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END NOTE

*Smallest graphs with distinct singleton centers**

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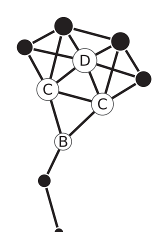
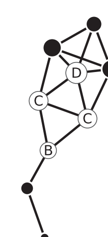
An incredible number of centrality indices has been proposed to date (Todeschini & Consonni, 2009). Four of them, however, can be considered prototypical because they operationalize distinct concepts of centrality and together cover the bulk of analyses and empirical uses: degree, closeness, betweenness, and eigenvector centrality.

In teaching any mathematical science, it is instructive to consider simple examples of extreme situations. While all of the four centrality indices mentioned yield the same ranking on star graphs, they can differ substantially on other graphs.

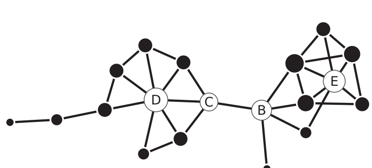
Krackhardt (1990) uses the example shown on the right, which he dubbed "kite structure," to explain differences between three of the indices. When Linton Freeman speculated in a post to SOCNET, the mailing list of the International Network for Social Network Analysis (INSNA), that Krackhardt's graph was the smallest graph of its kind, Martin Everett came up with a smaller kite. Borgatti (1999) provides the graph below as an example in which all four indices each have a single maximally central vertex, all of them distinct.



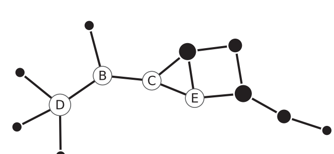
Krackhardt's kite ($n = 10, m = 18$)



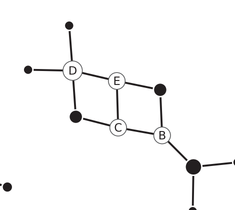
Everett's kite ($n = 9, m = 15$)



Borgatti ($n = 19, m = 32$)



Everett ($n = 13, m = 14$)



Potts ($n = 11, m = 12$)

Let us say that a (simple, connected, undirected) graph has the property $BCDE$, if each of the maximally central vertices under betweenness (B), closeness (C), degree (D), and eigenvector (E) centrality is unique. In other words, a $BCDE$ graph has distinct singleton centers with respect to the four canonical centrality indices. Note that Krackhardt's kite is not $BCDE$, because the degree and eigenvector center coincide and because the closeness center consists of two vertices; we hence refer to it as being BC_2D.

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On June 8, 1999, Everett followed up on Borgatti (1999) by posting also a smaller $BCDE$ graph with $n = 13$ vertices and $m = 14$ edges. He challenged the community to find an even smaller one, and two days later, Blyden Potts responded with an ($n = 10$, $m = 12$)-graph. To the best of our knowledge, no one has succeeded to date in finding a smaller example, but no one managed to prove its minimality, either.

Since there are less than 12 million non-isomorphic connected graphs with $n \leqslant 10$ vertices, we decided to resort to exhaustive search. Using the program nauty (McKay & Piperno, 2014), they can be listed in a few seconds on a standard laptop computer, and with code of our own we checked their centers for uniqueness and distinctness in a further few seconds. Figures 1–3 represent our findings.

It turns out that the smallest example depends on what we mean by small. The number of vertices n is usually referred to as the *order* of a graph, whereas the *size* is the sum $n + m$ where m denotes the number of edges.

There are seven order-minimal $BCDE$ graphs with $n = 10$ and $14 \leqslant m \leqslant 16$. Of the two graphs with $m = 14$ edges, we prefer the one shown in Figure 2(a) because it also has a singleton graph-theoretic center, i.e., the vertex with minimum eccentricity (maximum distance to any other vertex) is unique; it coincides with the vertex of maximum closeness, i.e., with minimum total distance. However, the size of these graphs is $10 + 14 = 24 > 23 = 11 + 12$ and thus larger than the size of Pott's graph.

So we also checked the few thousand non-isomorphic connected graphs with $n = 11$ and $10 \leqslant m < 12$ and found the one in Figure 2(b) to be the unique size-minimal $BCDE$ graph. Connected graphs with one more edge than necessary for connectedness, i.e., with exactly one cycle, are called *pseudotrees*, and we were curious to see how many vertices are needed for a tree to be $BCDE$. After testing about 13,000 non-isomorphic trees with at least 12 vertices, it turns out that the smallest $BCDE$ trees have $n = 15$ vertices, and there are seven of them all shown in Figure 3. Six of them have an induced path with all four singleton centers in the same order, while the other is the only *caterpillar* (a tree that is a path with pendant degree-one vertices).

Along the way we noted that the BCD graph of minimum order and size is unique. It has $n = 9$ vertices and even fewer edges than the smaller of the two BC_2D kites. In case you wondered, the smallest BCD trees have $n = 11$, and there are three of them.

Now that we know which are the smallest BCD and $BCDE$ graphs from enumeration, the obvious challenge is to provide analytic proofs.

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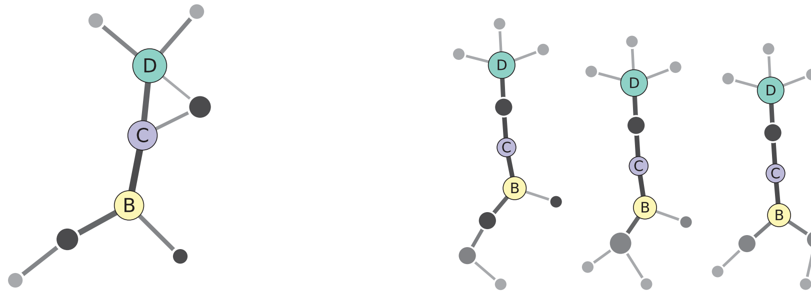
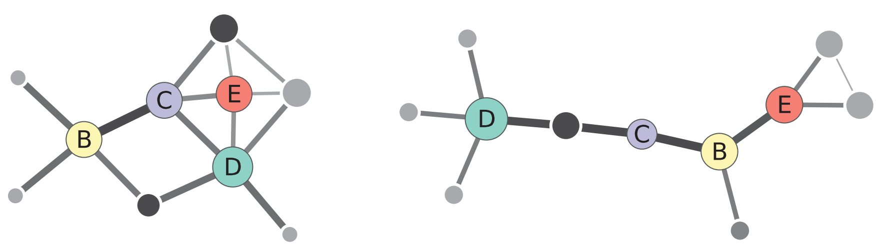


(a) minimum BCD graph ($n = 9$, $m = 9$)

(b) minimum BCD trees ($n = 11$, $m = 10$)

Fig. 1: The smallest BCD graph is unique, there are three minimum BCD trees (vertices: size–degree, brightness–eccentricity; edges: thickness/darkness–betweenness). (color online)

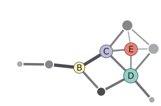


(a) minimum-order $BCDE$ graph ($n = 10$, $m = 14$)

(b) minimum-size $BCDE$ graph ($n = 11$, $m = 11$)

Fig. 2: The smallest $BCDE$ graphs. (color online)

Note that both graphs in Figure 2 have a singleton graph-theoretic center that coincides with C. While there is a second minimum-order $BCDE$ graph, it has three vertices of minimum eccentricity (incl. B, C).



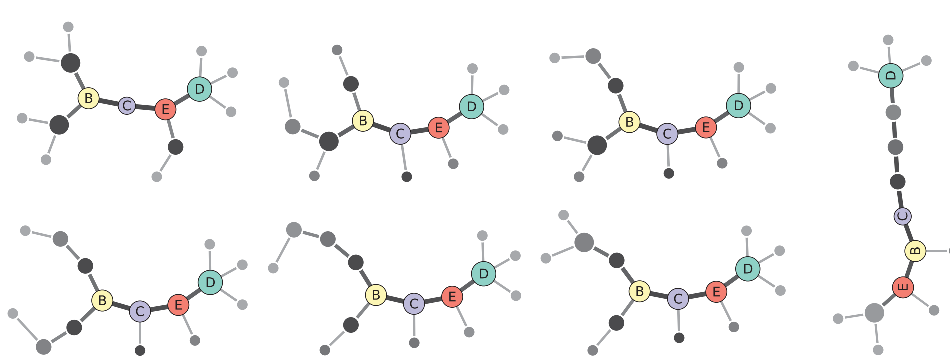


Fig. 3: The seven minimum $BCDE$ trees ($n = 15$, $m = 14$). (color online)