta_x \vartheta \in Y_0(J) \\ & \theta \in \mathrm{W}^{(3-1/p)\frac{1}{2}}_p(J; \mathrm{L}_p(\mathbb{R}^{n-1}; \mathbb{R}^{n-1})) \cap \mathrm{L}_p(J; \mathrm{W}^{3-1/p}_p(\mathbb{R}^{n-1}; \mathbb{R}^{n-1})), \quad \partial^\beta_x \theta \in Y_1(J; \mathbb{R}^{n-1}), \\ & \quad Y_i(J;E) := \mathrm{W}^{(2-i-1/p)\frac{1}{2}}_p(J; \mathrm{L}_p(\mathbb{R}^{n-1}; E)) \cap \mathrm{L}_p(J; \mathrm{W}^{2-i-1/p}_p(\mathbb{R}^{n-1}; E)), \quad i = 0, 1, \end{aligned}$$

where ∂^β_x with $\beta \in \mathbb{N}^n$, $|\beta| \le 2$, denotes tangential derivatives up to order 2. Moreover, the compatibility conditions take the form

$$\begin{aligned} u^n_{0|y=0} = \vartheta_{|t=0} \in \mathrm{W}^{4-3/p}_p(\mathbb{R}^n), \quad -[\partial_y u^t_0]_{|y=0} = \theta_{|t=0} \in \mathrm{W}^{3-3/p}_p(\mathbb{R}^n; \mathbb{R}^{n-1}), \\ \partial_t \vartheta_{|t=0} + \vartheta_{|t=0} - \tilde{\eta}[\Delta u^n_0]_{|y=0} = (\tilde{\lambda} + \tilde{\eta})[\partial_y \nabla\cdot u_0]_{|y=0} + f^n_{|t=0,y=0} \in \mathrm{W}^{2-3/p}_p(\mathbb{R}^n), \\ \partial_t \theta_{|t=0} + \theta_{|t=0} + \tilde{\eta}[\partial_y \Delta u^t_0]_{|y=0} = -(\tilde{\eta} + \tilde{\lambda})\partial_y \nabla_x \nabla\cdot u_{0|y=0} - \partial_y f^t_{|t=0,y=0} \in \mathrm{W}^{1-3/p}_p(\mathbb{R}^n; \mathbb{R}^{n-1}). \end{aligned} \tag{3.4}$$

As a first result, which can easily be verified by differentiating all equations of (3.3) with respect to ∂^β_x, we may claim $\partial^\beta_x u \in Z(J)$ and along with $u \in Z(J)$ this gives

$$u \in \mathrm{H}^1_p(J; \mathrm{H}^2_p(\mathbb{R}^n; \mathrm{L}_p(\mathbb{R}_+))) \cap \mathrm{L}_p(J; \mathrm{H}^2_p(\mathbb{R}^n; \mathrm{H}^2_p(\mathbb{R}_+))).$$

Hence it is left to show that the normal derivatives $\partial^j_y u$, $j \in \{1,2\}$, lie in $Z(J)$ as well as $\partial_t u \in \mathrm{H}^{1/2}_p(J; \mathrm{L}_p(\mathbb{R}^n_+; \mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}^2_p(\mathbb{R}^n_+; \mathbb{R}^n))$. To establish this regularity, we provide a solution formula of (3.3) from which the regularity can be read off. At first, it is useful to consider $v := \nabla\cdot u$ solving

$$\begin{aligned} \partial_t v - (2\tilde{\eta} + \tilde{\lambda})\Delta v = \nabla_x \cdot f^t + \partial_y f^n, \quad & t>0, \quad y>0, \quad x \in \mathbb{R}^{n-1}, \\ -\partial_y v = \psi, \quad & t>0, \quad y=0, \quad x \in \mathbb{R}^{n-1}, \\ v = \nabla\cdot u_0 =: v_0, \quad & t=0, \quad y>0, \quad x \in \mathbb{R}^{n-1} \end{aligned} \tag{3.5}$$

with $\psi = -\nabla_x \cdot \partial_y u^t_{|y=0} - \partial^2_y u^n_{|y=0} = \nabla_x \cdot \theta + (2\tilde{\eta} + \tilde{\lambda})^{-1}[f^n_{|y=0} - \partial_t \vartheta + \tilde{\eta}\Delta_x \vartheta - (\tilde{\eta} + \tilde{\lambda})\nabla_x \cdot \theta]$, in view of the identity $-(2\tilde{\eta} + \tilde{\lambda})\partial^2_y u^n = \tilde{\eta}\Delta_x u^n + (\tilde{\eta} + \tilde{\lambda})\nabla_x \cdot \partial_y u^t - \partial_t u^n + f^n$. Notice that ψ belongs to $\mathrm{W}^{1/2-1/4p}_p(J; \mathrm{L}_p(\mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{W}^{2-1/p}_p(\mathbb{R}^n))$ which comes from $f^n_{|y=0}$ having the least regularity. Further, the compatibility condition $-[\partial_y v_0]_{|y=0} = \psi_{|t=0}$ arises from the first three conditions of (3.4). A solution formula of (3.5) is very well-known, cf. [18], however we need a presentation allowing a verification of higher spatial regularity. In fact, the purpose is to establish the regularity $\partial_y v$, $B^{1/2}v \in \mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\mathbb{R}^n_+)) \cap \mathrm{L}_p(J;\mathrm{H}_p^2(\mathbb{R}^n_+))$, as these terms will appear at the right side in the model problem for u, see below. For this, let $B = -\Delta_x + I$ with domain $D(B) = \mathrm{H}_p^2(\mathbb{R}^n)$ and $A_{1/2} = \frac{1}{2}B - \partial_y^2$ with domain $D(A_{1/2}) = \{\varphi \in \mathrm{H}_p^2(\mathbb{R}^n_+) : \varphi_{|y=0} = 0\}$ and ϕ denote the unique solution of

$$
\begin{aligned}
-\partial_y^2\phi + \tfrac{1}{2}B\phi &= e^{-(B/2)^{1/2}y}g, \quad y>0, \quad g := g(v_0) := [\tfrac{1}{2}Bv_0 - \partial_y^2 v_0]_{|y=0} \qquad (3.6)\\
\phi(0) &= v_{0|y=0},
\end{aligned}
$$

which is given by

$$
\phi = \Phi(y)v_{0|y=0} + \tfrac{y}{2}B^{-1/2}\Phi(y)g \equiv \Phi(y)v_{0|y=0} + (D + (\tfrac{1}{2}B)^{1/2})^{-1}(\tfrac{1}{2}B)^{-1/2}\Phi(y)g, \quad (3.7)
$$

where Φ denotes the analytical semigroup $e^{-(B/2)^{1/2}y}$. Further, we have set $D = \partial_y$ with domain $D(D) = {}_0\mathrm{H}_p^1(\mathbb{R}_+;X)$, X any Banach space, and this operator is sectorial, invertible and belongs to $\mathcal{BIP}\,(\mathrm{L}_p(\mathbb{R}_+;X))$ with power angle $\pi/2$. Then ϕ belongs to $\mathrm{W}_p^{3-2/p}(\mathbb{R}^n_+)$ due to the regularities $v_{0|y=0} \in \mathrm{W}_p^{3-3/p}(\mathbb{R}^{n-1})$ and $g \in \mathrm{W}_p^{1-3/p}(\mathbb{R}^{n-1})$ as well as the mapping properties of Φ, see Proposition 6.1. In view of the construction of ϕ, we easily see that $v_0 - \phi \in A_{1/2}^{-1/2}D_{A_{1/2}}(1-1/p,p) \equiv \{\varphi \in \mathrm{W}_p^{3-2/p}(\mathbb{R}^n_+) : \varphi_{|y=0} = A_{1/2}\varphi_{|y=0} = 0\}$, if traces make sense. We further define $S_a(t) := e^{-aB/2t}$, $a = 2\tilde{\eta} + \tilde{\lambda}$, and $T_a(t) := e^{-aAt}$, where we have set $A := B - \partial_y^2$ with domain $D(A) = D(A_{1/2})$. Let $G = \partial_t$ with domain $D(G) = {}_0\mathrm{H}_p^1(J;X) := \{\varphi \in \mathrm{H}_p^1(J:X) : \varphi_{|t=0} = 0\}$ and $F_\alpha := (\alpha^{-1}G + B)^{1/2}$, any $\alpha > 0$, with domain $D(F_\alpha) = D(G^{1/2}) \cap D(B^{1/2}) \equiv {}_0\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J;\mathrm{H}_p^1(\mathbb{R}^{n-1}))$. These operators are sectorial, invertible and belong to $\mathcal{BIP}\left(\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^{n-1}))\right)$ with power angles $\theta_G \le \pi/2$ and $\theta_{F_\alpha} \le \pi/4$, respectively. Then v can be written as

$$
\begin{aligned}
v = v_1 + e^{-F_a y}F_a^{-1}[\psi - \partial_y v_{1|y=0}], \quad v_1 := T_a(t)[v_0 - \phi] + S_a(t)\phi +\\
T_a * \left\{\nabla\cdot f - e^{-F_{\tilde{\eta}}y}(\nabla\cdot f_{|y=0} - S_a(t)\nabla\cdot f_{|y=0,t=0}) - S_a(t)\Phi(y)\nabla\cdot f_{|y=0,t=0}\right\}(t) +\\
tS_a(t)\Phi(y)[\nabla\cdot f_{|y=0,t=0} - \tfrac{1}{2}Bv_{0|y=0} + \partial_y^2 v_{0|y=0}] + \tfrac{y}{2}e^{-F_a y}F_a^{-1}[\nabla\cdot f_{|y=0} - S_a(t)\nabla\cdot f_{|y=0,t=0}]
\end{aligned}
$$

To see that v possesses the regularity as mentioned, we remark that $\nabla\cdot f$ belongs to $\mathrm{H}_p^{1/4}(J;\mathrm{L}_p(\mathbb{R}^n_+)) \cap \mathrm{L}_p(J;\mathrm{H}_p^1(\mathbb{R}^n_+))$ and thus, by using trace theory, we obtain $\nabla\cdot f_{|y=0} \in \mathrm{W}_p^{1/4-1/4p}(J;\mathrm{L}_p(\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J;\mathrm{W}_p^{1-1/p}(\mathbb{R}^{n-1}))$, $\nabla\cdot f_{|t=0,y=0} \in \mathrm{W}_p^{1-5/p}(\mathbb{R}^{n-1})$. The verification of regularity for v is quite similar to u^t and can be adopted, see below.

The results above are very helpful to find a solution formula for u. More precisely, u can be considered as the unique solution of

$$
\begin{aligned}
\partial_t u^t - \tilde{\eta}\Delta u^t &= (\tilde{\lambda}+\tilde{\eta})\nabla_x v + f^t(t,y,x) =: h^t, && t>0, \quad y>0, \quad x\in\mathbb{R}^{n-1},\\
\partial_t u^n - \tilde{\eta}\Delta u^n &= (\tilde{\lambda}+\tilde{\eta})\partial_y v + f^n(t,y,x) =: h^n, && t>0, \quad y>0, \quad x\in\mathbb{R}^{n-1},\\
-\partial_y u^t = \theta(t,x), &\quad u^n = \vartheta(t,x), && t>0, \quad y=0, \quad x\in\mathbb{R}^{n-1},\\
u^t = u_0^t(y,x), &\quad u^n = u_0^n(y,x) && t=0, \quad y>0, \quad x\in\mathbb{R}^{n-1},
\end{aligned}
$$

where we splitted again $u = (u^t,u^n)$ and $f = (f^t,f^n)$, and consider ∇v, which is known by means of the results above, as an inhomogeneity. Therefore, the problem for u^t and u^n

decouples and each part can be represented in the following way

$$\begin{aligned} u^t &= u_1^t + e^{-F_{\tilde\eta} y} F_{\tilde\eta}^{-1}[\theta - \partial_y u^t_{1|y=0}], \quad u_1^t := T_{\tilde\eta}(t)[u_0^t - \phi^t] + \\ & T_{\tilde\eta} * \Big\{ h^t - e^{-F_{\tilde\eta} y}(h^t_{|y=0} - S_{\tilde\eta}(t) h^t_{|y=0,t=0}) - S_{\tilde\eta}(t)\Phi(y) h^t_{|y=0,t=0} \Big\}(t) + S_{\tilde\eta}(t)\phi^t \\ & + tS_{\tilde\eta}(t)\Phi(y)[h^t_{|y=0,t=0} - \tfrac{1}{2} B u^t_{0|y=0} + \partial_y^2 u^t_{0|y=0}] + \tfrac{y}{2} e^{-F_{\tilde\eta} y} F_{\tilde\eta}^{-1}[h^t_{|y=0} - S_{\tilde\eta}(t) h^t_{|y=0,t=0}] \end{aligned}$$

and

$$\begin{aligned} u^n &= u_1^n + T_{\tilde\eta} * \Big\{ h^n - e^{-F_{\tilde\eta} y}(h^n_{|y=0} - S_{\tilde\eta}(t) h^n_{|y=0,t=0}) - S_{\tilde\eta}(t)\Phi(y) h^n_{|y=0,t=0} \Big\}(t) \\ & + T_{\tilde\eta}(t)[u_0^n - \phi^n] + \tfrac{y}{2} e^{-F_{\tilde\eta} y} F_{\tilde\eta}^{-1}[h^n_{|y=0} - S_{\tilde\eta}(t) h^n_{|y=0,t=0}] + e^{-F_{\tilde\eta} y}[\vartheta - u^n_{1|y=0}], \\ & \qquad u_1^n := S_{\tilde\eta}(t)\phi^n + tS_{\tilde\eta}(t)\Phi(y)[h^n_{|y=0,t=0} - B/2 u^n_{0|y=0} + \partial_y^2 u^n_{0|y=0}]. \end{aligned}$$

Here, $\phi = (\phi^t, \phi^n) \in \mathrm{W}_p^{4-2/p}(\mathbb{R}^n_+; \mathbb{R}^{n-1}) \times \mathrm{W}_p^{4-2/p}(\mathbb{R}^n_+)$ denotes the unique solution of (3.6) with the data $(\Phi(y)g(u_0), u_{0|y=0})$, which then implies $u_0 - \phi \in A^{-1}D_A(1-1/p,p) \equiv \{\varphi \in \mathrm{W}_p^{4-2/p}(\mathbb{R}^n_+;\mathbb{R}^n) : \varphi_{|y=0} = A\varphi_{|y=0} = 0\}$. To understand where this regularity comes from, one has to make sure of $u_{0|y=0} \in \mathrm{W}_p^{4-3/p}(\mathbb{R}^{n-1};\mathbb{R}^n)$, $g(u_0) \in \mathrm{W}_p^{2-3/p}(\mathbb{R}^{n-1};\mathbb{R}^n)$ and $\Phi(\cdot)g(u_0) \in \mathrm{W}_p^{2-3/p}(\mathbb{R}^n_+;\mathbb{R}^n)$, by using Proposition 6.1. Further, due to the second representation of ϕ, see (3.7), derivatives concerning ∂_y correspond to $B^{1/2}$ and therefore $B\phi$, $\partial_y^2\phi \sim \Phi(y)Bu_{0|y=0}$ plus $B^{1/2}(D + (\frac{1}{2}B)^{1/2})^{-1}\Phi(y)g$, where both terms lie in $\mathrm{W}_p^{2-2/p}(\mathbb{R}^n_+;\mathbb{R}^n)$.

Observe that the compatibility conditions were incorporated in these formulas to the result: $\theta_{|t=0} = \partial_y u^t_{1|y=0,t=0}$, $\partial_t\theta_{|t=0} = \partial_t\partial_y u^t_{1|y=0,t=0}$, $\vartheta_{|t=0} = u^n_{1|y=0,t=0}$ and $\partial_t\vartheta_{|t=0} = \partial_t u^n_{1|y=0,t=0}$. Finally, this solution formula allows us to verify the additional regularity, that is, $\partial^k_{x_i}u$, $\partial^k_y u \in Z(J)$, $k = 1,2$, and $\partial_t u \in \mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\mathbb{R}^n_+;\mathbb{R}^n))$. Exemplarily, this is to be carried out by means of u^t. At first, we study $w := T_{\tilde\eta}(t)[u_0^t - \phi^t]$, where in this case ∂_t, $\partial^2_{x_i}$ and ∂^2_y correspond to A. Therefore, due to $A[u_0^t - \phi^t] \in D_A(1-1/p,p)$ we may conclude that $w \in \mathrm{H}^2_p(J;\mathrm{L}_p(\mathbb{R}^n_+;\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J;\mathrm{H}^4_p(\mathbb{R}^n_+;\mathbb{R}^{n-1}))$. The next part of u_1^t, namely $w := T_{\tilde\eta} * \{\ldots\}$, is more involved. To begin with, take note of $\{\ldots\}_{|y=0} = 0$ which along with $\{\ldots\} \in \mathrm{L}_p(J;\mathrm{H}^2_p(\mathbb{R}^n_+;\mathbb{R}^{n-1}))$ leads to $Aw \in Z(J)$. Moreover, it is easy to see that $S_{\tilde\eta}(t)\Phi(y)h^t_{|y=0,t=0} \in Z(J)$, $e^{-F_{\tilde\eta}y}(h^t_{|y=0} - S_{\tilde\eta}(t)h^t_{|y=0,t=0}) \in \mathrm{W}_p^{1/2+1/2p}(J;\mathrm{L}_p(\mathbb{R}^n_+;\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J;\mathrm{H}^2_p(\mathbb{R}^n_+;\mathbb{R}^{n-1}))$ and, since $h^t \in \mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\mathbb{R}^n_+;\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J;\mathrm{H}^2_p(\mathbb{R}^n_+;\mathbb{R}^{n-1}))$, we may conclude from the identity $\partial_t w = T * A\{\ldots\} + \{\ldots\}$ the wished regularity. The next term under investigation is $w := S_{\tilde\eta}(t)\phi^t$ and even lies in $Z^2(J) := \mathrm{H}^2_p(J;\mathrm{L}_p(\mathbb{R}^n_+;\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J;\mathrm{H}^4_p(\mathbb{R}^n_+;\mathbb{R}^{n-1}))$. To see this, take notice of the regularities $\phi^t \in \mathrm{W}_p^{4-2/p}(\mathbb{R}^n_+;\mathbb{R}^{n-1})$ and $S_{\tilde\eta}(t)[B\phi^t]_{|y=0} \in \mathrm{L}_p(J;\mathrm{W}_p^{2-1/p}(\mathbb{R}^n;\mathbb{R}^{n-1})) \cap \mathrm{W}_p^{1-1/2p}(J;\mathrm{L}_p(\mathbb{R}^n;\mathbb{R}^{n-1}))$ which follows from Proposition 6.1. Further, as we have the relations $\partial_t w \sim Bw$, $\partial^2_{x_i}w \sim Bw$ and $\partial^2_y w \sim Bw$ it is sufficient to consider $v := Bw$ solving

$$\begin{aligned} &\partial_t v - \tilde\eta\Delta v = 0, \quad t > 0, \quad y > 0, \quad x \in \mathbb{R}^{n-1}, \\ &\quad v_{|y=0} \in \mathrm{W}_p^{1-1/2p}(J;\mathrm{L}_p(\mathbb{R}^n;\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J;\mathrm{W}_p^{2-1/p}(\mathbb{R}^n;\mathbb{R}^{n-1})), \\ &\quad v_{|t=0} \in \mathrm{W}_p^{2-2/p}(\mathbb{R}^n_+;\mathbb{R}^{n-1}). \end{aligned}$$

Then by maximal L_p-regularity we know $v \in Z(J)$ and hence $w \in Z^2(J)$. The function $w := tS_{\tilde\eta}(t)\Phi(y)[h^t_{|y=0,t=0} - \frac{1}{2}Bu^t_{0|y=0} + \partial^2_y u^t_{0|y=0}] =: tS_{\tilde\eta}(t)\Phi(y)\tilde w_{00} =: t\tilde w$ belongs to $Z^2(J)$ as well. At first, observe that $\tilde w_{00} \in \mathrm{W}_p^{2-3/p}(\mathbb{R}^n;\mathbb{R}^{n-1})$ and $\tilde w \in Z(J)$, as this function solves the problem above with right side 0, boundary data $S(t)\tilde{w}_{00}$ and initial data $\Phi(y)\tilde{w}_{00}$, where the data possess the regularity stated above. Moreover, w solves the problem above with right side $\tilde{w}$, boundary data $tS(t)\tilde{w}_{00}$ and initial data 0. Because of this observation we are able to rewrite w as follows

$$\begin{aligned} w(t,y) &= e^{-F_{\tilde{\eta}}y} t S_{\tilde{\eta}}(t)\tilde{w}_{00} + \tfrac{1}{2} F_{\tilde{\eta}}^{-1/2} \int_0^\infty [e^{-F_{\tilde{\eta}}|y-s|} - e^{-F_{\tilde{\eta}}(y+s)}]\tilde{w}(t,s)\,ds \\ &\equiv e^{-F_{\tilde{\eta}}y}(G+1/2B)^{-1} S_{\tilde{\eta}}(t)\tilde{w}_{00} + \ldots, \end{aligned}$$

and this representation reveals the regularity. At last, we study the function $w(t,y) := \frac{y}{2} F_{\tilde{\eta}}^{-1} e^{-F_{\tilde{\eta}}y}[h^t_{|y=0} - S_{\tilde{\eta}}(t)h^t_{|t=0,y=0}]$ which, by using similar arguments, can be rewritten as

$$w(t,y) = \tfrac{1}{2}(D+F_{\tilde{\eta}})^{-1} F_{\tilde{\eta}}^{-1} e^{-F_{\tilde{\eta}}y}[h^t_{|y=0} - S_{\tilde{\eta}}(t)h^t_{|t=0,y=0}],$$

where we set $D = \partial_y$ with domain $D(D) = {}_0\mathrm{H}^1_p(\mathbb{R}_+;X)$, X any Banach space. Note that D is sectorial, invertible and belongs to $\mathcal{BIP}\,(\mathrm{L}_p(\mathbb{R}_+;X))$ with power angle $\pi/2$. Having in mind that $[h^t_{|y=0} - S_{\tilde{\eta}}(t)h^t_{|t=0,y=0}] \in {}_0\mathrm{W}_p^{1/2-1/4p}(J;\mathrm{L}_p(\mathbb{R}^n;\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J;\mathrm{W}_p^{2-1/p}(\mathbb{R}^n;\mathbb{R}^{n-1}))$ and thus $e^{-F_{\tilde{\eta}}y}[\ldots] \in {}_0\mathrm{W}_p^{1/2+1/2p}(J;\mathrm{L}_p(\mathbb{R}^n_+;\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J;\mathrm{H}^2_p(\mathbb{R}^n_+;\mathbb{R}^{n-1}))$ as well as the embedding

$${}_0\mathrm{W}_p^{1/2+1/2p}(J;\mathrm{L}_p(\mathbb{R}^n_+;\mathbb{R}^{n-1})) \hookrightarrow {}_0\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\mathbb{R}^n_+;\mathbb{R}^{n-1})),$$

it is an easy task to verify that $\partial_t w,\, Bw,\, \partial_y^2 w \in \mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\mathbb{R}^n_+;\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J;\mathrm{H}^2_p(\mathbb{R}^n_+;\mathbb{R}^{n-1}))$ which finally shows

$$w \in \mathrm{H}_p^{3/2}(J;\mathrm{L}_p(\mathbb{R}^n_+;\mathbb{R}^{n-1})) \cap \mathrm{H}^1_p(J;\mathrm{H}^2_p(\mathbb{R}^n_+;\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J;\mathrm{H}^4_p(\mathbb{R}^n_+;\mathbb{R}^{n-1})),$$

finishing the proof. □

3.2. The continuity equation

In this section the equation of conservation of mass is carefully studied in terms of the regularity dependency of u, that is, we are interested in the "maximal" regularity we can expect. Since a third order term of ρ appears in the Cahn-Hilliard equation, which is supposed to be in $\mathcal{X}_2(J)$, we need $\partial_{x_i}\partial_{x_j}\partial_{x_k}\rho \in \mathrm{L}_p(J;\mathrm{L}_p(\Omega))$, $i,j,k \le 3$, at least. A similar situation occurs in the Navier-Stokes equation, since here first order terms of ρ have to be in $\mathrm{L}_p(J;\mathrm{H}^2_p(\Omega))$. Surprisingly, this regularity and even more can be gained from the assumption $u \in \mathcal{Z}_1(J)$.

Lemma 3.1 *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \ge 1$, with boundary Γ, $J = [0,T]$ a compact time interval, and $p > \max\{4/3, n/2\}$. Further, assuming that $\rho_0 \in \mathrm{H}^3_p(\Omega)$ with $\rho_0(x) > 0$ for all $x \in \overline{\Omega}$, and $u \in \mathcal{Z}_1(J)$ satisfies $(u|\nu) \ge 0$ on Γ. Then problem (2.3) supplemented with initial condition $\rho(0) = \rho_0$ possesses a unique positive solution $\rho \in \mathcal{Z}_3(J)$ and there exists a constant $c_0(R)$ independent of T such that*

$$\|\rho\|_{\mathcal{Z}_3(J)} \le c_0, \tag{3.8}$$

provided that $\|\rho_0\|_{\mathrm{H}^3_p(\Omega)},\ \|u\|_{\mathcal{Z}_1(J)} \le r,\ r \in (0,R)$.

Proof. Step I – preliminaries. First of all we shall state some embeddings which will be crucial for all forthcoming estimates. These embeddings are due to the mixed derivative theorem and Sobolev embeddings, for which $p > \max\{1, n/2\}$ is sufficient. Let $\theta \in (0,1)$ and $1 < p < \infty$ then it holds

$$\mathcal{Z}_1(J) \hookrightarrow \mathrm{H}_p^{1+\theta/2}(J; \mathrm{H}_p^{(1-\theta)2}(\Omega; \mathbb{R}^n)) \cap \mathrm{H}_p^{\theta}(J; \mathrm{H}_p^{(1-\theta)2+2}(\Omega; \mathbb{R}^n)).$$

Setting $E_i := \mathbb{R}^{n^i}$, $i = 1, \dots, 4$ we infer from this embedding and the assumption $p > \max\{4/3, n/2\}$ that

$$\begin{aligned}
&u \in \mathrm{C}^{\alpha}(J; \mathrm{C}(\overline{\Omega}; E_1)), \quad \nabla u \in \mathrm{H}_p^{1/2}(J; \mathrm{C}(\overline{\Omega}; E_2)) \cap \mathrm{C}(J; \mathrm{H}_p^1(\Omega; E_2)), \quad \exists \alpha > 1/4 \\
&\partial_t u \in \mathrm{L}_p(J; \mathrm{C}(\overline{\Omega}; E_1)), \quad \nabla \partial_t u \in \mathrm{H}_p^{1/4}(J; \mathrm{L}_p(\Omega; E_2)) \cap \mathrm{L}_p(J; \mathrm{H}_p^1(\Omega; E_2)), \\
&\nabla^2 u \in \mathrm{L}_p(J; \mathrm{C}(\overline{\Omega}; E_3)), \quad \nabla^3 u \in \mathrm{H}_p^{1/2}(J; \mathrm{L}_p(\Omega; E_4)),
\end{aligned} \tag{3.9}$$

Further, a substantial argument will be that all subsequent constants C, C_i, c_i, $i \in \mathbb{N}$, which may differ from line to line, are always independent of u, ρ and T, but may depend on the constant R. Since this fact plays a decisive role and will be used several times, we are going to show that is always possible to estimate u (∇u, $\nabla^2 u$, etc.) independent of T in the function spaces being under considerations. For this, let $w \in \mathcal{Z}_1(\mathbb{R}_+)$ be any function such that $w(0) - u(0) = 0$ and $\|w\|_{\mathcal{Z}_1(\mathbb{R}_+)} \leq r$. Further, let $Y(J)$ denote a function space with the property $\mathcal{Z}_1(J) \hookrightarrow Y(J)$ as well as there exists a bounded extension operator E_+ from ${}_0Y(J)$ to ${}_0Y(\mathbb{R}_+)$ satisfying $\|E_+\|_{\mathcal{B}({}_0\mathcal{Z}_1(\mathbb{R}_+), {}_0\mathcal{Z}_1(J))} =: c_+ < \infty$. Using $\|u\|_{\mathcal{Z}_1} \leq r$, $r \in (0, R)$, and the assumptions above we can proceed as follows

$$\begin{aligned}
\|u\|_{Y(J)} &\leq \|u - w\|_{{}_0Y(J)} + \|w\|_{Y(\mathbb{R}_+)} \leq \|E_+(u - w)\|_{{}_0Y(\mathbb{R}_+)} + C_1 \|w\|_{\mathcal{Z}_1(\mathbb{R}_+)} \\
&\leq C_2 \|E_+(u - w)\|_{{}_0\mathcal{Z}_1(\mathbb{R}_+)} + C_1 r \leq C_2 c_+ \|u - w\|_{{}_0\mathcal{Z}_1(J)} + C_1 r \\
&\leq C_2 c_+ (\|u\|_{\mathcal{Z}_1(J)} + \|w\|_{\mathcal{Z}_1(\mathbb{R}_+)}) + C_1 r \leq 2 C_2 c_+ R + C_1 R,
\end{aligned}$$

which shows independency of T and r. Another preliminary consideration concerns estimates of solutions $\varrho : J \times \Omega \to E$, E any finite dimensional space, of

$$\begin{aligned}
\partial_t \varrho + \nabla \varrho \cdot v + \varrho \nabla \cdot v = f(t, x, \varrho), \quad &(t, x) \in J \times \Omega, \\
\varrho = \varrho_0(x), \quad &(t, x) \in \{0\} \times \Omega,
\end{aligned} \tag{3.10}$$

where $\varrho_0 \in \mathrm{L}_p(\Omega; E)$, and f is subject to the condition

$$\|f(t, \varrho)\|_{\mathrm{L}_p(\Omega; E)} \leq k_1(t) \|\varrho(t)\|_{\mathrm{L}_p(\Omega; E)} + k_2(t), \quad k_1(t), k_2(t) \in \mathrm{L}_1(J; \mathbb{R}_+). \tag{3.11}$$

Lemma 3.2 *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with boundary Γ, $J = [0, T]$ a compact time interval, and $p \in (1, \infty)$. Further, assuming that $\varrho_0 \in \mathrm{L}_p(\Omega; E)$, $v \in \mathrm{L}_1(J; \mathrm{C}^1(\overline{\Omega}; E))$ satisfies $(v|\nu) \geq 0$ on Γ, and f fulfils condition* (3.11)*. Then, every solution ϱ of problem* (3.10) *suffices the estimate*

$$\|\varrho\|_{\mathrm{B}(J; \mathrm{L}_p(\Omega; E))} \leq \exp\left\{ \|\nabla \cdot v\|_{\mathrm{L}_1(J; \mathrm{C}(\overline{\Omega}))} + \|k_1\|_{\mathrm{L}_1(J)} \right\} \left(\|\varrho_0\|_{\mathrm{L}_p(\Omega; E)} + \|k_2\|_{\mathrm{L}_1(J)} \right). \tag{3.12}$$

Moreover, assuming that $\varrho_0 \in \mathrm{L}_\infty(\Omega; E)$ and k_1, k_2 are independent of p, we even conclude

$$\|\varrho\|_{\mathrm{B}(J; \mathrm{L}_\infty(\Omega; E))} \leq \exp\left\{ \|\nabla \cdot v\|_{\mathrm{L}_1(J; \mathrm{C}(\overline{\Omega}))} + \|k_1\|_{\mathrm{L}_1(J)} \right\} \left(\|\varrho_0\|_{\mathrm{L}_\infty(\Omega; E)} + \|k_2\|_{\mathrm{L}_1(J)} \right). \tag{3.13}$$

Proof of Lemma 3.2. After multiplying the differential equation of (3.10) with $|\varrho(t)|^{p-2}\varrho(t)$, integrating over Ω and integration by parts, we obtain

$$\frac{d}{dt}\tfrac{1}{p}\|\varrho(t)\|^p_{\mathrm{L}_p(\Omega;E)} = \tfrac{1-p}{p}\int_\Omega \nabla\cdot v(t,x)|\varrho(t,x)|^p\,dx + \int_\Omega f(t,x,\varrho)\cdot\varrho(t,x)|\varrho(t,x)|^{p-2}\,dx$$
$$-\tfrac{1-p}{p}\int_\Gamma |\varrho(t,x)|^p(v|\nu)\,d\sigma \le \tfrac{p-1}{p}\|\nabla\cdot v(t)\|_{\mathrm{C}(\overline{\Omega})}\|\varrho(t)\|^p_{\mathrm{L}_p(\Omega;E)} + \|f(t,\varrho)\|_{\mathrm{L}_p(\Omega;E)}\|\varrho(t)\|^{p-1}_{\mathrm{L}_p(\Omega;E)}$$
$$\le \Big[\|\nabla\cdot v(t)\|_{\mathrm{C}(\overline{\Omega})} + k_1(t)\Big]\|\varrho(t)\|^p_{\mathrm{L}_p(\Omega;E)} + k_2(t)\|\varrho(t)\|^{p-1}_{\mathrm{L}_p(\Omega;E)}.$$

From this inequality we conclude $\frac{d}{dt}\|\varrho(t)\|_{\mathrm{L}_p(\Omega;E)} \le (\|\nabla\cdot v(t)\|_{\mathrm{C}(\overline{\Omega})}+k_1(t))\|\varrho(t)\|_{\mathrm{L}_p(\Omega;E)}+k_2(t)$, which, after employing Gronwall's Lemma, leads to

$$\|\varrho(t)\|_{\mathrm{L}_p(\Omega;E)} \le \exp\Big\{\int_0^t \|\nabla\cdot v(s)\|_{\mathrm{C}(\overline{\Omega})} + k_1(s)\,ds\Big\}\Big(\|\varrho_0\|_{\mathrm{L}_p(\Omega;E)}+$$
$$\int_0^t \exp\Big\{-\int_s^t \|\nabla\cdot v(\tau)\|_{\mathrm{C}(\overline{\Omega})} + k_1(\tau)\,d\tau\Big\}k_2(s)\,ds\Big)$$
$$\le \exp\big\{\|\nabla\cdot v\|_{\mathrm{L}_1(J;\mathrm{C}(\overline{\Omega}))} + \|k_1\|_{\mathrm{L}_1(J)}\big\}\big(\|\varrho_0\|_{\mathrm{L}_p(\Omega;E)} + \|k_2\|_{\mathrm{L}_1(J)}\big)$$

and thus estimate (3.12). Finally, letting go $p\to\infty$ in (3.12) we obtain the second estimate (3.13) as well. Notice that the assumptions $\varrho_0\in\mathrm{L}_\infty(\Omega;E)$ and k_1, k_2 being independent of p allow this limiting process. □

Step II - $u\in\mathcal{Z}_1(J)$ implies $\rho\in\mathrm{B}(J;\mathrm{H}^3_p(\Omega))$ and $\partial_t\rho\in\mathrm{B}(J;\mathrm{L}_p(\Omega))$. Due to Lemma 3.2 and the above remarks we are able to derive estimates for $\nabla^k\rho$, $k=0,\dots,3$, independent of J. In fact, since ρ solves (2.3) we may apply Lemma 3.2 with $f\equiv 0$ resulting in

$$\begin{aligned}\|\rho\|_{\mathrm{B}(J;\mathrm{L}_p(\Omega))} &\le \exp\big\{\|\nabla\cdot u\|_{\mathrm{L}_1(J;\mathrm{L}_\infty(\Omega))}\big\}\|\rho_0\|_{\mathrm{L}_p(\Omega)} \le c_1(R),\\ \|\rho\|_{\mathrm{B}(J;\mathrm{L}_\infty(\Omega))} &\le c_E c_1 =: c_2(R),\end{aligned}\tag{3.14}$$

where the embedding $\mathcal{Z}_1(J)\hookrightarrow\mathrm{L}_1(J;\mathrm{C}^1(\overline{\Omega};\mathbb{R}^n))$ entered. A similar estimate for $\nabla\rho$ can be derived, as $\varrho:=\nabla\rho$ solves

$$\partial_t\varrho + \nabla\varrho\cdot u + \varrho\nabla\cdot u = -\nabla u\cdot\varrho - \rho\nabla\nabla\cdot u =: f\tag{3.15}$$

and f satisfies the estimate

$$\|f(t)\|_{\mathrm{L}_p(\Omega;E_1)} \le \|u(t)\|_{\mathrm{C}^1(\overline{\Omega};E_2)}\|\varrho(t)\|_{\mathrm{L}_p(\Omega;E_1)} + c_2|\Omega|^{1/p}\|u(t)\|_{\mathrm{C}^2(\overline{\Omega};E_1)},$$

as $u(t)\in\mathrm{H}^4_p(\Omega;E_1)\hookrightarrow\mathrm{C}^2(\overline{\Omega};E_1)$ for $p>n/2$. Due to the remarks in *step II* and Lemma 3.2 we obtain

$$\begin{aligned}\|\nabla\rho\|_{\mathrm{B}(J;\mathrm{L}_p(\Omega;E_1))} &\le e^{2\|u\|_{\mathrm{L}_1(J;\mathrm{C}^1(\overline{\Omega};E_1))}}\Big(\|\nabla\rho_0\|_{\mathrm{L}_p(\Omega;E_1)} + c_2\|u\|_{\mathrm{L}_1(J;\mathrm{H}^2_p(\Omega;E_1))}\Big) \le c_3,\\ \|\nabla\rho\|_{\mathrm{B}(J;\mathrm{L}_\infty(\Omega;E_1))} &\le e^{2\|u\|_{\mathrm{L}_1(J;\mathrm{C}^1(\overline{\Omega};E_1))}}\Big(\|\nabla\rho_0\|_{\mathrm{L}_\infty(\Omega;E_1)} + c_2\|u\|_{\mathrm{L}_1(J;\mathrm{C}^2(\overline{\Omega};E_1))}\Big) \le c_4.\end{aligned}\tag{3.16}$$

Besides, the estimates (3.14), (3.16), (3.16) and the equation of mass entail the estimate $\|\partial_t\rho\|_{B(J;L_p(\Omega))} \leq (c_2+c_4)\|u\|_{C(J;H^1_p(\Omega;E_1))} \leq c_5(R)$. The higher spatial regularity of ρ can be proved in same manner as before. More precisely, at first one applies ∇^2 to (2.3) to obtain an equation for $\varrho := \nabla^2\rho$ having the form

$$\partial_t\varrho + \nabla\varrho\cdot u + \varrho\nabla\cdot u = -2\varrho\cdot\nabla u - \nabla\rho\cdot\nabla^2 u - 2\nabla\rho\otimes\nabla\nabla\cdot u - \rho\nabla^2\nabla\cdot u =: f,$$

where f suffices the inequality

$$\|f(t)\|_{L_p(\Omega;E_2)} \leq 2\|u(t)\|_{C^1(\overline{\Omega};E_1)}\|\varrho(t)\|_{L_p(\Omega;E_2)} + (3c_4+c_2)\|u(t)\|_{H^3_p(\Omega;E_1)}.$$

Reasoning in the same way as before gives rise to the estimate $\|\nabla^2\rho\|_{B(J;L_p(\Omega;E_2))} \leq c_6(R)$. In the next step ∇^3 is applied to (2.3), and in view of $\nabla^2\rho \in B(J;L_p(\Omega;E_2))$ and $\nabla^2 u \in L_p(J;C(\Omega;E_3))$ we may proceed as usual, leading to the result $\|\rho\|_{B(J;H^3_p(\Omega))} \leq c_7(R)$. We point out that $p > n/2$ is only required.

Step III - $\rho \in H_p^{2+\frac{1}{4}}(J;L_p(\Omega))$. This regularity is available due to $\nabla\cdot\partial_t u \in H_p^{1/4}(J;L_p(\Omega))$, which is caused by the embedding $\mathcal{Z}_1(J) \hookrightarrow H_p^{1+1/4}(J;H^1_p(\Omega;\mathbb{R}^n))$, and the equation

$$\partial_t^2\rho = -\nabla\partial_t\rho\cdot u - \rho\nabla\cdot\partial_t u - \partial_t\rho\nabla\cdot u - \nabla\rho\cdot\partial_t u. \tag{3.17}$$

The idea is based on studying the regularity of the right-hand side which is at most as mentioned above. Since no further information of $\nabla\partial_t\rho$ is at hand, we firstly aim at proving $\nabla\partial_t\rho \in B(J;L_p(\Omega;\mathbb{R}^n))$. This can easily be derived from the equation (3.15) in view of the estimate

$$\begin{aligned}\|\nabla\partial_t\rho\|_{B(J;L_p(\Omega;E_1))} &\leq \|\rho\|_{B(J;L_\infty(\Omega))}\|\nabla\nabla\cdot u\|_{C(J;L_p(\Omega;E_1))}\\ &+ 2\|\nabla\rho\|_{B(J;L_\infty(\Omega;E_1))}\|\nabla u\|_{C(J;L_p(\Omega;E_2))} + \|\nabla^2\rho\|_{B(J;L_p(\Omega;E_2))}\|u\|_{C(J;C(\overline{\Omega};E_1))}\\ &\leq (c_2+2c_4)\|u\|_{C(J;H^2_p(\Omega;E_1))} + c_7\|u\|_{C(J;C(\overline{\Omega};E_1))} \leq c_{10}(R).\end{aligned}$$

Now, on account of the regularities ρ, $\partial_{x_i}\rho \in B(J;L_\infty(\Omega))$ and $\partial_t\rho$, $\partial_{x_i}\partial_t\rho \in B(J;L_p(\Omega))$, we perceive that each term on the right-hand side of (3.17) lies in $L_p(J;L_p(\Omega))$ at least (without any additional restriction to p); thus we conclude, by using all previous results, $\rho \in Y := H^2_p(J;L_p(\Omega)) \cap H^1_\infty(J;H^1_p(\Omega)) \cap B(J;H^3_p(\Omega))$ and $\|\rho\|_Y \leq c_{11}(R)$. To make out more time regularity for ρ, we only have to put this newly-acquired regularity into the equation (3.15) and carry out similar estimates in "better" spaces. More precisely, the first purpose is to show $\nabla\partial_t\rho \in H_p^{1/4}(J;L_p(\Omega;E_1))$ which implies $\rho \in Y \cap H_p^{1+1/4}(J;H^1_p(\Omega))$. Taking into account the embeddings (3.9) and

$$Y \hookrightarrow H^1_s(J;H^1_p(\Omega)) \cap L_s(J;H^3_p(\Omega)) \hookrightarrow H^\theta_s(J;H_p^{1+(1-\theta)2}(\Omega)), \quad \forall s \in [1,\infty), \quad \theta \in (0,1),$$

which implies

$$H^\theta_s(J;H_p^{1+(1-\theta)2}(\Omega)) \hookrightarrow C^\alpha(J;H^2_p(\Omega)) \hookrightarrow C^\alpha(J;C(\overline{\Omega})), \quad \alpha \in [0,1),$$

we may proceed as follows

$$\begin{aligned}\|\nabla\partial_t\rho\|_{H_p^{1/4}(J;L_p(\Omega;E_1))} &\leq \|\rho\|_{C^\alpha(J;C(\overline{\Omega}))}\|\nabla\nabla\cdot u\|_{H_p^{1/4}(J;L_p(\Omega;E_1))} + 2\|\nabla\rho\|_{C^\alpha(J;L_p(\Omega;E_1))}\cdot\\ &\|\nabla u\|_{H_p^{1/4}(J;L_\infty(\Omega;E_2))} + \|\nabla^2\rho\|_{H_p^{1/4}(J;L_p(\Omega;E_2))}\|u\|_{C^\alpha(J;C(\overline{\Omega};E_1))} \leq c_{10}(R), \quad \alpha > 1/4.\end{aligned}$$

Now, according to the equation (3.17) and the regularity $\rho \in Y \cap \mathrm{H}_p^{5/4}(J;\mathrm{H}_p^1(\Omega))$ as well as the above embeddings, we easily infer

$$\|\partial_t^2\rho\|_{\mathrm{H}_p^{1/4}(J;\mathrm{L}_p(\Omega))} \le \|\nabla\partial_t\rho\|_{\mathrm{H}_p^{1/4}(J;\mathrm{L}_p(\Omega;E_1))}\|u\|_{\mathrm{C}^\alpha(J;\mathrm{C}(\overline{\Omega};E_1))} + \|\nabla\rho\|_{\mathrm{C}^\alpha(J;\mathrm{H}_p^1(\Omega;E_1))}\|\partial_t u\|_{\mathrm{H}_p^{1/4}(J;\mathrm{H}_p^1(\Omega;E_1))}$$
$$+ \|\partial_t\rho\|_{\mathrm{H}_p^{1/4}(J;\mathrm{H}_p^1(\Omega))}\|\nabla\cdot u\|_{\mathrm{C}^\alpha(J;\mathrm{H}_p^1(\Omega))} + \|\rho\|_{\mathrm{C}^\alpha(J;\mathrm{C}(\overline{\Omega}))}\|\partial_t\nabla\cdot u\|_{\mathrm{H}_p^{1/4}(J;\mathrm{L}_p(\Omega))} \le c_{11}(R),$$

which shows the estimate (3.8).

Step IV – existence and uniqueness. This is a very well-known result and a consequence of the theory of symmetric hyperbolic systems as well as the estimate (3.8), cf. [17].

The next lemma concerns the estimate of differences of solutions of the equation of mass.

Lemma 3.3 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 1$, be a bounded domain with smooth boundary Γ, $J = [0,T]$, $p > \hat{p}$ and $r \in [1,\infty)$. Assuming that (u_1,ρ_1), $(u_2,\rho_2) \in \mathcal{Z}_1(J) \times \mathcal{Z}_3(J)$ with $(u_i|\nu) \ge 0$ on Γ, $i = 1,2$, and $\|(u_i,\rho_i)\|_{\mathcal{Z}_1(J)\times\mathcal{Z}_3(J)} \le K_0$, $K_0 \in (0,K)$. If both (u_1,ρ_1) and (u_2,ρ_2) solve* (2.3) *with initial data (u_0,ρ_0), then there is a constant $\kappa(T,K) > 0$ with the property $\kappa(T,K) \to 0$ as $T \to 0$, such that*

$$\|\rho_1-\rho_2\|_{{}_0\mathrm{H}_r^1(J;\mathrm{L}_p(\Omega))} + \|\rho_1-\rho_2\|_{\mathrm{B}(J;\mathrm{H}_p^1(\Omega))} \le \kappa(T,K)\|u_1-u_2\|_{{}_0Z_1(J)}. \tag{3.18}$$

Proof. Supposing that $(u_i,\rho_i) \in \mathcal{Z}_1(J) \times \mathcal{Z}_3(J)$ with $\|u_i\|_{\mathcal{Z}_1(J)} + \|\rho_i\|_{\mathcal{Z}_3(J)} \le K_0 < K$, $i = 1,2$, solve the equation of conservation of mass. Let denote $\varrho := \rho_1 - \rho_2$ and $v := u_1 - u_2$ then (ϱ, v) satisfies

$$\begin{aligned} \partial_t\varrho + \nabla\cdot(\varrho u_1) &= -\nabla\cdot(\rho_2 v), && (t,x) \in J \times \Omega,\\ \varrho &= 0, && (t,x) \in \{0\} \times \Omega. \end{aligned} \tag{3.19}$$

Applying Lemma 3.2 we are able to derive the following estimate

$$\begin{aligned} \|\varrho\|_{\mathrm{B}(J;\mathrm{L}_p(\Omega))} &\le \exp\left\{\|u_1\|_{\mathrm{L}_1(J;\mathrm{C}^1(\overline{\Omega};\mathbb{R}^n))}\right\}\|\nabla\cdot(\rho_2 v)\|_{\mathrm{L}_1(J;\mathrm{L}_p(\Omega))}\\ &\le \exp\left\{c_E\|u_1\|_{\mathrm{L}_1(J;\mathrm{H}_p^4(\Omega;\mathbb{R}^n))}\right\}\Big(\|\rho_2\|_{\mathrm{B}(J;\mathrm{C}(\overline{\Omega}))}\|\nabla\cdot v\|_{\mathrm{L}_1(J;\mathrm{L}_p(\Omega))}\\ &\quad + c_E\|\nabla\rho_2\|_{\mathrm{B}(J;\mathrm{H}_p^1(\Omega;\mathbb{R}^n))}\|v\|_{\mathrm{L}_1(J;\mathrm{H}_p^1(\Omega;\mathbb{R}^n))}\Big) \end{aligned}$$

where in the latter inequality we exploited the embeddings $\mathrm{H}_p^4(\Omega) \hookrightarrow \mathrm{C}^1(\overline{\Omega})$, $\mathrm{H}_p^3(\Omega) \hookrightarrow \mathrm{C}(\overline{\Omega})$, for $p > n/3$, as well as inequality (6.1). Due to the boundedness $\|u_1\|_{\mathcal{Z}_1(J)}$, $\|\rho_2\|_{\mathcal{Z}_3(J)} \le K_0 < K$, we further conclude

$$\begin{aligned} \|\varrho\|_{\mathrm{B}(J;\mathrm{L}_p(\Omega))} &\le C(K)\|v(s)\|_{\mathrm{L}_1(J;\mathrm{H}_p^1(\Omega;\mathbb{R}^n))} \le C(K)T\|v\|_{\mathrm{C}(J;\mathrm{H}_p^1(\Omega;\mathbb{R}^n))}\\ &\le C(K)T\|v\|_{{}_0Z_1(J)}, \end{aligned} \tag{3.20}$$

due to the embedding $Z_1(J) \hookrightarrow \mathrm{H}_p^{1/2}(J;\mathrm{H}_p^1(\Omega;\mathbb{R}^n)) \hookrightarrow \mathrm{C}(J;\mathrm{H}_p^1(\Omega;\mathbb{R}^n))$, $p > 2$. Next we apply ∇ to (3.19) in order to derive an estimate for $\nabla\varrho$. The equation we have to study reads as

$$\partial_t\nabla\varrho + \nabla^2\varrho \cdot u_1 + \nabla\varrho\nabla\cdot u_1 = -\nabla u_1 \cdot \nabla\varrho - \nabla\nabla\cdot(\rho_2 v) - \varrho\nabla\nabla\cdot u_1 =: f, \tag{3.21}$$

where f fulfils the estimate

$$\begin{aligned}\|f(t)\|_{\mathrm{L}_p(\Omega;\mathbb{R}^n)} &\le \|u_1(t)\|_{\mathrm{C}^1(\overline{\Omega};\mathbb{R}^n)}\|\nabla\varrho(t)\|_{\mathrm{L}_p(\Omega;\mathbb{R}^n)} + \|\nabla\nabla\cdot(\rho_2 v)(t)\|_{\mathrm{L}_p(\Omega;\mathbb{R}^n)}\\ &\quad + c_E\|\varrho(t)\|_{\mathrm{H}^1_p(\Omega)}\|\nabla\nabla\cdot u_1(t)\|_{\mathrm{H}^2_p(\Omega;\mathbb{R}^n)}\\ &\le 2c_E\|u_1(t)\|_{\mathrm{H}^4_p(\Omega;\mathbb{R}^n)}\|\nabla\varrho(t)\|_{\mathrm{L}_p(\Omega;\mathbb{R}^n)} + C(K)\|v(t)\|_{\mathrm{H}^2_p(\Omega;\mathbb{R}^n)}\\ &\quad + c(K)T\|v\|_{{}_0Z_1(J)}\|u_1(t)\|_{\mathrm{H}^4_p(\Omega;\mathbb{R}^n)}.\end{aligned}$$

Using again Lemma 3.2 we arrive at the inequality

$$\begin{aligned}\|\nabla\varrho(t)\|_{\mathrm{B}(J;\mathrm{L}_p(\Omega;\mathbb{R}^n))} &\le e^{3c_E\|u_1\|_{\mathcal{Z}_1(J)}}\Big(c(K)T\|v\|_{{}_0Z_1(J)}\|u_1\|_{\mathcal{Z}_1(J)}\\ &\qquad + c(K)\|v\|_{\mathrm{L}_1(J;\mathrm{H}^2_p(\Omega;\mathbb{R}^n))}\Big) \le c(K)(T+T^{1-1/p})\|v\|_{{}_0Z_1(J)},\end{aligned}$$

and along with (3.20) this gives the first part of (3.18). So, it lefts to prove an estimate for $\partial_t\varrho$ in $\mathrm{L}_r(J;\mathrm{L}_p(\Omega))$. This is again a consequence of equation (3.19). In fact, having in mind the embedding $\mathrm{H}^2_p(\Omega)\hookrightarrow \mathrm{C}(\overline{\Omega})$ and inequality (6.1) we come up to

$$\begin{aligned}\|\partial_t\varrho\|_{\mathrm{L}_r(J;\mathrm{L}_p(\Omega))} &\le T^{1/r}\|\nabla\varrho\|_{\mathrm{B}(J;\mathrm{L}_p(\Omega;\mathbb{R}^n))}\|u_1\|_{\mathrm{C}(J;\mathrm{C}(\overline{\Omega};\mathbb{R}^n))}\\ &\quad + c_E\|\varrho\|_{\mathrm{B}(J;\mathrm{H}^1_p(\Omega))}\|\nabla\cdot u_1\|_{\mathrm{L}_r(J;\mathrm{H}^1_p(\Omega))} + \|\rho_2\|_{\mathrm{B}(J;\mathrm{L}_\infty(\Omega))}\|v\|_{\mathrm{L}_r(J;\mathrm{H}^1_p(\Omega;\mathbb{R}^n))}\\ &\quad + c_E\|\nabla\rho_2\|_{\mathrm{B}(J;\mathrm{L}_\infty(\Omega;\mathbb{R}^n))}\|v\|_{\mathrm{L}_r(J;\mathrm{H}^1_p(\Omega;\mathbb{R}^n))}\\ &\le c(T^{1/r}\|\varrho\|_{\mathrm{B}(J;\mathrm{H}^1_p(\Omega))}\|u_1\|_{\mathrm{C}(J;\mathrm{C}(\overline{\Omega};\mathbb{R}^n))} + \|\rho_2\|_{\mathrm{B}(J;\mathrm{H}^3_p(\Omega))}\|v\|_{\mathrm{L}_r(J;\mathrm{H}^1_p(\Omega;\mathbb{R}^n))}).\end{aligned}$$

In view of the embeddings $u_1\in \mathrm{H}^1_p(J;\mathrm{H}^2_p(\Omega))\hookrightarrow \mathrm{C}(J;\mathrm{C}(\overline{\Omega};\mathbb{R}^n))$, $p>n/2$, and $v\in Z_1(J)\hookrightarrow \mathrm{C}(J;\mathrm{H}^1_p(\Omega;\mathbb{R}^n))$, we may continue with the above estimate as follows

$$\begin{aligned}\|\partial_t\varrho\|_{\mathrm{L}_r(J;\mathrm{L}_p(\Omega))} &\le c(K)T^{1/r}\big(\|\varrho\|_{\mathrm{B}(J;\mathrm{H}^1_p(\Omega))} + \|v\|_{\mathrm{C}(J;\mathrm{H}^1_p(\Omega))}\big)\\ &\le c(K)T^{1/r}(T+T^{1-1/p}+1)\|v\|_{{}_0Z_1(J)},\end{aligned}$$

which finally implies the second part of (3.18), finishing the proof. □

Remark 3.1 This lemma or rather the approach of the proof is the real cause for working in different regularity classes with respect to selfmapping and contraction. To get the idea why this approach seems to be necessary, we have to take a closer look on the L_p-estimate for $\nabla\varrho=\nabla(\rho_1-\rho_2)$. For the time being, let us assume that selfmapping and contraction is proved in the same space, which for the sake of simplicity is to be $(u,\rho)\in \mathrm{L}_p(J;\mathrm{H}^2_p(\Omega;\mathbb{R}^n))\times \mathrm{B}(J;\mathrm{H}^1_p(\Omega))$. To obtain L_p-estimates for the density, we have always applied Lemma 3.2 and this is also the case for differences. However, the right-hand side of (3.21) has to satisfy the condition (3.11) of Lemma 3.2 meaning that $\nabla\nabla\cdot(\rho_2 v)\in \mathrm{L}_1(J;\mathrm{L}_p(\Omega))$ and thus $\rho_2\in \mathrm{L}_1(J;\mathrm{H}^2_s(\Omega))$ at least, $s\in(1,\infty)$ chosen appropriately. This contradicts the assumption $\rho_1,\rho_2\in \mathrm{B}(J;\mathrm{H}^1_p(\Omega))$. This lack of regularity always occurs and cannot be resolved by considering higher regularities.

3.3. A weak estimate for Cahn-Hilliard

This section is devoted to a "weak estimate" for solutions of the Cahn-Hilliard equation (2.2). This subject will play a decisive role in proving contraction for the fixed point equation. But first of all, we recall the reformulated Cahn-Hilliard equation (2.15)

$$\tfrac{\varepsilon_0\rho_0}{\gamma_0}\partial_t c - \nabla\cdot(\varepsilon_0\nabla(g-\nabla\cdot(\varepsilon_0\nabla c)) = \tfrac{\varepsilon_0}{\gamma_0}\big\{\partial_t\big([\rho_0-\rho]c\big) - \nabla\cdot\big(c\rho u - (\gamma_0-\gamma)\nabla[\nabla\cdot(\varepsilon_0\nabla c)+g]\big)\big\}$$
$$- \tfrac{\varepsilon_0^2}{\gamma_0}\nabla(\tfrac{\gamma_0}{\varepsilon_0})\cdot\nabla[\nabla\cdot(\varepsilon_0\nabla c) - g],$$
$$\partial_\nu c = 0, \quad \partial_\nu(g-\nabla\cdot(\varepsilon_0\nabla c)) = 0$$

with $g(\rho,c) := \nabla\cdot([\varepsilon-\varepsilon_0]\nabla c) + \rho^{-1}\varepsilon\nabla\rho\cdot\nabla c - \partial_c\psi$. Next we define $M := \frac{\varepsilon_0\rho_0}{\gamma_0}$ with domain $D(M) := Y := \mathrm{L}_p(J;X)$ and $X := \mathrm{L}_p(\Omega)$, $G = \partial_t$ with natural domain $D(G) = {}_0\mathrm{H}^1_p(J;X)$, and $\mathcal{A}_\varepsilon = -\nabla\cdot(\varepsilon\nabla)$ with domain $D(\mathcal{A}_\varepsilon) := \{v\in \mathrm{H}^2_p(\Omega) : \partial_\nu v = 0\}$. Further, let A_ε denote the natural extension to $\mathrm{L}_p(J;X)$ with domain $\mathrm{L}_p(J;D(\mathcal{A}_\varepsilon))$; then M, G, A_ε are sectorial operators and M, G are even invertible. Moreover, these operators belong to $\mathcal{BIP}(Y)$ with power angles $\theta_M = 0$, $\theta_G = \pi/2$ and $\theta_{A_\varepsilon} = 0$. Let (u_1,c_1), $(u_2,c_2)\in\Sigma$ and $\rho_i := L[u_i,\rho_0]$ solve problem (2.1)-(2.8). Then we introduce $\vartheta := c_1-c_2$ to distinguish differences appearing on the left-hand and right-hand side. This function satisfies

$$MG\vartheta + A_{\varepsilon_0}[\phi + A_{\varepsilon_0}\vartheta] = \tfrac{\varepsilon_0}{\gamma_0}G\Phi_1 - \tfrac{\varepsilon_0}{\gamma_0}\nabla\cdot\Phi_2 + A_{\gamma_0-\gamma_1}(\phi - A_{\varepsilon_0}[c_1-c_2])$$
$$- A_{\gamma_1-\gamma_2}(A_{\varepsilon_0}c_2 - g(\rho_2,c_2)) + \tfrac{\varepsilon_0^2}{\gamma_0}\nabla(\tfrac{\gamma_0}{\varepsilon_0})\cdot\nabla(\phi + A_{\varepsilon_0}[c_1-c_2]), \quad (3.22)$$

where we used again $\varepsilon_0 := \varepsilon_0(0,x,\rho_0(x),c_0(x))$ as well as

$$\phi := g(\rho_1,c_1) - g(\rho_2,c_2), \quad \Phi_1 := (\rho_0-\rho_1)(c_1-c_2) - (\rho_1-\rho_2)c_2, \quad \Phi_2 := c_1\rho_1u_1 - c_2\rho_2u_2$$

with $a_i := a(t,x,\rho_i,c_i)$, $a\in\{\gamma,\varepsilon\}$, $i=1,2$. In this setting the difference ϑ can be estimated in ${}_0Z_2(J)$ by means of differences on the right-hand side. This astounding result draws upon the divergence structure and appropriate boundary conditions as well as maximal regularity.

Lemma 3.4 *Let $\Omega\subset\mathbb{R}^n$, $n\geq 1$, be a bounded domain with smooth boundary Γ, and $J=[0,T]$ a compact time interval and $p\in(\hat p,\infty)$. Assuming that γ_0, ε_0 belong to $\mathrm{H}^3_p(\Omega)$ and $\gamma_0(x)$, $\varepsilon_0(x) > 0$ for all $x\in\overline{\Omega}$. Further, let $(u_i,c_i)\in\Sigma$ and $\rho_i := L[u_i,\rho_0]$, $i=1,2$, and $\vartheta\in{}_0\mathcal{Z}_2(J)$ solve (3.22). Then there exists a constant $M_2(R)>0$ ($R_0<R$), such that*

$$\|\vartheta\|_{{}_0Z_2(J)} \leq M_2\big(\|\phi\|_{\mathrm{L}_p(J;\mathrm{L}_p(\Omega))} + \|\Phi_1\|_{{}_0\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\Omega))} + \|\Phi_2\|_{\mathrm{L}_p(J;\mathrm{L}_p(\Omega))} +$$
$$\|\gamma_0-\gamma_1\|_{\mathrm{B}(J;\mathrm{C}^1(\overline{\Omega}))}\|c_1-c_2\|_{\mathrm{L}_p(J;\mathrm{H}^2_p(\Omega))} + \max\{T^{\frac14}, T^{\frac14+\frac1p}\}\|\gamma_1-\gamma_2\|_{{}_0\mathrm{B}(J;\mathrm{H}^1_p(\Omega))}\big). \quad (3.23)$$

Proof. At first, let us define $B := G^{1/2} + A_{\varepsilon_0}$. Having in mind that G and A_{ε_0} commute as well as G is invertible, the Dore-Venni theorem yields that B with domain

$$D(B) = D(G^{1/2})\cap D(A_{\varepsilon_0}) = [D(G),Y]_{1/2}\cap D(A_{\varepsilon_0}) = {}_0\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\Omega))\cap \mathrm{L}_p(J;D(\mathcal{A}_{\varepsilon_0}))$$

is invertible, sectorial, and belongs to $\mathcal{BIP}(Y)$ with power angle $\theta_B \leq \max\{\theta_G/2,\theta_A\} = \pi/4$. Furthermore, if we set $F := MG + A_{\varepsilon_0}A_{\varepsilon_0}$ with domain $D(F) = D(G)\cap D(A^2_{\varepsilon_0})$, then maximal L_p-regularity of the Cahn-Hilliard problem gives rise to $F\in\mathcal{L}is(D(F),Y)$. The next step consists in stating a weak formulation of (3.22). For this, we multiply (3.22) with ψ, integrate over $J\times\Omega$ and integrate by parts etc. to obtain

$$\langle (G^{1/2})'\psi | MG^{1/2}\vartheta\rangle + \langle A'_{\varepsilon_0}\psi | A_{\varepsilon_0}\vartheta + \phi\rangle = \langle (G^{1/2})'\psi | \tfrac{\varepsilon_0}{\gamma_0}G^{1/2}\Phi_1\rangle + \langle \nabla(\tfrac{\varepsilon_0}{\gamma_0}\psi) | \Phi_2\rangle$$
$$+ \langle A_{\gamma_0-\gamma_1}\psi | \phi - A_{\varepsilon_0}[c_1-c_2]\rangle + \langle [\gamma_1-\gamma_2]\nabla\psi | \nabla[A_{\varepsilon_0}c_2 - g(\rho_2,c_2)]\rangle$$
$$- \langle \nabla\cdot(\tfrac{\varepsilon_0^2}{\gamma_0}\nabla(\tfrac{\gamma_0}{\varepsilon_0})\psi) | \phi + A_{\varepsilon_0}[c_1-c_2]\rangle, \quad (3.24)$$

for all $\psi \in D((G^{1/2})') \cap D(A'_{\varepsilon_0})$. Here $\langle\cdot|\cdot\rangle$ denotes the duality pairing between Y and $Y' := \mathrm{L}_{p'}(J; X')$, $A'_{\varepsilon_0} \equiv A_{\varepsilon_0}$ with domain $D(A'_{\varepsilon_0}) := \{\varphi \in \mathrm{L}_{p'}(J; \mathrm{H}^2_{p'}(\Omega)) : \partial_\nu \varphi = 0\}$, and

$$(G^{1/2})'v := -\int_t^T g_{1/2}(\tau - t)v'(\tau)\, d\tau = -\partial_t \int_0^{T-t} g_{1/2}(\tau)v(\tau + t)\, d\tau, \quad g_{1/2} := \frac{t^{-1/2}}{\Gamma(1/2)},$$

with domain $D((G^{1/2})') := \{v \in \mathrm{H}^{1/2}_{p'}([0, T]; X') : v(T) = 0\}$. Notice that $(G^{1/2})'$ and A'_{ε_0} commute and belong to $\mathcal{BIP}(Y')$ with power angles $\theta_{(G^{1/2})'} \le \pi/2$ and $\theta_{A'_{\varepsilon_0}} = 0$, respectively. As above, we may conclude by the Dore-Venni theorem that $(G^{1/2})' + A'_{\varepsilon_0}$ is invertible and belongs to $\mathcal{BIP}(Y')$. In particular, there exists a constant $c > 0$ such that for all $\psi \in D((G^{1/2})') \cap D(A'_{\varepsilon_0})$ holds

$$\|(G^{1/2})'\psi\|_{Y'} + \|A'_{\varepsilon_0}\psi\|_{Y'} \le c\|(G^{1/2})'\psi + A'_{\varepsilon_0}\psi\|_{Y'}.$$

With these preliminary considerations we are able to find an estimate for ϑ in ${}_0Z_2(J)$. The idea is based on equivalence of the norms $\|\vartheta\|_{{}_0Z_2(J)}$ and $\|B^{-1}F\vartheta\|_Y$ as well as exploitation of duality relations. Therefore, we have to study the dual operator B'. Since B is densely defined, closable, and bounded invertible, we already know the existence of $(B^{-1})'$ as well as $(B^{-1})' = (B')^{-1} \in \mathcal{B}(Y', D(B'))$ where $D(B') = \{y' \in Y' : \exists z' \in Y' : (y'|By) = (z'|y)\, \forall y \in D(B)\}$. Furthermore, in view of the dense embedding $D(B) \hookrightarrow Y$, which implies uniqueness of $z' \in Y'$, the above characterisation of $D(B')$ takes the form

$$D(B') = \{y' \in Y' : (G^{1/2})'y' + A'_{\varepsilon_0}y' = z' \in Y'\}.$$

The equation $(G^{1/2})'y' + A'_{\varepsilon_0}y' = z'$ can uniquely be solved in Y', that is, for every $z' \in Y'$ there exists a unique $y' \in D((G^{1/2})') \cap D(A'_{\varepsilon_0})$; but this entails $D(B') = D((G^{1/2})') \cap D(A'_{\varepsilon_0})$. We are now prepared to tackle the estimation of ϑ in ${}_0Z_2(J)$. On condition that $\phi \in {}_0Z_2(J)$ and $\vartheta \in {}_0\mathcal{Z}_2(J)$ satisfy (3.22), which implies $\vartheta + B^{-1}\phi \in D(F)$ due to the regularity $B^{-1}\phi \in {}_0\mathcal{Z}_2(J)$ as well as $\partial_\nu\vartheta = \partial_\nu B^{-1}\phi = 0$ and $\partial_\nu A_{\varepsilon_0}(\vartheta + B^{-1}\phi) = \partial_\nu(A_{\varepsilon_0}\vartheta + \phi) - G^{1/2}\partial_\nu B^{-1}\phi = 0$, we may proceed as follows

$$\begin{aligned}\|\vartheta\|_{{}_0Z_2(J)} &\le C\|\vartheta\|_{D(B)} \le C(\|\vartheta + B^{-1}\phi\|_{D(B)} + \|B^{-1}\phi\|_{D(B)})\\ &\le C(\|B(\vartheta + B^{-1}\phi)\|_Y + \|\phi\|_Y) = C(\|LB^{-1}F(\vartheta + B^{-1}\phi)\|_Y + \|\phi\|_Y)\end{aligned}$$

with $L := BF^{-1}B$. Thus, we have to show boundedness of L, more precisely, for all $\psi \in D(B)$ it must hold $\|L\psi\|_Y \le c\|\psi\|_Y$. To see this, we firstly reform L according to

$$L = BG^{1/2}F^{-1} + BF^{-1}A_{\varepsilon_0} = B^2F^{-1} + BG^{1/2}F^{-1}\,[A_{\varepsilon_0}, M]_-\; G^{1/2}F^{-1}.$$

Computation of the commutator results in

$$[A_{\varepsilon_0}, M]_-\; \psi = 2\varepsilon_0\nabla(\tfrac{\varepsilon_0\rho_0}{\gamma_0}) \cdot \nabla\psi + \nabla\cdot(\varepsilon_0\nabla(\tfrac{\varepsilon_0\rho_0}{\gamma_0}))\psi =: K_1\psi$$

where K_1 is a differential operator of first order. Thus L can be represented as follows

$$L = B^2F^{-1} + [BF^{-1/2}][G^{1/2}F^{-1/2}][K_1F^{-1/2}][G^{1/2}F^{-1/2}],$$

from which boundedness, bearing in mind the regularity assumptions on ρ_0, γ_0, ε_0 as well as the condition on p, is obvious. After all we have achieved

$$\|\vartheta\|_{{}_0Z_2(J)} \leq c_1\|B^{-1}F(\vartheta + B^{-1}\phi)\|_Y + c_2\|\phi\|_Y.$$

Setting $\tilde{\vartheta} := \vartheta + B^{-1}\phi$, the first norm is equivalent to $\sup\{\langle\psi'|B^{-1}F\tilde{\vartheta}\rangle : \|\psi'\|_{Y'} \leq 1\} = \sup\{\langle\psi|F\tilde{\vartheta}\rangle : \psi = (B')^{-1}\psi' \in D(B'), \|B'\psi\|_{Y'} \leq 1\}$ and, due to relation (3.24), we obtain

$$\begin{aligned} \|B^{-1}F\tilde{\vartheta}\|_Y = \sup_{\|B'\psi\|_{Y'}\leq 1} \Big[\langle (G^{\frac{1}{2}})'\psi|MG^{\frac{1}{2}}\vartheta\rangle + \langle A'_{\varepsilon_0}\psi|A_{\varepsilon_0}\vartheta + \phi\rangle + \langle (G^{\frac{1}{2}})'\psi|MG^{\frac{1}{2}}B^{-1}\phi\rangle \\ -\langle A'_{\varepsilon_0}\psi|G^{\frac{1}{2}}B^{-1}\phi\rangle\Big] \leq C\|\phi\|_Y + \sup_{\|B'\psi\|_{Y'}\leq 1}\Big(\langle (G^{\frac{1}{2}})'\psi|\tfrac{\varepsilon_0}{\gamma_0}G^{1/2}\Phi_1\rangle + \langle\nabla(\tfrac{\varepsilon_0}{\gamma_0}\psi)|\Phi_2\rangle \\ + \langle A_{\gamma_0-\gamma_1}\psi|\phi - A_{\varepsilon_0}[c_1 - c_2]\rangle + \langle[\gamma_1-\gamma_2]\nabla\psi|\nabla[A_{\varepsilon_0}c_2 - g(\rho_2,c_2)]\rangle - \langle\nabla\cdot(\tfrac{\varepsilon_0^2}{\gamma_0}\nabla(\tfrac{\gamma_0}{\varepsilon_0})\psi)|\phi\rangle \\ - \langle\nabla\cdot[\tfrac{\varepsilon_0^2}{\gamma_0}\nabla(\tfrac{\gamma_0}{\varepsilon_0})\psi]|A_{\varepsilon_0}[c_1-c_2]\rangle\Big) \leq C\Big(\|\phi\|_Y + \|\Phi_1\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega))} + \|\Phi_2\|_{\mathrm{L}_p(J;\mathrm{L}_p(\Omega;\mathbb{R}^n))} \\ + \|\gamma_0-\gamma_1\|_{\mathrm{C}(J;\mathrm{C}^1(\overline{\Omega}))}\|c_1-c_2\|_{{}_0Z_2(J)}\Big) + \sup_{\|B'\psi\|_{Y'}\leq 1}|\langle\nabla\cdot[\tfrac{\varepsilon_0^2}{\gamma_0}\nabla(\tfrac{\gamma_0}{\varepsilon_0})\psi]|A_{\varepsilon_0}[c_1-c_2]\rangle| \\ + \sup_{\|B'\psi\|_{Y'}\leq 1}|\langle[\gamma_1-\gamma_2]\nabla\psi|\nabla[A_{\varepsilon_0}c_2 - g(\rho_2,c_2)]\rangle|. \end{aligned}$$

In the end, the remaining terms can be estimated by Hölder's inequality with exponents $s = \frac{4p}{4+p}$ and $s' = \frac{4p}{3p-4}$ due to the embeddings $|\nabla\psi| \in \mathrm{H}_{p'}^{1/4}(J;\mathrm{L}_{p'}(\Omega)) \hookrightarrow \mathrm{L}_{s'}(J;\mathrm{L}_{p'}(\Omega))$, $c_1 - c_2 \in {}_0Z_2(J) \hookrightarrow {}_0\mathrm{H}_p^{1/4}(J;\mathrm{H}_p^1(\Omega)) \hookrightarrow {}_0\mathrm{C}(J;\mathrm{C}(\overline{\Omega}))$, $c_2 \in \mathcal{Z}_2(J) \hookrightarrow \mathrm{H}_p^{1/4}(J;\mathrm{H}_p^3(\Omega)) \hookrightarrow \mathrm{C}(J;\mathrm{H}_p^3(\Omega))$ and $\rho_2 \in \mathcal{Z}_3(J) \hookrightarrow \mathrm{L}_\infty(J;\mathrm{C}^2(\overline{\Omega}))$. Thus, we end up

$$\begin{aligned} \sup_{\|B'\psi\|_{Y'}\leq 1}|\langle\nabla\cdot[\tfrac{\varepsilon_0^2}{\gamma_0}\nabla(\tfrac{\gamma_0}{\varepsilon_0})\psi]|A_{\varepsilon_0}[c_1-c_2]\rangle| \leq \|\varepsilon_0\|_{\mathrm{C}^1(\overline{\Omega})}\|\tfrac{\varepsilon_0^2}{\gamma_0}\nabla(\tfrac{\gamma_0}{\varepsilon_0})\|_{\mathrm{C}^1(\overline{\Omega})}\|\nabla\psi\|_{\mathrm{L}_{s'}(J;\mathrm{L}_{p'}(\Omega;\mathbb{R}^n))} \\ \cdot\|c_1-c_2\|_{\mathrm{L}_s(J;\mathrm{H}_p^2(\Omega))} \leq C(R_0,R_1)T^{\frac{1}{4}}\|c_1-c_2\|_{\mathrm{L}_p(J;\mathrm{H}_p^2(\Omega))} \end{aligned}$$

and

$$\begin{aligned} \sup_{\|B'\psi\|_{Y'}\leq 1}\langle[\gamma_1-\gamma_2]\nabla\psi|\nabla[A_{\varepsilon_0}c_2 - g(\rho_2,c_2)]\rangle \leq \|\gamma_1-\gamma_2\|_{\mathrm{L}_s(J;\mathrm{C}(\overline{\Omega}))}\|\nabla\psi\|_{\mathrm{L}_{s'}(J;\mathrm{L}_{p'}(\Omega;\mathbb{R}^n))}\cdot \\ \|\nabla[A_{\varepsilon_0}c_2 - g(\rho_2,c_2)]\|_{\mathrm{C}(J;\mathrm{L}_p(\Omega;\mathbb{R}^n))} \leq C(R_0,R_1)T^{\frac{1}{4}+\frac{1}{p}}\|\gamma_1-\gamma_2\|_{{}_0\mathrm{B}(J;\mathrm{H}_p^1(\Omega))}. \end{aligned}$$

Observe that $g(\rho_2, c_2)$ only consists of terms comprising ρ_2, c_2 and their derivatives up to order 2. Therefore, by exploiting the embeddings $\rho_2 \in \mathcal{Z}_3(J) \hookrightarrow B(J;\mathrm{C}^2(\overline{\Omega}))$, for $p > n$, and $c_2 \in \mathcal{Z}_2(J) \hookrightarrow \mathrm{L}_p(J;\mathrm{C}^3(\overline{\Omega})) \cap \mathrm{C}(J;\mathrm{C}^2(\overline{\Omega})) \cap \mathrm{C}(J;\mathrm{H}_p^3(\Omega))$, for $p > \hat{p}$, it is easy to show that $\nabla(A_{\varepsilon_0}c_2 - g(\rho_2,c_2))$ lies in $\mathrm{C}(J;\mathrm{L}_p(\Omega;\mathbb{R}^n))$. Finally, the estimates above bring forth (3.23). We also underline that the constants C above, arising from embedding inequalities, are independent of T, which is caused by working with differences having time trace 0 at $t = 0$. This fact can be seen by means of extending these functions to the whole positive line $\mathbb{R}_+$. □

4. Proof of Theorem 2.1

At first, keep in mind that problem (2.1)-(2.8) could equivalently rewritten as (2.17) with $\rho(t,x) = L[u,\rho_0](t,x)$ where

$$L : B_r(0) \subset \mathcal{Z}_{1,\Gamma}(J) \times \mathrm{H}^3_p(\Omega) \to B_{c(r)}(0) \subset \mathcal{Z}_3(J),$$

$B_r(0)$ denoting a ball with radius r and centre 0, and $\mathcal{Z}_{1,\Gamma}(J) := \{v \in \mathcal{Z}_1(J) : (v|\nu)_\Gamma \geq 0\}$, see Lemma 3.1. Furthermore, due to the Theorems 3.1 and 3.2 we have maximal L_p-regularity for the associated inear problem, that is, $\mathcal{L}$ is a continuous one-to-one mapping from the space of data to the class of maximal regularity,

$$\begin{aligned}
&\mathcal{L} \in \mathcal{L}is(\mathcal{Z}_{1,\mathcal{B}}(J) \times \mathcal{Z}_2(J), \mathcal{D}_1(J) \times \mathcal{D}_2(J)),\\
&\mathcal{D}_1(J) := \{\varphi \in \mathcal{X}_{1,\Gamma}(J) \times \mathcal{Y}_{0,d}(J;\mathbb{R}^n) \times \mathcal{Y}_{0,s}(J) \times \mathcal{Y}_{1,s}(J;\mathbb{R}^n) \times \mathrm{W}_p^{4-2/p}(\Omega;\mathbb{R}^n) :\\
&\qquad \varphi \text{ fulfils 4.-7. of Thereom 3.2}\}\\
&\mathcal{D}_2(J) := \{\varphi \in \mathcal{X}_2(J) \times \times \mathcal{Y}_1(J) \times \mathcal{Y}_2(J) \times \mathrm{W}_p^{4-4/p}(\Omega) : \varphi \text{ fulfils 4. of Theorem 3.1 }\}.
\end{aligned}$$

Using this property and Lemma 3.1, it is easy to verify that $\mathcal{F}_i$, see (2.18), maps $\mathcal{Z}(J)$ to $\mathcal{D}_i(J)$ and hence $\mathcal{L}(u',c') = (\mathcal{F}_1(u,c,\rho), u_0, \mathcal{F}_2(u,c,\rho), c_0)$ can be solved uniquely, meaning that the fixed point mapping $\mathcal{G}$ is well-defined. As elucidated in section 2.1 we shall prove selfmapping in $\Sigma' := \{(u,c) \in \Sigma : u \in \mathcal{Z}_{1,\Gamma}(J) \cap \mathcal{Z}_{1,\mathcal{B}}(J)\}$ and the contraction inequality with respect to the norm of $Z(J)$. Now, we give an answer to the choice of $(\overline{u},\overline{c})$ entering in the definition of Σ. Let $(\tilde{u},\tilde{c}) \in \mathcal{Z}_{1,\mathcal{B}}(\mathbb{R}_+) \cap \mathcal{Z}_{1,\Gamma}(\mathbb{R}_+) \times \mathcal{Z}_2(\mathbb{R}_+)$ be given, so that $(\tilde{u},\tilde{c})_{|t=0} = (u_0,c_0)$ and additionally $\partial_t \tilde{u}_{|t=0} = -\nabla u_0 \cdot u_0 + \rho_0^{-1} \nabla\cdot[\mathcal{S}_{|t=0} + \mathcal{P}_{|t=0}] + f_{|t=0}$, in view of higher regularities. We further set $R_2 := \|(\tilde{u},\tilde{c})\|_{\mathcal{Z}_1(\mathbb{R}_+)\times\mathcal{Z}_2(\mathbb{R}_+)}$. Then by $\tilde{\rho}$ we mean the unique solution of

$$\begin{aligned}
\partial_t \tilde{\rho} + \nabla\cdot(\tilde{\rho}\tilde{u}) = 0, &\quad (t,x) \in J \times \Omega,\\
\tilde{\rho}(0) = \rho_0, &\quad x \in \Omega.
\end{aligned}$$

This kind of approximation for ρ ensures $\partial_t^k \tilde{\rho}_{|t=0} = \partial_t^k \rho_{|t=0}$, $k = 0, 1$. Then we just put

$$\mathcal{L}(\overline{u},\overline{c}) = (\mathcal{F}_1(\tilde{u},\tilde{c},\tilde{\rho}), u_0, \mathcal{F}_2(u_0,c_0,\rho_0), c_0). \tag{4.1}$$

Notice that the right side $(\mathcal{F}_1(\tilde{u},\tilde{c},\tilde{\rho}), u_0)$ belongs to $\mathcal{D}_1(J)$ and, in view of the constraint $p > \hat{p}$, $(\mathcal{F}_2(u_0,c_0,\rho_0), c_0) \in \mathcal{D}_2(J)$ as well, in particular, all compatibility conditions are satisfied. Theorem 3.1 and 3.2 guarantee existence and uniqueness of $(\overline{u},\overline{c})$ in $\mathcal{Z}_{1,\mathcal{B}}(J)\times\mathcal{Z}_2(J)$ with $J = [0,T]$, any $T < \infty$. Hence $(\overline{u},\overline{c})$ can be considered as the value of "one fixed point iteration".

4.1. Contraction and selfmapping

Let us fix $R > 0$ and $T > 0$. We consider $R_0 \in (0,R)$, $T_0 \in (0,T)$ and set $J_0 = [0,T_0]$. It follows that for any $(u,c) \in \Sigma'$

$$\|u\|_{\mathcal{Z}_1(J_0)} + \|c\|_{\mathcal{Z}_2(J_0)} \leq R_0 + \|\overline{u}\|_{\mathcal{Z}_1(J_0)} + \|\overline{c}\|_{\mathcal{Z}_2(J_0)} \leq R + R_2,$$

and, due to Lemma 3.1, we also get $\|\rho\|_{\mathcal{Z}_3(J_0)} \leq c_0(\overline{R})$ with $\overline{R} := \max\{\|\rho_0\|_{\mathrm{H}^3_p(\Omega)}, R + R_2\}$ which is independent of R_0 and T_0. Also, notice that for any $(u,c) \in \Sigma'$ and any function space $Y(J_0)$ being continuously embedded into $\mathcal{Z}_1(J_0)\times\mathcal{Z}_2(J_0)$ we have

$$\begin{aligned}\|(u,c)\|_{Y(J_0)} &\le \|(u,c)-(\overline{u},\overline{c})\|_{_0Y(J_0)} + \|(\overline{u},\overline{c})\|_{Y([0,T])}\\ &\le c_E\|(u,c)-(\overline{u},\overline{c})\|_{_0\mathcal{Z}_1(J_0)\times{}_0\mathcal{Z}_2(J_0)} + c_E R_2,\end{aligned}$$

hence independent of R_0 and T_0, cf. the first remarks in the proof of Lemma 3.1.

Step 1: Contraction. Let $w_1=(u_1,c_1)$, $w_2=(u_1,c_2)\in\Sigma'$ be given and set $\rho_i = L[u_i,\rho_0]$. Then by $(u_1'-u_2',c_1'-c_2')$ we denote the unique solution of

$$\mathcal{L}(u_1'-u_2',c_1'-c_2') = (\mathcal{F}_1(w_1,\rho_1)-\mathcal{F}_1(w_2,\rho_2),0,\mathcal{F}_2(w_1,\rho_1)-\mathcal{F}_2(w_2,\rho_2),0).$$

Using the maximal regularity result 6.1 and 3.4 we obtain

$$\begin{aligned}&\|u_1'-u_2'\|_{_0\mathcal{Z}_1(J_0)} \le M_1\|(\mathcal{F}_1(w_1,\rho_1)-\mathcal{F}_1(w_2,\rho_2),0)\|_{_0\mathcal{D}_1(J_0)} \le M_1\big\{\|(\rho_1-\rho_2)\partial_t u_2\|_{X_1(J_0)}+\\ &\|(\rho_0-\rho_1)\partial_t(u_1-u_2)\|_{X_1(J_0)} + \|\rho_1\nabla u_1 u_1-\rho_2\nabla u_2 u_2\|_{X_1(J_0)} + \|\nabla\cdot([\mathcal{S}_1-\tilde{\mathcal{S}}](u_1-u_2))\|_{X_1(J_0)}\\ &+\|\nabla\cdot([\mathcal{S}_1-\mathcal{S}_2](u_2))\|_{X_1(J_0)} + \|(\rho_1-\rho_2)f_{ext}\|_{X_1(J_0)} + \|\nabla(\pi_1-\pi_2)\|_{X_1(J_0)}\\ &+\|(\rho_1^2\varepsilon_{\rho,1}\nabla c_1-\tilde{\rho}^2\tilde{\varepsilon}_\rho\nabla\tilde{c})\cdot\nabla^2[c_1-c_2]\|_{X_1(J_0)} + \|(\rho_1^2\varepsilon_{\rho,1}\nabla c_1-\rho_2^2\varepsilon_{\rho,2}\nabla c_2)\cdot\nabla^2 c_2\|_{X_1(J_0)}\\ &+\tfrac12\|\nabla(\rho_1^2\varepsilon_{\rho,1})|\nabla c_1|^2-\nabla(\rho_2^2\varepsilon_{\rho,2})|\nabla c_2|^2\|_{X_1(J_0)} + \|[\rho_1\varepsilon_1\nabla c_1-\tilde{\rho}\tilde{\varepsilon}\nabla\tilde{c}](\Delta\mathcal{I}+\nabla^2)[c_1-c_2]\|_{X_1(J_0)}\\ &+\|[\rho_1\varepsilon_1\nabla c_1-\rho_2\varepsilon_2\nabla c_2](\Delta c_2\mathcal{I}+\nabla^2 c_2)\|_{X_1(J_0)} + \|\nabla(\rho_1\varepsilon_1)\cdot\nabla c_1\nabla c_1-\nabla(\rho_2\varepsilon_2)\cdot\nabla c_2\nabla c_2\|_{X_1(J_0)}\\ &+\|[\tilde{\eta}-\eta_1]\mathcal{Q}\mathcal{D}(u_1-u_2)\cdot\nu_{|\Gamma_s}\|_{Y_{1,s}(J_0;\mathbb{R}^n)} + \|[\eta_1-\eta_2]\mathcal{Q}\mathcal{D}(u_2)\cdot\nu_{|\Gamma_s}\|_{Y_{1,s}(J_0;\mathbb{R}^n)}\big\},\end{aligned}$$

with $\mathcal{S}_i(v) := \mathcal{S}(v,\rho_i)$, and

$$\begin{aligned}&\|c_1'-c_2'\|_{_0Z_2(J_0)} \le M_2\Big\{\|\phi\|_{\mathrm{L}_p(J_0;\mathrm{L}_p(\Omega))} + \|\rho_0-\rho_1\|_{_0\mathrm{C}^{\frac12+\beta}(J_0;\mathrm{C}(\overline{\Omega}))}\|c_1-c_2\|_{_0\mathrm{H}_p^{1/2}(J_0;\mathrm{L}_p(\Omega))}+\\ &\|\rho_1-\rho_2\|_{_0\mathrm{H}_p^{1/2}(J_0;\mathrm{L}_p(\Omega))}\|c_2\|_{\mathrm{C}^{\frac12+\beta}(J_0;\mathrm{C}(\overline{\Omega}))} + \|\Phi_2\|_{\mathrm{L}_p(J_0;\mathrm{L}_p(\Omega))}+\\ &\|\gamma_0-\gamma_1\|_{\mathrm{L}_\infty(J_0;\mathrm{C}^1(\overline{\Omega}))}\|c_1-c_2\|_{\mathrm{L}_p(J_0;\mathrm{H}_p^2(\Omega))} + \max\{T_0^{\frac14},T_0^{\frac14+\frac1p}\}\|\gamma_1-\gamma_2\|_{_0\mathrm{B}(J_0;\mathrm{H}_p^1(\Omega))}\Big\}\end{aligned}$$

with an appropriate $\beta>0$. In view of Lemma 3.3, each difference $\rho_1-\rho_2$ can be estimated by means of u_1-u_2,

$$\|L[\rho_0,u_1]-L[\rho_0,u_2]\|_{_0Z_3(J_0)} \le \kappa(T_0,c(\overline{R}))\|u_1-u_2\|_{_0Z_1(J_0)}.$$

Recall that ε_i, γ_i and η_i were shortcuts for $\varepsilon(\rho_i,c_i)$, $\gamma(\rho_i,c_i)$ and $\eta(\rho_i,c_i)$, respectively. Subsequently, it is decisive that the operator norm of $\mathcal{L}^{-1}$ is independent of the time interval $J_0=[0,T_0]$, but might depend on $T>T_0$. This can only be achieved in case of null initial data, which is satisfied by considering differences. This fact will also be used in the upcoming estimates in which constants occur due to embedding and interpolation inequalities. The latter estimate exemplarily shows how contraction will be achieved, since $T_0\in(0,T)$ can be chosen freely and $\kappa(T_0)\to 0$ as $T_0\to 0$. To see that the two inequalities above can be estimated to a similar result, we will only demonstrate this proceedure by means of some selected terms.

Let us begin with a few examples from the first inequality. Using the identity $\rho_1(t)-\rho_0 = \int_0^t \nabla\cdot(\rho_1 u_1)ds$, the quasilinear term $(\rho_0-\rho_1)\partial_t u_1-(\rho_0-\rho_2)\partial_t u_2$ can be estimated in

$X_1(J_0) = \mathrm{L}_p(J_0; \mathrm{L}_p(\Omega; \mathbb{R}^n))$ by

$$\|\rho_0 - \rho_1\|_{\mathrm{C}(J_0;\mathrm{C}(\overline{\Omega}))}\|u_1 - u_2\|_{{}_0Z_1(J_0)} + \|\rho_1 - \rho_2\|_{\mathrm{B}(J_0;\mathrm{L}_p(\Omega))}\|\partial_t u_2\|_{\mathrm{L}_p(J_0;\mathrm{C}(\overline{\Omega};\mathbb{R}^n))} \leq \Big(T_0\|\rho_1\|_{\mathrm{C}(J_0;\mathrm{C}^1(\overline{\Omega}))}\|u_1\|_{\mathrm{C}(J_0;\mathrm{C}^1(\overline{\Omega};\mathbb{R}^n))} + \kappa(T_0)\|\partial_t u_2\|_{\mathrm{L}_q(J_0;\mathrm{C}(\overline{\Omega};\mathbb{R}^n))}\Big)\|u_1 - u_2\|_{{}_0Z_1(J_0)} \leq C(\overline{R})\max\{T_0, \kappa(T_0)\}\|u_1 - u_2\|_{{}_0Z_1(J_0)}.$$

Typical nonlinear differences involving c are $(\rho_1^2\varepsilon_{\rho,1}\nabla c_1 - \tilde{\rho}^2\tilde{\varepsilon}_\rho\nabla\tilde{c})\cdot\nabla^2[c_1 - c_2]$ and $(\rho_1^2\varepsilon_{\rho,1}\nabla c_1 - \rho_2^2\varepsilon_{\rho,2}\nabla c_2)\cdot\nabla^2 c_2$, the first one of highest order but with a "small factor" and the second one of lower order. The first difference can be estimated in $X_1(J_0)$ by

$$\|\rho_1^2\varepsilon_{\rho,1}\nabla c_1 - \tilde{\rho}^2\tilde{\varepsilon}_\rho\nabla\tilde{c}\|_{\mathrm{B}(J_0;\mathrm{C}(\overline{\Omega}))}\|\nabla^2(c_1 - c_2)\|_{\mathrm{L}_p(J_0;\mathrm{L}_p(\Omega;\mathbb{R}^{n\times n}))} \leq T_0^{1/4}\|\rho_1^2\varepsilon_{\rho,1}\nabla c_1 - \tilde{\rho}^2\tilde{\varepsilon}_\rho\nabla\tilde{c}\|_{{}_0\mathrm{C}^{1/4}(J_0;\mathrm{C}(\overline{\Omega}))}\|c_1 - c_2\|_{{}_0Z_2(J_0)} \leq C(\overline{R})T_0^{1/4}\|c_1 - c_2\|_{{}_0Z_2(J_0)},$$

where we used $c_1 - \tilde{c} \in {}_0\mathcal{Z}_2(J_0) \hookrightarrow {}_0\mathrm{C}^{1/2}(J_0; \mathrm{C}^1(\overline{\Omega}))$, $\rho_1 - \tilde{\rho} \in {}_0\mathcal{Z}_3(J_0) \hookrightarrow {}_0\mathrm{C}^{1/2}(J_0; \mathrm{C}^1(\overline{\Omega}))$ and triangle inequality along with differentiability of ε. The second difference can be approached in the same way,

$$\|(\rho_1^2\varepsilon_{\rho,1}\nabla c_1 - \rho_2^2\varepsilon_{\rho,2}\nabla c_2)\cdot\nabla^2 c_2\|_{X_1(J_0)} \leq T_0^{1/p}\|\rho_1^2\varepsilon_{\rho,1}\nabla c_1 - \rho_2^2\varepsilon_{\rho,2}\nabla c_2\|_{\mathrm{B}(J_0;\mathrm{L}_p(\Omega;\mathbb{R}^n))} \|c_2\|_{\mathrm{C}(J_0;\mathrm{C}^2(\overline{\Omega}))} \leq C(\overline{R})T_0^{1/p}(\|c_1 - c_2\|_{{}_0Z_2(J_0)} + \|\rho_1 - \rho_2\|_{Z_3(J_0)}),$$

as $c_1 - c_2 \in {}_0Z_2(J_0) \hookrightarrow \mathrm{C}(J_0; \mathrm{H}^1_p(\Omega))$ and ε is sufficient smooth, cf. below the treatment of differences of the form $a(\rho_1, c_1) - a(\rho_2, c_2)$ with a smooth. Next we study some norms issuing from the Cahn-Hilliard equation. To begin with, we briefly discuss the smallness of $\rho_1 - \rho_0$ and $\gamma_1 - \gamma_0$ appearing in front of $\|c_1 - c_2\|_{{}_0Z_2(J_0)}$. Due to the relation $\rho_1(t) - \rho_0 = \int_0^t \nabla\cdot(\rho_1(s)u_1(s))ds$, we are able to proceed as follows

$$\|\rho_1 - \rho_0\|_{\mathrm{C}^{1/2+\beta}(J_0;\mathrm{C}(\overline{\Omega}))} \leq T_0^{1-1/2-\beta}\|\nabla\cdot(\rho_1 u_1)\|_{\mathrm{C}(J_0;\mathrm{C}(\overline{\Omega}))} \leq T_0^{1/2-\beta}C(\overline{R}),$$

as $\mathcal{Z}_1(J_0)$, $\mathcal{Z}_3(J_0) \hookrightarrow \mathrm{C}(J_0; \mathrm{C}^1(\overline{\Omega}))$. Applying the mean value theorem to $\gamma_1 - \gamma_0$, smallness of $c_1 - c_0$ and $\rho_1 - \rho_0$ can be exploited to the result

$$\begin{aligned}\|\gamma_1 - \gamma_0\|_{\mathrm{L}_\infty(J_0;\mathrm{C}^1(\overline{\Omega}))} &\leq \|\gamma'(c_1 + \theta(c_0 - c_1), \rho_1 + \theta(\rho_0 - \rho_1))\|_{\mathrm{L}_\infty(J_0;\mathrm{C}^1(\overline{\Omega}))} \\ &\quad\cdot\Big(\|c_1 - c_0\|_{\mathrm{L}_\infty(J_0;\mathrm{C}^1(\overline{\Omega}))} + \|\rho_1 - \rho_0\|_{\mathrm{L}_\infty(J_0;\mathrm{C}^1(\overline{\Omega}))}\Big), \quad \theta \in (0,1), \\ &\leq \max_{(h,k)\in B_{r_1}(0)\times B_{r_2}(0)}\|\gamma'(h,k)\|_{\mathrm{L}_\infty(J_0;\mathrm{C}^1(\overline{\Omega}))}\Big(T_0^{1/4}\|c_1 - c_0\|_{\mathrm{C}^{1/4}(J_0;\mathrm{C}^1(\overline{\Omega}))} + C(\overline{R})T_0\Big) \\ &\leq C(T_0^{1/4}[R_0 + \|\overline{c} - c_0\|_{\mathrm{C}^{1/4}(J_0;\mathrm{C}^1(\overline{\Omega}))}] + T_0) \leq C(\overline{R})\max\{T_0^{1/4}, T_0\}.\end{aligned}$$

Here, we have set $r_1 := 2R_0 + \|\overline{c}\|_{\mathrm{L}_\infty(J_0;\mathrm{C}^1(\overline{\Omega}))} + \|c_0\|_{\mathrm{L}_\infty(J_0;\mathrm{C}^1(\overline{\Omega}))}$, $r_2 := 2c_0(\overline{R})$ and $B_r(0) \subset \mathrm{L}_\infty(J_0; \mathrm{C}^1(\overline{\Omega}))$ denotes a ball with radius r and centre 0. Further we have used the embeddings $\mathcal{Z}_1(J_0) \hookrightarrow \mathrm{C}(J_0; \mathrm{C}^2(\overline{\Omega}))$ and $\mathcal{Z}_2(J_0) \hookrightarrow \mathrm{C}^{1/4}(J_0; \mathrm{C}^1(\overline{\Omega}))$, both valid for $p > \hat{p}$. Next we consider the difference $\rho_1 - \rho_2$ in $\mathrm{H}_p^{1/2}(J_0; \mathrm{L}_p(\Omega))$. Since this difference has to be measured in $\mathrm{H}^1_r(J_0; \mathrm{L}_p(\Omega))$ with any $r \in [1, \infty)$, we may assume $r > 4$ at least; but this entails the embedding $\mathrm{H}^1_r(J_0; \mathrm{L}_p(\Omega)) \hookrightarrow \mathrm{C}^{3/4}(J_0; \mathrm{L}_p(\Omega))$ and thus the estimate

$$\|\rho_1 - \rho_2\|_{{}_0\mathrm{H}_p^{1/2}(J_0;\mathrm{L}_p(\Omega))} \leq T_0^{2/p}\|\rho_1 - \rho_2\|_{{}_0\mathrm{C}^{3/4}(J_0;\mathrm{L}_p(\Omega))} \leq CT_0^{2/p}\|\rho_1 - \rho_2\|_{{}_0Z_3(J_0)}.$$

As a last estimate we study the norm involving $\eta_1-\eta_2$ and $\mathcal{QD}(u_2)\cdot\nu$ on the boundary Γ_s. In this case one has to use $\eta_1-\eta_2\sim\rho_1-\rho_2\in {}_0Z_3(J_0)\hookrightarrow {}_0\mathrm{H}^\theta_r(J_0;\mathrm{H}^{1-\theta}_p(\Omega))\hookrightarrow {}_0\mathrm{C}^\beta(J_0;\mathrm{H}^{1-\theta}_p(\Omega))$ with $\theta\in(0,1)$, $0<\beta<3/4$ and $1-\theta-1/p>0$, and thus $[\rho_1-\rho_2]_{|\Gamma_s}\in\mathrm{C}^\beta(J_0;\mathrm{L}_p(\Gamma_s))$. This embedding as well as $\mathcal{Z}_1(J_0)\hookrightarrow\mathrm{H}^{1/2}_p(J_0;\mathrm{C}^1(\overline{\Omega};\mathbb{R}^n))\cap\mathrm{L}_p(J_0;\mathrm{C}^2(\overline{\Omega};\mathbb{R}^n))$ enables us to proceed as follows

$$\begin{aligned}\|[\eta_1-\eta_2]\mathcal{QD}(u_2)\cdot\nu_{|\Gamma_s}\|_{Y_{1,s}(J_0;\mathbb{R}^n)}&\le C(\eta')\Big(\|\rho_1-\rho_2\|_{\mathrm{C}^{1/2}(J;\mathrm{L}_p(\Gamma_s))}\|u_2\|_{\mathrm{W}^{1/2-1/2p}_p(J_0;\mathrm{C}(\overline{\Omega};\mathbb{R}^n))}\\&\quad+T_0^{1/p}\|\rho_1-\rho_2\|_{\mathrm{B}(J_0;\mathrm{W}^{1-1/p}_p(\Gamma_s))}\|u_2\|_{\mathrm{C}(J_0;\mathrm{C}^1(\overline{\Omega};\mathbb{R}^n))}\Big)\\&\le C(\overline{R})\max\{T_0^{\beta-\frac12},T_0^{1/p}\}\|\rho_1-\rho_2\|_{{}_0\mathcal{Z}_3(J_0)},\quad\beta\in(\tfrac12,\tfrac34).\end{aligned}$$

Step 2: Selfmapping. In this case a very similar approach is possible, however we are concerned with estimates in spaces including higher regularites (a drawback of spaces involving high regularities). To begin with, let $(u,c)\in\Sigma'$ be given. We have to show that (u',c') given as the solution of $\mathcal{L}(u',c')=\mathcal{F}(u,c,L[u,\rho_0])$ lies in Σ' as well, that is, $\|(u'-\overline{u},c'-\overline{c})\|_{{}_0\mathcal{Z}_{1,\mathcal{B}}(J_0)\times{}_0\mathcal{Z}_2(J_0)}\le R_0$. By the Theorems 3.1 and 3.2 the following estimate is again available

$$\begin{aligned}&\|(u',c')-(\overline{u},\overline{c})\|_{{}_0\mathcal{Z}_{1,\mathcal{B}}(J_0)\times{}_0\mathcal{Z}_2(J_0)}=\\&\quad\|\mathcal{L}^{-1}(\mathcal{F}_1(u,c,\rho)-\mathcal{F}_1(\tilde{u},\tilde{c},\tilde{\rho}),0,\mathcal{F}_2(u,c,\rho)-\mathcal{F}_2(u_0,c_0,\rho_0),0)\|_{{}_0\mathcal{Z}_{1,\mathcal{B}}(J_0)\times{}_0\mathcal{Z}_2(J_0)}\le\\&\qquad M\left\{\|F_1(u,c,\rho)-F_1(\tilde{u},\tilde{c},\tilde{\rho})\|_{{}_0\mathcal{X}_{1,\Gamma}(J_0)}+\|F_2(u,c,\rho)-F_2(u_0,c_0,\rho_0)\|_{{}_0\mathcal{X}_2(J_0)}\right.\\&\qquad\quad\left.+\|\sigma_s(u,\rho)-\sigma_s(\tilde{u},\tilde{\rho})\|_{{}_0\mathcal{Y}_{0,s}(J_0)\times{}_0\mathcal{Y}_{1,s}(J_0;\mathbb{R}^n)}+\|\partial_\nu[g_0(\rho,c)-g_0(\rho_0,c_0)]\|_{{}_0\mathcal{Y}_3(J_0)}\right\}.\end{aligned}$$

Using (2.16) and (2.18) we further obtain for the boundary norm

$$\begin{aligned}\|\sigma_s(u,\rho)-\sigma_s(\tilde{u},\tilde{\rho})\|_{{}_0\mathcal{Y}_{0,s}(J_0)\times{}_0\mathcal{Y}_{1,s}(J_0;\mathbb{R}^n)}&=\|\tfrac{\eta(\tilde{\rho})-\eta(\rho)}{\eta(\tilde{\rho})}\mathcal{Q}\tilde{\mathcal{S}}(u)\cdot\nu_{|\Gamma_s}\|_{{}_0\mathcal{Y}_{1,s}(J_0;\mathbb{R}^n)}\\&\le C\|\tfrac{\eta(\tilde{\rho})-\eta(\rho)}{\eta(\tilde{\rho})}\|_{{}_0\mathcal{Y}_{1,s}(J_0)}\|\mathcal{Q}\tilde{\mathcal{S}}(u)\cdot\nu_{|\Gamma_s}\|_{\mathcal{Y}_{1,s}(J_0;\mathbb{R}^n)}\\&\le C\|\tfrac{\eta(\tilde{\rho})-\eta(\rho)}{\eta(\tilde{\rho})}\|_{{}_0\mathrm{H}^{3/2}_p(J_0;\mathrm{L}_p(\Omega))\cap\mathrm{L}_p(J_0;\mathrm{H}^3_p(\Omega))}\|u\|_{\mathcal{Z}_{1,\mathcal{B}}(J_0)}\\\le C\overline{R}\left(T_0^{\frac1p}\|\tfrac{\eta(\tilde{\rho})-\eta(\rho)}{\eta(\tilde{\rho})}\|_{\mathrm{B}(J_0;\mathrm{H}^3_p(\Omega))}\right.&\left.+\max\{T_0,T_0^{\frac2p}\}\|\tfrac{\eta(\tilde{\rho})-\eta(\rho)}{\eta(\tilde{\rho})}\|_{{}_0\mathrm{W}^{2+\frac14}_p(J_0;\mathrm{L}_p(\Omega))}\right)\\&\le C\overline{R}k(T_0)\|\eta(\tilde{\rho})^{-1}\|_{\mathcal{Z}_3(J_0)}\|\eta(\tilde{\rho})-\eta(\rho)\|_{{}_0\mathcal{Z}_3(J_0)}\\&\le C\overline{R}k(T_0)\max_{\|\varrho\|_{\mathcal{Z}_3(J_0)}\le 2R_1}\|\eta'(\varrho)\|_{\mathcal{Z}_3(J)}\|\rho-\tilde{\rho}\|_{{}_0\mathcal{Z}_3(J_0)}\le Ck(T_0),\end{aligned}$$

because ${}_0\mathcal{Y}_{1,s}(J_0)$ and $\mathcal{Z}_3(J_0)$ form multiplication algebras, for $p>\hat{p}$ at least, and the embedding $\mathrm{W}^{2+\frac14}_p(J_0)\hookrightarrow\mathrm{C}^{\frac32+\frac1p}(J_0)$ holds for $p>4$. Here, $k(T_0)$ tends to 0 as $T_0\to0$. Observe that, in view of this estimate, viscosities of the form $\eta(\rho,c)$ are not admissible. Moreover, in this estimate we require that $\tilde{\rho}$, constructed in section 4, approximate ρ in $\mathcal{Z}_3(J_0)$ which indeed is not the case for $\tilde{\rho}:=\rho_0$. We continue with the estimate above by considering the other boundary norm,

$$\begin{aligned}&\|\partial_\nu[g(\rho,c)-g(\rho_0,c_0)]\|_{{}_0\mathcal{Y}_3(J_0)}\le C\|g(\rho,c)-g(\rho_0,c_0)\|_{{}_0\mathcal{X}_1(J_0;\mathbb{R})}\le C\,\{\\&\quad\|\nabla\cdot([\varepsilon_0-\varepsilon]\nabla c)\|_{{}_0\mathcal{X}_1(J_0;\mathbb{R})}+\|\rho^{-1}\varepsilon\nabla\rho\cdot\nabla c-\rho_0^{-1}\varepsilon_0\nabla\rho_0\cdot\nabla c_0\|_{{}_0\mathcal{X}_1(J_0;\mathbb{R})}\\&\quad+\|\overline{\psi}_c(\rho,c)-\overline{\psi}_c(\rho_0,c_0)\|_{{}_0\mathcal{X}_1(J_0;\mathbb{R})}+\tfrac12\|[\varepsilon_c(\rho,c)|\nabla c|^2-\varepsilon_c(\rho_0,c_0)|\nabla c_0|^2]\|_{{}_0\mathcal{X}_1(J_0;\mathbb{R})}\}\end{aligned}$$

where two kind of differences occur, the highest order term $\nabla\cdot([\varepsilon_0-\varepsilon]\nabla c)$ and lower order terms having additional time regularity. Let us exemplarily study this highest order term. Using again that $\mathcal{X}_1(J_0;\mathbb{R})$ forms a multiplication algebra for $p>\hat{p}$, we may estimate $\|\nabla\cdot([\varepsilon_0-\varepsilon]\nabla c)\|_{{}_0\mathcal{X}_1(J_0;\mathbb{R})}$ by

$$\begin{aligned}C\|\varepsilon_0-\varepsilon\|_{{}_0\mathrm{H}_p^{1/2}(J_0;\mathrm{H}_p^1(\Omega))\cap\mathrm{L}_p(J_0;\mathrm{H}_p^3(\Omega))}\|c\|_{\mathrm{H}_p^{1/2}(J_0;\mathrm{H}_p^2(\Omega))\cap\mathrm{L}_p(J_0;\mathrm{H}_p^4(\Omega))}\leq\overline{R}C(\\ T_0^{1/p}\|\varepsilon_0-\varepsilon\|_{\mathrm{B}(J_0;\mathrm{H}_p^3(\Omega))}+T_0^{\frac{2}{p}}\|\varepsilon_0-\varepsilon\|_{\mathrm{C}^{\frac12+\frac1p}(J_0;\mathrm{H}_p^1(\Omega))})\leq C(\varepsilon',\overline{R})\max\{T_0^{1/p},T_0^{2/p}\},\end{aligned}$$

where we used $\mathcal{Z}_3(J_0)\hookrightarrow\mathrm{C}^{\frac12+\frac1p}(J_0;\mathrm{H}_p^1(\Omega))\hookrightarrow\mathrm{C}^{\frac12+\frac1p}(J_0;\mathrm{C}(\overline{\Omega}))$, for $p>\hat{p}$. Since the terms of lower order possess more time regularity as needed, we are able to get similar results as above. Next we insert the definitions of F_1 and F_2, see (2.16), to obtain the estimates

$$\begin{aligned}&\|F_1(u,c,\rho)-F_1(\tilde{u},\tilde{c},\tilde{\rho})\|_{{}_0\mathcal{X}_{1,\Gamma}(J_0)}\leq\|(\tilde{\rho}-\rho)\partial_t u\|_{{}_0\mathcal{X}_1(J_0)}+\|\nabla\cdot[\mathcal{S}(u)-\tilde{\mathcal{S}}(u)]\|_{{}_0\mathcal{X}_1(J_0)}+\\ &\quad\|\pi'(\rho,c)\nabla(\rho,c)-\pi'(\tilde{c},\tilde{\rho})\nabla(\tilde{c},\tilde{\rho})\|_{{}_0\mathcal{X}_1(J_0)}+\|\nabla(\rho\varepsilon)\cdot\nabla c\nabla c-\nabla(\tilde{\rho}\tilde{\varepsilon})\cdot\nabla\tilde{c}\nabla\tilde{c})\|_{{}_0\mathcal{X}_1(J_0)}+\\ &\tfrac12\|\nabla(\rho^2\varepsilon_\rho)|\nabla c|^2-\nabla(\tilde{\rho}^2\tilde{\varepsilon}_\rho)|\nabla\tilde{c}|^2\|_{{}_0\mathcal{X}_1(J_0)}+\|[\rho-\tilde{\rho}]f_{ext}\|_{{}_0\mathcal{X}_1(J_0)}+\|\rho\nabla u\cdot u-\tilde{\rho}\nabla\tilde{u}\cdot\tilde{u}\|_{{}_0\mathcal{X}_1(J_0)},\end{aligned}$$

as $[F_1(u,c,\rho)-F_1(\tilde{u},\tilde{c},\tilde{\rho})]_{|t=0,\Gamma}=0$, and at last

$$\begin{aligned}&\|F_2(u,c,\rho)-F_2(u_0,c_0,\rho_0)\|_{{}_0\mathcal{X}_2(J_0)}\leq\|\tfrac{\varepsilon_0}{\gamma_0}\|_{\mathrm{C}^1(\overline{\Omega})}\big(\|[\rho_0-\rho]\partial_t c\|_{{}_0\mathcal{X}_2(J_0)}\\ &\quad+\|\rho u\cdot\nabla c-\rho_0u_0\cdot\nabla c_0\|_{{}_0\mathcal{X}_2(J_0)}+\|\nabla\cdot([\gamma_0-\gamma]\nabla(\nabla\cdot(\varepsilon_0\nabla c)+g(\rho,c)))\|_{{}_0\mathcal{X}_2(J_0)}\big)\\ &\qquad+\|\nabla\cdot(\varepsilon_0\nabla[g(\rho,c)-g(\rho_0,c_0)])\|_{{}_0\mathcal{X}_2(J_0)}\\ &\quad+\|\tfrac{\varepsilon_0^2}{\gamma_0}\nabla(\tfrac{\gamma_0}{\varepsilon_0})\|_{\mathrm{C}^1(\overline{\Omega})}\big(\|\nabla\nabla\cdot(\varepsilon_0\nabla[c-c_0])\|_{{}_0\mathcal{X}_2(J_0)}+\|\nabla[g(\rho,c)-g(\rho_0,c_0)]\|_{{}_0\mathcal{X}_2(J_0)}\big).\end{aligned}$$

We will not carry out all estimates in every detail, because it would go beyond the scope of this work, but the forthcoming procedure can be adopted to all other cases. First of all, notice that there are again two kinds of differences in the estimates above, higher order terms multiplied with a 'small' difference and lower order terms (l.o.t.) with more time regularity inducing a factor T_0^β with $\beta>0$. We start with the highest order difference $\nabla\cdot[\eta\mathcal{D}(u)-\tilde{\eta}\mathcal{D}(u)]$ being a part of $\nabla\cdot[\mathcal{S}(u)-\tilde{\mathcal{S}}(u)]$. Having in mind that $\mathcal{X}_1(J_0)$ forms a multiplication algebra, this difference can be treated as follows

$$\begin{aligned}&\|\nabla\cdot[\eta\mathcal{D}(u)-\tilde{\eta}\mathcal{D}(u)]\|_{{}_0\mathcal{X}_1(J_0)}\leq C\|\eta-\tilde{\eta}\|_{{}_0\mathrm{H}_p^{1/2}(J_0;\mathrm{H}_p^1(\Omega))\cap\mathrm{L}_p(J_0;\mathrm{H}_p^3(\Omega))}\|u\|_{\mathcal{Z}_1(J_0)}\\ &\quad\leq C\overline{R}\Bigg(\max_{\|\varrho\|_{\mathcal{Z}_3(J_0)}\leq 2c_0(R)}\|\eta'(\varrho)\|_{\mathrm{H}_p^{1/2}(J_0;\mathrm{H}_p^1(\Omega))}T_0^{2/p}\|\rho-\tilde{\rho}\|_{{}_0\mathrm{C}^1(J_0;\mathrm{H}_p^1(\Omega))}+\\ &\quad T_0^{1/p}\max_{\|\varrho\|_{\mathcal{Z}_3(J_0)}\leq 2c_0(R)}\|\eta'(\varrho)\|_{\mathcal{Z}_3(J_0)}\|\rho-\tilde{\rho}\|_{{}_0\mathrm{B}(J_0;\mathrm{H}_p^3(\Omega))}\Bigg)\leq C(\overline{R})\max\{T_0^{2/p},T_0^{1/p}\}.\end{aligned}$$

Next we pick up the lower order term (l.o.t.) $\rho\nabla u\cdot u-\tilde{\rho}\nabla\tilde{u}\cdot\tilde{u}$ to demonstrate how such terms can be dealt with. Using the above embedding for $\mathcal{Z}_3(J_0)$ and $\mathcal{Z}_1(J_0)\hookrightarrow\mathrm{C}^{\frac12+\frac1p}(J_0;\mathrm{H}_p^1(\Omega;\mathbb{R}^n))\cap\mathrm{C}(J_0;\mathrm{H}_p^2(\Omega;\mathbb{R}^n))$ as well as the fact that $\mathrm{H}_p^2(\Omega)$ forms a multiplication algebra, we may proceed as follows

$$\begin{aligned}\|\rho\nabla u\cdot u-\tilde{\rho}\nabla\tilde{u}\cdot\tilde{u}\|_{{}_0\mathcal{X}_1(J_0)}&\leq T_0^{\frac2p}\|l.o.t.\|_{{}_0\mathrm{C}^{\frac12+\frac1p}(J_0;\mathrm{L}_p(\Omega;\mathbb{R}^n))}+T_0^{\frac1p}\|l.o.t.\|_{{}_0\mathrm{B}(J_0;\mathrm{H}_p^2(\Omega;\mathbb{R}^n))}\\ &\leq C(\overline{R})\max\{T_0^{\frac2p},T_0^{\frac1p}\},\end{aligned}$$

As a last task we consider $\nabla\cdot(\gamma\nabla\nabla\cdot([\varepsilon_0-\varepsilon]\nabla c))$ appearing in $F_2(u_1,c_1,\rho_1)-F_2(u_0,c_0,\rho_0)$. This highest order term involves third derivatives of $\rho-\rho_0$ which indeed has more time regularity as actually needed. More precisely, it holds

$$\|\gamma\Delta\nabla[\varepsilon_0-\varepsilon]\cdot\nabla c\|_{_0\mathcal{X}_2(J_0)}\leq\|\gamma\|_{\mathrm{C}^{\frac14}(J_0;\mathrm{C}(\overline{\Omega}))\cap\mathrm{B}(J_0;\mathrm{C}^1(\overline{\Omega}))}\||\nabla c\||_{\mathrm{C}^{\frac14}(J_0;\mathrm{C}(\overline{\Omega}))\cap\mathrm{C}(J_0;\mathrm{C}^1(\overline{\Omega}))}\cdot$$
$$T_0^{1/p}\|\varepsilon_0-\varepsilon\|_{_0\mathrm{B}(J_0;\mathrm{H}^3_p(\Omega))}\leq C(\overline{R})T_0^{\frac1p}\cdot\|\varepsilon_0-\varepsilon\|_{_0\mathrm{B}(J_0;\mathrm{H}^3_p(\Omega))}\leq C(\overline{R})T_0^{\frac1p}.$$

Finally, putting together all estimates above and choosing $T_0\in(0,T)$ sufficiently small, we obtain the inequality

$$\|(u',c')-(\overline{u},\overline{c})\|_{_0\mathcal{Z}_1(J_0)\times{_0\mathcal{Z}_2(J_0)}}\leq C(\overline{R})r(T_0)$$

with $r(T_0)$ tending to 0 as $T_0\to0$. Choosing T_0 sufficiently small, such that $C(\overline{R})r(T_0)\leq R_0$, we have accomplished selfmapping. Therefore $\mathcal{G}:\Sigma'\mapsto\Sigma'$ is a strict contraction w.r.t. the topology of $Z(J_0)$, hence by Lemma 2.1 and the contraction mapping admits a unique fixed point in $(u,c)\in\mathcal{Z}_1(J_0)\times\mathcal{Z}_2(J_0)$ and thus $\rho=L[u,\rho]\in\mathcal{Z}_3(J_0)$ is unique as well. Repeating the above arguments we obtain solutions in the maximal regularity class on intervals $[t_i,t_{i+1}]$. Either after finitely many steps we reach T, or we have an infinite strictly increasing sequence which converges to some $T^*(u_0,c_0,\rho_0)<T$. In case $\lim_{i\to\infty}(u,c,\rho)(t_i)=:(u(T^*),c(T^*),\rho(T^*))$ exists in $\mathcal{V}_p$ and $\rho(T^*)>0$, we may continue the process, which shows that the maximal time is characterized by condition (2.12). □

5. The nonlinear problem with general boundary conditions and unbounded domains

We now consider (2.1)-(2.5) with general inhomogeneous boundary conditions, that is, we replace (2.6)-(2.8) by

$$u=\Theta_d(t,x),\quad(t,x)\in J\times\Gamma_d,\quad(u|\nu)_{|\Gamma_s}=\Theta_1(t,x),\quad(t,x)\in J\times\Gamma_s,$$
$$\mathcal{QS}\cdot\nu_{|\Gamma_s}=\Theta_2(t,x),\quad(t,x)\in J\times\Gamma_s,\quad\partial_\nu c=\theta_1(t,x),\quad\partial_\nu\mu=\theta_2(t,x),\quad(t,x)\in J\times\Gamma,$$

where Θ_d and $\Theta_s:=(\Theta_1,\Theta_2)$ are subject to the condition

$$(\Theta_d(t,x)|\nu(x))_{|\Gamma_d}\geq0,\quad\forall(t,x)\in J\times\Gamma_d,\quad\Theta_1(t,x)\geq0,\quad\forall(t,x)\in J\times\Gamma_s,$$

in order that the equation (2.3) can be uniquely solved by Lemma 3.1. The only modification in the proof of Theorem 2.1 is a natural alteration of boundary conditions in the fixed point equation, see section 2.2, which then takes effect in the defintion of $(\overline{u},\overline{c})$ as well. Comparing the setting in section (2.2) and the boundary conditions above, we now have to define $\sigma_d(u,\rho):=\Theta_d$ and $\sigma_s(u,\rho):=(\Theta_1,\Theta_2+\mathcal{Q}[\tilde{\mathcal{S}}-\mathcal{S}]\cdot\nu)$, and this changes $\overline{u}$ and $\overline{c}$ according to

$$\overline{u}:=\mathcal{L}_1^{-1}(\mathcal{F}_1(\tilde{u},\tilde{c},\tilde{\rho}),u_0)\equiv\mathcal{L}_1^{-1}(F_1(\tilde{u},\tilde{c},\tilde{\rho}),\Theta_d,\Theta_s,u_0),$$
$$\overline{c}:=\mathcal{L}_2^{-1}(\mathcal{F}_2(u_0,c_0,\rho_0),c_0)\equiv\mathcal{L}_2^{-1}(F_2(u_0,c_0,\rho_0),\theta_1,\theta_2+\partial_\nu g_0(\rho_0,c_0),c_0).$$

These modifications permit the same approach as before, such that the following result is available.

Theorem 5.1 *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with compact C^4-boundary Γ decomposing disjointly as $\Gamma = \Gamma_d \cup \Gamma_s$, $J = [0,T]$ with $T \in (0,\infty]$ and $p \in (\hat{p},\infty)$. Further, let $\overline{\psi} \in \mathrm{C}^{5-}(\mathbb{R}^2)$ and assume* (2.9), (2.10)*. Then for each $f_{ext} \in \mathcal{X}_1(J;\mathbb{R}^n)$ and initial data (u_0, c_0, ρ_0) in*

$$\mathcal{V} := \mathrm{W}_p^{4-\frac{2}{p}}(\Omega;\mathbb{R}^n) \times \mathrm{W}_p^{4-\frac{4}{p}}(\Omega) \times \{\varphi \in \mathrm{H}^3_p(\Omega;\mathbb{R}_+) : \varphi(x) > 0, \quad \forall x \in \overline{\Omega}\}$$

and boundary data

$$\Theta_d \in \mathcal{Y}_{0,d}(J;\mathbb{R}^n), \quad (\Theta_d|\nu) \geq 0, \quad \Theta_s = (\Theta_1,\Theta_2) \in \mathcal{Y}_{0,s}(J) \times \mathcal{Y}_{1,s}(J;\mathbb{R}^n), \quad \Theta_1 \geq 0,$$

satisfying the compatibility conditions

$$\begin{aligned}
&u_{0|\Gamma_d} = \Theta_{d|t=0} \in \mathrm{W}_p^{4-3/p}(\Gamma_d;\mathbb{R}^n), \quad (u_0|\nu)_{|\Gamma_s} = \Theta_{1|t=0} \in \mathrm{W}_p^{4-3/p}(\Gamma_s),\\
&\mathcal{QS}_{|t=0} \cdot \nu_{|\Gamma_s} = \Theta_{2|t=0} \in \mathrm{W}_p^{3-2/p}(\Gamma_s;\mathbb{R}^n),\\
&\partial_\nu c_{0|\Gamma} = \theta_{1|t=0} \in \mathrm{W}_p^{3-5/p}(\Gamma), \quad \partial_\nu \mu(\rho_0, c_0)_{|\Gamma} = \theta_{2|t=0} \in \mathrm{W}_p^{1-5/p}(\Gamma),\\
&\rho_{0|\Gamma_d}\partial_t\Theta_{d|t=0} - \nabla\cdot\mathcal{S}(u)_{|t=0,\Gamma_d} = (\nabla\cdot\mathcal{P}_{|t=0} - \rho_0\nabla u_0 \cdot u_0 + \rho_0 f_{ext|t=0})_{|\Gamma_d} \in \mathrm{W}_p^{2-\frac{3}{p}}(\Gamma_d;\mathbb{R}^n),\\
&\rho_{0|\Gamma_d}\partial_t\Theta_{1|t=0} - (\nabla\cdot\mathcal{S}(u)_{|t=0}|\nu)_{|\Gamma_s} = (\nabla\cdot\mathcal{P}_{|t=0} - \rho_0\nabla u_0 \cdot u_0 + \rho_0 f_{ext|t=0}|\nu)_{|\Gamma_s} \in \mathrm{W}_p^{2-\frac{3}{p}}(\Gamma_s),\\
&\partial_t\Theta_{2|t=0} - \mathcal{QS}(\rho^{-1}\nabla\cdot\mathcal{S}(u))_{|t=0,\Gamma_s} \cdot \nu_{|\Gamma_s} = [\tfrac{\partial_t \eta}{\eta}\Theta_2]_{|t=0}\\
&\qquad\qquad + \mathcal{QS}(\rho_0^{-1}\nabla\cdot\mathcal{P}_{|t=0} - \nabla u_0 \cdot u_0 + f_{ext|t=0})_{\Gamma_s} \cdot \nu_{|\Gamma_s} \in \mathrm{W}_p^{1-\frac{3}{p}}(\Gamma_s;\mathbb{R}^n),
\end{aligned}$$

there is a unique solution (u,c,ρ) of (2.1)-(2.8) *on a maximal time interval $[0,T^*)$, $T^* \leq T$. The solution (u,c,ρ) belongs to the class $\mathcal{Z}(J_0)$ for each interval $J_0 = [0,T_0]$ with $T_0 < T^*$. The maximal time interval is characterised by the property:*

$$\lim_{t\to T^*} w(t) \quad \textit{does not exist in } \mathcal{V}, \textit{ or } \quad \lim_{t\to T^*} \rho(t,x) \not> 0 \quad \forall x \in \overline{\Omega}.$$

The solution map $(u_0,c_0,\rho_0) \to (u,c,\rho)(t)$ generates a local semiflow on the phase space $\mathcal{V}_p := \{v \in \mathcal{V} :\ v$ satisfies (2.11) *$\}$ in the autonomous case.*

Another generalization concerns unbounded domains, which are treatable if the boundary is again smooth and compact, which of course includes $\mathbb{R}^n$. However, in these cases we further need assumptions to the coefficients which guarantee that the limit for $|x| \to \infty$ exists for all $t \in J$. We have to impose

$$\begin{aligned}
&\lim_{|x|\to\infty} a(t,x,v(t,x)) = a_\infty(t), \quad \forall t \in J, \quad v \in \mathrm{C}(J;\mathrm{C}^1(\overline{\Omega};E)), \quad E \in \{\mathbb{R},\mathbb{R}^2\},\\
&a \in \{\eta,\lambda,\gamma,\varepsilon\}, \quad a_\infty \in \{\eta_\infty,\lambda_\infty\} \subset \mathrm{C}^\beta(J), \quad \beta > 1/2, \quad a_\infty \in \{\gamma_\infty,\varepsilon_\infty\} \subset \mathrm{C}(J),
\end{aligned} \tag{5.1}$$

These assumptions make possible to solve the linear problems via maximal L_p-regularity, in particular, the Theorems 3.1 and 3.2 apply to unbounded domains as well.

Another discrepancy to the case of bounded domains concerns the assumptions of the initial value ρ_0. In fact, on the one hand we need $\rho_0(x) \geq \underline{\rho} > 0$ to avoid vacuum and on the other hand ρ_0 has to be in $\mathrm{H}^3_p(\Omega)$, which is impossible in case of unbounded domains. Therefore the initial condition for ρ has to be replaced by

$$\rho_{|t=0} = \rho_0 - \overline{\rho} \in \mathrm{H}^3_p(\Omega), \quad \overline{\rho} \in \mathbb{R}_+\backslash\{0\} \text{ and } \exists\, \underline{\rho} > 0, \text{ such that } \rho_0(x) \geq \overline{\rho} \quad \forall x \in \overline{\Omega}. \tag{5.2}$$

Then introducing the density fluctuation $\varrho := (\rho - \overline{\rho})/\overline{\rho}$, that is, using the identity $\rho = 1 + \varrho$ in (2.1)-(2.8), the system for (u, c, ϱ) can be treated in the same manner as before, leading to the following result.

Theorem 5.2 *Let Ω be a unbounded domain in $\mathbb{R}^n$, $n \geq 1$, with compact C^4-boundary Γ decomposing disjointly as $\Gamma = \Gamma_d \cup \Gamma_s$, $J = [0, T]$ with $T \in (0, \infty]$ and $p \in (\hat{p}, \infty)$. Further, let $\overline{\psi} \in \mathrm{C}^{5-}(\mathbb{R}^2)$ and assume (2.9), (2.10) and (5.1). Then, replacing the initial value ρ_0 by (5.2), the same assertions of Theorem 5.1 hold true for (u, c, ϱ) in unbounded domains Ω.*

6. Appendix

The following proposition can be found in [18]

Proposition 6.1 *Let $1 < p < \infty$, $1/p < \beta < 1$, suppose A is an invertible pseudo-sectorial operator in X with $\phi_A < \pi/2$, and set $u(t) = e^{-At}x$, $x \in X$. Then the following statements are equivalent:*

$$(i) \quad x \in D_A(\beta - 1/p, p); \quad (ii) \quad u \in \mathrm{L}_p(\mathbb{R}_+; D_A(\beta, p)); \quad (iii) \quad u \in \mathrm{W}^{\beta}_p(\mathbb{R}_+; X).$$

Lemma 6.1 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, $p > \max\{1, n/2\}$, γ, $\delta > 0$, and $\gamma + \delta \geq 2$. Then it holds*

$$\|fg\|_{\mathrm{L}_p(\Omega)} \leq c_E \|f\|_{\mathrm{W}^{\gamma}_p(\Omega)} \|g\|_{\mathrm{W}^{\delta}_p(\Omega)}, \quad \forall\, (f, g) \in \mathrm{W}^{\gamma}_p(\Omega) \times \mathrm{W}^{\delta}_p(\Omega). \tag{6.1}$$

Proof. W.l.o.g. we assume that $\gamma \geq 1 \geq \delta$. Hölder's inequality entails

$$\|fg\|_{\mathrm{L}_p(\Omega)} \leq \|f\|_{\mathrm{L}_{p\sigma}(\Omega)} \|g\|_{\mathrm{L}_{p\sigma'}(\Omega)}, \quad 1/\sigma + 1/\sigma' = 1,$$

so long as

$$\mathrm{W}^{\gamma}_p(\Omega) \hookrightarrow \mathrm{L}_{p\sigma}(\Omega), \quad \sigma := \begin{cases} \frac{n}{n-\gamma p} & p < \frac{n}{\gamma} \\ < \infty & p = \frac{n}{\gamma} \\ \infty & p > \frac{n}{\gamma} \end{cases} \quad \text{and } \mathrm{W}^{\delta}_p(\Omega) \hookrightarrow \mathrm{L}_q(\Omega), \quad q := \begin{cases} \frac{np}{n-\delta p} & p < \frac{n}{\delta} \\ < \infty & p = \frac{n}{\delta} \\ \infty & p > \frac{n}{\delta} \end{cases},$$

and $q \geq p\sigma'$. For $p < n/\gamma \leq n/\delta$ the later condition is equivalent to $p \geq n/(\gamma + \delta) \geq n/2$. Otherwise, $p \geq n/\gamma$, we have $\|fg\|_{\mathrm{L}_p(\Omega)} \leq \|f\|_{\mathrm{L}_{pq/(q-p)}(\Omega)} \|g\|_{\mathrm{L}_q(\Omega)}$ for $p < q$. □

The next theorem states maximal L_p-regularity of (2.14), a direct consequence of [5].

Theorem 6.1 *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with compact C^2-boundary Γ decomposing disjointly $\Gamma = \Gamma_d \cup \Gamma_s$, $J = [0, T]$, and $p \in (1, \infty)$ with $p \neq 3/2$, 3. Further, assume that $\tilde{\rho} \in \mathrm{C}(J; \mathrm{C}(\overline{\Omega}))$, $\tilde{\eta}$, $\tilde{\lambda} \in \mathrm{C}(J; \mathrm{C}^1(\overline{\Omega}))$ and $\tilde{\rho}(t, x)$, $\tilde{\eta}(t, x)$, $2\tilde{\eta}(t, x) + \tilde{\lambda}(t, x) > 0$ for all $(t, x) \in J \times \overline{\Omega}$. Then problem (2.14) possesses a unique solution*

$$u \in \mathrm{H}^1_p(J; \mathrm{L}_p(\Omega; \mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{H}^2_p(\Omega; \mathbb{R}^n)),$$

if and only if the data f, σ_d, σ_s, u_0 satisfy the following conditions

1. $f \in \mathrm{L}_p(J; \mathrm{L}_p(\Omega; \mathbb{R}^n))$
2. $(\sigma_d, \sigma_s) \in Y_{0,d}(J; \mathbb{R}^n) \times Y_{0,s}(J) \times Y_{1,s}(J; \mathbb{R}^n)$ *with* $\sigma_s := (\sigma_1, \sigma_2)$ *and* $Y_{i,k}(J; E) := \mathrm{W}_p^{(2-i-\frac{1}{p})\frac{1}{2}}(J; \mathrm{L}_p(\Gamma_k; E)) \cap \mathrm{L}_q(J; \mathrm{W}_p^{2-i-\frac{1}{p}}(\Gamma_k; E))$, $i = 0, 1$, $k = d, s$;
3. $u_0 \in \mathrm{W}_p^{2-\frac{2}{p}}(\Omega; \mathbb{R}^n)$;
4. $u_{0|\Gamma_d} = \sigma_{d|t=0}$ *in* $\mathrm{W}_p^{2-\frac{3}{p}}(\Gamma_d; \mathbb{R}^n)$ *if* $p > 3/2$;
5. $(u_0|\nu)_{|\Gamma_s} = \sigma_{1|t=0}$ *in* $\mathrm{W}_p^{2-\frac{3}{p}}(\Gamma_s)$, $\mathcal{Q}\tilde{\mathcal{S}}(u_0) \cdot \nu_{|\Gamma_s} = \sigma_{2|t=0}$ *in* $\mathrm{W}_p^{1-\frac{3}{p}}(\Gamma_s; \mathbb{R}^n)$ *if* $p > 3$.

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