Universität Konstanz

Traveling-Wave Phase Boundaries in Compressible Binary Fluid Mixtures

Heinrich Freistühler

Konstanzer Schriften in Mathematik

Nr. 272, Juli 2010

ISSN 1430-3558

Traveling-wave phase boundaries in compressible binary fluid mixtures

This abridged version sketches the isothermal case, temperature acting as an external parameter.

Consider the augmented Euler / Navier-Stokes system

$$\begin{aligned} \partial_t\rho + \nabla\cdot(\rho\mathbf{u}) &= 0, \\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u} + p(\rho,\chi)\mathbf{I}) &= \nabla\cdot\left(\nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) + (\lambda\nabla\cdot\mathbf{u})\mathbf{I} - \delta\rho\nabla\chi\otimes\nabla\chi\right), \\ \partial_t(\rho\chi) + \nabla\cdot(\rho\chi\,\mathbf{u}) &= \rho q(\rho,\chi) + \nabla\cdot(\delta\rho\nabla\chi) \end{aligned} \tag{1}$$

for a compressible isothermal, viscous or inviscid fluid. The fluid is assumed to be a locally homogeneous mixture of two components such that its local state is completely described by the mass fraction $\chi \in [0,1]$ of the components and the mass, per volume, of the mixture, $\rho > 0$. This density ρ is the reciprocal value,

$$\rho = 1/\tau,$$

of the fluid's specific volume τ. The behaviour of the fluid is described by

(H1) a thermodynamic potential

$$U(\tau,\chi,\nabla\chi) = \hat{U}(\tau,\chi) + \frac{1}{2}\delta|\nabla\chi|^2, \quad \hat{U}(\tau,\chi) = \theta W(\chi) + F(\tau,\chi),$$

with constant temperature $\theta > 0$ and mixing entropy

$$W(\chi) = \chi\log\chi + (1-\chi)\log(1-\chi) - \frac{1}{2}\chi^2,$$

from which the pressure p and the transformation rate q derive as

$$p(\rho,\chi) = -\frac{\partial\hat{U}}{\partial\tau}(\tau,\chi), \qquad q(\rho,\chi) = -k\frac{\partial\hat{U}}{\partial\chi}(\tau,\chi), \tag{2}$$

with some constant $k = k(\theta)$, and

(H2) a Stokes viscosity with appropriate coefficients $\nu, \lambda > 0$; or no viscosity: $\nu = \lambda = 0$.

System (1), also called the Navier-Stokes-Allen-Cahn equations, has been derived and studied in the literature. The modelling is, in particular, intended to describe two-phase configurations, i. e., partitionings

$$\Omega = \Omega^-(t) \cup \Sigma^\delta(t) \cup \Omega^+(t)$$

of spatial regions Ω into disjoint portions, of which $\Sigma^\delta(t)$ is a narrow transition zone and $\Omega^-(t), \Omega^+(t)$ carry „one phase" and „the other phase", respectively, the terms „one phase" and „the other phase" referring to two different limited ranges of the mass fraction χ.

The purpose of the present paper is to show that typically such configurations indeed occur in the sense of planar standing / traveling waves. We notably show the following two theorems.

Theorem 1. *(Maxwell states and standing phase boundaries.) Under generic assumptions, the following holds with a critical value θ_* of the temperature and some $\theta_1 \in (0, \theta_*)$. For every temperature $\theta \in (\theta_1, \theta_*]$, there are uniquely determined fluid states*

$$(\underline{\rho}_M, \underline{\chi}_M), (\mathring{\rho}, \mathring{\chi}), (\overline{\rho}_M, \overline{\chi}_M),$$

depending continuously on θ, such that

(i)

$$\begin{aligned} p(\underline{\rho}_M, \underline{\chi}_M) &= p(\mathring{\rho}, \mathring{\chi}) = p(\overline{\rho}_M, \overline{\chi}_M), \\ q(\underline{\rho}_M, \underline{\chi}_M) &= q(\mathring{\rho}, \mathring{\chi}) = q(\overline{\rho}_M, \overline{\chi}_M) = 0, \end{aligned}$$

and

$$\text{for } \theta = \theta_*, \quad (\underline{\rho}_M, \underline{\chi}_M) = (\mathring{\rho}, \mathring{\chi}) = (\overline{\rho}_M, \overline{\chi}_M).$$

while

(ii) for $\theta < \theta_*$,

$$\underline{\rho}_M < \mathring{\rho} < \overline{\rho}_M \quad \text{and} \quad \underline{\chi}_M < \mathring{\chi} < \overline{\chi}_M$$

and system (1) *admits a standing ($\mathbf{u} \equiv 0$) planar phase boundary*

$$(\overrightarrow{\rho}(x\cdot\mathbf{n}), \overrightarrow{\chi}(x\cdot\mathbf{n})) \quad \text{with} \quad (\overrightarrow{\rho}(-\infty), \overrightarrow{\chi}(-\infty)) = (\underline{\rho}_M, \underline{\chi}_M), \;\; (\overrightarrow{\rho}(\infty), \overrightarrow{\chi}(\infty)) = (\overline{\rho}_M, \overline{\chi}_M)$$

or (equivalently via $x \mapsto -x$)

$$(\overleftarrow{\rho}(x\cdot\mathbf{n}), \overleftarrow{\chi}(x\cdot\mathbf{n})) \quad \text{with} \quad (\overleftarrow{\rho}(-\infty), \overleftarrow{\chi}(-\infty)) = (\overline{\rho}_M, \overline{\chi}_M), \;\; (\overleftarrow{\rho}(\infty), \overleftarrow{\chi}(\infty)) = (\underline{\rho}_M, \underline{\chi}_M).$$

Theorem 2. *Let $m = \rho\mathbf{u}\cdot\mathbf{n}$ denote the possible temporal rate of mass flux through a non-standing planar phase boundary and $\zeta = 2\nu + \lambda$ the viscosity coefficient. For sufficiently small values of*

$$m > 0 \text{ and } m\zeta \geq 0,$$

(i) the (left endstate, right endstate, profile) triple

$$(\underline{\rho}_M, \underline{\chi}_M), (\overline{\rho}_M, \overline{\chi}_M), (\overrightarrow{\rho}, \overrightarrow{\chi})$$

perturbs regularly to a (left endstate, right endstate, profile) triple

$$(\overrightarrow{\rho}^{\,-}_m, \overrightarrow{\chi}^{\,-}_m), (\overrightarrow{\rho}^{\,+}_m, \overrightarrow{\chi}^{\,+}_m), (\overrightarrow{\rho}_{m,\zeta}, \overrightarrow{\chi}_{m,\zeta})$$

describing a traveling-wave phase boundary with densifying transformation;

(ii) the (left endstate, right endstate, profile) triple

$$(\overline{\rho}_M, \overline{\chi}_M), (\underline{\rho}_M, \underline{\chi}_M), (\overleftarrow{\rho}, \overleftarrow{\chi})$$

perturbs regularly to a (left endstate, right endstate, profile) triple

$$(\overleftarrow{\rho}^{\,-}_m, \overleftarrow{\chi}^{\,-}_m), (\overleftarrow{\rho}^{\,+}_m, \overleftarrow{\chi}^{\,+}_m), (\overleftarrow{\rho}_{m,\zeta}, \overleftarrow{\chi}_{m,\zeta})$$

describing a traveling-wave phase boundary with rarifying transformation.

The proofs of these theorems rely on Hamiltonian dynamics, hyperbolicity and transversality, and a transition of $D^2\hat{U}$ from convexity to non-convexity. The latter corresponds to what is known as a „spinodal" region.

As a byproduct, one finds:

Corollary 1. *System* (1) *admits planar „droplets"*

$$(\overleftrightarrow{\rho}(\xi), \overleftrightarrow{\chi}(\xi)) \quad \textit{with} \quad (\overleftrightarrow{\rho}(\pm\infty), \overleftrightarrow{\chi}(\pm\infty)) = (\rho_-, \chi_-), \quad \textit{with } (\rho_-, \chi_-) \approx (\underline{\rho}_M, \underline{\chi}_M),$$

and planar „bubbles"

$$(\overleftrightarrow{\rho}(\xi), \overleftrightarrow{\chi}(\xi)) \quad \textit{with} \quad (\overleftrightarrow{\rho}(\pm\infty), \overleftrightarrow{\chi}(\pm\infty)) = (\rho_+, \chi_+), \quad \textit{with } (\rho_+, \chi_+) \approx (\overline{\rho}_M, \overline{\chi}_M).$$

Konstanz, July 22, 2009

Heinrich Freistühler