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Weakly hyperbolic equations in domains with boundaries*

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Abstract: We consider weakly hyperbolic equations of the type $u_{tt}(t)+a(t)Au(t)=f(t,u(t))$, $u(0)=u_0$, $u_t(0)=u_1, u(t)\in D(A)$, $t\in[0,T]$, for a function $u: [0,T]\longrightarrow H$, $T\in[0,\infty]$, H a separable Hilbert space, A being a non-negative, self-adjoint operator with domain $D(A)$. The real function a is assumed to be non-negative, continuous and (piecewise) continuous differentiable, and the derivative a' will have to satisfy an integrability condition, which will admit infinitely many oscillations near the point of degeneration. For given initial data u_0, u_1 a global existence theorem in $C([0,T],D(A^s))$ is proved for the linear problem $f\equiv f(t)$. If a' does not change sign, the result can be improved, and finally a local (in time) existence theorem can be proved for nonlinearities f essentially satisfying the mapping property $f(\cdot,D(A^s))\subset D(A^s)$, where $s>0$ describes the regularity class. In the applications, A will be a uniformly elliptic operator in a domain Ω, Ω being a bounded domain with smooth boundary in $\mathbb{R}^n$, $n\geq 2$, for second-order operators then describing a weakly hyperbolic wave equation.

AMS subject classification: 35 L 05, 35 L 20, 35 L 70

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1 Introduction

We consider first abstract degenerate equations of the type

$$u_{tt}(t)+a(t)Au(t)=f(t,u(t)), \qquad t\in[0,T], \tag{1.1}$$

$$u(0)=u_0, \quad u_t(0)=u_1, \tag{1.2}$$

$$u(t)\in D(A), \qquad t\in[0,T], \tag{1.3}$$

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where u maps $[0,T]$ into a separable Hilbert space H, $0 \le T \le \infty$, A is a non-negative self-adjoint operator with domain $D(A)$. The real-valued function a is first assumed to satisfy the following conditions:

There exists a sequence $(t_j)_j \subset [0,\infty)$, decreasing (or increasing) to some $t_0 \in [0,\infty)$ such that

$$a \in C^1([0,\infty) \setminus \bigcup_j \{t_j\}, \mathbb{R}), \quad a \ge 0, \tag{1.4}$$

and

$$\sum_{j\in\mathbb{N}} \int_{t_j}^{t_{j+1}} \frac{|a'(r)|}{a(r)+\varepsilon}\, dr \le M|\log\varepsilon|, \tag{1.5}$$

where M is independent of $\varepsilon \in (0,\varepsilon_0]$ for some fixed $\varepsilon_0 \in (0,1)$. Typical examples will be the following ones:

$$a(t) = t\,(\sin(\log t) + (1+\delta)) \tag{1.6}$$

or

$$a(t) = \exp(-1/t)\ \left(\sin(1/t) + (1+\delta)\right), \tag{1.7}$$

where

$$0 < \delta < \sqrt{2} - 1 \tag{1.8}$$

is arbitrary, but fixed. As these examples show, infinitely many oscillations near a point of degeneration (here $t = 0$) are possible, also with infinite order or degeneracy. Moreover, the condition of Oleinik [19],

$$\exists\, B > 0 :\ a' + Ba \ge 0, \tag{1.9}$$

is not satisfied (see section 2).

Under the assumption

$$a \ge 0 \quad \text{and} \quad a \in C^1([0,\infty),\mathbb{R}), \quad a' \ge 0 \quad \text{or} \quad a' \le 0, \tag{1.10}$$

it will be possible to obtain better *a priori* estimates and to deal with the nonlinear case too. For the possible nonlinearities f we assume

$$f : [0,T] \times H \to H \text{ smooth} \tag{1.11}$$

and for a fixed $s > 0$, there is a non decreasing function $\varphi_s : [0,\infty) \to [0,\infty)$ and a function $\psi_s : [0,\infty) \times [0,\infty) \to [0,\infty)$, non-decreasing in each component, such that for

$$B := C^0([0,T], D(A^s))$$

we have

$$\forall w \in B \quad \forall t \in [0,T] : \|f(t,w(t))\|_{D(A^s)} \le \varphi_s(\|w(t)\|_{D(A^s)})\|w(t)\|_{D(A^s)}, \tag{1.12}$$

$$\forall u, w \in B \quad \|f(\cdot,u) - f(\cdot,w)\|_B \le \psi_s(\|u\|_B, \|w\|_B)\|u-w\|_B, \tag{1.13}$$

$\|\cdot\|_z$ denoting the norm in Z.

Since by (1.4) a is allowed to have zeros, we deal with a degenerate equation, typically a weakly hyperbolic equation, see section 5, while also, for example, degenerations of equations describing plates are included: $A = \Delta^2$, roughly spoken.

It should be noticed that (1.12) in particular requires that $f(t,\cdot)$ maps $D(A^s)$ into itself, thus restricting the admissible nonlinearities. As typical examples, we shall be able to deal with the case where A is the self-adjoint realization in $H := L^2(\Omega))$ of the formal uniformly elliptic operator of second order,

$$A = -\sum_{i,j=}^{n} \partial_i a_{ik}(x)\partial_k, \qquad \partial_i = \frac{\partial}{\partial x_i}, \tag{1.14}$$

subject to Dirichlet boundary conditions

$$w(x) = 0, \qquad x \in \partial\Omega, \tag{1.15}$$

or Neumann boundary conditions

$$\nu_i(x)a_{ik}(x)\partial_k w(x) = 0, \qquad x \in \partial\Omega, \tag{1.16}$$

where $\nu = (\nu_1, \ldots, \nu_n)'$ denotes the exterior normal. Here, $\Omega \subset \mathbb{R}^n$, $n \ge 2$, is a domain with smooth boundary, not necessarily bounded, and the coefficients a_{ik} are assumed to be smooth and to satisfy

$$a_{ik} = a_{ki} \quad \text{and} \quad \exists \gamma > 0 \quad \forall x \in \overline{\Omega} \quad \forall \xi \in \mathbb{C}^n \quad \sum_{k=1}^{n} a_{i_k}(x)\xi_i\overline{\xi}_k \ge \gamma|\xi|^2. \tag{1.17}$$

Typical examples of nonlinearities satisfying (1.12) are

$$f = f(t,x,u) = b(t,x,u)u^p, \tag{1.18}$$

with smooth b, or smooth f such that

$$|f(t,x,u)| \le \text{const.}\,|\text{dist}\,(x,\partial\Omega)|^p, \tag{1.19}$$

where $p > s$ for (1.15) and $p > s+1$ for (1.16) and

$$s > \frac{n}{4}. \tag{1.20}$$

The last condition stems from the desired continuous imbedding $H^{2s}(\Omega) \subset L^\infty(\Omega)$. The number s describes the regularity of the solution in the space variable. For $s \in \mathbb{N}$ the domain of A^s is included in the Sobolev space $H^{2s}(\Omega) = W^{2s,2}(\Omega)$, by the usual elliptic regularity theory and the smoothness assumptions on a_{ik} and on $\partial\Omega$. In particular, a_{ik} and $\partial\Omega$ need not be C^∞-smooth. We shall prove a global (T arbitrary) existence result for (1.1)–(1.3) in the linear case, f only depending on t, and a local ($T = T_*$ sufficiently small) existence result in the general case.
The assumption $a \geq 0$, even after replacing it by

$$a \in C^\infty([0,T],\mathbb{R}), \quad a \geq 0, \tag{1.21}$$

does not guarantee the well-posedness even locally of the linear problem in the class C^∞, as was shown by Colombini & Spagnolo [4], in contrast to the situation in the strictly hyperbolic case, where $a(t) \geq \delta > 0$ uniformly; the additional sufficient condition (1.10) is important. In [4], a is in $C^\infty((0,\infty),\mathbb{R})$, $a(t) > 0$ if $t > 0$ and $a(0) = 0$, with infinitely many oscillations near $t = 0$. Recently, Tarama [26] showed that the Cauchy problem for $u_{tt} - \exp(-2/(t^\alpha))\, b(1/t) u_{xx} = 0$ is well-posed in C^∞ if and only if $\alpha \geq 1/2$; here $b \in C^\infty([0,\infty))$ is not constant and 1-periodic; compare our example (1.7) above.

We mention that in spaces of smoother functions such as analytic or Gevrey functions, stronger results can be obtained, and in particular no assumption is necessary on the coefficient $a(t)$. Apart from the local existence in the analytic class, which is trivial thanks to the Cauchy-Kowalevsky theorem, one can prove results of analytic regularity or stability of solutions (see the papers of Spagnolo [23] and D'Ancona & Spagnolo [7] and the references therein), at least for the case $\Omega = \mathbb{R}^n$. Moreover weakly hyperbolic equations and systems are well posed in Gevrey spaces of suitable order (see e.g. Kajitani [9] and Kajitani & Wakabayashi [11] or Colombini, Jannelli & Spagnolo [3]). This should be contrasted with the solvability in C^∞ which always requires additional assumptions regarding the oscillations of $a(t)$. In the latter paper [3] the equivalent to the integrability condition (1.5) was proposed; Kajitani [10] considered a to be a C^1-function for solutions in $C^2([0,T]), C^\infty(\mathbb{R}^n))$ with a similar assumption, and also extended the results to some cases with coefficients depending also on x. Different conditions have been proposed by Oleinik [20] and Nishitani [18] (see also D'Ancona [6]) in the linear case; the local existence for some semilinear degenerate equations in $\Omega = \mathbb{R}^n$ was investigated by D'Ancona [5] and Manfrin [17]; see also the recent work of Reissig [22] for solutions in Sobolev spaces.
As far as the special application to mixed weakly hyperbolic initial-boundary value problems is concerned, we would like to mention that there exist numerous papers on the linear problem, discussing various boundary conditions, but not on the nonlinear case and being more restrictive with respect to the admissible degeneracy in time typically studying $a(t) = t^{2k}$ near $t = 0$, see

the papers by Baranovskii [2], Kimura [12], Kubo [13], [14], [15], Oleinik [19], Taniguchi [24], [25] and the references therein. Yamazaki [27] considered the linear case with $f = 0$. For this case our example (1.6) above is included, but the complicated proof — allowing finally for infinitely many degenerate points — can be replaced in the present case by ours; moreover, here the loss of derivatives can be computed explicitly.

In each case our results extend partial aspects of the above mentioned papers because of

- the simple assumption on a given in (1.5),
- the discussion of semilinear problems,
- the solution space being of Sobolev type in the applications,
- the discussion of domains with boundaries,
- the simplicity of the proofs using appropriate energy estimates.

The technique of proof reflects energy estimates known for the application in the case $\Omega = \mathbb{R}^n$, here using a general spectral mapping theorem replacing the Fourier transform.

The paper is organized as follows: In section 2 and 3 the global well-posedness of the abstract linear system (1.1)–(1.3), $f = f(t)$ is investigated under the assumption (1.5) and (1.10), respectively, in section 4, assuming (1.10), the local well-posedness of the abstract nonlinear system. Applications to initial boundary value problems will be discussed in section 5. In section 6 we conclude with remarks concerning the case when A has additional, negative eigenvalues.

2 Global solutions for the linear problem I

Under the assumptions (1.4),(1.5), we consider the linearized system, $f = h(t)$ in (1.1),

$$u_{tt}(t) + a(t)Au(t) = h(t), \quad t \geq 0, \tag{2.1}$$

$$u(0) = u_0, \quad u_t(0) = u_1, \tag{2.2}$$

$$u(t) \in D(A), \quad t \geq 0, \tag{2.3}$$

where $A \geq 0$ is self-adjoint, and shall prove

Theorem 2.1 *Let a satisfy the assumptions (1.4), (1.5), let $T > 0$ be arbitrary, but fixed. Let $h \in C^0([0,T], D(A^{s+M/2}))$ for some fixed $s \geq 1$, $u_1 \in D(A^{s+M/2})$ and $u_0 \in D(A^{s+(M+1)/2})$. Then there exists a unique solution u to (2.1)–(2.3) such that*

$$u \in C^2([0,T], D(A^{s-1})) \cap C^0([0,T], D(A^s)).$$

Moreover,

$$\exists c = c(\alpha, T) > 0 : \|u\|_{C^0([0,T],D(A^s))} \le c(\|u_0\|_{D(A^{s+(M+1)/2})} + \|u_1\|_{D(A^{s+M/2})} + \|h\|_{C^0([0,T],D(A^{s+M/2}))}), \tag{2.4}$$

where

$$\alpha := \sup_{t \ge 0} |a(t)|. \tag{2.5}$$

PROOF: We use the following version of the spectral theorem for self-adjoint operators (cf. [8], [16]): There exists a Hilbert space $\mathcal{H}$

$$\mathcal{H} = \int_{\oplus} \mathcal{H}(\lambda) d\mu(\lambda),$$

a direct integral of Hilbert spaces $\mathcal{H}(\lambda), \lambda \in \mathbb{R}$, with respect to a measure μ, and a unitary operator $\mathcal{A} : H \to \mathcal{H}$ such that for $s > 0$

$$D(A^s) = \{w \in H| \quad \lambda \mapsto \mathcal{A}w(\lambda) \in \mathcal{H}\}, \tag{2.6}$$

$$\mathcal{A}(A^s w)(\lambda) = \lambda^s \mathcal{A}w(\lambda), \tag{2.7}$$

$$\|A^s w\|^2 = \int_0^\infty \lambda^{2s} |\mathcal{A}w(\lambda)|^2_{\mathcal{H}(\lambda)} d\mu(\lambda), \tag{2.8}$$

for $w \in D(A^s)$. Here, $\|\cdot\|$ denotes the norm in H, while $|\cdot|_{\mathcal{H}(\lambda)}$ denotes the norm in $\mathcal{H}(\lambda)$; in the sequel the index $\mathcal{H}(\lambda)$ will be dropped for simplicity. Since $A \ge 0$, the integral in (2.8) can be taken over $[0, \infty)$ instead of $(-\infty, \infty)$. Now let

$$v_0 := \mathcal{A}u_0, \quad v_1 := \mathcal{A}u_1, \quad g(t) := \mathcal{A}h(t), \quad t \in [0, T],$$

and let $v = v(t, \lambda)$ be the solution to the ordinary differential equation

$$v''(t, \lambda) + a(t)\lambda v(t, \lambda) = g(t, \lambda), \tag{2.9}$$

$$v(0, \lambda) = v_0(\lambda), \quad v'(0, \lambda) = v_1(\lambda), \tag{2.10}$$

where a prime $'$ denotes differentiation with respect to t, and $\lambda \ge 0$ is regarded as a fixed parameter. Then $v(\cdot, \lambda)$ is a well-defined, twice continuously differentiable function on $[0, T]$. Let

$$E(t, \lambda) := |v'(t, \lambda)|^2 + (a(t)\lambda + 1)|v(t, \lambda)|^2. \tag{2.11}$$

Then, for $\lambda > 0$,

$$\begin{aligned} E'(t) &= 2\mathrm{Re}\, v''(t)\overline{v}'(t) + a'(t)\lambda|v(t)|^2 + 2\mathrm{Re}\,(a(t)\lambda+1)v(t)\overline{v}'(t) \\ &\leq |g(t)|^2 + \underbrace{(2 + \frac{|a'(t)|}{a(t)+1/\lambda})}_{=:\chi(t)} E(t), \end{aligned}$$

hence

$$E(t,\lambda) \leq \exp(\int_0^t \chi(r)\,dr)\, E(0,\lambda) + \int_0^t \exp(\int_r^t \chi(\sigma)\,d\sigma)|g(r,\lambda)|^2\,dr.$$

Using assumption (1.5), we obtain

$$\int_r^t \chi(\sigma)\,d\sigma \leq 2(t-r) + \log\lambda^M,$$

where, without loss of generality,

$$\lambda \geq \lambda_0 := 1/\varepsilon_0.$$

This implies

$$\begin{aligned} E(t,\lambda) &\leq c(T)\lambda^M \left\{ E(0,\lambda) + \int_0^t |g(r,\lambda)|^2 dr \right\} \\ &\leq c(\alpha,T) \left\{ |(\lambda^{(M+1)/2}+1)v_0|^2 + |(\lambda^{M/2}+1)v_1|^2 + \int_0^t |\lambda^{M/2} g(r,\lambda)|^2 dr \right\}, \end{aligned}$$

where

$$\alpha = \sup_{t\in[0,T]} |a(t)|.$$

We conclude

$$\int_0^\infty \lambda^{2s}(|v(t,\lambda)|^2 + |v'(t,\lambda)|^2) d\mu(\lambda) \leq c(\alpha,T) \left\{ \int_0^\infty \lambda^{2s+M+1}|v_0(\lambda)|^2 d\mu(\lambda) \right.$$

$$\left. + \int_0^\infty \lambda^{2s+M}|v_1(\lambda)|^2 d\mu(\lambda) + \int_0^t \int_0^\infty \lambda^{2s+M}|g|(r,\lambda)|^2 d\mu(\lambda) dr. \right. \tag{2.12}$$

The estimate (2.12) will imply (2.4) by (2.8), and to complete the proof of Theorem 2.1 it remains to show the classical regularity properties of u. For this purpose let, for $\tau > 0$, L^2_τ denote the following Banach space

$$L^2_\tau := \mathcal{A}D(A^\tau).$$

with obvious norm.

Then the *a priori* estimate (2.12) shows

$$v' \in L^\infty([0,T], L^2_s) \tag{2.13}$$

which implies

$$v \in C^0([0,T], L^2_s),$$

and by the differential equation

$$v'' \in C^0([0,T], L^2_{s-1}),$$

hence, by (2.8),

$$u \in C^2([0,T], D(A^{s-1})) \cap C^0([0,T], D(A^s)).$$

Q.E.D.

As typical example we now discuss that given in (1.6) above,

$$a(t) = t\left(\sin(\log t) + (1+\delta)\right),$$

with

$$0 < \delta < \sqrt{2} - 1$$

arbitrary, but fixed. The calculations for the second example (1.7) are analogous ones. The sequence $(t_j)_j$ postulated in connection with (1.4) and (1.5) is given by the sequence of zeros of a', i.e.

$$t_{2j} := t_j^+, \quad t_{2j-1} := t_j^-,$$

where

$$t_j^\pm := \underbrace{\mathrm{e}^{-5/4\pi}\mathrm{e}^{\pm\zeta}}_{=:d_1^\pm>0}\mathrm{e}^{-2\pi j},$$

with a fixed

$$\zeta \in (0, \pi/4)$$

depending on δ. In each interval, where a' does not change sign, $|a'|/(a+\varepsilon)$ is a logarithmic derivative. Hence, in order to fulfil (1.5), it is sufficient to estimate

$$\sum_j \log\frac{a(t_{j+1})+\varepsilon}{a(t_j)+\varepsilon} \quad \text{resp.} \quad \sum_j \log\frac{a(t_j)+\varepsilon}{a(t_{j+1})+\varepsilon}.$$

Observing that

$$\begin{aligned} a(t_j^\pm) &= -t_j^\pm\cos(\log t_j^\pm) \\ &= -t_j^\pm \underbrace{\cos(-5/4\pi - 2\pi j \pm \zeta)}_{=:d_2^\pm} \\ &= d_1^\pm d_2^\pm\, \mathrm{e}^{-2\pi j}, \end{aligned}$$

we have to estimate a sum of the type

$$\sum_j \log \frac{c_1 \mathrm{e}^{-2\pi j} + \varepsilon}{c_2 \mathrm{e}^{-2\pi j} + \varepsilon},$$

where, without loss of generality,

$$c_1 > c_2 > 0.$$

This sum can easily be estimated as follows:

$$\begin{aligned}
\sum_j \log \frac{c_1 \mathrm{e}^{-2\pi j} + \varepsilon}{c_2 \mathrm{e}^{-2\pi j} + \varepsilon} &= \sum_j \log \left(1 + \frac{c_1 - c_2}{c_2 + \varepsilon \mathrm{e}^{2\pi j}}\right) \\
&\le \sum_j \frac{c_1 - c_2}{c_2 + \varepsilon \mathrm{e}^{2\pi j}} \\
&\le (c_1 - c_2) \int_0^\infty \frac{dx}{c_2 + \varepsilon \mathrm{e}^{2\pi x}} \\
&= \frac{c_1 - c_2}{2\pi} \int_{\log \varepsilon}^\infty \frac{dy}{c_2 + \mathrm{e}^y} \\
&= \frac{c_1 - c_2}{2\pi} (\underbrace{\int_0^\infty \frac{dy}{c_2 + \mathrm{e}^y}}_{=: c_3 > 0} + \int_{\log \varepsilon}^0 \frac{dy}{c_2 + \mathrm{e}^y}) \\
&\le \frac{(c_1 - c_2) c_3}{2\pi} + \frac{c_1 - c_2}{2\pi c_2} |\log \varepsilon| \\
&\le \frac{c_1 - c_2}{\pi c_2} |\log \varepsilon|
\end{aligned}$$

if

$$\varepsilon \le \varepsilon_0 := \mathrm{e}^{-c_2 c_3}.$$

Therefore, (1.5) is satisfied and Theorem 2.1 applies.

Finally, we remark that the condition of Oleinik [19], i.e.

$$\exists\, B > 0 : \; a' + Ba \ge 0,$$

is violated in our example, namely, taking for $j \in \mathbb{N}$

$$\rho_j := \mathrm{e}^{-(5/4\pi + 2\pi j)},$$

we have, as $j \to \infty$,

$$\begin{aligned}
\frac{-a'(\rho_j)}{a(\rho_j)} &= -\frac{1}{\rho_j} \left(1 + \frac{\cos(\log \rho_j)}{\sin(\log \rho_j) + (1 + \delta)}\right) \\
&= -\frac{1}{\rho_j} \left(1 + \frac{\sqrt{2}}{\sqrt{2} - 2(1 + \delta)}\right) \\
&= \frac{-c_\delta}{\rho_j} \;\to\; \infty,
\end{aligned}$$

with some constant c_δ depending on δ, which is negative because of the assumption (1.8) on δ.

3 Global solutions for the linear problem II

Now we consider the linearized problem (2.1)–(2.3) under the assumption (1.10). The problem can be solved globally, and a suitable *a priori* estimate can be proved which will be useful for the nonlinear case. In particular, the estimates below show the same order of derivatives for the solution as for the right-hand side, which is important for the semilinear problem to be discussed in the next section.

Theorem 3.1 *Let a satisfy the assumption (1.10), let $T > 0$ be arbitrary, but fixed, let $h \in C^0([0,T], D(A^s))$ for some fixed $s \geq 1$, $u_1 \in D(A^s)$ and $u_0 \in D(A^s)$ if $a' \geq 0$, $u_0 \in D(A^{s+1})$ if $a' \leq 0$. Then there exists a unique solution u to (2.1)–(2.3) such that*

$$u \in C^2([0,T], D(A^{s-1})) \cap C^0([0,T], D(A^s)).$$

Moreover, if $a' \geq 0$,

$$\exists c = c(T) > 0 \qquad \|u\|_B \leq c(\|(u_0, u_1)\|_{D(A^s)} + \|h\|_B), \tag{3.1}$$

where $B = C^0([0,T], D(A^s))$ and, if $a' \leq 0$,

$$\exists c = c(\alpha, T) > 0 : \|u\|_B \leq c(\|u_0\|_{D(A^{s+1})} + \|u_1\|_{D(A^s)} + \|h\|_B), \tag{3.2}$$

where again

$$\alpha = \sup_{t \geq 0} |a(t)|.$$

Proof: As in the proof of Theorem 2.1 let

$$v_0 := \mathcal{A}u_0, \quad v_1 := \mathcal{A}u_1, \quad g(t) := \mathcal{A}h(t), \quad t \in [0,T],$$

and let $v = v(t,\lambda)$ be the solution to the ordinary differential equation

$$v''(t,\lambda) + a(t)\lambda v(t,\lambda) = g(t,\lambda), \tag{3.3}$$

$$v(0,\lambda) = v_0(\lambda), \quad v'(0,\lambda) = v_1(\lambda). \tag{3.4}$$

Case 1: $a' \geq 0$ on $[0, t_1]$, $\quad 0 < t_1 \leq T$.

To prove an *a priori* estimate, we define

$$w(t,\tau,\lambda) := \int_t^\tau v(r,\lambda)dr, \tag{3.5}$$

where $0 \le t \le \tau \le T$ (cf. [20] where the integrated function has been used too). Multiplying (3.3) by $\overline{w}$ and integrating from 0 to τ, we obtain, dropping λ for the moment in v and w,

$$\int_0^\tau v''(t)\overline{w}(t,\tau)dt + \int_0^\tau a(t)\lambda v(t)\overline{w}(t,\tau)dt = \int_0^\tau g(t)\overline{w}(t,\tau)dt. \tag{3.6}$$

Observing (Re: real parts)

$$\begin{aligned}
\mathrm{Re}\int_0^\tau v''(t)\overline{w}(t,\tau)dt &= \mathrm{Re}\,\{v'(0)\overline{w}(0,\tau) - \int_0^\tau v'(t)\partial_t\overline{w}(t,\tau)dt\} \\
&= \mathrm{Re}\,v'(0)\overline{w}(0,\tau) + \mathrm{Re}\int_0^\tau v'(t)\overline{v}(t)dt \\
&= \mathrm{Re}\,v'(0)\overline{w}(0,\tau) + \frac{1}{2}|v(\tau)|^2 - \frac{1}{2}|v(0)|^2
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\mathrm{Re}\int_0^\tau a(t)\lambda v(t)\overline{w}(t,\tau)dt &= -\mathrm{Re}\int_0^\tau a(t)\lambda\partial_t w(t,\tau)\overline{w}(t,\tau)dt \\
&= -\int_0^\tau a(t)\lambda\frac{1}{2}\partial_t|w(t,\tau)|^2dt \\
&= \frac{1}{2}a(0)\lambda|w(0,\tau)|^2 + \frac{1}{2}\int_0^\tau a'(t)\lambda|w(t,\tau)|^2dt,
\end{aligned} \tag{3.8}$$

we conclude from (3.6)–(3.8)

$$\begin{aligned}
|v(\tau,\lambda)|^2 &= |v_0(\lambda)|^2 - 2\mathrm{Re}\,v_1(\lambda)\overline{w}(0,\tau,\lambda) - \lambda a(0)|w(0,\tau)|^2 \\
&\quad - \int_0^\tau a'(t)\lambda|w(t,\tau,\lambda)|^2dt + 2\mathrm{Re}\int_0^\tau g(t,\lambda)\overline{w}(t,\tau,\lambda)dt.
\end{aligned} \tag{3.9}$$

We obtain from (3.9), the assumption (1.10) and the definition of w,

$$\begin{aligned}
|v(\tau,\lambda)|^2 &\le |v_0(\lambda)|^2 + |v_1(\lambda)|^2 + \int_0^\tau |v(t,\lambda)|^2dt + \int_0^\tau |g(t,\lambda)|^2dt \\
&\quad + \int_0^\tau \left|\int_t^\tau v(r,s)d\tau\right|^2 dt \\
&\le |v_0(\lambda)|^2 + |v_1(\lambda)|^2 + \int_0^\tau |g(t,\lambda)|^2dt + c(T)\int_0^\tau |v(t,\lambda)|^2dt
\end{aligned} \tag{3.10}$$

where $c(T)$ denotes a constant only depending on T. By Gronwall's inequality we conclude

$$|v(\tau,\lambda)|^2 \le c(T)\left\{|v_0(\lambda)|^2 + |v_1(\lambda)|^2 + \int_0^\tau |g(t,\lambda)|^2dt\right\} \tag{3.11}$$

which implies

$$\int_0^\infty \lambda^{2s}|v(\tau,\lambda)|^2 d\mu(\lambda) \le c(T)\left\{\int_0^\infty \lambda^{2s}|(v_0,v_1)(\lambda)|^2 d\mu(\lambda) + \int_0^\tau\int_0^\infty \lambda^{2s}|g(t(\lambda)|^2 d\mu(\lambda)dt\right\}. \quad (3.12)$$

In particular, for $\tau \in [0,T]$,

$$u(\tau) := \mathcal{A}^{-1}v(\tau) \in H$$

is well defined and by (2.8) we shall obtain from (3.12) the estimate (3.1). The regularity of u can now be obtained as follows. From (3.9) we see that for $\tau, \sigma \in [0,T]$

$$\begin{aligned} |v(\tau,\lambda)|^2 - |v(\sigma,\lambda)|^2 \;=\;& -2\mathrm{Re}\, v_1(\lambda)\int_\sigma^\tau \bar{v}(r,\lambda)dr - \lambda a(0)(|w(0,\tau,\lambda)|^2 - |w(0,\sigma,\lambda)|^2) \\ & -\lambda(\int_0^\tau a'(t)|w(t,\tau,\lambda)|^2 dt - \int_0^\sigma a'(t)|w(t,\sigma,\lambda)|^2 dt \\ & +2\mathrm{Re}\,(\int_0^\tau g(t,\lambda)\bar{w}(t,\tau,\lambda)dt - \int_0^\sigma g(t,\lambda)\bar{w}(t,\sigma,\lambda)dt), \end{aligned} \quad (3.13)$$

which implies, for fixed λ,

$$\lambda^{2s}|v(\tau,\lambda)|^2 \to \lambda^{2s}|v(\sigma,\lambda)|^2, \quad \text{as } \tau \to \sigma. \quad (3.14)$$

Using (3.12) and Lebesgue's theorem on dominated convergence, we conclude, as $\tau \to \sigma$,

$$\|v(\tau,\cdot)\|_{L^2_s} \to \|v(\sigma,\cdot)\|_{L^2_s}. \quad (3.15)$$

On the other hand, the *a priori* estimate (3.12) shows

$$v \in L^\infty([0,T], L^2_s) \quad (3.16)$$

which implies, using the differential equation for v,

$$v'' \in L^\infty([0,T], L^2_{s-1}),$$

hence

$$v \in C^0([0,T], L^2_{s-1}). \quad (3.17)$$

Let $(t_n)_n \subset [0,T], \quad t_n \to t$ (as $n \to \infty$), then by (3.17)

$$v(t_n) \to v(t) \text{ in } L^2_{s-1}, \quad (3.18)$$

and by (3.16) there is a subsequence $(\tilde{t}_n)_n$ and $w \in L^2_s$ such that

$$v(\tilde{t}_n) \rightharpoonup w, \text{ weakly in } L^2_s. \quad (3.19)$$

Since L_s^2 is continuously imbedded into L_{s-1}^2, we conclude from (3.18), (3.19)

$$w = v(t),$$

which implies

$$v(\tau, \cdot) \to v(\sigma, \dot{)} \quad \text{weakly in } L_s^2. \tag{3.20}$$

¿From (3.15) and (3.20) we get

$$v \in C^0([0,T], L_s^2),$$

and by the differential equation

$$v'' \in C^0([0,T], L_{s-1}^2),$$

hence, by (2.8),

$$u \in C^2([0,T], D(A^{s-1})) \cap C^0([0,T], D(A^s)).$$

Case 2: $a' \leq 0$ on $[0, t_1]$, $t \in [0, t_1]$.

As in the proof of Theorem 2.1 let

$$E(t,\lambda) := |v'(t,\lambda)|^2 + (a(t)\lambda + 1)|v(t,\lambda)|^2. \tag{3.21}$$

Then

$$\begin{aligned} E'(t) &= 2\mathrm{Re}\, v''(t)\overline{v}'(t) + a'(t)\lambda|v(t)|^2 + 2\mathrm{Re}\,(a(t)\lambda + 1)v(t)\overline{v}'(t) \\ &\leq 2\mathrm{Re}\, g(t)\overline{v}'(t) + 2\mathrm{Re}\, v(t)\overline{v}'(t) \\ &\leq |g(t)|^2 + E(t), \end{aligned}$$

hence

$$E(t,\lambda) \leq c(T)\left\{E(0,\lambda) + \int_0^t |g(r,\lambda)|^2 dr\right\}$$

which implies

$$\begin{aligned} \int_0^\infty \lambda^{2s}|v(t,\lambda)|^2 d\mu(\lambda) \quad &\leq \quad c(\alpha, T)\left\{\int_0^\infty \lambda^{2s+1}|v_0(\lambda)|^2 d\mu(\lambda)\right. \\ &\left. + \int_0^\infty \lambda^{2s}|v_1(\lambda)|^2 d\mu(\lambda) + \int_0^t\int_0^\infty \lambda^{2s}|g|(r,\lambda)|^2 d\mu(\lambda) dr. \right. \end{aligned} \tag{3.22}$$

The estimate (3.22) implies (3.2) by (2.8), and the regularity properties of u follow as in the proof of Theorem 2.1. Q.E.D.

Remarks:

1. The loss of derivatives encountered in Theorem 3.1 (or Theorem 2.1) seems to be natural for weakly hyperbolic equations, although we do not claim the optimaliy of our result with respect to regularity.

2. If a' changes sign only in a finite number of values $t_1, \ldots, t_n$, a solution u in $[0, T]$ can be constructed that looses regularity of order n by putting pieces of solutions together.

4 Local solutions for the nonlinear problem

For the nonlinear problem (1.1)–(1.3) we are now able to present a local existence theorem and prove it by a fixed point argument.

Theorem 4.1 *Let a satisfy the assumption (1.10), and let f satisfy the assumptions (1.11) – (1.13). Let $s \geq 1$, $u_1 \in D(A^s)$ and $u_0 \in D(A^s)$ if $a' \geq 0$, $u_0 \in D(A^{s+1})$ if $a' \leq 0$. Then there exists $T_* > 0$ and a unique solution u to (1.1)–(1.3) for $t \in [0, T_*]$ such that*

$$u \in C^2([0, T_*], D(A^{s-1})) \cap C^0([0, T_*], D(A^s)).$$

PROOF: For T_*, $R > 0$, to be determined below, define

$$X := \{w \in B := C^0([0, T*], D(A^s)) \quad | \quad \|u\|_B \leq R\}.$$

X is a Banach space with norm $\|\cdot\|_B$. For the given initial data u_0, u_1 we define a mapping

$$\Phi : X \to X, \qquad u \mapsto \overline{u} = \Phi(u),$$

by solving

$$\overline{u}_{tt}(t) + a(t)A\overline{u}(t) = f(t, u(t)),$$

$$\overline{u}(0) = u_0, \quad \overline{u}_t(0) = u_1,$$

$$\overline{u}(t) \in D(A).$$

$\Phi(u) = \overline{u}$ is well defined by Theorem 3.1 because h with

$$h(t) := f(t, u(t))$$

satisfies

$$h \in C^0([0, T_*], D(A^s))$$

by assumptions (1.11), (1.12) and since $u \in X$. From Theorem 3.1 we conclude that $\overline{u}$ satisfies

$$\overline{u} \in C^2([0,T_*], D(A^{s-1})) \cap C^0([0,T_*], D(A^s)),$$

and the estimate (cf. (3.12), (3.22))

$$\forall t \in [0,T_*]: \ \|\overline{u}(t)\|^2 \le c(\alpha, T)\{\|u_0, u_1)\|_Y^2 + \int_0^t \|f(r, u(r))\|^2_{D(A^s)} dr\}$$

holds, where

$$Y := D(A^s) \times D(A^s) \text{ resp. } Y := D(A^{s+1}) \times D(A^s),$$

depending naturally on the sign of a', and $0 < T_* \le T$ (fixed). The assumption (1.12) on f now implies

$$\|\overline{u}(t)\|^2_{D(A^s)} \le c(\alpha, T)\left\{\|(u_0, u_1)\|_y^2 + T_* \varphi_s(R) R^2\right\}. \tag{4.1}$$

Choosing R such that

$$R^2 \ge 2c(\alpha, T) \quad \|(u_0, u_1)\|_Y^2 \tag{4.2}$$

and choosing T_* such that

$$T_* \le 1/(2\varphi_s(R) R^2 c(\alpha, T)), \tag{4.3}$$

we obtain from (4.1)–(4.3)

$$\|\overline{u}\|_B \le R$$

which implifies

$$\Phi(X) \subset X.$$

Moreover, Φ can be shown to define a contraction if T_* is chosen appropriately:
Let $u, w \in X$ and $\overline{u} := \Phi(u), \overline{w} := \Phi(w)$. Then $\overline{u} - \overline{w}$ satisfies

$$(\overline{u} - \overline{w})_{tt}(t) + a(t) A(\overline{u} - \overline{w})(t) = f(t, u(t)) - f(t, w(t)),$$

$$(\overline{u} - \overline{w})(0) = 0, \quad (\overline{u} - \overline{w})_t(0) = 0,$$

$$(\overline{u} - \overline{w})(t) \in D(A).$$

The energy estimates (3.12), (3.22) imply

$$\|\overline{u} - \overline{w}\|_B^2 \le c(\alpha, T) T_* \|f(\cdot, u) - f(\cdot, w)\|_B^2.$$

Using assumption (1.13), we obtain

$$\begin{aligned}\|\overline{u}-\overline{w}\|_B^2 &\le c(\alpha,T)T_*\psi_s^2(\|u\|_B,\|w\|_B)\|u-w\|_B^2\\ &\le c(\alpha,T)T_*\psi_s^2(R,R)\|u-w\|^2.\end{aligned}$$

Choosing

$$T_* \le 1/(2c(\alpha,T)\psi_s^2(R,R)), \tag{4.4}$$

Φ becomes a contraction. Thus with R, T_* satisfying (4.2)–(4.4), there is a unique fixed point $u \in X$ of Φ being the desired solution.

Q.E.D.

5 Applications

Typical examples for operators A could be elliptic operators of order $2m, m \in \mathbb{N}$. For $m = 2$ for example, it might be appropriate to speak of weakly *parabolic* equations rather than of *hyperbolic* equations. In detail we look at the case $m = 1$, where we obtain a weakly hyperbolic wave equation.

Let a_{ik}, $i,k = 1,\ldots,n$, be smoth functions of $x \in \Omega \subset \mathbb{R}^n$, $n \ge 2$, when Ω is a domain with smooth boundary $\partial\Omega$. Let a_{ik} satisfy

$$a_{ik} = a_{ki} \text{ and } \exists\gamma > 0 \ \forall x \in \overline{\Omega} \ \forall\xi \in \mathbb{C}^n \ \sum_{i,k=1}^n a_{ik}(x)\xi_i\overline{\xi}_k \ge \gamma|\xi|^2.$$

Then

$$A : D(A) \subset H := L^2(\Omega) \to L^2(\Omega),$$

$$D(A) := H^2(\Omega) \cap H_0^1(\Omega),$$

resp.

$$D(A) := H^2(\Omega) \cap \{w \mid \nu_i(x)a_{ik}(x)\partial_k w(x) = 0, \ \ x \in \partial\Omega\}$$

$\nu(x)$ denoting the exterior normal in $x \in \partial\Omega$,

$$Aw := -\sum_{i,k=1}^n \partial_i a_{ik}(\cdot)\partial_k w,$$

defines *a* self-adjoint operator $A \ge 0$.

Remark: a_{ik} and $\partial\Omega$ are only assumed to be smooth enough to assure the H^{2s}-regularity of A. Now the abstract conditions (1.12), (1.13) can be interpreted. Assuming

$$\mathbb{N} \ni s > \frac{n}{4}$$

to assure the continuous imbedding

$$H^{2s}(\Omega) \subset L^\infty(\Omega), \tag{5.1}$$

the following examples are typical:

$$f = f(t,x,u) = b(t,x,u)u^p, \tag{5.2}$$

with smooth b, or smooth f with

$$|f(t,x,u)| \leq \text{const.}\,|\text{dist}\,(x,\partial\Omega)|^p, \tag{5.3}$$

where $p > s$ for Dirichlet boundary conditions and $p > s+1$ for Neumann boundary conditions. The condition on p guarantees that $f(t,\cdot,u(t,\cdot))$ satisfies the boundary conditions

$$f(t,\cdot,u(t)) \in D(A^s)) \text{ if } u(t,\cdot) \in D(A^s).$$

The estimates (1.12), (1.13) can be proved using Gagliardo-Nirenberg type estimates for composite functions and (5.1).

Thus we obtain the following — exemplary — application:

Theorem 5.1 *Under the assumptions made in this section above, Theorem 2.1 and Theorem 3.1 apply to the weakly hyperbolic initial-boundary value problem*

$$u_{tt}(t,x) - a(t)\sum_{i,k=1}^{n} \partial_i a_{ik}(x)\partial_k u(t,x) = f(t,x,u(t,x)),$$

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x),$$

$$u(t,\cdot)_{|\partial\Omega} = 0 \quad \textit{resp.} \quad \nu_i(\cdot)a_{ik}(\cdot)\partial_k u(t,\cdot)_{|\partial\Omega} = 0,$$

$t \geq 0, \quad x \in \Omega.$

6 Elliptic operators with potential

Continuing the discussion of applications to wave equations, we wish to consider the case where A has finitely many negative eigenvalues (counted with multiplicity), i.e. the spectrum $\sigma(A)$ of A consists of

$$\sigma(A) = \{\lambda_1, \ldots, \lambda_n\} \cup \sigma_1(A), \quad \sigma_1(A) \subset [0,\infty)$$

where $\lambda_j < 0$ is an eigenvalue of A corresponding to the eigenvector w_j:

$$Aw_j = \lambda_j w_j.$$

Then the space of eigenvectors H_E,

$$H_E := \text{span}\,\{w_1, \ldots, w_n\}$$

and its orthogonal complement in H reduce A

$$H = H_E \oplus H_E^{\perp}, \tag{6.1}$$

and we can solve the associated linear problem (2.1)–(2.3) by decomposing u_0, u_1 and $h(t)$ according to (6.1). For the part in $H_E^{\perp}$ A is non-negative, and Theorem 2.1 applies immediately, while for the part in H_E a simple ansatz

$$u(t,\cdot) = \sum_{j=1}^{n} d_j(t) w_j(\cdot),$$

where d_j is determined by

$$d_j''(t) + a(t) d_j(t) = h_j(t),$$

$$d_j(0) = u_{0,j}, \quad d_j'(0) = u_{1,j},$$

with

$$b_j := \langle b, w_j \rangle_H \quad \text{inner product}$$

for $b \in \{u_0, u_1, h(t)\}$ gives the solution in H_E.
A typical situation is given, when

$$A = -\Delta + q(x) \text{ in } L^2(\mathbb{R}^n), \quad q \in C_0^\infty(\mathbb{R}^n),$$

(cp. [21]).
We remark that for the more general case

$$A = -\sum_{i,k=1}^{n} \partial_i a_{ik}(x) \partial_k + g(x), \qquad g \in C_0^\infty(\Omega),$$

a_{ik} as in section 4, a local existence theorem for the *linear* case follows from Theorem 4.1 resp. Theorem 3.1, since

$$f(t,x,u) := a(t) g(x) u$$

obviously satisfies the assumptions.

References

[1] Arosio, A., Spagnolo, S.: Global existence for abstract evolution equations of weakly hyperbolic type. *J. Math. pures et appl.* **65** (1986), 263–305.

[2] Baranovskii, F.T.: A boundary-value problem for a hyperbolic equation with degenerate principal part. *Ukrain. Math. J.* **31** (1979), 177–184.

[3] Colombini, F., Jannelli, S., Spagnolo, S.: Well-posedness in the Gevrey classes of the Cauchy problem for a non-strictly hyperbolic equation with coefficients depending on time. *Ann. Scuola Norm. Sup. Pisa* **10** (1983), 291–312.

[4] Colombini, S., Spagnolo, S.: An example of a weakly hyperbolic Cauchy problem not well posed in C^∞. *Acta Math.* **148** (1982), 243–253.

[5] D'Ancona, P.: Local existence for semilinear weakly hyperbolic equations with time dependent coefficients. *Nonlinear Anal., T.M.A.* **21** (1993), 685–696.

[6] ———— Well posedness in C^∞ for a weakly hyperbolic second order equation. *Rend. Sem. Mat. Univ. Padova* **91** (1994), 65–83.

[7] D'Ancona, P., Spagnolo, S.: On the life span of the analytic solutions to quasilinear weakly hyperbolic equations. *Indiana Univ. Math. J.* **40** (1991), 71–99.

[8] Huet, D.: *Décomposition spectrale et opérateurs.* Presses Universitaires de France (1976).

[9] Kajitani, K.: Local solution of Cauchy problem for nonlinear hyperbolic systems in Gevrey classes. *Hokkaido Math. J.* **12** (1983), 434–460.

[10] Kajitani, K.: The well posed Cauchy problem for hyperbolic operators. Exposé au Séminaire de Vaillant (1989).

[11] Kajitani, K., Wakabayashi, S.: The hyperbolic mixed problem in Gevrey classes. *Japan J. Math.* **15** (1989), 309–383.

[12] Kimura, K.: A mixed problem for weakly hyperbolic equations of second order. *Comm. PDE* **6** (1981), 1335–1361.

[13] Kubo, A.: Mixed problems for some weakly hyperbolic second order equations. *Math. Japonica* **29** (1984), 721–751.

[14] ———— On the mixed problems for weakly hyperbolic equations of second order. *Comm. PDE* **9** (1984), 889–917.

[15] ———— Well posedness for the mixed problems of degenerate hyperbolic equations. *Funkcial. Ekvac.* **34** (1991), 95–102.

[16] Lions, J.L., Magenes, E.: *Problèmes aux limites non homogènes et applications, vol. 1.* Dunod, Paris (1968).

[17] Manfrin, R.: A solvability result for a nonlinear weakly hyperbolic equation of second order. *NoDEA* **2** (1995), 245–264.

[18] Nishitani, T.: The Cauchy problem for weakly hyperbolic equations of second order. *Comm. PDE* **5** (1980), 1273–1296.

[19] Oleinik, O.A.: The Cauchy problem and the boundary value problem for second-order hyperbolic equations degenerating in a domain and on its boundary. *Sov. Math. Dokl.* **7** (1966), 969–973.

[20] ——— On the Cauchy problem for weakly hyperbolic equations. Comm. Pure Appl. Math. **23** (1970), 569–586.

[21] Reed, M., Simon, B.: *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press, San Diego et al. (1978).

[22] Reissig, M.: Weakly hyperbolic equations with time degeneracy in Sobolev spaces. *Preprint* **96 – 07**, Techn. University of Freiberg (1996).

[23] Spagnolo, S.: Analytic solutions to nonlinear weakly hyperbolic equations. *Ric. Mat.* **40**, *Suppl.* (1991), 241–254.

[24] Taniguchi, M.: Mixed problem for weakly hyperbolic equations of second order with degenerate Neumann boundary condition. *Funkcial. Ekvac.* **27** (1984), 331–366.

[25] ——— Mixed problem for weakly hyperbolic equations of second order with degenerate first order boundary condition. *Tokyo J. Math.* **7** (1984), 61–98.

[26] Tarama, S.: On the second order hyperbolic equations degenerating in the infinite order — example —. *Math. Japonica* **42** (1995), 523–534.

[27] Yamazaki, T.: Unique existence of evolution equations of hyperbolic type with countably many singular of degenerate points. *J. Differential Equations* **77** (1989), 38–72.

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