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Matrices Having A Positive Determinant And All Other Minors Nonpositive

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Abstract

The class of square matrices of order n having a positive determinant and all their minors up to order $n-1$ nonpositive is considered. A characterization of these matrices based on the Cauchon algorithm is presented which provides an easy test for their recognition. Furthermore, it is shown that all matrices lying between two matrices of this class with respect to the checkerboard ordering are contained in this class, too.

Keywords: Sign regular matrix, Totally nonnegative matrix, Totally nonpositive matrix, Interval property, Checkerboard ordering, Cauchon algorithm.
MSC: 15B48, 65G99

1. Introduction

A real matrix is called *sign regular* if all its nonzero minors of the same order have identical sign. Sign regular matrices have found a variety of applications, e.g., in computer aided geometric design [31] and computer vision [27, Section 3.3]. If the sign of all minors of any order is nonnegative (nonpositive) then the matrix is called *totally nonnegative* (*totally nonpositive*). Totally nonnegative matrices arise in a variety of ways in mathematics and its applications. For background information the reader is referred to the monographs [32], [16], [24],[15] .

In [7], the second and the third author have considered the sign regular matrices which have a negative determinant and all of their other minors nonnegative. In the present paper, we investigate the dual class, viz. the matrices having all their minors nonpositive with the exception of the determinant which is positive, termed below *t.n.p.*$^+$ matrices. It turns out that the analysis of this matrix class is more involved than the one of the other class. Theorem 17 of Chapter V in [16] ascertains the existence of such matrices for any order.

In this paper we apply the so-called Cauchon algorithm [21],[26] for the study of the *t.n.p.*$^+$ matrices. This algorithm, also called *deleting derivations algorithm* and *Cauchon reduction algorithm*, was developed by Gérard Cauchon in [10] while studying quantum matrices. It turned out to be an important tool in the investigation of the connection between torus-invariant prime ideals, torus orbits of symplectic leaves, and cells of totally nonnegative matrices, see, e.g., [21]. By the Cauchon algorithm, we derive necessary and sufficient conditions for a matrix to be *t.n.p.*$^+$ which provide an easy test for a given n-by-n matrix requiring $O(n^3)$ arithmetic operations [5, Section 3.2]. Furthermore, employing the necessary and sufficient conditions we show that the *t.n.p.*$^+$ matrices possess the so-called *interval property*. To explain this property, consider the checkerboard ordering which is obtained from the usual entry-wise ordering in the set of the square real matrices of fixed order by reversing the inequality sign for each entry in a checkerboard fashion. Then all matrices lying with respect to this ordering between two *t.n.p.*$^+$ matrices with a negative entry in their bottom right position are *t.n.p.*$^+$, too. The motivation for considering such an interval property stems, e.g., from the investigation of the linear complementarity problem [14]. Often properties of this problem like solvability, uniqueness, convexity, and finite number of solutions are reflected by properties of the constraint matrix; for a large collection of respective matrix classes see [13]. In the case that one considers the linear complementarity problem with uncertain data modeled by intervals [8],[28], it is important to know whether the matrices obtained by choosing all possible values in the intervals are in the same matrix class. Then it is an enormous advantage if one could ascertain this containment by checking a finite set of matrices [22] - in the ideal case, from only two matrices. Collections of matrix classes which possess the interval property can be found in the survey articles [19], [18].
The class of the *t.n.p.*$^+$ matrices forms a subclass of the almost N_0-matrices [29] (also termed weak almost N-matrices [30]. A real matrix is called an almost N_0-matrix if all its proper principal minors are nonpositive and its determinant is positive. Such matrices appear in linear complementarity the-

ory [29] and in the theory of the global univalence of C^1 functions in $\mathbb{R}^n$ [30, Section 4]. To recognize whether a given matrix is in this class requires much more effort than the test for being $t.n.p.^+$.

The organization of our paper is as follows. In Section 2, we introduce our notation and give some auxiliary results which we use in the subsequent sections. In Section 3, we recall from [21] the Cauchon algorithm on which our results heavily rely. In Section 4, we firstly extend some results presented in [6] for the nonsingular totally nonpositive matrices to cover also the singular case. Then we apply the Cauchon algorithm to derive necessary and sufficient conditions for a matrix to be $t.n.p.^+$. Finally, we prove the interval property for $t.n.p.^+$ matrices and related classes of sign regular matrices.

2. Notation and auxilary results

2.1. Notation

We now introduce the notation used in our paper. For k, n, we denote by $Q_{k,n}$ the set of all strictly increasing sequences of k integers chosen from $\{1, 2, \ldots, n\}$. If $\alpha \in Q_{k,n}$, $\alpha = (\alpha_1, \ldots, \alpha_k)$, then $\alpha_{\hat{\kappa}}$ denotes the sequence α without its κ^{th} member. The *dispersion* of α, denoted by $d(\alpha)$, is defined to be $d(\alpha) = \alpha_k - \alpha_1 - (k-1)$; it represents a measure for the gaps in the sequence α. If $d(\alpha) = 0$, i.e., α is formed from consecutive integers, α is called *contiguous*. We use the set theoretic symbols $\cup$ and $\setminus$ to denote somewhat not precisely but intuitively the union and difference of two index sequences, where we consider the resulting sequence as strictly increasing ordered. For $\alpha, \beta \in Q_{k,n}$, we say that α is greater than or equal to β with respect to the *lexicographical order* [*colexicographical order*] denoted by $\beta \le \alpha$ [$\beta \le_c \alpha$], if the first non-zero entry in the sequence $(\alpha_1 - \beta_1, \alpha_2 - \beta_2, \ldots, \alpha_k - \beta_k)$ $\big[(\alpha_k - \beta_k, \alpha_2 - \beta_2, \ldots, \alpha_1 - \beta_1)\big]$ is positive or $\alpha = \beta$. We use the strict inequality sign if we exclude the equality.

Let A be a real $n \times m$ matrix. For $\alpha = (\alpha_1, \ldots, \alpha_k) \in Q_{k,n}$ and $\beta = (\beta_1, \ldots, \beta_l) \in Q_{l,m}$, we denote by $A[\alpha|\beta]$ the $k \times l$ submatrix of A contained in the rows indexed by $\alpha_1, \ldots, \alpha_k$ and columns indexed by $\beta_1, \ldots, \beta_l$. If $\alpha = \beta$, we denote the principal submatrix of A by $A[\alpha]$. We suppress the brackets when we enumerate the indices explicitly. If $d(\alpha) = d(\beta) = 0$, we call the submatrix $A[\alpha|\beta]$ as well as if $k = l$, its determinant *contiguous*. For any contiguous k-by-k submatrix $A[\alpha|\beta]$ of A, we call the submatrix

$$A[\alpha_1, \ldots, \alpha_k, \alpha_k + 1, \ldots, n|1, \ldots, \beta_1 - 1, \beta_1, \ldots, \beta_k]$$

of A having $A[\alpha|\beta]$ in its upper right corner the *left shadow* of $A[\alpha|\beta]$, and, analogously, we call the submatrix

$$A[1,\ldots,\alpha_1-1,\alpha_1,\ldots,\alpha_k|\beta_1,\ldots,\beta_k,\beta_k+1,\ldots,n]$$

having $A[\alpha|\beta]$ in its lower left corner the *right shadow* of $A[\alpha|\beta]$.
Let $\epsilon=(\epsilon_1,\ldots,\epsilon_n)$ be a signature sequence, i.e., $\epsilon\in\{1,-1\}^n$. The matrix A is called *sign regular* (abbreviated *SR*) *with signature* ϵ if $0\le\epsilon_k\det A[\alpha|\beta]$, for all $\alpha,\beta\in Q_{k,n},\ k=1,2,\ldots,n$. If A is *SR* with signature $\epsilon=(1,1,\ldots,1)$, then A is called *totally nonnegative* (abbreviated *TN*); if $\epsilon=(-1,-1,\ldots,-1)$, then A is termed *totally nonpositive* (abbreviated *t.n.p.*). If A is nonsingular and *SR* with signature $\epsilon=(-1,-1,\ldots,+1)$, then we denote this by *t.n.p.*$^+$. If A is in a certain class of *SR* matrices and in addition also nonsingular then we affix *Ns* to the name of the class. By E_{ij} we denote the matrix having a 1 in position (i,j) and all other entries zero. We reserve throughout the notation $T=(t_{ij})$ for the *backward identity* matrix with $t_{ij}:=\delta_{n+1-i,j}$, $i,j=1,\ldots,n$.

We endow $\mathbb{R}^{n,n}$, the set of real $n\times n$ matrices, with two partial orderings: Firstly, with the usual entry-wise ordering $\big(A=(a_{ij}),\ B=(b_{ij})\in\mathbb{R}^{n,n}\big)$

$$A\le B:\Leftrightarrow a_{ij}\le b_{ij}, i,j=1,\ldots,n.$$

The strict inequality $A<B$ is also understood entry-wise.
Secondly, with the *checkerboard ordering*, which is defined as follows. Let $S:=\operatorname{diag}\big(1,-1,\ldots,(-1)^{n+1}\big)$ and $A^*:=SAS, B^*:=SBS$. Then we define

$$A\le^* B:\Leftrightarrow A^*\le B^*.$$

2.2. Auxiliary results

The first lemma is a useful special case of Sylvester's Determinant Identity.

Lemma 1. *E.g.,[15, p.29] Partition* $A\in\mathbb{R}^{n,n}, n\ge 3$, *as follows:*

$$A=\begin{bmatrix} c & A_{12} & d\\ A_{21} & A_{22} & A_{23}\\ e & A_{32} & f\end{bmatrix},$$

where $A_{22} \in \mathbb{R}^{n-2,n-2}$ and c, d, e, f are scalars. Define the submatrices

$$C := \begin{bmatrix} c & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, D := \begin{bmatrix} A_{12} & d \\ A_{22} & A_{23} \end{bmatrix},$$
$$E := \begin{bmatrix} A_{21} & A_{22} \\ e & A_{32} \end{bmatrix}, F := \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & f \end{bmatrix}.$$

Then

$$\det A_{22} \det A = \det C \det F - \det D \det E.$$

The previous lemma is the key of deducing the following lemma.

Proposition 1. *[1, Lemma 1.7], Let $A \in \mathbb{R}^{n,m}, \alpha = (\alpha_1, \ldots, \alpha_k) \in Q_{k,n}$, and $\beta = (\beta_1, \ldots, \beta_{k-1}) \in Q_{k-1,m-1}$ with $0 < d(\beta)$. Then for all η such that $\beta_{k-1} < \eta \leq m, \kappa \in \{1, \ldots, k\}, s \in \{1, \ldots, h\}$, and $\beta_h < t < \beta_{h+1}$ for some $h \in \{1, \ldots, k-2\}$ or $\beta_{k-1} < t < \eta$ the following determinantal identity holds:*

$$\det A \left[\alpha_{\hat{\kappa}} \mid \beta_{\hat{s}} \cup \{t\}\right] \det A \left[\alpha \mid \beta \cup \{\eta\}\right] = \det A \left[\alpha_{\hat{\kappa}} \mid \beta_{\hat{s}} \cup \{\eta\}\right] \det A \left[\alpha \mid \beta \cup \{t\}\right] + \det A \left[\alpha_{\hat{\kappa}} \mid \beta\right] \det A \left[\alpha \mid \beta_{\hat{s}} \cup \{t, \eta\}\right].$$

3. The Cauchon algorithm and sign regular matrices

In this section we first recall from [21], [26] the Cauchon algorithm. In the second part we present properties of some class of sign regular matrices mainly based on the performance of the Cauchon algorithm.

3.1. The Cauchon algorithm

Definition 1. An $n \times m$ Cauchon diagram C is an $n \times m$ grid consisting of $n \cdot m$ squares colored black and white, where each black square has the property that either every square to its left (in the same row) or every square above it (in the same column) is black.

We denote by $C_{n,m}$ the set of the $n \times m$ Cauchon diagrams. We fix positions in a Cauchon diagram in the following way: For $C \in C_{n,m}$ and $i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$, $(i, j) \in C$ if the square in row i and column j is black. Here we use the usual matrix notation for the (i, j) position in a Cauchon diagram, i.e., the square in $(1, 1)$ position of the Cauchon diagram is in its top left corner.

Definition 2. Let $A \in R^{n,m}$ and let $C \in C_{n,m}$. We say that A is a Cauchon matrix associated with the Cauchon diagram C if for all $(i,j), i \in \{1,\ldots,n\}, j \in \{1,\ldots,m\}$, we have $a_{ij} = 0$ if and only if $(i,j) \in C$. If A is a Cauchon matrix associated with an unspecified Cauchon diagram, we just say that A is a *Cauchon matrix*.

Define the set $E^\circ := \{1,\ldots,n\} \times \{1,\ldots,m\} \setminus \{(1,1)\}, E := E^\circ \cup \{(n+1,2)\}$.
Let $(s,t) \in E^\circ$. Then $(s,t)^+ := \min\{(i,j) \in E \mid (s,t) \leq (i,j), (s,t) \neq (i,j)\}$; here the minimum is taken with respect to the lexicographical order.

Algorithm 1. *(Cauchon algorithm) [21, Algorithm 3.2] Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$. As r runs in decreasing order over the set E with respect to the lexicographical order, we define matrices*
$A^{(r)} = \left(a_{ij}^{(r)}\right) \in \mathbb{R}^{n,m}$ *as follows:*

1. Set $A^{(n+1,2)} := A$.
2. For $r = (s,t) \in E^\circ$, define the matrix $A^{(r)} = \left(a_{ij}^{(r)}\right)$ as follows:
(a) If $a_{st}^{(r^+)} = 0$, then put $A^{(r)} := A^{(r^+)}$.
(b) If $a_{st}^{(r^+)} \neq 0$, then put

$$a_{ij}^{(r)} := \begin{cases} a_{ij}^{(r^+)} - \frac{a_{it}^{(r^+)} a_{sj}^{(r^+)}}{a_{st}^{(r^+)}}, & \text{for } i < s \text{ and } j < t, \\ a_{ij}^{(r^+)}, & \text{otherwise.} \end{cases}$$

3. Set $\tilde{A} := A^{(1,2)}$; $\tilde{A}$ is called the matrix obtained from A (by the Cauchon algorithm).

Remark 1. *For practical computations, we advise to use instead of Algorithm 1 its condensed form [5, Section 3.2]. Then the number of arithmetic operations is reduced from $O(n^4)$ to $O(n^3)$.*

We recall from [26] the definition of a lacunary sequence associated with a Cauchon diagram.

Definition 3. Let $C \in \mathcal{C}_{n,m}$. We say that a sequence

$$\gamma := \left((i_k, j_k), \quad k = 0, 1, \ldots, t\right) \tag{1}$$

which is strictly increasing in both arguments is a lacunary sequence with respect to C if the following conditions hold:
(i) $(i_k, j_k) \notin C, k = 1, \dots, t$;
(ii) $(i, j) \in C$ for $i_t < i \leq n$ and $j_t < j \leq m$.
(iii) Let $s \in \{1, \dots, t-1\}$. Then $(i, j) \in C$ if
(a) either for all $(i, j), i_s < i < i_{s+1}$ and $j_s < j$.
or for all $(i, j), i_s < i < i_{s+1}$ and $j_0 \leq j < j_{s+1}$
and
(b) either for all $(i, j), i_s < i$ and $j_s < j < j_{s+1}$
or for all $(i, j), i < i_{s+1}$, and $j_s < j < j_{s+1}$.
We call t the length of γ.

The following procedure is beneficial to construct lacunary sequences.

Procedure 1. *[2] Let $A \in \mathbb{R}^{n,m}$ be a Cauchon matrix. Construct the lacunary sequence*

$$\gamma = \Big(\left(i_p, j_p\right), \dots, \left(i_0, j_0\right) \Big),$$

as follows: Put $(i_{-1}, j_{-1}) := (n+1, m+1)$. For $k = 0, 1, \dots, p$, define

$$M_k := \left\{ (i, j) \mid 1 \leq i < i_{k-1}, 1 \leq j < j_{k-1}, a_{ij} \neq 0 \right\}.$$

If $M_k = \phi$, put $p := k-1$.
Otherwise, put $(i_k, j_k) := \max M_k$, where the maximum is taken with respect to the lexicographical order.

Procedure 1 is useful to determine the rank of a given matrix A from $\tilde{A}$, provided that $\tilde{A}$ is a Cauchon matrix.

Theorem 3.1. *[2, Theorem 3.4] Let $A \in \mathbb{R}^{n,m}$ be such that $\tilde{A}$ is a Cauchon matrix. Then $\operatorname{rank} A = p+1$, where p is the length of the sequence which is obtained by application of Procedure 1 to $\tilde{A}$.*

The following theorem tells how to find the value of some minors of A by using lacunary sequences with respect to $\tilde{A}$, provided that $\tilde{A}$ is Cauchon matrix.

Theorem 3.2. *[2, Corollary 3.3] Let $A \in \mathbb{R}^{n,m}$ be such that $\tilde{A} = (\tilde{a}_{ij})$ is a Cauchon matrix and let $\gamma = \big((i_k, j_k),\ k = 0, 1, \dots, p \big)$ be a lacunary sequence. Then the following representation holds:*

$$\det A \left[i_0, \dots, i_p \mid j_0, \dots, j_p \right] = \tilde{a}_{i_0, j_0} \tilde{a}_{i_1, j_1} \cdot \ldots \cdot \tilde{a}_{i_p, j_p}.$$

Corollary 1. *[6] Let* $A \in \mathbb{R}^{n,n}$ *and assume that* $\tilde{A} = (\tilde{a}_{ij})$ *is a Cauchon matrix with* $\tilde{a}_{ii} \neq 0$, $i = 1, \ldots, n$. *Then the following equality holds*

$$\det A = \tilde{a}_{11} \cdot \ldots \cdot \tilde{a}_{nn}.$$

3.2. Sign regular matrices

Theorem 3.3. *E,g.,[9, Theorem 2.1] Let* $A \in \mathbb{R}^{n,m}$ *be of rank* r *and* ϵ *be a signature sequence. If* $0 \leq \epsilon_k \det A[\alpha \mid \beta]$ *for all* $\alpha, \beta \in Q_{k,n'}$, *where* $n' = \min\{n, m\}$, *is valid whenever* $d(\alpha) \leq n - r$, *then* A *is* SR *with signature* ϵ.

Lemma 2. *[9, Theorem 3.1] If* $A, B \in \mathbb{R}^{n,n}$ *are* SR *with signatures* $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ *and* $\delta = (\delta_1, \ldots, \delta_n)$*,respectively, then* AB *is* SR *with signature* $(\epsilon_1\delta_1, \ldots, \epsilon_n\delta_n)$.

Totally nonnegative matrices

Lemma 3. *[15, Lemma 2.3] Let* $A \in \mathbb{R}^{n,m}$ *be* TN. *Then* A *is a Cauchon matix.*

Theorem 3.4. *[4, Theorem 3.3],[26, Theorem 2.6] The matrix* A *is* TN *if and only if* $\tilde{A}$ *is an entry-wise nonnegative Cauchon matrix.* A *is in addition nonsingular if and only if all diagonal entries of* $\tilde{A}$ *are positive.*

Totally nonpositive matrices

The following lemma was originally presented for TN matrices as Proposition 1.15 in [32]. It was shown in [1, Proposition 2.5] by a similar proof that it is also valid for $t.n.p.$ matrices.

Lemma 4. *Let* $A \in \mathbb{R}^{n,m}$ *be t.n.p. and let* $\alpha = (i+1, \ldots, i+r), \beta = (j+1, \ldots, j+r)$ *for some* $i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$, *and* $2 \leq r < \min\{n, m\} - 1$. *If* $A[\alpha \mid \beta]$ *has rank* $r - 1$, *then*

(i) either the rows $i+1, \ldots, i+r$ *or the columns* $j+1, \ldots, j+r$ *of* A *are linearly dependent,*

or

(ii) the right or left shadow of $A[i+1, \ldots, i+r \mid j+1, \ldots, j+r]$ *has rank* $r - 1$.

The following theorem presents equivalent statements for a nonsingular matrix to be *t.n.p.* using a fixed number of minors.

Theorem 3.5. *[23] Let* $A = \big(a_{ij}\big) \in \mathbb{R}^{n,n}$ *with* $2 \leq n$ *be nonsingular. Then the following three statements are equivalent:*
(i) A *is t.n.p.*
(ii) For any $k \in \{1, \ldots, n-1\}$,

$$\begin{gathered} a_{11} \leq 0, \quad a_{nn} \leq 0, \quad a_{n1} < 0, \quad a_{1n} < 0, \\ \det A[\alpha \mid k+1, \ldots, n] \leq 0, \text{ for all } \alpha \in Q_{n-k,n}, \\ \det A[k+1, \ldots, n \mid \beta] \leq 0, \text{ for all } \beta \in Q_{n-k,n}, \\ \det A[k, \ldots, n] < 0. \end{gathered}$$

(iii) For any $k \in \{1, \ldots, n-1\}$,

$$\begin{gathered} a_{11} \leq 0, \quad a_{nn} \leq 0, \quad a_{n1} < 0, \quad a_{1n} < 0, \\ \det A[\alpha \mid 1, \ldots, k] \leq 0, \text{ for all } \alpha \in Q_{k,n}, \\ \det A[1, \ldots, k \mid \beta] \leq 0, \text{ for all } \beta \in Q_{k,n}, \\ \det A[1, \ldots, k+1] < 0. \end{gathered}$$

The following theorem tells that the entries of $\tilde{A}$ can be represented as ratios of contiguous minors if A is $Ns.t.n.p.$

Theorem 3.6. *[6] Let* $A = \big(a_{ij}\big) \in \mathbb{R}^{n,n}$ *be Ns.t.n.p. with* $a_{nn} < 0$. *Then the entries* $\tilde{a}_{kj}$ *of the matrix* $\tilde{A}$ *can be represented as* $(k, j = 1, \ldots, n)$

$$\tilde{a}_{kj} = \frac{\det A[k, \ldots, k+p \mid j, \ldots, j+p]}{\det A[k+1, \ldots, i+p \mid j+1, \ldots, j+p]},$$

with a suitable $0 \leq p \leq n-k$, *if* $j \leq k$ *and* $0 \leq p \leq n-j$, *if* $k < j$.

Theorem 3.7. *[6] Let* $A = \big(a_{ij}\big) \in \mathbb{R}^{n,n}$ *have all its entries negative except possibly* $a_{11} \leq 0$. *Then the following two properties are equivalent:*
(i) A *is a Ns.t.n.p. matrix.*
(ii) $\tilde{A}$ *is a Cauchon matrix and* $\tilde{A}[1, \ldots, n-1]$ *is a nonnegative matrix with positive diagonal entries.*

4. Main results

In this section, we present our results on the characterization of a special class of sign regular matrices by application of the Cauchon algorithm. In the first subsection, we will extend some results on nonsingular totally nonpositive matrices to the rectangular case. In the second subsection, we will introduce some characterizations and necessary and sufficent conditions for a given square matrix to have a positive determinant and all other minors nonpositive, i.e., to be *t.n.p.*$^+$. We conclude this paper by Subsection 4.3 wherein we show that the so-called interval property holds for the *t.n.p.*$^+$ matrices.

4.1. Totally nonpositve matrices and the Cauchon algorithm

Firstly we show that a *t.n.p.* matrix with a negative entry in its bottom right position either has all its entries in the last row and column negative or it contains a zero row or a zero column.

Proposition 2. *Let* $A = \big(a_{ij}\big) \in \mathbb{R}^{n,m}$ *be a t.n.p. matrix with* $a_{nm} < 0$. *Then the following implications hold.*

(i) If there exist $i \in \{1, \ldots, n-1\}$ *with* $a_{im} = 0$, *then* $a_{ij} = 0$ *for all* $j \in \{1, \ldots, m-1\}$.

(ii) If there exist $j \in \{1, \ldots, m-1\}$ *with* $a_{nj} = 0$, *then* $a_{ij} = 0$ *for all* $i \in \{1, \ldots, n-1\}$.

Proof. Firstly, assume $a_{im} = 0$ for some $i \in \{1, \ldots, n-1\}$, Then it follows from the total nonpositivity of A that for any $j \in \{1, \ldots, m-1\}$

$$\begin{aligned} 0 \geq \det A[i,n|j,m] &= a_{ij}a_{nm} - a_{im}a_{nj} \\ &= a_{ij}a_{nm} \geq 0. \end{aligned}$$

Because $a_{nm} < 0$, we conclude that $a_{ij} = 0$ for all $j = 1, \ldots m-1$, i.e., A has a zero row. The proof for (ii) is similar. □

In the following theorem, we are making extensive use of the following proposition on the relationship between the minors of the intermediate matrices of the Cauchon algorithm. The statements are composed of Propositions 3.7 and 3.11 and Lemma B.3 in [21].

Proposition 3. *[21] Let* $A = \big(a_{ij}\big) \in \mathbb{R}^{n,m}$ *and* $r = (s,t) \in E^\circ$.

(i) Let $a_{st} \neq 0$, $\alpha = (\alpha_1, \ldots, \alpha_l) \in Q_{l,n}$, *and* $\beta = (\beta_1, \ldots, \beta_l) \in Q_{l,m}$, *where* $l \leq \min\{n,m\}$ *and* $(\alpha_l, \beta_l) = r$. *Then*

$$\det A^{(r^+)}[\alpha \mid \beta] = \det A^{(r)}[\alpha_{\hat{s}} \mid \beta_{\hat{t}}] \cdot a_{st}.$$

(ii) Let $(\alpha_l, \beta_l) < r$. *If* $a_{st} = 0$, *or if* $\alpha_l = s$, *or if* $t \in \{\beta_1, \ldots, \beta_l\}$, *or if* $t < \beta_1$, *then*

$$\det A^{(r^+)}[\alpha \mid \beta] = \det A^{(r)}[\alpha \mid \beta].$$

(iii) Assume that $a_{st} \neq 0$ *and* $\alpha_l < s$ *while* $\beta_h < t < \beta_{h+1}$ *for some* $h \in \{1, \ldots, l\}$ *(by convention,* $\beta_{l+1} := m+1$*). Then*

$$\det A^{(r^+)}[\alpha|\beta] = \det A^{(r)}[\alpha|\beta] + \frac{1}{a_{st}} \sum_{k=1}^{h} (-1)^{k+h} \det A^{(r)}[\alpha|\beta_{\hat{k}} \cup \{t\}] a^{(r)}_{s,\beta_k}.$$

In the following theorem, we investigate the entries of the intermediate matrices that result by the application of the Cauchon algorithm to a given *t.n.p.* matrix. We proceed similarly to the case of a *Ns.t.n.p.* matrix which was considered in [6, Theorem 4.4].

Theorem 4.1. *Let* $A = (a_{ij}) \in \mathbb{R}^{n,m}$ *be t.n.p. with all the entries in its last row and column negative. If we apply the Cauchon algorithm to A, then the following properties hold.*

(i) All entries of $A^{(n,t)}[1, \ldots, n-1|1, \ldots, m-1]$ *are nonnegative for* $t = 2, \ldots, m$.

(ii) $A^{(n,t)}[1, \ldots, n-1|1, \ldots, t-1]$ *is TN for* $t = 2, \ldots, m$.

(iii) $A^{(n,t)}[1, \ldots, n-1|1, \ldots, m-1]$ *is TN for* $t = 2, \ldots, m$.

(iv) $A^{(n,2)}$ *is a Cauchon matrix.*

(v) For $t = 2, \ldots, m$, $\det A^{(n,t)}[\alpha|\beta] \leq 0$ *for all* $\alpha \in Q_{l,n-1}$, $\beta = (\beta_1, \beta_2, \ldots, \beta_l) \in Q_{l,m}$ *with* $\beta_l = m$ *and* $l = 1, \ldots, \min\{n-1, m\}$.

Proof. (i) If $t = m$, then set $r = (n,m) \in E^\circ$. By Proposition 3 (i), we have

$$\det A[i,n|j,m] = \det A^{(r^+)}[i,n|j,m] = \det A^{(r)}[i|j] \cdot a_{nm} = a^{(r)}_{ij} \cdot a_{nm},$$

for $i \in \{1, \ldots, n-1\}$, $j \in \{1, \ldots, m-1\}$. Since A is *t.n.p.*, whence $\det A[i,n|j,m] \leq 0$, and $a_{nm} < 0$, it follows that

$$a_{ij}^{(r)} \geq 0 \text{ for } i = 1, \ldots, n-1 \text{ and } j = 1, \ldots, m-1.$$

For the other cases, let $r = (n,t)$. Since the last row index in the underlying submatrices of the following minors equals n, we apply Proposition 3 (ii) to conclude that

$$\det A^{(r^+)}[i,n|j,t] = \cdots = \det A^{(n,m)}[i,n|j,t] = \det A[i,n|j,t] \text{ for } t \leq m-1. \quad (2)$$

By Proposition 3 (i), we obtain

$$\begin{aligned} \det A[i,n|j,t] = \det A^{(r^+)}[i,n|j,t] &= \det A^{(r)}[i|j] \cdot a_{nt} \\ &= a_{ij}^{(r)} \cdot a_{nt}. \end{aligned}$$

Since the left-hand side is nonpositive by the total nonpositivity of A, we conclude that $a_{ij}^{(r)} \geq 0$.

(ii) Let $r = (n,t)$, $t \in \{2, \ldots, m\}$, and $\alpha \in Q_{i,n-1}$, $\beta \in Q_{i,t-1}$. Since the last row index in the underlying submatrices of the following minors equals n, we apply Proposition 3 (ii) to obtain

$$\begin{aligned} \det A^{(r^+)}[\alpha_1, \ldots, \alpha_i, n|\beta_1, \ldots, \beta_i, t] &= \\ &\vdots \\ &= \det A^{(n,m)}[\alpha_1, \ldots, \alpha_i, n|\beta_1, \ldots, \beta_i, t] \\ &= \det A[\alpha_1, \ldots, \alpha_i, n|\beta_1, \ldots, \beta_i, t] \\ &\leq 0. \qquad (3) \end{aligned}$$

and by Proposition 3 (i) with $r = (n,t)$, we get

$$\det A^{(r^+)}[\alpha_1, \ldots, \alpha_i, n|\beta_1, \ldots, \beta_i, t] = \det A^{(r)}[\alpha_1, \ldots, \alpha_i|\beta_1, \ldots, \beta_i] \cdot a_{nt}.$$

Since $a_{nt} < 0$, we conclude by (3) that

$$\det A^{(n,t)}[\alpha_1, \ldots, \alpha_i|\beta_1, \ldots, \beta_i] \geq 0.$$

Hence $A^{(n,t)}[1, \ldots, n-1|1, \ldots, t-1]$ is TN, for all $t = 2, \ldots, m-1$.

(iii) We will prove this statement by decreasing primary induction on the step number t and secondary induction on the order l of the minors. For $t = m$, $A^{(n,m)}[1, \ldots, n-1|1, \ldots, m-1]$ is TN by (ii).

Suppose that $A^{(n,t+1)}[1, \ldots, n-1|1, \ldots, m-1]$ is TN. We want to show that $A^{(n,t)}[1, \ldots, n-1|1, \ldots, m-1]$ is TN, i.e.,

$$0 \leq \det A^{(n,t)}[\alpha|\beta] \text{ for all } \alpha \in Q_{l,n-1},\ \beta \in Q_{l,m-1}. \tag{4}$$

For the case $l = 1$, all entries of $A^{(n,t)}[1, \ldots, n-1|1, \ldots, m-1]$ are nonnegative for $t = 2, \ldots, m$ by (i).
Now assume that (4) is true for all steps $m-1, \ldots, t+1$ and all minors of order $1, \ldots, l-1$. We want to show the claim for step t and minors of order l.

Let $\alpha = (\alpha_1, \ldots, \alpha_l) \in Q_{l,n-1}$ and $\beta = (\beta_1, \ldots, \beta_l) \in Q_{l,m-1}$.
If $\beta_l < t$, then the matrix $A^{(n,t)}[\alpha|\beta]$ is a submatrix of $A^{(n,t)}[1, \ldots, n-1|1, \ldots, t-1]$ which is TN by (ii) and so $\det A^{(n,t)}[\alpha|\beta] \geq 0$.
If $t < \beta_1$ or t is contained in β, then by Proposition 3 (ii), we have

$$\det A^{(n,t+1)}[\alpha|\beta] = \det A^{(n,t)}[\alpha|\beta],$$

which implies by the induction hypothesis on t that

$$\det A^{(n,t)}[\alpha|\beta] \geq 0.$$

Hence it remains to consider the case, where there exists h, $1 \leq h \leq l-1$, such that $\beta_h < t < \beta_{h+1}$ which implies $d(\beta) > 0$.

In order to prove (4) in this case we simplify the notation by setting

$$[\alpha \mid \beta] := \det A^{(n,t)}[\alpha \mid \beta], \quad [\alpha \mid \beta]^{\dagger} := \det A^{(n,t+1)}[\alpha \mid \beta]$$

and for $j \in \{1, \ldots, h\}$,

$$\beta'_{\hat{j}} := \left(\beta_1, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_{l-1}\right), \tag{5}$$

such that $\beta'_{\hat{j}}$ has the length $l-2$.

Since $d(\beta) > 0$ and $\beta_h < t < \beta_{h+1}$, we use Lemma 1 to conclude that for $k = 1, \ldots, l$

$$\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right] \cdot [\alpha \mid \beta] =$$
$$\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{\beta_l\}\right] \cdot \left[\alpha \mid \beta'_{\hat{j}} \cup \left\{\beta_j, t\right\}\right] + \left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \left\{\beta_j\right\}\right] \cdot \left[\alpha \mid \beta'_{\hat{j}} \cup \{t, \beta_l\}\right] \tag{6}$$

It follows from the induction hypothesis on l that the minors $\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right]$, $\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{\beta_l\}\right]$, and $\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \left\{\beta_j\right\}\right]$ are nonnegative because they have order $l-1$.

Furthermore, since t is a column index of the submatrices underlying the following minors, we get by Proposition 3 (ii) that

$$\left[\alpha \mid \beta'_{\hat{j}} \cup \left\{\beta_j, t\right\}\right] = \left[\alpha \mid \beta'_{\hat{j}} \cup \left\{\beta_j, t\right\}\right]^{\dagger},$$

and

$$\left[\alpha \mid \beta'_{\hat{j}} \cup \{t, \beta_l\}\right] = \left[\alpha \mid \beta'_{\hat{j}} \cup \{t, \beta_l\}\right]^{\dagger}.$$

Hence by induction on t, these minors are also nonnegative.

Since the four minors on the right-hand side of (6) are nonnegative, the left-hand side is nonnegative, too. If $0 < \left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right]$ for some k and j, then $0 \leq [\alpha|\beta]$, as desired. If for all k, j, $\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right] = 0$, then it follows by Laplace expansion along column β_l that $\left[\alpha \mid \beta'_{\hat{j}} \cup \{\beta_l, t\}\right] = 0$. Then by Proposition 3 (iii) we have

$$\det A^{(n,t)^{\dagger}}[\alpha \mid \beta] = \det A^{(n,t)}[\alpha \mid \beta].$$

Hence we obtain by induction on t that $0 \leq \det A^{(n,t)}[\alpha \mid \beta]$, as desired. This completes the induction step for the proof of (iii).

(iv) By (iii), $A^{(n,2)}[1, \ldots, n-1|1, \ldots, m-1]$ is TN and therefore by Lemma 3 a Cauchon matrix. The negative entries in the last row and column of A do not alter during the run of the Cauchon algorithm, and hence we may conclude that $A^{(n,2)}$ is a Cauchon matrix.

(v) We prove the claim by decreasing primary induction on t and secondary induction on l as in the proof of statement (iii). For $l = 1$, due to $\beta_l = m$, we are referring to the entries in the last column which are negative by assumption and the fact that the entries of the last column and row do not change during the run of the Cauchon algorithm. If $t = m$, then by Proposition 3 (ii) we have $\det A^{(n,m)}[\alpha \mid \beta] = \det A[\alpha \mid \beta] \leq 0$ since $\beta_l = m$.

Suppose that the statement is true for all minors of order less than l and for all steps $t+1, \ldots, m-1$. Let $\alpha \in Q_{l,n-1}$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_l) \in Q_{l,m}$ with $\beta_l = m$.
If $t < \beta_1$ or $t = \beta_h$ for some $h = 1, \ldots, l$, we use similarly as in the proof of (iii) Proposition 3 (ii) to conclude that $\det A^{(n,t)}[\alpha \mid \beta] \leq 0$.

If $\beta_h < t < \beta_{h+1}$ for some $h = 1, \ldots, l-1$, we apply Lemma 1 and use (6) with the notation (5).

The minors $\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right]$, $\left[\alpha \mid \beta'_{\hat{j}} \cup \left\{\beta_j, t\right\}\right]$, $\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \left\{\beta_j\right\}\right]$ are nonnegative by (iii). $\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{\beta_l\}\right]$ is nonpositive by the induction hypothesis on l, $\left[\alpha \mid \beta'_{\hat{j}} \cup \{t, \beta_l\}\right] = \left[\alpha \mid \beta'_{\hat{j}} \cup \{t, \beta_l\}\right]^{\dagger}$ by Proposition 3 (ii), and by the induction hypothesis on t the latter minor is nonpositive. All of these inequalities yield

$$\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right] \cdot [\alpha \mid \beta] \leq 0.$$

If $0 < \left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right]$ for some k and j, then we have $[\alpha \mid \beta] \leq 0$. If for all k, j, $\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right] = 0$, then proceeding parallel to the last part of (iii) we get

$$\det A^{(n,t+1)}[\alpha \mid \beta] = \det A^{(n,t)}[\alpha \mid \beta].$$

Hence we obtain by induction on t that $0 \leq \det A^{(n,t)}[\alpha \mid \beta]$, as desired. □

By sequentially repeating the steps of the proof of Theorem 4.1, we obtain the following theorem.

Theorem 4.2. *Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$ be t.n.p. with all the entries in its last row and column negative. Then the following statements hold:*

(i) $A^{(s,t)}[1, \ldots, s-1 \mid 1, \ldots, t-1]$ *is TN for all* $s = 2, \ldots, n$ *and* $t = 2, \ldots, m$.

(ii) $A^{(s,2)}[1, \ldots, s-1|1, \ldots, t-1]$ *is* TN *for all* $s = 2, \ldots, n$ *and* $t = 2, \ldots, m$.

(iii) $\tilde{A}[1,\ldots,n-1|1,\ldots,m-1]$ *is a nonnegative matrix.*

(iv) $\tilde{A}$ *is a Cauchon matrix.*

4.2. Characterization of matrices having a positive determinant and all other minors nonpositive

In this subsection, we employ the results obtained so far to investigate and characterize the matrices having positive determinants and all other minors nonpositive, the *t.n.p.*$^+$ matrices, by using the Cauchon algorithm. By the nonsingularity of these matrices, it suffices to assume that the entry in the bottom right position is negative to conclude by Proposition 2 that all entries in their last rows and columns are negative. In the following theorem, we present some properties of the entries in the matrix that we obtain after application of the Cauchon algorithm to a *t.n.p.*$^+$ matrix.

Theorem 4.3. *Let* $A=(a_{ij})\in\mathbb{R}^{n,n}$ *be t.n.p.*$^+$ *and* $a_{nn}<0$. *Then application of the Cauchon Algorithm to A results in the following properties:*

(i) $\tilde{a}_{ii}\neq 0$ *for* $i=3,\ldots,n-1$.

(ii) *If* $\tilde{a}_{2j}=0$ *for* $j>2$, *then* $\tilde{a}_{2i}=0$ *for all* $i=2,\ldots,j-1$ *or* $\tilde{a}_{1j}=0$.

(iii) *If* $\tilde{a}_{i2}=0$ *for* $i>2$, *then* $\tilde{a}_{j2}=0$ *for all* $j=2,\ldots,i-1$ *or* $\tilde{a}_{i1}=0$.

Proof. Since A is *t.n.p.*$^+$ we conclude that $A[1,\ldots,n|2,\ldots,n]$ and $A[2,\ldots,n|1,\ldots,n]$ are *t.n.p.*. Let $\tilde{A}$ be the matrix obtained by the application of the Cauchon algorithm to A. By Theorem 4.2, $\tilde{A}[1,\ldots,n|2,\ldots,n]$ and $\tilde{A}[2,\ldots,n|1,\ldots,n]$ are Cauchon matrices since all entries of the above matrices coincide with the corresponding entries of the matrices obtained by the running the Cauchon algorithm on $A[1,\ldots,n|2,\ldots,n]$ and $A[2,\ldots,n|1,\ldots,n]$. The reason is that the entries of the first column and first row do not affect the calculation of the new entries of the latter matrices.

(i) We prove this statement by decreasing induction on i for $i=n-1,\ldots,3$. Suppose that $\tilde{a}_{jj}>0$ for $j=n-1,\ldots,i+1$ and $\tilde{a}_{ii}=0$. Since $\tilde{A}[1,\ldots,n|2,\ldots,n]$ and $\tilde{A}[2,\ldots,n|1,\ldots,n]$ are Cauchon matrices, we distinguish between the following three cases:

Case (1) $\tilde{a}_{is}=0$ for all $s=1,\ldots,i-1$. By application of Procedure 1 to $\tilde{A}[i,\ldots,n|1,\ldots,n]$, we construct the lacunary sequence

$\big((i+1,i+1),(i+2,i+2),\ldots,(n,n)\big)$ for the Cauchon matrix $\tilde{A}[i,\ldots,n|1,\ldots,n]$. By Theorem 3.1, the rank of the matrix $A[i,\ldots,n|1,\ldots,n]$ is $n-i$, which is a contradiction to the linear independence of the rows of A (note that A is nonsingular as a $t.n.p.^+$ matrix).

Case (2) $\tilde{a}_{ti}=0$ for all $t=1,\ldots,i-1$. By application of Procedure 1 to $\tilde{A}[1,\ldots,n|i,\ldots,n]$, we obtain the lacunary sequence $\big((i+1,i+1),(i+2,i+2),\ldots,(n,n)\big)$ and similarly as in case (1) a contradiction to the linear independence of the columns of A.

Case (3) $\tilde{a}_{is}=0$ for $s=2,\ldots,i-1$ and $\tilde{a}_{i1}\neq 0$ and $\tilde{a}_{ti}=0$ for $t=2,\ldots i-1$ and $\tilde{a}_{1i}\neq 0$.
Since $\tilde{A}[1,\ldots,n|2,\ldots,n]$ and $\tilde{A}[2,\ldots,n|1,\ldots,n]$ are Cauchon matrices, $\tilde{A}$ has the following form:

$$\tilde{A}=\begin{bmatrix}
\tilde{a}_{11} & \tilde{a}_{12} & \ldots & \tilde{a}_{1,i-1} & \tilde{a}_{1,i} & \tilde{a}_{1,i+1} & \ldots & \tilde{a}_{1n}\\
\tilde{a}_{21} & 0 & \ldots & 0 & 0 & \tilde{a}_{2,i+1} & \ldots & \tilde{a}_{2n}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\tilde{a}_{i-1,1} & 0 & \ldots & 0 & 0 & \tilde{a}_{i-1,i+1} & \ldots & \tilde{a}_{i-1,n}\\
\tilde{a}_{i1} & 0 & \ldots & 0 & 0 & \tilde{a}_{i,i+1} & \ldots & \tilde{a}_{in}\\
\tilde{a}_{i+1,1} & \tilde{a}_{i+1,2} & \ldots & \tilde{a}_{i+1,i-1} & \tilde{a}_{i+1,i} & \tilde{a}_{i+1,i+1} & \ldots & \tilde{a}_{i+1,n}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\tilde{a}_{n1} & \tilde{a}_{n2} & \ldots & \tilde{a}_{n,i-1} & \tilde{a}_{ni} & \tilde{a}_{n,i+1} & \ldots & \tilde{a}_{nn}
\end{bmatrix}.$$

By application of Procedure 1 to $\tilde{A}[2,\ldots,n|1,\ldots,n]$, we construct the lacunary sequence $\big((i,1),(i+1,i+1),\ldots,(n-1,n-1),(n,n)\big)$ for the Cauchon matrix $\tilde{A}[2,\ldots,n|1,\ldots,n]$. By Theorem 3.1, the rank of the matrix $A[2,\ldots,n|1,\ldots,n]$ is $n-i+1$. Since $i\geq 3$, we obtain $n-i+1\leq n-2$ which is a contradiction to the linear independence of the rows of this matrix which proves (i).

(ii) Let $\tilde{a}_{2j}=0$ for $j>2$. Then since $\tilde{A}[1,\ldots,n|2,\ldots,n]$ is a Cauchon matrix we get that $\tilde{a}_{2i}=0$ for $i=2,\ldots,j-1$, or $\tilde{a}_{1j}=0$.

(iii) Let $\tilde{a}_{i2}=0$ for $i>2$. Then since $\tilde{A}[2,\ldots,n|1,\ldots,n]$ is a Cauchon matrix we get that $\tilde{a}_{j2}=0$ for $j=2,\ldots,i-1$, or $\tilde{a}_{i1}=0$. □

Corollary 2. *Let $A=(a_{ij})\in\mathbb{R}^{n,n}$ be $t.n.p.^+$ and $a_{nn}<0$. Then*

(i) *If $\tilde{a}_{ij}=0$ for $i>2$ and $i<j$, then $\tilde{a}_{sj}=0$ for $s=1,\ldots,i-1$.*

(ii) *If* $\tilde{a}_{ij} = 0$ *for* $j > 2$ *and* $i > j$, *then* $\tilde{a}_{it} = 0$ *for* $t = 1, \ldots, j-1$.

Proof. By the hypotheses, $\tilde{A}[1, \ldots, n|2, \ldots, n]$ and $\tilde{A}[2, \ldots, n|1, \ldots, n]$ are Cauchon matrices as in the proof of Theorem 4.3. It suffices to prove only (i).

Suppose that $\tilde{a}_{ij} = 0$ for $i > 2$ and $i < j$. We want to show that $\tilde{a}_{sj} = 0$ for $s = 1, \ldots, i-1$. Suppose on the contrary that $\tilde{a}_{sj} \neq 0$ for some $s \in \{1, \ldots, i-1\}$. Then since $\tilde{a}_{ij} = 0$ and $\tilde{A}[1, \ldots, n|2, \ldots, n]$ is a Cauchon matrix, it follows that $\tilde{a}_{ik} = 0$ for all $k = 2, \ldots, j-1$. Now since $j > i$ we conclude that $\tilde{a}_{ii} = 0$ for $i > 2$ which provides a contradiction to Theorem 4.3 (i). □

Theorem 4.4. *Let* $A = (a_{ij}) \in \mathbb{R}^{n,n}$ *be t.n.p.*$^+$ *with* $a_{nn} < 0$, *and let* $\tilde{A} = (\tilde{a}_{ij})$ *be the matrix obtained by the Cauchon algorithm. Then*

(i) $\det A[1, \ldots, n-1|2, \ldots, n], \det A[2, \ldots, n|1, \ldots, n-1] < 0$;

(ii) $\det A[2, \ldots, n-1] < 0$;

(iii) $\tilde{a}_{i,i-1}, \tilde{a}_{i-1,i} > 0$ *for* $i = 2, \ldots, n-1$.

(iv) *If* $\tilde{a}_{2j} = 0$ *for some* $j \in \{4, \ldots, n-1\}$ *then* $\tilde{a}_{1j} = 0$, *and if* $\tilde{a}_{i2} = 0$ *for some* $i \in \{4, \ldots, n-1\}$ *then* $\tilde{a}_{i1} = 0$.

Proof. Since A is a *t.n.p.*$^+$ matrix, its submatrices $A[1, \ldots, n|2, \ldots, n]$ and $A[2, \ldots, n|1, \ldots, n]$ are *t.n.p.* with $a_{nn} < 0$. By Theorem 4.2 we get that $\tilde{A}[1, \ldots, n|2, \ldots, n]$ and $\tilde{A}[2, \ldots, n|1, \ldots, n]$ are Cauchon matrices.

(i) Set the matrix $C = \left(c_{ij}\right) := SA^{-1}S^{-1}$, where $S = \operatorname{diag}(1, -1 \ldots, (-1)^{n+1})$. For $\alpha, \beta \in Q_{l,n}$, $l = 1, \ldots, n$, we get

$$\det C[\alpha|\beta] = (-1)^s \det A^{-1}[\alpha|\beta] = \frac{\det A[\beta^c|\alpha^c]}{\det A}, \tag{7}$$

where $s = \sum_{i=1}^{l} (\alpha_i + \beta_i)$ and α^c and β^c denote the complement of α and β in $\{1, \ldots, n\}$, respectively. Since for $l = 1, \ldots, n-1$, $\det A[\beta^c|\alpha^c] \leq 0$ and $\det A > 0$, we obtain that $\det C[\alpha|\beta] \leq 0$ for $l = 1, \ldots, n-1$. Moreover, for $\alpha, \beta = (1, \ldots, n)$,

$$\det C = \frac{1}{\det A} > 0,$$

whence C is *t.n.p.*$^+$. By (7), we get that

$$c_{1n} = \frac{\det A[1,\ldots,n-1|2,\ldots,n]}{\det A}. \tag{8}$$

We claim that c_{1n} is negative by which we conclude that $\det A[1,\ldots,n-1|2,\ldots,n]$ is negative. Suppose on the contrary that $c_{1n} = 0$. Since C is nonsingular, it has neither a zero row nor a zero column. Hence we may assume that $c_{1j}, c_{in} \neq 0$ for some $i \in \{2,\ldots,n\}$, $j \in \{1,\ldots,n-1\}$. Then we obtain

$$0 \geq \det C[1,i|j,n] = c_{1j}c_{in} - c_{1n}c_{ij} = c_{1j}c_{in},$$

but since c_{1j}, c_{in} are negative, the right-hand side is positive which is a contradiction. Therefore, $c_{1n} < 0$.

The proof for $\det A[2,\ldots,n|1,\ldots,n-1] < 0$ is similar by using

$$c_{n1} = \frac{\det A[2,\ldots,n|1,\ldots,n-1]}{\det A}.$$

(ii) Since $\tilde{a}_{ii} \neq 0$ by Theorem 4.3 for $i = 3,\ldots,n$, we construct the lacunary sequence $\big((2,2),(3,3),\ldots,(n,n)\big)$ with respect to the Cauchon matrix $\tilde{A}[2,\ldots,n|1,\ldots,n]$, see Theorem 4.2 (iv). Since $A[2,\ldots,n]$ is *t.n.p.*, we obtain by Theorem 4.2 (iv) that $\tilde{A}[2,\ldots,n]$ is a Cauchon matrix. Moreover, by Theorem 3.2 we conclude that

$$\det A[2,\ldots,n] = \tilde{a}_{22} \cdot \tilde{a}_{33} \cdot \ldots \cdot \tilde{a}_{nn}. \tag{9}$$

If $\tilde{a}_{22} = 0$, application of Lemma 1 to the matrix A yields

$$\begin{aligned} \det[2,\ldots,n-1]\det A &= \det A[1,\ldots,n-1]\det A[2,\ldots,n] \\ &- \det A[1,\ldots,n-1|2,\ldots,n]\det A[2,\ldots,n|1,\ldots,n-1]. \end{aligned}$$

Since $\det A > 0$ and $\det A[1,\ldots,n-1|2,\ldots,n], \det A[2,\ldots,n|1,\ldots,n-1] < 0$ by (i), and since (9) gives $\det[2,\ldots,n] = 0$, we conclude that

$$\det[2,\ldots,n-1] < 0. \tag{10}$$

If $\tilde{a}_{22} > 0$, then (9) provides by Theorem 4.3 (i) and $\tilde{a}_{nn} = a_{nn} < 0$ that $A[2,\ldots,n]$ is nonsingular, and by Theorem 3.5 we conclude that (10) holds.

(iii) It suffices to prove only that $\tilde{a}_{i,i-1} > 0$ for $i = 2, 3, \ldots, n-1$. By (i) $A[2, \ldots, n|1, \ldots, n-1]$ is $Ns.t.n.p.$ Hence by Theorem 3.5 (ii), we obtain

$$\det A[i, \ldots, n|i-1, \ldots, n-1] < 0 \tag{11}$$

for $i = 2, \ldots, n-1$. Recall that $a_{n,n-1} = \tilde{a}_{n,n-1} < 0$. To prove that $\tilde{a}_{i,i-1} > 0$, we verify the following formula

$$\tilde{a}_{i,i-1} = \frac{\det A[i, \ldots, n|i-1, \ldots, n-1]}{\det A[i+1, \ldots, n|i, \ldots, n-1]}; \tag{12}$$

we will do this by decreasing induction on i for $i = n-1, \ldots, 2$. We construct the lacunary sequence $\big((n-1, n-2), (n, n-1)\big)$ for the matrix $\tilde{A}[n-1, n|n-2, n-1]$. By Theorem 3.2 we conclude that

$$\det A[n-1, n|n-2, n-1] = \tilde{a}_{n-1,n-2} \cdot \tilde{a}_{n,n-1},$$

or equivalently,

$$\tilde{a}_{n-1,n-2} = \frac{\det A[n-1, n|n-2, n-1]}{\det A[n|n-1]}.$$

Since by (11) $\det A[n-1, n|n-2, n-1] < 0$, we obtain $\tilde{a}_{n-1,n-2} > 0$. Suppose that $\tilde{a}_{i,i-1} > 0$, for $i = n-1, \ldots, k+1$, $k \geq 2$. Now, for $i = k$, $\tilde{a}_{k+1,k}, \tilde{a}_{k+2,k+1}, \ldots, \tilde{a}_{n-1,n-2} \neq 0$, by the induction hypothesis. So we find the lacunary sequence $\big((k, k-1), (k+1, k), \ldots, (n, n-1)\big)$ for the Cauchon matrix $\tilde{A}[k, \ldots, n|k-1, \ldots, n-1]$, and by Theorem 3.2 we conclude that

$$\det A[k, \ldots, n|k-1, \ldots, n-1] = \tilde{a}_{k,k-1} \cdot \tilde{a}_{k+1,k} \cdot \ldots \cdot \tilde{a}_{n,n-1}.$$

Moreover, it follows by the induction hypothesis that

$$\det A[k+1, \ldots, n|k, \ldots, n-1] = \tilde{a}_{k+1,k} \cdot \ldots \cdot \tilde{a}_{n,n-1},$$

which yields

$$\tilde{a}_{k,k-1} = \frac{\det A[k, \ldots, n|k-1, \ldots, n-1]}{\det A[k+1, \ldots, n|k, \ldots, n-1]},$$

where $\det A[k, \ldots, n|k-1, \ldots, n-1]$, $\det A[k+1, \ldots, n|k, \ldots, n-1] < 0$, by (11). Thus $\tilde{a}_{k,k-1} > 0$, as desired.

(iv) It suffices to prove only the first statement. Let $\tilde{a}_{2j} = 0$ for some $j \in \{4, \ldots, n-1\}$. Then by Theorem 4.3 (ii), we obtain that either $\tilde{a}_{2k} = 0$ for $k = 2, \ldots, j-1$ or $\tilde{a}_{1j} = 0$. Since $\tilde{a}_{23} \neq 0$ by (iii) which excludes $\tilde{a}_{2k} = 0$ for $k = 2, \ldots, j-1$, we get that $\tilde{a}_{1j} = 0$. □

The following two theorems will show that the entry $\tilde{a}_{22}$ is the most critical one. The next theorem presents necessary and sufficient conditions for a given matrix to be *t.n.p.*$^+$ under certain conditions.

Theorem 4.5. *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ with $a_{nn} < 0$ and let $\tilde{A} = (\tilde{a}_{ij})$ be the matrix obtained by the Cauchon algorithm satisfying $\tilde{a}_{22} > 0$.*
(a) If A is t.n.p.$^+$, then the following statements hold

(i) $\tilde{A}$ is a Cauchon matrix;

(ii) $\tilde{a}_{ii} > 0$ for all $i = 3, \ldots, n-1$;

(iii) $\tilde{a}_{11} < 0$;

(iv) $\tilde{a}_{ij} \geq 0$ for $i = 1, \ldots, n-1$, $j = 2, \ldots, n-1$, and $i = 2, \ldots, n-1$, $j = 1$;

(v) $\tilde{a}_{i,i-1}, \tilde{a}_{i-1,i} > 0$ for $i = 2, \ldots, n-1$.

(b) Conversely, assume that $\det A[1, \ldots, n-1] \leq 0$, $A[1,2] \leq 0$,
$a_{ni}, a_{in} < 0$ for $i = 1, \ldots, n-1$, and

(i) - (v) hold.
Then A is t.n.p.$^+$.

Proof. To prove the necessity, let A be *t.n.p.*$^+$ with $a_{nn} < 0$ and $\tilde{a}_{22} > 0$.

(i) Since A is *t.n.p.*$^+$ with $a_{nn} < 0$, then as in the proof of Theorem 4.3, $\tilde{A}[1, \ldots, n|2, \ldots, n]$ is a Cauchon matrix. By Corollary 2 if $\tilde{a}_{ij} = 0$ for some $i > 2$ and $i < j$, then $\tilde{a}_{sj} = 0$ for $s = 1, \ldots, i-1$ and if $\tilde{a}_{ij} = 0$ for $j > 2$ and $i > j$, then $\tilde{a}_{it} = 0$ for $t = 1, \ldots, j-1$. If $\tilde{a}_{2j} = 0$ for some $j > 2$ or $\tilde{a}_{i2} = 0$ for some $i > 2$, then by Theorem 4.3 and $\tilde{a}_{22} > 0$, we conclude that $\tilde{a}_{1j} = 0$ or $\tilde{a}_{i1} = 0$. Hence $\tilde{A}$ is a Cauchon matrix.

(ii) Since A is *t.n.p.*$^+$ and $a_{nn} < 0$, we get by Theorem 4.3 (i) that $\tilde{a}_{ii} \neq 0$ for $i = 3, \ldots, n-1$. To prove the positivity of these entries, we will proceed using decreasing induction for $i = n-1, \ldots, 3$.

For $i = n-1$, since $\tilde{A}$ is a Cauchon matrix, $\big((n-1, n-1), (n, n)\big)$ is a lacunary sequence for $\tilde{A}$. Since $\tilde{a}_{ii} \neq 0$ for $i = n-1, n$, it follows from Theorem 3.2 that

$$0 \geq \det A\,[n-1, n] = \tilde{a}_{n-1,n-1} \cdot \tilde{a}_{nn} \neq 0.$$

Since $\tilde{a}_{nn} = a_{nn} < 0$, we get that $\tilde{a}_{n-1,n-1} > 0$.

Suppose that $\tilde{a}_{n-1,n-1}, \ldots, \tilde{a}_{i+1,i+1}$ are positive. Since $\tilde{A}$ is a Cauchon matrix by (i), $\big((i,i), (i+1, i+1), \ldots, (n, n)\big)$ is a lacunary sequence for $\tilde{A}$. Since $\tilde{a}_{ii} \neq 0$ for $i = n, \ldots, 3$, by Theorem 3.2 we get that

$$0 \geq \det A\,[i, \ldots, n] = \tilde{a}_{ii} \cdot \tilde{a}_{i+1,i+1} \cdot \ldots \cdot \tilde{a}_{nn} \neq 0.$$

Since $\tilde{a}_{nn} = a_{nn} < 0$ and $\tilde{a}_{n-1,n-1}, \ldots, \tilde{a}_{i+1,i+1}$ are positive by the induction hypothesis, we conclude that $\tilde{a}_{ii} > 0$.

(iii) Since $\tilde{A}$ is a Cauchon matrix, $\tilde{a}_{22} > 0$, and $\tilde{a}_{ii} \neq 0$ for $i = 3, \ldots, n$, $\big((1,1), \ldots, (n,n)\big)$ is a lacunary sequence for $\tilde{A}$, and it follows from by Theorem 3.2 that

$$0 < \det A = \tilde{a}_{11} \cdot \tilde{a}_{22} \cdot \ldots \cdot \tilde{a}_{nn}.$$

Since $\tilde{a}_{nn} = a_{nn} < 0$, $\tilde{a}_{22}, \ldots, \tilde{a}_{n-1,n-1} > 0$, we conclude that $\tilde{a}_{11} < 0$.

(iv) Since A is $t.n.p.^{+}$, the matrices $A[1, \ldots, n|2, \ldots, n]$ and $A[2, \ldots, n|1, \ldots, n]$ are $t.n.p.$ and by Theorem 4.2 (iii) we conclude that the matrices $\tilde{A}\left[1, \ldots, n-1|2, \ldots, n-1\right]$ and $\tilde{A}[2, \ldots, n-1|1, \ldots, n-1]$ are nonnegative, from which (iv) follows.

(v) This is just Theorem 4.4 (iii).

For the converse direction, we will prove that $\det A[\alpha|\beta] \leq 0$, for all $\alpha, \beta \in Q_{k,n}$, $k = 1, \ldots, n-1$ and $\det A > 0$.
Since $\tilde{A}$ is a Cauchon matrix, $a_{nn} < 0$, $\tilde{a}_{22} > 0$, and by (ii) and (iii) we have that $\tilde{a}_{ii} \neq 0$ for all $i = 1, \ldots, n$. We conclude by Corollary 1 that

$$\det A[1, \ldots, n] = \det A = \tilde{a}_{11} \cdot \tilde{a}_{22} \cdot \ldots \cdot \tilde{a}_{n-1,n-1} \cdot \tilde{a}_{nn},$$

whence $\det A > 0$.

In the same manner we obtain from Theorem 3.2 that $\det A[k,\ldots,n] < 0$ for $k = 2,\ldots,n-1$. Moreover, by following the proof of Theorem 3.9 in [1], we have

$$detA[\alpha|\beta] \leq 0, \tag{13}$$

for all $\alpha,\beta \in Q_{l,n}$ with $\alpha_l = n$, $l = 1,\ldots,n-1$, and for all $\alpha,\beta \in Q_{l,n}$ with $\beta_l = n$, $l = 1,\ldots,n-1$. Hence $A[2,\ldots,n]$ is $Ns.t.n.p.$ by Theorem 3.5 and the fact that $\tilde{A}[2,\ldots,n] = A[\widetilde{2,\ldots},n]$ and the assumption that $a_{22} \leq 0$, $a_{nn}, a_{2,n}, a_{n,2} < 0$.

Furthermore, in the same manner and by (v) and $a_{n,n-1}, a_{n-1,n} < 0$, we have

$$\det A[i,\ldots,n|i-1,\ldots,n-1] < 0,$$
$$\det A[i-1,\ldots,n-1|i,\ldots,n] < 0,$$

for $i = 2,\ldots,n$, and by (13) we have for $k = 2,\ldots,n$,

$$\det A[k,\ldots,n|\beta] \leq 0, \text{ for all } \beta \in Q_{n-k+1,n-1},$$

$$\det A[\alpha|k,\ldots,n] \leq 0, \text{ for all } \alpha \in Q_{n-k+1,n-1}.$$

In addition, since $A[2,\ldots,n]$ is $Ns.t.n.p.$, we obtain for $k = 3,\ldots,n$, that

$$\det A[\alpha|k-1,\ldots,n-1] \leq 0, \text{ for all } \alpha \in Q_{n-k+1,\{2,\ldots,n\}},\ [1]$$

$$\det A[k-1,\ldots,n-1|\beta] \leq 0, \text{ for all } \beta \in Q_{n-k+1,\{2,\ldots,n\}}.$$

Hence by Theorem 3.5, $A[2,\ldots,n|1,\ldots,n-1]$ and $A[1,\ldots,n-1|2,\ldots,n]$ are $Ns.t.n.p.$

In order to complete the proof, by employing Theorem 3.3, it is sufficient to show that $\det A[\alpha|\beta] \leq 0$, for $\alpha = (s+1,\ldots,s+l)$, $\beta = (\beta_1,\ldots,\beta_l) \in$

[1] The notation $\{2,\ldots,n\}$ means that the underlying index range $\{1,\ldots,n\}$ is replaced by $\{2,\ldots,n\}$.

$Q_{l,n}$ with $s = 0, 1, \ldots, n-1-l$, $l = 1, \ldots, n-1$ and $\beta_l < n$. For $l = 1$, it is fulfilled by the fact that $A[1, \ldots, n-1|2, \ldots, n]$ and $A[2, \ldots, n|1, \ldots, n-1]$ are $Ns.t.n.p.$ and the assumption $A[1, 2] \leq 0$. If $s + 1 \geq 2$ and $\beta_1 \geq 2$, then $\det A[\alpha|\beta] \leq 0$ since $A[\alpha|\beta]$ is a submatrix of $A[2, \ldots, n]$. If $s + 1 \geq 2$ and $\beta_1 = 1$, then $A[\alpha|\beta]$ is a submatrix of $A[2, \ldots, n|1, \ldots, n-1]$; in this case $\det A[\alpha|\beta] \leq 0$ since $A[2, \ldots, n|1, \ldots, n-1]$ is $Ns.t.n.p.$ If $s + 1 = 1$ and $\beta_1 \geq 2$, then $\det A[\alpha|\beta] \leq 0$ since $A[\alpha|\beta]$ is a submatrix of the $Ns.t.n.p.$ matrix $A[1, \ldots, n-1|2, \ldots, n]$. In the following we will consider only the remaining case $s + 1 = 1$ and $\beta_1 = 1$.

By Lemma 1, properties of determinants, and rearrangement, we obtain

$$\begin{aligned} &\det A[1, \ldots, l|\beta] \det A[2, \ldots, l, t_1|\beta_{\hat{1}} \cup \{t_2\}] \\ &= \det A[2, \ldots, l|\beta_{\hat{1}}] \det A[1, \ldots, l, t_1|\beta \cup \{t_2\}] \\ &+ \det A[2, \ldots, l, t_1|\beta] \det A[1, \ldots, l|\beta_{\hat{1}} \cup \{t_2\}], \end{aligned} \tag{14}$$

for all $t_1 > l$ and $t_2 \in \{1, \ldots, n\} \backslash \beta$.
The minors $\det A[2, \ldots, l, t_1|\beta_{\hat{1}} \cup \{t_2\}]$, $\det A[2, \ldots, l|\beta_{\hat{1}}]$, $\det A[2, \ldots, l, t_1|\beta]$, and $\det A[1, \ldots, l|\beta_{\hat{1}} \cup \{t_2\}]$ are nonpositive since the corresponding submatrices lie in $A[2, \ldots, n]$ or $A[1, \ldots, n-1|2, \ldots, n]$ or $A[2, \ldots, n|1, \ldots, n-1]$ which are $t.n.p.$ matrices. For $t_1 = n$ or $t_2 = n$, $\det A[1, \ldots, l, t_1|\beta \cup \{t_2\}] \leq 0$.

In the following we first consider the case $l < n-1$.

If for $t_1 = n$ or $t_2 = n$, and $\det A[2, \ldots, l, t_1|\beta_{\hat{1}} \cup \{t_2\}] < 0$, then we conclude that $\det A[1, \ldots, l|\beta] \leq 0$, as desired. Otherwise, $\det A[2, \ldots, l, t_1|\beta_{\hat{1}} \cup \{t_2\}] = 0$ for $t_1 = n$ and $t_2 \in \{1, \ldots, n\} \backslash \beta$ or $t_2 = n$ and $t_1 > l$.
If $\det A[2, \ldots, l|\beta_{\hat{1}}] < 0$, then together with $\det A[2, \ldots, l, t_1|\beta_{\hat{1}} \cup \{n\}] = 0$ for $t_1 > l$, we conclude that $A[2, \ldots, n]$ is singular which is a contradiction.
Hence in the following we assume that $\det A[2, \ldots, l|\beta_{\hat{1}}] = 0$. By (14), $\det A[2, \ldots, l, t_1|\beta] \det A[1, \ldots, l|\beta_{\hat{1}} \cup \{t_2\}] = 0$ for all $t_1 > l$ and $t_2 \in \{1, \ldots, n\} \backslash \beta$.

If the rows of $A[2, \ldots, l|\beta]$ or the columns of $A[1, \ldots, l|\beta_{\hat{1}}]$ are linearly dependent, then $\det A[1, \ldots, l|\beta] = 0$, as desired.
Hence the rows of $A[2, \ldots, l|\beta]$ and the columns of $A[1, \ldots, l|\beta_{\hat{1}}]$ are linearly independent. Moreover, $\det A[2, \ldots, l, t_1|\beta] = 0$ for all $t_1 > l$ or $\det A[1, \ldots, l|\beta_{\hat{1}} \cup \{t_2\}] = 0$ for $t_2 \in \{1, \ldots, n\} \backslash \beta$, since otherwise by (14) we have a nonzero quantity equals a zero quantity.

Whence by $\det A[2,\ldots,l,t_1|\beta] = 0$ for all $t_1 > l$ and linear independence of the rows of $A[2,\ldots,l|\beta]$ we conclude that $\text{rank} A[2,\ldots,n|\beta] \le l-1$ which is a contradiction to the nonsingularity of $A[2,\ldots,n]$.

Now, if $l = n-1$, then $\det A[1,\ldots,l|\beta] = \det A[1,\ldots,n-1]$ which is nonpositive by assumption. This completes the proof of the theorem. $\square$

The condition that $\det A[1,\ldots,n-1] \le 0$ in Theorem 4.5, is necessary to conclude that a given matrix is *t.n.p.*$^+$ as the following example shows.

Example 1. *Let* $A = \begin{bmatrix} -6 & -10 & -2 \\ -4 & -7 & -1 \\ -5 & -8 & -1 \end{bmatrix}$. *The application of the Cauchon algorithm yields*

$$A^{(4,2)} = A, \quad A^{(3,3)} = \begin{bmatrix} 4 & 6 & -2 \\ 1 & 1 & -1 \\ -5 & -8 & -1 \end{bmatrix},$$

$$A^{(3,2)} = \begin{bmatrix} \frac{1}{4} & 6 & -2 \\ \frac{3}{8} & 1 & -1 \\ -5 & -8 & -1 \end{bmatrix} = A^{(3,1)}, \quad A^{(2,3)} = \begin{bmatrix} -\frac{1}{2} & 4 & -2 \\ \frac{3}{8} & 1 & -1 \\ -5 & -8 & -1 \end{bmatrix},$$

$$A^{(2,2)} = \begin{bmatrix} -2 & 4 & -2 \\ \frac{3}{8} & 1 & -1 \\ -5 & -8 & -1 \end{bmatrix} = A^{(2,1)} = A^{(1,3)} = A^{(1,2)} = \tilde{A}.$$

$\tilde{A}$ *satisfies all the conditions listed in Theorem 4.5, but the matrix* A *is not* *t.n.p.*$^+$ *since* $\det A[1,2] = \begin{vmatrix} -6 & -10 \\ -4 & -7 \end{vmatrix} = 2 > 0$.

In the following theorem, we present a property of the matrix that we obtain after application of the Cauchon algorithm to a *t.n.p.*$^+$ matrix, where the entry $(2,2)$ in the resulting matrix vanishes.

Theorem 4.6. *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be t.n.p.$^+$ with $a_{nn} < 0$, and let $\tilde{A} = (\tilde{a}_{ij})$ be the matrix obtained by the Cauchon algorithm satisfying $\tilde{a}_{22} = 0$. Then $\tilde{a}_{12}, \tilde{a}_{21} \neq 0$ holds.*

Proof. Suppose by contradiction that $\tilde{a}_{12} = 0$. By Theorem 4.3 (i) and similar arguments as in the proof of Theorem 4.5 (i), it follows that $\tilde{a}_{ii} > 0$ for $i = 3, \ldots, n-1$. By application of Procedure 1 to $\tilde{A}[1, \ldots, n|2, \ldots, n]$, we obtain the lacunary sequence $\big((3,3), \ldots, (n-1, n-1), (n,n)\big)$. Hence by Theorem 3.1, the rank of the matrix $A[1, \ldots, n|2, \ldots, n]$ is $n-2$ which is a contradiction to the linear independence of the columns of $A[1, \ldots, n|2, \ldots, n]$. For $\tilde{a}_{21} = 0$, we proceed similarly working with $\tilde{A}\left[2, \ldots, n|1, \ldots, n\right]$. □

4.3. Matrix intervals

We consider now the checkerboard ordering introduced in Subsection 2.1 and matrix intervals with respect to this partial ordering. For $A, B \in \mathbb{R}^{n,n}$, $A = \big(a_{ij}\big)$, $B = \big(b_{ij}\big)$, if $A \leq^* B$, i.e.,

$$(-1)^{i+j} a_{ij} \leq (-1)^{i+j} b_{ij},\ i, j = 1, \ldots, n,$$

we introduce

$$[A, B] := \left\{ Z \in \mathbb{R}^{n,n} \mid A \leq^* Z \leq^* B \right\}.$$

The matrices A and B are called the *corner matrices*. By $\mathbb{I}\left(\mathbb{R}^{n,n}\right)$ we denote the set of all matrix intervals of order n with respect to the checkerboard partial ordering. In [1], [4], [6], [7] matrix intervals of $NsTN$ (with a weakening of the nonsingularity [3]) and $Ns.t.n.p.$ matrices and matrices having a negative determinant and all of their other minors nonnegative have been studied.

Lemma 5. *[25, Corollary 3.5] Let $A, B, Z \in \mathbb{R}^{n,n}$ and let A and B be nonsingular with $0 \leq A^{-1}, B^{-1}$. If $A \leq Z \leq B$, then Z is nonsingular and $B^{-1} \leq Z^{-1} \leq A^{-1}$.*

Theorem 4.7. *[6, Theorem 6.7] Let $[A, B] \in \mathbb{I}\left(\mathbb{R}^{n,n}\right)$ with $Z \in [A, B]$. If A and B are Ns.t.n.p. with $b_{nn} < 0$, then $\tilde{A} \leq^* \tilde{B}$, $\tilde{Z} \in [\tilde{A}, \tilde{B}]$, and Z is $Ns.t.n.p.$*

Theorem 4.8. *Let $[A, B] \in \mathbb{I}\left(\mathbb{R}^{n,n}\right)$ with $Z \in [A, B]$. If $A = (a_{ij})$ and $B = (b_{ij})$ are t.n.p.$^+$ and $b_{nn} < 0$, then Z is t.n.p.$^+$.*

Proof. First of all, since $b_{nn} < 0$ and therefore $a_{nn} < 0$ we may conclude that all the entries in the last row and column of A and B are negative. For $Z = \big(z_{ij}\big) \in [A, B]$ we have

$$(-1)^{i+j} a_{ij} \le (-1)^{i+j} z_{ij} \le (-1)^{i+j} b_{ij}, \quad i, j = 1, \dots, n,$$

thus all the entries in the last row and column of Z are negative, too, and $Z[1.2] \le 0$.
The entries a_{ij}^{-1} of A^{-1} can be represented as

$$a_{ij}^{-1} = (-1)^{i+j} \frac{\det A\, \big[1, \dots, j-1, j+1, \dots, n | 1, \dots, i-1, i+1, \dots, n\big]}{\det A}. \quad (15)$$

Since A is $t.n.p.^{+}$, if $i + j$ is even, then $a_{ij}^{-1} \le 0$ and if $i + j$ is odd, then $a_{ij}^{-1} \ge 0$. Thus $-SA^{-1}S \ge 0$, and analogously $-SB^{-1}S \ge 0$, where $S = \mathrm{diag}(1, -1, \dots, (-1)^{n+1})$.

Since $-SAS \ge -SZS \ge -SBS$, we obtain by Lemma 5 that $-SZS$ is nonsingular and so Z is nonsingular. Since Z is choosen arbitrarily and $\det A, \det B > 0$, it follows that $\det Z > 0$. Moreover, we get

$$-SA^{-1}S \le -SZ^{-1}S \le -SB^{-1}S. \quad (16)$$

It follows from (15) and (16) that for $Z^{-1} = (z_{ij}^{-1})$

$$0 \ge \frac{\det A[1, \dots, n-1]}{\det A} = a_{nn}^{-1} \ge z_{nn}^{-1} = \frac{\det Z[1, \dots, n-1]}{\det Z},$$

which implies $\det Z[1, \dots, n-1] \le 0$.
We may conclude by Theorem 4.4 (i) that $A[1, \dots, n-1|2, \dots, n]$, $A[2, \dots, n|1, \dots, n-1]$ and $B[1, \dots, n-1|2, \dots, n]$, and$B[2, \dots, n|1, \dots, n-1]$ are $Ns.t.n.p.$ Since

$$B[1, \dots, n-1|2, \dots, n] \le^{*} Z[1, \dots, n-1|2, \dots, n] \le^{*} A[1, \dots, n-1|2, \dots, n],$$

$$B[2, \dots, n|1, \dots, n-1] \le^{*} Z[2, \dots, n|1, \dots, n-1] \le^{*} A[2, \dots, n|1, \dots, n-1],$$

it follows from Theorem 4.7 that $Z[1 \dots, n-1|2, \dots, n]$ and $Z[2, \dots, n|1, \dots, n-1]$ are $Ns.t.n.p.$, too.

To show that Z is $t.n.p.^{+}$ we employ Theorem 4.5 (b). We firstly assume that $A[2, \dots, n]$ and $B[2, \dots, n]$ are nonsingular. Then by Theorem 4.7 $Z[2, \dots, n]$ is $Ns.t.n.p.$, too, and by $\tilde{Z}[2, \dots, n] = Z[2, \dots, n]$, we have

$$(-1)^{i+j} \tilde{a}_{ij} \le (-1)^{i+j} \tilde{z}_{ij} \le (-1)^{i+j} \tilde{b}_{ij}, \; i = 2, \dots, n.$$

By Theorem 4.2 (iv), $\widetilde{A[2,\ldots,n]} = \tilde{A}[2,\ldots,n]$ is a Cauchon matrix, and thus by Theorems 3.2 and 3.5 (ii) $\tilde{a}_{22}$ can be represented as

$$\tilde{a}_{22} = \frac{\det A[2,\ldots,n]}{\det A[3,\ldots,n]}$$

which is positive. Analogously, we obtain that $\tilde{b}_{22} > 0$. Hence we may apply Theorem 4.5 (ii), (iv), and (v), to conclude that $\tilde{a}_{ii} > 0$ for $i = 2,\ldots,n-1$, $\tilde{a}_{ij}, \tilde{b}_{ij} \geq 0$ for $i,j = 2,\ldots,n-1$, and $\tilde{b}_{i,i-1}, \tilde{b}_{i-1,i} > 0$ for $i = 2,\ldots,n-1$, which implies that $\tilde{z}_{ii} > 0$ for $i = 2,\ldots,n-1$, $\tilde{z}_{ij} \geq 0$ for $i,j = 2,\ldots,n-1$, and $\tilde{z}_{i,i-1}, \tilde{z}_{i-1,i} > 0$ for $i = 3,\ldots,n-1$. Furthermore, by Theorem 4.2 (iv), $\tilde{Z}[2,\ldots,n]$ is a Cauchon matrix. To conclude the proof it remains to show that $\tilde{z}_{11} < 0$, $\tilde{z}_{12}, \tilde{z}_{21} > 0$, $\tilde{z}_{1j}, \tilde{z}_{i1} \geq 0$ for $i,j = 2,\ldots,n-1$, and $\tilde{Z}$ is a Cauchon matrix.

Claim: $\tilde{z}_{i1} \geq 0$ and $\tilde{z}_{i1} = 0$ if $\tilde{z}_{i2} = 0$, for $i = 3,\ldots,n-1$.
We proceed by decreasing induction on i.
For $i = n-1$, we consider the submatrix $\tilde{Z}[n-1,n|1,\ldots,n]$. which is obviously a Cauchon matrix. We choose the lacunary sequence $\big((n-1,1),(n,2)\big)$ since $\tilde{z}_{n2} = z_{n2} < 0$. Hence by Theorem 3.2, we get

$$\tilde{z}_{n-1,1} = \frac{\det Z[n-1,n|1,2]}{z_{n2}}.$$

Herein the numerator is nonpositive since the underlying submatrix lies in $Z[2,\ldots,n|1,\ldots,n-1]$, hence $\tilde{z}_{n-1,1} \geq 0$.

If $\tilde{z}_{n-1,2} = 0$, the sequence $\big((n-1,n),(2,3)\big)$ is lacunary for the matrix $\tilde{Z}[n-1,n|1,\ldots,n]$, whence $\det Z[n-1,n|2,3] = \tilde{z}_{n-1,2} \cdot \tilde{z}_{n3} = 0$. Thus, the matrix $Z[n-1,n|2,3]$ has rank 1. By Lemma 4, either the rows $n-1$ and n or the columns 2 and 3 are linearly dependent in $Z[2,\ldots,n|1,\ldots,n-1]$ which is a contradiction to the nonsingularity of this matrix, or the right shadow of $Z[n-1,n|2,3]$ which is $Z[2,\ldots,n|2,\ldots,n-1]$ has rank 1. Since $Z[2,\ldots,n]$ is $Ns.t.n.p.$, we obtain by Theorem 3.5 (iii) that $\det Z2,\ldots,n-1] < 0$, a contradiction. Hence the only option that is left is that the left shadow of $Z[n-1,n|2,3]$ which is $Z[n-1,n|1,2,3]$ has rank 1. The sequence $\big((n-1,1),(n,2)\big)$ is lacunary for the matrix $\tilde{Z}[n-1,n|1,\ldots,n]$ and by Theorem 3.2 we get that

$$0 = \det Z[n-1,n|1,2] = \tilde{z}_{n-1,1} \cdot \tilde{z}_{n2}.$$

Since $\tilde{z}_{n2} = z_{n2} < 0$, we obtain $\tilde{z}_{n-1,1} = 0$.

As the induction hypothesis, suppose that $\tilde{z}_{i1} \geq 0$ and $\tilde{z}_{i1} = 0$ if $\tilde{z}_{i2} = 0$ for $i = k+1, \ldots, n-1$. Then the matrix $\tilde{Z}[k, \ldots, n|1, \ldots, n]$ is a Cauchon matrix. Define for $s = 1, 2, \ldots$

$$(i_s, j_s) := \min\{(i,j) : i_{s-1} < i \leq n, j_{s-1} < j \leq n, \tilde{z}_{ij} \neq 0\},$$

where the minimum is taken with respect to the colexicographical order with $(i_0, j_0) = (k, 1)$. Assume that the sequence that is produced by this procedure is $\big((k,1), (i_1, j_1), \ldots, (i_r, j_r)\big)$. By construction, it is easy to see that this sequence is a lacunary sequence for the Cauchon matrix $\tilde{Z}[k, \ldots, n|1, \ldots, n]$, $i_r = n$ and $j_r \leq n-1$, since $\tilde{z}_{ii} > 0$ for $i = 2, \ldots, n-1$, and $z_{ni} < 0$ for $i = 1, \ldots, n$. Hence by Theorem 3.2, we have

$$\tilde{z}_{k1} = \frac{\det Z[k, i_1, \ldots, i_r|1, j_1, \ldots, j_r]}{\det Z[i_1, \ldots, i_r|j_1, \ldots, j_r]},$$

where the underlying submatrices in the numerator and denominator lie in $Z[2, \ldots, n|1, \ldots, n-1]$. Hence $\tilde{z}_{k1} \geq 0$, since the latter submatrix is $Ns.t.n.p.$

If $\tilde{z}_{k2} = 0$, then by the above procedure construct a lacunary sequence starting from $(k, 2)$ and call the resulting sequence $\big((k,2), (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)\big)$, where $\alpha_r = n$ and $\beta_r \leq n-1$. Then by Theorem 3.2, we get

$$\tilde{z}_{k2} = \frac{\det Z[k, \alpha_1, \ldots, \alpha_r|2, \beta_1, \ldots, \beta_r]}{\det Z[\alpha_1, \ldots, \alpha_r|\beta_1, \ldots, \beta_r]} = 0.$$

By Theorem 3.6, the above ratio can be written as a ratio of two contiguous minors as they lie in $\tilde{Z}[2, \ldots, n|1, \ldots, n-1]$. Hence we obtain

$$\tilde{z}_{k2} = \frac{\det Z[k, k+1, \ldots, k+r|2, 3, \ldots, 2+r]}{\det Z[k+1, \ldots, k+r|3, \ldots, 2+r]} = 0,$$

which implies that $Z[k, k+1, \ldots, k+r|2, 3, \ldots, r+2]$ has rank r.

Using Lemma 4, we proceed similarly as for $k = n-1$. Since neither the rows $k, \ldots, k+r$ nor the columns $2, \ldots, 2+r$ are linearly dependent in $Z[2, \ldots, n|1, \ldots, n-1]$, we consider the right shadow of $Z[k, \ldots, k+r|2, \ldots, 2+r]$ which is $Z[2, \ldots, k+r|2, \ldots, n-1]$. Since $k \geq 3$ and $Z[2, \ldots, n]$

is $Ns.t.n.p.$, we obtain by Theorem 3.5 (iii) $\text{rank}(Z[2,\ldots,k+r]) \geq r+2$. Thus the only option which is left is that the left shadow which is $Z[k,\ldots,n|1,\ldots,2+r]$ has rank r. By construction and since $\tilde{z}_{k2}=0$, it is easy to see that the sequence $\big((k,1),(\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r)\big)$ is lacunary for $\tilde{Z}[k,\ldots,n|1,\ldots,n]$.
Hence by Theorem 3.6, we get that

$$\tilde{z}_{k1} \quad = \quad \frac{\det Z[k,\alpha_1,\ldots,\alpha_r|1,\beta_1,\ldots,\beta_r]}{\det Z[\alpha_1,\ldots,\alpha_r|\beta_1,\ldots,\beta_r]} = 0,$$

since $Z[k,\alpha_1,\ldots,\alpha_r|1,\beta_1,\ldots,\beta_r]$ lies in the left shadow of $Z[k,\ldots,k+r|2,\ldots,2+r]$. This completes the proof of the claim.

Since $\tilde{Z}[2,\ldots,n]$ is a Cauchon matrix with nonzero diagonal entries and by the claim we conclude that $\tilde{Z}$ is a Cauchon matrix.

Furthermore, $\big((1,1),\ldots,(n,n)\big)$ is a lacunary sequence for the Cauchon matrix $\tilde{Z}$. By Theorem 3.2, we get

$$\tilde{z}_{11} = \frac{\det Z}{\det Z[2,\ldots,n]} < 0.$$

Finally, since $\tilde{z}_{i,i-1},\tilde{z}_{i-1,i} > 0$ for $i=3,\ldots,n-1$ the sequences $\big((2,1),(3,2),\ldots,(n,n-1)\big)$ and $\big((1,2),(2,3),\ldots,(n-1,n)\big)$ are lacunary for the Cauchon matrix $\tilde{Z}$. By Theorem 3.2, we get

$$\tilde{z}_{21} = \frac{\det Z[2,\ldots,n|1,\ldots,n-1]}{\det Z[3,\ldots,n|2,\ldots,n-1]} > 0,$$

and

$$\tilde{z}_{12} = \frac{\det Z[1,\ldots,n-1|2,\ldots,n]}{\det Z[2,\ldots,n-1|3,\ldots,n]} > 0.$$

Since all conditions of Theorem 4.5 (b) are satisfied, we conclude that Z is $t.n.p.^{+}$.

In the following we consider the case that $A[2,\ldots,n]$ or $B[2,\ldots,n]$ are singular. For $C \in \mathbb{R}^{n,n}$ we use the following notation $C_\epsilon := C + \epsilon E_{n,n}$, where ϵ is a small positive real number, $C_\epsilon = (c_{ij}(\epsilon))$. If $A[2,\ldots,n]$ or $B[2,\ldots,n]$ are singular, then we replace A, B, and Z by A_ϵ, B_ϵ, and Z_ϵ, respectively, and obviously,

$$A_\epsilon \leq^* Z_\epsilon \leq^* B_\epsilon.$$

Furthermore, A_ϵ and B_ϵ are $t.n.p^+$ for all sufficiently small real numbers ϵ and $A_\epsilon[2,\ldots,n], B_\epsilon[2,\ldots,n]$ are nonsingular. Indeed, application of Laplace expansion to A_ϵ and $A_\epsilon[2,\ldots,n]$ along their last row yields

$$\det A_\epsilon = \det A + \epsilon \det A[1,\ldots,n-1] > 0,$$

and

$$\det A_\epsilon[2,\ldots,n] \;=\; \det A[2,\ldots,n] + \epsilon \det A[2,\ldots,n-1] < 0,$$

for all sufficiently small positive numbers ϵ since by Theorem 4.4 (ii), $\det A[2,\ldots,n-1] < 0$. Moreover, for all $\alpha, \beta \in Q_{l,n}$ with $\alpha_l \neq n$ or $\beta_l \neq n$, the minors $\det A_\epsilon[\alpha|\beta]$ are also minors of A and therefore nonpositive. If $l < n$, $\alpha_l = n$, and $\beta_l = n$, Laplace expansion along the last row of $A_\epsilon[\alpha|\beta]$ yields

$$\det A_\epsilon[\alpha|\beta] = \det A[\alpha|\beta] + \epsilon \det A[\alpha_{\hat{l}}|\beta_{\hat{l}}] \leq 0.$$

In the same way we obtain that B_ϵ is $t.n.p.^+$ with $B_\epsilon[2,\ldots,n]$ nonsingular. By proceeding as above we may conclude that Z_ϵ is $t.n.p.^+$. for all sufficiently small positive numbers ϵ. Since $\det Z > 0$ and by letting $\epsilon \to 0$, we obtain that Z is $t.n.p.^+$. □

Remark 2.

(i) In the case that all minors of A and B are nonzero, the statement of Theorem 4.8 follows from [17, Theorem 1].

(ii) Reference [11] may be used to derive an interval property of the $t.n.p.^+$ matrices (without the assumption of Theorem 4.8 that $b_{nn} < 0$), however, at the expense of checking an exponentially (in n) growing number of vertex matrices: Let $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ and $D_\epsilon := \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$ for $\epsilon \in \{-1, 1\}^n$. If the vertex matrices $\frac{1}{2}\left[(A + B) + D_\epsilon(B - A)D_{\epsilon'}\right]$ are $t.n.p.^+$ for all $\epsilon, \epsilon' \in \{-1, 1\}^n$, then $Z \in [A, B]$ is $t.n.p.^+$, too. In fact, since A and B are vertex matrices, we obtain by the beginning of the proof of Theorem 4.8 that $\det(Z) > 0$. Furthermore, since all proper minors of the vertex matrices are nonpositive, we conclude by [11, Theorem 5.5] that all proper minors of Z are nonpositive.

Using Theorem 4.8, we obtain the interval property of further classes of $NsSR$ matrices.

Theorem 4.9. *Let $A, B \in \mathbb{I}(\mathbb{R}^{n,n})$ with $Z \in [A, B]$ and let $A = \big(a_{ij}\big)$ and $B = \big(b_{ij}\big)$ be $NsSR$ matrices with the same signature $\epsilon = (\epsilon_1, \dots, \epsilon_n)$. Let ϵ be one of the following signatures:*

(i) $\epsilon_i = (-1)^{i+1}, i = 1, \dots, n-1, \epsilon_n = (-1)^n$,

(ii) $\epsilon_i = (-1)^{\frac{i(i-1)}{2}+1}, i = 1, \dots, n-1, \epsilon_n = (-1)^{\frac{n(n-1)}{2}}$,

(iii) $\epsilon_i = (-1)^{\frac{i(i+1)}{2}+1}, i = 1, \dots, n-1, \epsilon_n = (-1)^{\frac{n(n+1)}{2}}$,

and assume that in case

(i) $b_{nn} > 0$,

(ii) $\max\{a_{1n}, b_{1n}\} < 0$,

(iii) $\min\{a_{1n}, b_{1n}\} > 0$.

Then Z is $NsSR$ with signature ϵ.

Proof. The matrices $D := \operatorname{diag}(-1, -1, \dots, -1)$ and T are $NsSR$ with signatures $\epsilon_i = (-1)^i$ and $\epsilon_i = (-1)^{\frac{i(i-1)}{2}}$, $i = 1, \dots, n$, respectively, and $D^{-1} = D$ and $T^{-1} = T$. Hence if A and B are $NsSR$ matrices with the same signature which is given in one of (i)$-$(iii), then by Lemma 2 the following hold. If ϵ is the signature in case (i), then DA and DB, (ii), then TA and TB, (iii), then DTA and DTB are *t.n.p.*$^+$ with negative entries in their bottom right position, and by Theorem 4.8, Z is an $NsSR$ matrix with the same signature. □

By Theorem 4.9 and by the respective interval properties of *NsTN* matrices [4], of *Nst.n.p.* matrices [6], and of *NsSR* matrices with further signatures [7, Theorems 5.2 and 5.3], [18, IP 3.2.6], we summarize the interval property of the following classes of $NsSR$ matrices. Let A, B, $Z \in \mathbb{R}^{n,n}$ (with a possible requirement on the sign of one coefficient of either A or B, see, e.g., Theorem 4.9) be such that $A \le^* Z \le^* B$ and let A and B be *NsSR* matrices with the same signature $\epsilon = (\epsilon_1, \dots, \epsilon_n)$. Let ϵ be one of the following eight periodic (of length 4) signatures

$$
\begin{gathered}
(1, 1, 1, 1, \dots), (-1, -1, -1, -1, \dots), \\
(1, -1, 1, -1, \dots), (-1, 1, -1, 1, \dots), \\
(1, 1, -1, -1, \dots), (-1, -1, 1, 1, \dots), \\
(1, -1, -1, 1, \dots), (-1, 1, 1, -1, \dots),
\end{gathered}
$$

or of one of the eight signatures obtained from these sequences by reversing the sign of ϵ_n. Then Z is $NsSR$ with the same signature ϵ. In particular, in case $n = 4$, all possible signatures are covered.

Conclusion In this paper, we have provided by using the Cauchon algorithm a characterization of the matrices having all their proper minors nonpositive and a positive determinant, the class of the $t.n.p.^{+}$ matrices. Using this characterization, we have shown that these and related sign regular matrices possess the so-called interval property. This result provides further classes of nonsingular sign regular matrices which have the interval property [6]. In a future paper, we will remove by preturbation methods our present restriction that the entry in the bottom right position in a $t.n.p.^{+}$ matrix is negative.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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