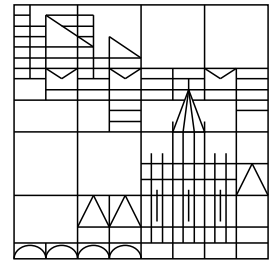


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Homology and Cohomology of Toric Varieties

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Homology and Cohomology of Toric Varieties

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Referent: Prof. Dr. Ludger Kaup

Referent: Prof. Dr. Gottfried Barthel

To the memory of Bettina

Preface

... the real mathematics, which must be justified as art if it can be justified at all.

GODFREY HAROLD HARDY (1877–1947) [Ha, p. 139]

The theory of toric varieties plays an important role as a bridge between algebraic geometry, combinatorial convex geometry, and commutative algebra. The interpretation and application of many notions and methods of these three theories in the case of toric varieties leads to fruitful examples and results. The relations between various algebraic, topological, and geometric properties of toric varieties fill extensive dictionaries.

From the point of view of combinatorial convexity, all the structure of a toric variety can be encoded in a fan, i.e., a finite collection of strictly convex cones spanned by finitely many vectors with integral coordinates and satisfying very natural incidence relations. Now the obvious question is how to extract algebro-geometric or topological invariants of the associated toric variety from these data.

A most striking example is given by the famous Theorem of Jurkiewicz-Danilov [Da, Theorem 10.8 and Remark 10.9]: the integral cohomology ring of a smooth compact toric variety can be explicitly computed in terms of the associated fan. Allowing mild singularities, the analogous result holds with rational coefficients. For toric varieties with arbitrary singularities, but still in the compact case, there is a spectral sequence relating data of the fan to integral cohomology that has been investigated by Stephan Fischli in his dissertation [Fi]. It admits the explicit computation of some integral cohomology groups in low and in high degrees; in particular, it yields complete results up to dimension three.

In the work presented here, we also use spectral sequences to determine (co-)homological data of a toric variety in terms of the associated fan, but in a much more general setting: we investigate the homology with closed supports and the cohomology with compact supports and with arbitrary (constant) coefficients for not necessarily compact toric varieties with arbitrary singularities. We do not consider toric varieties over arbitrary fields but restrict ourselves to the complex case:

The toric variety X_Δ associated to a fan Δ in a vector space \mathbb{R}^n consists of $(\mathbb{C}^*)^n$ -orbits O_σ corresponding to the cones $\sigma \in \Delta$; in particular, the full-

dimensional cones $\sigma \in \Delta^n$ correspond to fixed points x_σ . It turns out that the natural filtration of the toric variety X_Δ induced by its orbit structure provides convergent (co-)homology spectral sequences. An explicit calculation of the associated E^2 - or E_2 -terms yields formulæ — in low and in high degrees ℓ — for the homology groups $H_\ell^{\text{clid}}(X_\Delta; G)$, the cohomology groups $H_c^\ell(X_\Delta; G)$, and the integral local homology groups $\mathcal{H}_{\ell, x_\sigma}$ in the fixed points x_σ . All Betti numbers $b_\ell^{\text{clid}}(X_\Delta)$ are computable. For integral coefficients, the $E_{p,p}^2$ -terms are isomorphic to the Chow groups $A_p(X_\Delta)$, i.e., we obtain a generalization of the Theorem of Jurkiewicz-Danilov to the singular, noncompact case. Up to dimension four our method admits the computation of almost the whole homology $H_\bullet^{\text{clid}}(X_\Delta; \mathbb{Z})$ and almost the whole cohomology $H_c^\bullet(X_\Delta; \mathbb{Z})$; only the computation of the torsion part $\text{Tor } H_4^{\text{clid}}(X_\Delta; \mathbb{Z}) = \text{Tor } H_c^5(X_\Delta; \mathbb{Z})$ of a 4-dimensional toric variety X_Δ may fail.

It is impossible to mention all those whose direct or indirect influence contributed to the completion of this dissertation. I am grateful to Heinz Ebner, who encouraged me to take part in mathematical competitions and to study mathematics. Most especially, my thanks go to Professor Dr. Ludger Kaup, Professor Dr. Gottfried Barthel, and Professor Dr. Karl-Heinz Fieseler. I learned very much from them about complex analysis and algebraic geometry; they have always been very helpful advisers. Professor Dr. Günther Ewald and Professor Dr. Bernard Teissier gave me the chance to contribute two chapters to the notes about their “DMV-Seminar on Convex Geometry and Toric Varieties” held in Blaubeuren in spring 1993, I am thankful for that. Professor Dr. Volker Puppe and Professor Dr. Volker Strassen have earned my gratitude for offering me an assistant position during the time when this thesis was not supported by a grant of the State of Baden-Württemberg.

Last but not least, I would like to thank my parents and my sister for their tremendous support.

Konstanz, Autumn 1997

Arno Christian Jordan

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Zusammenfassung

Torische Varietäten bieten die Möglichkeit, komplizierte Konzepte der Algebraischen Geometrie oder Algebraischen Topologie an elementaren, aber keinesfalls trivialen Beispielen zu studieren. Gleichzeitig wird eine Brücke zur kombinatorischen Konvexgeometrie geschlagen. Die Theorie torischer Varietäten ist somit gekennzeichnet durch ein reizvolles Wechselspiel zwischen Algebra, Geometrie und Topologie, das sich in hohem Maße konkret ausdrücken und berechnen läßt.

Grundlage hierfür ist die Charakterisierung einer jeden torischen Varietät durch ein endliches System ganzzahliger Daten: Auf der konvexgeometrischen Seite ist dies der zugehörige Fächer, der aus endlich vielen spitzen konvexen Kegeln besteht, die von endlich vielen Gittervektoren mit ganzzahligen Koordinaten aufgespannt werden, und die natürlichen Inzidenzrelationen genügen. Den einzelnen Kegeln entsprechen nun affine algebraische Varietäten, die sich durch endlich viele Gleichungen vom Typ $\text{Monom} = \text{Monom}'$ beschreiben lassen. Die Verklebung dieser affinen torischen Varietäten zu der dem Fächer zugeordneten torischen Varietät geschieht wiederum auf monomiale Art und wird durch die Inzidenzrelationen zwischen den Kegeln des Fächers induziert.

Arbeitet man — wie im folgenden immer — über dem Koeffizientenkörper \mathbb{C} , so erhält man zu den Fächern im Vektorraum \mathbb{R}^n gerade diejenigen normalen komplex-algebraischen Varietäten, welche eine (effektive und) fast transitive Operation des komplexen algebraischen n -Torus zulassen, die auf einer dichten Zariski-offenen Teilmenge mit der Operation des n -Torus auf sich selbst vermöge komponentenweiser Multiplikation übereinstimmt.

Selbst bei dieser recht speziellen Klasse algebraischer Varietäten — z.B. sind torische Varietäten und ihre Singularitäten rational — ist man von einer zufriedenstellenden Klassifikation durch topologische oder algebro-geometrische Daten noch weit entfernt. Um jene zu erreichen, erscheint daher das Studium entsprechender Invarianten torischer Varietäten nützlich, zumal damit insbesondere in höheren Dimensionen die Aussicht auf eine Fülle algebraischer Varietäten mit berechenbaren Eigenschaften besteht.

Der berühmte Hauptsatz von Jurkiewicz-Danilov stellt ein hervorragendes Beispiel für das elegante Zusammenwirken von Algebraischer Topologie, Algebraischer Geometrie und Kommutativer Algebra innerhalb der Theorie torischer Varietäten dar, vgl. [Da, Theorem 10.8 und Remark 10.9]: Zwischen dem ganzzahligen Kohomologiering einer glatten, kompakten torischen Varietät und ihrem

Chowring besteht ein natürlicher, graduierter Isomorphismus. Beide Ringe lassen sich explizit aus Fächerdaten berechnen: Sie sind durch den Stanley-Reisner-Ring des Fächers modulo einem Ideal gegeben, das die torusinvarianten Hauptdivisoren auf der torischen Varietät beschreibt. Läßt man milde Singularitäten zu, so gelten analoge Aussagen mit rationalen Koeffizienten. Für eine allgemeine Beschreibung der Chowgruppen sphärischer Varietäten siehe [FMSS].

Stephan Fischli hat in seiner Dissertation [Fi] eine Spektralsequenz zur Berechnung der ganzzahligen Kohomologie kompakter torischer Varietäten — nun aber mit beliebigen Singularitäten — aus Fächerdaten untersucht. Seine Methode liefert explizite Beschreibungen der Kohomologiegruppen mit niedrigem bzw. hohem Grad, wobei für höchstens dreidimensionale, kompakte torische Varietäten vollständige Resultate erzielt werden.

Anknüpfend an diese Überlegungen werden in der vorliegenden Arbeit hauptsächlich die folgenden Problemkreise behandelt:

- Läßt sich Fischlis Vorgehen verallgemeinern, um auch einen Zugang zur Kohomologie nichtkompakter torischer Varietäten (mit beliebigen Singularitäten) zu gewinnen? Inwieweit können dabei durchgängig beliebige (konstante) Koeffizientengruppen betrachtet werden, d.h. die nachträgliche Verwendung universeller Koeffizientensätze vermieden werden?
- Wie sieht die duale, homologische Theorie aus?
- Besteht im singulären, nichtkompakten Fall noch ein gewisser Zusammenhang zwischen den Chowgruppen und den ganzzahligen (Ko-)homologiegruppen?

Der Hauptsatz von Jurkiewicz-Danilov legt es nahe, auch bei nichtkompakten torischen Varietäten mit milden Singularitäten torusinvariante algebraische Zyklen als natürliche algebro-geometrische Repräsentanten der Homologieklassen zu erwarten. Da diese Zyklen zwar abgeschlossen aber im allgemeinen nicht kompakt sind, ist es sinnvoll, den Untersuchungsrahmen wie folgt weit abzustecken:

Mit Hilfe von Spektralsequenzen, die von der Bahnenstruktur induziert werden, wird versucht, die Homologie mit abgeschlossenen Trägern und, dual dazu, die Kohomologie mit kompakten Trägern — jeweils mit beliebigen (konstanten) Koeffizienten — nicht notwendigerweise kompakter torischer Varietäten zu berechnen. Dabei sind beliebige Singularitäten zugelassen. Als Hauptergebnis erhalten wir (in Kapitel 3 ausführlich an Beispielen vorgeführt) Methoden zur expliziten Bestimmung mehrerer *Invarianten torischer Varietäten*, insbesondere:

- Alle Betti-Zahlen $b_\ell^{\text{cld}}(X_\Delta)$ einer torischen Varietät X_Δ .
- Ein weiterer Beweis dafür, daß die topologische Euler-Charakteristik $e(X_\Delta)$ einer torischen Varietät X_Δ mit der Anzahl der volldimensionalen Kegel im Fächer Δ übereinstimmt.

- Explizite Darstellungen der Homologiegruppen $H_\ell^{\text{cld}}(X_\Delta; G)$ sowie der Kohomologiegruppen $H_\ell^c(X_\Delta; G)$ mit niedrigem und hohem Grad ℓ in Abhängigkeit von Fächerdaten, wobei G eine abelsche Gruppe ist. Insbesondere lassen sich hiermit bis zur Dimension vier fast die gesamte Homologie $H_\bullet^{\text{cld}}(X_\Delta; \mathbb{Z})$ und fast die gesamte Kohomologie $H_c^\bullet(X_\Delta; \mathbb{Z})$ leicht ausrechnen; nur für die Berechnung des Torsionsanteils $\text{Tor } H_4^{\text{cld}}(X_\Delta; \mathbb{Z}) = \text{Tor } H_c^5(X_\Delta; \mathbb{Z})$ einer vierdimensionalen torischen Varietät X_Δ wird eine zusätzliche Voraussetzung benötigt.
- Identifikation aller Chowgruppen $A_p(X_\Delta; \mathbb{Z})$ einer torischen Varietät X_Δ mit den $E_{p,p}^2$ -Termen in einer Spektralsequenz zur Berechnung der Homologie $H_\bullet^{\text{cld}}(X_\Delta; \mathbb{Z})$, d.h. eine Verallgemeinerung des Hauptsatzes von Jurkiewicz-Danilov auf den singulären, nichtkompakten Fall; damit auch Kriterien für die Existenz von Monomorphismen $A_p(X_\Delta; \mathbb{Z}) \hookrightarrow H_{2p}^{\text{cld}}(X_\Delta; \mathbb{Z})$ oder gar Isomorphismen.
- Für niedrigen und hohen Grad ℓ , explizite Darstellungen der ganzzahligen lokalen Homologiegruppen $\mathcal{H}_{\ell, x_\sigma}$ in jedem (zu einem volldimensionalen Kegel σ gehörenden) Fixpunkt x_σ einer torischen Varietät X_Δ ; ferner, für $\ell \in \{0, 1, n-1, n\}$, Isomorphismen $\mathcal{H}_{2\ell, x_\sigma} \cong A_\ell(X_\sigma; \mathbb{Z})$ mit Chowgruppen der affinen torischen Varietät X_σ , die den Fixpunkt x_σ umgibt.

Die wesentlichen Schritte zur Erzielung dieser Ergebnisse sollen im folgenden skizziert werden: Im Kapitel 1 beschäftigen wir uns mit der *Topologie torischer Varietäten*. Nach Gordans Lemma sind die jedem Kegel σ eines Fächers Δ im Vektorraum \mathbb{R}^n zugeordneten Monoide $S_\sigma = \check{\sigma} \cap \mathbb{Z}^n$ mit $\check{\sigma} = \{y \in \mathbb{R}^n; \langle y, \sigma \rangle \geq 0\}$ endlich erzeugt. Durch Zusammenkleben der affinen torischen Varietäten $X_\sigma = \text{Spec } \mathbb{C}[S_\sigma]$ entsteht sodann die torische Varietät X_Δ . Diese Konstruktion ist die Grundlage für alle Wechselbeziehungen zwischen torischen Varietäten einerseits als Objekten der Algebraischen Geometrie oder Topologie und andererseits — in Form der zugehörigen Fächer — als Objekten der Konvexgeometrie.

Zum Beispiel zerfällt eine torische Varietät X_Δ in den Kegeln $\sigma \in \Delta$ zugeordnete Bahnen $O_\sigma \cong (\mathbb{C}^*)^k$, welche komplexe algebraische Tori der Dimension $k = n - \dim \sigma$ sind. Jeder solche komplexe algebraische Torus $(\mathbb{C}^*)^k$ ist — als topologischer Raum, ja sogar als reelle algebraische Varietät betrachtet — ein Produkt $\mathbb{R}_{>0}^k \times T_{\mathbb{R}}^k$, wobei $T_{\mathbb{R}}^k$ den kompakten reellen k -Torus $(S^1)^k$ bezeichnet. Indem MacPherson den Raum $\mathbb{R}_{>0}^k$ durch eine raffiniert konstruierte k -Zelle $\tilde{\sigma}$ ersetzte, ist es ihm gelungen, die Topologie torischer Varietäten mittels einer Art Polarkoordinaten zu beschreiben, wobei der Radialfaktor geeignet skaliert wird: In Verallgemeinerung der Momentabbildung [Fu, Section 4.2], die nur bei projektiven torischen Varietäten verwendet werden kann, ergibt sich eine stetige Projektionsabbildung $\pi: X_\Delta \twoheadrightarrow B_\Delta$ auf einen endlichen Zellenkomplex $B_\Delta = \bigcup_{\sigma \in \Delta} \tilde{\sigma}$, deren Fasern $\pi^{-1}(x)$ für $x \in \tilde{\sigma}$ kompakte reelle Tori $T(\sigma) \cong T_{\mathbb{R}}^k$

sind. Der Zellenkomplex B_Δ liegt innerhalb der abgeschlossenen reellen Einheitskugel B^n ; seine Zerlegung in die Zellen $\tilde{\sigma}$ entspricht der Zerlegung der torischen Varietät X_Δ in die Bahnen $O_\sigma = \pi^{-1}(\tilde{\sigma}) = \tilde{\sigma} \times T(\sigma)$.

Da wir an der *transzendenten* und nicht an der Zariski-Topologie interessiert sind, ist somit MacPhersons Beschreibung torischer Varietäten unseren Zwecken besonders gut angepaßt. Die Struktur der natürlichen Torusoperation auf einer torischen Varietät X_Δ wird daher nicht nur ausführlich mittels Fächerdaten, sondern auch mit Hilfe der Projektion $\pi: X_\Delta \rightarrow B_\Delta$ ausgedrückt.

In Kapitel 2 werden die Hauptwerkzeuge für die Berechnung der *Homologie und Kohomologie torischer Varietäten* eingeführt. Zuerst wird ein allgemeiner Rahmen für die Beschreibung der E^1 - und E^2 -Terme der später zu betrachtenden Spektralsequenzen bereitgestellt:

Jeder Fächer läßt sich als (endliche) Kategorie mit seinen Kegeln als Objekten und seinen Seitenrelationen als Morphismen auffassen. Es werden Orientierungskoeffizienten $o_{\sigma,\tau}$ für Kegel σ, τ eines Fächers Δ festgelegt, die es erlauben, jedem kovarianten Funktor F von der Kategorie Δ in die Kategorie der abelschen Gruppen einen Kettenkomplex $C_\bullet(\Delta, F)$ mit zugehöriger Homologie $H_\bullet(\Delta, F)$ zuzuordnen. In Examples 2.1.1 werden zwei solche — zueinander natürlich äquivalente — Funktoren H_\bullet^Δ und Λ_\bullet^Δ vorgestellt, mit denen die Homologie der Fasern der kanonischen Projektion $\pi: X_\Delta \rightarrow B_\Delta$ — also die Homologie kompakter reeller Tori $T(\sigma)$ — samt induzierten Homomorphismen explizit beschrieben und ausgerechnet werden kann.

Die Bahnstruktur einer torischen Varietät X legt eine natürliche Filtrierung $X_0 \subset X_1 \subset \dots \subset X_n$ fest, wobei X_k die Vereinigung aller Torusbahnen der Dimension höchstens k ist. Die von dieser Filtrierung induzierte konvergente Homologiespektralsequenz wird im folgenden torische Homologiespektralsequenz genannt. Für Homologie mit abgeschlossenen Trägern und Koeffizienten in einer abelschen Gruppe G sind die zugehörigen Moduln $E_{p,q}^1$ nach Lemma 2.3.1 durch die direkten Summen der $(p+q)$ -ten Homologiegruppen der p -dimensionalen Bahnen O_σ gegeben:

$$E_{p,q}^1 = H_{p+q}^{\text{cld}}(X_p, X_{p-1}; G) \cong \bigoplus_{\sigma \in \Delta^{(n-p)}} H_{p+q}^{\text{cld}}(O_\sigma; G).$$

Die Homologiegruppe $H_{p+q}^{\text{cld}}(O_\sigma; G) = H_{p+q}^{\text{cld}}(\tilde{\sigma} \times T(\sigma); G)$ läßt sich mit Hilfe von u.a. Poincaré-Dualität und Künnethformel weiter zerlegen:

$$H_{p+q}^{\text{cld}}(\tilde{\sigma} \times T(\sigma); G) \cong H_p^{\text{cld}}(\tilde{\sigma}; \mathbb{Z}) \otimes H_q(T(\sigma); G) \cong H_q(T(\sigma); G),$$

so daß letztendlich gemäß dem kommutativen Diagramm in Figure 2.2 und Theorem 2.4.1 das Differential $d_{p,q}^1: E_{p,q}^1 \rightarrow E_{p-1,q}^1$ mit den Randabbildungen δ_p der Kettenkomplexe $C_\bullet(\Delta, H_q^\Delta)$ und $C_\bullet(\Delta, \Lambda_q^\Delta)$ korrespondiert. Dadurch sind die Moduln

$$E_{p,q}^1 \cong C_p(\Delta, H_q^\Delta) \cong C_p(\Delta, \Lambda_q^\Delta) \cong G^b$$

mit $b = \binom{p}{q} \# \Delta^{n-p}$ und

$$E_{p,q}^2 \cong H_p(\Delta, H_q^\Delta) \cong H_p(\Delta, \Lambda_q^\Delta)$$

explizit berechenbar. Für Indexpaare (p, q) außerhalb des durch die Ungleichungen $0 \leq q \leq p \leq n$ festgelegten Dreiecks verschwinden sonach alle Moduln $E_{p,q}^r$ mit $r \geq 1$. Für Paare auf dem Dreiecksrand gibt es ebenfalls konkrete Darstellungen der Moduln $E_{p,q}^2$:

- Für alle Fächer $\Delta \neq \{0\}$ ist jeder Modul $E_{p,0}^2$ isomorph zur reduzierten Kohomologiegruppe $\widetilde{H}^{n-p-1}(S_\Delta; G)$ des endlichen CW-Komplexes $S_\Delta = \bigcup_{\sigma \in \Delta} (\sigma \cap S^{n-1})^\circ$, siehe Corollary 2.4.9.
- Bezeichnet $s = \dim_{\mathbb{R}} \text{lin } \Delta^1$ die Dimension des von den Strahlen des Fächers Δ aufgespannten Vektorraums, so gilt für ganzzahlige Koeffizienten $E_{n,q}^2(\mathbb{Z}) \cong \mathbb{Z}^b$ mit $b = \binom{n-s}{n-q}$, vgl. Proposition 2.4.10.
- Ebenfalls für ganzzahlige Koeffizienten läßt sich schließlich nach Theorem 3.3.1 jeder Modul $E_{p,p}^2(\mathbb{Z})$ als Chowgruppe $A_p(X_\Delta; \mathbb{Z})$ interpretieren. Dazu wird sowohl die explizite Darstellung der Chowgruppen torischer Varietäten nach Fulton und Sturmfels ausgenutzt als auch die Tatsache, daß jeder Bahnabschluß wiederum eine torische Varietät mit explizit angebbarem Fächer ist.

Alle Differentiale $d_{p,q}^r: E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ mit $r \geq 2$ verschwinden (bis auf ggf. Torsion) nach Proposition 2.4.5, d.h. es werden Koeffizienten $m_{p,q}^r \geq 1$ mit $m_{p,q}^r d_{p,q}^r = 0$ angegeben; diese Koeffizienten $m_{p,q}^r$ sind im übrigen meistens gleich eins, vgl. auch Corollary 2.4.7. Größtenteils liegt dies daran, daß, falls nicht $0 \leq q' \leq p' \leq n$, die Beziehung $E_{p',q'}^r = 0$ gilt. Manchmal hilft aber auch erst eine von Totaro ins Spiel gebrachte natürliche Operation des multiplikativen Monoids $\mathbb{N}_{>0}$ auf der torischen Varietät X_Δ weiter, die durch Skalarmultiplikation auf dem Fächer Δ induziert wird, und die die Operation $k \cdot (t_1, \dots, t_n) = (t_1^k, \dots, t_n^k)$ auf dem Torus $(\mathbb{C}^*)^n \subset X_\Delta$ fortsetzt.

In sehr vielen Fällen läßt sich daher zeigen, daß die Spektralfolge ab dem E^2 -Level konstant ist. Insbesondere ist dies bei divisiblen Koeffizientengruppen G immer der Fall, so daß sich zum Beispiel alle Betti-Zahlen $b_\ell^{\text{clid}}(X_\Delta)$ einer torischen Varietät X_Δ als Summe der Dimensionen $\dim E_{p,q}^2(\mathbb{Q})$ mit $p+q = \ell$ der rationalen torischen Homologiespektralsequenz berechnen lassen. Ferner werden in Theorem 2.4.12 die Homologiegruppen $H_\ell^{\text{clid}}(X_\Delta; G)$ mit niedrigem und hohem Grad ℓ aufgeführt, die sich allein aufgrund der Kenntnis von E^2 -Termen der torischen Homologiespektralsequenz angeben lassen.

Dual zur eben geschilderten Vorgehensweise kann sukzessive eine konvergente Spektralsequenz für die *Kohomologie torischer Varietäten* — nun natürlich mit kompakten anstelle von abgeschlossenen Trägern — ausgewertet werden;

entsprechende Überlegungen werden im Abschnitt 2.5 skizziert. Aufgrund universeller Koeffizientensätze sind die homologische bzw. kohomologische Sichtweise zueinander äquivalent. Allerdings findet sich zur Interpretation der $E_{p,p}^2(\mathbb{Z})$ -Terme als Chowgruppen $A_p(X_\Delta; \mathbb{Z})$ kein direktes Gegenstück in der torischen Kohomologiespektralsequenz. Ferner empfiehlt sich der homologische Standpunkt durch einfachere Formeln, da im ganzzahligen (nichtentarteten) Fall der Nachweis des Verschwindens oder zumindest der Freiheit der $E_{n,q}^2(\mathbb{Z})$ -Terme gelingt.

Chapter 0

Introduction

Combinations of different mathematical theories such as algebraic topology and algebraic geometry are charming as well as effective: In fact, it often turns out that the algebraic reformulation of a question that originally appeared in topology or geometry is the first step to its solution.

Since the foundation of the theory of toric varieties at the beginning of the 1970s, many notions, methods, and results of algebraic topology and algebraic geometry have been successfully applied to this fertile class of algebraic varieties. Important algebraic varieties such as the affine variety \mathbb{C}^n , the complex algebraic torus $(\mathbb{C}^*)^n$, and the projective space \mathbb{P}_n are toric; in particular, every quasi-projective algebraic variety “lives” in a toric variety. Especially in higher dimensions, the category of toric varieties is “rich enough” to provide interesting examples of algebraic varieties with prescribed properties — although toric varieties are very special in a general classification scheme. For instance, toric varieties and their singularities are rational.

The main motivation to study toric varieties is their elementary “monomial” structure, which admits detailed statements and descriptions. The coordinate ring of an affine toric variety is a subring of a polynomial ring generated by finitely many monomials; and arbitrary toric varieties arise by the gluing — again determined by finitely many monomials — of finitely many affine toric varieties. Thus the structure of a toric variety is given by a finite set of integral data.

In the language of convex geometry of real affine spaces, these data are *lattice cones* and *rational fans*: Lattice cones are strictly convex cones spanned by finitely many vectors with integral coordinates; they correspond to *affine* toric varieties. Rational fans are finite collections of lattice cones satisfying natural incidence relations such that the corresponding affine toric varieties form an affine open cover of the associated toric variety, i.e., the incidence relations prescribe the gluing.

Applying the fundamental notions, methods, and results of *algebraic* geometry to the category of toric varieties, the challenge is to find their counterparts in *convex* geometry and vice versa. An extensive dictionary between algebraic and

convex geometry in the case of toric varieties is known. For example, compact toric varieties correspond to complete fans; toric varieties associated to regular (simplicial) fans are smooth or possess at most quotient singularities, respectively. Precise definitions of these and other basic notions are collected in Section 1.1; for proofs, we refer the reader to the introductory texts or textbooks [Da, Ew, Fu, Od, Od₂]. During our investigations we will come across another type of geometry: The topological spaces underlying toric varieties may be described with the help of the *spherical* geometry of real closed unit balls.

It is amazing how many phenomena in algebraic geometry and algebraic topology are visible in the elementary structure of toric varieties, where almost everything is much more computable and concrete as usual. But nevertheless we are still far away from a satisfying understanding of toric varieties. To gradually reach this end, it is useful to learn more about some algebro-topological or algebro-geometric invariants of these varieties: the homology with closed supports, the cohomology with compact supports (both with coefficients in an abelian group), and the Chow groups of toric varieties are the main objects of investigation in this thesis.

The reason for our choice of supports lies in the famous Theorem of Jurkiewicz-Danilov [Da, Theorem 10.8 and Remark 10.9]: It says that, for a smooth compact toric variety X_Δ , there are isomorphisms of graded rings between the integral cohomology ring $H^\bullet(X_\Delta; \mathbb{Z}) \cong H_\bullet(X_\Delta; \mathbb{Z})$, the Chow ring $A^\bullet(X_\Delta; \mathbb{Z})$, and the Stanley-Reisner ring of the fan Δ modulo an ideal that describes the torus-invariant principal divisors. For compact toric varieties with at most quotient singularities, i.e., for complete simplicial fans, the analogous statement with rational coefficients is valid as well. (For a general description of the Chow groups of algebraic cycles on spherical varieties see [FMSS].) If we therefore expect torus-invariant cycles as natural algebro-geometric representatives for homology classes also in the noncompact case (and with mild singularities), then we have to note that these cycles are closed but in general not compact, i.e., we should consider homology with *closed* supports (and cohomology with *compact* supports as dual counterpart).

Moreover, Fischli's cohomology spectral sequence [Fi] admits the calculation of the integral cohomology groups in low and in high degrees of compact toric varieties with arbitrary singularities. Using contravariant universal coefficient formulæ, the corresponding homology groups may be expressed by data of the associated fans.

But for a deeper understanding of the homology groups themselves, a genuine homological theory is indispensable even in the compact case. So it was my aim to develop the homological counterpart of Fischli's approach, even for the noncompact toric varieties. In addition, Fischli's cohomological results are generalized to this setting. Furthermore, the coefficient module \mathbb{Z} is widely replaced by an abelian group G , i.e., the later use of universal coefficient formulæ is avoided. We hope that our results may serve as a good preparation for the study

of the homomorphisms between (co-)homology groups of toric varieties that are induced by toric morphisms.

Together with the explicit description of the Chow groups of toric varieties due to Fulton and Sturmfels the (co-)homology spectral sequences should help to clarify the question of generalizations of the Jurkiewicz-Danilov Theorem. In the noncompact case it is necessary to distinguish between homology with compact or closed supports. As expected, the latter one turns out to be suitable for our purpose: We shall see that the Chow group $A_p(X_\Delta; \mathbb{Z})$ appears as $E_{p,p}^2$ -term in the integral homology (with closed supports) spectral sequence associated to a filtration of the toric variety X_Δ that is induced by the natural torus action. The explicit computation of the E^2 -term of the toric homology spectral sequence is reduced to a solvable problem in multilinear algebra. We exploit this method by calculating the homology groups $H_\ell^{\text{cld}}(X_\Delta; G)$ in low and in high degrees ℓ . Since, for divisible groups G , the E^∞ -term of the homology spectral sequence turns out to coincide with the E^2 -term, we also can compute all the Betti numbers $b_\ell^{\text{cld}}(X_\Delta)$. Analogously, the toric cohomology (with compact supports) spectral sequence admits the computation of the cohomology groups $H_c^\ell(X_\Delta; G)$ in low and in high degrees ℓ .

In the sequel, it is our leitmotiv to emphasize the relations between the geometric and algebraic properties of toric varieties using the convex geometry of rational fans and the spherical geometry of the corresponding subcomplexes of closed real unit balls. We now outline the main methods and results of the following chapters:

Chapter 1 is devoted to the study of the *topology of toric varieties*. Such an algebraic variety X_Δ is usually constructed by gluing together the affine toric varieties $X_\sigma = \text{Spec } \mathbb{C}[\check{\sigma} \cap \mathbb{Z}^n]$ associated to the cones σ of a rational fan Δ in a vector space \mathbb{R}^n . This construction is the most important basis for the dictionary between algebraic and convex geometry in the case of toric varieties.

For our purpose, we are interested in the *transcendental* rather than the Zariski topology. Generalizing the *moment map* [Fu, Section 4.2] that only can be applied to projective toric varieties, there is an approach due to MacPherson to describe the topology of a toric variety in terms of “polar coordinates” (with a scaling of the radial factor): We recall that a toric variety consists of one open dense orbit, isomorphic to the n -dimensional complex algebraic torus $(\mathbb{C}^*)^n$, and its “boundary”, built up of finitely many orbits that are themselves again complex algebraic tori of lower dimension. As a topological space — even as a real algebraic variety — such a complex algebraic torus $(\mathbb{C}^*)^k$ is a product $\mathbb{R}_{>0}^k \times T_{\mathbb{R}}^k$, where $T_{\mathbb{R}}^k$ is the usual compact real k -torus $(S^1)^k$. Replacing $\mathbb{R}_{>0}^k$ by a k -cell, one may now associate to each toric variety X_Δ a continuous projection mapping $\pi: X_\Delta \rightarrow B_\Delta$ onto a finite cell complex B_Δ , consisting of the open unit ball in \mathbb{R}^n and of finitely many cells in the boundary sphere — one for each nonopen orbit of the complex algebraic torus action on X_Δ — such that the fibers are compact real tori $T_{\mathbb{R}}^k$.

The natural torus action on a toric variety X_Δ is discussed in terms of the fan Δ as well as in terms of MacPherson's description, i.e., with the help of the spherical complex $\tilde{\Delta}$ that is defined by means of the barycentric subdivision of the intersection of the fan Δ with the unit sphere S^{n-1} .

In Chapter 2 the main tools for the calculation of the *homology and cohomology of toric varieties* are introduced. Every fan Δ may be considered as a category with the cones as objects and the face relations as morphisms. By choosing appropriate orientation coefficients $o_{\sigma,\tau}$ for cones σ, τ in a fan Δ , covariant functors F from the category Δ to the category of abelian groups determine chain complexes $C_\bullet(\Delta, F)$ and homology theories $H_\bullet(\Delta, F)$. Two such functors H_\bullet^Δ and Λ_\bullet^Δ , which describe the homology groups of the fibers of the canonical projection $\pi: X_\Delta \rightarrow B_\Delta$ and the transition mappings between them, are of special interest. In fact, they help to calculate the homology of X_Δ :

There is a finite filtration $X_0 \subset X_1 \subset \cdots \subset X_n$, where X_k is the union of all orbits of dimension at most k . Associated to that filtration, there is a convergent homology spectral sequence, called *toric homology spectral sequence* in the sequel. For homology with closed supports and coefficients in an abelian group G , this is a first-quadrant spectral sequence, whose E^2 -terms are explicitly computable: $E_{p,q}^2 \cong H_p(\Delta, H_q^\Delta) \cong H_p(\Delta, \Lambda_q^\Delta)$ for all integers p, q , see Theorem 2.4.1.

Using a natural action of the multiplicative monoid $\mathbb{N}_{>0}$ on a toric variety that extends the action $k \cdot (t_1, \dots, t_n) = (t_1^k, \dots, t_n^k)$ on the "big" torus, we get vanishing results (up to torsion) for the differentials $d_{p,q}^r: E_{p,q}^r \rightarrow E_{p-r, q+r-1}^r$ for $r \geq 2$ that allow to describe the limit terms of the spectral sequence with the help of its E^2 -terms. The homology groups $H_\ell^{\text{cld}}(X_\Delta; G)$ in low and in high degrees ℓ and all the Betti numbers $b_\ell^{\text{cld}}(X_\Delta)$ will be calculated.

In addition, we translate the homology results of Chapter 2 into the language of cohomology, of course now with compact instead of closed supports.

In Chapter 3 we explain connections between different *invariants of toric varieties*. The toric homology spectral sequence yields another proof for the fact that the Euler characteristic for homology with closed supports $e^{\text{cld}}(X_\Delta)$ is given by the number of n -dimensional cones in the fan Δ , i.e., the coincidence of the Euler characteristic $e^{\text{cld}}(X_\Delta)$ with the topological Euler characteristic $e(X_\Delta)$ is verified.

As a main result, we prove a generalization of the Theorem of Jurkiewicz-Danilov: The Chow groups $A_p(X_\Delta; \mathbb{Z})$ of any toric variety X_Δ are canonically isomorphic to the terms $E_{p,p}^2(\mathbb{Z})$ of the integral toric homology spectral sequence, see Theorem 3.3.1, which leads to some monomorphisms (or even isomorphisms) of the kind $A_p(X_\Delta; \mathbb{Z}) \hookrightarrow H_{2p}^{\text{cld}}(X_\Delta; \mathbb{Z})$.

We collect special results for the Betti numbers and torsion coefficients of the integral (co-)homology groups of a toric variety and compute some integral local homology groups in its fixed points.

For the convenience of the reader, we recall in Appendix A the Fundamental

Theorem on *Finitely Generated Abelian Groups* that characterizes such groups by their respective ranks and torsion coefficients. The calculation of (co-)homology and Chow groups of toric varieties leads to quotients of lattices of finite rank by (necessarily) finitely generated sublattices, i.e., groups of the previous type. With the help of elementary divisors, we present a formula for the computation of the ranks and torsion coefficients of those quotients that depends on generators of the respective sublattice.

In Appendix B relations between the convex geometry of real affine spaces, the spherical geometry of the real closed unit ball B^n , and the algebraic geometry and topology of toric varieties are summarized in a *dictionary for the theory of toric varieties*.

Chapter 1

Topology of Toric Varieties

The class of toric varieties in algebraic geometry corresponds to basic objects in convex geometry: rational fans consisting of finitely many lattice cones. The usual way to construct the toric variety X_Δ associated to a fan Δ in a vector space \mathbb{R}^n is to glue together the affine toric varieties $X_\sigma = \text{Spec } \mathbb{C}[S_\sigma]$ for the cones $\sigma \in \Delta$, where the dual cones $\check{\sigma} = \{y \in \mathbb{R}^n; \langle y, \sigma \rangle \geq 0\}$ determine the monoids $S_\sigma = \check{\sigma} \cap \mathbb{Z}^n$, which are finitely generated by Gordan's lemma. As a consequence, algebro-geometric properties of toric varieties often admit a nice combinatorial interpretation in the convex geometry of real affine spaces.

But, as has been indicated in the introduction, MacPherson's description of the topological space X_Δ is more convenient for our purpose: It provides a canonical projection $\pi: X_\Delta \rightarrow B_\Delta$ onto a subspace B_Δ of the real closed unit ball $B^n \subset \mathbb{R}^n$, where the fibers are compact real tori. This projection will serve as our starting-point for the evaluation of spectral sequences that converge to the (co-)homology of the toric variety X_Δ .

Furthermore, the natural torus action on the toric variety X_Δ is discussed in terms of MacPherson's description, i.e., with the help of the spherical dual $\tilde{\Delta}$ that is defined by means of the barycentric subdivision of the intersection of the fan Δ with the unit sphere S^{n-1} .

In Section 1.1 fundamental notions and notations of the classical construction of toric varieties as algebraic varieties are fixed. Then, following MacPherson, every complete fan Δ in \mathbb{R}^n is associated with a spherical dual $\tilde{\Delta}$ that decomposes the unit ball B^n into open cells. The spherical dual $\tilde{\Delta}$ induces an equivalence relation \sim on the Cartesian product of the unit ball B^n and the real torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ such that the compact topological space X_Δ is homeomorphic to the quotient space $(B^n \times T^n)/\sim$. The canonical mapping $\pi: X_\Delta \approx (B^n \times T^n)/\sim \rightarrow B^n$ is induced by projection to the first coordinate of an equivalence class in $(B^n \times T^n)/\sim$; its fibers may be identified with quotients of the torus T^n . These results — as stated in [Fi, Section 1.1] — are generalized to the case of noncomplete fans, i.e., noncompact toric varieties, via completion of the fans, i.e., compactification of the varieties.

In Section 1.2 the natural torus action on the toric variety X_Δ is introduced as continuous extension of the action of the algebraic torus $(\mathbb{C}^*)^n$ by group multiplication on the dense affine open subset $X_{\{0\}} = \text{Spec } \mathbb{C}[\mathbb{Z}^n] \cong (\mathbb{C}^*)^n$. Each cone $\sigma \in \Delta$ corresponds to an orbit O_σ of this action. The orbit closure $V(\sigma) = \bar{O}_\sigma$ consists of all orbits O_τ such that the cone σ is a face of the cone $\tau \in \Delta$. The orbit closures are again toric varieties, whose associated fans are explicitly constructed. Finally, via a homeomorphism $(\mathbb{C}^*)^n \approx (B^n)^\circ \times T^n$, the structure of the orbits, their closures, and their relative boundaries is expressed in terms of MacPherson's description of toric varieties.

1.1 MacPherson's Description of Toric Varieties

A toric variety of dimension n may be defined as a normal algebraic variety endowed with an (effective and) almost transitive action of the complex algebraic n -torus that coincides on a dense Zariski open subset with the ordinary group multiplication. These algebraic varieties correspond to rational fans Δ in the vector space \mathbb{R}^n , i.e., finite nonempty complexes of strictly convex lattice cones with respect to the face relation. The underlying topological spaces may be described with the help of the spherical duals $\check{\Delta}$ that are constructed by means of the barycentric subdivision of the intersection of the fan Δ with the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

To fix notation, we recall some of the basic definitions and facts about toric varieties. For further details, we refer the reader to [Da, Ew, Fu, Od, Od₂]. Throughout the text, N and M always denote *dual lattices* of rank n , i.e., $N \cong \mathbb{Z}^n$ and $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^n$. Lattice cones and their dual cones are defined in the associated real vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$. A *lattice cone* σ in the vector space $N_{\mathbb{R}}$ is the *positive hull* spanned by finitely many *lattice points* v_1, \dots, v_k in the lattice N , i.e.,

$$\sigma = \text{pos} \{v_1, \dots, v_k\} := \sum_{i=1}^k \mathbb{R}_{\geq 0} v_i.$$

Its *dual cone*

$$\check{\sigma} := \{u \in M_{\mathbb{R}}; \langle u, \sigma \rangle \geq 0\},$$

where $\langle \cdot, \cdot \rangle: M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$ denotes the standard *dual pairing*, is a lattice cone in the vector space $M_{\mathbb{R}}$. If the lattice cone σ is *strictly convex*, i.e., $\sigma \cap (-\sigma) = \{0\}$, then its dual cone $\check{\sigma}$ is *full-dimensional*, i.e., $\dim \check{\sigma} := \dim_{\mathbb{R}} \text{lin } \check{\sigma} = n$, where the *linear hull* $\text{lin } \check{\sigma}$ is the minimal linear subspace of $M_{\mathbb{R}}$ that contains the cone $\check{\sigma}$. Furthermore, the submonoid

$$S_\sigma := \check{\sigma} \cap M = \{u \in \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z}); \langle u, \sigma \rangle \geq 0\}$$

of the lattice M is finitely generated by Gordan's lemma, see [Da, Lemma 1.3]. Hence, the corresponding *monoid algebra* $\mathbb{C}[S_\sigma]$ is a finitely generated \mathbb{C} -algebra that determines the *affine toric variety* $X_\sigma := \text{Spec } \mathbb{C}[S_\sigma]$.

As illustration of the monomial structure of affine toric varieties, though not needed in the sequel, we describe the minimal embedding $X_\sigma \hookrightarrow \mathbb{C}^d$, where d denotes the embedding-dimension: Choose *minimal* generators $0 \neq m_i, i = 1, \dots, d$ of the monoid S_σ , i.e., there is no relation $m_i = m_j + m_k$ with $1 \leq i, j, k \leq d$. Then the affine toric variety X_σ may be identified with the affine algebraic subvariety

$$\left\{ (z_1, \dots, z_d) \in \mathbb{C}^d; \prod z_i^{a_i} = \prod z_j^{b_j} \quad \text{if there is an additive relation} \right. \\ \left. \sum a_i m_i = \sum b_j m_j \text{ with } a_i, b_j \in \mathbb{N} \text{ and disjoint } \{m_i\}, \{m_j\} \right\}$$

of the affine algebraic variety \mathbb{C}^d .

By a *point* in the toric variety X_σ we always mean a *closed* point corresponding to a maximal ideal in the \mathbb{C} -algebra $\mathbb{C}[S_\sigma]$. It may be identified with a homomorphism of semigroups from the monoid S_σ to the multiplicative semigroup $\mathbb{C} = \mathbb{C}^* \cup \{0\}$: $\text{Spec } \mathbb{C}[S_\sigma] = \text{Hom}_{\text{sg}}(S_\sigma, \mathbb{C})$. Thus a basis of the \mathbb{C} -algebra $\mathbb{C}[S_\sigma]$ of regular functions on the affine variety X_σ is given by the regular functions χ^u for $u \in S_\sigma$, where $\chi^u(x) := x(u)$ for each point $x \in X_\sigma = \text{Hom}_{\text{sg}}(S_\sigma, \mathbb{C})$.

The affine toric variety X_σ has at most *quotient singularities* if the cone σ is *simplicial*, i.e., it is spanned by $\dim \sigma$ (necessarily linearly independent) lattice points.

The *faces* τ of the cone σ — indicated by $\tau \prec \sigma$ — are the cone σ itself and the nonempty intersections $\sigma \cap H$ of the cone σ with hyperplanes H in the vector space $N_{\mathbb{R}}$ such that the cone σ is contained in one of the corresponding half-spaces H^+ or H^- ; such hyperplanes are called *supporting hyperplanes* for σ . The faces of the lattice cone σ are again lattice cones; faces of codimension 1 are called *facets*. For each face $\tau \prec \sigma$, there exists a lattice point $u \in S_\sigma$ such that $\tau = \sigma \cap u^\perp$, where $u^\perp := \{v \in N_{\mathbb{R}}; \langle u, v \rangle = 0\}$. Hence, $S_\tau = S_\sigma + \mathbb{N} \cdot (-u)$, and we may identify the affine toric variety X_τ with the *principal open subset*

$$(X_\sigma)_{\chi^u} := \{x \in X_\sigma; \chi^u(x) = x(u) \neq 0\} \subset X_\sigma.$$

The affine toric varieties determined by lattice cones in $N_{\mathbb{R}}$ are the building blocks in the usual construction of general toric varieties. The objects on the combinatorial side that prescribe how these blocks are to be put together are the *rational fans* Δ in $N_{\mathbb{R}}$, i.e., finite nonempty complexes of strictly convex lattice cones in $N_{\mathbb{R}}$ satisfying the following conditions:

- (a) With each cone in the collection Δ all its faces belong to Δ .
- (b) If σ_1 and σ_2 are cones in Δ , then their intersection $\sigma_1 \cap \sigma_2$ is a (common) face of both.

Note that, for every natural number k , the k -skeleton

$$\Delta^{\leq k} := \{\sigma \in \Delta; \dim \sigma \leq k\}$$

of the fan Δ is again a fan. The number of elements in the set

$$\Delta^k := \{\sigma \in \Delta; \dim \sigma = k\} = \Delta^{\leq k} \setminus \Delta^{\leq k-1}$$

is denoted by $d_k := d_k(\Delta) := \#\Delta^k$.

By gluing together the affine toric varieties X_{σ_i} for $\sigma_i \in \Delta$ along the common principal open subsets $X_{\sigma_1 \cap \sigma_2} = X_{\sigma_1} \cap X_{\sigma_2}$ for $\sigma_1, \sigma_2 \in \Delta$, respectively, we obtain an algebraic variety: the *toric variety* X_Δ , which has the (canonical images of) the affine toric varieties X_{σ_i} as a finite open cover. The toric variety X_Δ is compact iff the fan Δ is *complete*, i.e., its *support* $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma$ coincides with the vector space $N_{\mathbb{R}}$.

By intersection with the unit sphere S^{n-1} in the Euclidean vector space $N_{\mathbb{R}}$, such a complete fan Δ determines a *spherical complex*

$$\mathcal{C}_\Delta := \{\sigma \cap S^{n-1}; \{0\} \neq \sigma \in \Delta\}.$$

This complex provides a finite CW-cell decomposition of the real closed unit ball B^n in the Euclidean vector space $N_{\mathbb{R}}$. The *barycentric subdivision* \mathcal{C}'_Δ of the spherical complex \mathcal{C}_Δ is defined to be the spherical complex whose vertices are the barycenters of the elements of \mathcal{C}_Δ and whose spherical simplices are spanned by finite nonempty collections of barycenters of elements of \mathcal{C}_Δ that are totally ordered by inclusion.



Figure 1.1: A spherical 2-simplex and its barycentric subdivision.

Now associate to each cone $\sigma \in \Delta$ a *spherical dual* $\hat{\sigma}$ as follows: The spherical dual of the *zero cone* $\{0\} \in \Delta$ is defined to be the real closed unit ball B^n . For each nonzero cone $\sigma \in \Delta$, let its spherical dual $\hat{\sigma}$ be the union of all spherical simplices in the barycentric subdivision \mathcal{C}'_Δ whose vertices are barycenters of elements $\tau \cap S^{n-1} \in \mathcal{C}_\Delta$ with $\sigma \prec \tau \in \Delta$, i.e., $\sigma \cap S^{n-1}$ is a spherical face of $\tau \cap S^{n-1}$. Then the *relative interior* $\tilde{\sigma} := (\hat{\sigma})^\circ$ of the closed subset $\hat{\sigma} \subset B^n$ is a cell of dimension $\dim \tilde{\sigma} = n - \dim \sigma$. The collection $\tilde{\Delta} := \{\tilde{\sigma}; \sigma \in \Delta\}$ is called the *spherical dual* of the fan Δ .

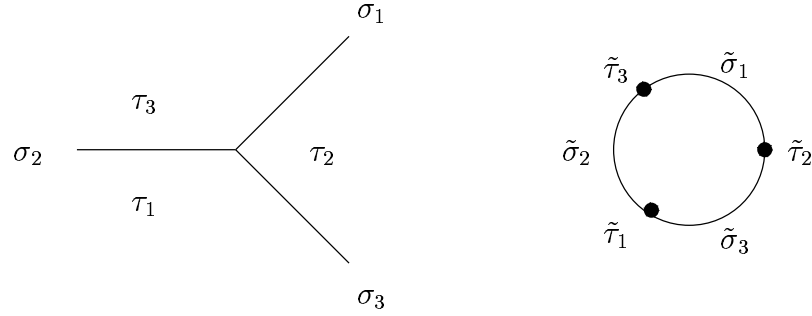


Figure 1.2: A complete fan Δ in \mathbb{R}^2 and its spherical dual $\tilde{\Delta}$.

Consider the canonical projection

$$N_{\mathbb{R}} \rightarrow T^n := N_{\mathbb{R}}/N \cong \mathbb{R}^n/\mathbb{Z}^n$$

of the vector space $N_{\mathbb{R}}$ onto the compact real n -dimensional torus T^n . For each cone σ of dimension d , its linear hull $\text{lin } \sigma$ is a d -dimensional linear subspace of $N_{\mathbb{R}}$ with a rational basis, whose image under the projection mapping is the d -dimensional subtorus

$$T_{\sigma} := (N + \text{lin } \sigma)/N \cong \text{lin } \sigma / (N \cap \text{lin } \sigma)$$

of the compact real torus T^n . Using these compact real tori, MacPherson gives the following description of the topology of a — for the present: compact — complex toric variety, cf. [Fi, Section 1.1]:

1.1.1 Theorem (MacPherson). *For every complete fan Δ in $N_{\mathbb{R}}$, the topological space X_{Δ} is homeomorphic to the quotient space $(B^n \times T^n)/\sim$, where the equivalence relation \sim on the Cartesian product $B^n \times T^n$ is given by $(x, t) \sim (x', t')$ iff $x = x'$ and $t - t' \in T_{\sigma}$ for the unique cone $\sigma \in \Delta$ with $x \in \tilde{\sigma}$. ■*

The fibers of the canonical projection

$$\pi: X_{\Delta} \approx (B^n \times T^n)/\sim \rightarrow B^n, \quad [(x, t)] \mapsto x$$

are quotients of T^n , in detail: The fibers $\pi^{-1}(x)$ of the points $x \in \tilde{\sigma}$ for $\sigma \in \Delta$ may be identified with the quotient

$$T(\sigma) := T^n/T_{\sigma} \cong N_{\mathbb{R}}/(N + \text{lin } \sigma).$$

Choosing a sublattice $L \subset N$ such that $N = (N \cap \text{lin } \sigma) \oplus L$, it follows that the quotient $T(\sigma) \cong (\text{lin } L)/L$ is a compact real torus of dimension $\dim T(\sigma) = n - \dim \sigma = \dim \tilde{\sigma}$ yielding a direct sum decomposition $T^n \cong T_{\sigma} \oplus T(\sigma)$.

Try to imagine the topological space X_{Δ} associated to the fan Δ of Figure 1.2 by gluing together for all points $x \in B^2$ the fibers $\pi^{-1}(x)$, which are compact real tori of dimension 0, 1, or 2!

In the case of *noncomplete* fans we may proceed as follows: Note that in MacPherson's description the topological space of the affine open subvariety $X_\sigma \subset X_{\bar{\Delta}}$ for a cone σ in a complete fan $\bar{\Delta}$ is given by the subquotient

$$(B_\sigma \times T^n) / \sim \subset (B^n \times T^n) / \sim ,$$

where $B_\sigma := \bigcup_{\tau \prec \sigma} \tilde{\tau} \subset B^n$. Moreover, for all cones $\sigma_1, \sigma_2 \in \bar{\Delta}$, two such subvarieties X_{σ_1} and X_{σ_2} intersect in the subvariety $X_{\sigma_1 \cap \sigma_2}$ given by the subquotient $(B_{\sigma_1 \cap \sigma_2} \times T^n) / \sim$, where $B_{\sigma_1 \cap \sigma_2} = B_{\sigma_1} \cap B_{\sigma_2}$. Hence, for each subfan $\Delta \subset \bar{\Delta}$, the toric variety X_Δ is homeomorphic to the subquotient $(B_\Delta \times T^n) / \sim$, where $B_\Delta := \bigcup_{\sigma \in \Delta} \tilde{\sigma} \subset B^n$ denotes the *spherical support* of the subfan Δ . By choosing any complete fan that contains a given noncomplete fan, i.e., via compactification of the toric variety, we may thus generalize the description of MacPherson:

1.1.2 Corollary. *For every fan Δ in $N_{\mathbb{R}}$, the topological space X_Δ is homeomorphic to the quotient space $(B_\Delta \times T^n) / \sim$, where the equivalence relation \sim on the Cartesian product $B_\Delta \times T^n$ is given by $(x, t) \sim (x', t')$ iff $x = x'$ and $t - t' \in T_\sigma$ for the unique cone $\sigma \in \Delta$ with $x \in \tilde{\sigma}$. ■*

Note that the construction of the spherical duals $\hat{\sigma}$, their relative interiors $\tilde{\sigma}$, the spherical support B_Δ , and the equivalence relation \sim in the preceding corollary depend in general on the chosen completion $\bar{\Delta}$ of the fan Δ . If the spherical duals are once determined, we may forget all additional cones that were needed to complete the fan Δ . But we do not see a more natural definition of the spherical duals that avoids the completion of the fan (Consider for instance subfans and their respective spherical duals of the fan in Figure 1.2)!

The fibers of the canonical projection

$$\pi: X_\Delta \approx (B_\Delta \times T^n) / \sim \rightarrow B_\Delta, \quad [(x, t)] \mapsto x$$

are quotients of T^n , namely $\pi^{-1}(x) = T(\sigma)$ for the points $x \in \tilde{\sigma}$ with $\sigma \in \Delta$.

The toric variety X_Δ is compact iff the fan Δ is complete iff its spherical support B_Δ coincides with the closed unit ball B^n .

1.2 The Natural Torus Action on Toric Varieties

On an affine toric variety, there is a natural action of the complex algebraic torus $(\mathbb{C}^*)^n$. As the gluing of the affine open pieces X_σ is automatically equivariant with respect to that action, every toric variety X_Δ is naturally endowed with a $(\mathbb{C}^*)^n$ -action.

The complex algebraic torus

$$\mathbb{T}_N := \text{Spec } \mathbb{C}[M] \cong \text{Hom}(M, \mathbb{C}^*) \cong N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$$

acts algebraically on the affine open subsets $X_\sigma \subset X_\Delta$ for $\sigma \in \Delta$: Consider each point $t \in \mathbb{T}_N$ as a group homomorphism $t: M \rightarrow \mathbb{C}^*$, and each point $x \in X_\sigma = \text{Spec } \mathbb{C}[S_\sigma]$ as a semigroup homomorphism $x: S_\sigma \rightarrow \mathbb{C}$, i.e., we have $x(0) = 1$ and $x(u_1 + u_2) = x(u_1)x(u_2)$ for $u_1, u_2 \in S_\sigma$. Then define the point $t \cdot x \in X_\sigma$ as a homomorphism $t \cdot x: S_\sigma \rightarrow \mathbb{C}$ of semigroups by the assignment $(t \cdot x)(u) := t(u)x(u)$ for $u \in S_\sigma$. For the zero cone $\sigma = \{0\}$, this action is nothing else but group multiplication in the algebraic torus $\mathbb{T}_N = X_{\{0\}}$.

By gluing together these \mathbb{T}_N -actions on the affine open subsets X_σ for $\sigma \in \Delta$, the *natural torus action* on the toric variety X_Δ arises. In the sequel, we summarize some well-known properties of the natural torus action, cf. [Od, Proposition 1.6 and Corollary 1.7] or [Fu, Section 3.1]:

The fan Δ is in one-to-one correspondence with the set of orbits of the \mathbb{T}_N -action on the toric variety X_Δ by associating each cone $\sigma \in \Delta$ with the \mathbb{T}_N -orbit

$$\begin{aligned} O_\sigma &:= \{x \in X_\sigma; x \in \text{Hom}_{\text{sg}}(S_\sigma, \mathbb{C}) \text{ with } x(u) = 0 \text{ for } u \notin \sigma^\perp\} \\ &\cong \text{Hom}(\sigma^\perp \cap M, \mathbb{C}^*), \end{aligned}$$

where $\sigma^\perp := \{u \in M_\mathbb{R}; \langle u, \sigma \rangle = 0\}$. Such an orbit O_σ has dimension $\dim_{\mathbb{C}} O_\sigma = n - \dim_{\mathbb{R}} \sigma = \dim_{\mathbb{R}} \hat{\sigma} = \dim_{\mathbb{R}} \tilde{\sigma}$. It is the unique closed orbit in the affine toric variety $X_\sigma = \bigcup_{\tau \prec \sigma} O_\tau$. On the other hand, the closure of the orbit O_σ in the toric variety $X_\Delta = \bigcup_{\sigma \in \Delta} O_\sigma$ is given by the set $V(\sigma) := \bar{O}_\sigma = \bigcup_{\sigma \prec \tau \in \Delta} O_\tau$, i.e., $O_\tau \subset \bar{O}_\sigma$ iff $\sigma \prec \tau$ for cones $\sigma, \tau \in \Delta$.

The closed subvarieties of the toric variety X_Δ that are invariant under the natural torus action are unions of such orbit closures $V(\sigma)$; and each orbit closure $V(\sigma)$ is a toric variety of dimension $(n - \dim \sigma)$: Denote by

$$N_\sigma := (N \cap \sigma) - (N \cap \sigma) = N \cap \text{lin } \sigma$$

the (saturated) sublattice of N generated by the submonoid $N \cap \sigma$. Then the orbit closure $V(\sigma)$ is isomorphic to the toric variety $X_{\Delta(\sigma)}$, where

$$\Delta(\sigma) := \{(\tau + \text{lin } \sigma) / \text{lin } \sigma; \sigma \prec \tau \in \Delta\}$$

is a rational fan in the vector space $N_\mathbb{R} / \text{lin } \sigma = (N / N_\sigma) \otimes_{\mathbb{Z}} \mathbb{R}$ with respect to the *quotient lattice* $N(\sigma) := N / N_\sigma$ of rank $(n - \dim \sigma)$.

Finally, since the algebraic torus \mathbb{T}_N is a *commutative* group, the *isotropy group* $I_\sigma := \{t \in \mathbb{T}_N; t \cdot x = x \text{ for all } x \in O_\sigma\}$ of the orbit O_σ coincides with the isotropy group $(\mathbb{T}_N)_{x_\sigma}$ of the *distinguished point* $x_\sigma \in O_\sigma \cong \text{Hom}(\sigma^\perp \cap M, \mathbb{C}^*)$ determined by $x_\sigma(u) := 1$ for $u \in \sigma^\perp \cap M$. Hence,

$$\begin{aligned} I_\sigma = (\mathbb{T}_N)_{x_\sigma} &= \{t \in \mathbb{T}_N; t \cdot x_\sigma = x_\sigma\} \\ &= \{t \in \text{Hom}(M, \mathbb{C}^*); t(u) = 1 \text{ for } u \in \sigma^\perp \cap M\}. \end{aligned}$$

For later use in the description of the (co-)homology spectral sequences, note that, for $\sigma \in \Delta$, the orbit O_σ is given by the subset

$$X_\sigma \setminus \bigcup_{\tau \prec \sigma, \tau \neq \sigma} X_\tau \subset X_\Delta$$

that corresponds to the set

$$\left((B_\sigma \times T^n) / \sim \right) \setminus \bigcup_{\tau \prec \sigma, \tau \neq \sigma} \left((B_\tau \times T^n) / \sim \right) = \tilde{\sigma} \times T(\sigma) = \pi^{-1}(\tilde{\sigma}),$$

i.e., the preimage of the relative interior of the spherical dual $\hat{\sigma}$ under the canonical projection $\pi: (B_\Delta \times T^n) / \sim \rightarrow B_\Delta$. Analogously, the orbit closure $V(\sigma) = \bar{O}_\sigma = \bigcup_{\sigma \prec \tau \in \Delta} O_\tau \subset X_\Delta$ corresponds to the set

$$\bigcup_{\sigma \prec \tau \in \Delta} \pi^{-1}(\tilde{\tau}) = \pi^{-1}(\hat{\sigma} \cap B_\Delta) = \pi^{-1}(\bar{\sigma}),$$

where $\bar{\sigma} := \hat{\sigma} \cap B_\Delta$ denotes the *trace* of the spherical dual $\hat{\sigma}$ with respect to the spherical support B_Δ . Moreover, the *relative boundary* $\partial O_\sigma := V(\sigma) \setminus O_\sigma \subset X_\Delta$ of the orbit O_σ corresponds to the set $\pi^{-1}(\bar{\sigma}) \setminus \pi^{-1}(\tilde{\sigma}) = \pi^{-1}(\partial \bar{\sigma})$, where $\partial \bar{\sigma} := \bar{\sigma} \setminus \tilde{\sigma}$ denotes the *relative boundary* of $\bar{\sigma}$ with respect to B_Δ .

In the rest of this section, we investigate in some detail how the natural torus action is given in terms of MacPherson's description. This question seems to be of independent interest, though it will not be used in the sequel. We just have to choose a homeomorphism $(\mathbb{C}^*)^n \approx (B^n)^\circ \times T^n$ between the algebraic torus $\mathbb{T}_N \cong (\mathbb{C}^*)^n$ and the quotient space $(B_{\{0\}} \times T^n) / \sim = (B^n)^\circ \times T^n$ associated to the *zero fan* $\{0\}$, where $(B^n)^\circ$ denotes the real open unit ball in the Euclidean vector space $N_{\mathbb{R}}$. Then we calculate the action of the Cartesian product $(B^n)^\circ \times T^n$ on the quotient space $(B_\Delta \times T^n) / \sim$ that is determined by continuous extension of the action of the product $(B^n)^\circ \times T^n$ on itself which corresponds to the ordinary multiplication in the group $(\mathbb{C}^*)^n$. For instance, we may proceed as follows: The open intervals $] -1, 1[$ and $]0, \infty[$ are homeomorphic via the mapping

$$\varphi:] -1, 1[\xrightarrow{\approx}]0, \infty[; \quad x \mapsto \varphi(x) := \frac{1+x}{1-x} = y$$

with inverse mapping

$$\varphi^{-1}:]0, \infty[\xrightarrow{\approx}] -1, 1[; \quad y \mapsto \varphi^{-1}(y) := \frac{y-1}{y+1} = x,$$

as can be seen by similar triangles in the following figure:

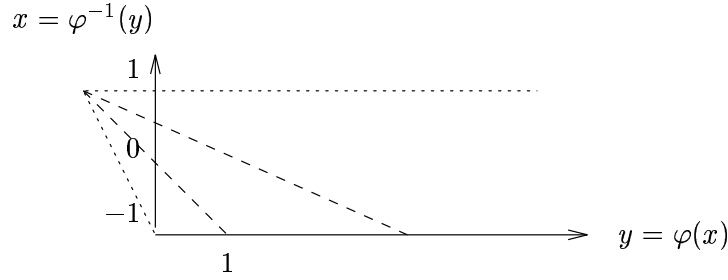


Figure 1.3: A homeomorphism $] - 1, 1[\approx]0, \infty[$ via stereographic projection.

Hence, using polar coordinates, the algebraic torus $\mathbb{C}^* = \{ye^{is}; 0 < y < \infty, 0 \leq s < 2\pi\}$ is homeomorphic to the Cartesian product $(B^1)^\circ \times T^1 =] - 1, 1[\times (\mathbb{R}/\mathbb{Z})$ via the mapping

$$ye^{is} \mapsto \left(\frac{y-1}{y+1}, \frac{s}{2\pi} + \mathbb{Z} \right).$$

The group multiplication

$$\mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathbb{C}^*; \quad (ye^{is}, y'e^{is'}) \mapsto yy'e^{i(s+s')}$$

thus induces the continuous action

$$((B^1)^\circ \times T^1) \times ((B^1)^\circ \times T^1) \longrightarrow (B^1)^\circ \times T^1$$

defined by

$$\begin{aligned} ((x, t + \mathbb{Z}), (x', t' + \mathbb{Z})) &\mapsto (\varphi^{-1}(\varphi(x) \cdot \varphi(x')), t + t' + \mathbb{Z}) \\ &= \left(\frac{x+x'}{xx'+1}, t + t' + \mathbb{Z} \right). \end{aligned}$$

The 1-dimensional case induces a suitable homeomorphism

$$(B^n)^\circ \times T^n \xrightarrow{\approx} ((B^1)^\circ \times T^1)^n \approx (\mathbb{C}^*)^n$$

for the general case: Note that in the vector space $N_{\mathbb{R}}$ the real open unit ball $(B^n)^\circ$ with respect to the Euclidean norm $\|\cdot\|$ and the real open unit ball $((B^1)^\circ)^n$ with respect to the maximum norm $|\cdot|$ are homeomorphic via the mapping

$$x \mapsto \frac{\|x\|}{|x|} \cdot x,$$

where $\frac{0}{0} := 1$. Together with the homeomorphism

$$T^n \xrightarrow{\approx} (T^1)^n; \quad (t_1, \dots, t_n) + \mathbb{Z}^n \mapsto (t_1 + \mathbb{Z}, \dots, t_n + \mathbb{Z}),$$

we therefore get a homeomorphism

$$(B^n)^\circ \times T^n \xrightarrow{\approx} ((B^1)^\circ)^n \times (T^1)^n \approx (\mathbb{C}^*)^n$$

via the mapping

$$(x, (t_1, \dots, t_n) + \mathbb{Z}^n) \mapsto \left(\frac{\|x\|}{|x|} \cdot x, t_1 + \mathbb{Z}, \dots, t_n + \mathbb{Z} \right).$$

Under this homeomorphism the multiplication in the group $(\mathbb{C}^*)^n$ corresponds to the continuous action

$$((B^n)^\circ \times T^n) \times ((B^n)^\circ \times T^n) \longrightarrow (B^n)^\circ \times T^n$$

defined by

$$((x, t + \mathbb{Z}^n), (x', t' + \mathbb{Z}^n)) \mapsto \left(\frac{|x \circ x'|}{\|x \circ x'\|} \cdot (x \circ x'), t + t' + \mathbb{Z}^n \right),$$

where, for vectors $x = (x_1, \dots, x_n)$ and $x' = (x'_1, \dots, x'_n)$, the k -th component of the vector $x \circ x'$ is given by

$$\frac{\|x\| \cdot |x'| \cdot x_k + \|x'\| \cdot |x| \cdot x'_k}{\|x\| \cdot \|x'\| \cdot x_k \cdot x'_k + |x| \cdot |x'|}.$$

By continuous extension, the natural action of the algebraic torus $(\mathbb{C}^*)^n \approx (B^n)^\circ \times T^n$ on the toric variety $X_\Delta \approx (B_\Delta \times T^n)/\sim$ — that contains the dense open subset $(B_{\{0\}} \times T^n)/\sim = (B^n)^\circ \times T^n$ — may be computed explicitly in terms of MacPherson's description.

Chapter 2

Homology and Cohomology of Toric Varieties

The investigation of the topology is a basic mean to distinguish between non-isomorphic varieties. The various (co-)homology theories provided by algebraic topology are known to be most useful tools for that purpose. Moreover, (co-)homology groups often play the role of obstruction to solving problems in algebraic or complex-analytic geometry. With universal coefficient formulæ in mind, the calculation of the integral homology of toric varieties is therefore an interesting problem. But although MacPherson's description provides a finite CW-cell decomposition for every compact toric variety X_Δ , see [Fi, Lemma 1.1] — which obviously leads to a finite cell decomposition in the noncompact case — it seems rather difficult to compute the cellular homology of X_Δ if $\dim X_\Delta \geq 3$.

We thus have to look for other means of computation. A powerful tool is provided by spectral sequences, e.g., the Leray-Serre spectral sequence associated to a fibration, the general Leray spectral sequence for proper mappings, and, most basically, the spectral sequence associated to a filtration.

The Leray-Serre spectral sequence as described in [Sp, Section 9.2] converges to the homology of the total space of a locally trivial fibration, and its E^2 -term is given by the homology of the base space with coefficients in the homology of the fiber. Unfortunately, except in the trivial case $\Delta = \{0\}$, that tool can not be applied to the canonical projection $\pi: X_\Delta \rightarrow B_\Delta$: The fibers of π are tori of different dimension, so π fails to be a fibration. On the other hand, the evaluation of the general Leray spectral sequence for proper mappings, see [Br, Section IV 6], proves to be too hard. But there is a way out:

The finite filtration of the toric variety X_Δ induced by the natural torus action on X_Δ leads to a convergent homology spectral sequence, whose E^1 -terms are direct sums of homology groups of pairs consisting of orbit closures and their relative boundaries. Looking at homology with *closed* supports, we may replace the homology groups of these pairs by the homology groups of the orbits themselves. Via Poincaré duality, these homology groups with closed supports are

identified with “usual” cohomology groups of the orbits. But the orbits are complex algebraic tori of suitable dimension d having the compact real d -torus as a deformation retract, and these compact real tori occur as the fibers of the natural projection $\pi: X_\Delta \rightarrow B_\Delta$. Hence, the cohomology of the orbits coincides with the (co-)homology of the fibers of π , and the E^2 -terms are determined by these groups and the natural transition mappings between them. It turns out that all $E_{p,q}^1$ -terms outside of a triangular region bounded by the positive p -axis and the diagonal $p = q$ vanish. Hence, the explicit computation of the E^2 -terms immediately yields the homology groups (with closed supports and coefficients in an abelian group G) of the toric variety X_Δ in low and in high degrees.

There is another important reason to study homology with *closed* supports: The geometry of a toric variety is essentially encoded in the invariant algebraic cycles, and hence, in cycles with closed supports. As a most interesting consequence, we shall see that expressions we obtain here admit a natural interpretation in Chow theory (see Chapter 3). With the help of an action of the multiplicative monoid $\mathbb{N}_{>0}$ on the toric variety X_Δ and the toric homology spectral sequence, it turns out that suitable multiples of the differentials $d_{p,q}^r: E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ with $r \geq 2$ vanish. In particular, for divisible groups G , the E^∞ -terms coincide with the E^2 -terms. Hence, we can compute all the Betti numbers $b_\ell^{\text{cld}}(X_\Delta)$.

In Section 2.1 we provide a general framework to describe the E^1 - and E^2 -terms of the toric homology spectral sequence: We study covariant functors F from the category given by a fan Δ to the category of abelian groups. By choosing appropriate orientation coefficients, each such functor F determines a chain complex $C_\bullet(\Delta, F)$ and a corresponding homology theory $H_\bullet(\Delta, F)$. With the naturally equivalent covariant functors H_\bullet^Δ and Λ_\bullet^Δ that describe and compute the homology groups of the fibers of the canonical projection $\pi: X_\Delta \rightarrow B_\Delta$ and the transition mappings between them, the E^2 -terms of the toric homology spectral sequence are given by the graded homology groups $H_\bullet(\Delta, H_\bullet^\Delta) \cong H_\bullet(\Delta, \Lambda_\bullet^\Delta)$. The homology $H_\bullet(\Delta, G^\Delta)$ is isomorphic to the reduced cellular cohomology $\tilde{H}^{n-\bullet-1}(S_\Delta; G)$ of the finite CW-complex $S_\Delta = |\Delta| \cap S^{n-1}$, where G^Δ denotes the constant functor on a fan $\Delta \neq \{0\}$ associated to an abelian group G .

In Section 2.2 we give a short introduction to the theory of spectral sequences. The exact couple built from the long exact homology sequences associated to a finite increasing filtration of a chain complex determines a homology spectral sequence. This sequence converges to the homology of the original chain complex. Every endomorphism of the chain complex that respects the given filtration induces endomorphisms of the associated exact couple and homology spectral sequence.

In Section 2.3 we discuss the spectral sequence associated to the finite increasing filtration of a toric variety X_Δ induced by the natural torus action. For homology with closed supports and coefficients in an abelian group, we express the associated differentials $d_{p,q}^1: E_{p,q}^1 \rightarrow E_{p-1,q}^1$ with the help of the homology of

the fibers of the projection $\pi: X_\Delta \rightarrow B_\Delta$ and the transition mappings between them.

In Section 2.4 we evaluate the toric homology spectral sequence for homology with closed supports and coefficients in an abelian group. The E^1 - and E^2 -terms of this first-quadrant spectral sequence are explicitly computable: $E_{p,q}^1 \cong C_p(\Delta, \Lambda_q^\Delta)$ and $E_{p,q}^2 \cong H_p(\Delta, \Lambda_q^\Delta)$ for all integers p, q . We study the question which multiples of the differentials $d_{p,q}^r: E_{p,q}^r \rightarrow E_{p-r, q+r-1}^r$ with $r \geq 2$ vanish. With the help of the E^2 -terms, we can calculate the homology groups $H_\ell^{\text{cld}}(X_\Delta; G)$ in low and in high degrees and all the Betti numbers $b_\ell^{\text{cld}}(X_\Delta)$.

In Section 2.5 we sketch the dual counterparts of Sections 2.1–2.4 for the computation of the cohomology (now with *compact* instead of closed supports) of toric varieties.

2.1 Covariant Functors and Associated Homology Theories

The covariant functor H_\bullet^Δ on the category Δ contains the whole information about the homology (with coefficients in an abelian group G) of the fibers of the projection $\pi: X_\Delta \rightarrow B_\Delta$ via the transition mappings $\pi_{\bullet}^{\sigma, \tau}: H_\bullet(T(\sigma); G) \rightarrow H_\bullet(T(\tau); G)$ for $\sigma \prec \tau \in \Delta$. This functor is naturally equivalent to the functor Λ_\bullet^Δ that associates to any “morphism” $\sigma \prec \tau \in \Delta$ the explicitly computable homomorphism $\nu_{\sigma, \tau}^\Delta: \Lambda^*N(\sigma) \otimes G \rightarrow \Lambda^*N(\tau) \otimes G$. The graded homology groups $H_\bullet(\Delta, H_\bullet^\Delta) \cong H_\bullet(\Delta, \Lambda_\bullet^\Delta)$ of the associated chain complexes are our main tools for the description and calculation of the E^2 - and E^∞ -terms of the toric spectral sequence that converges to the homology (with closed supports and coefficients in an abelian group G) of the toric variety X_Δ . In addition, we will use the graded homology groups $H_\bullet(\Delta, G^\Delta)$ that are isomorphic to the graded reduced cellular cohomology groups $\tilde{H}^{n-\bullet-1}(S_\Delta; G)$ of the (compact) finite CW-complex $S_\Delta = \bigcup_{\sigma \in \Delta} (\sigma \cap S^{n-1})^\circ$, where G^Δ denotes the constant functor on a nonzero fan Δ associated to the abelian group G .

In the sequel, a fan Δ will often be considered as a (finite) category with the cones $\sigma \in \Delta$ as objects and the face relations $\sigma \prec \tau$ in Δ as morphisms.

2.1.1 Examples. For a fixed abelian group G , which will be frequently omitted in our notation, consider the following covariant functors from the category Δ to the category of abelian groups:

- (a) Let H_\bullet^Δ be the covariant functor on the category Δ that associates to each cone $\sigma \in \Delta$ the homology module $H_\bullet(T(\sigma); G)$ of the compact real torus

$$T(\sigma) = T^n/T_\sigma \cong N_{\mathbb{R}}/(N + \text{lin } \sigma),$$

and to each face relation $\sigma \prec \tau$ in Δ the homomorphism

$$\pi_{\bullet}^{\sigma, \tau}: H_\bullet(T(\sigma); G) \rightarrow H_\bullet(T(\tau); G)$$

induced by the canonical projection $\pi^{\sigma,\tau}: T(\sigma) \twoheadrightarrow T(\tau)$.

- (b) Let $\Lambda_{\bullet}^{\Delta}$ be the covariant functor on the category Δ that associates to each cone $\sigma \in \Delta$ the graded abelian group $\Lambda^*N(\sigma) \otimes G$ given by the tensor products of the exterior powers of the quotient lattice

$$N(\sigma) = N/N_{\sigma} = N/(N \cap \text{lin } \sigma)$$

with the abelian group G , and to each face relation $\sigma \prec \tau$ in Δ the epimorphism

$$\nu_{\sigma,\tau}^{\Lambda} := \Lambda^* \nu_{\sigma,\tau} \otimes 1_G: \Lambda^*N(\sigma) \otimes G \twoheadrightarrow \Lambda^*N(\tau) \otimes G$$

induced by the natural epimorphism $\nu_{\sigma,\tau}: N(\sigma) \twoheadrightarrow N(\tau)$.

We may replace the calculation of the first functor — it will be used to express the homology of the toric variety X_{Δ} — by the computation of the second one:

2.1.2 Proposition. *For every fan Δ in $N_{\mathbb{R}}$, there is a natural equivalence $\Phi^{\Delta} = (\Phi(\sigma))_{\sigma \in \Delta}$ of degree 0 between the covariant functors H_{\bullet}^{Δ} and $\Lambda_{\bullet}^{\Delta}$ on the category Δ .*

Proof. A natural equivalence Φ^{Δ} between the covariant functors H_{\bullet}^{Δ} and $\Lambda_{\bullet}^{\Delta}$ consists of isomorphisms

$$\Phi(\sigma): H_{\bullet}(T(\sigma); G) \xrightarrow{\cong} \Lambda^*N(\sigma) \otimes G$$

for each cone $\sigma \in \Delta$ such that the diagram

$$(*) \quad \begin{array}{ccc} H_{\bullet}(T(\sigma); G) & \xrightarrow{\pi_{\bullet}^{\sigma,\tau}} & H_{\bullet}(T(\tau); G) \\ \Phi(\sigma) \downarrow \cong & & \cong \downarrow \Phi(\tau) \\ \Lambda^*N(\sigma) \otimes G & \xrightarrow{\nu_{\sigma,\tau}^{\Lambda}} & \Lambda^*N(\tau) \otimes G \end{array}$$

is commutative for each face relation $\sigma \prec \tau$ in Δ . First, consider the coefficient module $G := \mathbb{Z}$ and the face relation $\{0\} \prec \sigma$ for each cone $\sigma \in \Delta$. Choosing a direct sum decomposition $T^n \cong T_{\sigma} \oplus T(\sigma)$ induced by a direct sum decomposition $N = N_{\sigma} \oplus L$, we may assume that the canonical projection

$$\pi^{\sigma}: (S^1)^n \cong T^n \twoheadrightarrow T(\sigma) \cong (S^1)^{n-\dim \sigma}$$

is given by projection to the last $(n - \dim \sigma)$ factors S^1 . Then, for $0 \leq p \leq n$, the induced epimorphism

$$\pi_p^{\sigma}: H_p(T^n; \mathbb{Z}) \twoheadrightarrow H_p(T(\sigma); \mathbb{Z})$$

corresponds to the epimorphism

$$\Lambda^p \nu_\sigma: \mathbb{Z}^{\binom{n}{p}} \cong \Lambda^p N \twoheadrightarrow \Lambda^p N(\sigma) \cong \mathbb{Z}^{\binom{n-\dim \sigma}{p}}$$

induced by the natural epimorphism $\nu_\sigma: N \twoheadrightarrow N(\sigma)$, since the homology group $H_p((S^1)^k; \mathbb{Z}) \cong \mathbb{Z}^{\binom{k}{p}}$ is generated by the homology classes of the $\binom{k}{p}$ different Cartesian products consisting of p factors S^1 and $(k-p)$ factors $s \in S^1$. (Note that the homology group $H_0(S^1; \mathbb{Z}) \cong \mathbb{Z}$ is generated by the homology class $[s]$ of any point $s \in S^1$, the homology group $H_1(S^1; \mathbb{Z}) \cong \mathbb{Z}$ is generated by the homology class $[S^1]$, and apply the Künneth theorem!)

As a consequence, there are isomorphisms

$$\Phi := \Phi(\{0\}): H_\bullet(T(\{0\}); \mathbb{Z}) = H_\bullet(T^n; \mathbb{Z}) \xrightarrow{\cong} \Lambda^* N = \Lambda^* N(\{0\}) \otimes \mathbb{Z}$$

and

$$\Phi(\sigma): H_\bullet(T(\sigma); \mathbb{Z}) \xrightarrow{\cong} \Lambda^* N(\sigma)$$

such that the diagram

$$\begin{array}{ccc}
 H_\bullet(T^n; \mathbb{Z}) & \xrightarrow{\pi_\bullet^\sigma} & H_\bullet(T(\sigma); \mathbb{Z}) \\
 \Phi \downarrow \cong & & \cong \downarrow \Phi(\sigma) \\
 \Lambda^* N & \xrightarrow{\nu_\sigma^\Lambda} & \Lambda^* N(\sigma)
 \end{array}
 \quad (**)$$

is commutative, where the canonical projection $\pi^\sigma: T^n \twoheadrightarrow T(\sigma)$ induces the epimorphism $\pi_\bullet^\sigma: H_\bullet(T^n; \mathbb{Z}) \twoheadrightarrow H_\bullet(T(\sigma); \mathbb{Z})$, and ν_σ^Λ denotes the epimorphism $\Lambda^* \nu_\sigma: \Lambda^* N \twoheadrightarrow \Lambda^* N(\sigma)$.

Now the commutativity of the diagram (*) for $G := \mathbb{Z}$ follows by combining the diagram (**) for the cone σ with the analogous diagram for the cone τ : The facts that π_\bullet^σ is an epimorphism, $\pi_\bullet^\tau = \pi_\bullet^{\sigma, \tau} \circ \pi_\bullet^\sigma$, and $\nu_\tau^\Lambda = \nu_{\sigma, \tau}^\Lambda \circ \nu_\sigma^\Lambda$ — since $\pi^\tau = \pi^{\sigma, \tau} \circ \pi^\sigma$ and $\nu_\tau = \nu_{\sigma, \tau} \circ \nu_\sigma$, respectively — guarantee that $\Phi(\tau) \circ \pi_\bullet^{\sigma, \tau} = \nu_{\sigma, \tau}^\Lambda \circ \Phi(\sigma)$ for $\sigma \prec \tau \in \Delta$.

Finally, the case of a general coefficient module G is obtained by tensoring with the abelian group G : Since the integral homology of the torus $T(\sigma)$ is free abelian and of finite rank, $H_\bullet(T(\sigma); G) \cong H_\bullet(T(\sigma); \mathbb{Z}) \otimes G$ by the universal coefficient formula. \blacksquare

2.1.3 Remark. Since the natural equivalence Φ^Δ in the preceding proposition is of degree 0, it consists of natural equivalences Φ_p^Δ between the graded pieces H_p^Δ and Λ_p^Δ , i.e., natural equivalences between the covariant functors on the category Δ that associate to each cone $\sigma \in \Delta$ the abelian groups $H_p(T(\sigma); G)$ and $\Lambda^p N(\sigma) \otimes G$, and to each face relation $\sigma \prec \tau$ in Δ the canonical epimorphisms

$$\pi_p^{\sigma, \tau}: H_p(T(\sigma); G) \twoheadrightarrow H_p(T(\tau); G)$$

and

$$\nu_{\sigma,\tau}^p: \Lambda^p N(\sigma) \otimes G \rightarrow \Lambda^p N(\tau) \otimes G,$$

respectively.

In order to define a chain complex $C_\bullet(\Delta, F)$ for every covariant functor F from the category Δ to the category of abelian groups, we define as follows *orientation coefficients* $o_{\sigma,\tau} \in \{0, \pm 1\}$ for $\sigma, \tau \in \Delta$ that satisfy Lemma 2.1.4 below: For every cone $\sigma \in \Delta$, we arbitrarily fix an orientation $\text{or}(\sigma)$ of the vector space $\text{lin } \sigma$. (For instance, we may choose any numbering $\varrho_1, \dots, \varrho_d$ of the rays, i.e., 1-dimensional cones, in the fan Δ . For each ray ϱ_i , we denote with v_i the unique primitive lattice point on it. Then there are uniquely determined minimal integers $1 \leq i_1 < \dots < i_s \leq d$ with $s := \dim \text{lin } \sigma$ such that the s -tuple $B(\sigma) := (v_{i_1}, \dots, v_{i_s})$ is a basis of the vector space $\text{lin } \sigma$. Now the orientation $\text{or}(\sigma)$ is defined to be the orientation of the vector space $\text{lin } \sigma$ that is determined by the basis $B(\sigma)$.)

Observe that if the cone σ is a facet of a cone $\tau \in \Delta$, then the vector space $\text{lin } \sigma$ is a hyperplane of $\text{lin } \tau$, and all vectors $v \in \tau \setminus \sigma$ lie in the same half-space of $\text{lin } \tau$ determined by this hyperplane. Hence, the orientation $\text{or}(\sigma)$ of the vector space $\text{lin } \sigma$ — represented by a basis $B(\sigma)$ of $\text{lin } \sigma$ — induces the unique orientation $\text{or}(\sigma, \tau)$ of the vector space $\text{lin } \tau$ that is represented by the basis $B(\sigma, \tau) := (B(\sigma), v)$ of $\text{lin } \tau$, where v is any vector in the set $\tau \setminus \sigma$. Now we set

$$o_{\sigma,\tau} := \begin{cases} +1 & \text{if } \text{or}(\sigma, \tau) = \text{or}(\tau), \\ -1 & \text{otherwise.} \end{cases}$$

Furthermore, if the cone σ is not a facet of the cone τ , we set $o_{\sigma,\tau} := 0$.

Note that every 2-codimensional face ϱ of a cone τ is precisely the intersection of two facets σ_1, σ_2 of the cone τ , see for instance [Ew, Proposition II 1.10]. Then, for any vectors $v_1 \in \sigma_1 \setminus \sigma_2$ and $v_2 \in \sigma_2 \setminus \sigma_1$, we have $v_i \in \sigma_i \setminus \varrho$; i.e., the basis $(B(\varrho), v_i)$ of the vector space $\text{lin } \sigma_i$ represents the orientation

$$o_{\varrho,\sigma_i} \text{or}(\sigma_i)$$

for $i = 1, 2$. Moreover, we have $v_2 \in \tau \setminus \sigma_1$ and $v_1 \in \tau \setminus \sigma_2$; i.e., the bases $(B(\varrho), v_1, v_2)$ and $(B(\varrho), v_2, v_1)$ of the vector space $\text{lin } \tau$ represent the (obviously opposite) orientations

$$o_{\varrho,\sigma_i} \cdot o_{\sigma_i,\tau} \text{or}(\tau)$$

for $i = 1, 2$, respectively. In other words:

2.1.4 Lemma. *Let the cone ϱ be a 2-codimensional face of a cone $\tau \in \Delta$. Then*

$$\sum_{\varrho \prec \sigma \prec \tau} o_{\varrho,\sigma} \cdot o_{\sigma,\tau} = 0. \quad \blacksquare$$

2.1.5 Corollary and Definition. For every fan Δ in $N_{\mathbb{R}}$ and every covariant functor F from the category Δ to the category of abelian groups, the collection

$$C_{\bullet}(\Delta, F) := (C_p(\Delta, F), \delta_p)_{p \in \mathbb{Z}}$$

of the abelian groups

$$C_p(\Delta, F) := \bigoplus_{\sigma \in \Delta^{(n-p)}} F(\sigma)$$

associated with the boundary mappings

$$\delta_p := \sum_{\substack{\sigma \in \Delta^{(n-p)} \\ \tau \in \Delta^{(n-p+1)}}} o_{\sigma, \tau} F(\sigma \prec \tau) : C_p(\Delta, F) \longrightarrow C_{p-1}(\Delta, F)$$

defines a chain complex. The corresponding graded homology is denoted by $H_{\bullet}(\Delta, F)$.

Proof. For each integer p and each cone $\varrho \in \Delta^{n-p-1}$,

$$\delta_p \circ \delta_{p+1}|_{F(\varrho)} : F(\varrho) \longrightarrow C_{p-1}(\Delta, F) = \bigoplus_{\tau \in \Delta^{(n-p+1)}} F(\tau)$$

satisfies

$$\begin{aligned} \delta_p \circ \delta_{p+1}|_{F(\varrho)} &= \delta_p \circ \sum_{\varrho \prec \sigma \in \Delta^{(n-p)}} o_{\varrho, \sigma} F(\varrho \prec \sigma) \\ &= \sum_{\varrho \prec \sigma \in \Delta^{(n-p)}} o_{\varrho, \sigma} \delta_p|_{F(\sigma)} \circ F(\varrho \prec \sigma) \\ &= \sum_{\varrho \prec \sigma \in \Delta^{(n-p)}} \sum_{\sigma \prec \tau \in \Delta^{(n-p+1)}} (o_{\varrho, \sigma} \cdot o_{\sigma, \tau}) F(\sigma \prec \tau) \circ F(\varrho \prec \sigma) \\ &= \sum_{\varrho \prec \sigma \in \Delta^{(n-p)}} \sum_{\sigma \prec \tau \in \Delta^{(n-p+1)}} (o_{\varrho, \sigma} \cdot o_{\sigma, \tau}) F(\varrho \prec \tau) \\ &= \sum_{\varrho \prec \tau \in \Delta^{(n-p+1)}} \left(\sum_{\substack{\sigma \in \Delta^{(n-p)} \\ \varrho \prec \sigma \prec \tau}} o_{\varrho, \sigma} \cdot o_{\sigma, \tau} \right) F(\varrho \prec \tau) = 0 \end{aligned}$$

by Lemma 2.1.4. ■

2.1.6 Example. Let G^{Δ} be the covariant constant functor on the category Δ that associates to each cone σ of a fan Δ in $N_{\mathbb{R}}$ the abelian group G , and to each face relation $\sigma \prec \tau$ in Δ the identity mapping id_G .

(a) For the zero fan $\Delta = \{0\}$, the associated homology groups are

$$H_p(\Delta, G^{\Delta}) \cong C_p(\Delta, G^{\Delta}) = \begin{cases} 0 & \text{for } p \neq n, \\ G & \text{for } p = n. \end{cases}$$

(b) For nonzero fans Δ , the associated chain complex $C_\bullet(\Delta, G^\Delta)$ is isomorphic to the augmented cellular cochain complex $\tilde{C}^{n-\bullet-1}(S_\Delta; G)$ of the finite CW-complex $S_\Delta := |\Delta| \cap S^{n-1} = \bigcup_{\sigma \in \Delta} (\sigma \cap S^{n-1})^\circ$: By the definition of the chain complex $C_\bullet(\Delta, G^\Delta)$, we have

$$\begin{aligned} \delta_p: C_p(\Delta, G^\Delta) &= G^{d_{n-p}(\Delta)} \rightarrow G^{d_{n-p+1}(\Delta)} = C_{p-1}(\Delta, G^\Delta) \\ (g_\sigma)_{\sigma \in \Delta^{(n-p)}} &\mapsto \left(\sum_{\sigma \in \Delta^{(n-p)}} o_{\sigma, \tau} g_\sigma \right)_{\tau \in \Delta^{(n-p+1)}} \end{aligned}$$

for $0 \leq p \leq n$. On the other hand, for $0 \leq p \leq n-1$, the cellular chain group $C_{n-p-1}(S_\Delta)$ is a free abelian group of rank $d_{n-p}(\Delta)$, where a basis is given by the *oriented cells* $c_\sigma := (\sigma \cap S^{n-1})^\circ$ for $\sigma \in \Delta^{n-p}$ with orientations determined by the orientations $\text{or}(\sigma)$ of the vector spaces $\text{lin } \sigma$. The differential

$$\delta_{n-p}: C_{n-p}(S_\Delta) = \bigoplus_{\tau \in \Delta^{(n-p+1)}} \mathbb{Z} c_\tau \longrightarrow \bigoplus_{\sigma \in \Delta^{(n-p)}} \mathbb{Z} c_\sigma = C_{n-p-1}(S_\Delta)$$

is determined by the equations

$$\delta_{n-p}(c_\tau) = \sum_{\sigma \in \Delta^{(n-p)}} o_{\sigma, \tau} c_\sigma$$

for $\tau \in \Delta^{n-p+1}$. Now we have to apply the functor $\text{Hom}(\cdot, G)$: Since the CW-complex S_Δ has only finitely many cells, every homomorphism $C_{n-p-1}(S_\Delta) \rightarrow G$ is of the form

$$\sum_{\sigma \in \Delta^{(n-p)}} z_\sigma c_\sigma \mapsto \sum_{\sigma \in \Delta^{(n-p)}} z_\sigma g_\sigma$$

with $z_\sigma \in \mathbb{Z}$ and unique $g_\sigma \in G$, i.e., the cellular cochain group

$$C^{n-p-1}(S_\Delta; G) = \text{Hom}_{\mathbb{Z}}(C_{n-p-1}(S_\Delta), G) \cong \bigoplus_{\sigma \in \Delta^{(n-p)}} \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, G)$$

is isomorphic to the group $G^{d_{n-p}(\Delta)}$. Moreover, the induced differential

$$\delta^{n-p-1}: C^{n-p-1}(S_\Delta; G) \rightarrow C^{n-p}(S_\Delta; G)$$

maps the homomorphism

$$\sum_{\sigma \in \Delta^{(n-p)}} z_\sigma c_\sigma \mapsto \sum_{\sigma \in \Delta^{(n-p)}} z_\sigma g_\sigma$$

onto the homomorphism

$$\begin{aligned} \sum_{\tau \in \Delta^{(n-p+1)}} z_\tau c_\tau &\mapsto \sum_{\tau \in \Delta^{(n-p+1)}} z_\tau \sum_{\sigma \in \Delta^{(n-p)}} o_{\sigma, \tau} c_\sigma = \sum_{\sigma} \left(\sum_{\tau} z_\tau o_{\sigma, \tau} \right) c_\sigma \\ &\mapsto \sum_{\sigma \in \Delta^{(n-p)}} \left(\sum_{\tau \in \Delta^{(n-p+1)}} z_\tau o_{\sigma, \tau} \right) g_\sigma = \sum_{\tau \in \Delta^{(n-p+1)}} z_\tau g_\tau, \end{aligned}$$

where

$$g_\tau := \sum_{\sigma \in \Delta^{(n-p)}} o_{\sigma, \tau} g_\sigma.$$

Thus, for $0 \leq p \leq n-1$, the chain groups $C_p(\Delta, G^\Delta)$ with respective differentials δ_p correspond to the *reduced* cellular cochain groups $\tilde{C}^{n-p-1}(S_\Delta; G) = C^{n-p-1}(S_\Delta; G)$ with respective differentials $\tilde{\delta}^{n-p-1} = \delta^{n-p-1}$. Finally, the differential

$$\begin{aligned} \delta_n: C_n(\Delta, G^\Delta) = G &\rightarrow G^{d_1(\Delta)} = C_{n-1}(\Delta, G^\Delta) \\ g &\mapsto (g)_{\tau \in \Delta^{(1)}} \end{aligned}$$

corresponds to the differential

$$\begin{aligned} \tilde{\delta}^{-1}: \tilde{C}^{-1}(S_\Delta; G) = G &\rightarrow \text{Hom}\left(\bigoplus_{\tau \in \Delta^{(1)}} \mathbb{Z} c_\tau, G\right) = \tilde{C}^0(S_\Delta; G) \\ g &\mapsto (g)_{\tau \in \Delta^{(1)}}. \end{aligned}$$

As a consequence,

$$H_p(\Delta, G^\Delta) \cong \tilde{H}^{n-p-1}(S_\Delta; G),$$

which, for $p \neq n-1$, is $H^{n-p-1}(S_\Delta; G)$. In particular, $H_n(\Delta, G^\Delta) = 0$ for nonzero fans Δ .

(c) If the fan Δ is complete, then $S_\Delta = S^{n-1}$, and thus

$$H_p(\Delta, G^\Delta) = \begin{cases} G & \text{for } p = 0, \\ 0 & \text{for } p \neq 0, \end{cases}$$

sort of dual to the other “extreme” case where Δ is the zero fan in $N_{\mathbb{R}}$. \blacksquare

The chain complex $C_\bullet(\Delta, F)$ is natural with respect to the covariant functor F in the following sense:

2.1.7 Proposition. *Let F_1 and F_2 be covariant functors from the category Δ to the category of abelian groups. Then every natural transformation $\Phi: F_1 \rightarrow F_2$ induces a chain mapping $f: C_\bullet(\Delta, F_1) \rightarrow C_\bullet(\Delta, F_2)$. Furthermore, if Φ is a natural equivalence, then f is an isomorphism, in particular $H_\bullet(\Delta, F_1) \cong H_\bullet(\Delta, F_2)$.*

Proof. The natural transformation $\Phi: F_1 \rightarrow F_2$ associates to each cone $\sigma \in \Delta$ a homomorphism $\Phi(\sigma): F_1(\sigma) \rightarrow F_2(\sigma)$ such that $\Phi(\tau) \circ F_1(\sigma \prec \tau) = F_2(\sigma \prec \tau) \circ \Phi(\sigma)$ if $\sigma \prec \tau \in \Delta$. For all integers p , define now homomorphisms $f_p: C_p(\Delta, F_1) \rightarrow C_p(\Delta, F_2)$ by $f_p := \sum_{\sigma \in \Delta^{(n-p)}} \Phi(\sigma)$. Then $f := (f_p)_{p \in \mathbb{Z}}$ is a chain mapping, i.e., $f_{p-1} \circ \delta_p^1 = \delta_p^2 \circ f_p$ for all p . The mappings f_p are isomorphisms if the homomorphisms $\Phi(\sigma)$ are isomorphisms, i.e., if Φ is a natural equivalence. \blacksquare

2.2 The Homology Spectral Sequence of a Filtered Chain Complex

To a finite increasing filtration of a chain complex, one may associate a homology spectral sequence, i.e., a sequence $E = (E^r, d^r)_{r \geq 1}$ of (bigraded) differential modules such that E^{r+1} is the homology module of E^r . Under suitable conditions, the spectral sequence “converges” to a limit object E^∞ that allows to determine the homology of the given chain complex up to solving module extension problems. This spectral sequence associated to a filtered chain complex is easily derived from a more refined object called an “exact couple”. The construction is natural with respect to endomorphisms of the filtered chain complex, i.e., such an endomorphism induces an endomorphism of the associated exact couple and hence of the spectral sequence. The most important application for our purpose is the homology spectral sequence associated to a suitable finite increasing filtration of a topological space.

To fix notation and terminology, we recall the notion of a homology spectral sequence and the construction of the spectral sequence associated to a filtered chain complex. Our exposition mainly follows [Sp, Section 9.1] and [Be, Section 3.2]. In the sequel, we consider modules over a fixed commutative ring R ; in particular, abelian groups are included as \mathbb{Z} -modules. A *bigraded module* E over R is a family $(E_{p,q})_{p,q \in \mathbb{Z}}$ of R -modules. A *differential* $d: E \rightarrow E$ of *bidegree* $(-r, r-1)$ is a family

$$(d_{p,q}: E_{p,q} \rightarrow E_{p-r,q+r-1})_{p,q \in \mathbb{Z}}$$

of homomorphisms such that $d \circ d = 0$. The *homology module* $H(E, d)$ is defined to be the bigraded module with

$$H_{p,q}(E, d) := \ker d_{p,q} / \operatorname{im} d_{p+r,q-r+1}.$$

Such a bigraded differential module (E, d) determines a chain complex $(E_\bullet, \partial_\bullet)$ as follows: Set

$$E_\ell := \bigoplus_{p+q=\ell} E_{p,q} \quad \text{and} \quad \partial_\ell := \bigoplus_{p+q=\ell} d_{p,q}: E_\ell \rightarrow E_{\ell-1}.$$

Then there is an isomorphism of homology modules

$$H_\ell(E_\bullet, \partial_\bullet) \cong \bigoplus_{p+q=\ell} H_{p,q}(E, d).$$

A *homology spectral sequence* $(E^r, d^r)_{r \geq 1}$ is a sequence of bigraded modules E^r and differentials $d^r: E^r \rightarrow E^r$ of bidegree $(-r, r-1)$ such that there are isomorphisms $E^{r+1} \cong H(E^r, d^r)$ for $r \geq 1$. The module E^1 is called the *initial term* of the spectral sequence. In order to define its *limit term* E^∞ , identify the module E^{r+1} with the module $H(E^r, d^r)$. Construct a tower of submodules

$$B^1 \subset B^2 \subset \cdots \subset B^r \subset \cdots \subset B^\infty \subset Z^\infty \subset \cdots \subset Z^r \subset \cdots \subset Z^2 \subset Z^1$$

of the module E^1 such that $E^{r+1} = Z^r/B^r$ as follows: Define the bigraded modules $Z^1 := (Z_{p,q}^1)_{p,q \in \mathbb{Z}}$ by

$$Z_{p,q}^1 := \ker [d_{p,q}^1: E_{p,q}^1 \rightarrow E_{p-1,q}^1]$$

and $B^1 := (B_{p,q}^1)_{p,q \in \mathbb{Z}}$ by

$$B_{p,q}^1 := \text{im} [d_{p+1,q}^1: E_{p+1,q}^1 \rightarrow E_{p,q}^1].$$

Then $B^1 \subset Z^1$, since $d \circ d = 0$, and the module $E^2 = H(E^1, d^1) = Z^1/B^1$ is a subquotient of the module E^1 . Now consider the bigraded modules $Z(E^2) := (Z(E^2)_{p,q})_{p,q \in \mathbb{Z}}$ with

$$Z(E^2)_{p,q} := \ker [d_{p,q}^2: E_{p,q}^2 \rightarrow E_{p-2,q+1}^2]$$

and $B(E^2) := (B(E^2)_{p,q})_{p,q \in \mathbb{Z}}$ with

$$B(E^2)_{p,q} := \text{im} [d_{p+2,q-1}^2: E_{p+2,q-1}^2 \rightarrow E_{p,q}^2].$$

By the Noether isomorphism theorem, there exist bigraded submodules Z^2 and B^2 of the module Z^1 containing the module B^1 such that $Z(E^2)_{p,q} = Z_{p,q}^2/B_{p,q}^1$ and $B(E^2)_{p,q} = B_{p,q}^2/B_{p,q}^1$ for all integers p, q . Then $B^2 \subset Z^2$, since $B(E^2) \subset Z(E^2)$, and the tower of submodules may be constructed by induction. Finally, define the bigraded modules $Z^\infty := \bigcap_{r \in \mathbb{Z}} Z^r$ and $B^\infty := \bigcup_{r \in \mathbb{Z}} B^r$. Then $B^\infty \subset Z^\infty$, and the bigraded module $E^\infty := Z^\infty/B^\infty$ is called the *limit term* of the spectral sequence $(E^r, d^r)_{r \geq 1}$.

The spectral sequence $(E^r, d^r)_{r \geq 1}$ converges if, for each pair (p, q) of integers, there exists an integer $r(p, q) \geq 1$ such that the differentials $d_{p,q}^r: E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ vanish for $r \geq r(p, q)$. In this case the module $E_{p,q}^{r+1}$ is isomorphic to a quotient of the module $E_{p,q}^r$ for $r \geq r(p, q)$, and the limit term $E_{p,q}^\infty$ is isomorphic to the direct limit of the sequence

$$E_{p,q}^{r(p,q)} \rightarrow E_{p,q}^{r(p,q)+1} \rightarrow \dots$$

A *first-quadrant* homology spectral sequence $(E^r, d^r)_{r \geq 1}$ is a spectral sequence with the property that, for some r , $E_{p,q}^r = 0$ if $p < 0$ or $q < 0$. Such a spectral sequence converges even in a stronger sense: For $r' > \max(r-1, p, q+1)$, we have

$$0 = E_{p+r',q-r'+1}^{r'} \xrightarrow{d^{r'}} E_{p,q}^{r'} \xrightarrow{d^{r'}} E_{p-r',q+r'-1}^{r'} = 0,$$

and hence

$$E_{p,q}^{r'} \cong E_{p,q}^{r'+1} \cong \dots \cong E_{p,q}^\infty.$$

Furthermore, for every integer ℓ , there are only a finite number of nonzero modules $E_{p,q}^\infty$ with $p+q = \ell$.

Every filtration of a chain complex induces a homology spectral sequence. In order to avoid detailed discussions of convergence problems, we restrict ourselves to the following setting: We suppose that

$$0 = \mathbf{C}_{-1} \subset \mathbf{C}_0 \subset \cdots \subset \mathbf{C}_{n-1} \subset \mathbf{C}_n = \mathbf{C}$$

is a *finite* increasing filtration of a chain complex \mathbf{C} by subcomplexes \mathbf{C}_p , where we set $\mathbf{C}_p := 0$ for $p < -1$ and $\mathbf{C}_p := \mathbf{C}$ for $p > n$.

(For us, the most important example is obtained as follows: A finite increasing filtration

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_{n-1} \subset X_n = X$$

of a topological space X by closed subspaces X_p determines a finite increasing filtration (\mathbf{C}_p) of the chain complex $\mathbf{C} := C_{\bullet}^{\text{cld}}(X; G)$ of locally finite singular chains on X with values in an abelian group G by the subcomplexes $\mathbf{C}_p := C_{\bullet}^{\text{cld}}(X_p; G)$.)

Then the long exact homology sequences

$$\cdots \xrightarrow{\delta} H_{\ell}(\mathbf{C}_{p-1}) \xrightarrow{i} H_{\ell}(\mathbf{C}_p) \xrightarrow{j} H_{\ell}(\mathbf{C}_p, \mathbf{C}_{p-1}) \xrightarrow{\delta} H_{\ell-1}(\mathbf{C}_{p-1}) \xrightarrow{i} \cdots$$

combine to form a commutative diagram:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
 \cdots & \xrightarrow{\delta} & H_{\ell}(\mathbf{C}_{p-1}) & \longrightarrow & H_{\ell}(\mathbf{C}_{p-1}, \mathbf{C}_{p-2}) & \longrightarrow & H_{\ell-1}(\mathbf{C}_{p-2}) \longrightarrow \cdots \\
 & & i \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & H_{\ell}(\mathbf{C}_p) & \xrightarrow{j} & H_{\ell}(\mathbf{C}_p, \mathbf{C}_{p-1}) & \xrightarrow{\delta} & H_{\ell-1}(\mathbf{C}_{p-1}) \longrightarrow \cdots \\
 & & \downarrow & & \downarrow i & & \\
 \cdots & \longrightarrow & H_{\ell}(\mathbf{C}_{p+1}) & \longrightarrow & H_{\ell}(\mathbf{C}_{p+1}, \mathbf{C}_p) & \longrightarrow & H_{\ell-1}(\mathbf{C}_p) \xrightarrow{j} \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
 \end{array}$$

Figure 2.1: Homology diagram associated to a filtered chain complex.

Any sequence in this diagram consisting of a vertical mapping i followed by two horizontal mappings j and δ (and iteration of this) is exact. That situation may very conveniently be expressed in the language of “exact couples”: We set

$$E_{p,q} := H_{p+q}(\mathbf{C}_p, \mathbf{C}_{p-1})$$

and

$$D_{p,q} := H_{p+q}(\mathbf{C}_p)$$

and form the triangle of bigraded modules

$$\begin{array}{ccc} D_{\bullet,\bullet} & \xrightarrow{i} & D_{\bullet,\bullet} \\ \delta \swarrow & & \searrow j \\ & E_{\bullet,\bullet} & \end{array}$$

with $\deg(i) = (1, -1)$, $\deg(j) = (0, 0)$, and $\deg(\delta) = (-1, 0)$. Then this is an *exact triangle*, i.e., any two consecutive arrows form an exact sequence. (D, E) or, more precisely, $(D, E; i, j, \delta)$ is also called an *exact couple*.

We now sketch how an exact couple (D, E) of bigraded modules determines a *derived couple* (D', E') as follows: Setting $d := j \circ \delta$, we have $d \circ d = 0$, since $\delta \circ j = 0$. Thus we may define the homology $E' := H(E, d) := \ker(d)/\text{im}(d)$ as a subquotient of E . Furthermore, we set

$$D' := \text{im}(i) \subset D,$$

$$i' := i|_{D'},$$

$$j'(i(x)) := j(x) + \text{im}(d) \in \ker(d)/\text{im}(d),$$

and

$$\delta'(z + \text{im}(d)) := \delta(z)$$

with $\deg(i') = (1, -1)$, $\deg(j') = (-1, 1)$, and $\deg(\delta') = (-1, 0)$. By diagram chase it is easy to check that these mappings are well-defined, and that the derived couple is again an exact couple. Hence, starting from $(D^1, E^1; i^1, j^1, \delta^1) := (D, E; i, j, \delta)$ and iterating the construction, we may define higher derived couples

$$(D^r, E^r; i^r, j^r, \delta^r) := ((D^{r-1})', (E^{r-1})'; (i^{r-1})', (j^{r-1})', (\delta^{r-1})')$$

for $r \geq 1$. We claim that the sequence $(E^r, d^r := j^r \circ \delta^r)$ is a homology spectral sequence. In fact, since

$$\begin{aligned} \deg(i^r) &= \deg(i^{r-1}) = (1, -1), \\ \deg(j^r) &= \deg(j^{r-1}) - \deg(i^{r-1}) = (1 - r, r - 1), \\ \deg(\delta^r) &= \deg(\delta^{r-1}) = (-1, 0), \end{aligned}$$

we have

$$\deg(d^r) = \deg(j^r) + \deg(\delta^r) = (-r, r - 1)$$

and $E^{r+1} = H(E^r, d^r)$, i.e., we have defined a homology spectral sequence as introduced at the beginning of this section. It is called the spectral sequence determined by the exact couple (D, E) , and it converges to the homology of the chain complex \mathbf{C} , see [Be, remarks preceding Proposition 3.2.11, p. 107]:

2.2.1 Theorem. *For every finite increasing filtration*

$$0 = \mathbf{C}_{-1} \subset \mathbf{C}_0 \subset \cdots \subset \mathbf{C}_{n-1} \subset \mathbf{C}_n = \mathbf{C}$$

of a chain complex \mathbf{C} , we have

$$E_{p,q}^1 = H_{p+q}(\mathbf{C}_p, \mathbf{C}_{p-1}) \implies H_{p+q}(\mathbf{C}),$$

i.e., the limit term E^∞ of the associated homology spectral sequence $(E^r, d^r)_{r \geq 1}$ is isomorphic to the bigraded module associated to the filtration of the homology module $H_\bullet(\mathbf{C})$ defined by

$$F_p H_\bullet(\mathbf{C}) := \text{im}[H_\bullet(\mathbf{C}_p) \rightarrow H_\bullet(\mathbf{C})],$$

that is $E_{p,q}^\infty \cong F_p H_{p+q}(\mathbf{C}) / F_{p-1} H_{p+q}(\mathbf{C})$ for all integers p, q . ■

Under some additional conditions we can recover the homology module $H_\ell(\mathbf{C})$ from the limit modules $E_{p,q}^\infty$ with $p + q = \ell$:

2.2.2 Lemma. *Let ℓ be a natural number. Assume that there are natural numbers $p \leq \tilde{p}$ for the homology spectral sequence in the preceding theorem such that the modules $E_{k,\ell-k}^\infty$ are free for $p < k \leq \tilde{p}$, and $E_{k,\ell-k}^\infty = 0$ for $k \notin [p, \tilde{p}]$. Then*

$$H_\ell(\mathbf{C}) \cong \bigoplus_{k=p}^{\tilde{p}} E_{k,\ell-k}^\infty.$$

Proof. Since $\mathbf{C}_{p'} = \mathbf{C}$ for $p' \geq n$, and

$$0 = E_{p',\ell-p'}^\infty \cong \text{im}[H_\ell(\mathbf{C}_{p'}) \rightarrow H_\ell(\mathbf{C})] / \text{im}[H_\ell(\mathbf{C}_{p'-1}) \rightarrow H_\ell(\mathbf{C})]$$

for $p' > \tilde{p}$, it follows that

$$F_{p'} H_\ell(\mathbf{C}) = \text{im}[H_\ell(\mathbf{C}_{p'}) \rightarrow H_\ell(\mathbf{C})] \cong H_\ell(\mathbf{C})$$

for $p' \geq \tilde{p}$. Analogously, since $\mathbf{C}_{p'} = 0$ for $p' \leq -1$, and $0 = E_{p',\ell-p'}^\infty$ for $p' < p$, it follows that $F_{p'} H_\ell(\mathbf{C}) \cong 0$ for $p' < p$. Hence, if $\tilde{p} = p$, then

$$E_{p,\ell-p}^\infty \cong F_p H_\ell(\mathbf{C}) / F_{p-1} H_\ell(\mathbf{C}) \cong H_\ell(\mathbf{C}) / 0 \cong H_\ell(\mathbf{C}).$$

If $\tilde{p} = p + 1$, then $E_{p,\ell-p}^\infty \cong F_p H_\ell(\mathbf{C})$ and $E_{p+1,\ell-p-1}^\infty \cong H_\ell(\mathbf{C})/F_p H_\ell(\mathbf{C})$. The canonical exact sequence

$$0 \longrightarrow F_p H_\ell(\mathbf{C}) \longrightarrow H_\ell(\mathbf{C}) \longrightarrow H_\ell(\mathbf{C})/F_p H_\ell(\mathbf{C}) \longrightarrow 0$$

splits, i.e.,

$$H_\ell(\mathbf{C}) \cong E_{p,\ell-p}^\infty \oplus E_{p+1,\ell-p-1}^\infty,$$

since the module $E_{p+1,\ell-p-1}^\infty$ is free. If $\tilde{p} = p + 2$, then $E_{p,\ell-p}^\infty = F_p H_\ell(\mathbf{C})$, $E_{p+1,\ell-p-1}^\infty \cong F_{p+1} H_\ell(\mathbf{C})/E_{p,\ell-p}^\infty$, and $E_{p+2,\ell-p-2}^\infty \cong H_\ell(\mathbf{C})/F_{p+1} H_\ell(\mathbf{C})$. The canonical exact sequence

$$0 \longrightarrow F_{p+1} H_\ell(\mathbf{C}) \longrightarrow H_\ell(\mathbf{C}) \longrightarrow H_\ell(\mathbf{C})/F_{p+1} H_\ell(\mathbf{C}) \longrightarrow 0$$

splits, i.e.,

$$H_\ell(\mathbf{C}) \cong F_{p+1} H_\ell(\mathbf{C}) \oplus E_{p+2,\ell-p-2}^\infty,$$

since the module $E_{p+2,\ell-p-2}^\infty$ is free. Furthermore, the canonical exact sequence

$$0 \longrightarrow F_p H_\ell(\mathbf{C}) \longrightarrow F_{p+1} H_\ell(\mathbf{C}) \longrightarrow F_{p+1} H_\ell(\mathbf{C})/E_{p,\ell-p}^\infty \longrightarrow 0$$

splits, i.e.,

$$F_{p+1} H_\ell(\mathbf{C}) \cong E_{p,\ell-p}^\infty \oplus E_{p+1,\ell-p-1}^\infty,$$

since the module $E_{p+1,\ell-p-1}^\infty$ is free. Finally, we may argue analogously if $\tilde{p} \geq p+3$. ■

Every endomorphism $\kappa: \mathbf{C} \rightarrow \mathbf{C}$ that respects the filtration of the chain complex \mathbf{C} , i.e., $\kappa(\mathbf{C}_p) \subset \mathbf{C}_p$ for all integers p , induces an endomorphism of the associated exact couple and hence of the spectral sequence as follows: By the induced endomorphisms

$$\kappa_* : H_\bullet(\mathbf{C}_*) \rightarrow H_\bullet(\mathbf{C}_*)$$

and

$$\bar{\kappa}_* : H_\bullet(\mathbf{C}_*, \mathbf{C}_{*-1}) \rightarrow H_\bullet(\mathbf{C}_*, \mathbf{C}_{*-1}),$$

we may combine two copies of the big commutative homology diagram from Figure 2.1 (p. 34) to a “sandwich”. Looking at the consecutive mappings i , j , and δ and again writing $E_{p,q} := H_{p+q}(\mathbf{C}_p, \mathbf{C}_{p-1})$ and $D_{p,q} := H_{p+q}(\mathbf{C}_p)$, we obtain a long exact “ladder”

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\delta} & D_{p-1,q+1} & \xrightarrow{i} & D_{p,q} & \xrightarrow{j} & E_{p,q} & \xrightarrow{\delta} & D_{p-1,q} & \xrightarrow{i} & \cdots \\ & & \downarrow \kappa_{p-1} & & \downarrow \kappa_p & & \downarrow \bar{\kappa}_p & & \downarrow \kappa_{p-1} & & \\ \cdots & \xrightarrow{\delta} & D_{p-1,q+1} & \xrightarrow{i} & D_{p,q} & \xrightarrow{j} & E_{p,q} & \xrightarrow{\delta} & D_{p-1,q} & \xrightarrow{i} & \cdots \end{array}$$

In other words, we have an endomorphism of the corresponding exact couple:

$$\begin{array}{ccc} \kappa \circ D_{\bullet,\bullet} & \xrightarrow{i} & D_{\bullet,\bullet} \circ \kappa \\ & \delta \swarrow \quad \searrow j & \\ & E_{\bullet,\bullet} & \\ & \circ \bar{\kappa} & \end{array}$$

By diagram chase it is easy to check that the mappings

$$\kappa' := \kappa|_{D'},$$

$$\bar{\kappa}'(z + \text{im}(d)) := \bar{\kappa}(z) + \text{im}(d)$$

provide a well-defined endomorphism of the derived exact couple:

$$\begin{array}{ccc} \kappa' \circ D'_{\bullet,\bullet} & \xrightarrow{i'} & D'_{\bullet,\bullet} \circ \kappa' \\ & \delta' \swarrow \quad \searrow j' & \\ & E'_{\bullet,\bullet} & \\ & \circ \bar{\kappa}' & \end{array}$$

By induction, we may thus define endomorphisms $\bar{\kappa}^r : E^r \rightarrow E^r$ of the homology spectral sequence $(E^r, d^r)_{r \geq 1}$ that, in particular, commute with the differentials $d^r = j^r \circ \delta^r : E^r \rightarrow E^r$.

2.3 The Toric Homology Spectral Sequence

The orbit structure of the natural torus action on a toric variety X_Δ induces a finite increasing filtration of X_Δ . Hence, for every homology theory, there is a spectral sequence that converges to the respective homology of X_Δ . We study this toric spectral sequence for homology with closed supports and coefficients in an abelian group. We show that the differentials $d_{p,q}^1 : E_{p,q}^1 \rightarrow E_{p-1,q}^1$ reflect the homology of the fibers of the canonical projection $\pi : X_\Delta \rightarrow B_\Delta$ and the transition mappings between them.

For every fan Δ in $N_{\mathbb{R}}$, there is a canonical finite increasing filtration

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_{n-1} \subset X_n = X_\Delta$$

of the toric variety X_Δ given by the closed subvarieties

$$X_p := \bigcup_{\sigma \in \Delta^{(n-p)}} \bar{O}_\sigma = \bigcup_{\sigma \in \Delta^{(n-p)}} \pi^{-1}(\bar{\sigma}) = \bigcup_{\sigma \in \Delta^{(\geq n-p)}} \overset{\bullet}{O}_\sigma = \bigcup_{\sigma \in \Delta^{(\geq n-p)}} \pi^{-1}(\bar{\sigma})$$

for all integers p , where $X_p = \emptyset$ for $p \leq -1$ and $X_p = X_\Delta$ for $p \geq n$, i.e., the closed subvariety X_p is the union of the closures of the p -dimensional orbits of

the natural torus action on X_Δ or, equivalently, it is the disjoint union of the orbits of dimension at most p .

By Theorem 2.2.1, the corresponding finite increasing filtration of the chain complex $C_\bullet^{\text{cld}}(X_\Delta; G)$ given by the subcomplexes $C_\bullet^{\text{cld}}(X_p; G)$ for homology with closed supports and coefficients in an abelian group G induces a convergent homology spectral sequence $(E^r, d^r)_{r \geq 1}$ with initial term

$$E_{p,q}^1 = H_{p+q}^{\text{cld}}(X_p, X_{p-1}; G) \cong H_{p+q}^{\text{cld}}(X_p \setminus X_{p-1}; G)$$

and differential $d_{p,q}^1: E_{p,q}^1 \rightarrow E_{p-1,q}^1$ given by the connecting homomorphism $j^1 \circ \delta^1$ in the long exact homology sequence of the triple of spaces (X_p, X_{p-1}, X_{p-2}) , where

$$\delta^1: E_{p,q}^1 = H_{p+q}^{\text{cld}}(X_p, X_{p-1}; G) \longrightarrow D_{p-1,q}^1 = H_{p+q-1}^{\text{cld}}(X_{p-1}; G)$$

and

$$j^1: D_{p-1,q}^1 = H_{p+q-1}^{\text{cld}}(X_{p-1}; G) \longrightarrow E_{p-1,q}^1 = H_{p+q-1}^{\text{cld}}(X_{p-1}, X_{p-2}; G).$$

The limit term E^∞ of this *toric homology spectral sequence* is isomorphic to the bigraded module associated to the filtration of the homology module $H_\bullet^{\text{cld}}(X_\Delta; G)$ defined by

$$F_p H_\bullet^{\text{cld}}(X_\Delta; G) := \text{im} [H_\bullet^{\text{cld}}(X_p; G) \rightarrow H_\bullet^{\text{cld}}(X_\Delta; G)],$$

that is $E_{p,q}^\infty \cong F_p H_{p+q}^{\text{cld}}(X_\Delta; G) / F_{p-1} H_{p+q}^{\text{cld}}(X_\Delta; G)$ for all integers p, q .

In the sequel, we shall show how the differential $d^1: E^1 \rightarrow E^1$ may be described with the help of the isomorphic chain complexes $C_\bullet(\Delta, H_\bullet^\Delta) \cong C_\bullet(\Delta, \Lambda_\bullet^\Delta)$. We begin with a look at the relative homology groups $H_{p+q}^{\text{cld}}(X_p, X_{p-1}; G) = E_{p,q}^1$:

2.3.1 Lemma. *For all integers p, q , the module $E_{p,q}^1$ may be written as a direct sum of absolute homology groups of orbits:*

$$E_{p,q}^1 = H_{p+q}^{\text{cld}}(X_p, X_{p-1}; G) \cong \bigoplus_{\sigma \in \Delta^{(n-p)}} H_{p+q}^{\text{cld}}(O_\sigma; G).$$

Proof.

$$H_{p+q}^{\text{cld}}(X_p, X_{p-1}) \cong H_{p+q}^{\text{cld}}(X_p \setminus X_{p-1}) = H_{p+q}^{\text{cld}}\left(\bigcup_{\sigma \in \Delta^{(n-p)}} O_\sigma\right) \cong \bigoplus_{\sigma \in \Delta^{(n-p)}} H_{p+q}^{\text{cld}}(O_\sigma) \blacksquare$$

Thus we may decompose the differential $d_{p,q}^1: E_{p,q}^1 \rightarrow E_{p-1,q}^1$ into homomorphisms

$$H_{p+q}^{\text{cld}}(O_\sigma) = H_{p+q}^{\text{cld}}(\tilde{\sigma} \times T(\sigma)) \xrightarrow{\delta_\bullet} H_{p+q-1}^{\text{cld}}(O_\tau) = H_{p+q-1}^{\text{cld}}(\tilde{\tau} \times T(\tau))$$

for $\Delta^{n-p} \ni \sigma \prec \tau \in \Delta^{n-p+1}$. Recall that we have written the complex orbit O_σ of complex dimension p corresponding to the cone $\sigma \in \Delta^{n-p}$ as a product $\tilde{\sigma} \times T(\sigma)$ of an open p -cell $\tilde{\sigma}$ and a compact real p -torus $T(\sigma)$.

Step by step, these induced homomorphisms δ_\bullet will be investigated until the following big commutative diagram arises, which expresses the differential $d_{p,q}^1: E_{p,q}^1 \rightarrow E_{p-1,q}^1$ in terms of transition mappings $\pi_{\bullet}^{\sigma,\tau}: H_q(T(\sigma)) \rightarrow H_q(T(\tau))$ between homology groups of compact real tori. Note that the coefficient module of all homology groups is assumed to be the abelian group G unless otherwise stated:

$$\begin{array}{ccc}
H_{p+q}^{\text{clid}}(\tilde{\sigma} \times T(\sigma)) & \xrightarrow{\delta_\bullet} & H_{p+q-1}^{\text{clid}}(\tilde{\tau} \times T(\tau)) \\
\cong \searrow & & \nearrow \\
& (1) & \\
& H_{p+q-1}^{\text{clid}}(\tilde{\tau} \times T(\sigma)) & \\
\cong \left| & (2) & \left| \cong & (3) & \left| \cong \\
H_p^{\text{clid}}(\tilde{\sigma}; \mathbb{Z}) \otimes H_q(T(\sigma)) & \xrightarrow{\cong} & H_{p-1}^{\text{clid}}(\tilde{\tau}; \mathbb{Z}) \otimes H_q(T(\sigma)) & \longrightarrow & H_{p-1}^{\text{clid}}(\tilde{\tau}; \mathbb{Z}) \otimes H_q(T(\tau)) \\
\cong \left| & (4) & \left| \cong & (5) & \left| \cong \\
H_q(T(\sigma)) & \xrightarrow[\cong]{o_{\sigma,\tau} \cdot \text{id}} & H_q(T(\sigma)) & \xrightarrow{\pi_{\bullet}^{\sigma,\tau}} & H_q(T(\tau))
\end{array}$$

Figure 2.2: Diagram for the explicit computation of the differential $d_{p,q}^1: E_{p,q}^1 \rightarrow E_{p-1,q}^1$ of the toric homology spectral sequence.

First, consider for each cone $\sigma \in \Delta$ the continuous mapping

$$f^\sigma: \bar{\sigma} \times T(\sigma) \longrightarrow V(\sigma)$$

defined by

$$f^\sigma(x, t) := (x, \pi^{\sigma, \sigma_0}(t)),$$

where $\sigma_0 \in \Delta$ denotes the unique cone with $x \in \tilde{\sigma}_0$. (Recall that $\bar{\sigma} = \bigcup_{\sigma \prec v \in \Delta} \tilde{v}$, $V(\sigma) = \bigcup_{\sigma \prec v \in \Delta} \tilde{v} \times T(v)$, and $\pi^{\sigma, \sigma_0}: T(\sigma) \rightarrow T(\sigma_0)$ is the canonical projection.) Since

$$f^\sigma(\partial \bar{\sigma} \times T(\sigma)) = \partial O_\sigma$$

and

$$f^\sigma((\partial\bar{\sigma} \setminus \tilde{\tau}) \times T(\sigma)) = \partial O_\sigma \setminus O_\tau,$$

the mapping f^σ induces the following commutative diagram:

$$\begin{array}{ccc} H_{p+q}^{\text{clid}}(V(\sigma), \partial O_\sigma) & \xrightarrow{\delta_\bullet} & H_{p+q-1}^{\text{clid}}(\partial O_\sigma, \partial O_\sigma \setminus O_\tau) \\ \uparrow f_\bullet & (1') & \uparrow (f^\sigma|_{\partial\bar{\sigma} \times T(\sigma)})_\bullet \\ H_{p+q}^{\text{clid}}((\bar{\sigma}, \partial\bar{\sigma}) \times T(\sigma)) & \xrightarrow{\delta_\bullet} & H_{p+q-1}^{\text{clid}}((\partial\bar{\sigma}, \partial\bar{\sigma} \setminus \tilde{\tau}) \times T(\sigma)), \end{array}$$

where δ_\bullet always denotes the connecting homomorphism in the long exact homology sequence of the respective triple of spaces.

2.3.2 Lemma. *The commutative diagram (1') induces the commutative diagram*

$$\begin{array}{ccc} H_{p+q}^{\text{clid}}(\tilde{\sigma} \times T(\sigma)) & \xrightarrow{\delta_\bullet} & H_{p+q-1}^{\text{clid}}(\tilde{\tau} \times T(\tau)) \\ \delta_\bullet \searrow & (1) & \nearrow (\text{id}_{\tilde{\tau}} \times \pi^{\sigma, \tau})_\bullet \\ & & H_{p+q-1}^{\text{clid}}(\tilde{\tau} \times T(\sigma)) \end{array}$$

Proof. Apply the natural isomorphism $H_\bullet^{\text{clid}}(A, B) \cong H_\bullet^{\text{clid}}(A \setminus B)$ and use the fact that $f^\sigma|_{\tilde{\sigma} \times T(\sigma)} = \text{id}_{\tilde{\sigma} \times T(\sigma)}$ and $f_\bullet^\sigma|_{\tilde{\tau} \times T(\sigma)} = \text{id}_{\tilde{\tau}} \times \pi^{\sigma, \tau}$. \blacksquare

Second, we calculate the homology groups $H_k^{\text{clid}}(\tilde{\alpha} \times T(\beta); G)$ via Poincaré duality and the Künneth formula for cohomology:

2.3.3 Proposition. *For all cones $\alpha, \beta \in \Delta$ and all integers k , there is a natural isomorphism*

$$H_k^{\text{clid}}(\tilde{\alpha} \times T(\beta)) \cong H_{\dim \tilde{\alpha}}^{\text{clid}}(\tilde{\alpha}; \mathbb{Z}) \otimes H_{k - \dim \tilde{\alpha}}(T(\beta)).$$

Proof. We set $a := \dim \tilde{\alpha} = n - \dim \alpha$ and $b := \dim T(\beta) = n - \dim \beta$. Since $\tilde{\alpha}$ and $T(\beta)$ are smooth, we may apply Poincaré duality:

$$H_k^{\text{clid}}(\tilde{\alpha} \times T(\beta)) \cong H^{a+b-k}(\tilde{\alpha} \times T(\beta)).$$

Furthermore, since $\tilde{\alpha}$ is in addition contractible,

$$H_\ell^{\text{clid}}(\tilde{\alpha}; \mathbb{Z}) \cong H^{a-\ell}(\tilde{\alpha}; \mathbb{Z}) \cong \begin{cases} 0 & \text{for } \ell \neq a, \\ \mathbb{Z} & \text{for } \ell = a. \end{cases}$$

Hence, the Künneth formula for cohomology implies

$$H^{a+b-k}(\tilde{\alpha} \times T(\beta)) \cong H^0(\tilde{\alpha}; \mathbb{Z}) \otimes H^{a+b-k}(T(\beta)).$$

Finally, Poincaré duality yields

$$H^0(\tilde{\alpha}; \mathbb{Z}) \otimes H^{a+b-k}(T(\beta)) \cong H_a^{\text{cld}}(\tilde{\alpha}; \mathbb{Z}) \otimes H_{k-a}(T(\beta)). \quad \blacksquare$$

Now we apply this natural isomorphism to the homology groups in the Diagram (1) above and obtain the commutative diagrams

$$\begin{array}{ccc} H_{p+q}^{\text{cld}}(\tilde{\sigma} \times T(\sigma)) & \xrightarrow{\delta_\bullet} & H_{p+q-1}^{\text{cld}}(\tilde{\tau} \times T(\sigma)) \\ \cong \Big| & (2) & \Big| \cong \\ H_p^{\text{cld}}(\tilde{\sigma}; \mathbb{Z}) \otimes H_q(T(\sigma)) & \xrightarrow{\delta_\bullet \otimes \text{id}} & H_{p-1}^{\text{cld}}(\tilde{\tau}; \mathbb{Z}) \otimes H_q(T(\sigma)) \end{array}$$

and

$$\begin{array}{ccc} H_{p+q-1}^{\text{cld}}(\tilde{\tau} \times T(\sigma)) & \xrightarrow{(\text{id}_{\tilde{\tau}} \times \pi^{\sigma, \tau})_\bullet} & H_{p+q-1}^{\text{cld}}(\tilde{\tau} \times T(\tau)) \\ \cong \Big| & (3) & \Big| \cong \\ H_{p-1}^{\text{cld}}(\tilde{\tau}; \mathbb{Z}) \otimes H_q(T(\sigma)) & \xrightarrow{\text{id} \otimes \pi^{\sigma, \tau}_\bullet} & H_{p-1}^{\text{cld}}(\tilde{\tau}; \mathbb{Z}) \otimes H_q(T(\tau)) \end{array} .$$

Third and last, observe that

$$H_p^{\text{cld}}(\tilde{\sigma}; \mathbb{Z}) \otimes H_q(T(\sigma)) \cong \mathbb{Z} \otimes G_q^{\binom{p}{q}} \cong H_q(T(\sigma))$$

and

$$H_{p-1}^{\text{cld}}(\tilde{\tau}; \mathbb{Z}) \otimes H_q(T(\sigma)) \cong \mathbb{Z} \otimes G_q^{\binom{p}{q}} \cong H_q(T(\sigma)),$$

i.e., the homomorphism

$$\delta_\bullet \otimes \text{id}: H_p^{\text{cld}}(\tilde{\sigma}; \mathbb{Z}) \otimes H_q(T(\sigma)) \longrightarrow H_{p-1}^{\text{cld}}(\tilde{\tau}; \mathbb{Z}) \otimes H_q(T(\sigma))$$

in the Diagram (2) above is an isomorphism, namely $\pm \text{id}_{H_q(T(\sigma))}$. (As a consequence, the homomorphism

$$\delta_\bullet: H_{p+q}^{\text{cld}}(\tilde{\sigma} \times T(\sigma)) \longrightarrow H_{p+q-1}^{\text{cld}}(\tilde{\tau} \times T(\sigma))$$

in the same commutative diagram is an isomorphism, too.) In fact, we may assume that the diagram

$$\begin{array}{ccc}
 H_p^{\text{cld}}(\tilde{\sigma}; \mathbb{Z}) \otimes H_q(T(\sigma)) & \xrightarrow[\cong]{\delta_{\bullet} \otimes \text{id}} & H_{p-1}^{\text{cld}}(\tilde{\tau}; \mathbb{Z}) \otimes H_q(T(\sigma)) \\
 \cong \Big| & (4) & \Big| \cong \\
 H_q(T(\sigma)) & \xrightarrow[\cong]{o_{\sigma, \tau} \cdot \text{id}} & H_q(T(\sigma))
 \end{array}$$

is commutative, namely if, for a fixed orientation of the sphere S^{n-1} , we choose the isomorphisms

$$H_p^{\text{cld}}(\tilde{\sigma}; \mathbb{Z}) \otimes H_q(T(\sigma)) \cong H_q(T(\sigma))$$

and

$$H_{p-1}^{\text{cld}}(\tilde{\tau}; \mathbb{Z}) \otimes H_q(T(\sigma)) \cong H_q(T(\sigma))$$

corresponding to the unique orientations of the cells $\tilde{\sigma}, \tilde{\tau}$ induced by the orientations $o(\sigma), o(\tau)$ of the vector spaces $\text{lin } \sigma, \text{lin } \tau$. Finally, the commutativity of the diagram

$$\begin{array}{ccc}
 H_{p-1}^{\text{cld}}(\tilde{\tau}; \mathbb{Z}) \otimes H_q(T(\sigma)) & \xrightarrow{\text{id} \otimes \pi_{\bullet}^{\sigma, \tau}} & H_{p-1}^{\text{cld}}(\tilde{\tau}; \mathbb{Z}) \otimes H_q(T(\tau)) \\
 \cong \Big| & (5) & \Big| \cong \\
 H_q(T(\sigma)) & \xrightarrow{\pi_{\bullet}^{\sigma, \tau}} & H_q(T(\tau))
 \end{array}$$

is obvious.

2.4 Evaluation of the Toric Homology Spectral Sequence

For homology with closed supports and coefficients in an abelian group G , the toric homology spectral sequence $(E^r, d^r)_{r \geq 1}$ associated to a toric variety X_{Δ} is a first-quadrant spectral sequence. Its modules $E_{p,q}^2$ are computable as homology groups $H_p(\Delta, H_q^{\Delta}) \cong H_p(\Delta, \Lambda_q^{\Delta})$, since the differentials $d_{p,q}^1: E_{p,q}^1 \rightarrow E_{p-1,q}^1$ correspond to the differentials $\delta_p: C_p(\Delta, H_q^{\Delta}) \rightarrow C_{p-1}(\Delta, H_q^{\Delta})$ for all integers p, q . If the coefficient group G is divisible — e.g., for rational coefficients —, then the spectral sequence degenerates at the E^2 -level, i.e., all differentials d^r vanish for $r \geq 2$. In that case, the E^2 -term already coincides with the limit term E^{∞} . This

degeneration property is proved by studying the action of the multiplicative monoid $\mathbb{N}_{>0}$ on the toric variety X_Δ that is induced by scalar multiplication on the lattice N . The consideration of this action was inspired by the preprint [To] of Burt Totaro. Besides all the Betti numbers $b_i^{\text{clid}}(X_\Delta)$, this method gives much information about the torsion coefficients.

The following description of the E^1 - and E^2 -terms of the toric homology spectral sequence may be easily extracted from the first and last row of the big commutative diagram from Figure 2.2 (p. 40):

2.4.1 Theorem. *For every fan Δ in $N_{\mathbb{R}}$ and every abelian group G , the differentials $d_{p,q}^1: E_{p,q}^1 \rightarrow E_{p-1,q}^1$ of the associated toric homology spectral sequence correspond to the boundary mappings δ_p of the chain complexes $C_\bullet(\Delta, H_q^\Delta)$ and $C_\bullet(\Delta, \Lambda_q^\Delta)$, i.e., the diagram*

$$\begin{array}{ccccc} E_{p,q}^1 & \cong & C_p(\Delta, H_q^\Delta) & \cong & C_p(\Delta, \Lambda_q^\Delta) \\ d_{p,q}^1 \downarrow & & \downarrow \delta_p & & \downarrow \delta_p \\ E_{p-1,q}^1 & \cong & C_{p-1}(\Delta, H_q^\Delta) & \cong & C_{p-1}(\Delta, \Lambda_q^\Delta) \end{array}$$

is commutative. Thus the E^2 -term of the toric homology spectral sequence is explicitly computable by the isomorphisms

$$E_{p,q}^2 \cong H_p(\Delta, H_q^\Delta) \cong H_p(\Delta, \Lambda_q^\Delta)$$

for all integers p, q .

Proof. Under the isomorphism of Lemma 2.3.1 the differential

$$d_{p,q}^1: E_{p,q}^1 = H_{p+q}^{\text{clid}}(X_p, X_{p-1}; G) \longrightarrow H_{p+q-1}^{\text{clid}}(X_{p-1}, X_{p-2}; G) = E_{p-1,q}^1$$

of the toric homology spectral sequence decomposes into the homomorphisms

$$\delta_\bullet: H_{p+q}^{\text{clid}}(\tilde{\sigma} \times T(\sigma); G) \longrightarrow H_{p+q-1}^{\text{clid}}(\tilde{\tau} \times T(\tau); G)$$

for $\Delta^{n-p} \ni \sigma \prec \tau \in \Delta^{n-p+1}$ of the first row of the diagram from Figure 2.2 (p. 40). By the isomorphisms of the first and last column, these homomorphisms correspond to the homomorphisms

$$o_{\sigma,\tau} \cdot \pi_{\bullet}^{\sigma,\tau}: H_q(T(\sigma); G) \longrightarrow H_q(T(\tau); G)$$

in the last row of the same diagram, i.e., they correspond to the homomorphisms into which the differential

$$\delta_p: C_p(\Delta, H_q^\Delta) \longrightarrow C_{p-1}(\Delta, H_q^\Delta)$$

decomposes. Hence, there are isomorphisms $E_{p,q}^1 \cong C_p(\Delta, H_q^\Delta)$ and $E_{p,q}^2 \cong H_p(\Delta, H_q^\Delta)$. The natural equivalence between the covariant functors H_q^Δ and Λ_q^Δ by Remark 2.1.3 implies the isomorphisms $C_p(\Delta, H_q^\Delta) \cong C_p(\Delta, \Lambda_q^\Delta)$ and $H_p(\Delta, H_q^\Delta) \cong H_p(\Delta, \Lambda_q^\Delta)$ by Proposition 2.1.7. ■

As a consequence of the description of the E^1 -term in the preceding theorem, we see that the toric homology spectral sequence $(E^r, d^r)_{r \geq 1}$ is a first-quadrant spectral sequence. In fact, all modules $E_{p,q}^r$ outside the triangle determined by the inequalities $0 \leq q \leq p \leq n$ are zero:

2.4.2 Corollary. *For every fan Δ in $N_{\mathbb{R}}$, every abelian group G , and all integers p, q , the module $E_{p,q}^1$ of the associated toric homology spectral sequence is described by the isomorphisms*

$$E_{p,q}^1 \cong C_p(\Delta, H_q^\Delta) \cong C_p(\Delta, \Lambda_q^\Delta) = \bigoplus_{\sigma \in \Delta^{(n-p)}} \Lambda^q N(\sigma) \otimes G \cong G^{\binom{p}{q} d_{n-p}},$$

where $d_{n-p} = \#\Delta^{n-p}$; in particular, $E_{p,q}^1 = 0$ (and thus $E_{p,q}^r = 0$ for $r \geq 1$) unless $0 \leq q \leq p \leq n$. ■

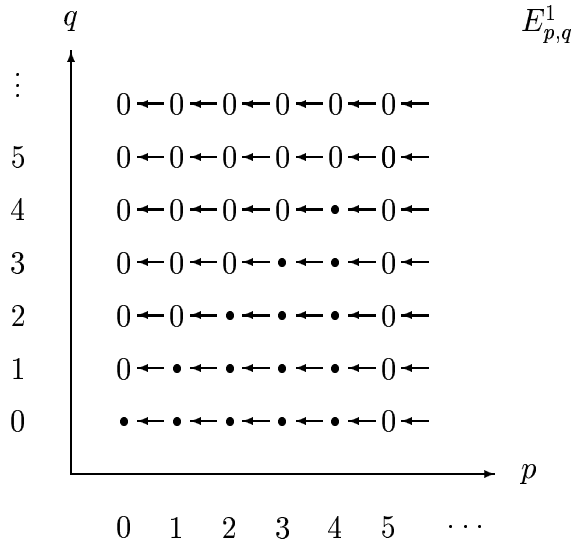


Figure 2.3: The differentials $d_{p,q}^1: E_{p,q}^1 \rightarrow E_{p-1,q}^1$ of the toric homology spectral sequence associated to a 4-dimensional toric variety.

We now want to show that the multiplicative monoid $\mathbb{N}_{>0}$ acts on the toric variety X_Δ and hence on the associated spectral sequence. That action will be the key to the degeneration properties mentioned at the beginning of this section. We start with recalling the notion of a morphism of fans: Given rational fans Δ and

Δ' with respect to lattices N and N' of rank n and n' , respectively, a *morphism of fans*

$$\psi: (N, \Delta) \rightarrow (N', \Delta')$$

is a homomorphism $\psi: N \rightarrow N'$ of lattices such that for each cone $\sigma \in \Delta$ there is a cone $\sigma' \in \Delta'$ with

$$(\psi \otimes_{\mathbb{Z}} \text{id}_{\mathbb{R}})(\sigma) \subset \sigma'.$$

Such a morphism of fans induces an algebraic mapping

$$\psi_*: X_{\Delta} \rightarrow X_{\Delta'},$$

whose restriction to the open dense subsets $\mathbb{T}_N \subset X_{\Delta}$ and $\mathbb{T}_{N'} \subset X_{\Delta'}$ coincides with the homomorphism of complex algebraic tori

$$\psi \otimes_{\mathbb{Z}} \text{id}_{\mathbb{C}^*}: \mathbb{T}_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \longrightarrow \mathbb{T}_{N'} = N' \otimes_{\mathbb{Z}} \mathbb{C}^*.$$

Moreover, the mapping ψ_* is equivariant with respect to this homomorphism and the natural actions of the tori \mathbb{T}_N and $\mathbb{T}_{N'}$ on the toric varieties X_{Δ} and $X_{\Delta'}$, respectively, i.e.,

$$\psi_*(t \cdot x) = (\psi \otimes_{\mathbb{Z}} \text{id}_{\mathbb{C}^*})(t) \cdot \psi_*(x)$$

for $t \in \mathbb{T}_N$ and $x \in X_{\Delta}$, see [Od, Theorem 1.13] (or [Fu, Section 3.1, p. 56] where that is stated as an exercise).

We use that to define an action of the multiplicative monoid $\mathbb{N}_{>0}$ on any toric variety X_{Δ} as follows: There is an obvious action ψ of $\mathbb{N}_{>0}$ on the lattice N , assigning to $k \in \mathbb{N}_{>0}$ the endomorphism $\psi(k)$, given by scalar multiplication

$$(k, v) \mapsto \psi(k)(v) := kv.$$

It immediately induces endomorphisms of fans

$$\psi(k): (N, \Delta) \rightarrow (N, \Delta)$$

and hence endomorphisms

$$\psi(k)_*: X_{\Delta} \rightarrow X_{\Delta}$$

of the associated toric variety that clearly respect the orbit structure and hence, the canonical filtration: Observing that $\psi(k)(\sigma) \subset \sigma$ for each cone $\sigma \in \Delta$, we obtain the mapping $\psi(k)_*: X_{\Delta} \rightarrow X_{\Delta}$ by gluing together the mappings $\psi(k)_*: X_{\sigma} \rightarrow X_{\sigma}$ that are constructed as follows, cf. the proof of [Od, Theorem 1.13]: The homomorphism $\psi(k)^*: M \rightarrow M$ that is dual to the homomorphism $\psi(k): N \rightarrow N$ is given by the assignment $u \mapsto ku$ for $u \in M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. By identifying the (closed) points of the affine toric variety X_{σ} with the elements of $\text{Spec } \mathbb{C}[S_{\sigma}] = \text{Hom}_{\text{sg}}(S_{\sigma}, \mathbb{C})$, the image $\psi(k)_* x$ of a point $x \in X_{\sigma}$ is given by the homomorphism of semigroups

$$\psi(k)_* x: S_{\sigma} \rightarrow \mathbb{C}, u \mapsto (\psi(k)_* x)(u) := x(\psi(k)^* u) = x(ku) = (x(u))^k.$$

In particular, the distinguished point $x_\sigma \in O_\sigma \cong \text{Hom}(\sigma^\perp \cap M, \mathbb{C}^*) \cong (\mathbb{C}^*)^{n-\dim \sigma}$ determined by $x_\sigma(u) = 1$ for $u \in \sigma^\perp \cap M$ is mapped onto itself; and the orbit O_σ is invariant via the assignment

$$t \cdot x_\sigma \mapsto \psi(k)_*(t \cdot x_\sigma) = (\psi(k) \otimes_{\mathbb{Z}} \text{id}_{\mathbb{C}^*})(t) \cdot \psi(k)_* x_\sigma = t^k \cdot x_\sigma$$

for $t = (t_1, \dots, t_n) \in \mathbb{T}_N \cong (\mathbb{C}^*)^n$, where $t^k := (t_1^k, \dots, t_n^k)$.

Since the endomorphism $\psi(k)_*: X_\Delta \rightarrow X_\Delta$ respects the canonical filtration $\emptyset = X_{-1} \subset \dots \subset X_p \subset \dots \subset X_n = X_\Delta$ of the toric variety X_Δ , it induces an endomorphism $\kappa: \mathbf{C} \rightarrow \mathbf{C}$ of the chain complex $\mathbf{C} := C_\bullet^{\text{clid}}(X_\Delta; G)$ that respects the canonical filtration of \mathbf{C} by the subcomplexes $\mathbf{C}_p := C_\bullet^{\text{clid}}(X_p; G)$. As pointed out at the end of Section 2.2, the endomorphism $\kappa: \mathbf{C} \rightarrow \mathbf{C}$ induces endomorphisms $\bar{\kappa}^r: E^r \rightarrow E^r$ of the associated toric homology spectral sequence; they commute with the differentials $d^r: E^r \rightarrow E^r$. The so-defined action of the multiplicative monoid $\mathbb{N}_{>0}$ on the toric homology spectral sequence is easy to describe:

2.4.3 Proposition. *For every fan Δ in $N_{\mathbb{R}}$, every abelian group G , and all integers p, q , the multiplicative monoid $\mathbb{N}_{>0}$ acts on the module $E_{p,q}^1$ of the associated toric homology spectral sequence by assigning to $k \in \mathbb{N}_{>0}$ multiplication by the factor k^q , i.e., the diagram*

$$\begin{array}{ccc} E_{p,q}^1 & \cong & G^{(q)^{d_{n-p}}} \\ \bar{\kappa}^1 \downarrow & & \downarrow \cdot k^q \\ E_{p,q}^1 & \cong & G^{(q)^{d_{n-p}}} \end{array}$$

is commutative.

Proof. We may assume that $0 \leq q \leq p \leq n$, since otherwise $E_{p,q}^1 = 0$. First, observe that the endomorphism $\bar{\kappa}^1$ of the module

$$E_{p,q}^1 \cong \bigoplus_{\sigma \in \Delta^{(n-p)}} H_{p+q}^{\text{clid}}(O_\sigma; G) \cong \bigoplus_{\sigma \in \Delta^{(n-p)}} H_q(T(\sigma); G) \cong G^{(q)^{d_{n-p}}}$$

is induced by the endomorphism

$$T(\sigma) \cong (S^1)^p \rightarrow (S^1)^p \cong T(\sigma), \quad (e^{it_1}, \dots, e^{it_p}) \mapsto (e^{ikt_1}, \dots, e^{ikt_p})$$

of the deformation retract $T(\sigma)$ of the $\psi(k)_*$ -invariant orbit O_σ for $\sigma \in \Delta^{n-p}$. For $p = 1$, the induced endomorphisms of the homology groups $H_0(S^1; G)$ and $H_1(S^1; G)$ are the identity mapping and multiplication by the factor k , respectively. Now the general case follows from the functoriality of the Künneth formula. ■

As the endomorphism $\bar{\kappa}^{r+1}: E^{r+1} \rightarrow E^{r+1}$ is nothing else but restriction of the endomorphism $\bar{\kappa}^r: E^r \rightarrow E^r$ to the subquotient E^{r+1} of the E^r -term, the

multiplicative monoid $\mathbb{N}_{>0}$ analogously acts on the modules $E_{p,q}^r$ with $r \geq 1$ of the toric homology spectral sequence:

2.4.4 Corollary. *For every fan Δ in $N_{\mathbb{R}}$, every abelian group G , all integers p, q , and every integer $r \geq 1$, the multiplicative monoid $\mathbb{N}_{>0}$ acts on the module $E_{p,q}^r$ of the associated toric homology spectral sequence by assigning to $k \in \mathbb{N}_{>0}$ multiplication by the factor k^q . ■*

The fact that this action on the spectral sequence commutes with the differentials yields striking vanishing results for these differentials (including the case $G = \mathbb{Q}$, which is considered in [To]):

2.4.5 Proposition. *For every fan Δ in $N_{\mathbb{R}}$, every abelian group G , all integers p, q , and every integer $r \geq 2$, there are strictly positive integers $m_{p,q}^r$ such that the differential $d_{p,q}^r: E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ satisfies*

$$m_{p,q}^r d_{p,q}^r = 0.$$

More precisely, the integer $m_{p,q}^r$ is given as follows:

$$m_{p,q}^r = \begin{cases} \gcd\{k^q(k^{r-1} - 1); k \in \mathbb{N}_{>0}\} & \text{if } 2r - 1 \leq q + 2r - 1 \leq p \leq n, \\ 1 & \text{otherwise.} \end{cases}$$

In particular, for divisible groups G , all differentials $d_{p,q}^r$ with $r \geq 2$ vanish. Furthermore, all differentials $d_{p,q}^r$ with $2r - 1 > n$ vanish.

Proof. We may assume that $2r - 1 \leq q + 2r - 1 \leq p \leq n$, i.e., the point (p, q) lies in the triangle given by the vertices $(2r - 1, 0)$, $(n, 0)$, and $(n, n - 2r + 1)$ with $2r - 1 \leq n$, since otherwise $E_{p,q}^r = 0$ or $E_{p-r,q+r-1}^r = 0$, i.e., $d_{p,q}^r = 0$. Since the action of the multiplicative monoid $\mathbb{N}_{>0}$ on the modules $E_{p,q}^r$ and $E_{p-r,q+r-1}^r$ commutes with the differential $d_{p,q}^r$, and since the number k acts on $\alpha \in E_{p,q}^r$ by multiplication by the factor k^q , we have the equations

$$k^{q+r-1} d_{p,q}^r(\alpha) = k \cdot d_{p,q}^r(\alpha) = d_{p,q}^r(k \cdot \alpha) = d_{p,q}^r(k^q \alpha) = k^q d_{p,q}^r(\alpha),$$

i.e., $k^q(k^{r-1} - 1) d_{p,q}^r = 0$ for $k \in \mathbb{N}_{>0}$. These equations imply the equation $m_{p,q}^r d_{p,q}^r = 0$ with the help of the Euclidean algorithm. ■

The following criterion is useful for the computation of the numbers $m(r, q) := \gcd\{k^q(k^{r-1} - 1); k \in \mathbb{N}_{>0}\}$ that were used to define the numbers $m_{p,q}^r$:

2.4.6 Proposition. *Let a, b be positive integers. Then the number of roots of the congruence*

$$x^a \equiv 1 \pmod{p^b}$$

$$\text{is exactly } \left\{ \begin{array}{l} \gcd(a, p^{b-1}(p-1)) \\ \gcd(a, 2^{b-1}) \\ 2 \gcd(a, 2^{b-2}) \\ \gcd(a, 2^{b-2}) \end{array} \right\} \text{ if } \left\{ \begin{array}{l} p \text{ is an odd prime} \\ p = 2, b = 1, 2 \\ p = 2, b \geq 3, a \text{ even} \\ p = 2, b \geq 3, a \text{ odd} \end{array} \right\}.$$

Proof. Recall that *Euler's function* $\varphi(j)$ denotes the number of positive integers not exceeding j and relatively prime to j . The numbers $j = p^b, 2^1$, or 2^2 have *primitive roots*, i.e., there are integers c such that their powers

$$c^0, c^1, c^2, \dots, c^{\varphi(j)-1}$$

form a *reduced residue system modulo* j , i.e., they are distinct modulo j and are all relatively prime to j , see [Le, Theorem 4-11]. For every x relatively prime to j , an *index* $\text{ind}_c x$ denotes any number such that

$$c^{\text{ind}_c x} \equiv x \pmod{j};$$

it is unique modulo $\varphi(j)$. The function ind_c is a bijection of any reduced residue system modulo j onto the set $\{0, 1, \dots, \varphi(j) - 1\}$, i.e., for x, x' relatively prime to j , we have

$$x \equiv x' \pmod{j} \quad \text{iff} \quad \text{ind}_c x \equiv \text{ind}_c x' \pmod{\varphi(j)}.$$

Moreover,

$$\text{ind}_c x^a \equiv a \cdot \text{ind}_c x \pmod{\varphi(j)},$$

see [Le, Theorem 4-12]. The roots of the congruence $x^a \equiv 1 \pmod{j}$ are relatively prime to j . As a consequence, they are given by the $\gcd(a, \varphi(j))$ roots of the congruence

$$\text{ind}_c x^a \equiv \text{ind}_c 1 \pmod{\varphi(j)},$$

i.e.,

$$a \cdot \text{ind}_c x \equiv 0 \pmod{\varphi(j)}.$$

The numbers $j = 2^b$ with $b \geq 3$ have no primitive roots. But as a reduced residue system modulo j is given by the powers

$$5^0, 5^1, 5^2, \dots, 5^{\varphi(j)-1}, -5^0, -5^1, -5^2, \dots, -5^{\varphi(j)-1},$$

we may proceed in a similar way, cf. [Vi, Chapter VI §6]: For every x relatively prime to j , an *index system* $(\text{ind}_- x; \text{ind}_5 x)$ denotes any pair of numbers such that

$$x \equiv (-1)^{\text{ind}_- x} \cdot 5^{\text{ind}_5 x} \pmod{j}.$$

For x, x' relatively prime to j , we have $x \equiv x' \pmod{j}$ iff

$$\text{ind}_- x \equiv \text{ind}_- x' \pmod{2} \quad \text{and} \quad \text{ind}_5 x \equiv \text{ind}_5 x' \pmod{\varphi(j)/2}.$$

Furthermore,

$$\text{ind}_- x^a \equiv a \cdot \text{ind}_- x \pmod{2}$$

and

$$\text{ind}_5 x^a \equiv a \cdot \text{ind}_5 x \pmod{\varphi(j)/2}.$$

Hence, the roots of the congruence $x^a \equiv 1 \pmod{j}$ are given by the common roots of the congruences

$$\text{ind}_- x^a \equiv \text{ind}_- 1, \quad \text{i.e.,} \quad a \cdot \text{ind}_- x \equiv 0 \pmod{2}$$

and

$$\text{ind}_5 x^a \equiv \text{ind}_5 1, \quad \text{i.e.,} \quad a \cdot \text{ind}_5 x \equiv 0 \pmod{\varphi(j)/2}.$$

Obviously, there are exactly $2 \gcd(a, \varphi(j)/2)$ common roots of these congruences if a is even, and there are exactly $\gcd(a, \varphi(j)/2)$ common roots if a is odd. \blacksquare

2.4.7 Corollary. *For all natural numbers $r \geq 2$ and $q \geq 0$, the decomposition of the number $m(r, q) = \gcd\{k^q(k^{r-1} - 1); k \in \mathbb{N}_{>0}\}$ into primes is as follows:*

(a) For all $r \geq 2$, $m(r, 0) = 1$.

(b) If r is even and $q \geq 1$, then $m(r, q) = 2$.

(c) If r is odd and $q \geq 1$, then

(c₁) $2|m(r, q)$,

(c₂) for $\alpha \geq 2$, $2^\alpha|m(r, q)$ iff $q \geq \alpha$ and $2^{\alpha-2}|r-1$,

(c₃) for p odd prime and $\alpha \geq 1$, $p^\alpha|m(r, q)$ iff $q \geq \alpha$ and $p^{\alpha-1}(p-1)|r-1$.

In particular, if $p|m(r, q)$, then $p \leq r$.

Proof.

(a) For every prime p , $p^0(p^{r-1} - 1) \equiv -1 \not\equiv 0 \pmod{p}$, hence $p \nmid m(r, 0)$.

(b) If k is even (odd), then k^q is even (odd), and $k^{r-1} - 1$ is odd (even), hence $2|m(r, q)$. Furthermore, $4|k^q(k^{r-1} - 1)$ iff $4|k^q$ or $4|k^{r-1} - 1$. Hence, $4 \nmid m(r, q)$, as the congruence $x^q \equiv 0 \pmod{4}$ has at most two roots (namely, $x \equiv 0$ or $x \equiv 2 \pmod{4}$), and the congruence $x^{r-1} \equiv 1 \pmod{4}$ has exactly $\gcd(r-1, 2) = 1$ root by Proposition 2.4.6. If $p|m(r, q)$ for any odd prime p , then the congruence $x^q(x^{r-1} - 1) \equiv 0 \pmod{p}$ would have p roots, namely the root $x \equiv 0 \pmod{p}$ from the factor x^q , and the $p-1$ roots $x \equiv 1, 2, \dots$, or $p-1 \pmod{p}$ from the factor $x^{r-1} - 1$. But the congruence $x^{r-1} - 1 \equiv 0 \pmod{p}$ has exactly $\gcd(r-1, p-1)$ roots by Proposition 2.4.6. Hence, $p-1 = \gcd(r-1, p-1)$, i.e., $p-1|r-1$, a contradiction to the facts that p is odd and r is even.

(c₁) If k is even (odd), then k^q is even (odd), and $k^{r-1} - 1$ is odd (even), hence $2|m(r, q)$.

(c₂) The congruence $x^{r-1} \equiv 1 \pmod{2^\alpha}$ has exactly $2 \gcd(r-1, 2^{\alpha-2})$ roots by Proposition 2.4.6; i.e., it has the $2^{\alpha-1}$ roots

$$x \equiv 1, 3, \dots, \text{ or } 2^\alpha - 1 \pmod{2^\alpha}$$

if $2^{\alpha-2}|r-1$, and it has less than $2^{\alpha-1}$ roots if $2^{\alpha-2} \nmid r-1$. Hence, $2^\alpha|m(r, q)$ iff $2^{\alpha-2}|r-1$ and the congruence $x^q \equiv 0 \pmod{2^\alpha}$ has the “other” $2^{\alpha-1}$ roots

$$x \equiv 2, 4, \dots, \text{ or } 2^\alpha \pmod{2^\alpha}$$

iff $2^{\alpha-2}|r-1$ and $q \geq \alpha$.

(c₃) The congruence $x^{r-1} \equiv 1 \pmod{p^\alpha}$ has exactly $\gcd(r-1, p^{\alpha-1}(p-1))$ roots by Proposition 2.4.6; i.e., it has the $p^\alpha - p^{\alpha-1}$ roots

$$x \equiv 1, 2, \dots, p-1, p+1, p+2, \dots, 2p-1, 2p+1, \dots, \text{ or } p^\alpha - 1 \pmod{p^\alpha}$$

if $p^{\alpha-1}(p-1)|r-1$, and it has less than $p^\alpha - p^{\alpha-1}$ roots if $p^{\alpha-1}(p-1) \nmid r-1$. Hence, $p^\alpha | m(r, q)$ iff $p^{\alpha-1}(p-1)|r-1$ and the congruence $x^q \equiv 0 \pmod{p^\alpha}$ has the “other” $p^{\alpha-1}$ roots

$$x \equiv p, 2p, 3p, \dots, \text{ or } p^\alpha \pmod{p^\alpha}$$

iff $p^{\alpha-1}(p-1)|r-1$ and $q \geq \alpha$. In particular, if $p|m(r, q)$, then $p-1|r-1$, i.e., $p \leq r$.

■

2.4.8 Remark. With the help of the results obtained so far, the following procedure admits in many cases the computation of a homology group $H_\ell^{\text{cld}}(X_\Delta; G)$, where ℓ is a natural number:

- (1) Calculate explicitly the modules $E_{p,q}^2 \cong H_p(\Delta, \Lambda_q^\Delta)$ of the associated toric homology spectral sequence for all natural numbers p, q , where $E_{p,q}^2 = 0$ unless $0 \leq q \leq p \leq n$; see Theorem 2.4.1 and Corollary 2.4.2.
- (2) Try to show that $E_{p,q}^2 \cong E_{p,q}^3 \cong \dots \cong E_{p,q}^\infty$ for all natural numbers p, q with $p+q = \ell$, using the fact that $E_{p,q}^{r+1} \cong E_{p,q}^r$ if the differentials $d_{p,q}^r$ and $d_{p+r, q-r+1}^r$ vanish.

Remember that $m_{p',q'}^r d_{p',q'}^r = 0$ by Proposition 2.4.5, where $m_{p',q'}^r = 1$ unless $2r-1 \leq q' + 2r-1 \leq p' \leq n$. Note that $m_{p',q'}^r d_{p',q'}^r = 0$ implies $d_{p',q'}^r = 0$ if the abelian group G is $m_{p',q'}^r$ -divisible. In particular, $E_{p,q}^\infty \cong E_{p,q}^2$ for all natural numbers p, q if the abelian group G is φ -divisible for all primes φ with $2\varphi-1 \leq n$, cf. Corollary 2.4.7(c₃).

- (3) If $E_{p',q'}^2 \cong E_{p',q'}^\infty$ for all natural numbers p', q' with $p'+q' = \ell$, then

$$H_\ell^{\text{cld}}(X_\Delta; G) \cong \bigoplus_{k=p}^{\tilde{p}} E_{k, \ell-k}^2$$

under the additional conditions that the modules $E_{k, \ell-k}^2$ are free for $p < k \leq \tilde{p}$, and $E_{k, \ell-k}^2 = 0$ for $k \notin [p, \tilde{p}]$, where p and \tilde{p} are natural numbers; see Lemma 2.2.2.

Obviously, this procedure works for all homology groups with rational coefficients, and the Betti numbers $b_\ell^{\text{cld}}(X_\Delta) = \dim H_\ell^{\text{cld}}(X_\Delta; \mathbb{Q})$ may be calculated as sum over the dimensions $\dim E_{p,q}^2(\mathbb{Q})$ with $p+q = \ell$ of the associated toric homology spectral sequence. ■

In the sequel, we collect some general results about the E^2 -terms of the toric homology spectral sequence and the homology of toric varieties in arbitrary dimensions. For special results in low dimensions see Chapter 3. The modules $E_{p,0}^2$ are described by Example 2.1.6:

2.4.9 Corollary. *For every fan Δ in $N_{\mathbb{R}}$, every abelian group G , and all integers p , there are isomorphisms*

$$E_{p,0}^2 \cong H_p(\Delta, H_0^\Delta) \cong H_p(\Delta, \Lambda_0^\Delta) \cong H_p(\Delta, G^\Delta).$$

In particular,

(a) for the zero fan $\Delta = \{0\}$ in $N_{\mathbb{R}}$, i.e., for the complex algebraic torus $(\mathbb{C}^*)^n$,

$$E_{p,0}^2 = \begin{cases} 0 & \text{for } p \neq n, \\ G & \text{for } p = n, \end{cases}$$

(b) for every nonzero fan Δ in $N_{\mathbb{R}}$, i.e., for every n -dimensional toric variety $X_\Delta \not\cong (\mathbb{C}^*)^n$,

$$E_{p,0}^2 \cong \tilde{H}^{n-p-1}(S_\Delta; G),$$

(c) for every complete fan Δ in $N_{\mathbb{R}}$, i.e., for every compact toric variety X_Δ ,

$$E_{p,0}^2 = \begin{cases} G & \text{for } p = 0, \\ 0 & \text{for } p \neq 0, \end{cases}$$

(d) for every fan Δ with contractible $|\Delta| \setminus \{0\}$, e.g. fan of all faces of a nonzero cone σ in $N_{\mathbb{R}}$, and thus for every n -dimensional affine variety $X_\Delta \not\cong (\mathbb{C}^*)^n$,

$$E_{p,0}^2 = 0 \text{ for all } p,$$

(e)

$$E_{0,0}^2 = \begin{cases} G & \text{if } \Delta \text{ is complete,} \\ 0 & \text{if } \Delta \text{ is noncomplete,} \end{cases}$$

and

$$E_{n,0}^2 = \begin{cases} 0 & \text{if } \Delta \text{ is nonzero,} \\ G & \text{if } \Delta \text{ is zero.} \end{cases}$$

Proof. By Theorem 2.4.1, $E_{p,0}^2 \cong H_p(\Delta, H_0^\Delta) \cong H_p(\Delta, \Lambda_0^\Delta)$. Note that $\Lambda^0 N(\sigma) \otimes G \cong G$ and $\nu_{\sigma,\tau}^0 \cong \text{id}_G$ for $\sigma \prec \tau \in \Delta$; hence, $H_p(\Delta, \Lambda_0^\Delta) \cong H_p(\Delta, G^\Delta)$. Now see Example 2.1.6 for (a), (b), and (c). (d) follows from (b), since here the finite CW-complex $S_\Delta = |\Delta| \cap S^{n-1}$ is contractible. (e) follows from (b), since if $n > 1$ and the fan Δ is not complete, then there exists a deformation retract of S_Δ of dimension less than $n - 1$, i.e., $H_0(\Delta, G^\Delta) \cong H^{n-1}(S_\Delta; G) = 0$. ■

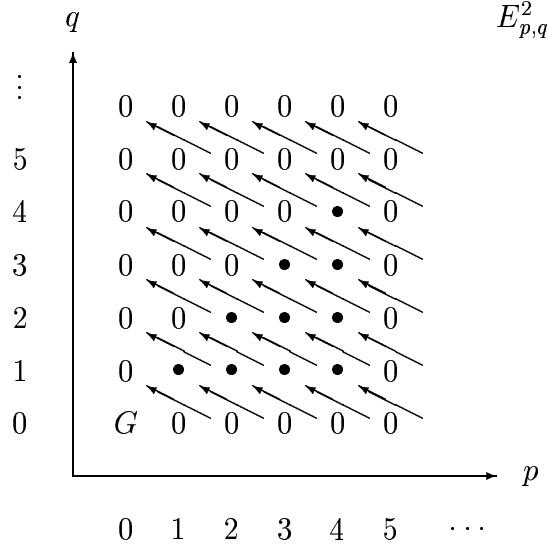


Figure 2.4: The differentials $d_{p,q}^2 : E_{p,q}^2 \rightarrow E_{p-2,q+1}^2$ of the toric homology spectral sequence associated to a compact 4-dimensional toric variety.

For an interpretation of the modules $E_{p,p}^2(\mathbb{Z})$ of the integral toric homology spectral sequence as Chow groups $A_p(X_\Delta)$ see Theorem 3.3.1. The spectral terms $E_{n,q}^2(\mathbb{Z})$ of the integral toric homology spectral sequence are easy to calculate:

2.4.10 Proposition. *For every fan Δ in $N_{\mathbb{R}}$ and every integer q , there are isomorphisms*

$$E_{n,q}^2(\mathbb{Z}) \cong H_n(\Delta, H_q^\Delta) \cong H_n(\Delta, \Lambda_q^\Delta) \cong \mathbb{Z}^{\binom{n-s}{n-q}},$$

where $s := s(\Delta) := \dim_{\mathbb{R}} \text{lin } \Delta = \dim_{\mathbb{R}} \text{lin } \Delta^1$.

Proof. By Theorem 2.4.1, we have

$$E_{n,q}^2(\mathbb{Z}) \cong H_n(\Delta, H_q^\Delta) \cong H_n(\Delta, \Lambda_q^\Delta),$$

where

$$H_n(\Delta, \Lambda_q^\Delta) = \ker [\delta_n : C_n(\Delta, \Lambda_q^\Delta) \rightarrow C_{n-1}(\Delta, \Lambda_q^\Delta)],$$

since $C_{n+1}(\Delta, \Lambda_q^\Delta) = 0$ and, hence,

$$\text{im} [\delta_{n+1} : C_{n+1}(\Delta, \Lambda_q^\Delta) \rightarrow C_n(\Delta, \Lambda_q^\Delta)] = 0.$$

In order to compute

$$\ker \delta_n = \ker \left[\sum_{\varrho \in \Delta^1} o_{\{0\},\varrho} \nu_{\{0\},\varrho}^q : \Lambda^q N \rightarrow \bigoplus_{\varrho \in \Delta^1} \Lambda^q N(\varrho) \right],$$

note that

$$\ker [\nu_{\{0\},\varrho}^q: \Lambda^q N \rightarrow \Lambda^q N(\varrho)] = N_\varrho \wedge \Lambda^{q-1} N,$$

since

$$\ker [\nu_{\{0\},\varrho}: N \rightarrow N(\varrho)] = N_\varrho;$$

where $N_\varrho \wedge \Lambda^{q-1} N$ is to be interpreted as 0 or N_ϱ for $q = 0$ or 1, respectively. As a consequence,

$$\ker \delta_n = \bigcap_{\varrho \in \Delta^1} (N_\varrho \wedge \Lambda^{q-1} N);$$

in particular, $\ker \delta_n = \Lambda^q N \cong \mathbb{Z}^{\binom{n-s}{n-q}}$ for $s(\Delta) = 0$, i.e., for the zero fan $\Delta = \{0\}$. Now observe that

$$\bigcap_{i=1}^k (N_{\mathbb{R}_{\geq 0} v_i} \wedge \Lambda^{k-1} N)$$

is a rational multiple of $\Lambda_{i=1}^k N_{\mathbb{R}_{\geq 0} v_i}$ if the primitive lattice points $v_1, \dots, v_k \in N$ are linearly independent, where equality holds if v_1, \dots, v_k form a part of a basis of the lattice N . As a consequence,

$$\bigcap_{\varrho \in \Delta^1} (N_\varrho \wedge \Lambda^{q-1} N)$$

is a rational multiple of

$$(\Lambda^s N_s) \wedge (\Lambda^{q-s} N),$$

where the sublattice $N_s := N \cap \text{lin } \Delta^1 \subset N$ has rank s and, hence, $\Lambda^s N_s$ has rank 1. The sublattice N_s is a direct factor of the lattice N , i.e., there exists a sublattice $N'_s \subset N$ of rank $n - s$ such that $N_s \oplus N'_s = N$. Choosing bases, we see that

$$(\Lambda^s N_s) \wedge (\Lambda^{q-s} N) = (\Lambda^s N_s) \wedge (\Lambda^{q-s} N'_s)$$

is a lattice of rank $\binom{n-s}{q-s} = \binom{n-s}{n-q}$; i.e.,

$$\ker \delta_n \cong (\Lambda^s N_s) \wedge (\Lambda^{q-s} N) \cong \mathbb{Z}^{\binom{n-s}{n-q}}. \quad \blacksquare$$

Observe that, for every fan Δ in $N_{\mathbb{R}}$, there is a unique decomposition $X_\Delta \cong \mathbb{T} \times Y$ where \mathbb{T} is an algebraic torus of complex dimension $n - s(\Delta)$ and Y is a toric variety of complex dimension $s(\Delta)$, namely, the toric variety associated to Δ considered as a fan with respect to the lattice $N \cap \text{lin } \Delta$. The fan Δ and the associated toric variety X_Δ are called *nondegenerate* iff $s(\Delta) = n$, i.e., the rays of the fan Δ span the vector space $N_{\mathbb{R}}$. Via Künneth formulæ, we may thus calculate the homology of the toric variety $X_\Delta \cong \mathbb{T} \times Y$ from the well-known homology of the algebraic torus \mathbb{T} and the homology of the (nondegenerate) toric variety Y . We frequently deal with the general case and mention the simpler formulæ in the nondegenerate integral case as remarks.

2.4.11 Remark. If in addition the fan Δ in the preceding proposition is *non-degenerate*, then

$$E_{n,q}^2(\mathbb{Z}) \cong \begin{cases} 0 & \text{for } q \neq n, \\ \mathbb{Z} & \text{for } q = n. \end{cases} \quad \blacksquare$$

The knowledge of the E^2 -terms of the toric homology spectral sequence enables us to compute the following homology groups according to the procedure described in Remark 2.4.8 above:

2.4.12 Theorem. For every fan Δ in $N_{\mathbb{R}}$ and every abelian group G , the homology groups in low and in high degrees of the toric variety X_{Δ} may be expressed via the modules $E_{p,q}^2 \cong H_p(\Delta, H_q^{\Delta}) \cong H_p(\Delta, \Lambda_q^{\Delta})$ of the associated toric homology spectral sequence as follows:

- (a) $H_0^{\text{clid}}(X_{\Delta}; G) \cong E_{0,0}^2 = \begin{cases} G & \text{if } \Delta \text{ is complete,} \\ 0 & \text{if } \Delta \text{ is noncomplete,} \end{cases}$
- (b) $H_1^{\text{clid}}(X_{\Delta}; G) \cong E_{1,0}^2,$
- (c) $H_2^{\text{clid}}(X_{\Delta}; G) \cong E_{1,1}^2 \oplus E_{2,0}^2$ if the module $E_{2,0}^2$ is free,
- (d) $H_3^{\text{clid}}(X_{\Delta}; G) \cong E_{2,1}^2 \oplus E_{3,0}^2$ if the module $E_{3,0}^2$ is free,
- (e) $H_{2n-4}^{\text{clid}}(X_{\Delta}; G) \cong E_{n-2,n-2}^2 \oplus E_{n-1,n-3}^2$ if the module $E_{n-1,n-3}^2$ is free and $E_{n,n-3}^2 = E_{n,n-4}^2 = 0,$
- (f) $H_{2n-3}^{\text{clid}}(X_{\Delta}; G) \cong E_{n-1,n-2}^2 \oplus E_{n,n-3}^2$ if the module $E_{n,n-3}^2$ is free,
- (g) $H_{2n-2}^{\text{clid}}(X_{\Delta}; G) \cong E_{n-1,n-1}^2 \oplus E_{n,n-2}^2$ if the module $E_{n,n-2}^2$ is free,
- (h) $H_{2n-1}^{\text{clid}}(X_{\Delta}; G) \cong E_{n,n-1}^2,$
- (i) $H_{2n}^{\text{clid}}(X_{\Delta}; G) \cong E_{n,n}^2 \cong G.$

Proof. For the computation of the respective homology group $H_{\ell}^{\text{clid}}(X_{\Delta}; G)$ in the cases (a) – (i), there are at most two nonzero modules $E_{p,q}^2$ such that $p + q = \ell$, since we know that $E_{p,q}^2 = 0$ unless $0 \leq q \leq p \leq n$. Furthermore, we have $E_{p,q}^2 \cong E_{p,q}^3 \cong \dots \cong E_{p,q}^{\infty}$ for all p, q with $p + q = \ell$, since the respective differentials $d_{p,q}^r: E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ and $d_{p+r,q-r+1}^r: E_{p+r,q-r+1}^r \rightarrow E_{p,q}^r$ vanish for $r \geq 2$. (Remember that $m_{p,q}^r d_{p,q}^r = 0$ by Proposition 2.4.5 and $m(r, 0) = 1$ by Corollary 2.4.7(a).) Finally, the asserted isomorphisms follow by Lemma 2.2.2. \blacksquare

2.4.13 Remark. If in addition the fan Δ in the preceding theorem is *nondegenerate*, then

$$(e') \quad H_{2n-4}^{\text{clid}}(X_\Delta; \mathbb{Z}) \cong E_{n-2, n-2}^2(\mathbb{Z}) \oplus E_{n-1, n-3}^2(\mathbb{Z}) \text{ if the module } E_{n-1, n-3}^2(\mathbb{Z}) \text{ is free,}$$

$$(f') \quad H_{2n-3}^{\text{clid}}(X_\Delta; \mathbb{Z}) \cong E_{n-1, n-2}^2(\mathbb{Z}),$$

$$(g') \quad H_{2n-2}^{\text{clid}}(X_\Delta; \mathbb{Z}) \cong E_{n-1, n-1}^2(\mathbb{Z}),$$

$$(h') \quad H_{2n-1}^{\text{clid}}(X_\Delta; \mathbb{Z}) = 0. \quad \blacksquare$$

A toric variety X_Δ whose orbits are at most 1-codimensional, i.e., $\Delta = \Delta^{\leq 1}$, is smooth, and its whole integral homology is easy to compute:

2.4.14 Proposition. *For every fan Δ in $N_{\mathbb{R}}$ with $\Delta = \Delta^{\leq 1}$, the integral homology of the toric variety X_Δ is determined by the E^2 -terms of the associated toric homology spectral sequence:*

$$H_{n+q}^{\text{clid}}(X_\Delta; \mathbb{Z}) \cong \begin{cases} E_{n,n}^2(\mathbb{Z}) & \text{for } q = n, \\ E_{n, n-1}^2(\mathbb{Z}) & \text{for } q = n-1, \\ E_{n-1, q+1}^2(\mathbb{Z}) \oplus E_{n, q}^2(\mathbb{Z}) & \text{for } 0 \leq q \leq n-2, \\ E_{n-1, 0}^2(\mathbb{Z}) & \text{for } q = -1, \\ 0 & \text{else,} \end{cases}$$

where $E_{n,q}^2(\mathbb{Z}) \cong \mathbb{Z}^{\binom{n-s}{n-q}}$ with $s = s(\Delta) = \dim_{\mathbb{R}} \text{lin } \Delta^1$,

$$E_{n-1, q+1}^2(\mathbb{Z}) \cong \bigoplus_{\varrho \in \Delta^1} \Lambda^{q+1} N(\varrho) \Big/ \text{im} \left[\sum_{\varrho \in \Delta^1} \nu_{\{0\}, \varrho}^{q+1} : \Lambda^{q+1} N \rightarrow \bigoplus_{\varrho \in \Delta^1} \Lambda^{q+1} N(\varrho) \right],$$

and $E_{n-1, 0}^2(\mathbb{Z}) \cong \mathbb{Z}^{\#\Delta^1 - 1}$ if Δ is nonzero.

Proof. Since $\Delta = \Delta^{\leq 1}$, the modules $E_{p,q}^2(\mathbb{Z}) \cong E_{p,q}^3(\mathbb{Z}) \cong \dots \cong E_{p,q}^\infty(\mathbb{Z})$ are zero unless $p \in \{n-1, n\}$. Now proceed as in the proof of Theorem 2.4.12 using the fact that the modules $E_{n,q}^\infty(\mathbb{Z}) \cong E_{n,q}^2(\mathbb{Z})$ are free. \blacksquare

Let us check these results in the simplest case, namely, the torus, where the homology is of course well-known.

2.4.15 Example. For every (nonzero) abelian group G , the (nonzero) homology groups of the algebraic torus \mathbb{T}_N are, for $0 \leq q \leq n$,

$$H_{n+q}^{\text{clid}}(\mathbb{T}_N; G) \cong E_{n,q}^2 \cong C_n(\Delta, H_q^\Delta) \cong H_q(T^n; G) \cong \Lambda^q N \otimes G \cong G^{\binom{n}{q}}.$$

2.4.16 Proposition. *For every fan Δ in $N_{\mathbb{R}}$ and every integer q , the module $E_{n-1,q}^2(\mathbb{Z})$ of the associated integral toric homology spectral sequence is free if $E_{n,q}^2(\mathbb{Z}_{\varphi}) = 0$ for all primes φ .*

Proof. For integral coefficients, the chain groups

$$C_p(\Delta, \Lambda_q^{\Delta}(\mathbb{Z})) = \bigoplus_{\sigma \in \Delta^{(n-p)}} \Lambda^q N(\sigma)$$

are free abelian, i.e., the chain complex $C_{\bullet}(\Delta, \Lambda_q^{\Delta}(\mathbb{Z}))$ is free. Furthermore, for every abelian group G ,

$$\Lambda_q^{\Delta}(G) = \Lambda_q^{\Delta}(\mathbb{Z}) \otimes G$$

and

$$C_{\bullet}(\Delta, \Lambda_q^{\Delta}(G)) = C_{\bullet}(\Delta, \Lambda_q^{\Delta}(\mathbb{Z})) \otimes G.$$

Hence, by the universal coefficient theorem for homology, see [Sp, Theorem 5.2.8], there is a short exact sequence

$$0 \rightarrow H_n(\Delta, \Lambda_q^{\Delta}(\mathbb{Z})) \otimes G \rightarrow H_n(\Delta, \Lambda_q^{\Delta}(G)) \rightarrow H_{n-1}(\Delta, \Lambda_q^{\Delta}(\mathbb{Z})) * G \rightarrow 0.$$

Since, by assumption,

$$H_n(\Delta, \Lambda_q^{\Delta}(\mathbb{Z}_{\varphi})) \cong E_{n,q}^2(\mathbb{Z}_{\varphi}) = 0,$$

the exact sequence implies $H_{n-1}(\Delta, \Lambda_q^{\Delta}(\mathbb{Z})) * \mathbb{Z}_{\varphi} = 0$ for all primes φ . In other words,

$$E_{n-1,q}^2(\mathbb{Z}) * \mathbb{Z}_{\varphi} \cong H_{n-1}(\Delta, \Lambda_q^{\Delta}(\mathbb{Z})) * \mathbb{Z}_{\varphi} = 0$$

for all primes φ , i.e., the module $E_{n-1,q}^2(\mathbb{Z})$ has no φ -torsion. \blacksquare

Note that, even for complete fans, we may not assume in general that all modules $E_{n-1,q}^2(\mathbb{Z})$ are free for $1 \leq q \leq n-1$, cf. Example 3.4.4(a) below.

2.4.17 Corollary. *For every nondegenerate fan Δ in $N_{\mathbb{R}}$ with $\Delta = \Delta^{\leq 2}$, the integral homology of the toric variety X_{Δ} is determined by the E^2 -terms of the associated toric homology spectral sequence:*

$$H_{n+q}^{\text{cld}}(X_{\Delta}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } q = n, \\ 0 & \text{for } q = n-1, \\ E_{n-1,q+1}^2(\mathbb{Z}) & \text{for } q = n-2, n-3, \\ E_{n-2,q+2}^2(\mathbb{Z}) \oplus E_{n-1,q+1}^2(\mathbb{Z}) & \text{for } -1 \leq q \leq n-4 \text{ if} \\ & E_{n,q+1}^2(\mathbb{Z}_{\varphi}) = 0 \text{ for all primes } \varphi, \\ E_{n-2,0}^2(\mathbb{Z}) & \text{for } q = -2, \\ 0 & \text{else.} \end{cases}$$

Proof. Since $\Delta = \Delta^{\leq 2}$, we have $E_{p,q}^2(\mathbb{Z}) = 0$ unless $p \in \{n-2, n-1, n\}$. In addition, since the fan Δ is nondegenerate, $E_{n,q}^2(\mathbb{Z}) = 0$ for $q \leq n-1$ by Proposition 2.4.10. Hence, $E_{p,q}^2(\mathbb{Z}) \cong E_{p,q}^3(\mathbb{Z}) \cong \cdots \cong E_{p,q}^\infty(\mathbb{Z})$ for all integers p, q . Now proceed as in the proof of Theorem 2.4.12 using the fact that the modules $E_{n-1,q+1}^\infty(\mathbb{Z}) \cong E_{n-1,q+1}^2(\mathbb{Z})$ are free for $-1 \leq q \leq n-4$ by assumption and Proposition 2.4.16. ■

2.5 Cohomology of Toric Varieties

For a smooth compact toric variety, the famous Theorem of Jurkiewicz-Danilov provides an explicit description of the integral cohomology ring in terms of generators and relations given by data of the associated fan; furthermore, an analogous result holds with rational coefficients in the case of a complete simplicial fan, i.e., when the variety is a compact rational homology manifold. Through the isomorphism between cohomology and homology coming from the Poincaré duality theorem, there are analogous results for homology. For compact toric varieties with arbitrary singularities, Fischli constructed an integral cohomology spectral sequence that allows to determine some cohomology groups in low and in high degrees, enough to completely handle the case of surfaces and threefolds. In the present section, we generalize Fischli's results to obtain a spectral sequence for cohomology with compact supports of arbitrary toric varieties. In particular, with rational coefficients, the whole cohomology can be thus computed.

Since cohomology with compact supports is largely dual to homology with closed supports, the contents of this section is essentially parallel to the contents of the previous Sections 2.1–2.4. To avoid a lengthy repetition, we thus refer the reader to the introduction of the present chapter (see pp. 23–25) for an overview. Moreover, we restrict ourselves to sketch the definitions and results for the general setting.

The cohomology groups of the fibers of the projection $\pi: X_\Delta \rightarrow B_\Delta$ and the transition mappings between them are described by the following contravariant functors H_Δ^\bullet and Λ_Δ^\bullet from the category Δ to the category of abelian groups:

2.5.1 Proposition. *For every fan Δ in $N_\mathbb{R}$, there is natural equivalence of degree 0 between the contravariant functors H_Δ^\bullet and Λ_Δ^\bullet on the category Δ , where*

- (a) *the contravariant functor H_Δ^\bullet associates to each cone $\sigma \in \Delta$ the cohomology ring $H^\bullet(T(\sigma); G)$ of the compact real torus*

$$T(\sigma) = T^n/T_\sigma \cong N_\mathbb{R}/(N + \text{lin } \sigma),$$

and to each face relation $\sigma \prec \tau$ in Δ the homomorphism

$$\pi_{\sigma,\tau}^\bullet: H^\bullet(T(\tau); G) \rightarrow H^\bullet(T(\sigma); G)$$

induced by the canonical projection $\pi_{\sigma,\tau}: T(\sigma) \rightarrow T(\tau)$;

- (b) the contravariant functor $\Lambda_{\Delta}^{\bullet}$ associates to each cone $\sigma \in \Delta$ the graded abelian group $\Lambda^*M(\sigma) \otimes G$ given by the tensor product of the exterior powers of the sublattice

$$M(\sigma) := \sigma^{\perp} \cap M := \{u \in M; \langle u, \sigma \rangle = 0\} \subset M$$

with the abelian group G , and to each face relation $\sigma \prec \tau$ in Δ the monomorphism

$$\mu_{\sigma, \tau}^{\Lambda} := \Lambda^* \mu_{\sigma, \tau} \otimes 1_G: \Lambda^*M(\tau) \otimes G \hookrightarrow \Lambda^*M(\sigma) \otimes G$$

induced by the inclusion $\mu_{\sigma, \tau}: M(\tau) \subset M(\sigma)$. ■

With the help of the orientation coefficients $o_{\sigma, \tau}$ as defined in Section 2.1 (see p. 28), such contravariant functors lead to cochain complexes as follows:

2.5.2 Definition. For every fan Δ in $N_{\mathbb{R}}$ and every contravariant functor F from the category Δ to the category of abelian groups, the collection

$$C^{\bullet}(\Delta, F) := (C^p(\Delta, F), \delta^p)_{p \in \mathbb{Z}}$$

of the abelian groups

$$C^p(\Delta, F) := \bigoplus_{\tau \in \Delta^{(n-p)}} F(\tau)$$

associated with the coboundary mappings

$$\delta^p := \sum_{\substack{\sigma \in \Delta^{(n-p-1)} \\ \tau \in \Delta^{(n-p)}}} o_{\sigma, \tau} F(\sigma \prec \tau): C^p(\Delta, F) \longrightarrow C^{p+1}(\Delta, F)$$

defines a cochain complex. The corresponding graded cohomology is denoted by $H^{\bullet}(\Delta, F)$. ■

With these ingredients, we can explicitly describe and calculate the E_1 - and E_2 -terms of the *toric cohomology spectral sequence* that is induced by the filtration of a toric variety X_{Δ} by the subvarieties $X_p = \bigcup_{\sigma \in \Delta^{(n-p)}} \bar{O}_{\sigma}$ consisting of the orbits of dimension at most p , and that converges to the cohomology of X_{Δ} with compact supports and coefficients in an abelian group G :

2.5.3 Theorem. For every fan Δ in $N_{\mathbb{R}}$ and every abelian group G , the differentials $d_1^{p,q}: E_1^{p,q} \rightarrow E_1^{p+1,q}$ of the associated toric cohomology spectral sequence correspond to the coboundary mappings δ^p of the cochain complexes $C^\bullet(\Delta, H_\Delta^q)$ and $C^\bullet(\Delta, \Lambda_\Delta^q)$, i.e., the diagram

$$\begin{array}{ccccc} E_1^{p,q} & \cong & C^p(\Delta, H_\Delta^q) & \cong & C^p(\Delta, \Lambda_\Delta^q) \\ d_1^{p,q} \downarrow & & \downarrow \delta^p & & \downarrow \delta^p \\ E_1^{p+1,q} & \cong & C^{p+1}(\Delta, H_\Delta^q) & \cong & C^{p+1}(\Delta, \Lambda_\Delta^q) \end{array}$$

is commutative. Thus the E_2 -term of the toric cohomology spectral sequence is explicitly computable by the isomorphisms

$$E_2^{p,q} \cong H^p(\Delta, H_\Delta^q) \cong H^p(\Delta, \Lambda_\Delta^q)$$

for all integers p, q . ■

The toric cohomology spectral sequence $(E_r, d_r)_{r \geq 1}$ is a first-quadrant spectral sequence: Since $E_1^{p,q} \cong G^b$ where $b = \binom{p}{q} \# \Delta^{n-p}$, all modules $E_r^{p,q}$ outside the triangular region determined by the inequalities $0 \leq q \leq p \leq n$ are zero.

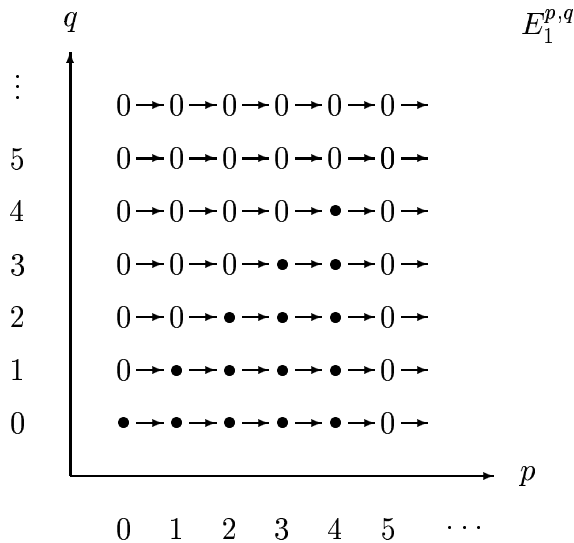


Figure 2.5: The differentials $d_1^{p,q}: E_1^{p,q} \rightarrow E_1^{p+1,q}$ of the toric cohomology spectral sequence associated to a 4-dimensional toric variety.

Furthermore, the spectral sequence degenerates at the E_2 -level if the coefficient group G is divisible, e.g., for rational coefficients:

2.5.4 Proposition. *For every fan Δ in $N_{\mathbb{R}}$, every abelian group G , all integers p, q , and every integer $r \geq 2$, there are strictly positive integers $m_r^{p,q}$ such that the differential $d_r^{p,q}: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ satisfies*

$$m_r^{p,q} d_r^{p,q} = 0.$$

More precisely, the integer $m_r^{p,q}$ is given as follows:

$$m_r^{p,q} = \begin{cases} \gcd\{k^q(1 - k^{-r+1}); k \in \mathbb{N}_{>0}\} & \text{if } r - 1 \leq q \leq p \leq n - r, \\ 1 & \text{otherwise.} \end{cases}$$

In particular, for divisible groups G , all differentials $d_r^{p,q}$ with $r \geq 2$ vanish. Furthermore, all differentials $d_r^{p,q}$ with $2r - 1 > n$ vanish. ■

As a consequence, the whole rational cohomology of a toric variety X_{Δ} can be computed: Each Betti number $b_c^{\ell}(X_{\Delta}) = \dim H_c^{\ell}(X_{\Delta}; \mathbb{Q})$ equals the sum $\sum_{p+q=\ell} \dim E_2^{p,q}(\mathbb{Q})$.

Moreover, the explicit knowledge of the E_2 -level allows to compute many cohomology groups:

2.5.5 Theorem. *For every fan Δ in $N_{\mathbb{R}}$ and every abelian group G , the cohomology groups in low and in high degrees of the toric variety X_{Δ} may be expressed via the modules $E_2^{p,q} \cong H^p(\Delta, H_{\Delta}^q) \cong H^p(\Delta, \Lambda_{\Delta}^q)$ of the associated toric cohomology spectral sequence as follows:*

- (a) $H_c^0(X_{\Delta}; G) \cong E_2^{0,0} = \begin{cases} G & \text{if } \Delta \text{ is complete,} \\ 0 & \text{if } \Delta \text{ is noncomplete,} \end{cases}$
- (b) $H_c^1(X_{\Delta}; G) \cong E_2^{1,0}$,
- (c) $H_c^2(X_{\Delta}; G) \cong E_2^{1,1} \oplus E_2^{2,0}$ if the module $E_2^{1,1}$ is free or $E_2^{2,0} = 0$,
- (d) $H_c^3(X_{\Delta}; G) \cong E_2^{2,1} \oplus E_2^{3,0}$ if the module $E_2^{2,1}$ is free or $E_2^{3,0} = 0$,
- (e) $H_c^{2n-2}(X_{\Delta}; G) \cong E_2^{n-1, n-1} \oplus E_2^{n, n-2}$ if the module $E_2^{n-1, n-1}$ is free or $E_2^{n, n-2} = 0$,
- (f) $H_c^{2n-1}(X_{\Delta}; G) \cong E_2^{n, n-1}$,
- (g) $H_c^{2n}(X_{\Delta}; G) \cong E_2^{n, n} \cong G$.

Note that the module

$$E_2^{p,p} = \ker [d_1^{p,p} : E_1^{p,p} \cong G^{d_{n-p}} \longrightarrow E_1^{p+1,p} \cong G^{(p+1)d_{n-p-1}}]$$

is free if G is free abelian. ■

With our method, the integral cohomology of an arbitrary toric variety of dimension at most four is almost completely computable. In addition, for every nonzero fan Δ , the terms $E_2^{p,0}$ of the toric cohomology spectral sequence may be interpreted as reduced homology groups $\tilde{H}_{n-p-1}(S_\Delta; G)$ of the finite CW-complex $S_\Delta = |\Delta| \cap S^{n-1}$, whose calculation is often very easy:

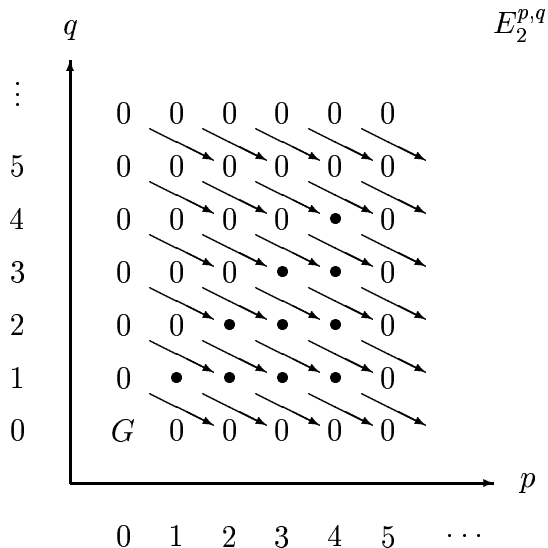


Figure 2.6: The differentials $d_2^{p,q} : E_2^{p,q} \rightarrow E_2^{p+2,q-1}$ of the toric cohomology spectral sequence associated to a compact 4-dimensional toric variety.

Chapter 3

Invariants of Toric Varieties

In this chapter we have a closer look at (co-)homological invariants of toric varieties. To compute the topological Euler characteristic, the Betti numbers, or the torsion coefficients explicitly in terms of data of the fan is a most natural problem. Another problem is to understand the relation between topology and algebraic geometry. For smooth compact toric varieties, the famous Theorem of Jurkiewicz-Danilov states that the integral homology groups are generated by algebraic cycles and in fact coincide with corresponding groups provided by Chow theory, and the latter admit a complete description in terms of cycles that are invariant under the torus action. It is thus a natural question to what extent such a result can be generalized.

In Section 3.1 we give another proof that the Euler characteristic for homology with closed supports of a toric variety X_Δ equals the number of n -dimensional cones in the fan Δ , i.e., $e^{\text{clid}}(X_\Delta) = d_n(\Delta)$. Hence, the coincidence of the topological Euler characteristic $e(X)$ and the Euler characteristic for homology with closed supports $e^{\text{clid}}(X)$ for complex algebraic varieties X is verified in the special case of toric varieties; nevertheless, the Betti numbers $b_k(X)$ and $b_k^{\text{clid}}(X)$ usually are different.

In Section 3.2 we recall the description, due to Fulton and Sturmfels, of the Chow groups of toric varieties: The Chow group $A_p(X_\Delta)$ is generated by the rational equivalence classes of the p -dimensional orbit closures, and the relations among them are generated by the torus-invariant principal divisors on the $(p+1)$ -dimensional orbit closures.

In Section 3.3 we prove with the help of the preceding description that the Chow group $A_p(X_\Delta)$ is canonically isomorphic to the term $E_{p,p}^2$ of the integral toric homology spectral sequence. Thus a generalization of the Theorem of Jurkiewicz-Danilov is found.

In Section 3.4 – 3.6 we present formulæ for the explicit computation — in low and in high degrees — of the integral (co-)homology groups and the integral local homology groups in the fixed points of a toric variety. For toric varieties X_Δ with $\dim X_\Delta \leq 4$, our method provides almost the whole homology $H_\bullet^{\text{clid}}(X_\Delta; \mathbb{Z})$ and

almost the whole cohomology $H_c^\bullet(X_\Delta; \mathbb{Z})$; only the computation of the torsion part $\text{Tor } H_4^{\text{clid}}(X_\Delta; \mathbb{Z}) \cong \text{Tor } H_c^5(X_\Delta; \mathbb{Z})$ of a 4-dimensional toric variety X_Δ may fail. In particular, we come across many monomorphisms (or even isomorphisms) of the kind $A_p(X_\Delta) \hookrightarrow H_{2p}^{\text{clid}}(X_\Delta; \mathbb{Z})$.

3.1 Euler Characteristics of Toric Varieties

The Euler characteristic for homology with closed supports $e^{\text{clid}}(X_\Delta)$ equals the number of full-dimensional cones in the fan Δ . Hence, it is a combinatorial invariant and coincides with the topological Euler characteristic $e(X_\Delta)$.

Various spectral sequences were used to compute the topological Euler characteristic of a toric variety, for instance the Hodge-de Rham spectral sequence in [Da, Corollary 12.8], the Čech cohomology spectral sequence in [Fu, Section 3.2, p. 59], or (for complete fans) the cohomology spectral sequence in [Fi, Theorem 2.2]:

3.1.1 Theorem. *For every fan Δ in $N_\mathbb{R}$, the topological Euler characteristic*

$$e(X_\Delta) := \sum_{k=0}^{\infty} (-1)^k \text{rk } H_k(X_\Delta; \mathbb{Z}) = \sum_{k=0}^{\infty} (-1)^k \text{rk } H^k(X_\Delta; \mathbb{Z})$$

of the n -dimensional toric variety X_Δ is given by the number $d_n(\Delta)$ of n -dimensional cones in the fan Δ , i.e., $e(X_\Delta) = \#\Delta^n = d_n(\Delta)$ equals the number of fixed points of the natural torus action on the toric variety X_Δ . ■

For a noncompact complex algebraic variety X , the homology Betti numbers $b_k(X)$ and $b_k^{\text{clid}}(X)$ of the same degree k , but with different supports usually are different — for instance, we have $b_0(X) = 1$ (if X is connected) and $b_0^{\text{clid}}(X) = 0$. On the other hand, their alternating sums $e(X)$ and $e^{\text{clid}}(X)$ coincide, i.e., the topological Euler characteristic $e(X)$ equals the Euler characteristic for homology with closed supports $e^{\text{clid}}(X)$, see [Fu, Section 4.5, Exercise p. 95]. Here is another proof for toric varieties:

3.1.2 Proposition. *For every fan Δ in $N_\mathbb{R}$, the Euler characteristic for homology with closed supports*

$$e^{\text{clid}}(X_\Delta) := \sum_{k=0}^{\infty} (-1)^k \text{rk } H_k^{\text{clid}}(X_\Delta; \mathbb{Z}) = \sum_{k=0}^{\infty} (-1)^k \text{rk } H_c^k(X_\Delta; \mathbb{Z})$$

of the n -dimensional toric variety X_Δ is given by the number $d_n(\Delta)$ of n -dimensional cones in the fan Δ , i.e., $e^{\text{clid}}(X_\Delta) = d_n(\Delta) = e(X_\Delta)$.

Proof. We define the *Euler characteristic* $e(E^r)$ of each spectral term E^r of the *integral* (i.e., $G := \mathbb{Z}$) toric homology spectral sequence by

$$e(E^r) := \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \operatorname{rk} E_{p,q}^r.$$

In fact, this number is independent of r , i.e., $e(E^r) = e(H(E^r, d^r)) = e(E^{r+1})$ for $r \geq 1$: The differential $d_{p,q}^r: E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$ provides the equation

$$\operatorname{rk} E_{p,q}^r = \operatorname{rk} (\operatorname{im} d_{p,q}^r) + \operatorname{rk} (\operatorname{ker} d_{p,q}^r),$$

and the isomorphism $E_{p,q}^{r+1} \cong \operatorname{ker} d_{p,q}^r / \operatorname{im} d_{p+r,q-r+1}^r$ implies that

$$\operatorname{rk} (\operatorname{ker} d_{p,q}^r) = \operatorname{rk} E_{p,q}^{r+1} + \operatorname{rk} (\operatorname{im} d_{p+r,q-r+1}^r)$$

for all $p, q \in \mathbb{Z}$. Eliminating $\operatorname{rk} (\operatorname{ker} d_{p,q}^r)$, we have

$$\operatorname{rk} E_{p,q}^r = \operatorname{rk} E_{p,q}^{r+1} + \operatorname{rk} (\operatorname{im} d_{p,q}^r) + \operatorname{rk} (\operatorname{im} d_{p+r,q-r+1}^r).$$

Multiplying this equation by $(-1)^{p+q}$ for all $p, q \in \mathbb{Z}$ and summing up yields the desired equation.

The Euler characteristic $e^{\operatorname{cld}}(X_\Delta)$ coincides with the *Euler characteristic*

$$e(E^\infty) := \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \operatorname{rk} E_{p,q}^\infty,$$

as the isomorphisms

$$E_{p,q}^\infty \cong F_p H_{p+q}^{\operatorname{cld}}(X_\Delta; \mathbb{Z}) / F_{p-1} H_{p+q}^{\operatorname{cld}}(X_\Delta; \mathbb{Z})$$

imply the equations

$$\operatorname{rk} H_k^{\operatorname{cld}}(X_\Delta) = \sum_{p=0}^n \operatorname{rk} (F_p H_k^{\operatorname{cld}}(X_\Delta) / F_{p-1} H_k^{\operatorname{cld}}(X_\Delta)) = \sum_{p+q=k} \operatorname{rk} E_{p,q}^\infty$$

for all $k \geq 0$. Finally, since the spectral sequence degenerates, i.e., $E^\infty \cong E^r$ for some r , it suffices to compute $e(E^1) = e(E^r) = e^{\operatorname{cld}}(X_\Delta)$: Corollary 2.4.2 implies that

$$\begin{aligned} e(E^1) &= \sum_{p=0}^n \sum_{q=0}^p (-1)^{p+q} \binom{p}{q} d_{n-p}(\Delta) \\ &= d_n(\Delta) + \sum_{p=1}^n (-1)^p (-1+1)^p d_{n-p}(\Delta) = d_n(\Delta), \end{aligned}$$

where $d_{n-p}(\Delta) = \#\Delta^{n-p}$ denotes the number of $(n-p)$ -dimensional cones in the fan Δ . ■

The facts that $b_k^{\text{clid}}(X_\Delta) = \dim H_k^{\text{clid}}(X_\Delta; \mathbb{Q})$ and $E^\infty \cong E^2$ for the *rational* (i.e., $G = \mathbb{Q}$) toric homology spectral sequence admit a slightly different proof of the preceding proposition. Analogously, we could use the (integral or rational) toric cohomology spectral sequence and the fact that $b_k^{\text{clid}}(X_\Delta) = \dim H_c^k(X_\Delta; \mathbb{Q}) = \text{rk } H_c^k(X_\Delta; \mathbb{Z}) = b_c^k(X_\Delta)$.

3.1.3 Remark. Obviously, for $n \geq 2$, all natural numbers occur as Euler characteristic of a n -dimensional toric variety. In the case of *compact* toric varieties, however, there is a lower bound $e(X) \geq \dim(X) + 1$, and that bound is sharp:

It is realized by the projective space \mathbb{P}_n . The cones in the associated complete fan are generated by the proper subsets of the set $\{v_1, \dots, v_n, -\sum_{i=1}^n v_i\}$, where v_1, \dots, v_n is a basis of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

On the other hand, it follows by induction that every complete fan Δ in \mathbb{R}^n satisfies $\#\Delta^n \geq n + 1$: This is clear for $n = 1$. Now let Δ be a complete fan in \mathbb{R}^{n+1} with $n \geq 1$. We choose a cone $\sigma \in \Delta^{n+1}$ and a *rational supporting hyperplane* $H \subset \mathbb{R}^{n+1}$ for σ (i.e., a hyperplane generated by lattice points such that the cone σ is contained in one of the associated half-spaces) that does not contain any n -dimensional cone of the fan Δ . Then

$$\Delta' := \Delta \cap H := \{\tau \cap H; \tau \in \Delta\}$$

is obviously a complete fan in \mathbb{R}^n with

$$(\Delta')^n \subset \{\tau \cap H; \sigma \neq \tau \in \Delta^{n+1}\},$$

i. e., $\#\Delta^{n+1} \geq \#(\Delta')^n + 1 \geq n + 2$ by the induction hypothesis. ■

3.2 The Chow Groups of Toric Varieties

The computation of the torus-invariant principal divisors induced by the rational functions on the $(p + 1)$ -dimensional orbit closures leads to an explicit description of the Chow group $A_p(X_\Delta)$ as a quotient of a free abelian group of rank $d_{n-p}(\Delta)$ modulo a subgroup generated by $(p + 1)d_{n-p-1}(\Delta)$ elements.

We first describe the torus-invariant principal divisors on an orbit closure $V(\varrho) = \bar{O}_\varrho = \bigcup_{\varrho \prec \sigma \in \Delta} O_\sigma \subset X_\Delta$ of dimension $(p + 1)$, i.e., $\varrho \in \Delta^{n-p-1}$. As already mentioned in Section 1.2 (see p. 19), the orbit closure $V(\varrho)$ is a $(p + 1)$ -dimensional toric variety with the orbit O_ϱ as “big torus”. Namely, $V(\varrho) \cong X_{\Delta(\varrho)}$, where

$$\Delta(\varrho) = \left\{ [\sigma] := (\sigma + \text{lin } \varrho) / \text{lin } \varrho; \varrho \prec \sigma \in \Delta \right\}$$

is a finite rational fan with respect to the quotient lattice $N(\varrho) = N/N_\varrho \cong \mathbb{Z}^{p+1}$, and we may identify the orbits $O_\sigma \subset V(\varrho)$ with the orbits $O_{[\sigma]} \subset X_{\Delta(\varrho)}$ for

$\varrho \prec \sigma \in \Delta$, as well as their respective closures. Thus, the \mathbb{C} -algebra of rational functions on the orbit closure $V(\varrho)$ is given by the \mathbb{C} -algebra of rational functions on the algebraic torus $O_\varrho = O_{[\varrho]}$, i.e., it is generated by the regular functions χ^u where u is an element of the lattice

$$\mathrm{Hom}_{\mathbb{Z}}(N(\varrho), \mathbb{Z}) \cong N_\varrho^\perp := \{u \in M; \langle u, N_\varrho \rangle = 0\} = M(\varrho).$$

In order to compute the torus-invariant principal divisor $\mathrm{div}(\chi^u)$ on the orbit closure $V(\varrho)$ induced by such a nonzero regular function χ^u on the algebraic torus O_ϱ with $u \in M(\varrho) \setminus \{0\}$, we choose (primitive) lattice points $v_{\varrho, \sigma} \in \sigma \cap N$ for $\varrho \prec \sigma \in \Delta^{n-p}$ that generate the 1-dimensional cones $[\sigma] \in \Delta(\varrho)$, i.e., $\mathbb{Z}v_{\varrho, \sigma} + N_\varrho = N_\sigma$.

3.2.1 Proposition. *For $\varrho \in \Delta^{n-p-1}$ and $u \in M(\varrho) \setminus \{0\}$, the nonzero rational function χ^u on the orbit closure $V(\varrho)$ determines the torus-invariant principal divisor*

$$\mathrm{div}(\chi^u) = \sum_{\varrho \prec \sigma \in \Delta^{(n-p)}} \langle u, v_{\varrho, \sigma} \rangle \cdot V(\sigma),$$

where the (primitive) lattice points $v_{\varrho, \sigma} \in \sigma \cap N$ are chosen such that $\mathbb{Z}v_{\varrho, \sigma} + N_\varrho = N_\sigma$.

Proof. See [FuSt, Equation (1.1)]. ■

The Chow “homology” group $A_p(X) = Z_p(X)/\mathrm{Rat}_p(X)$ of an algebraic variety X consists by definition of rational equivalence classes of p -cycles, i.e., it is the quotient of the free abelian group $Z_p(X)$ generated by the p -dimensional irreducible closed subvarieties of X modulo the subgroup $\mathrm{Rat}_p(X)$ generated by the principal divisors on the $(p+1)$ -dimensional irreducible closed subvarieties of X . In general, it is difficult to compute Chow groups, since the free abelian group $Z_p(X)$ has infinitely many generators for $0 \leq p < \dim X$.

But in the case of a toric variety X_Δ , we may introduce the torus-invariant Chow group $A_p^\mathbb{T}(X_\Delta) := Z_p^\mathbb{T}(X_\Delta)/\mathrm{Rat}_p^\mathbb{T}(X_\Delta)$ as a subquotient of the Chow group $A_p(X_\Delta)$, where the free abelian group $Z_p^\mathbb{T}(X_\Delta)$ generated by the p -dimensional orbit closures is divided modulo rational equivalence, i.e., the subgroup $\mathrm{Rat}_p^\mathbb{T}(X_\Delta)$ is generated by the torus-invariant principal divisors on the $(p+1)$ -dimensional orbit closures. Since the orbit structure is finite, the torus-invariant Chow groups are finitely generated and can be computed explicitly; even better, the natural inclusion $A_p^\mathbb{T}(X_\Delta) \subset A_p(X_\Delta)$ turns out to be an isomorphism:

3.2.2 Proposition. *For every fan Δ in $N_\mathbb{R}$ and every natural number p , there is a canonical isomorphism*

$$A_p^\mathbb{T}(X_\Delta) \cong A_p(X_\Delta).$$

In other words, the Chow group $A_p(X_\Delta)$ is generated by the rational equivalence classes of the p -dimensional orbit closures, i.e., the classes $[V(\sigma)]$ for $\sigma \in \Delta^{n-p}$;

and the group of relations on these generators is generated by the torus-invariant principal divisors on the $(p+1)$ -dimensional orbit closures, i.e, by all the relations of Proposition 3.2.1 above, where ϱ runs over the set Δ^{n-p-1} , and u runs over a generating set of the lattice $M(\varrho)$.

Proof. See [FuSt, Proposition 1.1]. ■

For an explicit computation of the Chow group $A_p(X_\Delta)$, we thus have to calculate a quotient of a free abelian group of rank $d_{n-p}(\Delta)$ modulo a subgroup generated by $(p+1)d_{n-p-1}(\Delta)$ elements. Corollary A.3 provides a formula for the computation of the associated rank and torsion coefficients.

3.2.3 Example. The Chow group $A_{n-1}(X_\Delta)$ is nothing else but the group $\text{ClDiv}_W^\mathbb{T}(X_\Delta)$ of rational equivalence classes of torus-invariant Weil divisors on the toric variety X_Δ . It is isomorphic to the quotient of the lattice \mathbb{Z}^d modulo the sublattice generated by the columns of the transposed matrix $(v_1, v_2, \dots, v_d)^{\text{tr}}$, where $d := d_1(\Delta)$ and v_1, \dots, v_d denote the primitive lattice points that generate the 1-dimensional cones of the fan Δ . This description follows directly from Proposition 3.2.2, since the lattice $M(\{0\}) \cong M$ is generated by the basis e_1^*, \dots, e_n^* dual to the canonical basis e_1, \dots, e_n of the lattice N ; hence,

$$A_{n-1}(X_\Delta) \cong \left(\bigoplus_{\sigma \in \Delta^1} \mathbb{Z} \cdot [V(\sigma)] \right) / \left\langle \sum_{j=1}^d \langle e_i^*, v_j \rangle [V(\mathbb{R}_{\geq 0} v_j)]; i = 1, \dots, n \right\rangle.$$

3.2.4 Remark. It is shown in [BBFK, Théorème principal 2.1] that there is an isomorphism $H_{2n-2}^{\text{clid}}(X_\Delta; \mathbb{Z}) \cong \text{ClDiv}_W^\mathbb{T}(X_\Delta) = A_{n-1}(X_\Delta)$ iff $s(\Delta) = \dim_{\mathbb{R}} \text{lin } \Delta \geq n-1$. We will verify this statement and look for related results in the following sections.

3.3 The Terms $E_{p,p}^2$ of the Integral Toric Homology Spectral Sequence

The best result about the relation between Chow theory and homology that one could hope for is that the natural homomorphism $A_\bullet(X) \rightarrow H_{2\bullet}(X; \mathbb{Z})$ is an isomorphism. By the remarkable Theorem of Jurkiewicz-Danilov, see [Da, Theorem 10.8], that holds in fact if X is a smooth compact toric variety. In the case of arbitrary toric varieties X_Δ , there is a canonical isomorphism $A_{n-1}(X_\Delta) \oplus \mathbb{Z}^b \cong H_{2n-2}^{\text{clid}}(X_\Delta; \mathbb{Z})$, where $b := \binom{n-s(\Delta)}{2}$, see [BBFK, Théorème principal 2.1]. More general, the Chow group $A_p(X_\Delta)$ appears as term $E_{p,p}^2$ of the integral toric homology spectral sequence:

3.3.1 Theorem. *For every fan Δ in $N_{\mathbb{R}}$ and every natural number p , there is a canonical isomorphism between the Chow group $A_p(X_\Delta)$ and the term $E_{p,p}^2 \cong H_p(\Delta, H_p^\Delta) \cong H_p(\Delta, \Lambda_p^\Delta)$ of the integral toric homology spectral sequence.*

Proof. By Proposition 3.2.2, the Chow group $A_p(X_\Delta)$ is isomorphic to the quotient of the free abelian group

$$\bigoplus_{\sigma \in \Delta^{(n-p)}} \mathbb{Z} \cdot [V(\sigma)]$$

modulo the subgroup generated by the elements

$$\sum_{\varrho \prec \sigma \in \Delta^{(n-p)}} \langle u, v_{\varrho, \sigma} \rangle \cdot [V(\sigma)],$$

where ϱ runs over the set Δ^{n-p-1} , u runs over a generating set of the lattice $M(\varrho)$, and $v_{\varrho, \sigma} \in \sigma \cap N$ denotes a lattice vector with $\mathbb{Z}v_{\varrho, \sigma} + N_\varrho = N_\sigma$.

On the other hand, by Theorem 2.4.1,

$$E_{p,p}^2 \cong H_p(\Delta, H_p^\Delta) \cong H_p(\Delta, \Lambda_p^\Delta) = \ker \delta_p / \text{im } \delta_{p+1}$$

with the differentials

$$\delta_p: C_p(\Delta, \Lambda_p^\Delta) \rightarrow C_{p-1}(\Delta, \Lambda_p^\Delta)$$

and

$$\delta_{p+1}: C_{p+1}(\Delta, \Lambda_p^\Delta) \rightarrow C_p(\Delta, \Lambda_p^\Delta).$$

Now

$$C_{p-1}(\Delta, \Lambda_p^\Delta) = \bigoplus_{\tau \in \Delta^{(n-p+1)}} \underbrace{\Lambda^p(N/N_\tau)}_{\cong \mathbb{Z}^{p-1}} = 0$$

yields $\ker \delta_p = C_p(\Delta, \Lambda_p^\Delta)$. Moreover, we have a canonical isomorphism

$$C_p(\Delta, \Lambda_p^\Delta) = \bigoplus_{\sigma \in \Delta^{(n-p)}} \underbrace{\Lambda^p(N/N_\sigma)}_{\cong \mathbb{Z}^p} \cong \bigoplus_{\sigma \in \Delta^{(n-p)}} \mathbb{Z} \cdot [V(\sigma)]$$

as follows: For each cone $\sigma \in \Delta^{n-p}$, we choose a direct sum decomposition $N = N_\sigma \oplus N'_\sigma$ and a basis $v_{\sigma,1}, \dots, v_{\sigma,p}$ of the lattice N'_σ . Then the assignment $v_{\sigma,1} \wedge \dots \wedge v_{\sigma,p} \mapsto [V(\sigma)]$ induces an isomorphism between

$$\bigoplus_{\sigma \in \Delta^{(n-p)}} \Lambda^p N'_\sigma \cong \bigoplus_{\sigma \in \Delta^{(n-p)}} \Lambda^p(N/N_\sigma) \quad \text{and} \quad \bigoplus_{\sigma \in \Delta^{(n-p)}} \mathbb{Z} \cdot [V(\sigma)].$$

Since

$$C_{p+1}(\Delta, \Lambda_p^\Delta) = \bigoplus_{\varrho \in \Delta^{(n-p-1)}} \Lambda^p N(\varrho),$$

the subgroup $\text{im } \delta_{p+1} \subset C_p(\Delta, \Lambda_p^\Delta)$ is generated by the subgroups

$$\delta_{p+1}(\Lambda^p N(\varrho)) = \text{im} \left[\sum_{\varrho \prec \sigma \in \Delta^{(n-p)}} o_{\varrho, \sigma} \nu_{\varrho, \sigma}^p: \Lambda^p N(\varrho) \rightarrow \bigoplus_{\sigma \in \Delta^{(n-p)}} \Lambda^p N(\sigma) \right]$$

for $\varrho \in \Delta^{n-p-1}$. Hence, it suffices to show for each cone $\varrho \in \Delta^{n-p-1}$ that the subgroup

$$\delta_{p+1}(\Lambda^p N(\varrho)) \subset C_p(\Delta, \Lambda_p^\Delta)$$

corresponds to the subgroup of $\bigoplus_{\sigma \in \Delta^{(n-p)}} \mathbb{Z} \cdot [V(\sigma)]$ generated by the elements

$$\sum_{\varrho \prec \sigma \in \Delta^{(n-p)}} \langle u, v_{\varrho, \sigma} \rangle \cdot [V(\sigma)],$$

where u runs over a generating set of the lattice $M(\varrho)$. In order to do this, we choose lattice vectors v_0, \dots, v_p such that

$$N = N_\varrho \oplus \mathbb{Z}v_0 \oplus \dots \oplus \mathbb{Z}v_p.$$

Hence, for $k = 0, \dots, p$, the wedge products $v_0 \wedge \dots \wedge \widehat{v}_k \wedge \dots \wedge v_p$ may be interpreted as a basis of the exterior power $\Lambda^p N(\varrho)$. Furthermore, for $k = 0, \dots, p$, the lattice vectors $u_k \in M$ determined by $u_k(N_\varrho) = 0$ and $u_k(v_l) = \delta_{kl}$ form a basis of the lattice $M(\varrho) \cong N_\varrho^\perp$. It now suffices to show that, for $k = 0, \dots, p$, the mapping

$$o_{\varrho, \sigma} \nu_{\varrho, \sigma}^p: \Lambda^p N(\varrho) \rightarrow \Lambda^p N(\sigma)$$

maps $v_0 \wedge \dots \wedge \widehat{v}_k \wedge \dots \wedge v_p$ onto $(-1)^k \langle u_k, v_{\varrho, \sigma} \rangle v_{\sigma, 1} \wedge \dots \wedge v_{\sigma, p}$: In the quotient lattice $N(\varrho)$, we have the equations

$$v_{\sigma, i} = \sum_{j=0}^p \langle u_j, v_{\sigma, i} \rangle v_j$$

for $\sigma \in \Delta^{n-p}$ and $i = 0, \dots, p$, where $v_{\sigma, 0} := v_{\varrho, \sigma}$. In other words,

$$\begin{pmatrix} v_{\sigma, 0} \\ \vdots \\ v_{\sigma, p} \end{pmatrix} = C \cdot \begin{pmatrix} v_0 \\ \vdots \\ v_p \end{pmatrix}$$

with the matrix $C := (\langle u_j, v_{\sigma, i} \rangle)_{0 \leq i, j \leq p}$. Since v_0, \dots, v_p as well as $v_{\sigma, 0}, \dots, v_{\sigma, p}$ are bases of the lattice $N(\varrho)$, we know that $\det C = \pm 1$. In fact, we may assume that $\det C = +o_{\varrho, \sigma}$ for all cones σ that have the cone ϱ as a facet. (Remember that $v_{\sigma, 0} = v_{\varrho, \sigma}$ is a lattice vector in the cone σ !) Since

$$\begin{pmatrix} v_0 \\ \vdots \\ v_p \end{pmatrix} = C^{-1} \cdot \begin{pmatrix} v_{\sigma, 0} \\ \vdots \\ v_{\sigma, p} \end{pmatrix},$$

the mapping

$$\nu_{\varrho, \sigma}: N(\varrho) \cong \mathbb{Z}v_0 \oplus \dots \oplus \mathbb{Z}v_p \twoheadrightarrow N(\sigma) \cong \mathbb{Z}v_{\sigma, 1} \oplus \dots \oplus \mathbb{Z}v_{\sigma, p}$$

is given by the matrix C' , where we get the matrix C' by elimination of the 0th column of the matrix C^{-1} . As a consequence, the mapping $\nu_{\varrho,\sigma}^p: \Lambda^p N(\varrho) \rightarrow \Lambda^p N(\sigma)$, i.e.,

$$\nu_{\varrho,\sigma}^p: \bigoplus_{0 \leq k \leq p} \mathbb{Z} v_0 \wedge \dots \wedge \widehat{v}_k \wedge \dots \wedge v_p \twoheadrightarrow \mathbb{Z} v_{\sigma,1} \wedge \dots \wedge v_{\sigma,p},$$

maps $v_0 \wedge \dots \wedge \widehat{v}_k \wedge \dots \wedge v_p$ onto $\det C'' v_{\sigma,1} \wedge \dots \wedge v_{\sigma,p}$, where we get the matrix C'' by elimination of the k th row of the matrix C' , cf. [Bo, III §8.5 Proposition 10]. Since $(C^{-1})^{-1} = C$, we get the equation $(-1)^k \det C'' / \det(C^{-1}) = C_{0,k}$, which implies the equation

$$o_{\varrho,\sigma} \det C'' v_{\sigma,1} \wedge \dots \wedge v_{\sigma,p} = (-1)^k \langle u_k, v_{\varrho,\sigma} \rangle v_{\sigma,1} \wedge \dots \wedge v_{\sigma,p}. \quad \blacksquare$$

3.4 Toric Varieties of Dimension 1 or 2

As there are only three different 1-dimensional toric varieties — \mathbb{C}^* , \mathbb{C} , and \mathbb{P}_1 —, it is immediate to write down completely their integral (co-)homology $H_{\bullet}^{\text{cld}}(X; \mathbb{Z})$ and $H_c^{\bullet}(X; \mathbb{Z})$; in particular, these groups are always free abelian. In the 2-dimensional case, we can also give explicit formulæ for computing all integral (co-)homology in terms of the fan. Torsion may occur, namely, in the groups $H_2^{\text{cld}}(X_{\Delta}; \mathbb{Z})$ and $H_c^3(X_{\Delta}; \mathbb{Z})$. For a nondegenerate toric surface, all Betti numbers are combinatorial invariants, in contrary to the torsion coefficients. Furthermore, for every toric surface $X \not\cong (\mathbb{C}^*)^2$, we have isomorphisms $A_{\bullet}(X) \cong H_{2\bullet}^{\text{cld}}(X; \mathbb{Z})$.

The integral (co-)homology of the 1-dimensional toric varieties is well-known. All (co-)homology groups are free abelian. We can distinguish the three 1-dimensional toric varieties even by their Euler characteristic:

X_{Δ}	$H_0^{\text{cld}} \cong H_c^0$	$H_1^{\text{cld}} \cong H_c^1$	$H_2^{\text{cld}} \cong H_c^2$	$e(X_{\Delta}) = d_1(\Delta)$
\mathbb{C}^*	0	\mathbb{Z}	\mathbb{Z}	0
\mathbb{C}	0	0	\mathbb{Z}	1
\mathbb{P}_1	\mathbb{Z}	0	\mathbb{Z}	2

Table 3.1: Integral (co-)homology groups and Euler characteristics of the 1-dimensional toric varieties.

Furthermore, $A_0(X_{\Delta}) \cong H_0^{\text{cld}}(X_{\Delta}; \mathbb{Z})$ and $A_1(X_{\Delta}) \cong H_2^{\text{cld}}(X_{\Delta}; \mathbb{Z})$ for all 1-dimensional toric varieties X_{Δ} .

A fan Δ in $N_{\mathbb{R}}$ and the associated toric variety X_{Δ} are called *simplicial* if each cone $\sigma \in \Delta$ is simplicial. In particular, toric varieties of dimension 1 or 2 are simplicial. Simplicial toric varieties have at most quotient singularities. The

Betti numbers $b_k := b_k(X_\Delta) = \text{rk } H_k(X_\Delta; \mathbb{Z})$ of a compact simplicial toric variety X_Δ are *combinatorial invariants*, i.e., two fans that are isomorphic as partially ordered (by inclusion) sets lead to the same Betti numbers:

3.4.1 Theorem. *For every complete simplicial fan Δ in $N_{\mathbb{R}}$, the compact simplicial toric variety X_Δ has the Betti numbers*

$$b_{2p}(X_\Delta) = \sum_{i=p}^n (-1)^{i-p} \binom{i}{p} d_{n-i}(X_\Delta)$$

and

$$b_{2p+1}(X_\Delta) = 0$$

for all natural numbers p .

Proof. Cf. [Da, Theorem 10.8, Remark 10.9, Proposition 12.11]. ■

For 2-dimensional compact toric varieties X_Δ , these formulæ are verified in the following two theorems, since $b_k(X) = b_k^{\text{cld}}(X) = b_c^k(X)$ for all compact varieties X and integers k . The Betti numbers $b_k^{\text{cld}}(X_\Delta) = b_c^k(X_\Delta)$ are combinatorial invariants for $k \in \{0, 1, 4\}$, for $k = 2, 3$, they are combinatorial invariants if in addition $s(\Delta)$ is fixed. Torsion may occur in the (co-)homology groups $H_2^{\text{cld}}(X_\Delta; \mathbb{Z})$ and $H_c^3(X_\Delta; \mathbb{Z})$ even if the toric variety X_Δ is compact:

3.4.2 Theorem. *For every nonzero (only needed in (c) below) fan Δ in $N_{\mathbb{R}} \cong \mathbb{R}^2$, the integral homology with closed supports of the 2-dimensional toric variety X_Δ may be expressed by the E^2 -terms of the associated toric homology spectral sequence as follows:*

- (a) $H_0^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{0,0}^2 \cong A_0(X_\Delta) \cong \mathbb{Z}^{b_0^{\text{cld}}}$, where $b_0^{\text{cld}} = 0$ if Δ is noncomplete, and $b_0^{\text{cld}} = 1 = d_2 - d_1 + d_0$ if Δ is complete,
- (b) $H_1^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{1,0}^2 \cong \mathbb{Z}^{b_1^{\text{cld}}}$, where $b_1^{\text{cld}} = 0$ if Δ is zero or complete, and $b_1^{\text{cld}} = -d_2 + d_1 - d_0$ else,
- (c) $H_2^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{1,1}^2 \cong A_1(X_\Delta) \cong \mathbb{Z}^{b_2^{\text{cld}}} \oplus \mathbb{Z}_t$, where $b_2^{\text{cld}} = d_1 - s(\Delta)$ and $t = \text{gcd} \{ \det(v_i, v_j); 1 \leq i < j \leq d_1 \}$ if Δ^1 is generated by the primitive lattice points v_1, \dots, v_{d_1} ; for the zero fan, $0 = A_1((\mathbb{C}^*)^2)$ is a proper subgroup of $H_2^{\text{cld}}((\mathbb{C}^*)^2; \mathbb{Z}) \cong E_{2,0}^2 \cong \mathbb{Z}$,
- (d) $H_3^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{2,1}^2 \cong \mathbb{Z}^{b_3^{\text{cld}}}$, where $b_3^{\text{cld}} = 2 - s(\Delta)$,
- (e) $H_4^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{2,2}^2 \cong A_2(X_\Delta) \cong \mathbb{Z}$.

Proof. The formulæ are easy consequences of Theorem 2.4.12 in connection with Corollary 2.4.9 and Corollary A.3. For complete fans Δ , they may also be achieved by computing the cellular homology of the finite CW-complex X_Δ , cf. [Fi, Lemma 3.1]. ■

Note that the homology group $H_3^{\text{cld}}(X_\Delta; \mathbb{Z})$ is zero iff the fan Δ is nondegenerate. Moreover, the cohomology $H_c^\bullet(X_\Delta; \mathbb{Z})$ may be computed directly from the homology $H_\bullet^{\text{cld}}(X_\Delta; \mathbb{Z})$ via the exact sequence

$$0 \rightarrow \text{Ext}(H_c^{p+1}(X_\Delta), \mathbb{Z}) \rightarrow H_p^{\text{cld}}(X_\Delta) \rightarrow \text{Hom}(H_c^p(X_\Delta), \mathbb{Z}) \rightarrow 0,$$

see [Br, Exercise V 29, p. 252], i.e., via the formulæ

$$b_c^p(X_\Delta) = b_p^{\text{cld}}(X_\Delta) \quad \text{and} \quad \text{Tor } H_c^{p+1}(X_\Delta; \mathbb{Z}) = \text{Tor } H_p^{\text{cld}}(X_\Delta; \mathbb{Z})$$

for all integers p . Via the toric cohomology spectral sequence, we obtain the following expressions:

3.4.3 Theorem. *For every nonzero (only needed in (c) below) fan Δ in $N_{\mathbb{R}} \cong \mathbb{R}^2$, the integral cohomology with compact supports of the 2-dimensional toric variety X_Δ may be expressed by the E_2 -terms of the associated toric cohomology spectral sequence as follows:*

- (a) $H_c^0(X_\Delta; \mathbb{Z}) \cong E_2^{0,0} \cong \mathbb{Z}^{b_c^0}$, where $b_c^0 = 0$ if Δ is noncomplete, and $b_c^0 = 1 = d_2 - d_1 + d_0$ if Δ is complete,
- (b) $H_c^1(X_\Delta; \mathbb{Z}) \cong E_2^{1,0} \cong \mathbb{Z}^{b_c^1}$, where $b_c^1 = 0$ if Δ is zero or complete, and $b_c^1 = -d_2 + d_1 - d_0$ else,
- (c) $H_c^2(X_\Delta; \mathbb{Z}) \cong E_2^{1,1} = \ker [d_1^{1,1}: E_1^{1,1} \rightarrow E_1^{2,1}] \cong \mathbb{Z}^{b_c^2}$, where $b_c^2 = d_1 - s(\Delta)$; for the zero fan, $H_c^2((\mathbb{C}^*)^2; \mathbb{Z}) \cong E_2^{2,0} \cong \mathbb{Z}$,
- (d) $H_c^3(X_\Delta; \mathbb{Z}) \cong E_2^{2,1} = E_1^{2,1} / \text{im} [d_1^{1,1}: E_1^{1,1} \rightarrow E_1^{2,1}] \cong \mathbb{Z}^{b_c^3} \oplus \mathbb{Z}_t$, where $b_c^3 = 2 - s(\Delta)$ and $t = \text{gcd} \{ \det(v_i, v_j); 1 \leq i < j \leq d_1 \}$ if Δ^1 is generated by the primitive lattice points v_1, \dots, v_{d_1} ; in particular, $t = 1$ if $s(\Delta) \leq 1$,
- (e) $H_c^4(X_\Delta; \mathbb{Z}) \cong E_2^{2,2} \cong \mathbb{Z}$. ■

3.4.4 Examples.

- (a) The homology $H_\bullet^{\text{cld}}(X_\Delta; \mathbb{Z})$ of a compact 2-dimensional toric variety X_Δ is completely characterized by the homology group $H_2^{\text{cld}}(X_\Delta; \mathbb{Z})$, i.e., the Betti number $b_2^{\text{cld}} = d_1 - 2 \geq 1$ and the torsion coefficient $t \geq 1$ of the cyclic group $\text{Tor } H_2^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong \mathbb{Z}_t$. In fact, all finitely generated abelian groups $\mathbb{Z}^b \oplus \mathbb{Z}_t$ with $b, t \geq 1$ appear as homology group $H_2^{\text{cld}}(X_\Delta; \mathbb{Z})$: Consider for

instance the complete fan $\Delta_{b,t}$ in \mathbb{R}^2 whose 1-dimensional cones are spanned by the primitive lattice points $(-1, 0)$, $(1, -t)$, and $(1, t)$, $(1, 2t), \dots, (1, bt)$! In addition, it is clear that the torsion coefficient t is not combinatorially invariant.

- (b) The homology $H_{\bullet}^{\text{cld}}(X_{\Delta}; \mathbb{Z})$ of a *nondegenerate noncompact* 2-dimensional toric variety is completely characterized by the following homology groups: $H_1^{\text{cld}}(X_{\Delta}; \mathbb{Z}) \cong \mathbb{Z}^{-d_2+d_1-d_0}$ and $H_2^{\text{cld}}(X_{\Delta}; \mathbb{Z}) \cong \mathbb{Z}^{d_1-2} \oplus \mathbb{Z}_t$. Obviously, all possible combinations are already realized by appropriate subfans of the fans $\Delta_{b,t}$ in Example (a).
- (c) Not even the 2-dimensional smooth compact toric varieties are classified up to isomorphism by their homology: For instance, look at the smooth compact *Hirzebruch surfaces* $\Sigma_{\ell} := X_{\Delta(\ell)}$ associated to the complete fans $\Delta(\ell)$ in \mathbb{R}^2 that are spanned by the primitive lattice points $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(\ell, -1)$ for $\ell \geq 0$. The isomorphism class of Σ_{ℓ} corresponds to the following *weighted dual graph*, which describes the self-intersection numbers of the 1-dimensional orbit closures, cf. [Od, Cor. 1.29]:

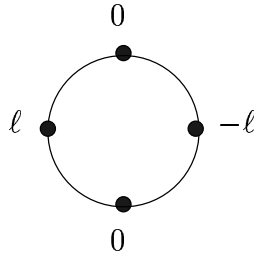


Figure 3.1: The weighted dual graph of the Hirzebruch surface Σ_{ℓ} .

Hence, the Hirzebruch surfaces Σ_{ℓ} are pairwise nonisomorphic, whereas all homology groups $H_2(\Sigma_{\ell}; \mathbb{Z}) \cong \mathbb{Z}^2$ coincide.

For the computation of the torsion coefficient t of the group $H_2^{\text{cld}}(X_{\Delta}; \mathbb{Z})$ it is sufficient to calculate the greatest common divisor of the determinants $\det(v, w)$ where the pair (v, w) ranges over the primitive generators of adjacent 1-dimensional cones in the fan Δ . The number $\det(v, w)$, which is called the *multiplicity* $\text{mult } \sigma$ of the cone $\sigma := \text{pos}\{v, w\}$, has the following interpretations:

- (a) The $\det(v, w)$ lattice points in the half-open parallelogram $[0, 1[v + [0, 1[w$ form a complete system of representatives of the cyclic quotient group $\mathbb{Z}^2/(\mathbb{Z}v + \mathbb{Z}w)$:

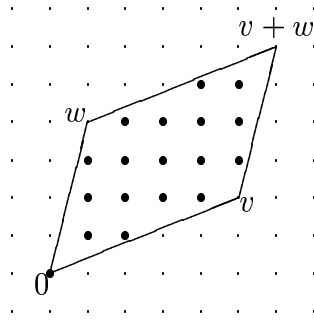


Figure 3.2: Complete system of $\det(v, w)$ representatives of the cyclic quotient group $\mathbb{Z}^2 / (\mathbb{Z}v + \mathbb{Z}w)$.

(b) The intersection number of the associated orbit closures is

$$V(\mathbb{R}_{\geq 0}v) \cdot V(\mathbb{R}_{\geq 0}w) = \det(v, w).$$

(c) The affine toric variety X_σ associated to the cone $\sigma = \text{pos}\{v, w\}$ is a quotient of the complex plane \mathbb{C}^2 by an action of the cyclic group \mathbb{Z}_d , where $d := \det(v, w)$, i.e., X_σ is a *cyclic quotient singularity*, cf. [Fu, Section 2.2, p. 32].

If the fan Δ' is a subdivision of the fan Δ , for instance, if we consider a resolution of singularities $X_{\Delta'} \rightarrow X_\Delta$, then the associated torsion coefficient does not increase. But even if the set Δ^1 consists of many rays, i.e., the associated Betti number b_2^{cld} is large, the associated torsion coefficient t need not be small or even vanish, see Examples 3.4.4 (a), (b).

Finally, the torsion coefficient t of a nondegenerate 2-dimensional toric variety may be interpreted as the order of the intersection of the isotropy groups of the (at most) 1-dimensional orbits of the natural torus action:

3.4.5 Proposition. *If Δ is a nondegenerate fan in $N_{\mathbb{R}} \cong \mathbb{R}^2$, then*

$$\text{Tor } H_2^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong \text{Tor } H_c^3(X_\Delta; \mathbb{Z}) \cong \bigcap_{\sigma \in \Delta^{\geq 1}} I_\sigma = \bigcap_{\varrho \in \Delta^1} I_\varrho.$$

Proof. The claim follows from Theorem 3.4.2(c) observing that we have $I_\sigma = \{t \in \text{Hom}(M, \mathbb{C}^*); t(u) = 1 \text{ for } u \in \sigma^\perp \cap M\}$, i.e., $I_\sigma \cong (\mathbb{C}^*)^2$ for $\sigma \in \Delta^2$ and $I_\varrho \cong \{(t_1, t_2) \in (\mathbb{C}^*)^2; t_1^{v_2} t_2^{-v_1} = 1\} \cong \mathbb{C}^*$ if $\varrho \in \Delta^1$ is generated by the primitive lattice point (v_1, v_2) . ■

3.5 Toric Varieties of Dimension 3

For every 3-dimensional toric variety, we can explicitly compute all integral (co-)homology in terms of the fan. The Betti numbers $b_k^{\text{cld}}(X_\Delta) = b_c^k(X_\Delta)$ are combinatorial invariants for $k \in \{0, 1, 6\}$; for $k = 4, 5$, they are combinatorial invariants if in addition $s(\Delta)$ is fixed. Torsion may occur, namely, in the groups $H_k^{\text{cld}}(X_\Delta; \mathbb{Z})$ and $H_c^{k+1}(X_\Delta; \mathbb{Z})$ for $k \in \{2, 3, 4\}$, even if the toric variety X_Δ is compact. If the set S_Δ is connected, then $A_1(X_\Delta) \cong H_2^{\text{cld}}(X_\Delta; \mathbb{Z})$; otherwise, the Chow group $A_1(X_\Delta)$ is a proper subgroup of the homology group $H_2^{\text{cld}}(X_\Delta; \mathbb{Z})$. We have isomorphisms $A_2(X_\Delta) \cong H_4^{\text{cld}}(X_\Delta; \mathbb{Z})$ unless X_Δ is isomorphic to $(\mathbb{C}^*)^3$, $\mathbb{C} \times (\mathbb{C}^*)^2$, or $\mathbb{P}_1 \times (\mathbb{C}^*)^2$.

3.5.1 Theorem. *For every nonzero fan Δ in $N_{\mathbb{R}} \cong \mathbb{R}^3$, the integral homology with closed supports of the 3-dimensional toric variety X_Δ may be expressed by the E^2 -terms of the associated toric homology spectral sequence as follows:*

- (a) $H_0^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{0,0}^2 \cong A_0(X_\Delta) \cong \mathbb{Z}^{b_0^{\text{cld}}}$, where $b_0^{\text{cld}} = 0$ if Δ is noncomplete, and $b_0^{\text{cld}} = 1$ if Δ is complete,
- (b) $H_1^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{1,0}^2 \cong H^1(S_\Delta; \mathbb{Z}) \cong \mathbb{Z}^{b_1^{\text{cld}}}$,
- (c) $H_2^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{1,1}^2 \oplus E_{2,0}^2 \cong A_1(X_\Delta) \oplus \mathbb{Z}^k$, where $k + 1$ equals the number of connected components of the set S_Δ ,
- (d) $H_3^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{2,1}^2$,
- (e) $H_4^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{2,2}^2 \oplus E_{3,1}^2 \cong A_2(X_\Delta) \oplus \mathbb{Z}^{\binom{3-s(\Delta)}{2}}$ with $b_4^{\text{cld}} = d_1(\Delta) - s(\Delta) + \binom{3-s(\Delta)}{2}$; in particular, $H_4^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong A_2(X_\Delta)$ iff $s(\Delta) > 1$, i.e., unless X_Δ is isomorphic to $(\mathbb{C}^*)^3$, $\mathbb{C} \times (\mathbb{C}^*)^2$, or $\mathbb{P}_1 \times (\mathbb{C}^*)^2$,
- (f) $H_5^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{3,2}^2 \cong \mathbb{Z}^{b_5^{\text{cld}}}$, where $b_5^{\text{cld}} = 3 - s(\Delta)$,
- (g) $H_6^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{3,3}^2 \cong A_3(X_\Delta) \cong \mathbb{Z}$.

Proof. The formulæ are easy consequences of Theorem 2.4.12 in connection with Corollary 2.4.9 and Proposition 2.4.10. ■

3.5.2 Theorem. *For every nonzero fan Δ in $N_{\mathbb{R}} \cong \mathbb{R}^3$, the integral cohomology with compact supports of the 3-dimensional toric variety X_Δ may be expressed by the E_2 -terms of the associated toric cohomology spectral sequence as follows:*

- (a) $H_c^0(X_\Delta; \mathbb{Z}) \cong E_2^{0,0} \cong \mathbb{Z}^{b_c^0}$, where $b_c^0 = 0$ if Δ is noncomplete, and $b_c^0 = 1$ if Δ is complete,
- (b) $H_c^1(X_\Delta; \mathbb{Z}) \cong E_2^{1,0} \cong H_1(S_\Delta; \mathbb{Z}) \cong \mathbb{Z}^{b_c^1}$,

- (c) $H_c^2(X_\Delta; \mathbb{Z}) \cong E_2^{1,1} \oplus E_2^{2,0} \cong E_2^{1,1} \oplus \mathbb{Z}^k$, where $k + 1$ equals the number of connected components of the set S_Δ ,
- (d) $H_c^3(X_\Delta; \mathbb{Z}) \cong E_2^{2,1}$,
- (e) $H_c^4(X_\Delta; \mathbb{Z}) \cong E_2^{2,2} \oplus E_2^{3,1}$, where $b_c^4 = d_1(\Delta) - s(\Delta) + \binom{3-s(\Delta)}{2}$,
- (f) $H_c^5(X_\Delta; \mathbb{Z}) \cong E_2^{3,2}$, where $b_c^5 = 3 - s(\Delta)$,
- (g) $H_c^6(X_\Delta; \mathbb{Z}) \cong E_2^{3,3} \cong \mathbb{Z}$. ■

For all 3-dimensional compact toric varieties X_Δ , the formulæ in Theorem 3.4.1 remain correct with the exception of $b_2(X_\Delta)$ and $b_3(X_\Delta)$. Furthermore, the differences

$$b_{2p}(X_\Delta) - b_{2p+1}(X_\Delta) = \sum_{i=p}^n (-1)^{i-p} \binom{i}{p} d_{n-i}(X_\Delta)$$

are combinatorial invariants for all natural numbers p :

3.5.3 Proposition. *For every 3-dimensional compact toric variety X_Δ , the Betti numbers $b_p := b_p(X_\Delta)$ may be expressed by the numbers $d_i := d_i(\Delta)$ of i -dimensional cones in the fan Δ as follows:*

- (a) $b_0 = 1 = d_3 - d_2 + d_1 - 1$,
- (b) $b_1 = 0$,
- (c) $b_2 = d_2 - 2d_1 + 3 + b_3$,
- (d) $b_3 = -d_2 + 2d_1 - 3 + b_2$,
- (e) $b_4 = d_1 - 3$,
- (f) $b_5 = 0$,
- (g) $b_6 = 1 = d_0$.

Proof. Consider the cell decomposition of the unit sphere S^2 induced by intersection with the cones of the complete fan Δ . Then *Euler's formula* says that $d_3 - d_2 + d_1 = 2$, i.e., $d_3 - d_2 + d_1 - d_0 = 1$. Now the equations (a), (b), (e), (f), and (g) follow from Theorem 3.5.1. Finally, by Theorem 3.1.1, $e(X_\Delta) = \sum_{p=0}^{\infty} (-1)^p b_p = d_3$, what implies that the difference $b_3 - b_2 = -d_2 + 2d_1 - 3$ is a combinatorial invariant. ■

Note that, in general, the Betti numbers $b_2(X_\Delta)$ and $b_3(X_\Delta)$ are not combinatorial invariants. Furthermore, $b_3(X_\Delta) = 0$ does not imply the simplicity of the fan Δ :

3.5.4 Examples.

- (a) Let Δ be the (nonsimplicial) complete fan in $N_{\mathbb{R}} \cong \mathbb{R}^3$ generated by the faces of the prism with vertices $v_1 := e_1 + e_2 + e_3$, $v_2 := -e_1 + e_2 + e_3$, $v_3 := -e_2 + e_3$, and $v_i := v_{i-3} - 2e_3$ for $i = 4, 5, 6$ according to the following figure, where e_1, e_2, e_3 denotes the canonical basis of the lattice $N \cong \mathbb{Z}^3$:

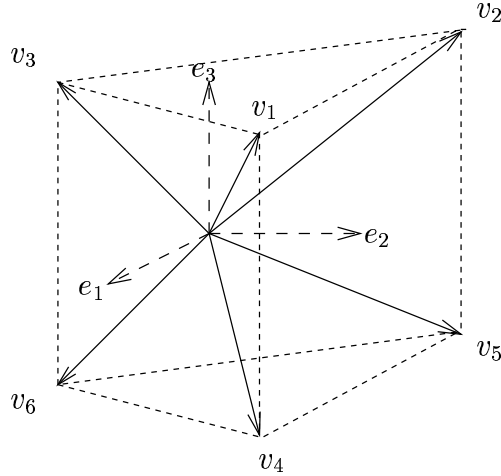


Figure 3.3: A (nonsimplicial) complete fan Δ in $N_{\mathbb{R}} \cong \mathbb{R}^3$ with Betti number $b_3(X_\Delta) \neq 0$.

By [BBFK, Exemple 3.5] or by Theorem 3.5.1, the toric variety X_Δ has the Betti numbers $b_2(X_\Delta) = 1$ and $b_3(X_\Delta) = 1$. With $d_2 = 9$ and $d_1 = 6$, the combinatorial invariant $b_2 - b_3 = d_2 - 2d_1 + 3 = 0$ is easily verified.

- (b) If the lattice point v_1 is replaced by $v'_1 := 2v_1 - v_2 = 3e_1 + e_2 + e_3$, then the so-defined fan Δ' is combinatorially equivalent to the fan Δ . But by [BBFK, Exemple 3.5] or by Theorem 3.5.1, the toric variety $X_{\Delta'}$ has different Betti numbers: $b_2(X_{\Delta'}) = 0$ and $b_3(X_{\Delta'}) = 0$, although the fan Δ' is not simplicial.
- (c) The combinatorial invariant $b_2 - b_3 = d_2 - 2d_1 + 3$ may take positive and negative values: Let Δ'' be the (simplicial) complete fan in $N_{\mathbb{R}} \cong \mathbb{R}^3$ generated by the faces of the tetrahedron with vertices $v_i := e_i$ for $i = 1, 2, 3$, and $v_4 := -e_1 - e_2 - e_3$ according to the following figure:

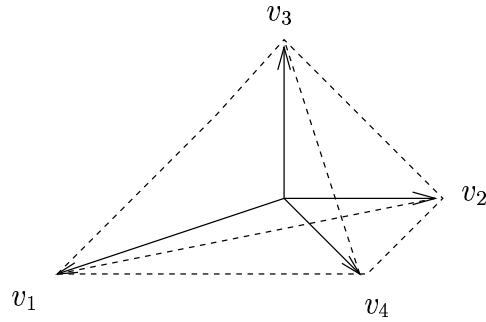


Figure 3.4: A (simplicial) complete fan Δ'' in $N_{\mathbb{R}} \cong \mathbb{R}^3$ with $X_{\Delta''} \cong \mathbb{P}_3$ and $b_2(X_{\Delta''}) - b_3(X_{\Delta''}) > 0$.

By Theorem 3.4.1, the toric variety $X_{\Delta''} \cong \mathbb{P}_3$ has the Betti numbers $b_2(\mathbb{P}_3) = d_2 - 2d_1 + 3 = 1$ and $b_3(\mathbb{P}_3) = 0$, i.e., the difference $b_2(\mathbb{P}_3) - b_3(\mathbb{P}_3) = 1$ is positive.

Let Δ''' be the (nonsimplicial) complete fan in $N_{\mathbb{R}} \cong \mathbb{R}^3$ generated by the faces of the cube with vertices $v_1 := e_1 - e_2 + e_3$, $v_2 := e_1 + e_2 + e_3$, $v_3 := -e_1 + e_2 + e_3$, $v_4 := -e_1 - e_2 + e_3$, and $v_i := v_{i-4} - 2e_3$ for $i = 5, \dots, 8$ according to the following figure:

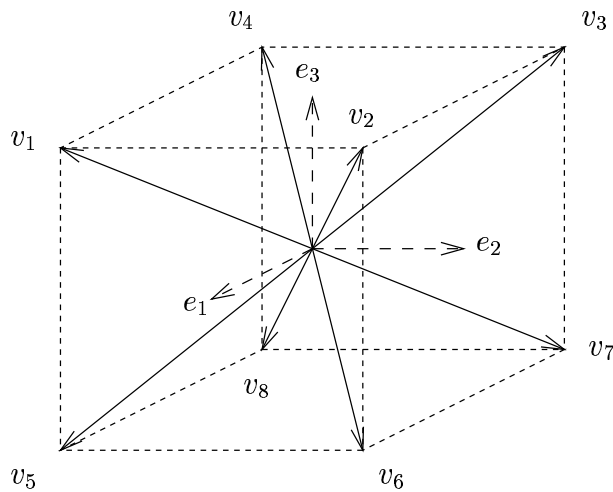


Figure 3.5: A (nonsimplicial) complete fan Δ''' in $N_{\mathbb{R}} \cong \mathbb{R}^3$ with $b_2(X_{\Delta'''}) - b_3(X_{\Delta'''}) < 0$.

We have $b_2(X_{\Delta'''}) - b_3(X_{\Delta'''}) = d_2 - 2d_1 + 3 = -1 < 0$ by Proposition 3.5.3(c); in fact, $b_2(X_{\Delta'''}) = 1$ and $b_3(X_{\Delta'''}) = 2$ by Theorem 3.5.1(c),(d).

The Betti numbers $b_2(X_\Delta)$ and $b_3(X_\Delta)$ of a 3-dimensional compact toric variety X_Δ possess the following combinatorial bounds:

3.5.5 Corollary. *For every complete fan Δ in $N_{\mathbb{R}} \cong \mathbb{R}^3$, we have*

- (a) $d_2(\Delta) - 2d_1(\Delta) + 3 \leq b_2(X_\Delta) \leq d_2(\Delta)$,
- (b) $-d_2(\Delta) + 2d_1(\Delta) - 3 \leq b_3(X_\Delta) \leq 2d_1(\Delta) - 3$.

Proof. By Theorem 3.5.1(c),

$$b_2(X_\Delta) = \text{rk } E_{1,1}^2 = \text{rk } H_1(\Delta, \Lambda_\Delta^1) \leq \text{rk } C_1(\Delta, \Lambda_\Delta^1) = d_2(\Delta),$$

what implies that

$$b_3(X_\Delta) = -d_2(\Delta) + 2d_1(\Delta) - 3 + b_2(X_\Delta) \leq 2d_1(\Delta) - 3$$

by Proposition 3.5.3(d). The lower bounds follow from the same equality and the fact that $b_2(X_\Delta), b_3(X_\Delta) \geq 0$. ■

3.6 Toric Varieties of Dimension 4 or more

The toric (co-)homology spectral sequence admits the calculation — in low and in high degrees — of the integral homology groups with closed supports and the integral cohomology groups with compact supports of arbitrary toric varieties. Furthermore, in low and in high degrees, the integral local homology groups in the fixed points are computable by our method.

3.6.1 Theorem. *For every nonzero fan Δ in $N_{\mathbb{R}} \cong \mathbb{R}^n$ with $n \geq 4$, the following integral homology groups with closed supports of the n -dimensional toric variety X_Δ may be expressed by the E^2 -terms of the associated toric homology spectral sequence:*

- (a) $H_0^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{0,0}^2 \cong A_0(X_\Delta) \cong \mathbb{Z}^{b_0^{\text{cld}}}$, where $b_0^{\text{cld}} = 0$ if Δ is noncomplete, and $b_0^{\text{cld}} = 1$ if Δ is complete,
- (b) $H_1^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{1,0}^2 \cong H^{n-2}(S_\Delta; \mathbb{Z})$; in particular, $E_{1,0}^2 \cong 0$ if X_Δ is affine or compact,
- (c) $H_2^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{1,1}^2 \oplus E_{2,0}^2 \cong A_1(X_\Delta) \oplus H^{n-3}(S_\Delta; \mathbb{Z})$ if the group $H^{n-3}(S_\Delta; \mathbb{Z})$ is free abelian; in particular, $H_2^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong A_1(X_\Delta)$ if X_Δ is affine or compact,
- (d) $H_3^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{2,1}^2 \oplus E_{3,0}^2$ if $E_{3,0}^2$ is free abelian; in particular, for $n = 4$, $E_{3,0}^2 \cong \mathbb{Z}^k$ where $k + 1$ equals the number of connected components of the set S_Δ ; and, for $n > 4$, $E_{3,0}^2 \cong H^{n-4}(S_\Delta; \mathbb{Z})$; and $H_3^{\text{cld}}(X_\Delta; \mathbb{Z}) \cong E_{2,1}^2$ if X_Δ is affine or compact,

- (e) $H_{2n-4}^{\text{clid}}(X_\Delta; \mathbb{Z}) \cong E_{n-2, n-2}^2 \oplus E_{n-1, n-3}^2 \cong A_{n-2}(X_\Delta) \oplus E_{n-1, n-3}^2$ if $E_{n-1, n-3}^2$ is free abelian and $s(\Delta) \geq n-2$,
- (f) $H_{2n-3}^{\text{clid}}(X_\Delta; \mathbb{Z}) \cong E_{n-1, n-2}^2$ if $s(\Delta) \geq n-2$,
- (g) $H_{2n-2}^{\text{clid}}(X_\Delta; \mathbb{Z}) \cong E_{n-1, n-1}^2 \oplus E_{n, n-2}^2 \cong A_{n-1}(X_\Delta) \oplus \mathbb{Z}^{\binom{n-s(\Delta)}{2}}$ with $b_{2n-2}^{\text{clid}}(X_\Delta) = d_1(\Delta) - s(\Delta) + \binom{n-s(\Delta)}{2}$;
in particular, $H_{2n-2}^{\text{clid}}(X_\Delta; \mathbb{Z}) \cong A_{n-1}(X_\Delta)$ iff $s(\Delta) > n-2$,
- (h) $H_{2n-1}^{\text{clid}}(X_\Delta; \mathbb{Z}) \cong E_{n, n-1}^2 \cong \mathbb{Z}^{n-s(\Delta)}$,
- (i) $H_{2n}^{\text{clid}}(X_\Delta; \mathbb{Z}) \cong E_{n, n}^2 \cong A_n(X_\Delta) \cong \mathbb{Z}$.

Proof. The formulæ are easy consequences of Theorem 2.4.12 in connection with Corollary 2.4.9 and Proposition 2.4.10. ■

3.6.2 Remark. With exception of the group $H_4^{\text{clid}}(X_\Delta; \mathbb{Z})$ (if $E_{3,1}^2$ is not free abelian) the whole homology $H_\bullet^{\text{clid}}(X_\Delta; \mathbb{Z})$ of a 4-dimensional toric variety X_Δ (and hence its whole cohomology $H_c^\bullet(X_\Delta; \mathbb{Z})$ with exception of $H_c^5(X_\Delta; \mathbb{Z})$) is computable by our method: For $s(\Delta) < 2$, i.e., for the smooth varieties $X_\Delta \cong (\mathbb{C}^*)^4, \mathbb{C} \times (\mathbb{C}^*)^3, \mathbb{P}_1 \times (\mathbb{C}^*)^3$ with $\Delta = \Delta^{\leq 1}$, see Proposition 2.4.14. For $s(\Delta) \geq 2$, see the preceding theorem and note that $E_{2,0}^2 \cong H^1(S_\Delta; \mathbb{Z})$ and $E_{3,0}^2 \cong \tilde{H}^0(S_\Delta; \mathbb{Z})$ are free abelian.

In order to compute the integral local homology in the fixed points x_σ for $\sigma \in \Delta^n$ of a toric variety X_Δ , it is sufficient to know the integral homology with closed supports of the affine toric varieties X_σ , since $\mathcal{H}_{q, x_\sigma} \cong H_q^{\text{clid}}(X_\sigma; \mathbb{Z})$, see [BBFK, Formula 0.A.1]. In particular, for toric varieties of dimension at most four, all these local homology groups with exception of $\mathcal{H}_{4, x_\sigma}$ for a 4-dimensional toric variety (if $E_{3,1}^2(X_\sigma)$ is not free abelian) are computable by our formulæ:

3.6.3 Corollary. For every fan Δ in $N_{\mathbb{R}} \cong \mathbb{R}^n$ and each cone $\sigma \in \Delta^n$, there are the following isomorphisms between the integral local homology groups $\mathcal{H}_{\bullet, x_\sigma}$ in the fixed point x_σ of the toric variety X_Δ , the Chow groups $A_\bullet(X_\sigma)$ of the affine toric variety X_σ , and the modules $E_{\bullet, \bullet}^2(X_\sigma)$ of the integral toric homology spectral sequence associated to the variety X_σ :

$$\mathcal{H}_{2q, x_\sigma} \cong A_q(X_\sigma) \cong E_{q, q}^2(X_\sigma) \quad \text{for } q \in \{0, 1, n-1, n\},$$

$$\mathcal{H}_{2n-4, x_\sigma} \cong A_{n-2}(X_\sigma) \oplus E_{n-1, n-3}^2(X_\sigma) \quad \text{if } E_{n-1, n-3}^2(X_\sigma) \text{ is free abelian, and}$$

$$\mathcal{H}_{2q+1, x_\sigma} \cong E_{q+1, q}^2(X_\sigma) \quad \text{for } q \in \{0, 1, n-2, n-1, n\}.$$

In particular, $\mathcal{H}_{0, x_\sigma} \cong \mathcal{H}_{1, x_\sigma} \cong \mathcal{H}_{2n-1, x_\sigma} \cong 0$ and $\mathcal{H}_{2n, x_\sigma} \cong \mathbb{Z}$. ■

Appendix A

Finitely Generated Abelian Groups

As we have seen, the calculation of integral (co-)homology and Chow groups of toric varieties leads to quotients of lattices of finite rank by (necessarily) finitely generated sublattices. By the *Fundamental Theorem on Finitely Generated Abelian Groups*, such a quotient is characterized by its rank, corank, and torsion coefficients. Using the *Theorem on Elementary Divisors*, we present a formula for the computation of these numbers that depends on generators of the respective sublattice.

The fundamental theorem gives a complete classification of the isomorphism classes of finitely generated abelian groups:

A.1 Fundamental Theorem on Finitely Generated Abelian Groups.

For every finitely generated abelian group G , there exist natural numbers ϱ , r , and elements $g_1, \dots, g_\varrho, \tilde{g}_1, \dots, \tilde{g}_r \in G$ with the following properties:

- (a) *The group G is the inner direct sum of the cyclic subgroups generated by those elements:*

$$G = \langle g_1 \rangle \oplus \cdots \oplus \langle g_\varrho \rangle \oplus \langle \tilde{g}_1 \rangle \oplus \cdots \oplus \langle \tilde{g}_r \rangle.$$

- (b) *For $i = 1, \dots, \varrho$, the subgroups $\langle g_i \rangle$ are of infinite order.*

- (c) *For $i = 1, \dots, r$, the subgroups $\langle \tilde{g}_i \rangle$ are of finite order but not trivial. Their orders $t_i := \text{ord } \tilde{g}_i \geq 2$ form a chain of divisors $t_1 \mid t_2 \mid \cdots \mid t_r$, i.e., t_i divides t_{i+1} for $i = 1, \dots, r - 1$.*

- (d) *There is no set of generators for G with less than $\varrho + r$ elements.*

The numbers ϱ , r , and t_1, \dots, t_r are uniquely determined by the Properties (a) – (d).

Proof. See for instance [StZi, Satz und Definition 8.1.14]. ■

Hence, every finitely generated abelian group G may be uniquely written up to isomorphism in the form

$$G \cong \mathbb{Z}^{\varrho} \oplus \mathbb{Z}_{t_1} \oplus \cdots \oplus \mathbb{Z}_{t_r}$$

with $\varrho \in \mathbb{N}$ and a chain of divisors $2 \leq t_1 | t_2 | \cdots | t_r$, where \mathbb{Z}_t denotes the finite cyclic group with t elements. Thus the group G is uniquely characterized by its rank ϱ , its corank $r + \varrho$, and its torsion coefficients t_1, \dots, t_r .

In order to compute the rank and the torsion coefficients of the quotient of a lattice of finite rank modulo a sublattice generated by finitely many lattice points, we will make use of the following theorem:

A.2 Theorem on Elementary Divisors. *Let $N' := \langle v_1, \dots, v_k \rangle$ be the sublattice of rank $n' := \text{rk } N'$ of the lattice $N = \mathbb{Z}^n$ that is generated by the lattice points $v_1, \dots, v_k \in N$. Then there exist a basis $e'_1, \dots, e'_{n'}$ of the lattice N' , a basis e_1, \dots, e_n of the lattice N , and elementary divisors $\varepsilon_1, \dots, \varepsilon_{n'}$ such that $e'_i = \varepsilon_i e_i$ for $i = 1, \dots, n'$ and $\varepsilon_i | \varepsilon_{i+1}$ for $i = 1, \dots, n' - 1$. The elementary divisor ε_i is the quotient of the two subsequent determinant divisors δ_i and δ_{i-1} , i.e., $\delta_i = \delta_{i-1} \varepsilon_i$, where the determinant divisor*

$$\delta_i := \text{gcd} \{ \det V; V \text{ minor of the matrix } (v_1, \dots, v_k) \text{ of order } i \}$$

denotes the greatest common divisor of the determinants of $(i \times i)$ -submatrices of the matrix (v_1, \dots, v_k) .

Proof. See [Wa, §85 Elementarteilersatz]. ■

A.3 Corollary. *Let $N' = \langle v_1, \dots, v_k \rangle$ be the sublattice of rank $n' := \text{rk } N'$ of the lattice $N = \mathbb{Z}^n$ that is generated by the lattice points $v_1, \dots, v_k \in N$. Denote the uniquely determined elementary divisors according to Theorem A.2 by ε_i for $i = 1, \dots, n'$. Then the quotient N/N' is a finitely generated abelian group with rank $\varrho = n - n'$, corank $\varrho + r = n - n' + \#\{i; 1 \leq i \leq n', \varepsilon_i \neq 1\}$, and torsion coefficients $t_j := \varepsilon'_{n-r+j}$ for $j = 1, \dots, r$.*

Proof. Obviously,

$$N/N' \cong \bigoplus_{i=1}^{n'} \mathbb{Z}e_i / \mathbb{Z}\varepsilon_i e_i \oplus \bigoplus_{i=n'+1}^n \mathbb{Z}e_i \cong \bigoplus_{i=1}^{n'} \mathbb{Z}_{\varepsilon_i} \oplus \mathbb{Z}^{n-n'}$$

The claim follows from Theorem A.1, since $1 \leq \varepsilon_1 | \varepsilon_2 | \cdots | \varepsilon_{n'}$. ■

Appendix B

Dictionary for the Theory of Toric Varieties

In our exposition of the (co-)homology of toric varieties we have encountered a phenomenon that is typical for this theory: Many notions, methods, and results may be expressed in different languages. For instance, in *algebraic geometry*, toric varieties X_Δ are algebraic varieties characterized by the existence of a special torus action. In the *convex geometry of real affine spaces*, toric varieties correspond to rational fans Δ consisting of lattice cones; and in the *spherical geometry of the closed unit ball B^n* , they correspond to the spherical complexes $\tilde{\Delta}$. Yet the Wechselspiel between these theories makes it possible to compute important invariants of the *algebraic topology* of toric varieties. In the following dictionary for the theory of toric varieties we collect relations between these different points of view:

	Convex Geometry of Real Affine Spaces	Spherical Geometry of the Closed Unit Ball B^n
1)	Vector space $N_{\mathbb{R}} \cong \mathbb{R}^n$	Closed unit ball $B^n \subset \mathbb{R}^n$
2)	Lattice cone σ in $N_{\mathbb{R}}$, dual cone $\check{\sigma}$ in $M_{\mathbb{R}}$	Spherical dual $\hat{\sigma}$
3)	Lattice point u in monoid $S_{\sigma} = \check{\sigma} \cap M$	
4)	Lattice cone $\{0\}$ in $N_{\mathbb{R}}$	Open unit ball $(B^n)^{\circ} \subset \mathbb{R}^n$
5)	Face τ of lattice cone σ	Spherical face $\hat{\sigma}$ of $\hat{\tau}$
6)	Rational fan Δ	Spherical dual $\tilde{\Delta}$
6a)	— complete: $\bigcup_{\sigma \in \Delta} \sigma = N_{\mathbb{R}}$	— complete: $\bigcup_{\hat{\sigma} \in \tilde{\Delta}} \hat{\sigma} = S^{n-1}$
6b)	— regular	
6c)	— simplicial	
7)	Intersection of cones — $\sigma \cap \tau$	Intersection of duals — $\hat{\sigma} \cap \hat{\tau}$
8)	Fan $\Delta(\sigma)$, lattice $N(\sigma)$	$\pi^{-1}(\bar{\sigma})$
8a)	— $\sigma \prec \tau$	— $\pi^{-1}(\hat{\tau}) \subset \pi^{-1}(\hat{\tau}) \subset \pi^{-1}(\bar{\sigma})$
8b)		— $\pi^{-1}(\partial\bar{\sigma}) = \pi^{-1}(\bar{\sigma}) \setminus \pi^{-1}(\bar{\sigma})$
9)		
10)	Number of cones $d_n(\Delta)$	
11)		

	Algebraic Geometry of Toric Varieties	Algebraic Topology of Toric Varieties
1)		
2)	Affine toric variety $X_\sigma = \text{Spec } \mathbb{C}[\check{\sigma} \cap M]$	
3)	Regular function χ^u in \mathbb{C} -algebra $\mathbb{C}[S_\sigma]$	
4)	Algebraic torus $(\mathbb{C}^*)^n$	
5)	Principal open subset $X_\tau \subset X_\sigma$	
6)	Toric variety X_Δ	Toric variety X_Δ
6a)		— compact
6b)	— smooth	
6c)	— at most quotient singularities	
7)	Intersection of subvarieties	
	— $X_{\sigma \cap \tau} = X_\sigma \cap X_\tau$	
8)	Orbit closure $V(\sigma) = \bar{O}_\sigma$	
8a)	— $O_\tau \subset V(\tau) \subset V(\sigma)$	
8b)	— $\partial O_\sigma = V(\sigma) \setminus O_\sigma$	
9)	Chow group $A_p(X_\Delta)$	Module $E_{p,p}^2$
10)	Number of fixed points	Euler characteristic $e(X_\Delta)$
11)	$A_p(X_\sigma)$ for $p \in \{0, 1, n-1, n\}$	$\mathcal{H}_{2p, x_\sigma} \cong H_{2p}^{\text{clid}}(X_\sigma; \mathbb{Z})$

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Glossary of Notation

\mathbb{N}	natural numbers including 0
\mathbb{Z}	integers
\mathbb{Q}	rational numbers
\mathbb{R}	real numbers
\mathbb{C}	complex numbers
\mathbb{C}^*	complex numbers excluding 0
$A := B, B =: A$	A is defined to be equal to B
■	end of proof or absence of proof
N	lattice of rank n , 14
M	dual lattice $\text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, 14
$N_{\mathbb{R}}$	vector space $N \otimes_{\mathbb{Z}} \mathbb{R}$, 14
$M_{\mathbb{R}}$	vector space $M \otimes_{\mathbb{Z}} \mathbb{R}$, 14
σ	lattice cone in $N_{\mathbb{R}}$, 14
$\text{pos}\{v_1, \dots, v_k\}$	positive hull, 14
$\langle, \rangle: M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$	dual pairing, 14
$\check{\sigma}$	dual cone, 14
$\text{lin } \check{\sigma}$	linear hull, 14
$\dim \check{\sigma}$	dimension, 14
S_{σ}	monoid $\check{\sigma} \cap M$, 14
$\mathbb{C}[S_{\sigma}]$	monoid algebra, 15
X_{σ}	affine toric variety $\text{Spec } \mathbb{C}[S_{\sigma}]$, 15
χ^u	regular function, 15
$\tau \prec \sigma$	τ is a face of σ , 15
u^{\perp}	$\{v \in N_{\mathbb{R}}; \langle u, v \rangle = 0\}$, 15
$(X_{\sigma})_{\chi^u}$	principal open subset, 15
Δ	rational fan in $N_{\mathbb{R}}$, 15
$\Delta^{\leq k}$	k -skeleton, 16

Δ^k	$\{\sigma \in \Delta; \dim \sigma = k\}$, 16
$d_k(\Delta)$	$\#\Delta^k$, 16
X_Δ	toric variety, 16
$ \Delta $	support of Δ , 16
S^{n-1}	unit sphere in $N_{\mathbb{R}}$, 16
\mathcal{C}_Δ	spherical complex, 16
\mathcal{C}'_Δ	barycentric subdivision of \mathcal{C}_Δ , 16
$\hat{\sigma}$	spherical dual of σ , 16
B^n	closed unit ball in $N_{\mathbb{R}}$, 16
$\tilde{\sigma}$	relative interior of $\hat{\sigma}$, 16
$\tilde{\Delta}$	spherical dual of Δ , 16
T^n	real torus $N_{\mathbb{R}}/N$, 17
T_σ	real torus $(N + \text{lin } \sigma)/N$, 17
\sim	equivalence relation on $B^n \times T^n$, 17
$\pi: (B^n \times T^n)/\sim \rightarrow B^n$	canonical projection, 17
$T(\sigma)$	real torus T^n/T_σ , 17
B_σ	$\bigcup_{\tau \prec \sigma} \tilde{\tau}$, 18
B_Δ	spherical support of Δ , 18
\sim	equivalence relation on $B_\Delta \times T^n$, 18
$\pi: (B_\Delta \times T^n)/\sim \rightarrow B_\Delta$	canonical projection, 18
\mathbb{T}_N	algebraic torus $\text{Spec } \mathbb{C}[M]$, 18
$t: M \rightarrow \mathbb{C}^*$	group homomorphism, 19
$x: S_\sigma \rightarrow \mathbb{C}$	semigroup homomorphism, 19
O_σ	\mathbb{T}_N -orbit in X_Δ , 19
σ^\perp	$\{u \in M_{\mathbb{R}}; \langle u, \sigma \rangle = 0\}$, 19
$V(\sigma)$	orbit closure \bar{O}_σ in X_Δ , 19
N_σ	lattice $N \cap \text{lin } \sigma$, 19
$N(\sigma)$	quotient lattice N/N_σ , 19
$\Delta(\sigma)$	fan associated to $V(\sigma)$, 19
I_σ	isotropy group of orbit O_σ , 19
x_σ	distinguished point of O_σ , 19
$\bar{\sigma}$	trace $\hat{\sigma} \cap B_\Delta$, 20
∂O_σ	relative boundary $V(\sigma) \setminus O_\sigma$, 20
$\partial \bar{\sigma}$	relative boundary $\bar{\sigma} \setminus \tilde{\sigma}$, 20

$\ \cdot\ $	Euclidean norm, 21
$ \cdot $	maximum norm, 21
G	abelian group, 25
H_{\bullet}^{Δ}	covariant functor, 25
$\pi^{\sigma,\tau}: T(\sigma) \rightarrow T(\tau)$	canonical projection, 26
$\Lambda_{\bullet}^{\Delta}$	covariant functor, 26
$\nu_{\sigma,\tau}: N(\sigma) \rightarrow N(\tau)$	canonical epimorphism, 26
$\nu_{\sigma,\tau}^{\Lambda}$	epimorphism $\Lambda^* \nu_{\sigma,\tau} \otimes 1_G$, 26
Φ^{Δ}	natural equivalence, 26
$\pi^{\sigma}: T^n \rightarrow T(\sigma)$	canonical projection, 26
$\nu_{\sigma}: N \rightarrow N(\sigma)$	canonical projection, 27
$\text{or}(\sigma)$	orientation of $\text{lin } \sigma$, 28
$B(\sigma)$	basis of $\text{lin } \sigma$, 28
$\text{or}(\sigma, \tau)$	orientation of $\text{lin } \tau$, 28
$o_{\sigma,\tau}$	orientation coefficient, 28
$C_p(\Delta, F)$	chain group, 29
$\delta_p: C_p(\Delta, F) \rightarrow C_{p-1}(\Delta, F)$	differential, 29
$H_{\bullet}(\Delta, F)$	homology of complex $C_{\bullet}(\Delta, F)$, 29
G^{Δ}	constant functor, 29
S_{Δ}	finite CW-complex $ \Delta \cap S^{n-1}$, 30
c_{σ}	oriented cell, 30
$E = (E_{p,q})_{p,q \in \mathbb{Z}}$	bigraded module, 32
$d: E \rightarrow E$	differential of bidegree $(-r, r-1)$, 32
$(E^r, d^r)_{r \geq 1}$	homology spectral sequence, 32
E^1	initial term, 32
E^{∞}	limit term, 32
\mathbf{C}	chain complex, 34
$\mathbf{C}_{-1} \subset \cdots \subset \mathbf{C}_n$	filtration of \mathbf{C} , 34
$E_{p,q}$	$H_{p+q}(\mathbf{C}_p, \mathbf{C}_{p-1})$, 35
$D_{p,q}$	$H_{p+q}(\mathbf{C}_p)$, 35
$F_p H_{\bullet}(\mathbf{C})$	filtration of $H_{\bullet}(\mathbf{C})$, 36
$X_{-1} \subset \cdots \subset X_n$	filtration of X_{Δ} , 38
$\psi(k): (N, \Delta) \rightarrow (N, \Delta)$	morphism of fans, 46
$m_{p,q}^r$	constant, 48

$m(r, q)$	constant, 48
φ	Euler's function, 49
$\text{ind}_c x$	index, 49
$(\text{ind}_- x; \text{ind}_5 x)$	index system, 49
$b_\ell^{\text{cld}}(X_\Delta)$	Betti number, 51
$s(\Delta)$	$\dim_{\mathbb{R}} \text{lin } \Delta^1$, 53
N_s	lattice $N \cap \text{lin } \Delta^1$, 54
H_Δ^\bullet	contravariant functor, 58
$\pi_{\sigma, \tau}: T(\sigma) \twoheadrightarrow T(\tau)$	canonical projection, 58
Λ_Δ^\bullet	contravariant functor, 59
$M(\sigma)$	lattice $\sigma^\perp \cap M$, 59
$\mu_{\sigma, \tau}: M(\tau) \subset M(\sigma)$	inclusion, 59
$C^p(\Delta, F)$	cochain group, 59
$\delta^p: C^p(\Delta, F) \rightarrow C^{p+1}(\Delta, F)$	differential, 59
$H^\bullet(\Delta, F)$	cohomology of complex $C^\bullet(\Delta, F)$, 59
$(E_r, d_r)_{r \geq 1}$	cohomology spectral sequence, 60
$m_r^{p, q}$	constant, 61
$b_c^\ell(X_\Delta)$	Betti number, 61
$e(X_\Delta)$	topological Euler characteristic, 64
$e^{\text{cld}}(X_\Delta)$	Euler characteristic, 64
N_ρ^\perp	lattice, 67
$\text{div}(\chi^u)$	principal divisor, 67
$v_{\rho, \sigma}$	lattice point, 67
$A_p(X)$	Chow group, 67
$Z_p(X)$	group of p -cycles, 67
$\text{Rat}_p(X)$	subgroup of $Z_p(X)$, 67
$A_p^{\mathbb{T}}(X_\Delta)$	\mathbb{T} -invariant Chow group, 67
$\text{mult } \sigma$	multiplicity, 74
Σ_ℓ	Hirzebruch surface, 74
$\mathcal{H}_{q, x_\sigma}$	local homology group, 81
\mathbb{Z}_t	cyclic group with t elements, 84
ε_i	elementary divisor, 84
δ_i	determinant divisor, 84

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