

ON THE NUMBER OF SQUARE CLASSES  
OF A FIELD OF FINITE LEVEL

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ABSTRACT. The *level question* is, whether there exists a field  $F$  with finite *square class number*  $q(F) := |F^\times/F^{\times 2}|$  and finite level  $s(F)$  greater than four. While an answer to this question is still not known, one may ask for lower bounds for  $q(F)$  when the level is given.

For a nonreal field  $F$  of level  $s(F) = 2^n$ , we consider the filtration of the groups  $D_F(2^i)$ ,  $0 \leq i \leq n$ , consisting of all the nonzero sums of  $2^i$  squares in  $F$ . Developing further ideas of A. Pfister, P. L. Chang and D. Z. Djoković and by the use of combinatorics, we obtain lower bounds for the invariants  $\bar{q}_i := |D_F(2^i)/D_F(2^{i-1})|$ , for  $1 \leq i \leq n$ , in terms of  $s(F)$ . As a consequence, a field with finite level  $\geq 8$  will have at least 512 square classes. Further we give lower bounds on the cardinalities of the Witt ring and of the 2-torsion part of the Brauer group of such a field.

1 INTRODUCTION

Let  $F$  be a field. The *level of  $F$* , denoted by  $s(F)$ , is defined as the least positive integer  $m$  such that  $-1$  is a sum of  $m$  squares in  $F$  whenever such an integer exists and  $\infty$  otherwise. For fields of positive characteristic this invariant can take only the values 1 and 2, depending just on whether  $-1$  is a square in  $F$  or not. Fields of level  $\infty$ , i.e. in which  $-1$  is not a sum of squares, are called *real fields* and an equivalent condition to  $s(F) = \infty$  is the existence of an ordering on  $F$ . Fields of finite level are also called *nonreal fields*.

For a long time it has been an open question which values exactly occur as the level of some field. The complete solution to this problem was given by A. Pfister in [10] and it inspired a big part of later advances in the theory of quadratic forms, e.g. the development of the theory of *Pfister forms* and the investigation of isotropy behaviors of quadratic forms under function field extensions.

Pfister proved that the level of a nonreal field is always a power of 2 [10, Satz 4] and further that, if  $F$  is any real field (e.g.  $\mathbb{Q}$  or  $\mathbb{R}$ ) and  $n \geq 0$ , then the function field of the projective quadric  $X_0^2 + \cdots + X_{2^n}^2 = 0$  over  $F$  has level  $2^n$  [10, Satz 5]. These were the first examples of nonreal fields of level greater than 4 and, actually, still no examples of an essentially different kind are known.

In general it remains a difficult problem to determine the level of a given field of characteristic zero. For an overview on what is known about levels of common types of fields we refer to [8, Chap. XI, Section 2]. In the same book T. Y. Lam also mentions the following question [8, p. 333]:

1.1. LEVEL QUESTION. *Does there exist a field  $F$  such that  $4 < s(F) < \infty$  and such that  $F^\times/F^{\times 2}$  is finite?*

Here and in the sequel we denote by  $F^\times$  the multiplicative group of  $F$  and by  $F^{\times 2}$  the subgroup of nonzero squares in  $F$ . The quotient  $F^\times/F^{\times 2}$  is called the *square class group of  $F$* . We call  $q(F) := |F^\times/F^{\times 2}|$  the *square class number of  $F$* . Another subgroup of  $F^\times$  of importance is the group of nonzero sums of squares in  $F$ , denoted as  $\sum F^{\times 2}$ .

Further, for any  $m \in \mathbb{N}$  we denote by  $D_F(m)$  the set of elements of  $F^\times$  which can be written as a sum of  $m$  squares over  $F$ . Pfister has shown that  $D_F(m)$  is a group whenever  $m$  is a power of 2 [10, Satz 9]. We thus have the following group filtration for  $\sum F^{\times 2}$ :

$$F^{\times 2} \subsetneq D_F(2) \subsetneq D_F(4) \subsetneq \cdots \subsetneq D_F(2^{i-1}) \subsetneq D_F(2^i) \subsetneq \cdots \subset \sum F^{\times 2}. \quad (1.2)$$

If  $F$  is nonreal of level  $2^n$  then we actually have  $D_F(2^n + 1) = \sum F^{\times 2} = F^\times$ . For  $i \geq 1$  we define  $\bar{q}_i(F) := |D_F(2^i)/D_F(2^{i-1})|$ . Note that the quotients  $F^\times/F^{\times 2}$  and  $D_F(2^i)/D_F(2^{i-1})$  are 2-elementary abelian groups. So  $q(F)$  and  $\bar{q}_i(F)$  are each either a power of 2 or  $\infty$ .

From (1.2) we see that the inequality

$$q(F) \geq \bar{q}_1(F) \cdots \bar{q}_n(F) \quad (1.3)$$

holds for any  $n \geq 1$ . We will use this in particular when  $s(F) = 2^n$ .

While an answer to the level question is still not known, one may look for lower bounds on  $|F^\times/F^{\times 2}|$  in terms of  $s(F)$ .

One approach is to search for lower bounds on the invariants  $\bar{q}_i(F)$  and to use then (1.3) to obtain a bound for  $q(F)$ . Following this idea, A. Pfister obtained in [11, Satz 18.d] the following estimate for a field  $F$  of level  $2^n$ :

$$q(F) \geq 2^{\frac{n(n+1)}{2}}. \quad (1.4)$$

His proof (see also [8, p. 325]) actually shows for  $1 \leq i \leq n$  that

$$\bar{q}_i(F) \geq 2^{n+1-i}. \quad (1.5)$$

Our standard examples of fields of level 1, 2 and 4, respectively, are the field of complex numbers  $\mathbb{C}$ , the finite field  $\mathbb{F}_3$  and  $\mathbb{Q}_2$ , the field of dyadic numbers. These examples show that (1.4) is best possible for  $n \leq 2$ . For higher  $n$ , however, P. L. Chang has improved the bound using combinatorics. In [1] he shows that  $q(F) \geq 128$  for a field  $F$  of level eight and further that  $q(F) \geq 16 \cdot \frac{2^s}{s^2}$  for any nonreal field  $F$  of level  $s \geq 16$ . His approach has been refined by D. Ž. Djoković in [2], leading to the following estimate:

$$q(F) \geq 2 \cdot \sum_{i=1}^{s/2} \frac{1}{s+2-i} \binom{s+1}{i} > \frac{2^s}{s}. \quad (1.6)$$

Their method does not provide any information about the invariants  $\bar{q}_i(F)$ .

The aim of the present work is to extend this method and to get lower bounds for the invariants  $\bar{q}_i(F)$  with respect to  $s(F)$  which improve (1.5). The combinatorial aspect is postponed to the two appendices where a certain coloring problem for (hyper-)graphs is considered.

We use common notations and results from quadratic form theory; the standard references are [8] and [12]. (Note that the uncomfortable case of characteristic 2 is implicitly excluded whenever we deal with a field of level greater than 1.) For isometry of quadratic forms we use the symbol  $\cong$ . For a quadratic form  $\varphi$  over  $F$  we denote by  $D_F(\varphi)$  the set of nonzero elements of  $F$  represented by  $\varphi$ . We sometimes say just “form” or “quadratic form” to mean “non-degenerate quadratic form”.

A diagonalized quadratic form over  $F$  with coefficients  $a_1, \dots, a_m \in F^\times$  is denoted by  $\langle a_1, \dots, a_m \rangle$ . An  $m$ -fold Pfister form is a quadratic form of the shape  $\langle 1, a_1 \rangle \otimes \dots \otimes \langle 1, a_m \rangle$  and shortly written as  $\langle\langle a_1, \dots, a_m \rangle\rangle$ ; its dimension is  $2^m$ . A neighbor of an  $m$ -fold Pfister form  $\pi$  is a quadratic form  $\varphi$  which is similar to a subform of  $\pi$  and of dimension greater than  $2^{m-1}$ . We know that in this situation  $\varphi$  is isotropic if and only if  $\pi$  is hyperbolic.

By  $W(F)$  we denote the Witt ring of  $F$ , further by  $\text{Br}(F)$  the Brauer group and by  $\text{Br}_2(F)$  its 2-torsion part. In (3.1), (5.4) and (5.5) we shall use Milnor  $K$ -theory. For definitions and properties of the Milnor ring  $k_*F$  and its homogenous components  $k_mF$  ( $m \geq 0$ ) we refer to [9] and [3]. However, we use the notation  $\{a_1, \dots, a_m\}$  instead of  $\ell(a_1) \cdots \ell(a_m)$  for a symbol in  $k_mF$ . We recall that this symbol is zero in  $k_mF$  if and only if the corresponding  $m$ -fold Pfister form  $\langle\langle -a_1, \dots, -a_m \rangle\rangle$  over  $F$  is hyperbolic (see [3, Main Theorem 3.2]). In particular,  $s(F) = 2^n$  is equivalent to  $\{-1\}^n \neq 0$  and  $\{-1\}^{n+1} = 0$  in  $k_*F$ . Everywhere else in the text,  $\{x_1, \dots, x_n\}$  stands simply for the set of elements  $x_1, \dots, x_n$ .

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## 2 SUMS OF SQUARES IN FIELDS

Let  $F$  be a field. For an element  $x \in F$  we define its *length (over  $F$ )* to be the least positive integer  $m$  such that  $x$  can be written as a sum of  $m$  nonzero squares over  $F$  if such an integer exists and  $\infty$  otherwise (i.e. if  $x$  is not a nontrivial sum of nonzero squares over  $F$ ). We denote this value in  $\mathbb{N} \cup \{\infty\}$  by  $\ell_F(x)$ , or just by  $\ell(x)$  whenever the context makes clear over which field  $F$  we are working. Obviously  $\ell_F(x)$  depends on  $x$  only up to multiplication by a nonzero square in  $F$ ; in other words,  $\ell_F(x)$  is an invariant of the square class  $xF^{\times 2}$  whenever  $x \neq 0$ .

For  $m \geq 1$ ,  $D_F(m)$  is by definition the set  $\{x \in F^\times \mid \ell(x) \leq m\}$ . Our investigation into lengths of field elements is based on the following famous result [10, Satz 2]:

2.1. THEOREM (PFISTER). *For any  $i \geq 0$ ,  $D_F(2^i)$  is a subgroup of  $F^\times$ .*

A simple proof within the theory of Pfister forms can be found in [12, 4.4.1. Lemma]. As a consequence of this theorem one gets an inequality linking the lengths of two elements to the length of their product. We include a proof of this result, which is [10, Satz 3].

2.2. LEMMA. *For any  $x, y \in F$  we have the inequalities  $\ell(x + y) \leq \ell(x) + \ell(y)$  and  $\ell(xy) \leq \ell(x) + \ell(y) - 1$ .*

*Proof:* The first inequality is obvious from the definition of the length. The second inequality is trivial if  $xy$  is zero or if  $x$  or  $y$  is not a sum of squares. So we may suppose that both  $x$  and  $y$  are nonzero sums of squares in  $F$ . Let then  $r$  be the least nonnegative integer such that  $x, y \in D_F(2^r)$ . We will prove  $\ell(xy) < \ell(x) + \ell(y)$  by induction on  $r$ . If  $r = 0$  then  $x, y$  and  $xy$  are squares in  $F$  and the inequality is clear. Suppose now that  $r > 0$ . Since  $D_F(2^r)$  is a group we know that  $\ell(xy) \leq 2^r$ . So the inequality is clear if  $2^r < \ell(x) + \ell(y)$ . Otherwise, we may suppose that  $\ell(y) \leq 2^{r-1}$ . By the choice of  $r$  we then have  $2^{r-1} < \ell(x) \leq 2^r$  and may therefore write  $x = a + z$  with  $a, z \in F^\times$  such that  $\ell(a) = 2^{r-1}$  and  $\ell(z) = \ell(x) - 2^{r-1} \leq 2^{r-1}$ . By the induction hypothesis we have  $\ell(zy) < \ell(y) + \ell(z)$ . As  $D_F(2^{r-1})$  is a group we have  $\ell(ay) \leq 2^{r-1}$ . Since  $xy = ay + zy$ , using the first inequality of the statement we obtain finally  $\ell(xy) \leq \ell(ay) + \ell(zy) < 2^{r-1} + \ell(y) + \ell(z) = \ell(x) + \ell(y)$ .  $\square$

According to the definition we gave in the introduction, the *level* of  $F$  is the length of  $-1$  in  $F$ . We may also conclude that  $\ell_F(0) = s(F) + 1$ . Therefore, from any of the inequalities of the lemma we obtain immediately:

2.3. COROLLARY. *For any  $x \in F$  we have  $\ell(x) + \ell(-x) \geq s(F) + 1$ .*  $\square$

2.4. COROLLARY. *Let  $a_1, \dots, a_m \in F^\times$ . If the quadratic form  $\langle a_1, \dots, a_m \rangle$  over  $F$  represents the element  $x \in F$  nontrivially then  $\ell(a_1) + \dots + \ell(a_m) \geq \ell(x)$ .*

*Proof:* If the form  $\langle a_1, \dots, a_m \rangle$  represents  $x \in F$  nontrivially, this means that there are  $x_1, \dots, x_m \in F$ , not all zero, such that  $a_1x_1^2 + \dots + a_mx_m^2 = x$ . We may suppose that  $x_i$  is nonzero for  $1 \leq i \leq m'$  and zero for  $m' < i \leq m$ . From the first inequality of the lemma we obtain  $\ell(x) \leq \ell(a_1x_1^2) + \dots + \ell(a_{m'}x_{m'}^2) = \ell(a_1) + \dots + \ell(a_{m'})$ .  $\square$

For  $i \geq 0$ , we say that the elements  $a_1, \dots, a_m \in F^\times$  are *independent modulo*  $D_F(2^i)$  if in  $F^\times/D_F(2^i)$ , considered as an  $\mathbb{F}_2$ -vectorspace, the classes represented by  $a_1, \dots, a_m$  are  $\mathbb{F}_2$ -linear independent.

**2.5. PROPOSITION.** *For  $i \geq 2$ , let  $a, b \in D_F(3 \cdot 2^{i-2}) \setminus D_F(2^{i-1})$  and  $c \in D_F(2^i)$  such that  $\ell(a + b + c) > 2^{i+1}$ . Then the elements  $a, b$  and  $c$  of  $D_F(2^i)$  are independent modulo  $D_F(2^{i-1})$ .*

*Proof:* We have to show that  $a, b, c, ab, ac, bc, abc \notin D_F(2^{i-1})$ . For  $a$  and  $b$  this is already given. We put  $x := a + b + c$ . Each of the quadratic forms  $\langle a, b, c \rangle$ ,  $\langle a, b, abc \rangle$ ,  $\langle 1, ab, ac \rangle$  and  $\langle ac, bc, 1 \rangle$  over  $F$  represents one of the elements  $x, abx, ax$  and  $cx$  and neither of these elements lies in the group  $D_F(2^{i+1})$ . We obtain from (2.4) that each of the numbers  $\ell(a) + \ell(b) + \ell(c)$ ,  $\ell(a) + \ell(b) + \ell(abc)$ ,  $1 + \ell(ab) + \ell(ac)$  and  $\ell(ac) + \ell(bc) + 1$  is greater than  $2^{i+1}$ . Since  $\ell(a) + \ell(b) \leq 3 \cdot 2^{i-1}$  and  $ab, ac, bc \in D_F(2^i)$  we obtain  $\ell(c), \ell(abc) \geq 2^{i-1}$  and further  $\ell(ab) = \ell(ac) = \ell(bc) = 2^i$ .  $\square$

For the rest of this section we fix a sum of squares

$$x = x_1^2 + \dots + x_l^2 \tag{2.6}$$

with  $x_1, \dots, x_l \in F^\times$ ,  $x \in F$  and  $l = \ell_F(x)$ . For a subset  $I \subset \{1, \dots, l\}$  we denote  $x_I := \sum_{i \in I} x_i^2$ . If  $I$  is not empty then we have  $\ell(x_I) = |I|$ . For a real number  $z$  we denote by  $\lceil z \rceil$  the least integer  $\geq z$ .

**2.7. THEOREM.** *Let  $I$  and  $J$  be nonempty proper subsets of  $\{1, \dots, l\}$ . Let  $r$  be a nonnegative integer such that  $x_I x_J \in D_F(2^r)$ . Then the following hold:*

- (i)  $\lceil \frac{|I|}{2^r} \rceil = \lceil \frac{|J|}{2^r} \rceil$ , in particular  $||I| - |J|| < 2^r$ ,
- (ii)  $|I \setminus J|, |J \setminus I| \leq 2 \ell(x_I x_J) - 1 < 2^{r+1}$ ,
- (iii)  $|I \cup J| - |I \cap J| \leq 2^{r+1} + \ell(x_I x_J) - 1 \leq 3 \cdot 2^r - 1$ .

*Proof:* The hypothesis implies that  $x_I$  and  $x_J$  are nonzero elements of  $F$ . We set  $m := \ell(x_I x_J)$  and  $a := \frac{x_J}{x_I}$ . Then  $\ell(a) = m \leq 2^r$ . If  $\nu$  is an integer such that  $|I| \leq \nu 2^r$  then we can write  $x_I$  as a sum of  $\leq \nu$  elements of  $D_F(2^r)$ . As  $D_F(2^r)$  is a group,  $x_J = ax_I$  can also be written as a sum of  $\leq \nu$  elements of  $D_F(2^r)$  which means that  $|J| = \ell(x_J) \leq \nu 2^r$ . By symmetry we obtain for any  $\nu \in \mathbb{N}$  that  $|I| \leq \nu 2^r$  if and only if  $|J| \leq \nu 2^r$ . This shows (i).

We compute  $x_{I \cup J} = x_{I \setminus J} + x_J = (1+a)x_{I \setminus J} + ax_{I \cap J}$  and then substitute  $y := (1+a)x_{I \setminus J}$  and  $z := ax_{I \cap J}$  to have  $x_{I \cup J} = y + z$ .

If  $y \neq 0$  then we have  $\ell(y) \leq m + |I \setminus J|$  by (2.2), but also  $\ell(y) \leq 2^{r+1}$  since  $D_F(2^{r+1})$  is a group. If  $z \neq 0$  then (2.2) yields  $\ell(z) \leq m + |I \cap J| - 1$ . Therefore, if at least one of  $y$  and  $z$  is nonzero then we obtain the inequalities  $\ell(y+z) \leq |I| + 2m - 1$  and  $\ell(y+z) \leq 2^{r+1} + m + |I \cap J| - 1$ . Both inequalities remain valid in the case  $y = z = 0$ , since then necessarily  $a = -1$ , whence  $\ell(y+z) = \ell(0) = m+1$ . As  $|I \cup J| = \ell(y+z)$  we obtain (ii) by symmetry from the first and (iii) from the second inequality.  $\square$

For  $m = 1$  this leads to an observation made in the proof of [1, Theorem 1]:

2.8. COROLLARY (CHANG). *Let  $I$  and  $J$  be as in the theorem. If  $x_I$  and  $x_J$  lie in the same square class then both sets have the same cardinality and differ by at most one element.*  $\square$

2.9. COROLLARY. *Let  $I$  and  $J$  be as in the theorem with  $|I| = |J| = 2^i$ ,  $i \geq 2$ . If  $x_I$  and  $x_J$  represent the same class modulo  $D_F(2^{i-1})$  then  $|I \cap J| \geq 2^{i-2} + 1$ .*

*Proof:* If  $x_I$  and  $x_J$  lie in the same class modulo  $D_F(2^{i-1})$  then  $\ell(x_I x_J) \leq 2^{i-1}$ . Applying part (iii) of the theorem for  $r = i - 1$  we obtain  $|I \cup J| - |I \cap J| \leq 3 \cdot 2^{i-1} - 1$ . But our hypothesis here gives  $|I \cup J| = 2 \cdot 2^i - |I \cap J|$ . This together implies  $|I \cap J| > 2^{i-2}$ .  $\square$

### 3 THE INVARIANTS $\bar{q}_i$

For a nonreal field  $F$  of level  $2^n$  we are going to study the invariants  $\bar{q}_i(F) = |D_F(2^i)/D_F(2^{i-1})|$  for  $1 \leq i \leq n$ . In particular, we are interested to know whether Pfister's bounds (1.5) can be improved.

First we note that the bound  $\bar{q}_n(F) \geq 2$ , obtained from (1.5) for  $i = n$ , just takes into account that  $-1$  represents a nontrivial class in the group  $D_F(2^n)/D_F(2^{n-1})$ . In spite of the simple argument, this bound is optimal for every  $n \geq 1$ . More precisely, for any  $n \geq 1$  there is a field  $F$  of level  $2^n$  such that  $F^\times = D_F(2^{n-1}) \cup -D_F(2^{n-1})$ . The construction of such an example will be included in a forthcoming paper of the author.

We now turn to consider  $\bar{q}_{n-1}(F)$ . For  $i = n - 1$ , (1.5) gives  $\bar{q}_{n-1}(F) \geq 4$ . The example  $F = \mathbb{Q}_2$  shows that this bound is optimal for  $n = 2$ .

3.1. THEOREM. *Let  $F$  be a field of level  $2^n$  with  $n \geq 3$ . Then  $\bar{q}_{n-1}(F) \geq 16$ .*

*Proof:* Since  $\ell(0) = 2^n + 1$  and  $n \geq 3$ , we may choose elements  $a_1, a_2, a_3 \in F^\times$  such that  $a_1 + a_2 + a_3 = 0$  and  $2^{n-2} + 1 \leq \ell(a_i) \leq 3 \cdot 2^{n-3}$  for  $i = 1, 2, 3$ . Then by (2.5),  $a_1, a_2$  and  $a_3$  are independent modulo  $D_F(2^{n-2})$ . Let  $H$  be the subgroup of  $D_F(2^{n-1})$  generated by  $D_F(2^{n-2})$  and the elements  $a_1, a_2$  and  $a_3$ . Since  $|H/D_F(2^{n-2})| = 8$  it remains to show that  $H \neq D_F(2^{n-1})$ .

To this aim, we will calculate in the Milnor ring  $k_*F$ . For  $i = 1, 2, 3$  we fix the symbols  $\beta_i := \{a_1a_2a_3, a_i\}$  and  $\gamma_i := \{-a_1a_2a_3, -a_i\}$  in  $k_2F$ . Let  $\varepsilon$  denote the element  $\{-1\}$  in  $k_1F$ . Since  $s(F) = 2^n$  we have  $\varepsilon^n \neq 0$ . As  $a_1, a_2, a_3 \in D_F(2^{n-1})$  we observe that  $\beta_1 + \beta_2 + \beta_3 = \{-1, a_1a_2a_3\}$  is annihilated by  $\varepsilon^{n-2}$  and that  $\varepsilon^{n-2}(\beta_i + \gamma_i) = \varepsilon^{n-2}(\{a_1a_2a_3, -1\} + \{-1, a_i\} + \{-1, -1\}) = \varepsilon^n$  for  $i = 1, 2, 3$ .

If  $\varepsilon^{n-2}\beta_i \neq 0$  in  $k_nF$  for some  $i$  then by the above relations we may suppose that  $\varepsilon^{n-2}\beta_i \neq 0$  for  $i = 1, 2$  and  $\varepsilon^{n-2}\beta_3 \neq \varepsilon^n$ , i.e.  $\varepsilon^{n-2}\gamma_3 \neq 0$ . Using that  $a_1 + a_2 + a_3 = 0$  we compute  $\{-a_2, -a_3\} = \{a_1, -a_2a_3\} = \beta_1$  and equally  $\{-a_1, -a_3\} = \beta_2$ . Since none of  $\beta_1, \beta_2$  and  $\gamma_3$  is annihilated by  $\varepsilon^{n-2}$ , the symbols  $\varepsilon^{n-2}\{-a_2, -a_3\}, \varepsilon^{n-2}\{-a_1, -a_3\}$  and  $\varepsilon^{n-2}\{-a_1a_2, -a_3\}$  in  $k_nF$  are all nonzero. Therefore the Pfister forms  $2^{n-2} \times \langle\langle a_2, a_3 \rangle\rangle, 2^{n-2} \times \langle\langle a_1, a_3 \rangle\rangle$  and  $2^{n-2} \times \langle\langle a_1a_2, a_3 \rangle\rangle$  are anisotropic. Further,  $2^{n-2} \times \langle\langle 1, a_3 \rangle\rangle \cong 2^n \times \langle 1 \rangle$  is anisotropic since  $s(F) = 2^n$ . This shows that  $-1, -a_1, -a_2, -a_1a_2 \notin D_F(2^{n-2} \times \langle\langle a_3 \rangle\rangle)$ . As the group  $D_F(2^{n-2} \times \langle\langle a_3 \rangle\rangle)$  contains the subgroup  $D_F(2^{n-2})$  and the element  $a_3$  we conclude that  $D_F(2^{n-2} \times \langle\langle a_3 \rangle\rangle) \cap -H = \emptyset$ . On the other hand, since  $\ell(-a_3) \leq \ell(a_1) + \ell(a_2) \leq 3 \cdot 2^{n-2}$  we can write  $-a_3 = x + y$  with  $x \in D_F(2^{n-1}), y \in D_F(2^{n-2})$  and obtain  $-x = y + a_3 \in D_F(2^{n-2} \times \langle\langle a_3 \rangle\rangle) \cap -D_F(2^{n-1})$ .

Now we study the case where  $\varepsilon^{n-2}\beta_i = 0$  for  $i = 1, 2, 3$ . As  $\varepsilon^{n-2}\beta_i = \varepsilon^{n-2}\{-a_1a_2a_3, a_i\}$ , this means that the Pfister form  $2^{n-2} \times \langle\langle a_1a_2a_3, -a_i \rangle\rangle$  is hyperbolic for  $i = 1, 2, 3$ . We conclude that  $H \subset D_F(2^{n-2} \times \langle\langle a_1a_2a_3 \rangle\rangle)$ . As the Pfister form  $2^{n-1} \times \langle\langle a_1a_2a_3 \rangle\rangle \cong 2^n \times \langle 1 \rangle$  is anisotropic we have  $-1 \notin D_F(2^{n-2} \times \langle\langle a_1a_2a_3 \rangle\rangle)$  and therefore  $D_F(2^{n-2} \times \langle\langle a_1a_2a_3 \rangle\rangle) \cap -H = \emptyset$ . Since  $-a_1a_2a_3 = a_1^2a_2 + a_2^2a_1$  we have  $\ell(-a_1a_2a_3) \leq \ell(a_2) + \ell(a_1) \leq 3 \cdot 2^{n-2}$  and may therefore write  $-a_1a_2a_3 = x + y$  with  $x \in D_F(2^{n-1})$  and  $y \in D_F(2^{n-2})$  to obtain this time  $-x = y + a_1a_2a_3 \in D_F(2^{n-2} \times \langle\langle a_1a_2a_3 \rangle\rangle) \cap -D_F(2^{n-1})$ .

In both cases we have found an element  $x \in D_F(2^{n-1}) \setminus H$ . □

While the lower bound on  $\bar{q}_{n-1}$  of the last theorem is based upon several algebraic arguments, the improvement (with respect to (1.5)) for the lower bounds on  $\bar{q}_i(F)$  for  $2 \leq i \leq n-2$  which we present now, is obtained by combinatorial reasoning, developed in appendix A.

For integers  $0 \leq k \leq l$  we denote by  $\mathcal{P}_k^l$  the set of subsets of  $\{1, \dots, l\}$  with exactly  $k$  elements.

3.2. THEOREM. *Let  $F$  be a field of level  $2^n$ . Then*

$$\bar{q}_i(F) \geq \begin{cases} 2^7 & \text{for } i = n-2 \geq 3, \\ 2^{(n-i)(2^{n-i}+1)+1} & \text{for } \frac{n+1}{2} < i \leq n-3, \\ 2^{(n-i)(2^{i-2}+1)+1} & \text{for } 2 \leq i \leq \frac{n+1}{2}. \end{cases}$$

*Proof:* We fix elements  $x_1, \dots, x_{2^n} \in F^\times$  such that  $x_1^2 + \dots + x_{2^n}^2 = -1$ . For a subset  $J \subset \{1, \dots, 2^n\}$  we denote  $x_J := \sum_{j \in J} x_j^2$ .

Let  $2 \leq i \leq \frac{n+1}{2}$ . We consider the map  $f : \mathcal{P}_i^{2^n} \rightarrow D_F(2^i)/D_F(2^{i-1})$  which sends a  $2^i$ -subset  $J \subset \{1, \dots, 2^n\}$  to the class  $x_J D_F(2^{i-1})$ . By (2.9), if  $J_1, J_2 \in$

$\mathcal{P}_2^{2^n}$  are such that  $f(J_1) = f(J_2)$  then  $|J_1 \cap J_2| \geq 2^{i-2} + 1$ . Therefore (A.8) in appendix A shows  $|D_F(2^i)/D_F(2^{i-1})| \geq |Im(f)| > 2^r$  for  $r := (n-i)(2^{i-2}+1)$ . Since  $D_F(2^i)/D_F(2^{i-1})$  is a 2-elementary abelian group it must then have at least  $2^{r+1}$  elements. This establishes the third case in the statement.

In the remaining cases we cannot apply (A.8) directly for  $i$  and  $m := n$ . In the case  $\frac{n+1}{2} < i \leq n-3$  we have  $n \geq 8$  and  $i \geq 5$  and define  $n' := 2(n-i+1)$  and  $i' := n-i+2 = \frac{n'}{2} + 1$ . In the case  $i = n-2$  and  $n \geq 5$  we set instead  $n' := 5$  and  $i' := 3 = \frac{n'+1}{2}$ . Note that in both cases  $n' - i' = n - i$ .

For  $1 \leq \nu \leq 2^{n'}$  let  $J_\nu := \{(\nu-1) \cdot 2^{n-n'} + 1, \dots, \nu \cdot 2^{n-n'}\}$  and  $y_\nu := x_{J_\nu}$ . This yields  $y_1 + \dots + y_{2^{n'}} = -1$  and  $\ell(y_\nu) = |J_\nu| = 2^{n-n'}$  for  $1 \leq \nu \leq 2^{n'}$ . Now we consider the map  $f' : \mathcal{P}_2^{2^{n'}} \rightarrow D_F(2^i)/D_F(2^{i-1})$  which sends a  $2^{i'}$ -subset  $N \subset \{1, \dots, 2^{n'}\}$  to the class  $(\sum_{\nu \in N} y_\nu) D_F(2^{i-1})$ .

Suppose that  $f'(N_1) = f'(N_2)$  for  $N_1, N_2 \in \mathcal{P}_2^{2^{n'}}$ . For  $k = 1, 2$  let  $I_k := \bigcup_{\nu \in N_k} J_\nu \in \mathcal{P}_2^{2^n}$ . Since by hypothesis  $\sum_{\nu \in N_1} y_\nu = x_{I_1}$  and  $\sum_{\nu \in N_2} y_\nu = x_{I_2}$  lie in the same class of  $D_F(2^i)/D_F(2^{i-1})$ , (2.9) shows that  $|I_1 \cap I_2| \geq 2^{i-2} + 1$  and it follows that  $|N_1 \cap N_2| \geq 2^{i-2-(n-n')} + 1 = 2^{i'-2} + 1$ .

Having established this intersection property of  $f'$ , we obtain from (A.8) that  $|D_F(2^i)/D_F(2^{i-1})| \geq |Im(f')| > 2^{r'}$  holds for  $r' := (n' - i')(2^{i'-2} + 1)$ . As before, we conclude that  $|D_F(2^i)/D_F(2^{i-1})| \geq 2^{r'+1}$ . This finishes the proof since  $r' = 6$  in case  $i = n-2$  and  $r' = (n-i)(2^{n-i} + 1)$  otherwise.  $\square$

#### 4 NONREAL FIELDS WITH $\bar{q}_1$ EQUAL TO THE LEVEL

From (1.5) we know that  $\bar{q}_1(F) \geq s(F)$  holds for any nonreal field  $F$ . This bound is optimal for fields of level 1, 2 and 4 as the standard examples show (see introduction). For nonreal fields of higher level, however, there is still no known example where  $\bar{q}_1(F) < \infty$ .

We show that  $\bar{q}_1(F) = s(F) < \infty$  is a rather strong condition, with several consequences on the quadratic form structure of  $F$ . In particular, for  $s(F) \geq 8$  it implies that  $\bar{q}_2(F) \geq \frac{s(F)^2}{2}$  (4.9).

Let  $\xi$  be an element of length  $l \geq 3$  of  $F$ . We fix a representation of  $\xi$  as a sum of  $l$  squares

$$\xi = x_1^2 + \dots + x_l^2 \tag{4.1}$$

with  $x_1, \dots, x_l \in F^\times$ . Let  $f : \mathcal{P}_2^l \rightarrow D_F(2)/F^{\times 2}$  be the function which sends a (nonordered) pair of distinct  $i, j \leq l$  to the square class of  $x_i^2 + x_j^2$ . Considering the elements of  $D_F(2)/F^{\times 2}$  as a set of colors, we can interpret  $f$  as an edge-coloring of a complete graph in  $l$  vertices  $v_1, \dots, v_l$ . We denote this graph together with its edge-coloring  $f$  by  $\mathcal{G}$ . If in this graph two edges  $[v_i, v_j]$  and  $[v_{i'}, v_{j'}]$  are of the same color (with  $\{i, j\}, \{i', j'\} \in \mathcal{P}_2^l$ ) this means that  $x_i^2 + x_j^2$

and  $x_i^2 + x_j^2$  lie in the same square class of  $F$ , which by (2.8) implies that the sets  $\{i, j\}$  and  $\{i', j'\}$  intersect. In other words, two edges of the same color in  $\mathcal{G}$  need to have a vertex in common, i.e.  $\mathcal{G}$  is a *CC-graph* in the terminology of appendix B.

We get from (B.1) that at least  $l - 2$  colors appear in  $\mathcal{G}$ . Furthermore, since  $x_1^2 + \dots + x_l^2$  is of length  $l$ , no sum  $x_i^2 + x_j^2$  with  $i \neq j$  can be a square. This gives a proof of [13, Theorem 1]:

4.2. PROPOSITION (TORT). *In (4.1), the partial sums  $x_i^2 + x_j^2$  with  $1 \leq i < j \leq l$  represent at least  $l - 2$  different nontrivial classes of  $D_F(2)/F^{\times 2}$ .  $\square$*

Let now  $F$  be a nonreal field of level  $s = 2^n$ . We then can choose  $\xi := 0$ , which is of length  $s + 1$  over  $F$ , and write (4.1) as

$$0 = x_1^2 + \dots + x_{s+1}^2. \quad (4.3)$$

By the above proposition the partial sums  $x_i^2 + x_j^2$  (with  $1 \leq i < j \leq s + 1$ ) represent at least  $s - 1$  nontrivial classes of  $D_F(2)/F^{\times 2}$ . This shows:

4.4. COROLLARY. *Let  $F$  be a nonreal field of level  $s$ . Then  $\bar{q}_1(F) \geq s$ . Moreover, if  $\bar{q}_1(F) = s$  then, given any representation (4.3) of zero as a sum of  $s + 1$  nonzero squares over  $F$ , every nontrivial class of  $D_F(2)/F^{\times 2}$  is represented by a partial sum  $x_i^2 + x_j^2$  with  $1 \leq i < j \leq s + 1$ .  $\square$*

Given a subgroup  $G \subset F^\times/F^{\times 2}$  of finite order  $2^m$  we may choose an irredundant set of representatives  $a_1, \dots, a_{2^m} \subset F^\times$  of the square classes in  $G$  and define the quadratic form  $\pi_G := \langle a_1, \dots, a_{2^m} \rangle$ . Up to isometry, this form does only depend on  $G$  and not on the particular choice of the  $a_i$ . If we choose the  $a_i$  such that  $a_1, \dots, a_m$  are independent modulo  $F^{\times 2}$  then  $\pi_G$  is equal to  $\langle\langle a_1, \dots, a_m \rangle\rangle$ , hence  $\pi_G$  is an  $m$ -fold Pfister form. If  $\bar{q}_1(F)$  is finite we write  $\pi_{D(2)}$  for  $\pi_G$  with  $G := D_F(2)/F^{\times 2}$ .

4.5. PROPOSITION. *Let  $F$  be a nonreal field with  $s(F) > 1$  and  $\bar{q}_1(F) < \infty$ . Then  $\pi_{D(2)}$  is hyperbolic.*

*Proof:* Let  $s := s(F)$ . Given a representation (4.3) of zero as sum of  $s + 1$  squares over  $F$  we define  $a_i := x_{2i-1}^2 + x_{2i}^2$  for  $1 \leq i \leq s/2$ . By (2.8) the  $a_i$  lie in distinct nontrivial square classes. Since  $a_1 + \dots + a_{s/2} + x_{s+1}^2 = 0$  the form  $\langle 1, a_1, \dots, a_{s/2} \rangle$  is isotropic. On the other hand, this is a subform of the Pfister form  $\pi_{D(2)}$ , which then must be hyperbolic.  $\square$

4.6. LEMMA. *Let  $H$  be a subgroup of  $F^\times$  containing  $F^{\times 2}$  such that  $H/F^{\times 2}$  is of order  $2^m$  with  $m \geq 2$ . If  $a, b, c, d \in H$ , lie in distinct square classes then there are  $a_3, \dots, a_m \in H$  such that  $\pi_H = \langle a, b, c, d \rangle \otimes \langle\langle a_3, \dots, a_m \rangle\rangle$ .*

*Proof:* It is easy to verify that, given four distinct elements  $t, u, v, w$  in a 2-elementary abelian group  $G$  there exists a subgroup  $K$  of index 4 in  $G$  such that  $t, u, v, w$  represent the four classes of  $G/K$ .

We apply this fact to the square classes  $aF^{\times 2}, bF^{\times 2}, cF^{\times 2}$  and  $dF^{\times 2}$  in  $G := H/F^{\times 2}$ . A subgroup  $K$  with the stated property must have order  $2^{m-2}$ . We choose elements  $a_3, \dots, a_m \in F^\times$  such that their square classes form an  $\mathbb{F}_2$ -basis of  $K$ . The rest is clear.  $\square$

4.7. PROPOSITION. *Let  $F$  be a field with  $\bar{q}_1(F) = s(F) = 2^n$ ,  $n \geq 2$ , and let  $a, b, c, d$  be elements of  $D_F(2)$  which lie in distinct square classes.*

- (a) *If  $a \notin F^{\times 2}$  then  $D_F(\langle 1, 1 \rangle) \cap D_F(\langle 1, a \rangle) = \{1, a\}F^{\times 2}$ .*
- (b) *If  $x \in D_F(\langle 1, a \rangle) \cap D_F(\langle 1, b \rangle) \cap D_F(\langle 1, c \rangle)$  then  $\ell(-x) = 2^n$ .*
- (c)  *$D_F(\langle a, b \rangle) \cap D_F(\langle c, d \rangle) = \emptyset$ .*
- (d) *If  $n \geq 3$  then  $D_F(\langle a, b \rangle) \cap D_F(\langle a, c \rangle) \cap D_F(\langle b, c \rangle) = \emptyset$ .*
- (e) *If  $x \in D_F(\langle 1, a \rangle) \cap D_F(\langle 1, b \rangle)$  then  $\ell(cx) = 4$  or  $\ell(-x) \geq 2^n - 1$ .*

*Proof:* (a) Given  $a$  and  $b$  lying in distinct nontrivial classes of  $D_F(2)/F^{\times 2}$  we may choose  $a_3, \dots, a_{2^n-1} \in D_F(2)$  such that  $\varphi := \langle 1, a, b, a_3, \dots, a_{2^n-1} \rangle$  is a neighbor of the Pfister form  $\pi_{D(2)}$  which is hyperbolic by the last proposition. So  $\varphi$  is isotropic. Now  $b \in D_F(\langle 1, a \rangle)$  would imply that  $\varphi$  is isometric to  $\langle 1, 1, ab, a_3, \dots, a_{2^n-1} \rangle$  which is a subform of  $2^n \times \langle 1 \rangle$ . This is impossible since the latter form is anisotropic by the hypothesis that  $s(F) = 2^n$ .

(b) Let  $x \in D_F(\langle 1, a \rangle) \cap D_F(\langle 1, b \rangle) \cap D_F(\langle 1, c \rangle)$  where  $a, b, c \in D_F(2)$  are distinct modulo squares. Then clearly  $\ell(x) \leq 3$  and we have also  $x \in D_F(\langle 1, abc \rangle)$  (with  $-a, -b$  and  $-c$  also  $-abc$  lies in  $D_F(\langle 1, -x \rangle)$ ). It follows from (a) that  $\ell(x) \neq 2$ . If  $x$  is a square then  $\ell(-x) = \ell(-1) = 2^n$ . Otherwise we must have  $\ell(x) = 3$ . Then none of  $a, b, c, abc$  can be a square. Further  $\ell(-x) \geq 2^n - 2$  by (2.3). Thus (4.2) shows that, in a representation of  $-x$  as sum of  $\ell(-x)$  squares over  $F$ , the partial sums of length two lie in at least  $2^n - 4$  distinct nontrivial square classes. As  $|D_F(2)/F^{\times 2}| = 2^n$  by hypothesis, at least one of these square classes must also be represented by one of  $a, b, c$  or  $abc$ . Without loss of generality we may suppose that  $-x = y + at^2$  with  $\ell(y) = \ell(-x) - 2$ . Writing  $x = u^2 + av^2$  yields  $0 = x - x = y + u^2 + a(t^2 + v^2)$ . Thus  $2^n + 1 \leq \ell(y) + 3$  and  $2^n \leq \ell(y) + 2 = \ell(-x)$ . Then  $-x = (-1) \cdot x \in D_F(2^n)$  implies  $\ell(-x) = 2^n$ .

(c) By the above lemma there are  $a_3, \dots, a_n \in D_F(2)$  such that  $\pi_{D(2)}$  is equal to  $\langle a, b, c, d \rangle \otimes \langle a_3, \dots, a_n \rangle$ .

Suppose now that there exists an  $x \in D_F(\langle a, b \rangle) \cap D_F(\langle c, d \rangle)$ . Then  $\langle a, b, c, d \rangle \cong \langle x, abx, x, cdx \rangle$ , which is similar to  $\langle 1, 1, 1, abcd \rangle$ . Hence  $\pi_{D(2)}$  is similar to  $\langle 1, 1, 1, abcd \rangle \otimes \langle a_3, \dots, a_n \rangle \cong 2^{n-1} \times \langle 1 \rangle \perp \langle abcd, a_3, \dots, a_n \rangle$ . It follows that the form  $(2^{n-1} + 1) \times \langle 1 \rangle$  is a Pfister neighbor of  $\pi_{D(2)}$ , hence isotropic since  $\pi_{D(2)}$  is hyperbolic. This is a contradiction to  $s(F) = 2^n$ .

(d) After multiplying by  $a$  in the statement we may suppose that  $a = 1$ . Suppose that there exists  $x \in D_F(\langle 1, b \rangle) \cap D_F(\langle 1, c \rangle) \cap D_F(\langle b, c \rangle)$ . It follows  $-b, -c \in D_F(\langle 1, -x \rangle)$ , thus  $bc \in D_F(\langle 1, -x \rangle) \cap D_F(\langle 1, 1 \rangle) \subset D_F(\langle 1, x \rangle)$ . Therefore we have  $\langle 1, b, c, bc \rangle \cong \langle 1, x, bcx, bc \rangle \cong \langle bc, bcx, bcx, bc \rangle$ , whence  $\langle 1, b, c, bc \rangle \cong \langle 1, 1, x, x \rangle$ . Next we choose  $a_3, \dots, a_n \in D_F(2)$  such that  $\pi_{D(2)} \cong \langle 1, b, c, bc \rangle \otimes \langle a_3, \dots, a_n \rangle$  and obtain  $\pi_{D(2)} \cong \langle 1, x, a_3, \dots, a_n \rangle \cong 2^{n-1} \times \langle x \rangle \cong 2^n \times \langle 1 \rangle$ , since  $a_3, \dots, a_n \in D_F(2)$ ,  $n \geq 3$  and  $x \in D_F(4)$ . This is contradictory since  $\pi_{D(2)}$  is hyperbolic but  $s(F) = 2^n$ .

(e) Let  $x \in D_F(\langle 1, a \rangle) \cap D_F(\langle 1, b \rangle)$ . Then, certainly,  $x$  and  $cx$  belong to  $D_F(4)$ . If  $\ell(cx) \leq 2$  then  $\ell(x) \leq 2$  and (2.3) yields  $\ell(-x) \geq 2^n - 1$ . Suppose now  $\ell(cx) = 3$  and write  $cx = e + t^2$  with  $t \in F^\times$  and  $e \in D_F(2)$ . We have  $cx \in D_F(\langle c, ac \rangle) \cap D_F(\langle c, bc \rangle) \cap D_F(\langle 1, e \rangle)$ . Since  $1, c, ac$  and  $bc$  represent distinct square classes, we conclude with (c) that  $e$  and  $c$  lie in the same square class. Therefore  $x \in D_F(\langle 1, a \rangle) \cap D_F(\langle 1, b \rangle) \cap D_F(\langle 1, c \rangle)$ , which by (b) implies  $\ell(-x) = 2^n$ . □

4.8. THEOREM. *Let  $F$  be a nonreal field of level  $s$ , equal to  $\bar{q}_1(F)$ . Any representation (4.3) of zero as a nontrivial sum of  $s + 1$  squares over  $F$  may be reordered in such way that the following holds: for  $\{i, j\}, \{i', j'\} \in \mathcal{P}_2^{s+1}$  the partial sums  $x_i^2 + x_j^2$  and  $x_{i'}^2 + x_{j'}^2$  lie in the same square class if and only if  $\max\{i, j, 3\} = \max\{i', j', 3\}$ .*

*Proof:* Let  $\mathcal{G}$  be a complete graph in  $s + 1$  vertices  $v_1, \dots, v_{s+1}$  and with the edge-coloring given by  $f : \mathcal{P}_2^{s+1} \rightarrow D_F(2)/F^{\times 2}, \{i, j\} \mapsto (x_i^2 + x_j^2)F^{\times 2}$  (see at the beginning of this section). We know from (4.4) that exactly  $s - 1$  colors appear in  $\mathcal{G}$ . Further,  $\mathcal{G}$  does not contain any triangle with three different colors; indeed, such a triangle would correspond to a partial sum of three squares  $x := x_i^2 + x_j^2 + x_k^2$  with  $1 \leq i < j < k \leq s + 1$  where  $a := x_i^2 + x_j^2$ ,  $b := x_i^2 + x_k^2$  and  $c := x_j^2 + x_k^2$  lie in three distinct square classes which is impossible by part (b) of the last proposition since  $\ell(-x) = s - 2$ . Therefore by (B.3),  $\mathcal{G}$  is a total CC-graph.

Since  $\mathcal{G}$  has precisely  $(s + 1) - 2$  colors we obtain from the definition of a total CC-graph in appendix B and the subsequent remarks: the vertices in  $\mathcal{G}$  (and at the same time the  $x_i$ ) may be renumbered in such way that for  $\{i, j\} \in \mathcal{P}_2^{s+1}$  the color of the edge between  $v_i$  and  $v_j$  (i.e. the square class of  $x_i^2 + x_j^2$ ) depends precisely on  $\max\{i, j, 3\}$ . □

4.9. COROLLARY. *Let  $F$  be a nonreal field of level  $s = \bar{q}_1(F) \geq 8$ . Then  $\bar{q}_2(F) \geq \frac{s^2}{2}$ .*

*Proof:* Let  $0 = x_1^2 + \dots + x_{s+1}^2$  be a representation of zero as a nontrivial sum of  $s + 1$  squares over  $F$ . By the theorem we may, after reordering the indices, suppose that for  $\{i, j\} \in \mathcal{P}_2^{s+1}$  the square class of  $x_i^2 + x_j^2$  depends precisely on  $\max\{i, j, 3\}$ .

Defining  $a_i := x_{i+1}^2 + x_{i+2}^2$  for  $1 \leq i \leq s-1$ , we get a system of representatives  $a_1, \dots, a_{s-1}$  of the  $s-1$  nontrivial classes of  $D_F(2)/F^{\times 2}$ . Further we set  $c_{jk} := x_1^2 + x_{j+2}^2 + x_{k+2}^2$  for  $1 \leq j < k \leq s-1$ .

Suppose now that  $b c_{jk} = c_{j'k'}$  for  $b \in D_F(2)$  and  $1 \leq j' < k' \leq s-1$ . Then  $c_{j'k'} \in D_F(\langle 1, a_{j'} \rangle) \cap D_F(\langle 1, a_{k'} \rangle) \cap D_F(\langle b, b a_j \rangle) \cap D_F(\langle b, b a_k \rangle)$ . In view of (b), (c) and (d) of the proposition this is only possible if  $b \in F^{\times 2}$ ,  $j = j'$  and  $k = k'$ .

This shows that the elements  $c_{jk}$  for  $1 \leq j < k \leq s-1$  represent distinct nontrivial classes of  $D_F(4)/D_F(2)$ . Therefore  $\bar{q}_2(F) > \binom{s-1}{2}$ . Since  $s$  is a power of 2, at least 8, and  $\bar{q}_2(F)$  is a power of 2 or infinite we obtain  $\bar{q}_2(F) \geq \frac{s^2}{2}$ .  $\square$

## 5 LOWER BOUNDS FOR THE SQUARE CLASS NUMBER

We start this section with Djoković's proof of his bound (1.6), rephrased in the terminology of appendix A.

5.1. THEOREM (DJOKOVIĆ). *If  $F$  is a nonreal field of level  $s \geq 8$  then*

$$q(F) \geq 2 \cdot |D_F(s/2)/F^{\times 2}| \geq 2 \cdot \sum_{i=1}^{s/2} \frac{1}{s+2-i} \binom{s+1}{i}.$$

*Proof:* The first inequality is clear since  $|F^{\times}/D_F(s/2)| \geq 2$ .

Next we consider a representation  $0 = x_1^2 + \dots + x_{s+1}^2$  of zero as a sum of  $s+1$  nonzero squares over  $F$ . We denote by  $\mathcal{P}$  the set of nonempty subsets of  $\{1, \dots, s+1\}$  of cardinality not greater than  $s/2$ . We define  $f : \mathcal{P} \rightarrow D_F(s/2)/F^{\times 2}$ ,  $J \mapsto (\sum_{j \in J} x_j^2)F^{\times 2}$ . For  $1 \leq k \leq s/2$  we write  $f_k$  for the restriction of  $f$  to  $\mathcal{P}_k^{s+1}$ . By (2.8), for  $k \neq k'$  the images of  $f_k$  and  $f_{k'}$  are disjoint. Also by (2.8),  $f_k$  is  $(k-1)$ -connected for any  $k \leq s/2$  and therefore  $|Im(f_k)| \geq \frac{1}{(s+1)-k+1} \binom{s+1}{k}$  by (A.4, c). All together we obtain

$$|D_F(s/2)/F^{\times 2}| \geq \sum_{k=1}^{s/2} |Im(f_k)| \geq \sum_{k=1}^{s/2} \frac{1}{s-k+2} \binom{s+1}{k}$$

which shows the second inequality.  $\square$

5.2. REMARK. For an integer  $s \geq 8$ , let  $\sum(s)$  denote the term on the right hand side in the inequality of the above theorem. Djoković showed by an elementary counting argument that  $\sum(s) > \frac{2^s}{s}$  [2]. As was pointed out by David B. Leep, the argument may be improved to obtain the bound  $\sum(s) > \frac{2^{s+1}}{s}$  for every even  $s \geq 8$ . Under the hypothesis of the last theorem one has thus  $q(F) > \frac{2^{s+1}}{s}$ ; further, since  $s = s(F)$  is a power of 2 and  $q(F)$  is also a power of 2 or infinite, it follows that  $q(F) \geq \frac{2^{s+2}}{s}$ .

Our calculations have shown that, at least for  $s$  a power of 2 in the range between 8 and  $2^{13}$ , actually one has  $\frac{2^{s+1}}{s} < \sum(s) \leq \frac{2^{s+2}}{s}$ .

However, for level 8 and 16 we get stronger bounds on  $q(F)$ .

5.3. THEOREM. *Let  $F$  be a field. If  $s(F) = 8$  then  $q(F) \geq 512$ . If  $s(F) = 16$  then  $q(F) \geq 2^{15}$ .*

*Proof:* Under the hypothesis  $s(F) = 8$  we have  $\bar{q}_3(F) \geq 2$ ,  $\bar{q}_2(F) \geq 16$  (3.1) and  $\bar{q}_1(F) \geq 8$  (1.5). Moreover, by (4.9) one of the last two inequalities must be proper. From  $|F^\times/F^{\times 2}| \geq \bar{q}_1(F) \cdot \bar{q}_2(F) \cdot \bar{q}_3(F)$  we get therefore  $q(F) \geq 512$ , since  $F^\times/F^{\times 2}$  is an elementary abelian 2-group.

For  $s(F) = 16$  we have by the previous sections  $\bar{q}_4(F) \geq 2$ ,  $\bar{q}_3(F) \geq 16$ ,  $\bar{q}_2(F) \geq 32$  and  $\bar{q}_1(F) \geq 16$  and one of the last two inequalities must be proper. As  $|F^\times/F^{\times 2}| \geq \bar{q}_1(F) \cdots \bar{q}_4(F)$  this leads to  $q(F) \geq 2^{15}$ .  $\square$

For  $s(F) = 2^n$  with  $n \geq 5$  the analogous arguments are not sufficient to improve Djoković's result. For  $s(F) = 32$ , for example, we may get in this way  $q(F) \geq 2^{25}$  while (5.1) yields  $q(F) \geq 2^{29}$ .

5.4. THEOREM. *Let  $F$  be a field of level  $2^n$  with  $n \geq 3$ . Then  $|k_{n-1}F| \geq 128$ . More precisely, the subgroup  $\{-1\}^{n-2}k_1F$  of  $k_{n-1}F$  is of index at least 4 and order at least 32.*

*Proof:* Again, we use the notation  $\varepsilon := \{-1\} \in k_1F$ . The homomorphism  $F^\times \rightarrow \{-1\}^{n-2}k_1F$  which maps  $x \in F^\times$  to the symbol  $\varepsilon^{n-2} \cdot \{x\}$ , has kernel  $D_F(2^{n-2})$ . Since  $\bar{q}_n(F) \geq 2$  and  $\bar{q}_{n-1}(F) \geq 16$  by (3.1), we have  $|F^\times/D_F(2^{n-2})| \geq \bar{q}_n(F) \cdot \bar{q}_{n-1}(F) \geq 32$ . Therefore  $\{-1\}^{n-2}k_1F$  has at least 32 elements.

To show that the index of this group in  $k_{n-1}F$  is at least 4 we just need to find  $\alpha, \beta, \gamma \in k_{n-1}F \setminus \{-1\}^{n-2}k_1F$  such that  $\alpha + \beta + \gamma \in \{-1\}^{n-2}k_1F$ .

By the hypothesis there are  $a, b, c \in D_F(3 \cdot 2^{n-3}) \setminus D_F(2^{n-2})$  such that  $a + b + c = 0$ . In  $k_2F$  we compute  $\{-a, -b\} + \{-a, -c\} + \{-b, -c\} = \{-a, bc\} + \{a, -bc\} = \{-1, abc\}$ . Therefore we are finished if we show that none of the symbols  $\varepsilon^{n-3}\{-a, -b\}$ ,  $\varepsilon^{n-3}\{-a, -c\}$  and  $\varepsilon^{n-3}\{-b, -c\}$  in  $k_{n-1}F$  lies actually in  $\{-1\}^{n-2}k_1F$ .

If this is not true we may by case symmetry suppose that  $\varepsilon^{n-3}\{-a, -b\} = \varepsilon^{n-2}\{-x\}$  for some  $x \in F^\times$ . Then the  $(n-1)$ -fold Pfister forms  $2^{n-3} \times \langle\langle a, b \rangle\rangle$  and  $2^{n-2} \times \langle\langle x \rangle\rangle$  over  $F$  are isometric, i.e. the quadratic form  $\varphi := 2^{n-3} \times \langle 1, x, x, -a, -b, -ab \rangle$  over  $F$  is hyperbolic. It follows that any subform of  $\varphi$  of dimension greater than  $\frac{1}{2} \dim(\varphi) = 3 \cdot 2^{n-3}$  is isotropic. In particular, the form  $2^{n-2} \times \langle -ax \rangle \perp 2^{n-3} \times \langle 1 \rangle \perp \langle b \rangle$ , similar to a subform of  $\varphi$ , must be isotropic. It follows that  $ax \in D_F(2^{n-2}) \cdot D_F(2^{n-3} \times \langle 1 \rangle \perp \langle b \rangle) \subset D_F(2^{n-1})$  whence  $x \in D_F(2^{n-1})$ . On the other hand,  $\varphi \cong 2^{n-3} \times \langle 1, x, x, c, abc, -ab \rangle$  shows that  $2^{n-2} \times \langle x \rangle \perp 2^{n-3} \times \langle 1 \rangle \perp \langle c \rangle$  is isotropic. This in turn implies that  $-x \in D_F(2^{n-2}) \cdot D_F(2^{n-3} \times \langle 1 \rangle \perp \langle c \rangle) \subset D_F(2^{n-1})$ . Together this leads to  $-1 \in D_F(2^{n-1})$  which contradicts  $s(F) = 2^n$ .  $\square$

5.5. COROLLARY. *Let  $F$  be a nonreal field with  $s(F) \geq 8$ . Then  $|\mathrm{Br}_2(F)| \geq 128$  and  $|W(F)| \geq 2^{18}$ .*

*Proof:* If  $s(F) = 8$  then the theorem shows  $|k_2F| \geq 128$ . But this is also true if  $s(F) = 2^n > 8$  since then already the subgroup  $\{-1\}k_1F$ , isomorphic to  $F^\times/D_F(2)$ , has order at least  $\bar{q}_n(F) \cdot \bar{q}_{n-1}(F) \cdot \bar{q}_{n-2}(F)$  which is sufficiently large by the results of section 3. By Merkuriev's theorem,  $\mathrm{Br}_2(F)$  is isomorphic to  $k_2F$ , so in particular we have  $|\mathrm{Br}_2(F)| \geq 128$ . (In fact, the arguments to estimate the size of  $k_2F$  work similarly for  $\mathrm{Br}_2(F)$ , so it is not necessary to invoke Merkuriev's theorem here.)

Let  $I$  denote the fundamental ideal of  $W(F)$  and let  $\bar{I}^i := I^i/I^{i+1}$  for  $i \geq 0$ . For  $i = 0, 1, 2$  it follows from [9] that  $\bar{I}^i \cong k_iF$ . Thus  $|\bar{I}^0| = 2$ ,  $|\bar{I}^1| = q(F) \geq 512$  and  $|\bar{I}^2| \geq 128$ . Moreover,  $s(F) \geq 8$  implies  $|\bar{I}^3| \geq 2$ . Therefore  $|W(F)| \geq |\bar{I}^0| \cdot |\bar{I}^1| \cdot |\bar{I}^2| \cdot |\bar{I}^3| \geq 2^{18}$ .  $\square$

## A HYPERGRAPHS WITH CONNECTED COLORINGS

In this appendix  $t, k$  and  $n$  denote nonnegative integers with  $t \leq k \leq n$ . We briefly say  $k$ -set for a set of cardinality  $k$ . A  $k$ -hypergraph is a system  $\mathcal{H} = (V, \mathcal{E})$  where  $V$  is a set whose elements are called *vertices* and  $\mathcal{E}$  a collection of distinct  $k$ -subsets of  $V$  called *edges*. A graph in the usual sense is then just a 2-hypergraph.

Let  $\mathcal{H} = (V, \mathcal{E})$  be a  $k$ -hypergraph. Its number of vertices  $|V|$  is called the *order* of  $\mathcal{H}$ . We say that  $\mathcal{H}$  is *complete* if each  $k$ -subset of  $V$  is actually an edge, i.e. if  $\mathcal{E} = \{E \subset V \mid |E| = k\}$ . By an *edge-coloring* of  $\mathcal{H}$  we mean a function  $f : \mathcal{E} \rightarrow C$ . We consider the elements of  $C$  as *colors* and for  $E \in \mathcal{E}$  we call  $f(E)$  the *color of  $E$* . For  $t > 0$  we say that the edge-coloring  $f$  is  *$t$ -connected* if any two edges of the same color meet in at least  $t$  vertices, i.e. if for any  $E, E' \in \mathcal{E}$  with  $f(E) = f(E')$  we have  $|E \cap E'| \geq t$ .

A.1. PROBLEM. *Let  $t, k, n$  be nonnegative integers with  $t \leq k \leq n$ . Let  $\mathcal{H} = (V, \mathcal{E})$  be a complete  $k$ -hypergraph of order  $n$ . What is the least integer  $m$  such that there exists a  $t$ -connected edge-coloring  $f : \mathcal{E} \rightarrow C$  on  $\mathcal{H}$  with  $|C| = m$ ?*

The integer  $m$  which meets the condition in the problem depends only on the values of  $t, k$  and  $n$  and will be denoted by  $M(t, k, n)$ . We recall our notation  $\mathcal{P}_k^n$  for the set of all  $k$ -subsets of  $\{1, \dots, n\}$ . A complete  $k$ -hypergraph of order  $n$  is then given by  $\mathcal{K}_k^n := (\{1, \dots, n\}, \mathcal{P}_k^n)$ . So  $M(t, k, n)$  is just the least integer  $m$  such that there exists a function  $f : \mathcal{P}_k^n \rightarrow C$  where  $|C| = m$  and such that  $f(X) = f(X')$  implies  $|X \cap X'| \geq t$  for any  $X, X' \in \mathcal{P}_k^n$ . To study  $M(t, k, n)$  as a function in  $t, k$  and  $n$  we use the theory of *intersecting families* in combinatorics.

Let  $\mathcal{F}$  be a family of sets. We write  $\bigcup \mathcal{F}$  (resp.  $\bigcap \mathcal{F}$ ) for the union (resp. the intersection) of all sets belonging to  $\mathcal{F}$ . If  $|U \cap V| \geq t$  holds for every  $U, V \in \mathcal{F}$  then we say that the family  $\mathcal{F}$  is  *$t$ -intersecting* (just *intersecting* for  $t = 1$ ). A

coloring  $f : \mathcal{E} \rightarrow C$  of a  $k$ -hypergraph  $\mathcal{H} = (V, \mathcal{E})$  is thus  $t$ -connected if and only if  $f^{-1}(\{c\})$  is a  $t$ -intersecting family for every  $c \in C$ .

The crucial result on intersecting families is the Erdős-Ko-Rado theorem [4] which we state in the slightly stronger version of [14]:

A.2. THEOREM (ERDŐS-KO-RADO). *Let  $n \geq (k - t + 1)(t + 1)$ . If  $\mathcal{F}$  is a  $t$ -intersecting family of  $k$ -subsets of an  $n$ -set then  $|\mathcal{F}| \leq \binom{n-t}{k-t}$ .*

This theorem gives the optimal bound. Indeed, if  $N$  is an  $n$ -set and  $T$  a  $t$ -subset then  $\mathcal{F} := \{U \subset N \mid |U| = k, T \subset U\}$  is a  $t$ -intersecting family with precisely  $\binom{n-t}{k-t}$  elements. However, under the additional condition  $|\bigcap \mathcal{F}| < t$ , better bounds on  $|\mathcal{F}|$  can be given. In the case  $t = 1$  this is the following main result of [6]. (A short proof of this can be found in [5] where the case  $t > 1$  is also treated.)

A.3. THEOREM (HILTON-MILNER). *Let  $\mathcal{F}$  be a family of pairwise intersecting  $k$ -subsets of an  $n$ -set such that  $\bigcap \mathcal{F} = \emptyset$ . Then  $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ .*

Now we begin with the investigation  $M(t, k, n)$  as a function in  $t, k$  and  $n$  with  $0 < t \leq k \leq n$ . We first treat the easy cases when  $t$  and  $k$  take extremal values. Part (c) is implicitly shown in [2].

A.4. PROPOSITION. (a)  $M(t, k, n) = 1$  is equivalent to  $n \leq 2k - t$ .

(b)  $M(t, k, n) = \binom{n}{k}$  is equivalent to  $k = t$ .

(c)  $M(k-1, k, n) = M(n-k-1, n-k, n) \geq \frac{1}{n-k+1} \binom{n}{k}$  for  $1 \leq k \leq n/2$ .

*Proof:* (a)  $M(t, k, n)$  is equal to 1 if and only if  $\mathcal{P}_k^n$  is  $t$ -intersecting; this is the case if and only if  $n \leq 2k - t$ .

(b) Each condition holds if and only if any nonempty  $t$ -intersecting family of  $k$ -subsets of  $\{1, \dots, n\}$  consists of just one  $k$ -set.

(c) It is quite obvious that a family  $\mathcal{F} \subset \mathcal{P}_k^n$  is  $(k-1)$ -intersecting if and only if the family of complement sets  $\{\{1, \dots, n\} \setminus U \mid U \in \mathcal{F}\}$  is  $(n-k-1)$ -intersecting. So  $f : \mathcal{P}_k^n \rightarrow C$  is  $(k-1)$ -connected if and only if  $f' : \mathcal{P}_{n-k}^n \rightarrow C, V \mapsto f(\{1, \dots, n\} \setminus V)$  is  $(n-k-1)$ -connected. This shows in particular  $M(k-1, k, n) = M(n-k-1, n-k, n)$ .

For a  $(k-1)$ -intersecting family  $\mathcal{F} \subset \mathcal{P}_k^n$  it is easy to check that either  $|\bigcap \mathcal{F}| \geq k-1$  or  $|\bigcup \mathcal{F}| \leq k+1$ . In the first case we conclude  $|\mathcal{F}| \leq n - k + 1$  and in the second case  $|\mathcal{F}| \leq k + 1 \leq n - k + 1$ . If now  $f : \mathcal{P}_k^n \rightarrow C$  is  $(k-1)$ -connected then  $\mathcal{P}_k^n$  is covered by the  $(k-1)$ -intersecting families  $f^{-1}(\{c\})$  for  $c \in C$ , which implies that  $\binom{n}{k} = |\mathcal{P}_k^n| \leq (n - k + 1) \cdot |C|$ .  $\square$

A.5. EXAMPLES. (1) The function  $f : \mathcal{P}_k^n \rightarrow \mathcal{P}_t^{n-k+t}$  which associates to  $X \in \mathcal{P}_k^n$  the set of the  $t$  smallest numbers in  $X$  is a  $t$ -connected edge-coloring of  $\mathcal{K}_k^n$ .

(2) If  $n \geq 2k - 1$  then a 1-connected edge-coloring of  $\mathcal{K}_k^n$  is given by

$$f : \mathcal{P}_k^n \longrightarrow \{1, \dots, n - 2k + 2\}, \quad X \longmapsto \max(X \cup \{2k - 1\}) - 2k + 2.$$

(3) Let  $t < k < n$ . If  $f : \mathcal{P}_k^n \rightarrow C$  be a  $t$ -connected edge-coloring of  $\mathcal{K}_k^n$  and  $g : \mathcal{P}_{k+1}^n \rightarrow C'$  is a  $(t+1)$ -connected edge-coloring of  $\mathcal{K}_{k+1}^n$ , where  $C$  and  $C'$  are disjoint sets, then a  $(t+1)$ -connected edge-coloring of  $\mathcal{K}_{k+1}^{n+1}$  is defined by

$$h : \mathcal{P}_{k+1}^{n+1} \longrightarrow C \cup C', \quad X \longmapsto \begin{cases} f(X \setminus \{n+1\}) & \text{if } n+1 \in X, \\ g(X) & \text{otherwise.} \end{cases}$$

From these examples we conclude:

A.6. PROPOSITION. (a)  $M(t, k, n) \leq \binom{n-k+t}{t}$ .

(b) If  $n \geq 2k - 1$  then  $M(1, k, n) \leq n - 2k + 2$ .

(c) If  $t < k < n$  then  $M(t+1, k+1, n+1) \leq M(t, k, n) + M(t+1, k+1, n)$ . □

For lower bounds on  $M(t, k, n)$  we first consider the case  $t \geq 2$ .

A.7. THEOREM. Let  $2 \leq t < k$ . Then for  $n \geq (k - t + 1)(t + 1)$  we have

$$M(t, k, n) \geq \prod_{i=0}^{t-1} \frac{n-i}{k-i} > \left(\frac{n}{k}\right)^t.$$

*Proof:* Let  $f : \mathcal{P}_k^n \rightarrow C$  be a  $t$ -connected edge-coloring of  $\mathcal{K}_k^n$  with  $n \geq (k - t + 1)(t + 1)$ . For each  $c \in C$  we have then by the Erdős-Ko-Rado theorem  $|f^{-1}(\{c\})| \leq \binom{n-t}{k-t}$ . As  $\mathcal{P}_k^n = \bigcup_{c \in C} f^{-1}(\{c\})$  we get  $\binom{n}{k} \leq |C| \cdot \binom{n-t}{k-t}$ . Therefore  $|C| \geq \frac{n}{k} \cdot \frac{n-1}{k-1} \cdots \frac{n-t+1}{k-t+1}$  and an easy computation shows the second inequality. □

For the purposes of section 3 we state the following particular case:

A.8. COROLLARY. Let  $i$  and  $m$  be positive integers satisfying either  $2 \leq i \leq \frac{m}{2}$  or  $3 \leq i = \frac{m+1}{2}$  or  $5 \leq i = \frac{m}{2} + 1$ . Then  $M(2^{i-2}+1, 2^i, 2^m) > 2^{(m-i)(2^{i-2}+1)}$ . □

Now we come to the case  $t = 1$ .

A.9. LEMMA. For  $k > 1$  we define the polynomial

$$F_k(X) := \prod_{i=0}^{k-1} (X-i) - k(X-2k+1) \left( \prod_{i=1}^{k-1} (X-i) - \prod_{i=1}^{k-1} (X-k-i) + (k-1)! \right).$$

If  $k \leq n$  and  $f : \mathcal{P}_k^n \rightarrow C$  is such that  $\bigcap f^{-1}(\{c\}) = \emptyset$  for every  $c \in C$  then either  $|C| \geq n - 2k + 2$  or  $F_k(n) \leq 0$ .

*Proof:* Suppose that  $f$  has the stated property. Then the Hilton-Milner theorem implies  $\binom{n}{k} \leq |C| \cdot \left[ \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 \right]$ . On the other hand,  $(k!)^{-1} \cdot F_k(n) = \binom{n}{k} - (n-2k+1) \cdot \left[ \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 \right]$ . Thus  $F_k(n) > 0$  implies  $|C| > (n-2k+1)$ .  $\square$

A.10. REMARK. The polynomial  $F_k$  defined in the lemma is monic of degree  $k$ . In particular, we have  $F_k(n) > 0$  for all  $n$  sufficiently large. Computation for small values of  $k$  yields:  $F_2(X) = X^2 - 7X + 18$ ,  $F_3(X) = X^3 - 21X^2 + 140X - 240$  and  $F_4(X) = X^4 - 54X^3 + 731X^2 - 3534X + 5880$ . Thus we have  $F_2(n) > 0$  for any  $n \in \mathbb{N}$ ,  $F_3(n) > 0$  for  $n \geq 3$  and  $F_4(n) > 0$  for  $n \geq 37$  whereas  $F_4(36) < 0$ .

A.11. THEOREM. *For any  $k \geq 1$  there is a constant  $c_k \geq 2k - 2$  such that for all  $n \in \mathbb{N}$  sufficiently large we have*

$$M(1, k, n) = n - c_k.$$

*For  $k \leq 3$  we have, more precisely,  $M(1, k, n) = n - 2k + 2$  for  $n \geq 2k - 1$ .*

*Proof:* For  $k = 1$  there is nothing to show since  $M(1, 1, n) = n$ . For  $k \geq 2$  let  $F_k(X)$  be defined as in the lemma. By the above remark we may choose the least integer  $n_k \geq 2k - 1$  such that  $F_k(n) > 0$  for all  $n \geq n_k - 1$ . In particular we have  $n_2 = 3$  and  $n_3 = 5$ . Let  $c_k := n_k - M(1, k, n_k)$ . Then (A.6, b) implies  $c_k \geq 2k - 2$  and we check that equality holds for  $k = 2, 3$ .

We want to prove by induction that  $M(1, k, n) = n - c_k$  for  $n \geq n_k$ . For  $n = n_k$  this is trivial statement. Suppose it is true for  $n - 1 \geq n_k$ . Let  $f : \mathcal{P}_k^n \rightarrow C$  be a 1-connected edge-coloring of  $\mathcal{K}_k^n$ . If  $\bigcap f^{-1}(\{c\}) = \emptyset$  for each  $c \in C$  then by the lemma we have  $|C| \geq n - 2k + 2 \geq n - c_k$ . On the other hand, if there is  $c \in C$  such that the intersection  $\bigcap f^{-1}(\{c\})$  is not empty then we may suppose that it contains the element  $n$ . Then the restriction  $f' : \mathcal{P}_k^{n-1} \rightarrow C \setminus \{c\}$  of  $f$  to  $\mathcal{P}_k^{n-1}$  is a 1-connected edge-coloring of  $\mathcal{K}_k^{n-1}$ . By the induction hypothesis we have  $|C \setminus \{c\}| \geq M(1, k, n-1) = (n-1) - c_k$  and thus  $|C| \geq n - c_k$ . This implies  $M(1, k, n) \geq n - c_k$ . But (A.6, c) shows  $M(1, k, n) \leq M(1, k, n-1) + M(0, k-1, n-1) = n - c_k$  since  $M(0, k-1, n-1) = 1$ . Hence  $M(1, k, n) \geq n - c_k$  which finishes the induction step.  $\square$

A.12. QUESTION. *Does  $M(1, k, n) = n - 2k + 2$  hold for all  $n \geq 2k - 1$ , even if  $k > 3$  ?*

## B CC-GRAPHS

In this appendix we study connected edge-colorings for usual complete graphs. Here we are not only interested in the minimal number of colors but also in the distribution of the colors in the graph.

Let  $\mathcal{G}$  denote a complete graph of order  $n$  with vertices  $v_1, \dots, v_n$  and colored edges. The distribution of colors in  $\mathcal{G}$  can be equivalently represented by an edge-coloring of  $\mathcal{K}_2^n$  (see appendix A), i.e. by a function  $f : \mathcal{P}_2^n \rightarrow C$ , where  $C$  stands for the set of colors in  $\mathcal{G}$  and  $f$  associates to  $\{i, j\} \in \mathcal{P}_2^n$  the color of the edge between the vertices  $v_i$  and  $v_j$ .

A set of all the edges of a certain color shall be called a *color-component*. If such a color-component consists of  $r \geq 3$  edges all together having a vertex  $x$  in common we call it an *r-star* and  $x$  its *center*. By a *triangle* in  $\mathcal{G}$  we mean a complete subgraph of order 3 of  $\mathcal{G}$ . A *triangle* is said to be *monochrome* (resp. *three-colored*) if the three edges are of the same color (resp. of three different colors). A second complete colored graph  $\mathcal{G}'$  of order  $n$  is said to be *equivalent to  $\mathcal{G}$*  if there is a bijection between the sets of vertices of  $\mathcal{G}$  and  $\mathcal{G}'$  such that the induced bijection on the sets of edges preserves the color-components (in both directions).

We call  $\mathcal{G}$  *color-connected* or a *CC-graph* if in  $\mathcal{G}$  any two edges of the same color are adjacent. This is equivalent to the edge-coloring  $f$  being 1-connected. The only possible color-components in  $\mathcal{G}$  are then single edges, pairs of edges with a vertex in common, stars and monochrome triangles.

Theorem (A.11) says that  $M(1, 2, n) = n - 2$  for  $n \geq 3$ . This corresponds to a result of [13]. We rephrase it as follows and give a direct proof.

**B.1. PROPOSITION (TORT).** *A CC-graph of order  $n \geq 3$  has at least  $n - 2$  colors.*

*Proof:* For  $n = 3$  the statement is trivial. If  $n > 3$  and  $\mathcal{G}$  has less than  $n$  colors then one of its color-components must be a star. Deleting the center of this star yields a CC-graph  $\mathcal{G}'$  of order  $n - 1$  with less colors. By induction hypothesis  $\mathcal{G}'$  has at least  $n - 3$  and therefore  $\mathcal{G}$  at least  $n - 2$  colors.  $\square$

For any  $n \geq 3$  the complete graph  $\mathcal{K}_2^n$ , whose vertices are the integers  $1, \dots, n$ , together with the 1-connected coloring  $f_n : \mathcal{P}_2^n \rightarrow \{1, \dots, n - 2\}$ ,  $\{i, j\} \mapsto \max\{i, j, 3\} - 2$  defines a particular CC-graph  $\mathcal{G}_n$  of order  $n$  with  $n - 2$  colors (compare with example (A.5, 2)). The color-components of  $\mathcal{G}_n$  are one monochrome triangle and one  $i$ -star for each  $3 \leq i \leq n - 1$ . For  $3 \leq n \leq 5$ , every CC-graph with  $n - 2$  colors is equivalent to  $\mathcal{G}_n$ . This is not true for  $n = 6$ , since there is a CC-graph of order 6 with color-components a triangle and three 4-stars.

**B.2. PROPOSITION.** *Let  $\mathcal{G}$  be a CC-graph with  $n \geq 3$  vertices and  $n - 2$  colors. Then  $\mathcal{G}$  has as color-components one monochrome triangle and  $n - 3$  stars. Moreover, each vertex of  $\mathcal{G}$  lies either on the monochrome triangle or is the center of exactly one star.*

*Proof:* Let  $\mathcal{G}'$  be the complete subgraph spanned by all vertices of  $\mathcal{G}$  which are not the center of a star in  $\mathcal{G}$ . We want to show that  $\mathcal{G}'$  is a monochrome triangle. Then the vertices of  $\mathcal{G}$  outside of  $\mathcal{G}'$  will be the centers of  $n - 3$  stars and as  $\mathcal{G}$  has just  $n - 2$  colors the entire statement follows.

Let  $n'$  be the order of  $\mathcal{G}'$ . The  $n - n'$  vertices of  $\mathcal{G}$  outside of  $\mathcal{G}'$  are all centers of stars whose colors do not appear in  $\mathcal{G}'$ . As a consequence,  $\mathcal{G}'$  has at least  $n - n'$  colors less than  $\mathcal{G}$ . Then by (B.1),  $\mathcal{G}'$  has exactly  $n' - 2$  colors. Since  $\mathcal{G}'$  is a graph without stars each color appears at most three times, counting the edges yields  $3(n' - 2) \geq \frac{n'(n'-1)}{2}$  whence  $n' \leq 5$ . As  $\mathcal{G}'$  has  $n' - 2$  colors and contains no star, we have  $n' = 3$  and  $\mathcal{G}'$  is a monochrome triangle.  $\square$

A CC-graph  $\mathcal{G}$  will be called *total* if there is a permutation  $\sigma \in \mathcal{S}_n$  such that for any  $\{i, j\} \in \mathcal{P}_2^n$  the color of the edge between  $v_i$  and  $v_j$  depends only on  $\max\{\sigma(i), \sigma(j)\}$ . After renumbering the vertices  $\mathcal{G}$  we may then suppose that the permutation  $\sigma$  is the identity on  $\{1, \dots, n\}$ .

Let  $\mathcal{G}$  be a total CC-graph of order  $n$  with vertices  $v_1, \dots, v_n$  enumerated in such a way that the color of any edge linking  $v_i$  and  $v_j$  depends only on  $\max\{i, j\}$ . Then  $\mathcal{G}$  has at most  $n - 1$  different colors. From (B.1) it follows that the number of colors in  $\mathcal{G}$  is either  $n - 2$  or  $n - 1$ . Further, by (B.2) the number of colors is  $n - 2$  if and only if  $v_1, v_2$  and  $v_3$  form a monochrome triangle and then the color of the edge between  $v_i$  and  $v_j$  depends precisely on  $\max\{i, j, 3\}$ . In both cases the enumeration of the vertices is unique up to changing the first three respectively the first two indices. Moreover,  $\mathcal{G}$  contains exactly  $n - 3$  stars. More precisely, for each  $4 \leq i \leq n$  there is exactly one  $(i - 1)$ -star in  $\mathcal{G}$  whose center is  $v_i$ . It is clear from the definition that a complete subgraph of a total CC-graph is also a total CC-graph.

**B.3. PROPOSITION.** *A CC-graph  $\mathcal{G}$  is total if and only if it contains no three-colored triangle.*

*Proof:* The necessity of the condition follows from the definition of a total CC-graph. Suppose now that  $\mathcal{G}$  is a CC-graph with  $n$  vertices with no three-colored triangle. We show by induction on  $n$  that  $\mathcal{G}$  is total. For  $n \leq 3$  this is evident. If  $n \geq 4$  then any complete subgraph with 4 vertices contains a star since otherwise it would contain a three-colored triangle. So we can choose an  $r$ -star in  $\mathcal{G}$  where  $r$  is as large as possible. For the ease of imagination say, it is of red color. We may suppose that  $v_n$  is the center of this star. Let  $\mathcal{G}'$  be the complete subgraph of  $\mathcal{G}$  with all the vertices of  $\mathcal{G}$  except  $v_n$ . Then  $\mathcal{G}'$  is also a CC-graph with  $n - 1$  vertices and contains no three-colored triangle. So, by the induction hypothesis,  $\mathcal{G}'$  is total, i.e. its vertices can be enumerated as  $v_1, \dots, v_{n-1}$  in such a way that the color of an edge connecting vertices  $v_i$  and  $v_j$  depends just on  $\max\{i, j\}$ . This would still be true for the enumeration of the vertices  $v_1, \dots, v_n$  of  $\mathcal{G}$ , if  $v_n$  is connected with each of the  $v_1, \dots, v_{n-1}$  by an edge of red color. So we just have to show that  $r = n - 1$ . Suppose that  $r < n - 1$ . Then certainly  $n > 4$  since  $r \geq 3$  by the definition of an  $r$ -star. But  $v_{n-1}$  is the center of an  $n - 2$ -star in  $\mathcal{G}'$ , say of blue color. By the maximality of  $r$  we see that the edge between  $v_{n-1}$  and  $v_n$  cannot be blue and that  $r = n - 2$ . So there must be exactly one vertex  $v_k$  with  $1 \leq k \leq n - 1$  which is connected with  $v_n$  with an edge of color different from red. It cannot be of blue color either so say that its color is green. Now we see that there is a triangle of colors

red, blue and green contained in  $\mathcal{G}$ , formed by  $v_k, v_{n-1}, v_n$  if  $k < n - 1$  and by  $v_1, v_{n-1}, v_n$  if  $k = n - 1$ , which gives the desired contradiction.  $\square$

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