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Classical Negation and Expansions of Belnap–Dunn Logic

Abstract. We investigate the notion of classical negation from a non-classical perspective. In particular, one aim is to determine what classical negation amounts to in a paracomplete and paraconsistent four-valued setting. We first give a general semantic characterization of classical negation and then consider an axiomatic expansion **BD+** of four-valued Belnap–Dunn logic by classical negation. We show the expansion complete and maximal. Finally, we compare **BD+** to some related systems found in the literature, specifically a four-valued modal logic of Béziau and the logic of classical implication and a paraconsistent de Morgan negation of Zaitsev.

Keywords: First-degree entailment, Belnap–Dunn logic, Classical negation, Many-valued logic, Paraconsistency, Paracompleteness, Maximality.

1. Introduction

One of the initial motivations behind the system of first-degree entailment, or Belnap–Dunn logic (**BD**), was to avoid fallacies of classical material implication, such as the so-called fallacies of relevance. Two such instances are embodied in the following theorems of classical logic:

$$A \rightarrow (B \rightarrow A),\ (A \wedge \neg A) \rightarrow B.$$

It would seem, then, that paraconsistent and relevant logicians should have no interest in classical material implication or classical negation, at least not if they wish to avoid fallacies such as these. There are, however, plenty of good reasons for paraconsistent logicians to be interested in these classical notions. One prominent example of such interest goes back to Routley and Meyer in the series of articles (see [16,17]) where they consider adding classical negation to relevant logic, resulting in what they there called *classical relevant logic*. Their reason was mainly a technical curiosity: Does adding classical negation to relevant logic result in what they called "breakdown", i.e. a collapse to classical logic? The answer, they showed, was "No" and, better, that the expansion to classical negation is even conservative.

Are there other reasons for a paraconsistentist to be interested in classical negation? Let us first consider a related question: Why would (revisionist) paraconsistists care about metalogical results concerning paraconsistent logics couched in a classical metatheory? R. K. Meyer's answer was "to preach to the gentiles in their own tongue".[1] If classicists claim not to grasp paraconsistent negation as being a genuine negation,[2] why not deliver the logic to them in terms of a semantics expressed entirely within a classical metatheory? And that is precisely what relevantists did. We think it is even better to go one step further by having all classical notions expressible in the *object language itself*. What better way to preach to the gentiles in their own tongue?[3] If a classical notion is coherently expressible in your language, why not help yourself to it? Paraconsistentists might be interested in classical notions simply because they find them coherent.

We are aware that a good number of paraconsistentists have an aversion to certain classical notions, especially negation and implication. We think, however, that this aversion stems from a misunderstanding that classical notions—and in particular negation—somehow lead to triviality in a suitably rich language. First, paraconsistent logics with classical negation need not collapse into classical logic, and so naive theories of truth or sets couched in these languages needn't be trivial provided they are formulated in the right way.[4]

Second, it is hard to see how certain paraconsistentists can deny the coherence of classical negation, in which case there is no reason to deny it its place in one's formal (object) language. Certainly if such a language is to serve as one in which much of natural language can be regimented, then if classical negation is coherent, one must allow it into their formal language for the sake of expressive adequacy.

Finally, a language which succeeds in evading paradox only because it lacks expressive resources serves as no solution to paradox. One cannot hope to avoid untoward consequences of one's theory by simply ignoring notions one deems problematic, at least not if those notions are coherent (or one has not shown that they are incoherent). Since this is no place to defend

[1] See [15, p. 1].

[2] On which, see [28].

[3] Indeed, there is no better way if you deny the object-/meta-language distinction in the first place, as some well-known paraconsistent logicians do (see e.g. [24]).

[4] See [20].

the coherence of classical negation, for present purposes we simply assume it. We will, however, say more about what classical negation amounts to in a non-classical setting, a topic we come to in the following section.

The aim of this paper is three-fold. First, to motivate a semantic characterization of classical negation applicable to non-classical logics (Sect. 2). Second, to investigate an axiomatization of the extension of **BD** by said classical negation (Sect. 3). Third, to compare the resulting extension to related systems found in the literature (Sect. 3.5).

2. What is Classical Negation?

We have been using "classical negation" within the context of non-classical logics without saying exactly what we mean by it. Before extending **BD** by classical negation, we need to say precisely what we take classical negation to be. One typically finds a definition according to which classical negation is any operation $\neg$ satisfying certain "characteristic" laws, e.g.

$$(A \wedge \neg A) \rightarrow B, \ \neg\neg A \rightarrow A.$$

But such a characterization depends crucially on what sort of conditional $\rightarrow$ is, and which laws are taken to be characteristic of classical negation. For instance, if $\rightarrow$ is not the classical material conditional, then even if $\neg$ satisfies the above laws, it may not satisfy others thought to be characteristically classical, such as the following form of contraposition[5]:

$$(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A).$$

The point is that it is especially difficult to say *in a purely syntactic way* which laws are characteristic of negation, depending on which other sentential operators we have in the background. If we have only a relevant conditional around, how should classical negation interact with it? It is for this reason that we examine the notion of classicality from a semantic perspective.[6]

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[5]See e.g. [16], where what is there called 'classical negation' fails precisely this law when $\rightarrow$ is a relevant conditional.

[6]Additional reasons for preferring a semantic over syntactic characterization of negation are given in [9, Chap. 2].

2.1. Contradictoriness

Since we will be working with **BD**, let us begin with the truth tables for its connectives:

A	$\sim A$	$A \wedge B$	**t**	**b**	**n**	**f**	$A \vee B$	**t**	**b**	**n**	**f**
t	**f**	**t**	**t**	**b**	**n**	**f**	**t**	**t**	**t**	**t**	**t**
b	**b**	**b**	**b**	**b**	**f**	**f**	**b**	**t**	**b**	**t**	**b**
n	**n**	**n**	**n**	**f**	**n**	**f**	**n**	**t**	**t**	**n**	**n**
f	**t**	**f**	**f**	**f**	**f**	**f**	**f**	**t**	**b**	**n**	**f**

Note here that designated values are **t** ("truth only") and **b** ("both truth and falsity"), and that $\sim$ is a paraconsistent negation. The values **f** and **n** are to be taken as "falsity only" and "neither truth nor falsity". Thus, when we speak of a sentence being true, we mean it takes either the value **t** or **b**, and when we speak of a sentence being false, we mean it takes either the value **b** or **f**. Indeed we take there to be only two genuine truth values, truth and falsity, that are neither exhaustive nor exclusive. Thus, for instance, by the assignment of the value **b** to A we are to understand that A is related to both truth and falsity, not that there is some further truth value, "both-truth-and-falsity", in relation to which A stands. If helpful, the reader may think of the four values as sets of values consisting of just truth and falsity so that, e.g., $\mathbf{b} = \{\text{truth}, \text{falsity}\}$.

Typically one thinks of **t** and **f** as classical values, and the others as non-classical. As such, we may then say of a pair of formulas A and B that:

Contra: A and B are *classically contradictory* if and only if $A \vee B$ is always true *and not false* and $A \wedge B$ is always false *and not true*.

Contra generalizes the usual notion of contradictoriness to allow for a non-classical understanding of the relation between truth and falsity, i.e. whether they're exhaustive or exclusive. It also uniquely determines a contradictory-forming connective whose truth table is given by

A	$\neg A$
t	**f**
b	**n**
n	**b**
f	**t**

Indeed, from an algebraic viewpoint $\neg$ is boolean complementation.[7]

[7] We criticized syntactic characterizations of negation since they rely crucially on what properties certain connectives have. Here we have proposed a characterization, Contra, which relies on the properties $\wedge$ and $\vee$ have. The difference is that we are assuming

An alternative notion of classical negation in the context of **BD**, often called *exclusion negation*, has the following truth table:

A	$\neg^e A$
t	**f**
b	**f**
n	**t**
f	**t**

It is often read as 'It is not true that…', keeping in mind that taking the value **b** means being both true and false.[8] It is thought of as classical since the logic in $\wedge, \vee$ and $\neg^e$ is precisely classical logic.[9] It is easily verified that the logic in $\wedge, \vee$, and $\neg$ too coincides with classical logic. Note, however, that $\neg^e$ fails to satisfy Contra, and so is not a contradictory forming operator in that sense, since $A \vee \neg^e A$ takes the value **b** (instead of **t**) when **b** is assigned to A. Moreover, once we add the paraconsistent negation $\sim$ to the language, $\neg^e$ and $\neg$ come apart, as witnessed by Proposition 4 below.

One problem with the above characterization of classical negation as one satisfying Contra is that the definition does not generalize, e.g. to a three-valued setting. For suppose we have only three values, **t**, **b**, and **f**. (It will not matter whether **b** is designated.) In order to meet Contra, a classical negation $\neg\!\!\!\!\rightharpoondown$ must take **t** to **f** and conversely. Now what to do with **b**? It can't go to **f**, lest $A \vee \neg\!\!\!\!\rightharpoondown A$ not always take **t**. And it can't go to **b** or **t** either, lest $A \vee \neg\!\!\!\!\rightharpoondown A$ not always take **t** or $A \wedge \neg\!\!\!\!\rightharpoondown A$ not always take **f**. In other words, there is no operation satisfying Contra in a three-valued setting. If we wish our characterization of classical negation to apply when there are arbitrarily many values, we need a more general notion of classical contradictoriness. For that, we propose the following account; call it *Liberal*.

Contrariety: two sentences are contraries if one of them is not true whenever the other is true;

Footnote 7 continued
with most others that $\wedge$ and $\vee$ behave classically—algebraically as meet and join—and we already know how classical negation interacts with these connectives. Apart from any semantic considerations, it is not clear, however, how classical negation should interact with e.g. a relevant arrow.

[8] The expression 'exclusion negation' typically refers to a connective in a three-valued setting with the same reading, 'It is not true that'. We have lifted the terminology to the four-valued case.

[9] For our purposes, a logic is a set of formulae closed under an appropriate relation of deducibility satisfying e.g. transitivity, reflexivity and substitution. Note that the criterion of classicality here is entirely proof-theoretic, as is assumed e.g. in [8].

Subcontrariety: two sentences are subcontraries if one of them is true whenever the other is not true.

Call two sentences *contradictories* if they are contraries and subcontraries.

Liberal generalizes the usual classical notion of contradictoriness so that it applies in a non-classical setting. In a classical setting, where truth and falsity are exclusive and exhaustive, falsity is simply untruth, and thus the above account is equivalent to the more familiar traditional account according to which:

- two sentences are contraries if they cannot be true together;
- two sentences are subcontraries if they cannot be false together.

Liberal uniquely secures classical negation when truth and falsity are exclusive and exhaustive—as they are classically—but it is not by itself enough to secure a single unary operation when truth and falsity interact non-classically. It can therefore serve only as a *necessary* condition on classical negation. Indeed it can be seen as merely one component of a definition of classical negation that generalizes to a non-classical setting. We now come to the further components of this definition.

2.2. Negations Satisfying Liberal in a Non-classical Setting

If truth and falsity are exhaustive but not exclusive (meaning we are in a paraconsistent setting where the only available values are **t**, **f**, and **b**), then the following two candidates satisfy Liberal:

A	$\sim A$	$\neg^1 A$	$\neg^2 A$
t	**f**	**f**	**f**
b	**b**	**f**	**f**
f	**t**	**t**	**b**

The operator $\neg^1$ is the familiar paraconsistent three-valued exclusion negation (read, 'It is not true that') while $\neg^2$ (which has no intuitive reading) appears not to be a negation at all. Each operation takes a designated value to an undesignated one and conversely—a necessary requirement of any classical negation—but only $\neg^2$ has the unusual property that $\sim\neg^2 A$ is valid *for arbitrary* A. But no double negation composed only of de Morgan and classical negations should be valid: for any such statement intuitively has the force of an assertion (or something slightly weaker). Moreover, $\neg^2$ makes any sentence false (thinking relationally) regardless of the value of that sentence. Such an operation seems to us not to be a *classical* negation. We therefore need an additional constraint that rules out $\neg^2$ from qualifying as classical.

Let us now consider the paracomplete case where truth and falsity are exclusive but not exhaustive (meaning the only available values are **t**, **f**, and **n**). Here again we have two candidate negations satisfying Liberal:

A	$\sim A$	$\neg^3 A$	$\neg^4 A$
t	**f**	**f**	**n**
n	**n**	**t**	**t**
f	**t**	**t**	**t**

The operator $\neg^3$ is the familiar three-valued paracomplete exclusion negation while the other, $\neg^4$, again appears not to be a negation at all; for $\neg^4{\sim}\neg^4 A$ is valid for arbitrary A. But no triple negation composed only of de Morgan and classical negations should be valid: for any such statement intuitively has the force of a denial (or something slightly weaker). Moreover, if not every double negation ${\sim}\neg^4 A$ is to be valid for arbitrary A (as we argued above), then for some invalid such double negation ${\sim}\neg^4 B$, appending B with a classical negation won't yield a validity either. Hence $\neg^4{\sim}\neg^4 A$ shouldn't be valid for arbitrary A. We therefore need an additional constraint that rules out $\neg^4$ from qualifying as classical.[10]

Finally, let us consider the paraconsistent and paracomplete case where truth and falsity are neither exclusive nor exhaustive (meaning all four truth values are available). Here we have sixteen candidate unary operations satisfy Liberal. Since the negation has to be undesignated (designated) when the negand is designated (undesignated), there are two possibilities for each

[10]By our definition, $\neg^3$ is classical since it satisfies both Liberal and Toggle. An anonymous referee questions this by considering two interesting cases. The first involves the weak Kleene interpretation of the connectives. On that interpretation, $A \vee \neg^3 A$ is not a theorem (though it is when $\vee$ is interpreted according to the strong Kleene tables), which is to suggest that $\neg^3$ cannot therefore be classical. We disagree. No operation $\otimes$ is such that $A \vee \otimes A$ will be a theorem of weak Kleene logic simply because of the *non-classical* interpretation given to $\vee$. A classical interpretation requires the truth of $A \vee B$ when either A is true or B is and this fails for $\vee$ in weak Kleene logic. We still maintain that $\neg^3$ is classical even in the weak Kleene setting since it meets the required semantic criteria and since all the usual classical theorems are valid when the other connectives such as $\vee$ are read classically. The second case involves giving a non-standard interpretation to the consequence relation (as preservation of non-falsity) so that it is not taken as the smallest relation preserving truth over all models. Relative to some such consequence relations, the inference from e.g. A to $\neg^3\neg^3 A$ may fail. This does not by our lights show that $\neg^3$ fails to be classical simply because the consequence relation does not cohere with the usual understanding of the truth values in the present non-classical setting.

input. And since there are four kinds of inputs, we obtain $2^4 = 16$ possibilities. However, again not each of these possibilities counts as a classical negation; for example, just as in the paraconsistent case, the following four-valued counterpart $\neg^5$ of $\neg^2$ satisfies Liberal:

A	$\neg^5 A$
t	**f**
b	**f**
n	**b**
f	**b**

As such, $\sim\neg^5 A$ is valid for arbitrary A, and this rules it out as being a classical negation on the same grounds as $\neg^2$.

2.3. The Classical Triad

In each the three-valued paraconsistent and paracomplete case, while Liberal does not by itself secure only classical negations, it does leave us with only two, one of which is clearly not classical. The following constraint seems a natural additional to Liberal in securing classical negation in a three-valued setting. We call it *Toggle*:

Toggle: an operation on truth values is a *classical* negation only if it toggles between the classical values **t** and **f**, i.e. it takes **t** to **f** and conversely.

This condition seems to us a reasonable constraint on classicality. First, because it is often taken as a *definition* of classical negation, though we have generalized it to a non-classical setting.[11] Second and more generally, classical operations that receive classical inputs should have classical outputs. This is already enough (in conjunction with Liberal) to secure a unique negation in the three-valued cases. Third, because there is no intuitive reading of an operation which takes a sentence that is true only or false only to one that is both true and false, or one that is neither. Yet classical negation presumably has an intuitive reading.

With Liberal and Toggle a unique classical negation is secured in the three-valued cases. The four-valued case presents a separate challenge, since these constraints do not uniquely secure a single negation. In fact, they determine the following four candidates:

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[11] Toggle is endorsed as a definition of classical negation, e.g., in [7,22,29]. Toggle and Liberal are equivalent under the assumption that there are no truth value gaps or gluts, and hence the two conditions are classically equivalent. In [23], Priest takes Liberal to be a definition of classical negation.

A	$\neg A$	$\neg^1 A$	$\neg^2 A$	$\neg^e A$
t	**f**	**f**	**f**	**f**
b	**n**	**n**	**f**	**f**
n	**b**	**t**	**b**	**t**
f	**t**	**t**	**t**	**t**

So the question is whether there are additional, natural semantic constraints that, in conjunction with Liberal and Toggle, secure a unique candidate from these four. We propose the following:

Involution: an operation $\neg$ on truth values $\mathbf{i}$ is a classical negation only if $\neg\neg\mathbf{i} = \mathbf{i}$,

i.e. the *semantic* equivalence of a sentence with its double negation. The triad, Liberal, Toggle and Involution, uniquely secure $\neg$ as the unique classical negation in the four-valued setting.

Note that, of the four candidate negations, only boolean negation is surjective. This implies that, for the non-boolean negations, A and its double negation will never be semantically equivalent for some values of A. For if some value $\mathbf{i}$ is not in the image of a negation, then the double negation of $\mathbf{i}$ must be something other than $\mathbf{i}$. In particular, we have that when (the value of) A is $\mathbf{b}$, $\neg^1\neg^1 A, \neg^2\neg^2 A$ and $\neg^e\neg^e A$ are all $\mathbf{t}$.

Why think Involution should be a constraint on classicality? Certainly for $\neg$ to be classical, $\neg\neg A$ must imply A and conversely. This may hold even when Involution fails. But the idea that this equivalence holds is surely grounded in the idea that, for any possible interpretation of the language, $\neg\neg A$ and A have the same value. This is, after all, a consequence of the intended interpretation of classical negation. We should be careful here, however. Any negation satisfying Involution in a three-valued setting will have to fail either Liberal or Toggle, and these latter two conditions strike us as more to the core of classical negation than does Involution. For note that this triad of constraints is in fact equivalent to Contra, which can be taken as a definition of boolean complementation (assuming $\wedge$ and $\vee$ are boolean meet and join), so one of them must go if they are to serve as a characterization of classical negation in a general setting. Involution is therefore applicable only in certain cases where Liberal and Toggle fail to be uniquely determining.

Here is a final considerations for siding with boolean negation as classical in a four-valued setting. The first is that, among the four candidates, only boolean negation is uniquely determined by the following thoroughly classical truth and falsity conditions:

- $\neg A$ is true iff A is not true.
- $\neg A$ is false iff A is not false.

It is important that these conditions together treat truth and falsity on a par, in the sense that both values are essentially appealed to in the truth and falsity conditions of negated sentences. It is this symmetry of the standard connectives of **BD** and classical logic that is absent with the three other candidate classical negations. To see this, note that while the four negations have the same *truth* conditions, they differ with respect to their *falsity* conditions, which are given as follows:

- $\neg^e A$ is false iff A is true.
- $\neg^1 A$ is false iff A is true and not false (iff A is true only).
- $\neg^2 A$ is false iff A is true or not false (iff A is not false only).

Only the falsity condition for exclusion negation strikes us as natural. Yet it does not treat truth and falsity on a par, as does the falsity condition for boolean negation. We take this to be another reason, in the context of **BD**, to regard only boolean negation as classical.

Let us briefly summarize the discussion so far. First, the notion of classical negation in the context of **BD** can be partially captured by the conditions Liberal and Toggle. In particular, Liberal and Toggle together uniquely determine the notion of classical negation in the two- and three-valued settings. In the four-valued setting, further constraints are needed if a negation is to be *uniquely* secured as classical. We proposed one such condition, Involution. These three conditions, Liberal, Toggle and Involution, uniquely secure boolean negation in any bounded distributive lattice, hence in the matrix for **BD**. Indeed, Contra is decomposable into precisely these three conditions. Finally, boolean negation is the only negation in the four-valued setting whose truth and falsity conditions, like two-valued classical negation, treat truth and falsity on a par. We took this as a final consideration for regarding boolean negation as classical in the context of **BD**. With that said, we do think that exclusion negation, $\neg^e$, comes a close second.

Now that we have a grasp on the notion of classical negation, we turn to the proof-theoretic details of extending **BD** by classical negation.

3. Expanding BD with Classical Negation

We present a Hilbert style system **BD+** which is the expansion of **BD** by classical negation. We show it (i) complete with respect to the semantics discussed in the previous section, and (ii) maximal with respect to classical logic.[12]

3.1. Formulation

The language $\mathcal{L}$ consists of the set of logical symbols S and a denumerable set, *Prop*, of propositional letters whose members we denote by p, q, etc. In the following, we assume that $\{\sim, \wedge, \vee\} \subseteq \mathsf{S}$, and indicate the inclusion of other logical symbols of S using subscripts. For example, the language $\mathcal{L}_{\to,\neg}$ of **BD+** includes, besides $\{\sim, \wedge, \vee\}$, also $\{\to, \neg\}$. Furthermore, we denote by $Form_{\mathcal{L}}$ the set of formulas defined as usual in $\mathcal{L}$. If $\mathcal{L}$ is e.g. $\mathcal{L}_{\to,\neg}$, we may denote its set of formulas similarly by $Form_{\to,\neg}$. We use uppercase Greek letters $(\Gamma, \Delta,$ etc.) to denote sets of formulas and uppercase Roman letters $(A, B,$ etc.) to denote formulas.

In **BD+**, the classical implication $A \to B$ is definable by $\neg A \vee B$ but we have chosen to take it as primitive to simplify the comparison of **BD+** with other systems we discuss in what follows. We provide its truth table for convenience.

$A \to B$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**t**	**t**	**n**	**n**
n	**t**	**b**	**t**	**b**
f	**t**	**t**	**t**	**t**

Definition 1. The system **BD+** consists of the following axiom schemata and a rule of inference, where $A \leftrightarrow B$ abbreviates $(A \to B) \wedge (B \to A)$.

(Ax1) $A \to (B \to A)$

(Ax2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

(Ax3) $((A \to B) \to A) \to A$

(Ax4) $(A \wedge B) \to A$

(Ax5) $(A \wedge B) \to B$

(Ax6) $(C \to A) \to ((C \to B) \to (C \to (A \wedge B)))$

(Ax7) $A \to (A \vee B)$

(Ax8) $B \to (A \vee B)$

(Ax9) $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$

(Ax10) $A \vee \neg A$

[12] We will not be too careful to distinguish a Hilbert system (i.e. a set of axiom schemata and rules of inference) from the logic it generates (i.e. a set of formulas, for our purposes), though we typically use 'system' for the former and 'logic' for the latter.

(Ax11)	$(A \wedge \neg A) \rightarrow B$
(Ax12)	$\sim\neg A \leftrightarrow \neg\sim A$
(Ax13)	$\sim\sim A \leftrightarrow A$
(Ax14)	$\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$
(Ax15)	$\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$
(Ax16)	$\sim(A \rightarrow B) \leftrightarrow (\neg\sim A \wedge \sim B)$
(MP)	$\frac{A \quad A \rightarrow B}{B}$

Finally, we write $\Gamma \vdash_{\mathbf{BD+}} A$ if there is a sequence of formulas $\langle B_1, \ldots, B_n, A\rangle$ ($n \geq 0$), called *a derivation*, such that every formula in the sequence either (i) belongs to Γ; (ii) is an axiom of **BD+**; (iii) is obtained by (MP) from formulas preceding it in the sequence. As usual, we write $\Gamma, A_1, \ldots, A_n \vdash_{\mathbf{BD+}} B$ for $\Gamma \cup \{A_1, \ldots, A_n\} \vdash_{\mathbf{BD+}} B$. We call A a theorem of **BD+** when $\emptyset \vdash_{\mathbf{BD+}} A$.

Remark 2. Consider the subsystem of **BD+** consisting of axioms (Ax1) through (Ax9) together with the rule of inference (MP). This system is equivalent to the negation-less fragment of **CL**, and we call it $\mathbf{CL}^+$. Moreover, the subsystem of $\mathbf{CL}^+$ obtained by dropping (Ax3) is the negation-less fragment of intuitionistic logic, and we call it $\mathbf{IL}^+$. Note also that the above axiomatization is redundant in a sense that some of the axioms are provable by means of others. We present it this way to ease the comparison of **BD+** with other systems.

Proposition 3. *The deduction theorem for* **BD+** *holds with respect to* $\rightarrow$*, that is,* $\Gamma, A \vdash_{\mathbf{BD+}} B$ *iff* $\Gamma \vdash_{\mathbf{BD+}} A \rightarrow B$.

Proof. The left-to-right direction can be proved in the usual manner in the presence of axioms (Ax1) and (Ax2), and (MP) the sole rule of inference. For the other direction, suppose $\Gamma \vdash_{\mathbf{BD+}} A \rightarrow B$, i.e. that there is a derivation $\langle B_1, \ldots, B_n, A \rightarrow B\rangle$. But then $\langle B_1, \ldots, B_n, A, A \rightarrow B\rangle$ is a derviation witnessing $\Gamma, A \vdash_{\mathbf{BD+}} A \rightarrow B$. By (MP), $\langle B_1, \ldots, B_n, A, A \rightarrow B, B\rangle$ is a derivation witnessing $\Gamma, A \vdash_{\mathbf{BD+}} B$. ■

3.2. Soundness and Completeness

We now turn to prove the soundness and completeness of **BD+** with respect to the semantics we considered earlier. We begin with a definition of a **BD+**-valuation.

Definition 4. A **BD+**-*valuation* is a homomorphism from the set $Form_{\mathcal{L}}$ of **BD+**-formulas to the set $\{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$ of truth values, induced by the following matrices:

A	$\sim A$	$\neg A$	$A \wedge B$	**t**	**b**	**n**	**f**	$A \vee B$	**t**	**b**	**n**	**f**
t	**f**	**f**	**t**	**t**	**b**	**n**	**f**	**t**	**t**	**t**	**t**	**t**
b	**b**	**n**	**b**	**b**	**b**	**f**	**f**	**b**	**t**	**b**	**t**	**b**
n	**n**	**b**	**n**	**n**	**f**	**n**	**f**	**n**	**t**	**t**	**n**	**n**
f	**t**	**t**	**f**	**f**	**f**	**f**	**f**	**f**	**t**	**b**	**n**	**f**

Note here that the designated values are **t** and **b**.

Definition 5. A formula A is a **BD+**-*tautology* iff, for any **BD+**-valuation $v, v(A)$ is always designated.

Theorem 1. (Soundness) *All the theorems of* **BD+** *are* **BD+**-*tautologies and (MP) is sound.*

Proof. By a straightforward verification that each instance of each axiom schema always takes a designated value, and that (MP) preserves designationhood. ∎

We now turn to completeness. We adopt the constructive method of Kalmár, used also in [6,27] for the paraconsistent logic $\mathbf{P^1}$ of Sette, and the logics **LFI1** and **LFI2** of Carnielli, Marcos and de Amo (see also Mendelson [14]).

For convenience, we list some formulas that are provable in **BD+**.

Lemma 6. *The following formulas are provable in the system* **BD+**.

(T1) $\neg\neg A \leftrightarrow A$ (T2) $\neg{\sim}\neg A \leftrightarrow {\sim} A$

(T3) $\neg{\sim}(A \to B) \leftrightarrow ({\sim} A \vee \neg{\sim} B)$ (T4) $\neg(A \to B) \leftrightarrow (A \wedge \neg B)$

(T5) $(A \wedge \neg{\sim} A) \vee (A \wedge {\sim} A) \vee (\neg A \wedge \neg{\sim} A) \vee (\neg A \wedge {\sim} A)$

Proof. Left as an exercise for the reader. ∎

Based on this lemma, we prove the key lemma for our completeness proof.

Lemma 7. *Given a* **BD+**-*valuation v, we define for each formula A an associated formula A^v:*

$$(*) \qquad A^v = \begin{cases} A \wedge \neg{\sim} A & \textit{if } v(A) = \mathbf{t} \\ A \wedge {\sim} A & \textit{if } v(A) = \mathbf{b} \\ \neg A \wedge \neg{\sim} A & \textit{if } v(A) = \mathbf{n} \\ \neg A \wedge {\sim} A & \textit{if } v(A) = \mathbf{f} \end{cases}$$

Now, let F be a formula whose set of atomic variables is $\{p_1, p_2, \dots, p_n\}$, and let Δ^v be the set $\{p_1^v, p_2^v, \dots, p_n^v\}$. Then $\Delta^v \vdash F^v$.

Proof. By induction on the number n of connectives. Outline of the proof is provided in the appendix. ∎

THEOREM 2. (Completeness) *All the* **BD+***-tautologies are theorems of* **BD+**.

PROOF. Let F be any **BD+**-tautology and Δ the set of propositional variables occurring in F. Then by Lemma 7 above, we have $\Delta^v \vdash F^v$. Furthermore, since F is a **BD+**-tautology, F^v is always $F \wedge \neg{\sim}F$ or $F \wedge {\sim}F$, so either $\Delta^v \vdash (F \wedge \neg{\sim}F)$ or $\Delta^v \vdash (F \wedge {\sim}F)$ holds. In either case, we obtain $\Delta^v \vdash F$.

Now, let Δ_k^v be the set $\Delta^v \setminus p_k$, and suppose the four valuations v_1, v_2, v_3 and v_4 are such that $\Delta_k^{v_1} = \Delta_k^{v_2} = \Delta_k^{v_3} = \Delta_k^{v_4} (=_{\text{def.}} \Delta_k)$ and $v_1(p_k) = \mathbf{t}, v_2(p_k) = \mathbf{b}, v_3(p_k) = \mathbf{n}$ and $v_4(p_k) = \mathbf{f}$.

Then, for $v_1, \Delta^{v_1} \vdash F$ is $\Delta_k^{v_1}, \{p_k \wedge \neg{\sim}p_k\} \vdash F$ by the definition of Δ^{v_1}. By the deduction theorem, we have $\Delta_k^{v_1} \vdash (p_k \wedge \neg{\sim}p_k) \rightarrow F$. Similarly we obtain $\Delta_k^{v_2} \vdash (p_k \wedge {\sim}p_k) \rightarrow F, \Delta_k^{v_3} \vdash (\neg p_k \wedge \neg{\sim}p_k) \rightarrow F$ and $\Delta_k^{v_4} \vdash (\neg p_k \wedge {\sim}p_k) \rightarrow F$ for v_2, v_3 and v_4 respectively. Putting these four results together by making use of (Ax9) and the fact that $\Delta_k^{v_1} = \Delta_k^{v_2} = \Delta_k^{v_3} = \Delta_k^{v_4} = \Delta_k$, we have $\Delta_k \vdash ((p_k \wedge \neg{\sim}p_k) \vee (p_k \wedge {\sim}p_k) \vee (\neg p_k \wedge \neg{\sim}p_k) \vee (\neg p_k \wedge {\sim}p_k)) \rightarrow F$, where $\Delta_k = \Delta \setminus p_k$. By (T5), we conclude that $\Delta_k \vdash F$. Repeating this procedure $k - 1$ more times gives us $\vdash F$ as desired. ■

3.3. Maximality

Maximality was given as a criterion of paraconsistency independently by Jaśkowski and da Costa, and we here show that **BD+** is indeed maximal. Jaśkowski required that paraconsistent systems "be rich enough to enable practical inference" ([12, p. 38]), while da Costa requires them to "contain the most part of the schemata and rules of $\mathbf{C}_0$" ([8, p. 498]) where $\mathbf{C}_0$ is the classical propositional calculus. Although these criteria are rather vague, it is common to interpret them as a maximality constraint, defined as follows.[13]

DEFINITION 8. Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be logics taken as sets of formulas closed under an appropriate relation of deducibility. Then, $\mathbf{L}_1$ is said to be *maximal relative to* $\mathbf{L}_2$ if the following holds:

- The languages of $\mathbf{L}_1$ and $\mathbf{L}_2$ are the same;
- $\mathbf{L_1} \subseteq \mathbf{L_2}$;
- $\mathbf{L_1} \cup \{G\} = \mathbf{L_2}$ for any theorem G of $\mathbf{L}_2$ which is not a theorem of $\mathbf{L}_1$.

The maximality of three-valued paraconsistent logics is relatively well discussed in the literature. In particular, in [2], some comprehensive results are presented by Arieli, Avron and Zamansky.[14] More general many-valued

[13] We here adopt the definition employed in [6, p. 135].

[14] Note that in [2], maximality with respect to consequence relations is also considered.

logics are considered in [1] in the context of what they call "ideal paraconsistent logics", though the results contained therein do not cover our case, and so we deal with it here.[15] For the purpose of stating the result, we introduce the following logic.

Definition 9. Let **ECL** be classical propositional logic with two classical negations in the language $\mathcal{L}_{\to,\neg}$. Namely, we obtain **ECL** by replacing (Ax12)–(Ax16) by the two axioms $A \vee {\sim}A$ and $(A \wedge {\sim}A) \to B$ in the formulation of **BD+**.

Then the following maximality result holds.

Theorem 3. **BD+** *is maximal relative to* **ECL**.

Since the first two conditions of Definition 8 are obviously satisfied, we only prove the third condition. For this purpose, we make use of an idea employed in [10].

Lemma 10. *Let G be a formula containing only one propositional variable p. Then one of the following four formulas is provable in the system* **BD+**:

$$\text{(I)}\quad (p \leftrightarrow {\sim}p) \to G \qquad\qquad \text{(II)}\quad (p \leftrightarrow {\sim}p) \to \neg G$$
$$\text{(III)}\quad (p \leftrightarrow {\sim}p) \to (G \leftrightarrow \neg p) \qquad \text{(IV)}\quad (p \leftrightarrow {\sim}p) \to (G \leftrightarrow p)$$

Proof. By induction on the complexity of G. ■

Lemma 11. *Let G be a formula containing only one propositional variable p such that $\vdash_{\mathbf{ECL}} G$ and $\nvdash_{\mathbf{BD+}} G$. Then $\nvdash_{\mathbf{BD+}} (p \leftrightarrow {\sim}p) \to G$.*

Proof. By $\nvdash_{\mathbf{BD+}} G$ and completeness, there is a **BD+**-valuation v_0 such that $v_0(G) \in \{\mathbf{f}, \mathbf{n}\}$. Since $\vdash_{\mathbf{ECL}} G$, it follows that $v_0(p) \in \{\mathbf{b}, \mathbf{n}\}$. In either case, $v_0(p \leftrightarrow {\sim}p) = \mathbf{t}$. So whether $v_0(G) = \mathbf{f}$ or $v_0(G) = \mathbf{n}$, $v_0((p \leftrightarrow {\sim}p) \to G) \notin \{\mathbf{t}, \mathbf{b}\}$. By soundness, $\nvdash_{\mathbf{BD+}} (p \leftrightarrow {\sim}p) \to G$. ■

Lemma 12. *Let G be a formula containing only one propositional variable p such that $\vdash_{\mathbf{ECL}} G$ and $\nvdash_{\mathbf{BD+}} G$. Then the system* **S** *obtained from the system* **BD+** *by adjoining G as an axiom schema is equivalent to* **ECL**.

[15] More precisely, [1, p. 55, Theorem 3] provides a sufficient condition for a many-valued logic's being ideal, a condition which requires maximality in our sense. However, ideality assumes the presence of the conditional $\supset$ defined as follows:

$$v(A \supset B) = \begin{cases} v(B) & \text{if } v(A) \text{ is designated;} \\ \mathbf{t} & \text{otherwise} \end{cases}$$

However, this conditional is not definable in **BD+**, as observed in Corollary 16.

Proof. By Lemmas 10 and 11, one of the following formulas are provable: (II) $(p \leftrightarrow \sim p) \to \neg G$, (III) $(p \leftrightarrow \sim p) \to (G \leftrightarrow \neg p)$, (IV) $(p \leftrightarrow \sim p) \to (G \leftrightarrow p)$. We here prove that in any of these cases, we have $\vdash_{\mathbf{S}} \neg p \leftrightarrow \sim p$ is provable which is sufficient for the desired result.

First, if (II) is provable, then we have $\vdash_{\mathbf{S}} (p \leftrightarrow \sim p) \to \neg G$, and therefore by contraposition with respect to $\neg$, we get $\vdash_{\mathbf{S}} G \to \neg(p \leftrightarrow \sim p)$ which is equivalent to $\vdash_{\mathbf{S}} G \to (\neg p \leftrightarrow \sim p)$. Since G is assumed as an axiom, we obtain $\vdash_{\mathbf{S}} \neg p \leftrightarrow \sim p$. Second, if (III) is provable then we have $\vdash_{\mathbf{S}} (p \leftrightarrow \sim p) \to (G \leftrightarrow \neg p)$ and, therefore, $\vdash_{\mathbf{S}} (p \leftrightarrow \sim p) \to (G \to \neg p)$. By permutation and contraposition with respect to $\neg$, we get $\vdash_{\mathbf{S}} G \to (p \to (\neg p \leftrightarrow \sim p))$. Since G is assumed as an axiom, we obtain $\vdash_{\mathbf{S}} p \to (\neg p \leftrightarrow \sim p)$. But then we also have $\vdash_{\mathbf{S}} \neg p \to (\neg\neg p \leftrightarrow \sim\neg p)$, which is equivalent to $\vdash_{\mathbf{S}} \neg p \to (\neg p \leftrightarrow \sim p)$ in view of (Ax12). Hence, together with (Ax7) and (Ax10), we reach $\vdash_{\mathbf{S}} \neg p \leftrightarrow \sim p$ as desired. Finally, if (IV) is provable then the proof is similar to the previous case, the details of which are left to the reader. This completes the proof. ■

Lemma 13. *Let $G(p_1, \dots, p_n)$ be a formula containing no propositional variables except $p_1, \dots, p_n$, such that $\vdash_{\mathbf{ECL}} G$ and $\not\vdash_{\mathbf{BD+}} G$. Then there are formulas $\varphi_1(p)$, ..., $\varphi_n(p)$ containing no propositional variable except p, such that $\vdash_{\mathbf{ECL}} G(\varphi_1(p)), \dots, \varphi_n(p))$ and $\not\vdash_{\mathbf{BD+}} G(\varphi_1(p)), \dots, \varphi_n(p))$.*

Proof. Since $\vdash_{\mathbf{ECL}} G$, it is clear that for any formulas $\varphi_1(p), \dots, \varphi_n(p)$, $G(\varphi_1(p), \dots, \varphi_n(p))$ is a tautology. Therefore, we shall only prove that we can construct formulas $\varphi_1(p), \dots, \varphi_n(p)$ so that $\not\vdash_{\mathbf{BD+}} G(\varphi_1(p)), \dots, \varphi_n(p))$.

By the assumption $\not\vdash_{\mathbf{BD+}} G$ and the completeness theorem, there is a **BD+**-valuation v_0 such that $v_0(G(p_1, \dots, p_n)) \notin \{\mathbf{t}, \mathbf{b}\}$. Given this valuation, we define $\varphi_k(p)$ as follows

$$\varphi_k(p) = \begin{cases} p \leftrightarrow \sim p & \text{if } v_0(p_k) = \mathbf{t} \\ p & \text{if } v_0(p_k) = \mathbf{b} \\ \neg p & \text{if } v_0(p_k) = \mathbf{n} \\ \neg p \leftrightarrow \sim p & \text{if } v_0(p_k) = \mathbf{f} \end{cases}$$

Then we have $v_0(\varphi_k(p)) = v_0(p_k)$ when $v_0(p) = \mathbf{b}$. Indeed,

- If $v_0(p_k) = \mathbf{t}$, then $v_0(\varphi_k(p)) = v_0(p \leftrightarrow \sim p) = \mathbf{t} = v_0(p_k)$;
- If $v_0(p_k) = \mathbf{b}$, then $v_0(\varphi_k(p)) = v_0(p) = \mathbf{b} = v_0(p_k)$;
- If $v_0(p_k) = \mathbf{n}$, then $v_0(\varphi_k(p)) = v_0(\neg p) = \mathbf{n} = v_0(p_k)$;
- If $v_0(p_k) = \mathbf{f}$, then $v_0(\varphi_k(p)) = v_0(\neg p \leftrightarrow \sim p) = \mathbf{f} = v_0(p_k)$.

Hence, we have $v_0(G(\varphi_1(p), \ldots, \varphi_n(p))) \notin \{\mathbf{t}, \mathbf{b}\}$ when $v_0(p) = \mathbf{b}$, and thus $\nvdash_{\mathbf{BD+}} G(\varphi_1(p)), \ldots, \varphi_n(p))$ as desired. ■

We are now in a position to prove Theorem 3.

PROOF OF THEOREM 3. Let G be a formula such that $\vdash_{\mathbf{ECL}} G$ and $\nvdash_{\mathbf{BD+}} G$. Let **S** be the system obtained from the system **BD+** by adjoining G as an axiom schema. In view of Lemma 12, it is sufficient to prove that there is a formula G' which contains only one propositional variable such that $\vdash_{\mathbf{ECL}} G'$ and $\nvdash_{\mathbf{BD+}} G'$. Given Lemma 13, we can construct such a formula from G, concluding the proof of Theorem 3. ■

REMARK 14. If we consider maximality with respect to consequence relations, then **BD+** is *not* maximal relative to **ECL**. Indeed, the addition of the rule $A, {\sim}A \vdash B$ to **BD+** gives us a strictly stronger system compared to **BD+**, and also a strictly weaker system compared to **ECL**. This may be observed in the truth tables that are exactly like those for the connectives of **BD+** except that the designated value is restricted to only **t**.

3.4. The Definability of Other Negations in BD+

In light of our discussion in Sect. 2, a natural question that arises is whether any of the other candidate classical negations discussed therein (i.e. precisely those satisfying Liberal and Toggle) are definable in **BD+**. The answer turns out to be negative; i.e., none of $\neg^e$, $\neg^1$ and $\neg^2$ are definable in **BD+**. For the purpose of proving this result, we need the following lemma.

LEMMA 15. *Let $\varphi(p)$ be any formula in* **BD+** *whose only propositional variable is p. Then, in terms of the four valued semantics, there are only the following four cases when we assign the values* **b** *and* **n** *respectively to p:*

(i) *values of $\varphi(p)$ are both* **f***,*

(ii) *values of $\varphi(p)$ are both* **t***,*

(iii) *values of $\varphi(p)$ are* **b** *and* **n** *respectively,*

(iv) *values of $\varphi(p)$ are* **n** *and* **b** *respectively.*

PROOF. We proceed by induction on the complexity of $\varphi(p)$. For the base case, if $\varphi(p)$ is p or $\bot$, then it satisfies the condition (iii) or (i) respectively. For the induction step, we cover only two of the four cases, as the others are similar.

Case 1: let $\varphi(p)$ be of the form ${\sim}\psi(p)$. Then, by induction hypothesis, $\psi(p)$ satisfies one of the four conditions. And with the truth table for $\sim$ in mind,

$\varphi(p)$ satisfies (ii), (i), (iii) and (iv) when $\psi(p)$ satisfies (i), (ii), (iii) and (iv) respectively.

Case 2: let $\varphi(p)$ be of the form $\psi(p) \wedge \xi(p)$. Then, by induction hypothesis, $\psi(p)$ and $\xi(p)$ both satisfy one of the four conditions. And with the truth table for $\wedge$ in mind, $\varphi(p)$ behaves as follows:

$\psi(p) \wedge \xi(p)$	(i)	(ii)	(iii)	(iv)
(i)	(i)	(i)	(i)	(i)
(ii)	(i)	(ii)	(iii)	(iv)
(iii)	(i)	(iii)	(iii)	(i)
(iv)	(i)	(iv)	(i)	(iv)

This completes the proof. ■

Based on this lemma, we can prove the desired result as follows.

Theorem 4. *The negations $\neg^e, \neg^1$ and $\neg^2$ (of Sect. 2) are not definable in* **BD+***.*

Proof. Suppose that $\neg^e$ is definable in **BD+**. This implies that there is a formula whose values are **f** and **t** when we assign the values **b** and **n** to p respectively. But this contradicts the previous lemma. The proof for other two cases are similar. ■

Corollary 16. *The conditional $\supset$ is not definable in* **BD+***.*

Proof. If $\supset$ is definable in **BD+**, then it follows that $\neg^e A$ is definable in **BD+** by $A \supset (A \wedge \neg A)$. But this contradicts Theorem 4. ■

Corollary 17. **BD+** *is not functionally complete.*[16]

Proof. This follows immediately by Theorem 4. ■

The following proof-theoretic considerations may be of interest to some readers. Consider the systems obtained by adding, respectively, $\neg^e, \neg^1$ and $\neg^2$ to **BD**, which we refer to as **BDe**, **BD1** and **BD2**. For ease of exposition, we rewrite all negations and conditionals uniformly as $\neg$ and $\rightarrow$. These systems are obtained by modifying only axioms (Ax12) and (Ax16) of **BD+** as follows:

	(Ax12)	(Ax16)
BD+	$\sim\neg A \leftrightarrow \neg\sim A$	$\sim(A \rightarrow B) \leftrightarrow (\neg\sim A \wedge \sim B)$
BDe	$\sim\neg A \leftrightarrow A$	$\sim(A \rightarrow B) \leftrightarrow (A \wedge \sim B)$
BD1	$\sim\neg A \leftrightarrow (A \wedge \neg\sim A)$	$\sim(A \rightarrow B) \leftrightarrow ((A \wedge \neg\sim A) \wedge \sim B)$
BD2	$\sim\neg A \leftrightarrow (A \vee \neg\sim A)$	$\sim(A \rightarrow B) \leftrightarrow ((A \vee \neg\sim A) \wedge \sim B)$

[16] We will say that a logic is functionally complete when at least one of its corresponding or characteristic matrices is functionally complete.

The following observations are of interest. The exclusion negation $\neg^e$ is definable in both **BD1** and **BD2** by $\neg^1\neg^1\neg^1 A$ and $\neg^2\neg^2\neg^2 A$ respectively. Therefore, **BD1** and **BD2** are extensions of **BDe**. Moreover, this extension is strict in the sense that neither $\neg^1$ nor $\neg^2$ is definable in **BDe**. Finally, $\neg$ is definable in neither **BD1** and **BD2**.

REMARK 18. **BD1** and **BD2** do not occur in the literature as far as we know. However, **BDe** is not a new system in the sense that there are at least three systems in the literature that are equivalent to it. These include i.e. $\mathbf{B}_4^{\rightarrow}$ of Odintsov [18], **BD**△ of Sano and Omori [25], and **BS4** of Omori and Waragai [21]. Moreover, the subsystem of **BDe** in the language $\mathcal{L}_{\supset}$ obtained by eliminating (Ax10), (Ax11) and (Ax12) is called **HBe** by Avron [3]. Closely related to **HBe** is the subsystem obtained by replacing $\mathbf{CL}^+$ by $\mathbf{IL}^+$ and it is known as **N4**, the paraconsistent version of Nelson logic. It has been investigated by Kamide and Wansing [13], Odinstov [19] and others.

3.5. Relations Between BD+ and Other Systems

We now wish to draw some comparisons between **BD+** and some closely related systems. That these systems turn out so closely related may be surprising given the vast difference in motivation from which these systems arose. We remark on some further interesting curiosities along the way.

3.5.1. A Comparison of PM4N with BD+. In [5], Béziau considers a four-valued modal logic inspired by a four-valued modal logic **Ł** of Łukasiewicz. One problem faced by **Ł** was that it validated inferences such as $p \rightarrow q \models \Box p \rightarrow \Box q$ and $\Diamond p \wedge \Diamond q \models \Diamond(p \wedge q)$ which are quite counter-intuitive given a necessity and possibility reading of the modalities. According to Béziau, these counterintuitive inferences are avoided in a preferred system **PM4N**. However, **PM4N** can be viewed as an expansion of **BD** and, in fact, as nothing more than the expansion **BD+**. Thus the extensional many-valued logic **BD+** may be seen as a modal logic that avoids certain modal fallacies faced by its cousin **Ł**.

The language of **PM4N** consists of the set of logical symbols $\{\wedge, \vee, \neg, \Box\}$, where $\Box$ obeys the following truth table:

A	$\Box A$
t	**t**
b	**f**
n	**f**
f	**f**

Note here that the designated values are **t** and **b**.

PROPOSITION 19. *Both* $\sim$ *and* $\to$ *of* **BD+** *are definable in* **PM4N**, *e.g.* $A \to B$ *is definable by* $\neg A \vee B$, *and* $\sim A$ *by* $\neg A \leftrightarrow (\Box A \vee \Box \neg A)$.

PROPOSITION 20. $\Box A$ *is definable in* **BD+** *by* $A \wedge (A \leftrightarrow \neg\sim A)$.

Combining these results, we obtain the following.

THEOREM 5. **PM4N** *and* **BD+** *are equivalent.*

3.5.2. A comparison of FDEP with BD+. We could have presented **BD+** in the language with set of logical symbols $\{\sim, \to\}$, i.e. de Morgan negation and classical implication. In [30], Zaitsev presents a logic **FDEP** in the same language. As we show below, **FDEP** and **BD+** are equivalent theorem-wise. Since **BD+** can be viewed as an expansion of classical logic by de Morgan negation, and since **BD+** and **FDEP** are equivalent, **FDEP** can be viewed in the same light. Yet **FDEP** is got by *weakening* classical negation to the strictly weaker de Morgan negation which does not e.g. satisfy *ex falso quodlibet*, from $A \wedge \sim A \vdash B$.[17] That weakening the negation yields an expansion of classical logic can be seen by the fact the classical negation $\neg A$ of A is definable by $A \to \sim(A \to A)$. Some will find this result analogous to what Béziau's has called a "translation paradox" in [4]. Whether paradoxical or not, it is somewhat surprising that what is clearly an expansion of classical logic turns out equivalent to an apparent weakening of classical logic.

DEFINITION 21. The system **FDEP** consists of the following axioms together with rules of inference:

(A1)	$A \to (B \to A)$
(A2)	$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
(A3)	$((A \to B) \to A) \to A$
(A4)	$\sim(A \to A) \to B$
(A5)	$\sim\sim A \to A$
(A6)	$A \to \sim\sim A$
(A7)	$\sim(A \to \sim(B \to B)) \to (\sim A \to \sim(B \to B))$
(A8)	$(\sim A \to \sim(B \to B)) \to \sim(A \to \sim(B \to B))$
(R1)	If $\vdash A$ and $\vdash A \to B$ then $\vdash B$
(R2)	If $\vdash A \to B$ then $\vdash \sim B \to \sim A$

[17] The observation that **FDEP** extends classical logic was already made in [30].

We write $\Gamma \vdash_{\mathbf{FDEP}} A$ iff there is a finite subset Γ' of Γ such that $\vdash_{\mathbf{FDEP}} \bigwedge \Gamma' \to A$, where $\bigwedge \Gamma'$ is a conjunction of members of Γ'.

Remark 22. Consider the subsystem of **FDEP** which consists of axioms (A1)–(A4) together with (R1). In this subsystem we can define classical negation since $\sim(A \to A)$ is a bottom according to (A4); whence $\neg A := A \to \sim(A \to A)$ defines classical negation. One may check that $\neg$ and $\to$ satisfy the axioms for classical logic. For example, if we take the axiomatization of Mendelson (cf. [14, p. 35]), then it suffices to show that $(\neg A \to B) \to ((\neg A \to \neg B) \to A)$ is provable. We make use of the classicality of $\neg$ in the following proofs.

Our original presentation of **BD+** had (MP) as the only rule of inference, so a comparison to **FDEP** would be made easier if there were an equivalent presentation of **FDEP** without (R2). We give such a presentation shortly.

Definition 23. The system **FDEP′** consists of the following axioms in addition to (A1)–(A6), along with sole rule of inference (R1):

$$\text{(A7}'\text{)} \qquad \sim(A \to B) \to (\sim A \to \sim(B \to B))$$

$$\text{(A8}'\text{)} \qquad \sim(A \to B) \to \sim B$$

$$\text{(A9}'\text{)} \qquad (\sim A \to \sim(B \to B)) \to (\sim B \to \sim(A \to B))$$

We define $\vdash_{\mathbf{FDEP'}}$ in the same manner as $\vdash_{\mathbf{FDEP}}$.

Remark 24. The above system **FDEP′** is a natural variation of **BD+** in the following sense. Schemas (A5) and (A6) replace (Ax13) of **BD+**, and (A7)′, (A8)′ and (A9)′ replace (Ax16) of **BD+** since $\mathcal{L}_{\sim,\to}$ is without primitive conjunction.

The following lemmas connect the above two systems, proofs of which are provided in the appendices.

Lemma 25. *(A7) and (A8) of* **FDEP** *are provable in* **FDEP′**.

Lemma 26. *(A7)′, (A8)′ and (A9)′ of* **FDEP′** *are provable in* **FDEP**.

We then obtain the following result.

Lemma 27. **FDEP′** *and* **FDEP** *are equivalent.*

Proof. Lemma 26 proves that **FDEP′** is a sublogic of **FDEP**. For the other direction, on the basis of Lemma 25, what remains to be shown is that

(R2) is valid in **FDEP′**, and this follows by a result of Scroggs in [26, Theorem 3, p. 118] showing that provable formulas of **FDEP′** always take the value **t**. ■

Proposition 28. **FDEP′** *and* **BD+** *are equivalent.*

Proof. We need to show that (i) axioms (A1) through (A9′) of **FDEP′** are **BD+**-derivable and that (R1) is **BD+**-admissible, and (ii) axioms (Ax1) through (Ax16) of **BD+** are **FDEP′**-derivable and that (MP) is a derivable rule in **FDEP′**.

For (i), (A1) through (A3) are identical to (Ax1) through (Ax3), while (A5) and (A6) correspond in an obvious way to (Ax13), and (A7′) through (A9′) correspond to (Ax16). (A4) is **BD+**-derivable by substituting A for B in (Ax16), and then applying (Ax11). Finally, (R1) is **BD+**-admissible given (MP).

For (ii), we need to show that defining $A \wedge B$ and $A \vee B$ as $\neg(A \to \neg B)$ and $\neg A \to B$ respectively in **FDEP′** (recalling that $\neg A$ is defined by $A \to {\sim}(A \to A)$) yields the **FDEP′**-derivability of (Ax4) through (Ax16), and that (MP) is a derivable rule in **FDEP′**. For the latter, we have that $\Gamma \vdash_{\mathbf{FDEP'}} A \to ((A \to B) \to B)$ by (A1), (A2) and (R1). By the definition of $\vdash_{\mathbf{FDEP'}}$ it immediately follows that $\Gamma, A, A \to B \vdash_{\mathbf{FDEP'}} B$. For the former, it is straightforward by the fact that $\neg$ is classical negation. ■

Corollary 29. **FDEP** *and* **BD+** *are equivalent.*

Proof. The result follows immediately from Lemma 27 and Proposition 28. ■

4. Concluding Remarks

We provided a general semantic analysis of classical negation so that it is applicable in a non-classical setting. We hope this serves to settle debates concerning whether certain negations are classical or not, or whether certain negations are "genuine" or not, depending on whether e.g. they are contradictory-forming operators (in the sense of Sect. 2), a condition we take to be minimally necessary for any operator's being deemed a negation. (For an excellent updated survey on this topic, see [11].) Our main interest of this endeavour, however, was to expand four-valued Belnap-Dunn logic by classical negation.

We then presented a Hilbert-style axiom system for the system **BD+**, an expansion of Belnap-Dunn logic by classical negation, which we showed is complete and maximal. We went on to compare **BD+** to two related system in the literature that we find of particular interest, the modal logic of [4] and the apparent weakening of classical logic of [30]. We showed these three logics are equivalent despite their being obtained through disparate motivations. Indeed, there are a good number of interesting connections between **BD+** and other systems of the literature, a topic we shall have to leave for another occasion.

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Appendix

Outline of the Proof of Lemma 7

We proceed by induction on the number n of connectives. For the base case, if $n = 0$, then F is p_i so we need to prove $p^v \vdash p^v$, but this holds in **BD+**. For the induction step, assume the conclusion for the cases where the number of connectives is less than $k + 1$. We split the cases depending on the main connective, and here we will only deal with $\sim, \neg$ and $\rightarrow$.

Case 1. If $F = {\sim}G$, then by induction hypothesis, we have $\Delta^v \vdash G^v$. We split the cases further depending on the value of G.

$v(G)$	$\Delta^v \vdash G^v$	$v(F)(= v({\sim}G))$	$\Delta^v \vdash F^v$, i.e. $\Delta^v \vdash ({\sim}G)^v$
t	$\Delta^v \vdash G \wedge \neg{\sim}G$	**f**	$\Delta^v \vdash \neg{\sim}G \wedge {\sim}{\sim}G$
b	$\Delta^v \vdash G \wedge {\sim}G$	**b**	$\Delta^v \vdash {\sim}G \wedge {\sim}{\sim}G$
n	$\Delta^v \vdash \neg G \wedge \neg{\sim}G$	**n**	$\Delta^v \vdash \neg{\sim}G \wedge \neg{\sim}{\sim}G$
f	$\Delta^v \vdash \neg G \wedge {\sim}G$	**t**	$\Delta^v \vdash {\sim}G \wedge \neg{\sim}{\sim}G$

Then, in all four cases, $\Delta^v \vdash G^v$ implies $\Delta^v \vdash F^v$ by (Ax13). Therefore, since we have $\Delta^v \vdash G^v$ as induction hypothesis, we obtain $\Delta^v \vdash F^v$ as desired.

Case 2. If $F = \neg G$, then by induction hypothesis, we have $\Delta^v \vdash G^v$. We split the cases further depending on the value of G.

$v(G)$	$\Delta^v \vdash G^v$	$v(F)(= v(\neg G))$	$\Delta^v \vdash F^v$, i.e. $\Delta^v \vdash (\neg G)^v$
t	$\Delta^v \vdash G \wedge \neg{\sim}G$	**f**	$\Delta^v \vdash \neg\neg G \wedge {\sim}\neg G$
b	$\Delta^v \vdash G \wedge {\sim}G$	**n**	$\Delta^v \vdash \neg\neg G \wedge \neg{\sim}\neg G$
n	$\Delta^v \vdash \neg G \wedge \neg{\sim}G$	**b**	$\Delta^v \vdash \neg G \wedge {\sim}\neg G$
f	$\Delta^v \vdash \neg G \wedge {\sim}G$	**t**	$\Delta^v \vdash \neg G \wedge \neg{\sim}\neg G$

Then, $\Delta^v \vdash G^v$ implies $\Delta^v \vdash F^v$ by (T1), (T1) and (T2), (Ax12) and (T2) when $v(G)$ takes the value $\mathbf{t}, \mathbf{b}, \mathbf{n}$ and $\mathbf{f}$ respectively. Therefore, since we have $\Delta^v \vdash G^v$ as induction hypothesis, we obtain $\Delta^v \vdash F^v$ as desired.

Case 3. If $F = G \to H$, then by induction hypothesis, we have $\Delta^v \vdash G^v$ and $\Delta^v \vdash H^v$. We split the cases depending on the values of G and H.

Cases	$v(G)$	$v(H)$	G^v	H^v	$v(F)$	F^v
(i)	**f**	any	$\neg G \wedge {\sim}G$	—	**t**	$F \wedge \neg{\sim}F$
(ii)	any	**t**	—	$H \wedge \neg{\sim}H$	**t**	$F \wedge \neg{\sim}F$
(iii)	**b**	**b**	$G \wedge {\sim}G$	$H \wedge {\sim}H$	**t**	$F \wedge \neg{\sim}F$
(iv)	**n**	**n**	$\neg G \wedge \neg{\sim}G$	$\neg H \wedge \neg{\sim}H$	**t**	$F \wedge \neg{\sim}F$
(v)	**t** or **n**	**b**	$\neg{\sim}G$	$H \wedge {\sim}H$	**b**	$F \wedge {\sim}F$
(vi)	**n**	**f**	$\neg G \wedge \neg{\sim}G$	$\neg H \wedge {\sim}H$	**b**	$F \wedge {\sim}F$
(vii)	**t** or **b**	**n**	G	$\neg H \wedge \neg{\sim}H$	**n**	$\neg F \wedge \neg{\sim}F$
(viii)	**b**	**f**	$G \wedge {\sim}G$	$\neg H \wedge {\sim}H$	**n**	$\neg F \wedge \neg{\sim}F$
(ix)	**t**	**f**	$G \wedge \neg{\sim}G$	$\neg H \wedge {\sim}H$	**f**	$\neg F \wedge {\sim}F$

- For (i) and (ii), we get $\Delta^v \vdash (G{\to}H) \wedge ({\sim}G \vee \neg{\sim}H)$, and hence by (T3) $\Delta^v \vdash (G{\to}H) \wedge \neg{\sim}(G{\to}H)$, i.e. $\Delta^v \vdash F \wedge \neg{\sim}F$. Thus $\Delta^v \vdash F^v$ by $(*)$.
- For (iii) and (iv), we have $\Delta^v \vdash H \wedge {\sim}G$ and $\Delta^v \vdash \neg G \wedge \neg{\sim}H$ respectively which both imply $\Delta^v \vdash (G{\to}H) \wedge ({\sim}G \vee \neg{\sim}H)$, and by (T3), we get $\Delta^v \vdash (G{\to}H) \wedge \neg{\sim}(G{\to}H)$, i.e. $\Delta^v \vdash F \wedge \neg{\sim}F$. Thus $\Delta^v \vdash F^v$ by $(*)$.
- For (v) and (vi), we have $\Delta^v \vdash H \wedge (\neg{\sim}G \wedge {\sim}H)$ and $\Delta^v \vdash \neg G \wedge (\neg{\sim}G \wedge {\sim}H)$ respectively which both imply $\Delta^v \vdash (G \to H) \wedge {\sim}(G \to H)$, i.e. $\Delta^v \vdash F \wedge {\sim}F$, by (Ax16). Thus $\Delta^v \vdash F^v$ by $(*)$.
- For (vii) and (viii), we have $\Delta^v \vdash (G \wedge \neg H) \wedge \neg{\sim}H$ and $\Delta^v \vdash (G \wedge \neg H) \wedge {\sim}G$ respectively which both imply $\Delta^v \vdash \neg(G{\to}H) \wedge \neg{\sim}(G{\to}H)$, i.e. $\Delta^v \vdash \neg F \wedge \neg{\sim}F$, by (T4) and (T3). Thus $\Delta^v \vdash F^v$ by $(*)$.
- For (ix), we obtain $\Delta^v \vdash (G \wedge \neg H) \wedge (\neg{\sim}G \wedge {\sim}H)$ and hence by (T4) and (Ax16), we get $\Delta^v \vdash \neg(G{\to}H) \wedge {\sim}(G{\to}H)$, i.e. $\Delta^v \vdash \neg F \wedge {\sim}F$. Thus $\Delta^v \vdash F^v$ by $(*)$.

This completes the proof. ∎

Proofs of Lemma 25 and Lemma 26

We use the following formulas which are well-known theorems of $\mathbf{CL}^+$:

$$\begin{array}{rc} \text{(Suffixing)} & (A \to B) \to ((B \to C) \to (A \to C)) \\ \text{(Permutation)} & (A \to (B \to C)) \to (B \to (A \to C)) \\ \text{(Prefixing)} & (B \to C) \to ((A \to B) \to (A \to C)) \\ \text{(Identity)} & A \to A \end{array}$$

Moreover, we write formulas of the form $\sim(A \to A)$ as $\bot$ when it is useful.

Proof of Lemma 25. For (A7), we obtain $\sim(A{\to}\bot){\to}(\sim A{\to}\sim(\bot{\to}\bot))$ by (A7′). We also get $(\sim A{\to}\sim(\bot{\to}\bot)){\to}(\sim A{\to}\bot)$ by (A4) and (Prefixing). Thus, by applying (Suffixing), we get $\sim(A{\to}\bot){\to}(\sim A{\to}\bot)$. For (A8), we have $(\sim A{\to}\sim(\bot{\to}\bot)){\to}(\sim\bot{\to}\sim(A{\to}\bot))$ by (A9′). And by (A4) and (Prefixing), we have $(\sim A{\to}\bot){\to}(\sim A{\to}\sim(\bot{\to}\bot))$. Thus, by applying (Suffixing), (Permutation) to these, we obtain $\sim\bot{\to}((\sim A{\to}\bot){\to}\sim(A{\to}\bot))$. Moreover, we get $\sim\bot$ by (Identity) and (A6). Therefore, by (R1), we obtain $(\sim A{\to}\bot){\to}\sim(A{\to}\bot)$. This completes the proof. ■

Proof of Lemma 26. For (A7′), we obtain $(A \to \bot) \to (A \to B)$ by (A4) and (Prefixing). Then, by an application of gives us $\sim(A \to B) \to \sim(A \to \bot)$, and therefore, by (A7) we obtain $\sim(A \to B) \to (\sim A \to \bot)$. For (A8′), it is immediate by (A1) and (R2). Finally (A9′) is provable as follows.

1	$(\sim(\sim B \to \sim A) \to B) \to (\sim(\sim B \to \sim A) \to \bot)$	[(A7′), (A6), (A2)]
2	$\sim(\sim B \to \sim A) \to A$	[(A1), (R2), (A5)]
3	$(A \to B) \to (\sim(\sim B \to \sim A) \to B)$	[2, (Suffixing)]
4	$(A \to B) \to \sim((\sim B \to \sim A) \to \bot)$	[1, 3, (A8)]
5	$(\neg(\sim B \to \sim A)) \to \sim(A \to B)$	[4, (R2), (A6), def. of ¬]
6	$(\sim B \to \neg(\sim B \to \sim A)) \to (\sim B \to \sim(A \to B))$	[5, (Prefixing)]
7	$\sim B \to ((\sim B \to \sim A) \to \sim A)$	[(Identity), (Permutation)]
8	$\neg\sim A \to (\sim B \to \neg(\sim B \to \sim A))$	[7, (Suffixing), (Permutation)]
9	$\neg\sim A \to (\sim B \to \sim(A \to B))$	[6, 8, (Suffixing)]

This completes the proof. ■

References

[1] Arieli, O., A. Avron, and A. Zamansky, Ideal paraconsistent logics, *Studia Logica* 99:31–60, 2011.

[2] Arieli, O., A. Avron, and A. Zamansky, Maximal and premaximal paraconsistency in the framework of three-valued semantics, *Studia Logica* 97:31–60, 2011.

[3] Avron, A., Natural 3-valued logics–characterization and proof theory, *Journal of Symbolic Logic* 56:276–294, 1991.

[4] Béziau, J.-Y., Classical negation can be expressed by one of its halves, *Logic Journal of the IGPL* 7(2):145–151, 1999.

[5] Béziau, J.-Y., A new four-valued approach to modal logic, *Logique et Analyse* 54(213):109–121, 2011.

[6] Carnielli, W., J. Marcos, and S. de Amo, Formal inconsistency and evolutionary databases, *Logic and Logical Philosophy* 8:115–152, 2000.

[7] Copeland, B. J., What is a semantics for classical negation?, *Mind* 95(380):478–490, 1986.

[8] Da Costa, N. C. A., On the theory of inconsistent formal systems, *Notre Dame Journal of Formal Logic* 15:497–510, 1974.

[9] De, M., *Negation in context*, Ph.D. thesis, University of St Andrews, Scotland, 2011.

[10] Hanazawa, M., A characterization of axiom schema playing the rôle of tertium non datur in intuitionistic logic, *Proceedings of the Japan Academy* 42:1007–1010, 1966.

[11] Horn, L. R., and H. Wansing, Negation, *The Stanford Encyclopedia of Philosophy.* Forthcoming.

[12] Jaśkowski, S., A propositional calculus for inconsistent deductive systems, *Logic and Logical Philosophy* 7:35–56, 2000.

[13] Kamide, N., and H. Wansing, Proof theory of Nelson's paraconsistent logic: A uniform perspective, *Theoretical Computer Science* 415:1–38, 2012.

[14] Mendelson, E., *Introduction to Mathematical Logic*, 4 edn., Chapman and Hall/CRC, Boca Raton, 1997.

[15] Meyer, R. K., Proving semantical completeness 'relevantly' for **R**, *Australian National University Research School of Social Sciences Logic Group Research Paper*, 23 1985.

[16] Meyer, R. K., and R. Routley, Classical relevant logics I, *Studia Logica* 32(1):51–66, 1973.

[17] Meyer, R. K., and R. Routley, Classical relevant logics II, *Studia Logica* 33(2):183–194, 1974.

[18] Odintsov, S. P., The class of extensions of Nelson paraconsistent logic, *Studia Logica* 80:291–320, 2005.

[19] Odintsov, S. P., *Constructive Negations and Paraconsistency*, Springer-Verlag, Dordrecht, 2008.

[20] Omori, H., Remarks on naive set theory based on **LP**, *The Review of Symbolic Logic.* Forthcoming.

[21] Omori, H., and T. Waragai, Some observations on the systems **LFI1** and **LFI1***, in *Proceedings of Twenty-Second International Workshop on Database and Expert Systems Applications (DEXA2011)*, 2011, pp. 320–324.

[22] Priest, G., Can contradictions be true?, *Proceedings of the Aristotelian Society, Supplementary Volumes* 67:34–54, 1993.

[23] Priest, G., *Doubt Truth to be a Liar*, Oxford University Press, New York, 2006.

[24] Priest, G., *In Contradiction: A Study of the Transconsistent*, 2nd edn., Oxford University Press, Oxford, 2006.

[25] Sano, K., and H. Omori, An expansion of first-order Belnap–Dunn logic, *Logic Journal of the IGPL* 22(3):458–481, 2014.

[26] Scroggs, S. J., Extensions of the Lewis system S5, *The Journal of Symbolic Logic* 16(2):112–120, 1951.

[27] Sette, A., On the propositional calculus P^1, *Mathematica Japonicae* 16:173–180, 1973.

[28] Slater, B. H., Paraconsistent logics?, *Journal of Philosophical Logic* 24(4):451–454, 1995.

[29] Smiley, T., Can contradictions be true?, *Proceedings of the Aristotelian Society, Supplementary Volumes* 67:17–33, 1993.

[30] Zaitsev, D., Generalized relevant logic and models of reasoning, *Moscow State Lomonosov University doctoral (Doctor of Science) dissertation*, 2012.

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