On the Spectral Stability of Internal Solitary Waves in Fluids with Density Stratification

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Abstract

In this thesis we investigate the spectral stability of internal solitary waves occurring frequently in large natural water bodies, such as lakes and oceans, where they play an important role in mechanisms of mixing and energy transport. A commonly used mathematical model in this context is given by the Euler equations for a heterogeneous incompressible fluid in a two-dimensional channel. The rest state supports localized, rapidly decaying disturbances called *internal solitary waves* travelling horizontally with constant speed and unaltered in shape. While mathematical results on the existence of internal solitary waves for quite general stratifications are well known, their stability analysis has received only little attention on a rigorous level.

In order to study the spectral stability of such waves we consider the associated eigenvalue problem, which is obtained from the Euler equations by linearizing about the internal solitary wave, specializing the time-dependence to an exponential factor $\mathrm{e}^{\kappa t}$ and eliminating this factor subsequently. If the eigenvalue problem has a bounded solution for some $\kappa \in \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ then the wave under consideration is called spectrally unstable; otherwise, the wave is called spectrally stable. The central subject of this thesis is a method for the investigation of spectral stability of internal solitary waves. Our approach consists in five steps: (i) reformulation of the eigenvalue problem as an infinite-dimensional spatial-dynamical system (EVP), (ii) procedure to obtain finite-dimensional truncations (EVP_N) of (EVP), (iii) definition of an Evans function $D_N(\kappa)$ for the finite-dimensional problems (EVP_N), (iv) investigation of D_N for zeros $\kappa \in \mathbb{C}_+$, (v) identification or preclusion of eigenvalues $\kappa \in \mathbb{C}_+$ of the infinite-dimensional system (EVP). While steps (i)-(iv) are carried out within this thesis, step (v) is left to future work.

In the following we outline the results of this thesis. We prove in Theorem I that the eigenvalue problem permits a spatial-dynamics formulation, which is a dynamical system (EVP) on $\left(L^2(0,1)\right)^4$ endowed with a special scalar product. After constructing a suitable Hilbert basis for this state space we project the dynamical system to the span of $4N+4$ basis vectors and thus obtain a sequence of truncated problems (EVP_N) which are ordinary differential equations on finite-dimensional spaces. Assuming the prototypical exponential density stratification, Theorem II states that the truncated problems (EVP_N) permit the construction of Evans functions $D_N(\kappa)$ which allow to decide whether these systems have bounded solutions for some $\kappa \in \mathbb{C}_+$. We currently know no way to generally decide whether the Evans functions possess roots with positive real part. For small-amplitude waves, however, we show in Theorem III that the Evans function does not have roots with positive real part in a neighbourhood of the origin; one part of the proof is based on the relationship between small-amplitude internal solitary waves and soliton solutions of the Korteweg-deVries equation (KdV) and exploits the spectral stability of KdV solitons.

Zusammenfassung

In dieser Dissertation widmen wir uns der spektralen Stabilität interner Solitärwellen. Das weit verbreitete mathematische Modell, das wir hier zugrundelegen, ist durch die Euler-Gleichungen für ein inhomogenes, inkompressibles Fluid in einem zweidimensionalen Gebiet gegeben. Der Ruhezustand dieses Systems erlaubt die Existenz räumlich lokalisierter, schnell abklingender Störungen, die man als *interne Solitärwellen* bezeichnet. Während mathematische Ergebnisse zur Existenz dieser Wellen seit Längerem bekannt sind, hat die rigorose Untersuchung ihrer Stabilität kaum Aufmerksamkeit erfahren.

Um zu untersuchen, ob eine interne Solitärwelle spektral stabil ist, betrachten wir das zugehörige Eigenwertproblem, das man aus den Euler-Gleichungen erhält, indem man zuerst um die Solitärwelle linearisiert, dann die Zeitabhängigkeit auf einen exponentiellen Faktor $\mathrm{e}^{\kappa t}$ einschränkt und diesen anschließend herauskürzt. Eine Solitärwelle heißt spektral instabil, falls das Eigenwertproblem für irgendein $\kappa \in \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ eine räumlich beschränkte Lösung besitzt; andernfalls heißt sie spektral stabil. In dieser Arbeit schlagen wir eine Methode zur Untersuchung der spektralen Stabilität interner Solitärwellen vor, die aus den folgenden fünf Schritten besteht: (i) Formulierung des Eigenwertproblems als unendlich-dimensionales räumlich-dynamisches System (EVP), (ii) Approximation von (EVP) durch endlich-dimensionale räumlich-dynamische Systeme (EVP_N) für beliebiges $N \in \mathbb{N}$, (iii) Definition von Evans-Funktionen D_N für die endlich-dimensionalen Systeme (EVP_N), (iv) Untersuchung der Evans-Funktionen D_N auf Nullstellen $\kappa \in \mathbb{C}_+$, (v) Auffinden oder Ausschluss von Eigenwerten $\kappa \in \mathbb{C}_+$ des unendlich-dimensionalen Systems (EVP). Von diesem umfangreichen Programm werden in dieser Arbeit nur die Schritte (i) bis (iv) durchgeführt, während Schritt (v) in Vorbereitung befindlichen Arbeiten überlassen bleibt.

Unter Annahme der prototypischen exponentiellen Dichte-Schichtung im Hauptteil der Arbeit wurden folgende Ergebnisse erzielt. Wir beweisen in Theorem I, dass sich das Eigenwertproblem als räumlich-dynamisches System (EVP) schreiben lässt. Dabei handelt es sich um ein dynamisches System auf $\left(L^2(0,1)\right)^4$, versehen mit einem speziellen Skalarprodukt. Nach Bestimmung einer geeigneten Hilbert-Basis für diesen Zustandsraum erhalten wir für jedes $N \in \mathbb{N}$ ein endlich-dimensionales Teilproblem (EVP_N), indem wir das System (EVP) auf einen von nur $4N + 4$ Basisvektoren aufgespannten Unterraum projizieren. Auf diese Weise konstruieren wir eine Folge $(\mathrm{EVP}_N)_{N\in\mathbb{N}}$ approximierender Probleme, die gewöhnliche Differentialgleichungen auf endlich-dimensionalen Räumen sind. In Theorem II zeigen wir, dass man für jedes dieser Probleme eine holomorphe Funktion, die Evans-Funktion, auf $\mathbb{C}_+$ definieren kann, die genau dann bei einem $\kappa \in \mathbb{C}_+$ eine Nullstelle besitzt, wenn das Problem (EVP_N) für dieses κ eine beschränkte Lösung besitzt. Im Allgemeinen erhalten wir keine Aussagen über An- oder Abwesenheit von Nullstellen mit positivem Realteil. Für Wellen genügend kleiner Amplitude jedoch können wir in Theorem III solche Nullstellen in einer kleinen Umgebung des Ursprungs ausschließen; ein Teil des Beweises basiert auf der bekannten Tatsache, dass sich Wellen kleiner Amplitude durch Solitonen-Lösungen der KdV-Gleichung annähern lassen, und nutzt die spektrale Stabilität der KdV-Solitonen aus.

Contents

1. Introduction

In this thesis we investigate the spectral stability of internal solitary waves. We begin with a brief account on internal waves in general, on their mathematical description and on spectral stability before approaching the core subject of the thesis.

An *internal wave* corresponds to a time-dependent displacement of fluid elements within central parts of the fluid body, as opposed to a displacement of fluid elements at or near its surface. An *internal travelling wave* is an internal wave which is stationary with respect to a reference frame moving with constant speed. An *internal solitary wave* is an internal travelling wave which decays to the rest state at either spatial infinity. Waves of this kind occur frequently in fluidic media that are stratified according to varying density, as for example large natural water bodies (like oceans and lakes) and large natural gas bodies (atmospheres). Interesting physical effects, for instance the *dead water phenomenon* and the appearance of *morning glory clouds*, have been attributed to internal solitary waves. Notably since internal solitary waves provide important mechanisms for mixing and energy transport and thus have direct ecological implications, the disciplines of oceanography, limnology, and atmosphere science have devoted considerable attention to their observation and description, cf. [29, 5, 14, 32, 52].

For a widely used mathematical model of internal waves one neglects compressibility, viscosity, bottom topography as well as surface effects and restricts to one horizontal and one vertical space variable, from now on denoted by x and y, and thus considers the Euler equations for an incompressible fluid, with gravity included as an external force, in the two-dimensional channel $\mathbb{R} \times [0, 1]$ together with boundary conditions that keep the fluid from leaving the domain. Motivated by the examples from oceanography and limnology, one assumes, moreover, that the fluid at rest exhibits a known density stratification given by some sufficiently smooth function $\bar{\rho}(y)$ depending only on the height $y \in [0, 1]$ (or rather on the *depth* $1 - y$).

First and fundamental rigorous mathematical contributions to the theory of internal travelling waves go back to Dubreil-Jacotin, Long, and Yih (see [21, 47, 61]). These authors have shown most notably that internal travelling waves can be found in the system of the Euler equations by solving a single nonlinear elliptic equation, the so-called *Dubreil-Jacotin-Long* equation or the related *Long-Yih* equation.

Various authors have obtained general existence results for one of these two equations.

The wave speed, which appears as a parameter in either equation, has to be determined together with the wave itself. This problem can be viewed as a nonlinear eigenvalue problem in which the eigenvalue is closely related to the wave speed. We refer to [13, 4, 58, 43] for different existence proofs along these lines.

A different rigorous way to study analytically the diversity of internal travelling waves has become available with the advent of Kirchgässner's spatial-dynamics approach (see [38]). This approach is tailored for equations that live on a cylinder, like the channel $\mathbb{R} \times [0, 1]$, and consists in rewriting the equation as a dynamical system with the unbounded spatial variable assuming the role of "time". This reformulation permits the application of various methods from the theory of dynamical systems, especially the centre-manifold reduction, which allows to considerably decrease the dimension of the problem, often from infinite to finite. The spatial-dynamics idea can be applied successfully to the Dubreil-Jacotin equation and the Long-Yih equation and we refer to [38, 39, 34] for a number of results concerning existence and structure of internal travelling waves.

A simpler, practical approach to capture internal travelling waves and to find approximate expressions for them consists in deriving model equations which are valid under certain assumptions concerning the ratio between wave amplitude and fluid depth. Two popular models in this vein are the Korteweg-deVries equation for shallow water and the Benjamin-Ono equation for deep water (see [10, 12] for KdV and [50] for BO). These are better understood, as they only contain the independent variables x and t but not y any more. For the same reason, they cannot properly represent all features of the spatially two-dimensional situation of the fully nonlinear waves.

Since the full Euler equations cannot be solved explicitly, even for special stratifications, it was an important advance when Turkington and collaborators devised and implemented an algorithm for the numerical computation of these waves (see [58]). For progress regarding the numerical identification of internal waves since then, see e.g. [55, 15] and references therein.

An important first step towards nonlinear stability of waves is the investigation of their spectral stability. One fruitful approach is based on the Evans function which is a tool to detect the point spectrum of differential operators. The main idea underlying this approach is that, under certain assumptions, eigenvalues of the linearized operator can be found as the roots of an analytic function, namely the Evans function. It was originally defined for travelling waves in reaction-diffusion equations, and has been extended substantially to cover conservation laws and dispersive equations as well, see [22, 3, 51, 27, 36, 25, 26] and references therein; we refer to [54] for an extensive introduction. In particular, the spectral stability of solitons in the Korteweg-deVries equation is well known due to Pego and Weinstein, see [51] for a proof relying on the Evans function.

The Evans function approach emphasizes the dynamical aspect of spectral stability by reformulating the eigenvalue problem associated with a travelling wave as a dynamical system. For example, in a system with *one* space dimension the eigenvalue problem can be formulated as a non-autonomous linear system of ordinary differential equations and, accordingly, eigenfunctions correspond with bounded solutions of the latter.

In situations where small-amplitude waves can be approximated by waves of a known model equation it often occurs that the eigenvalue problem associated with the model wave appears in the full eigenvalue problem, at least in a certain scaling regime regarding the smallness parameter and the spectral parameter. In certain circumstances the Evans function approach then provides a framework which allows to carry over the stability or instability of the underlying model wave to the wave of interest. For instance, Freistühler and Szmolyan (see [25]) have achieved stability results for small-amplitude viscous shock waves which are approximately described by the viscous Burgers equation by proving that the stability of the wave of interest is determined by the stability of some underlying viscous shock wave in Burgers equation.

When studying equations posed on a cylinder, the reformulated eigenvalue problem is typically a differential equation on an *infinite-dimensional* state space. Such a situation has been treated, for example, by Haragus and Scheel in [31] in the context of small-amplitude surface water waves which can be approximated by means of the Korteweg-deVries equation; in this article the authors have obtained their results on the point spectrum close to the origin by initially reducing the eigenvalue problem to a finite-dimensional centre manifold and by exploiting the spectral stability of the Korteweg-deVries soliton in treating the reduced problem. In recent years, a systematic study of infinite-dimensional systems arising from eigenvalue problems has begun and, notably, the construction of an infinite-dimensional Evans function has been accomplished in a number of cases, we refer to [19, 28, 44] and, for a more computational point of view, to [45]. A feasible, natural approach to the infinite-dimensional problem consists in finding finite-dimensional approximations. This approach is beautifully illustrated in [49]: In considering the spectral stability of periodic travelling waves in a cylinder, the authors of [49] use a Galerkin procedure to obtain a sequence of finite-dimensional approximations, for which a parity index based on the Evans function can be defined, and show that this actually approximates a parity index for the infinite-dimensional problem. Finite-dimensional approximations also play a crucial role in [28, 44, 45].

In this thesis we explore the spectral stability of internal solitary waves viewed as exact solutions of the Euler equations. We start from the eigenvalue problem

$$\begin{aligned} -\kappa\rho &= (u^c - c)\rho_\xi + v^c\rho_y + u\rho^c_\xi + v\rho^c_y, \\ -\rho^c\kappa u &= \rho^c\left((u^c-c)u_\xi + uu^c_\xi + v^c u_y + vu^c_y\right) \\ &\quad + \rho\left((u^c-c)u^c_\xi + v^c u^c_y\right) + p_\xi, \\ -\rho^c\kappa v &= \rho^c\left((u^c-c)v_\xi + uv^c_\xi + v^c v_y + vv^c_y\right) \\ &\quad + \rho\left((u^c-c)v^c_\xi + v^c v^c_y\right) + p_y + g\rho, \\ 0 &= u_\xi + v_y, \end{aligned} \tag{EVP-Euler}$$

which is obtained by linearizing the Euler equations, in co-moving coordinates, about a regular internal solitary wave $(\rho^c(\xi,y), u^c(\xi,y), v^c(\xi,y), p^c(\xi,y))^\mathsf{T}$ of speed c and searching for solutions of the form

$$\mathrm{e}^{\kappa t}(\rho(\xi,y), u(\xi,y), v(\xi,y), p(\xi,y))^\mathsf{T}$$

with an exponentially time-dependent factor $\mathrm{e}^{\kappa t}$ and some time-independent function $(\rho(\xi,y), u(\xi,y), v(\xi,y), p(\xi,y))^\mathsf{T}$. Such a solution exists if the eigenvalue problem (EVP-Euler), which depends on the spectral parameter $\kappa \in \mathbb{C}$, possesses a bounded solution for some κ. The wave under consideration is called *spectrally stable* provided there does not exist a bounded solution of (EVP-Euler) for any κ with $\operatorname{Re}\kappa > 0$, and *spectrally unstable* otherwise. In order to determine spectral stability we have, therefore, to examine whether the system (EVP-Euler) does possess a bounded solution for some $\kappa \in \mathbb{C}_+ := \{z : \operatorname{Re} z > 0\}$. It is this question we pursue. To this end, we propose an approach consisting of five steps: (i) a reformulation of the eigenvalue problem as an infinite-dimensional dynamical system, (ii) a procedure to obtain finite-dimensional truncations, (iii) a definition of an Evans function for the truncations, (iv) an investigation of the Evans functions for zeros with positive real part, (v) preclusion or identification of eigenvalues in the infinite-dimensional problem. While steps (i)-(iv) are carried out in this thesis, step (v) has been left to future work. The rest of the introduction is devoted to an explanation of these steps.

Theorem I, our first result, is a precise implementation of step (i) and states that the eigenvalue problem can be considered as a dynamical system on an infinite-dimensional state space $\mathcal{W}$, which turns out to be a Hilbert space. More precisely, with ψ denoting a stream function for (u,v), i.e. $\psi_y = u$ and $\psi_\xi = -v$:
The eigenvalue problem (EVP-Euler) *can be written as an abstract ordinary differential*

equation, posed on the function space

$$\mathcal{W} = \left(L^2(0,1)\right)^4,$$

of the form

$$W'(\xi) = \mathbb{A}(\xi;\kappa)W(\xi), \tag{EVP}$$

in which the dependent variable assumes, at "time" ξ, a value

$$W(\xi) = (\rho(\xi,\cdot), \psi(\xi,\cdot), \psi_\xi(\xi,\cdot), \psi_{\xi\xi}(\xi,\cdot))^\mathsf{T} \in \mathcal{W}$$

and the coefficient $\mathbb{A}$ is of the form

$$\mathbb{A}(\xi;\kappa) = \begin{pmatrix} R_1 & R_2 & R_3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ S_1 & S_2 & S_3 & S_4 \end{pmatrix},$$

where $R_1, \dots, S_4$ denote appropriate linear operators on $L^2(0,1)$.

After explicitly constructing a suitable Hilbert basis for $\mathcal{W}$ we obtain, in the spirit of [49], a sequence of finite-dimensional truncated problems of order N, given by

$$\hat{W}_N'(\xi) = \hat{\mathbb{A}}_N(\xi;\kappa)\hat{W}_N(\xi), \quad \text{for } N = 0, 1, 2, \dots \tag{EVP$_N$}$$

on $\mathbb{C}^{4N+4}$, by projecting the dynamical system (EVP) to the span of finitely many (exactly $4N+4$) basis vectors and investigate whether the truncated problems have bounded solutions for some $\kappa \in \mathbb{C}_+$. This constitutes step (ii).

Assuming an exponential stratification in step (iii) and afterwards, we show in Theorem II that these truncated problems permit the definition of Evans functions $D_N(\kappa)$. Each function D_N is analytic on the closed right complex half-plane and has the property that $D_N(\kappa) = 0$ for some $\kappa \in \mathbb{C}_+$ if and only if the truncated problem of order N has a bounded solution. This can be expressed more detailed as follows:
For a regular internal solitary wave of speed $c > c_0$ and for any $N \in \mathbb{N}$ there exist an open neighbourhood $\Omega = \Omega(c,N) \supset \overline{\mathbb{C}_+}$ and an analytic mapping

$$D_N(\cdot) : \Omega \to \mathbb{C}, \kappa \mapsto D_N(\kappa)$$

with the property that, for any $\kappa \in \mathbb{C}_+$, system (EVP$_N$) *has a bounded solution iff $D_N(\kappa) = 0$.*

Step (iv) consists in applying the Evans function approach to small-amplitude waves which are approximated by solitons of the KdV equation, which is known, e.g., from [34]. In Theorem III we state that associated Evans functions do not have zeros $\kappa \in \mathbb{C}_+$ in a neighbourhood of the origin thus suggesting spectral stability of small-amplitude waves. More precisely:
For all $N \in \mathbb{N}$ there exist $R_0 > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ the Evans function $D_{N,\varepsilon}(\kappa)$ associated with an internal solitary wave of speed $c_0 + \varepsilon^2$ satisfies

$$D_{N,\varepsilon}(0) = D'_{N,\varepsilon}(0) = 0 \quad \textit{and} \quad D_{N,\varepsilon}(\kappa) \neq 0 \, \textit{for all } \kappa \in \mathbb{C}_+ \textit{ with } 0 < |\kappa| \leq R_0.$$

One part of the proof is obtained by exploiting the facts that (1) in the truncated problems the eigenvalue problem associated with a KdV soliton is recovered and that (2) the latter is known to be spectrally stable. This idea is motivated by [31, 25].

This thesis is organized as follows. Chapter 2 contains the mathematical model under consideration, a derivation of the Dubreil-Jacotin-Long equation governing the travelling wave profiles, and a review of some known results on the existence of travelling waves in general as well as on their approximation in the small-amplitude case. In Chapter 3 we present the first three steps of our five-step approach to spectral stability, the starting point of which is the Euler eigenvalue problem, and we prove Theorems I and II. In Chapter 4 we consider small-amplitude internal solitary waves together with their associated truncated problems and we prove Theorem III stating that the associated Evans functions do not have zeros with positive real part in a neighbourhood of the origin. Chapter 5 contains conclusions and directions of on-going and future research. In the Appendix, Chapter A contains the derivation of the truncated problems in case of a small-amplitude wave and shows the relationship to the eigenvalue problem of a KdV soliton, and Chapter B provides some well-known background material on the Newton polygon and on the spectral stability approach due to Pego and Weinstein.

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2. The channel model for internal waves in stratified fluids

In this chapter we introduce the underlying mathematical model which describes the motion of fluids in a channel. We derive the Dubreil-Jacotin-Long equation, the profile equation for travelling waves, and present some known results on the existence of both small- and large-amplitude waves.

2.1. Euler equations on a strip: the channel model

In the mathematical modelling of internal waves it is common practice to consider a two-dimensional channel,

$$\mathcal{C} = \{(x, y) : x \in \mathbb{R}, 0 < y < 1\},$$

that is entirely filled with a non-homogeneous, inviscid, incompressible fluid.

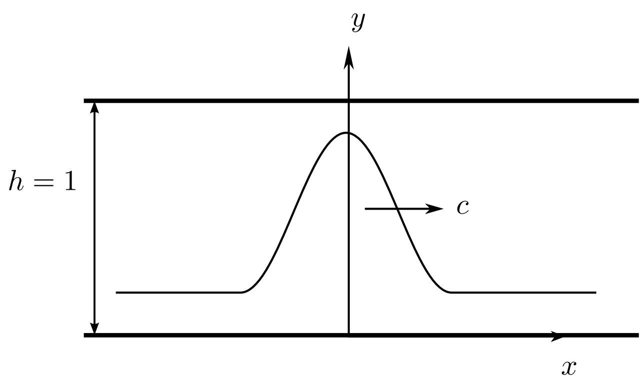

Figure 2.1.: The domain is the two-dimensional channel $\mathbb{R} \times [0, 1]$.

The motion of the fluid is assumed to be governed by the Euler equations (see [46]),

$$\rho_t + u\rho_x + v\rho_y = 0, \tag{2.1a}$$

$$\rho\,(u_t + uu_x + vu_y) = -p_x, \tag{2.1b}$$

$$\rho\left(v_t + uv_x + vv_y\right) = -p_y - g\rho, \tag{2.1c}$$

complemented by the incompressibility constraint

$$u_x + v_y = 0, \tag{2.1d}$$

with t, x and y denoting the time, horizontal and vertical position, respectively, whereas the sought functions, occasionally collected in the vector $U(t,x,y)$, are given by density $\rho(t,x,y)$, velocity field $(u(t,x,y), v(t,x,y))$, and hydrostatic pressure $p(t,x,y)$; the constant g denotes acceleration due to gravity.

The requirement that the fluid cannot leave the domain $\mathcal{C}$ is encoded in the boundary conditions

$$v(t,x,0) = 0 \quad \text{and} \quad v(t,x,1) = 0, \tag{2.2}$$

the second of which is often referred to as the *rigid lid* condition and expresses the fact that in typical applications free-surface displacements are appropriately neglected.

Motivated by the examples from oceanography and limnology, we assume that the fluid at rest exhibits a known density stratification given by a twice continuously differentiable function $\bar{\rho} \in C^2([0,1])$ which depends only on the depth y and which is supposed to satisfy

$$\bar{\rho}(y) > 0 \quad \text{and} \quad \bar{\rho}'(y) < 0; \tag{2.3}$$

such a stratification is called a *stable stratification* (cf. [20]). The two conditions (2.3) have a clear meaning: The first ascertains that the density be positive and the second requirement reflects the fact that in a natural water body the fluid density typically increases with depth.

With this notation the quiescent state, characterized by

$$\rho(t,x,y) = \bar{\rho}(y), \quad u(t,x,y) = v(t,x,y) = 0, \quad p(t,x,y) = \bar{p}(y) := -g\int_0^y \bar{\rho}(\eta)\mathrm{d}\eta, \tag{2.4}$$

is indeed a stationary solution of (2.1).

Remark 2.1. (i) *We would like to emphasize that we consider the incompressible motion of a fluid with a spatially non-homogeneous density distribution. This is an important difference to the typical treatment of incompressible fluid motion where density is usually supposed to be constant at some instant, hence for all time.*

(ii) *The exponential density stratification, given by*

$$\bar{\rho}(y) = \mathrm{e}^{-\delta y}, \ \textit{for some } \delta > 0, \tag{2.5}$$

is a prototypical stratification which clearly fulfils the conditions above, in particular differentiability is not an issue. The results in Section xxxx are achieved under the assumption of an exponentially stratified fluid.

(iii) *We describe two well-known symmetries of the Euler equations which will be exploited later. Suppose*

$$U(t,x,y) = (\rho(t,x,y), u(t,x,y), v(t,x,y), p(t,x,y))$$

is a solution of (2.1)*. A direct calculation shows then that*

$$U_1(t,x,y) = (\rho(t,-x,y), -u(t,-x,y), v(t,-x,y), p(t,-x,y))$$

and

$$U_2(t,x,y) = (\rho(-t,x,y), -u(-t,x,y), -v(-t,x,y), p(-t,x,y))$$

are also solutions of (2.1)*. The first symmetry is reflectional symmetry, the second means time-reversibility.*

(iv) *The initial-value problem for system* (2.1) *(in the full space and in bounded domains) has been considered by a number of authors, e.g. by Marsden [48], by daVeiga and Valli [7, 9, 8], and quite recently by Zhou [62] and by Danchin [17, 18], to name just a few. We do not go into details since we do not even touch on the question of nonlinear stability in this thesis.*

In the present two-dimensional setting it is possible to replace the two equations (2.1b), (2.1c) by only one equation which no longer contains the pressure $p(t,x,y)$. This is accomplished by using a stream function formulation. In the following we describe this alternative form of the Euler equations (2.1) that was proposed by Benjamin in [11]. The incompressibility constraint (2.1d) implies that the vector field $(-v,u)$ is integrable and thus possesses a potential ψ (in the mathematical sense), called the *stream function*, which satisfies

$$\psi_y = u \quad \text{and} \quad \psi_x = -v.$$

Using ψ one can further define a weighted vorticity by

$$\sigma = -\nabla \cdot (\rho \nabla \psi)\,. \tag{2.6a}$$

Now the system (2.1) can be written in terms of the variables (ρ, σ) yielding a non-local evolutionary system without the pressure p:

$$\rho_t = -\{\rho, \psi\}, \tag{2.6b}$$

$$\sigma_t = -\{\sigma, \psi\} - \left\{\rho, gy - \frac{1}{2}|\nabla\psi|^2\right\}, \tag{2.6c}$$

with the Poisson bracket $\{A, B\} := A_x B_y - A_y B_x$. The auxiliary variable ψ is coupled to (ρ, σ) by the uniformly elliptic equation (2.6a). This equation, augmented with suitable boundary conditions, determines ψ uniquely for given (ρ, σ) (see [11, p. 35]).

The derivation of Benjamin's form shows that the Euler equations (2.1) are *formally* equivalent to the system (2.6) in the sense that, under the assumption of sufficient regularity, a solution to the system (2.1) corresponds to a solution of the system (2.6), and vice versa.

We finally mention two important properties of Benjamin's formulation: First, it endows the Euler equations with a Hamiltonian formulation (see [11]), namely

$$\begin{pmatrix} \rho \\ \sigma \end{pmatrix}_t = J\left(\delta H(\rho, \sigma)\right) \tag{2.7}$$

with the skew-symmetric operator

$$J = \begin{pmatrix} 0 & -\{\rho, \cdot\} \\ -\{\rho, \cdot\} & -\{\sigma, \cdot\} \end{pmatrix}$$

and δH denoting the variational gradient of the energy functional

$$H(\rho, \sigma) = \int_{\mathbb{R}} \int_0^1 \frac{1}{2}\rho|\nabla\psi|^2 + gy(\rho - \bar{\rho})\mathrm{d}x\mathrm{d}y. \tag{2.8}$$

Second, it is suited for a straightforward derivation of the *Dubreil-Jacotin-Long equation.*

2.2. Existence of travelling waves: the Dubreil-Jacotin-Long equation

The search for travelling waves in system (2.6) can be reduced to solving a single nonlinear elliptic equation, the *Dubreil-Jacotin-Long (DJL) equation*, for the stream function. This observation goes back to Dubreil-Jacotin and Long, see [21, 47]; for the derivation of a different, but equivalent equation, the so-called *Long-Yih equation*, we refer to [61, 42].

We first review the derivation of the DJL equation for general internal travelling waves (following [58]) and then specialize to the case of *solitary* travelling waves. These types of waves are defined as follows.

Definition 2.2. (i) A solution $U(t,x,y) = (\rho(t,x,y), u(t,x,y), v(t,x,y), p(t,x,y))$ of Equation (2.1) is called an *internal travelling wave (ITW)* of speed c if it has the special form

$$U(t,x,y) = U^c(x-ct,y)$$

with some *travelling wave profile* $U^c(\xi,y)$.

(ii) An *internal solitary wave (ISW)* is an ITW $U(t,x,y) = U^c(x-ct,y)$ with a profile $U^c(\xi,y)$ which decays (uniformly in y) to the quiescent state $(\bar\rho(y), 0, 0, \bar p(y))$ as $|\xi| \to \infty$, i.e.

$$\lim_{\xi\to\pm\infty} U^c(\xi,y) = (\bar\rho(y), 0, 0, \bar p(y)).$$

(iii) A *regular* internal solitary wave additionally satisfies: (1) Each curve of constant density, or *isopycnal*, implicitly defined by $\rho^c(\xi,y) = \varrho = \text{const}$, can be written as a graph $y = Y(\xi)$ with some differentiable function $Y : \mathbb{R} \to [0,1]$ (which necessarily attains the same limits $\bar\rho^{-1}(\varrho)$ at $\pm\infty$. (2) The profile $U^c(\xi,y)$ and all of its partial derivatives up to third order decay exponentially as $|\xi| \to \infty$.

Remark 2.3. (i) *According to the preceding section a travelling wave solution* $U(t,x,y) = U^c(x-ct,y)$ *of system* (2.1) *corresponds to a travelling wave solution*

$$\begin{pmatrix}\rho(t,x,y)\\ \sigma(t,x,y)\end{pmatrix} = \begin{pmatrix}\rho^c(x-ct,y)\\ \sigma^c(x-ct,y)\end{pmatrix}$$

of system (2.6) *with some associated stream function* $\psi(t,x,y) = \psi^c(x-ct,y)$*. If* $U^c(\xi,y)$ *is a solitary wave, then the quantities* $\psi^c(\xi,y)$ *and* $\sigma^c(\xi,y)$ *attain the limits*

$$\lim_{\xi\to\pm\infty} \psi^c(\xi,y) = 0 \text{ and } \lim_{\xi\to\pm\infty} \sigma^c(\xi,y) = 0. \tag{2.9}$$

(ii) *For a regular ISW the equation* $\rho^c(\xi,y) = \text{const}$ *can, by definition, be solved for* y *globally. In particular, the condition for local solvability,* $\partial_y\rho^c(\xi,y) \neq 0$*, holds at all points of the domain. Moreover, as it will turn out, see Equation* (2.12) *and* (2.14)*, that* ρ^c *and* ψ^c *are related in the form*

$$\rho^c(\xi,y) = \bar\rho\left(y - \frac{\psi^c(\xi,y)}{c}\right),$$

there is a one-to-one correspondence between isopycnals and streamlines with the latter being implicitly defined by $\psi^c(\xi, y) - cy = \text{const}$. *Consequently, any streamline is also a differentiable graph over* $\mathbb{R}$, *i.e.* $\partial_y \psi^c(\xi, y) - c \neq 0$, *and connects to the same point for* $\xi \to -\infty$ *and* $\xi \to +\infty$. *Plotting the streamlines, or equivalently the isopycnals, for different values is a widely used means for an appealing visual representation of internal travelling waves (see Fig. 2.2).*

This relationship between ψ^c *and* ρ^c *also implies that the density is bounded, namely*

$$0 < \bar{\rho}(1) \leq \rho^c(\xi, y) \leq \bar{\rho}(0).$$

(iii) *The exponential decay that we require for regular ISWs will be needed for the construction of Evans functions.*

It is a natural assumption to make since exponential decay is true for generic small-amplitude solitary waves (see Lemma 2.8) and, moreover, experiments and numerical considerations (see [41, 55] and references therein) suggest that ISWs are indeed regular up to some large amplitude far beyond the small-amplitude regime. Cf. also [13] for an analytical result on exponential-decay of non-small ISWs.

Internal solitary waves exhibiting a merely algebraic decay are beyond the scope of this thesis (see [34, p. 87ff.] and cf. the Benjamin-Ono case [56]).

As all internal solitary waves we consider are regular, we will often drop this adjective.

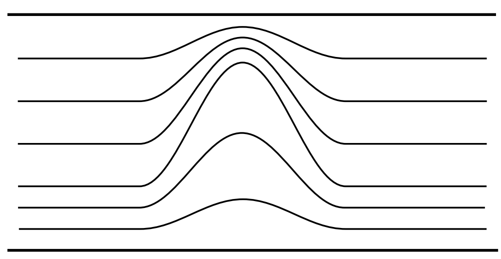

Figure 2.2.: Internal solitary wave of elevation visualized by its streamlines or isopycnals.

It will turn out that all relevant quantities of a travelling wave, i.e. σ^c and the entries of U^c, can be expressed in terms of the stream function ψ^c, thus a travelling wave is completely determined as soon as ψ^c is known. Consequently, we will also refer to ψ^c as *travelling wave of speed c*.

We begin the derivation of the Dubreil-Jacotin-Long equation from system (2.6). In co-moving coordinates $(t, \xi = x - ct, y)$ it has the form

$$\rho_t = -\{\rho, \psi - cy\}, \tag{2.10a}$$

$$\sigma_t = -\{\sigma, \psi - cy\} - \left\{\rho, gy - \frac{1}{2}|\nabla\psi|^2\right\}, \tag{2.10b}$$

$$\sigma = -(\rho\psi_\xi)_\xi - (\rho\psi_y)_y, \tag{2.10c}$$

with the Poisson bracket $\{A, B\} = A_\xi B_y - A_y B_\xi$ (now, due to the change of coordinates, involving derivation with respect to ξ and y instead of x and y as above).

Working in these coordinates is profitable since a travelling wave profile (ρ^c, σ^c) of Equation (2.6) is a steady state of Equation (2.10). It thus satisfies

$$0 = \{\rho^c, \psi^c - cy\}, \tag{2.11a}$$

$$0 = \{\sigma^c, \psi^c - cy\} + \left\{\rho^c, gy - \frac{1}{2}|\nabla\psi^c|^2\right\}, \tag{2.11b}$$

$$\sigma^c = -(\rho^c\psi^c_\xi)_\xi - (\rho^c\psi^c_y)_y. \tag{2.11c}$$

In the course of deriving the DJL equation it is helpful to use the notion of *functional dependency*: Two real-valued quantities $A(\xi, y)$ and $B(\xi, y)$ are called *functionally dependent* if there exists a sufficiently smooth function $F : \mathbb{R} \to \mathbb{R}$ such that

$$B(\xi, y) = F(A(\xi, y)).$$

Clearly, this implies $\{A, B\} = 0$, since

$$\{A, B\} = A_\xi F'(A) A_y - A_y F'(A) A_\xi = 0.$$

What is more, the converse is true as well, at least under the assumption of appropriate differentiability of A and B: *If two quantities A and B as above satisfy $\{A, B\} = 0$ then they are functionally dependent.* For a proof and further information, see [57, Lemma 4.1].

With this statement at hand we conclude from Equation (2.11a) that there exists some function $F : \mathbb{R} \to \mathbb{R}$ such that

$$\rho^c(\xi, y) = F(\psi^c(\xi, y) - cy). \tag{2.12}$$

After plugging this into Equation (2.11b) one finds

$$\left\{\sigma^c - F'(\psi^c - cy)\left(gy - \frac{1}{2}|\nabla\psi^c|^2\right), \psi^c - cy\right\} = 0,$$

hence, by expressing σ^c in terms of ρ^c and ψ^c, i.e. by writing

$$\sigma^c = -\nabla(\rho^c \cdot \nabla\psi^c) = -\rho^c\Delta\psi^c - \nabla\rho^c \cdot \nabla\psi^c$$

we see that

$$-F(\psi^c - cy)\Delta\psi^c - F'(\psi^c - cy)\left(\frac{1}{2}|\nabla\psi^c|^2 - c\psi^c_y + gy\right) \quad \text{and} \quad \psi^c - cy$$

are functionally dependent too, i.e. there exists a function $G : \mathbb{R} \to \mathbb{R}$ such that

$$\Delta\psi^c + \frac{F'(\psi^c - cy)}{F(\psi^c - cy)}\left(\frac{1}{2}|\nabla\psi^c|^2 - c\psi^c_y + gy\right) = G(\psi^c - cy). \tag{2.13}$$

This equation was first discovered by Dubreil-Jacotin (see [21]) and, independently, by Long (see [47]).

Depending on the type (e.g. periodic or solitary) of travelling waves under investigation it is possible to determine the functions F and G explicitly. We illustrate this for internal solitary waves in the rest of this section.

We recall from Definition 2.2, and the remark thereafter, that $\rho^c(\xi, y)$ and $\psi^c(\xi, y)$ attain limits as $\xi \to \pm\infty$, namely

$$\lim_{\xi\to\pm\infty} \rho^c(\xi, y) = \bar{\rho}(y) \text{ and } \lim_{\xi\to\pm\infty} \psi^c(\xi, y) = 0.$$

This permits a precise determination of F and G as follows. The identity Equation (2.12) holds in the entire domain $\mathcal{C}$ and we can take the limit as $|\xi| \to \infty$ yielding, by continuity,

$$\bar{\rho}(y) = \lim_{\xi\to\pm\infty} \rho^c(\xi, y) = \lim_{\xi\to\pm\infty} F(\psi^c(\xi, y) - cy) = F\left(\lim_{\xi\to\pm\infty} \psi^c(\xi, y) - cy\right) = F(-cy).$$

We have thus found the function F to be

$$F(z) = \bar{\rho}\left(-\frac{z}{c}\right). \tag{2.14}$$

We can determine G from Equation (2.13) in a similar fashion by taking the limits of both sides for $|\xi| \to \infty$; this yields

$$\frac{F'(-cy)}{F(-cy)} gy = G(-cy)$$

from which we obtain

$$G(z) = -\frac{g}{c}\frac{zF'(z)}{F(z)} = \frac{g}{c^2}\frac{z\bar{\rho}'\left(-\frac{z}{c}\right)}{\bar{\rho}\left(-\frac{z}{c}\right)}.$$

Using these concrete expressions for F and G, the DJL equation for an ISW finally takes the form:

$$\Delta\psi^c - \frac{\bar{\rho}'(y - \frac{\psi^c}{c})}{\bar{\rho}(y - \frac{\psi^c}{c})}\left(\frac{1}{2c}\left|\nabla\psi^c\right|^2 - \psi^c_y + \frac{g}{c^2}\psi^c\right) = 0. \tag{2.15}$$

From a solution $\psi^c(\xi, y)$ to this equation, we readily compute an exact travelling wave solution $U(t, x, y)$ of the Euler equations (2.1) by recalling how ρ^c, u^c and v^c are related to ψ^c; we find

$$U(t,x,y) = \left(\bar{\rho}\left(y - \frac{1}{c}\psi^c(x-ct,y)\right), \psi^c_y(x-ct,y), -\psi^c_\xi(x-ct,y), p^c(x-ct,y)\right)^\mathsf{T}.$$

Note that $p^c(\xi, y)$ is a solution of the uniformly elliptic equation

$$-\operatorname{div}\left(\frac{1}{\rho^c}\nabla p\right) = 2\det\left(D^2\psi^c\right),$$

which can be readily derived from the stationary Euler equations in coordinates ξ, y, hence it is uniquely determined by ψ^c, too, up to an unimportant additive constant.

By changing from ψ^c to $\hat{\psi} := -\frac{\psi^c}{c}$ Equation (2.15) looks a little simpler and the problem reads as a nonlinear elliptic boundary value problem consisting of the equation

$$\Delta\hat{\psi} + \frac{\bar{\rho}'(y+\hat{\psi})}{\bar{\rho}(y+\hat{\psi})}\left(\frac{1}{2}\left|\nabla\hat{\psi}\right|^2 + \hat{\psi}_y - \lambda\hat{\psi}\right) = 0, \tag{2.16a}$$

the boundary conditions

$$\hat{\psi}(\xi, 0) = \hat{\psi}(\xi, 1) = 0 \tag{2.16b}$$

and the decay condition

$$\lim_{|\xi|\to\infty} \hat{\psi}(\xi, y) = 0 \tag{2.16c}$$

with parameter $\lambda := \frac{g}{c^2}$ being the inverse square of the *Froude number*. In Section 2.4 we will point out that λ can actually be considered as an “eigenvalue”.

Strictly speaking, a *solution* to system (2.16) is a pair

$$(\lambda, \hat{\psi}) \in \mathbb{R} \times \mathcal{X}$$

with a real number λ and a function $\hat{\psi} = \hat{\psi}(\xi, y)$, which belongs to some appropriate function space $\mathcal{X}$, such that Eqs. (2.16a), (2.16b) and (2.16c) are satisfied. The trivial flow $\hat{\psi}(\xi, y) = 0$, which corresponds to the quiescent state (given in Equation (2.4)), obviously solves (2.16a) for any $\lambda \in \mathbb{R}$, hence $(\lambda, 0)$ is a solution of (2.16) for all $\lambda \in (0, \infty)$. Therefore, searching solutions of this problem means investigating a local or global bifurcation depending on whether the wave amplitude is assumed to be small or not.

The Sections 2.3 and 2.4 review known results about the existence of both small-amplitude and large-amplitude solutions to problem (2.16) and about approximate expressions for the solutions in the small-amplitude case.

2.3. Small-amplitude waves: approximation by KdV

In this section we discuss solutions of Equation (2.16) that have small amplitudes. Benjamin and Benney (see [10, 12]) appear to be the first who have noticed that long non-linear internal waves of sufficiently small amplitude can be described approximately by means of the Korteweg-deVries equation (KdV) in the following way: A small solution $\hat{\psi}(\xi, y)$ of system (2.16) is to leading order a product,

$$\hat{\psi}(\xi, y) = A(\xi) \times \varphi(y) + \text{ higher order terms,}$$

with separated variables consisting of a travelling wave solution $A(\xi)$ to (an appropriately scaled) KdV equation and a vertical mode $\varphi(y)$, which arises from a Sturm-Liouville problem.

The rigorous proofs of this fact which were given by Kirchgässner in [38], by Lankers and Kirchgässner in [39], and by James in [34] are based on Kirchgässner's spatial-dynamics approach to elliptic equations on cylindrical domains (see [38]). This approach, which is by now a standard method for problems of this kind, consists in rewriting the problem as an abstract ODE in which the unbounded space variable assume the role of "time". Although the elliptic initial-value problem is ill-posed, this approach allows to apply methods from the theory of dynamical systems in studying the behaviour of solutions. In particular, several theorems on centre-manifold reduction are available and often allow to reduce the problem of searching for small solutions to the investigation of an ODE on a finite-dimensional space.

In the rest of this section we present a result due to James on the existence of small-amplitude solutions of the DJL equation and some preliminary material; we follow closely his work [34]. For a corresponding treatment of small solutions of the Long-Yih equation we refer to Kirchgässner and Lankers, see [38, 39].

The DJL equation can be formulated as a dynamical system in which the unbounded variable ξ assumes the role of the time variable. Letting $\psi_1 = \psi$ and $\psi_2 = \psi_\xi$ Equation (2.16a) takes the form

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_\xi = \begin{pmatrix} \psi_2 \\ -\psi_{1,yy} - \frac{\bar\rho'(y+\psi_1)}{\bar\rho(y+\psi_1)} \left(\frac{1}{2}(\psi_{1,y}^2 + \psi_2^2) + \psi_{1,y} - \lambda\psi_1 \right) \end{pmatrix}, \tag{2.17}$$

or shorter

$$\Psi_\xi = \mathcal{F}(\Psi; \lambda), \tag{2.18}$$

with the parameter $\lambda \in \mathbb{R}$ (proportional to the inverse square of the speed c) as before. For a proper functional analytic setting for this equation (see [34, p. 70]) define

$$\mathcal{Y}^1 := (H^2(0,1) \cap H_0^1(0,1)) \times H^1(0,1) \quad \text{and} \quad \mathcal{Y} := H^1(0,1) \times L^2(0,1).$$

Then $\mathcal{F}(\cdot; \lambda) : \mathcal{Y}^1 \to \mathcal{Y}$ is defined in a neighbourhood of 0. In this context, an element $\Psi \in C^0(\mathbb{R}, \mathcal{Y}^1) \cap C^1(\mathbb{R}, \mathcal{Y})$ satisfying Equation (2.18) is called a *solution*.

Clearly, $\Psi = 0$ is a solution for all λ, hence Equation (2.18) forms a bifurcation problem and we search for the set of *bifurcation values*, i.e., the set of those λ where additional solutions appear. Candidates for bifurcation values are found, like in the finite-dimensional setting, as those values λ for which the linearized operator

$$L_\lambda := \frac{d\mathcal{F}}{d\Psi}(0; \lambda),$$

explicitly given by

$$L_\lambda \Psi = \begin{pmatrix} \psi_2 \\ -\psi_{1,yy} - \frac{\bar\rho'(y)}{\bar\rho(y)} (\psi_{1,y} - \lambda\psi_1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \mathcal{T}_\lambda & 0 \end{pmatrix} \Psi,$$

has imaginary eigenvalues; here

$$\mathcal{T}_\lambda \psi_1 := -\psi_{1,yy} - \frac{\bar\rho'(y)}{\bar\rho(y)} (\psi_{1,y} - \lambda\psi_1).$$

This question can be answered easily by investigating the spectrum Σ_λ of L_λ. In doing so, one is first of all led to analyze the spectrum of $\mathcal{T}_\lambda$, which is given in the following lemma.

Lemma 2.4. *The operator*

$$\mathcal{T}_\lambda : H_0^1(0,1) \cap H^2(0,1) \subseteq L^2_{\bar\rho}(0,1) \to L^2_{\bar\rho}(0,1)$$

is formally self-adjoint and uniformly elliptic. Its spectrum consists of an increasing sequence of real, simple eigenvalues,

$$\nu_0(\lambda) < \nu_1(\lambda) < \cdots \to +\infty,$$

which accumulate only at $+\infty$, and there are corresponding eigenfunctions $\{\chi_n\}_{n\in\mathbb{N}}$, which are normalized and mutually orthogonal, i.e.

$$\int_0^1 \bar\rho \chi_n \chi_m \mathrm{d}y = \delta_{nm}. \tag{2.19}$$

A sketch of the proof can be found in [34, p. 71]. Since the eigenvalue problem is a regular Sturm-Liouville problem, this statement follows directly from classical Sturm-Liouville theory, as presented, e.g., in [53, Ch. 8.6].

The following lemma describes the set of λ for which $\mathcal{T}_\lambda$ has a zero eigenvalue.

Lemma 2.5. *Let $\bar\rho$ denote a stable stratification. The operator*

$$\mathcal{S} := \frac{1}{\bar\rho'}\partial_y(\bar\rho\partial_y) : H_0^1(0,1) \cap H^2(0,1) \subseteq L^2_{-\bar\rho'}(0,1) \to L^2_{-\bar\rho'}(0,1),$$

is positive, formally self-adjoint and uniformly elliptic. Its spectrum consists of an increasing sequence of real, simple eigenvalues,

$$0 < \lambda_0 < \lambda_1 < \cdots \to \infty,$$

which accumulate only at $+\infty$, and there are corresponding eigenfunctions $\{\varphi_n\}_{n\in\mathbb{N}}$, which are normalized and mutually orthogonal, i.e.

$$\int_0^1 (-\bar\rho')\varphi_n \varphi_m \mathrm{d}y = \delta_{nm}. \tag{2.20}$$

The eigenvalue problem for $\mathcal{S}$ is a regular Sturm-Liouville problem as well, thus this statement also follows from classical Sturm-Liouville theory (see e.g. [53, Ch. 8.6]). A (sketch of the) proof can be found [12, 34].

Remark 2.6. (i) *Note that the preceding lemma actually shows that $\mathcal{T}_\lambda$ has a zero eigenvalue for $\lambda = \lambda_n$, namely $\nu_n(\lambda_n) = 0$. This is demonstrated by the following calculation:*

$$\mathcal{T}_{\lambda_n}\varphi_n = -\frac{\bar\rho'(y)}{\bar\rho(y)}(\mathcal{S} - \lambda_n \mathrm{Id})\varphi_n = -\frac{\bar\rho'(y)}{\bar\rho(y)}(\lambda_n\varphi_n - \lambda_n\varphi_n) = 0.$$

Consequently, $\nu_n(\lambda_n) = 0$ and, moreover, $\chi_n = C\varphi_n$. Another calculation, similar to the one performed in (ii), shows $C^{-2} = \int_0^1 \bar\rho\varphi_n^2 \mathrm{d}y$.

(ii) *For the exponential stratification $\bar\rho(y) = \mathrm{e}^{-\delta y}$, eigenfunctions of $\mathcal{S}$ are also eigenfunctions of $\mathcal{T}_\lambda$ because*

$$\mathcal{T}_\lambda = -\frac{\bar\rho'(y)}{\bar\rho(y)}(\mathcal{S} - \lambda\mathrm{Id}) = \delta(\mathcal{S} - \lambda\mathrm{Id});$$

consequently

$$\mathcal{T}_\lambda\varphi_n = \delta(\lambda_n - \lambda)\varphi_n \quad \textit{for any } n \in \mathbb{N}.$$

Thus $\chi_n = C \cdot \varphi_n$ where C is determined from the normalization of χ_n and φ_n as follows:

$$1 = \int_0^1 \bar\rho\chi_n^2 \mathrm{d}y = \frac{C^2}{\delta}\int_0^1 (-\bar\rho')\varphi_n^2 \mathrm{d}y = \frac{C^2}{\delta}.$$

To sum up, we have found $\nu_n(\lambda) = \delta(\lambda_n - \lambda)$ and $\chi_n = \sqrt{\delta}\varphi_n$.

Combining the two preceding lemmas gives a rather complete description of Σ_λ and its evolution under varying λ.

Lemma 2.7. *The spectrum Σ_λ of L_λ has the following properties:*

(i) $-\Sigma_\lambda = \Sigma_\lambda$.

(ii) $\Sigma_\lambda \subset \mathbb{R} \cup i\mathbb{R}$ *consists entirely of eigenvalues with finite multiplicity accumulating only at infinity.*

(iii) Σ_{λ_n} *consists of countably infinitely many eigenvalues among which there are one double zero eigenvalue and $2n$ imaginary non-zero simple eigenvalues whereas all other eigenvalues are real and different from zero.*

A proof is given in [34, p. 71].

To sum up, Lemma 2.7 shows that L_λ has a zero eigenvalue iff $\lambda = \lambda_n$ for some $n \in \mathbb{N}$. In this case 0 is a double eigenvalue, hence the dimension of the generalized central eigenspace (i.e. the span of all, possibly generalized, eigenvectors associated with any imaginary eigenvalue) is $2n + 2$. Given the assumptions on $\mathcal{F}$ (see [34, p. 70]), a theorem on centre-manifold reduction (due to Vanderbauwhede and Iooss, see [60] and references therein) is applicable and implies the existence of a $(2n+2)$-dimensional invariant manifold containing all solutions of sufficiently small amplitude. Then it suffices to find solutions of the reduced equation, which is an ODE on this finite-dimensional manifold. The structure of the solution can be studied by converting the reduced equation to normal form. We will not enter technical details of this procedure, instead we refer to the extensive monographs [33] and [30].

Various authors have performed this reduction procedure in order to obtain existence and structure of small-amplitude internal waves, e.g. see Kirchgässner and James in [38, 34] for the case $n = 0$ and Kirchgässner and Lankers in [39] for the cases $n = 0$ and $n = 1$. For small $\epsilon := \lambda_0 - \lambda > 0$ and in the case of lowest order (i.e. $p = 2$ in his notation), James obtains that the normal form, which is an ordinary differential equation on the two-dimensional centre manifold, is given by

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}\xi} &= B, \\ \frac{\mathrm{d}B}{\mathrm{d}\xi} &= -\left(a_{11} + O(\epsilon)\right)\epsilon A + \left(a_{20} + O(\epsilon)\right)A^2 + R(A, B, \epsilon), \end{aligned} \tag{2.21}$$

with the coefficients

$$a_{11} = \frac{\int_0^1 \bar{\rho}'\varphi_0^2 \mathrm{d}y}{\int_0^1 \bar{\rho}\varphi_0^2} \quad \text{and} \quad a_{20} = \frac{3}{2}\frac{\int_0^1 \bar{\rho}(\varphi_0')^3 \mathrm{d}y}{\int_0^1 \bar{\rho}\varphi_0^2 \mathrm{d}y} \tag{2.22}$$

and with an error term $R(A, B, \epsilon)$ of higher order.

From formula (2.22) it follows directly that $a_{11} < 0$ whereas, depending on the concrete form of $\bar{\rho}(y)$, a_{20} can be positive or negative or even zero. In case of $a_{20} \neq 0$ it suffices to consider the *truncated* version of Equation (2.21), obtained therefrom by formally setting $R \equiv 0$, in order to find an approximate solution to Equation (2.18). On the other hand, if $a_{20} = 0$ then this truncated version does not suffice to describe small solutions to Equation (2.18), and one has to take higher order terms into account; James investigates possible solutions for this case in detail.

In the following we restrict to stratifications satisfying $a_{20} \neq 0$, which is a generic property; in particular, the exponential stratification satisfies this assumption (see below). Under this assumption analyzing the phase portrait of the truncated version of Equation (2.21) yields: There are two fixed points, namely one hyperbolic saddle and one centre. The system is Hamiltonian, hence orbits are contained in its level sets. One thus finds that periodic solutions around the centre form a continuum the outer boundary of which consists of the hyperbolic fixed point and one homoclinic orbit connecting this point to itself (see Fig. 2.3). Suppose $(A(\xi), B(\xi))$ is a solution of (2.21). Then, due

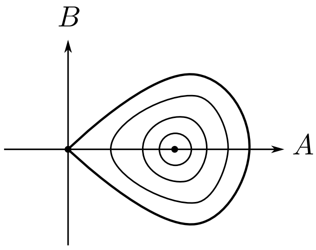

Figure 2.3.: Sketch of the phase portrait (known as the KdV fish*).*

to [34, Theorems 4.1., 4.2], the corresponding solution of Equation (2.18) is of the form

$$\Psi(\xi) = A(\xi) \begin{pmatrix} \varphi_0 \\ 0 \end{pmatrix} + B(\xi) \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix} + \text{nonlinear terms}$$

with $(\varphi_0, 0)^\mathsf{T}$ and $(0, \varphi_0)^\mathsf{T}$ spanning the generalized central space of L_λ. For $\psi(\xi, y)$, the first component of Ψ, this means

$$\psi(\xi, y) = A(\xi) \times \varphi_0(y) + \text{higher order terms}$$

confirming the claim in the section's beginning that small solutions have product structure to leading order.

We will express this result more precisely in the next lemma after slightly changing the notation to a form, which is more suitable for later purposes.

According to Benney (see [12]) we introduce the constants

$$r = -\frac{3}{4} \frac{\int_0^1 \bar{\rho}(\varphi_0')^3 \mathrm{d}y}{\int_0^1 \bar{\rho}(\varphi_0')^2 \mathrm{d}y} = \frac{a_{20}}{2\lambda_0 a_{11}},$$

and

$$s = -\frac{c_0}{2} \frac{\int_0^1 \bar{\rho}\varphi_0^2 \mathrm{d}y}{\int_0^1 \bar{\rho}(\varphi_0')^2 \mathrm{d}y} = \frac{c_0}{2\lambda_0 a_{11}}$$

and consider the small parameter

$$\varepsilon^2 := c - c_0$$

(instead of $\epsilon = \lambda_0 - \lambda$ as above, which corresponds to $-\mu$ in [34]).

Adopting this notation we state the announced lemma on the existence of small-amplitude waves.

Lemma 2.8. *Suppose $\bar{\rho}$ satisfies the condition*

$$\int_0^1 \bar{\rho}(\varphi_0')^3 \mathrm{d}y \neq 0. \tag{2.23}$$

Then there is some $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a unique symmetric regular ISW $\psi^c(\xi, y)$ of speed $c = c_0 + \varepsilon^2$. This wave is approximated by

$$\psi^c(\xi, y) = a_\varepsilon(\xi)\varphi_0(y) + O\left(|a_\varepsilon|^2 + |a_\varepsilon'|^2 + \left|\varepsilon^2 a_\varepsilon\right|\right)$$

with $a_\varepsilon(\xi)$ solving

$$a_\varepsilon''(\xi) = \left(-\frac{\varepsilon^2}{s}\right) a_\varepsilon + \left(-\frac{r}{s}\right) a_\varepsilon^2 + R(a_\varepsilon, a_\varepsilon', \varepsilon^2), \tag{2.24}$$

where

$$R(a_\varepsilon, a_\varepsilon', \varepsilon^2) = O(|a_\varepsilon|^3 + |a_\varepsilon|\,|a_\varepsilon'|^2 + |a_\varepsilon'|^4)$$

denotes some residual term which is even with respect to a_ε' and vanishes for $a_\varepsilon(\xi) = a_\varepsilon'(\xi) = 0$.

For a proof of this result, see e.g. [34] and references therein.

Remark 2.9. (i) *For a given stratification $\bar{\rho}$ fixing s and r let us denote by $A_*(\Xi)$ the reference soliton defined as the unique symmetric soliton solution of*

$$\ddot{A}_*(\Xi) = \left(-\frac{1}{s}\right) A_* + \left(-\frac{r}{s}\right) A_*^2$$

connecting the origin to itself. In terms of A_* the leading order of a_ε can be expressed as

$$a_\varepsilon(\xi) = \underbrace{\varepsilon^2 A_*(\varepsilon\xi)}_{=:\hat{a}_\varepsilon(\xi)} \left(1 + O(\varepsilon^2)\right)$$

where $\hat{a}_\varepsilon(\xi)$, which is an exact solution of

$$\hat{a}_\varepsilon''(\xi) = \left(-\frac{\varepsilon^2}{s}\right) \hat{a}_\varepsilon + \left(-\frac{r}{s}\right) \hat{a}_\varepsilon^2,$$

is given by the well-known explicit formula

$$\hat{a}_\varepsilon(\xi) = -\frac{3}{2r}\varepsilon^2 \operatorname{sech}^2\left(\sqrt{-\frac{1}{s}}\varepsilon\xi\right).$$

This formula serves to point out three important properties of $\hat{a}_\varepsilon(\xi)$, hence of $a_\varepsilon(\xi)$: The amplitude of $\hat{a}_\varepsilon(\xi)$ is of order ε^2 (because of the leading factor ε^2); the argument is of order ε (because of the factor ε inside the brackets), thus the function changes slowly in terms of ξ; the sign of r determines the sign of $\hat{a}_\varepsilon(\xi)$ and, thus, whether the resulting solution is a wave of elevation or a wave of depression.

(ii) *In Chapter 4 we will expediently use two rescaled variants of $a_\varepsilon(\xi)$ which are defined as*

$$a_\varepsilon(\xi) = \varepsilon^2 A_\varepsilon(\xi) \quad \textit{and} \quad A_\varepsilon(\xi) = \tilde{A}_\varepsilon(\varepsilon\xi).$$

Both of them can be described approximately in terms of $A_*(\Xi)$ by

$$A_\varepsilon(\xi) = A_*(\varepsilon\xi)\left(1 + O(\varepsilon^2)\right) \quad \textit{and} \quad \tilde{A}_\varepsilon(\Xi) = A_*(\Xi)\left(1 + O(\varepsilon^2)\right). \tag{2.25}$$

(iii) *From the solution $a_\varepsilon(\xi)$ we find a solution to Equation* (2.21) *in James's notation by the transformation $A(\xi) = -\frac{1}{c_0}a_\varepsilon(\xi)$, which results from different scalings: James considers waves travelling with speed* 1 *to the left, and Benney considers waves travelling with speed $c \approx c_0$ to the right.*

For the sake of concreteness we cite the explicit expressions for the exponential stratification $\bar{\rho}(y) = \mathrm{e}^{-\delta y}$ (see [12, p. 60]).

$$\begin{aligned}
c_0 &= \frac{\delta}{\frac{1}{4}\delta^2 + \pi^2},\\
\varphi_0(y) &= \sqrt{2}\,\mathrm{e}^{\frac{\delta}{2}y}\sin(\pi y),\\
r &= -\frac{3\delta\pi^3(\mathrm{e}^{\delta/2} + 1)}{2\left(\frac{1}{4}\delta^2 + \pi^2\right)\left(\frac{1}{4}\delta^2 + 9\pi^2\right)} < 0,\\
s &= -\frac{c_0}{2\delta\lambda_0} < 0.
\end{aligned}$$

For the higher modes associated with $n \geq 1$ much richer dynamics may occur since the dimension of the centre manifold increases and the normal form becomes more complicated. For an arbitrary stratification it is, therefore, no longer true that the expression $A(\xi) \times \varphi_n(y)$ is the leading order term of a persistent small ISW, even for $n = 1$. Kirchgässner and Lankers have provided a criterion for $\bar{\rho}$ which decides whether this persistence property is true or not (see [39]); in case it is not fulfilled then, typically, small-amplitude wave solutions do not decay to zero at infinity but develop so-called *oscillatory tails* in agreement with experimental results (see [2]).

2.4. Existence of large travelling waves

In the preceding section we presented results on the existence of small-amplitude ISWs and on approximate expressions for them. In this section we cite a result on the existence of general ISWs, i.e. *without* the requirement of small amplitudes, which is an interesting and complicated question by itself as indicated by the diversity of methods employed. Several authors have contributed results to this subject, see e.g. [4, 13, 43, 42] for theoretical results and, too, [58] for a numerical algorithm which serves to compute these solutions practically.

A central result in each of these publications is of the following kind.

For a certain set of density stratifications there exists a connected branch $\mathcal{K} \subset \mathbb{R} \times \mathcal{X}$ of solutions with $(\lambda_0, 0) \in \overline{\mathcal{K}}$.

This set of stratifications contains the exponential stratification in all cases. The solution space $\mathcal{X}$ depends on the method chosen, for instance $\mathcal{X} = H_0^1(\mathcal{C}) \cap C_0(\mathcal{C})$ in [4] and $\mathcal{X} = C^2(\mathcal{C}) \cap C_0(\mathcal{C})$ in [43].

Instead of an extensive overview of the various ways to prove the existence of large-amplitude waves, we only provide a quick glance on one such result relying on a variational formulation. The presentation follows closely [13] and [58] concerning statements on the existence of solutions and their properties.

Before stating the result, we would like to point out a peculiarity of the Euler equations. The formulation due to Benjamin, as given in (2.7), shows that the Euler equations can be viewed as a Hamiltonian system. This, however, does not automatically yield a variational principle as we explain a little more detailed. After changing to co-moving coordinates Equation (2.7) takes the form

$$\begin{pmatrix} \rho \\ \sigma \end{pmatrix}_t = J\left(\delta(H - cI)(\rho, \sigma)\right) \tag{2.26}$$

with

$$I(\sigma) = \int_{\mathbb{R}} \int_0^1 y\sigma \mathrm{d}\xi \mathrm{d}y \tag{2.27}$$

denoting the momentum functional. Now, a travelling wave (ρ^c, σ^c) is a stationary solution of this equation, hence it satisfies

$$J\left(\delta(H-cI)(\rho^c,\sigma^c)\right)=0; \tag{2.28}$$

however, (ρ^c,σ^c) is *not* a critical point of $H-cI$. This is a consequence of the fact that $J=J(\rho^c,\sigma^c)$, which depends itself on the yet unknown solution, has a non-trivial kernel. In the context of infinite-dimensional Hamiltonian systems, this is a familiar situation, and one possible way out consists in searching for conserved quantities $\tilde{H}$ and $\tilde{I}$ such that (ρ^c,σ^c) becomes a critical point of $(H+\tilde{H})-c(I+\tilde{I})$. In doing so, one finds

$$\begin{aligned}(H+\tilde{H})(\rho,\sigma)&=\int_{\mathbb{R}}\int_0^1 \frac{1}{2}\rho\,|\nabla\psi|^2+g\left\{\int_{\bar{\rho}}^{\rho}\left[y-\bar{\rho}^{-1}(\tilde{r})\right]\mathrm{d}\tilde{r}\right\}\mathrm{d}\xi\mathrm{d}y,\\(I+\tilde{I})(\rho,\sigma)&=\int_{\mathbb{R}}\int_0^1 \sigma\left[y-\bar{\rho}^{-1}(\rho)\right]\mathrm{d}\xi\mathrm{d}y.\end{aligned}$$

We refer to [58, p. 125] for these expressions and to [1] for comprehensive material on infinite-dimensional Hamiltonian systems arising in fluid mechanics and on the use of so-called Casimir functionals in systematically deriving variational principles.

It is also possible to find a variational principle for travelling waves in a more direct way starting from Equation (2.16a). This has been done e.g. in [58] resulting in the following characterization: A travelling wave ψ^c is a minimizer of the functional

$$E(\hat{\psi})=\int_{\mathbb{R}}\int_0^1\frac{1}{2}\bar{\rho}(y+\hat{\psi})\left|\nabla\hat{\psi}\right|^2\mathrm{d}\xi\mathrm{d}y$$

under the constraint that

$$F(\hat{\psi})=\int_{\mathbb{R}}\int_0^1\int_0^{\hat{\psi}}\left[\bar{\rho}(y+\eta)-\bar{\rho}(y+\hat{\psi})\right]\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}y$$

be constant. Therefore, ψ^c solves

$$\delta\left[E(\hat{\psi})-\lambda F(\hat{\psi})\right]=0,$$

where the parameter $\lambda=\frac{g}{c^2}$ appears as a Lagrange multiplier (see [58]). This equation can, likewise, be viewed as a nonlinear eigenvalue problem with eigenvalue λ. Turkington and co-workers also show that both variational principles are closely related $E-\lambda F$ and $(H+\tilde{H})-c(I+\tilde{I})$ only differing by a constant factor when evaluated on travelling wave solutions; see [58, p. 124ff.].

By using the variational principle based on $E-\lambda F$, the authors of [13] and [58] first construct periodic travelling waves and second obtain solitary waves as the limits when the period tends to infinity; we do not go into details of the sophisticated arguments needed to turn this idea into a rigorous proof. Applied to the exponential stratification the authors obtain the following result.

Lemma 2.10. *Consider the exponential stratification* $\bar{\rho} = \mathrm{e}^{-\delta y}$*. For any* $\lambda \in (0, \lambda_0)$ *there is a solution* $\psi \in C^{2,\alpha}(\mathcal{C})$ *to problem* (2.16) *with the properties:*

(i) *Symmetry:* $\psi(-\xi, y) = \psi(\xi, y)$ *holds for all* $(\xi, y) \in \mathcal{C}$*.*

(ii) *Exponential decay: There are constants* $C, C_1, C_2 > 0$ *such that* $|\psi(\xi, y)| \leq C_1 \mathrm{e}^{-C|\xi|}$ *and* $|\nabla\psi(\xi, y)| \leq C_2 \mathrm{e}^{-C|\xi|}$ *hold.*

(iii) *Constant sign: Either* $\psi(\xi, y) \geq 0$ *and* $\psi_\xi(\xi, y) \leq 0$ *for all* $\xi \geq 0$ *or* $\psi(\xi, y) \leq 0$ *and* $\psi_\xi(\xi, y) \geq 0$ *for all* $\xi \geq 0$*.*

The result in this form and a proof can be found in [13, Theorem 5.1].

Remark 2.11. (i) *The relation* $\lambda = \frac{g}{c^2}$ *implies that these waves become arbitrarily fast as* λ *tends to zero. In [43] it was shown that these waves become arbitrarily large as well.*

(ii) *It is not clear whether all of these waves are regular in the sense of Definition 2.2, whereas it is clear that they are regular for* λ *close to* λ_0 *due to the small-amplitude result in the previous section. However, elliptic regularity theory seems to indicate that these waves are regular ISWs for some larger finite range* $\lambda \in (\lambda_R, \lambda_0)$*, with some* $\lambda_R \gg \lambda_0$*, but we do not go into details here.*

3. Evans function approach to spectral stability

The purpose of this chapter is to present three central results of this thesis. After a precise statement of the Euler eigenvalue problem associated with a travelling wave solution, we present Theorem I on a spatial-dynamics formulation thereof in a suitable Hilbert space. Then, specializing to the exponential stratification, we show that finite-dimensional truncations of this infinite-dimensional system can be obtained which have a special structure. Finally, we prove in Theorem II that each of the truncated systems permits the proper definition of an Evans function providing a convenient tool to detect unstable modes in the truncated system.

3.1. The Euler eigenvalue problem

Considering the linearized problem and restricting to temporally exponential solutions of the form

$$\mathrm{e}^{\kappa t} U(\xi, y),$$

for some $\kappa \in \mathbb{C}$, one obtains the following eigenvalue problem associated with some internal travelling wave $U^c(\xi, y)$:

$$\begin{aligned}
-\kappa\rho &= (u^c - c)\rho_\xi + v^c\rho_y + u\rho^c_\xi + v\rho^c_y, \\
-\rho^c\kappa u &= \rho^c\left((u^c - c)u_\xi + uu^c_\xi + v^c u_y + vu^c_y\right) \\
&\quad + \rho\left((u^c - c)u^c_\xi + v^c u^c_y\right) + p_\xi, \\
-\rho^c\kappa v &= \rho^c\left((u^c - c)v_\xi + uv^c_\xi + v^c v_y + vv^c_y\right) \\
&\quad + \rho\left((u^c - c)v^c_\xi + v^c v^c_y\right) + p_y + g\rho, \\
0 &= u_\xi + v_y.
\end{aligned} \tag{3.1}$$

For the spectral stability of the underlying wave U^c one has to decide whether this problem possesses any bounded solution for some $\kappa \in \mathbb{C}$ with $\operatorname{Re}\kappa > 0$. For investigating this question we do not begin from Equation (3.1), but use instead a version corresponding to Benjamin's formulation which is given by

$$\kappa\rho = -\{\rho^c, \psi\} - \{\rho, \psi^c - cy\}, \tag{3.2a}$$

$$\begin{aligned}\kappa\sigma = &-\{\sigma^c, \psi\} - \{\sigma, \psi^c - cy\} \\ &- \left\{\rho, -\frac{1}{2}|\nabla\psi^c|^2 + gy\right\} - \{\rho^c, -\nabla\psi^c \cdot \nabla\psi\}, \end{aligned} \tag{3.2b}$$

$$\sigma = -\nabla \cdot (\rho^c \nabla\psi + \rho\nabla\psi^c), \tag{3.2c}$$

as our starting point. Assuming enough regularity of the functions involved, it is easy to see by elementary manipulations of the equations that solutions of (3.2) in fact correspond to solutions of (3.1). For brevity we write

$$\mathcal{L}^c \begin{pmatrix} \rho \\ \sigma \end{pmatrix} = \kappa \begin{pmatrix} \rho \\ \sigma \end{pmatrix}.$$

Note that the appearance of ψ in the definition of $\mathcal{L}^c$ means that, given a pair (ρ, σ), one has to determine ψ according to Equation (3.2c) with zero boundary conditions in order to evaluate $\mathcal{L}^c(\rho, \sigma)^\mathsf{T}$.

If a wave U^c admits a solution of (3.2) for some $\kappa \in \mathbb{C}$ with $\operatorname{Re}\kappa > 0$ then the linearized problem admits an exponentially growing mode and the wave is called *spectrally unstable*. On the other hand, if none such solution exists for any $\kappa \in \mathbb{C}$ with $\operatorname{Re}\kappa > 0$ the wave is called *spectrally stable*. This is the basic idea underlying spectral stability but, since we will not discuss functional-analytic aspects of the eigenvalue problem nor deal with appropriate spaces on which it may be posed, we refrain from giving a precise definition and refer, e.g. to [54].

Remark 3.1. *(i) Results of this thesis yield information on the eigenvalues of $\mathcal{L}^c$, but hardly on the essential spectrum of $\mathcal{L}^c$. We point out why instabilities from the essential spectrum are not expected. For the asymptotic operator $\mathcal{L}^c_\infty$, at least, the following heuristic argument indicates that there is no essential spectrum outside the imaginary axis assuming an exponential stratification. To this end, we consider the resolvent equation*

$$(\mathcal{L}^c_\infty - \kappa \mathrm{Id}) \begin{pmatrix} \rho \\ \sigma \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix} \tag{3.3}$$

*in the original variables x (*not ξ*) and y with any $\kappa \in \mathbb{C} \setminus i\mathbb{R}$. Applying the Fourier transform with respect to the unbounded variable x and denoting the Fourier variable by $k \in \mathbb{R}$ leads to the system*

$$\begin{aligned} -\kappa\hat{\rho}(k,y) + ik\bar{\rho}'(y)\hat{\psi}(k,y) &= \hat{f}(k,y) \\ -\kappa\hat{\sigma}(k,y) - ikg\hat{\rho}(k,y) &= \hat{h}(k,y) \\ -\hat{\sigma}(k,y) &= (\bar{\rho}(y)\hat{\psi}_y(k,y))_y - k^2\bar{\rho}(y)\hat{\psi}. \end{aligned} \tag{3.4}$$

Elimination of $\hat{\rho}$ *and* $\hat{\sigma}$ *yields the single equation*

$$\mathcal{S}\hat{\psi}(k,y) - \alpha(k)\hat{\psi}(k,y) = F(k,y) \tag{3.5}$$

with

$$\begin{aligned}\alpha(k) &= -k^2\left(\frac{g}{\kappa^2} + \frac{1}{\delta}\right), \\ F(k,y) &= \frac{1}{\bar{\rho}'(y)}\left(\frac{1}{\kappa}\hat{h}(k,y) - \frac{ikg}{\kappa^2}\hat{f}(k,y)\right).\end{aligned}$$

This is, for fixed $k \in \mathbb{R}$*, the resolvent equation of the operator* $\mathcal{S} = (\bar{\rho}')^{-1}\partial_y(\bar{\rho}\partial_y)$ *from Lemma 2.5 the spectrum of which is an increasing sequence of positive reals. From the formula for* $\alpha(k)$ *it follows that for* $\operatorname{Re}\kappa \neq 0$ *we never have* $\alpha(k) \in (0,\infty)$*, thus Equation 3.5 has a unique solution, and so Equation* (3.4) *has a unique solution as well. Consequently, also Equation* (3.3) *is uniquely solvable, hence any* κ *with* $\operatorname{Re}\kappa \neq 0$ *appears to be an element of the resolvent set (this is, of course, not a proof, as appropriate spaces have not been defined and the boundedness of the inverse of the resolvent operator has been left out).*

(ii) The time-reversibility of the Euler equation (see Remark 2.1) implies that if $\kappa \in \sigma(\mathcal{L}^c)$ *then also* $-\kappa \in \sigma(\mathcal{L}^c)$*. This means the usual definition of spectral stability, namely*

$$\sigma(\mathcal{L}^c) \subset \{z : \operatorname{Re} z \leq 0\},$$

is equivalent to

$$\sigma(\mathcal{L}^c) \subset i\mathbb{R}$$

in our situation.

In what follows, we are solely interested in eigenvalues. In the following section we present a reformulation of the eigenvalue problem upon which we build our approach to stability.

We finally remark that $\mathcal{L}^c$ has a double zero eigenvalue due to translational invariance of the travelling wave and due to the presence of a continuum of travelling waves as stated in the next lemma.

Lemma 3.2. $\mathcal{L}^c$ *has the algebraically double eigenvalue* $\kappa = 0$*.*

Proof. Suppose (ρ^c, σ^c) is an internal travelling wave. Recalling the profile equation from (2.11) we know that the equations

$$0 = \{\rho^c, \psi^c - cy\},$$

$$
\begin{aligned}
0 &= \{\sigma^c, \psi^c - cy\} + \left\{\rho^c, gy - \frac{1}{2}|\nabla\psi^c|^2\right\}, \\
-\sigma^c &= \nabla \cdot (\rho^c \nabla \psi^c)
\end{aligned}
$$

hold with some uniquely determined ψ^c. Deriving these equations by ξ yields

$$
\begin{aligned}
0 &= \{\partial_\xi \rho^c, \psi^c - cy\} + \{\rho^c, \partial_\xi \psi^c\}, \\
0 &= \{\partial_\xi \sigma^c, \psi^c - cy\} + \{\sigma^c, \partial_\xi \psi^c\} + \left\{\partial_\xi \rho^c, gy - \frac{1}{2}|\nabla\psi^c|^2\right\} \\
&\quad + \{\rho^c, -\nabla\psi^c \cdot \nabla\partial_\xi\psi^c\} \\
-\partial_\xi \sigma^c &= \nabla \cdot (\partial_\xi \rho^c \nabla \psi^c) + \nabla \cdot (\rho^c \nabla(\partial_\xi \psi^c)),
\end{aligned}
$$

this means

$$
\mathcal{L}^c \begin{pmatrix} \partial_\xi \rho^c \\ \partial_\xi \sigma^c \end{pmatrix} = 0,
$$

i.e. $(\partial_\xi \rho^c, \partial_\xi \sigma^c)^\mathsf{T}$ is an eigenvector to the eigenvalue $\kappa = 0$.

Moreover, deriving the profile equation by c yields

$$
\begin{aligned}
0 &= \{\partial_c \rho^c, \psi^c - cy\} + \{\rho^c, \partial_c \psi^c - y\}, \\
0 &= \{\partial_c \sigma^c, \psi^c - cy\} + \{\sigma^c, \partial_c \psi^c - y\} + \left\{\partial_c \rho^c, gy - \frac{1}{2}|\nabla\psi^c|^2\right\} \\
&\quad + \{\rho^c, -\nabla\psi^c \cdot \nabla\partial_c\psi^c\} \\
-\partial_c \sigma^c &= \nabla \cdot (\partial_c \rho^c \nabla \psi^c) + \nabla \cdot (\rho^c \nabla(\partial_c \psi^c)),
\end{aligned}
$$

which is, after rearranging,

$$
\begin{aligned}
\partial_\xi \rho^c &= \{\partial_c \rho^c, \psi^c - cy\} + \{\rho^c, \partial_c \psi^c\}, \\
\partial_\xi \sigma^c &= \{\partial_c \sigma^c, \psi^c - cy\} + \{\sigma^c, \partial_c \psi^c\} + \left\{\partial_c \rho^c, gy - \frac{1}{2}|\nabla\psi^c|^2\right\} + \{\rho^c, -\nabla\psi^c \cdot \nabla\partial_c\psi^c\}
\end{aligned}
$$

and this means

$$
\mathcal{L}^c \begin{pmatrix} \partial_c \rho^c \\ \partial_c \sigma^c \end{pmatrix} = \begin{pmatrix} \partial_\xi \rho^c \\ \partial_\xi \sigma^c \end{pmatrix},
$$

i.e. $(\partial_c \rho^c, \partial_c \sigma^c)^\mathsf{T}$ is a generalized eigenvector to the eigenvalue $\kappa = 0$. □

Remark 3.3. *In the case of* regular *internal solitary waves the two generalized eigenvectors are in fact exponentially decaying by the definition of "regular".*

3.2. Spatial dynamics for the eigenvalue problem

In this section we formulate the eigenvalue problem associated with an internal travelling wave as a spatial-dynamical system. As announced in the preceding section, the starting point is the formulation of the Euler equations due to Benjamin described in Section 2.2. The eigenvalue problem, obtained by first linearizing the system about a travelling wave and then plugging in $\mathrm{e}^{\kappa t}(\rho(\xi,y),\sigma(\xi,y))$ with the spectral parameter $\kappa\in\mathbb{C}$, is given as follows:

$$\begin{aligned} \kappa\rho &= -\left\{\rho^c,\psi\right\} - \left\{\rho,\psi^c - cy\right\}, \\ \kappa\sigma &= -\left\{\sigma^c,\psi\right\} - \left\{\sigma,\psi^c - cy\right\} \\ &\quad - \left\{\rho, -\frac{1}{2}\left|\nabla\psi^c\right|^2 + gy\right\} - \left\{\rho^c, -\nabla\psi^c\cdot\nabla\psi\right\}, \\ \sigma &= -\nabla\cdot\left(\rho^c\nabla\psi + \rho\nabla\psi^c\right). \end{aligned} \tag{3.6}$$

The following theorem states that the eigenvalue problem can be rewritten as a dynamical system in which ξ plays the role of the evolution parameter and a state $W(\xi)$ at “time” ξ is given by some function

$$W(\xi) = \left[y\mapsto (W_1(\xi,y),W_2(\xi,y),W_3(\xi,y),W_4(\xi,y))^{\mathsf{T}}\right]$$

mapping from the interval $(0,1)$ to $\mathbb{C}^4$.

Theorem I. *Given a regular ISW $\psi^c(\xi,y)$ of speed $c>0$, the associated eigenvalue problem* (3.6) *can be written as the abstract ordinary differential equation*

$$W'(\xi) = \mathbb{A}(\xi;\kappa)W(\xi), \tag{EVP}$$

on the Hilbert space

$$\mathcal{W} = L^2(0,1)\times L^2(0,1)\times L^2(0,1)\times L^2(0,1)$$

with

$$W = \begin{pmatrix}\rho\\ \psi\\ \psi_\xi\\ \psi_{\xi\xi}\end{pmatrix}\in\mathcal{W} \quad \textit{and} \quad \mathbb{A}(\xi;\kappa) = \begin{pmatrix} R_1 & R_2 & R_3 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ S_1 & S_2 & S_3 & S_4\end{pmatrix},$$

where

$$R_i = \frac{\tilde{R}_i}{\psi^c_y - c} \quad \textit{and} \quad S_j = \frac{\tilde{S}_j}{\psi^c_y - c}$$

for $i \in \{1,2,3\}$ and $j \in \{1,2,3,4\}$ with

$$\tilde{R}_1 = -\kappa + \psi^c_\xi \partial_y, \quad \tilde{R}_2 = -\rho^c_\xi \partial_y, \quad \tilde{R}_3 = \rho^c_y$$

and

$$\begin{aligned}
\tilde{S}_1 = {} & \big[-(\psi^c_y - c)\left(\psi^c_{\xi yy} + \psi^c_{\xi\xi\xi}\right) + \psi^c_\xi \left(\psi^c_{yyy} + \psi^c_{\xi\xi y}\right) \\
& - (\psi^c_y - c)^{-1}\{g\kappa + \kappa\psi^c_\xi \psi^c_{\xi y}\} + \kappa\psi^c_{\xi\xi}\big] \\
& + \big[-(\psi^c_y - c)\psi^c_{\xi y} + \psi^c_\xi \psi^c_{yy} - \psi^c_\xi \psi^c_{\xi\xi} + (\psi^c_y - c)^{-1}(g\psi^c_\xi + (\psi^c_\xi)^2 \psi^c_{\xi y})\big]\partial_y, \\
\tilde{S}_2 = {} & \big[-\rho^c_y \psi^c_{\xi y} - \rho^c \psi^c_{\xi yy} - \rho^c \psi^c_{\xi\xi\xi} - (\psi^c_y - c)^{-1}(g\rho^c_\xi + \rho^c_\xi \psi^c_\xi \psi^c_{\xi y}) - \kappa\rho^c_y\big]\partial_y \\
& + [\rho^c_y \psi^c_\xi - \kappa\rho^c]\partial_{yy} + [\rho^c \psi^c_\xi]\partial_{yyy}, \\
\tilde{S}_3 = {} & [\rho^c_y \psi^c_{yy} + \rho^c_\xi \psi^c_{\xi y} + \rho^c \psi^c_{yyy} + \rho^c \psi^c_{\xi\xi y} - \rho^c_y \psi^c_{\xi\xi} \\
& + (\psi^c_y - c)^{-1}(g\rho^c_y + \rho^c_y \psi^c_\xi \psi^c_{\xi y}) - \kappa\rho^c_\xi] \\
& + [-\rho^c_y(\psi^c_y - c) + \rho^c_\xi \psi^c_\xi]\partial_y + [-\rho^c(\psi^c_y - c)]\partial_{yy}, \\
\tilde{S}_4 = {} & [-\rho^c_\xi(\psi^c_y - c) - \kappa\rho^c] + [\rho^c \psi^c_\xi]\partial_y.
\end{aligned}$$

Proof. First, we note that the denominators in R_i and S_k do not vanish, since the wave ψ^c is regular (see Def. 2.2 and the subsequent remark).

To derive the expressions, we solve the first and the second equation in (3.6) for ρ_ξ and σ_ξ, respectively, and obtain:

$$(\psi^c_y - c)\rho_\xi = (-\kappa + \psi^c_\xi \partial_y)\rho + \rho^c_y \psi_\xi - \rho^c_\xi \partial_y \psi, \tag{3.7}$$

$$\begin{aligned}
(\psi^c_y - c)\sigma_\xi = {} & (-\psi^c_\xi \psi^c_{\xi\xi} - \psi^c_y \psi^c_{\xi y})\partial_y \rho \\
& + (\rho^c_\xi(\psi^c_{yy}\partial_y + \psi^c_y \partial_{yy}) - \sigma^c_\xi \partial_y - \rho^c_y \psi^c_{\xi y}\partial_y)\psi \\
& + (\rho^c_\xi(\psi^c_{\xi y} + \psi^c_\xi \partial_y) + \sigma^c_y - \rho^c_y(\psi^c_{\xi\xi} + \psi^c_y \partial_y))\psi_\xi \\
& + (-\kappa + \psi^c_\xi \partial_y)\sigma \\
& + (-g + \psi^c_\xi \psi^c_{\xi y} + \psi^c_y \psi^c_{yy})\rho_\xi - \rho^c_y \psi^c_\xi \psi_{\xi\xi}.
\end{aligned} \tag{3.8}$$

After introducing

$$W_1 = \rho, \ W_2 = \psi, \ W_3 = \psi_\xi, \ W_4 = \psi_{\xi\xi},$$

the first line becomes

$$(\psi^c_y - c)W_1' = (-\kappa + \psi^c_\xi \partial_y)W_1 - \rho^c_\xi \partial_y W_2 + \rho^c_y W_3$$

from which we may read off the expressions R_1, R_2, R_3 as given above.

In treating the second equation we have to express all of the quantities $\rho_\xi, \sigma, \sigma_y, \sigma_\xi$ in terms of $\rho, \psi, \psi_\xi, \psi_{\xi\xi}, \psi_{\xi\xi\xi}$ and their derivatives with respect to y. For ρ_ξ this has already been accomplished. Concerning σ one finds, by definition,

$$\begin{aligned}
-\sigma &= (\rho^c\psi_\xi + \rho\psi^c_\xi)_\xi + (\rho^c\psi_y + \rho\psi^c_y)_y \\
&= (\psi^c_{\xi\xi} + \psi^c_{yy} + \psi^c_y\partial_y)\rho + (\rho^c_y\partial_y + \rho^c\partial_{yy})\psi + \rho^c_\xi\psi_\xi + \rho^c\psi_{\xi\xi} + \psi^c_\xi\rho_\xi \\
&= (\psi^c_{\xi\xi} + \psi^c_{yy} + \psi^c_y\partial_y + \psi^c_\xi R_1)\rho + (\rho^c_y\partial_y + \rho^c\partial_{yy} + \psi^c_\xi R_2)\psi \\
&\quad + (\rho^c_\xi + \psi^c_\xi R_3)\psi_\xi + \rho^c\psi_{\xi\xi},
\end{aligned}$$

which is an expression of the desired form for σ. Consequently, σ_ξ and σ_y can be expressed in the desired form as well by deriving this equation by ξ and y, respectively. By using these expressions Equation (3.8) can be written in the form

$$\begin{aligned}
\rho^c(\psi^c_y - c)\psi_{\xi\xi\xi} = \left(\tilde{S}^0_1 + \tilde{S}^1_1\partial_y\right)W_1 + \left(\tilde{S}^1_2\partial_y + \tilde{S}^2_2\partial^2_y + \tilde{S}^3_2\partial^3_y\right)W_2 \\
+ \left(\tilde{S}^0_3 + \tilde{S}^1_3\partial^1_y + \tilde{S}^2_3\partial^2_y\right)W_3 + \left(\tilde{S}^0_4 + \tilde{S}^1_4\partial_y\right)W_4
\end{aligned}$$

where the functions $\tilde{S}^j_k$ are given as follows.

$$\begin{aligned}
\tilde{S}^0_1 &= -(\psi^c_y - c)\left(\psi^c_{\xi yy} + \psi^c_{\xi\xi\xi}\right) + \psi^c_\xi\left(\psi^c_{yyy} + \psi^c_{\xi\xi y}\right) \\
&\quad - (\psi^c_y - c)^{-1}\{g\kappa + \kappa\psi^c_\xi\psi^c_{\xi y}\} + \kappa\psi^c_{\xi\xi}, \\
\tilde{S}^1_1 &= -(\psi^c_y - c)\psi^c_{\xi y} + \psi^c_\xi\psi^c_{yy} - \psi^c_\xi\psi^c_{\xi\xi} + (\psi^c_y - c)^{-1}(g\psi^c_\xi + (\psi^c_\xi)^2\psi^c_{\xi y}), \\
\tilde{S}^1_2 &= -\rho^c_y\psi^c_{\xi y} - \rho^c\psi^c_{\xi yy} - \rho^c\psi^c_{\xi\xi\xi} - (\psi^c_y - c)^{-1}(g\rho^c_\xi + \rho^c_\xi\psi^c_\xi\psi^c_{\xi y}) - \kappa\rho^c_y, \\
\tilde{S}^2_2 &= \rho^c_y\psi^c_\xi - \kappa\rho^c, \\
\tilde{S}^3_2 &= \rho^c\psi^c_\xi, \\
\tilde{S}^0_3 &= \rho^c_y\psi^c_{yy} + \rho^c_\xi\psi^c_{\xi y} + \rho^c\psi^c_{yyy} + \rho^c\psi^c_{\xi\xi y} - \rho^c_y\psi^c_{\xi\xi} \\
&\quad + (\psi^c_y - c)^{-1}(g\rho^c_y + \rho^c_y\psi^c_\xi\psi^c_{\xi y}) - \kappa\rho^c_\xi, \\
\tilde{S}^1_3 &= -\rho^c_y(\psi^c_y - c) + \rho^c_\xi\psi^c_\xi, \\
\tilde{S}^2_3 &= -\rho^c(\psi^c_y - c), \\
\tilde{S}^0_4 &= -\rho^c_\xi(\psi^c_y - c) - \kappa\rho^c, \\
\tilde{S}^1_4 &= \rho^c\psi^c_\xi.
\end{aligned}$$

This yields the asserted expressions for the operators

$$\begin{aligned}
\tilde{S}_1 &= \tilde{S}^0_1 + \tilde{S}^1_1\partial_y, \\
\tilde{S}_2 &= \tilde{S}^1_2\partial_y + \tilde{S}^2_2\partial^2_y + \tilde{S}^3_2\partial^3_y,
\end{aligned}$$

$$\tilde{S}_3 = \tilde{S}_3^0 + \tilde{S}_3^1 \partial_y^1 + \tilde{S}_3^2 \partial_y^2,$$
$$\tilde{S}_4 = \tilde{S}_4^0 + \tilde{S}_4^1 \partial_y,$$

and finishes the proof. □

Remark 3.4. (i) *In the proof we have not used that the wave under consideration is solitary but only that it is a travelling wave and, hence, the corresponding stream function satisfies the Dubreil-Jacotin-Long equation. This is, in particular, true for periodic travelling waves, so there is a similar formulation for the associated eigenvalue problem as well.*

(ii) *Concerning the underlying functional analytic setting, which is not discussed in this thesis, we mention that* $\mathbb{A}(\xi;\kappa)$ *can be defined as a mapping from the domain*

$$D := H_0^1(0,1) \times (H^3(0,1) \cap H_0^1(0,1)) \times (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$$

taking values in

$$X := L^2(0,1) \times H^2(0,1) \times H^1(0,1) \times L^2(0,1) \subset \mathcal{W},$$

and the formal limit $\mathbb{A}^\infty(\kappa) := \lim_{|\xi|\to\infty} \mathbb{A}(\xi;\kappa)$ *can be defined as a mapping from*

$$D^1 := L^2(0,1) \times H^2(0,1) \times H^2(0,1) \times L^2(0,1)$$

to X*. Since the spectrum of the operator* $\mathbb{A}^\infty(\kappa)$ *is not bounded from below nor from above (see Lemma 3.8 on the spectrum of the finite-dimensional truncations of* $\mathbb{A}^\infty(\kappa)$ *and note that, for the exponential stratification,* $\mathbb{A}^\infty(0)$ *has the eigenvalues* $\pm\sqrt{\delta(\lambda_n - \lambda)}, n \in \mathbb{N}$*, which accumulate at* $+\infty$ *and* $-\infty$*), well-posedness of Equation* (EVP) *is not expected.*

(iii) *The spaces* D *and* D^1 *proposed in (ii) already indicate an unpleasent property of the equation* $W' = (\mathbb{A}^\infty + \mathbb{B})W$*: Since* $\mathbb{B}$ *involves higher derivatives (up to third order) than* $\mathbb{A}^\infty$ *(up to second order), it is not a relatively bounded perturbation of* $\mathbb{A}^\infty$*. This is one reason preventing us from drawing the usual conclusion that the essential spectrum is already determined by considering the autonomous part* $W' = \mathbb{A}^\infty W$*.*

3.3. Finite-dimensional truncations of the eigenvalue problem

In the previous section we formulated the eigenvalue problem as a dynamical system on an infinite-dimensional state space. This state space is a Hilbert space and it possesses an easily accessible Hilbert basis which emanates from the eigenfunctions $\{\chi_M\}_{M\in\mathbb{N}}$ of the operator $\mathcal{T}_\lambda$, defined and discussed in Lemma 2.4. Therefore, it is straightforward to employ a Galerkin-like procedure, in the spirit of [49], in order to obtain finite-dimensional truncations of the eigenvalue problem. For a regular ISW we describe this approach explicitly in the present section.

We begin with the fact that the state space is indeed a Hilbert space.

Lemma 3.5. (i) *The space $\mathcal{W}$ endowed with the scalar product*

$$\langle U, V\rangle := \int_0^1 \bar{\rho}\left(\frac{U_1 V_1}{\bar{\rho}_y^2} + U_2V_2 + U_3V_3 + U_4V_4\right) \mathrm{d}y$$

is a Hilbert space.

(ii) *Define for $M \in \mathbb{N}$*

$$U_M^1 = \begin{pmatrix} \bar{\rho}'\chi_M \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; U_M^2 = \begin{pmatrix} 0 \\ \chi_M \\ 0 \\ 0 \end{pmatrix}, \; U_M^3 = \begin{pmatrix} 0 \\ 0 \\ \chi_M \\ 0 \end{pmatrix}, \; U_M^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \chi_M \end{pmatrix}.$$

With this definition the set

$$\mathfrak{B} := \{U_M^k : k \in \{1, 2, 3, 4\}, M \in \mathbb{N}\}$$

forms an orthonormal Hilbert basis for $(\mathcal{W}, \langle\cdot,\cdot\rangle)$.

Proof. (i) follows from the fact that $\langle\cdot,\cdot\rangle$ is equivalent the standard L^2-scalar product and (ii) is a direct consequence of Lemma 2.4. □

By using the basis $\mathfrak{B}$ we introduce for an arbitrary, fixed $N \in \mathbb{N}$ the finite-dimensional subspace $\mathcal{W}_N \subset \mathcal{W}$, to which we will project the equation (EVP). Let us define

$$X_M := \operatorname{span}\{U_M^1, U_M^2, U_M^3, U_M^4\} \quad \text{and} \quad \mathcal{W}_N := \bigoplus_{0\le M\le N} X_M,$$

and denote by Q_N the orthogonal projection onto the space $\mathcal{W}_N$. Clearly, the space $\mathcal{W}_N$ has the dimension $d_N := 4N + 4 < \infty$.

We split the linear operator $\mathbb{A}$ in two parts,

$$\mathbb{A}(\xi;\kappa) = \mathbb{A}^\infty(\kappa) + \mathbb{B}(\xi;\kappa),$$

with

$$\mathbb{A}^{\infty}(\kappa) := \lim_{\xi \to \pm\infty} \mathbb{A}(\xi;\kappa) = \begin{pmatrix} \frac{\kappa}{c} & 0 & -\frac{\bar{\rho}'}{c} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{g\kappa}{c^2\bar{\rho}} & \frac{\kappa}{c}\frac{\bar{\rho}'(y)}{\bar{\rho}(y)}\mathcal{S} & \mathcal{T}_\lambda & \frac{\kappa}{c} \end{pmatrix}$$

denoting the formal asymptotic limit of $\mathbb{A}(\xi;\kappa)$

From now on we focus on the exponential stratification $\bar{\rho}(y) = \mathrm{e}^{-\delta y}$. In this case, as we recall from Remark 2.6, it is $\mathcal{T}_\lambda = \delta\,(\mathcal{S} - \lambda \mathrm{Id})$, hence

$$\nu_n(\lambda) = \delta(\lambda_n - \lambda) \quad \text{and} \quad \chi_n = \sqrt{\delta}\varphi_n.$$

Then $\mathbb{A}^{\infty}(\kappa)$ takes the form

$$\mathbb{A}^{\infty}(\kappa) = \begin{pmatrix} \frac{\kappa}{c} & 0 & -\frac{\bar{\rho}'}{c} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\lambda\kappa}{\bar{\rho}} & -\frac{\delta\kappa}{c}\mathcal{S} & \delta\,(\mathcal{S} - \lambda \mathrm{Id}) & \frac{\kappa}{c} \end{pmatrix}$$

that yields

$$\begin{aligned} \mathbb{A}^{\infty}(\kappa)U_M^1 &= \frac{\kappa}{c}U_M^1 + \delta\lambda\kappa U_M^4, \\ \mathbb{A}^{\infty}(\kappa)U_M^2 &= -\delta\lambda_M\frac{\kappa}{c}U_M^4, \\ \mathbb{A}^{\infty}(\kappa)U_M^3 &= -\frac{1}{c}U_M^1 + U_M^2 + \delta(\lambda - \lambda_M)U_M^4, \\ \mathbb{A}^{\infty}(\kappa)U_M^4 &= U_M^3 + \frac{\kappa}{c}U_M^4, \end{aligned}$$

implying that X_M for $M \in \mathbb{N}$, hence $\mathcal{W}_N$ for $N \in \mathbb{N}$, is invariant under $\mathbb{A}^{\infty}(\kappa)$.

Next, we turn to the truncated problem defined by

$$W'(\xi) = (\mathbb{A}^{\infty}(\kappa) + Q_N\mathbb{B}(\xi;\kappa))W(\xi).$$

We notice that $\mathcal{W}_N = \operatorname{Im} Q_N$ is an invariant subspace for this equation. We therefore decompose

$$W = Q_N W + (\mathrm{Id} - Q_N)W =: W_N + \hat{W}_N$$

and obtain the system

$$\begin{pmatrix} W_N \\ \hat{W}_N \end{pmatrix}' = \begin{pmatrix} \mathbb{A}^\infty|_{\operatorname{Im} Q_N} + Q_N\mathbb{B} & Q_N\mathbb{B} \\ 0 & \mathbb{A}^\infty|_{\ker Q_N} \end{pmatrix} \begin{pmatrix} W_N \\ \hat{W}_N \end{pmatrix}.$$

The equation for $\hat{W}_N$ has constant coefficients and we assume here that the second equation, $\hat{W}_N' = \mathbb{A}^\infty|_{\ker Q_N}\hat{W}_N$, has no bounded solution different from $\hat{W}_N \equiv 0$. For this reason we focus on the first equation, which is an ODE on the finite-dimensional space $\mathcal{W}_N$ called the *truncated problem of order* N.

After expanding $W_N \in \mathcal{W}_N$ in the basis $\mathfrak{B}$, i.e.

$$W_N(\xi) = \sum_{M=0}^{N}\sum_{k=1}^{4} w_M^k(\xi) U_M^k,$$

and introducing the notation

$$\mathcal{A}_{M\tilde{M}}^{k\tilde{k}}(\kappa) := \left\langle U_M^k, \mathbb{A}^\infty(\kappa) U_{\tilde{M}}^{\tilde{k}} \right\rangle \quad \text{and} \quad \mathcal{B}_{M\tilde{M}}^{k\tilde{k}}(\xi;\kappa) := \left\langle U_M^k, \mathbb{B}(\xi;\kappa) U_{\tilde{M}}^{\tilde{k}} \right\rangle,$$

with $0 \le M, \tilde{M} \le N$ and $1 \le k, \tilde{k} \le 4$, we obtain a representation of the truncated problem in terms of the coordinates w_M^k with $0 \le M \le N$ and $1 \le k \le 4$:

$$\begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_N \end{pmatrix}_\xi = \begin{pmatrix} \mathcal{A}_{0,0} + \mathcal{B}_{0,0} & \mathcal{A}_{0,1} + \mathcal{B}_{0,1} & \cdots & \mathcal{A}_{0,N} + \mathcal{B}_{0,N} \\ \mathcal{A}_{1,0} + \mathcal{B}_{1,0} & \mathcal{A}_{1,1} + \mathcal{B}_{1,1} & \cdots & \mathcal{A}_{N,1} + \mathcal{B}_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{N,0} + \mathcal{B}_{N,0} & \mathcal{A}_{N,1} + \mathcal{B}_{N,1} & \cdots & \mathcal{A}_{N,N} + \mathcal{B}_{N,N} \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_N \end{pmatrix} \tag{3.9}$$

with $w_M = (w_M^1, w_M^2, w_M^3, w_M^4)^\mathsf{T}$. Note that since each space X_M is $\mathbb{A}^\infty$-invariant, we obtain that $\mathcal{A}_{M\tilde{M}} = 0$ whenever $M \neq \tilde{M}$. Hence, defining $\mathcal{A}_M := \mathcal{A}_{MM}$ leads to the form

$$\begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_N \end{pmatrix}_\xi = \begin{pmatrix} \mathcal{A}_0 + \mathcal{B}_{0,0} & \mathcal{B}_{0,1} & \cdots & \mathcal{B}_{0,N} \\ \mathcal{B}_{1,0} & \mathcal{A}_1 + \mathcal{B}_{1,1} & \cdots & \mathcal{B}_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}_{N,0} & \mathcal{B}_{N,1} & \cdots & \mathcal{A}_N + \mathcal{B}_{N,N} \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_N \end{pmatrix}. \tag{3.10}$$

We write this equation more concisely as

$$w'(\xi) = \mathcal{A}(\xi;\kappa) w(\xi) = (\mathcal{A}^\infty(\kappa) + \mathcal{B}(\xi;\kappa))\, w(\xi) \tag{3.11}$$

and refer to it by (EVP$_N$). We emphasize that $\mathcal{A}^\infty(\kappa)$ is given by the block diagonal matrix

$$\mathcal{A}^{\infty}(\kappa) = \mathcal{A}_0(\kappa) \oplus \cdots \oplus \mathcal{A}_N(\kappa), \tag{3.12}$$

which will be crucial later in determining the eigenvalues of $\mathcal{A}^{\infty}(\kappa)$, see Section 3.4.

Remark 3.6. *In the case of a general stratification the space $\mathcal{W}_N$ is* not *$\mathbb{A}^{\infty}(\kappa)$-invariant and, consequently, we have $\mathcal{A}_{M\tilde{M}} \neq 0$ for $M \neq \tilde{M}$. It seems one better splits the operator in a different way, namely*

$$\mathbb{A}(\xi;\kappa) = \mathbb{A}^{\infty}(0) + \tilde{\mathbb{B}}(\xi;\kappa),$$

because $\mathbb{A}^{\infty}(0)$ still has the property to leave $\mathcal{W}_N$ invariant. An investigation of this question, however, is not part of this thesis.

3.4. Evans function for the truncated problems

3.4.1. Construction of the Evans function

We restrict to the exponential stratification and consider, for any fixed $N \in \mathbb{N}$, the truncated eigenvalue problem

$$w'(\xi) = \mathcal{A}(\xi;\kappa)w(\xi), \tag{3.13}$$

as derived in the previous section.

The purpose of the current section is to define a function $D_N(\kappa)$ which serves to locate those values κ with $\operatorname{Re}\kappa > 0$ for which the truncated eigenvalue problem (3.13) has a non-zero solution decaying as $\xi \to \pm\infty$. More precisely, we will prove the following theorem.

Theorem II. *For a regular ISW $\psi^c(\xi,y)$ of speed $c > c_0$ and for any $N \in \mathbb{N}$ there is an open region $\Omega = \Omega(N,c) \supset \overline{\mathbb{C}_+}$ such that the following two statements hold.*

(i) *There are two unique analytic mappings*

$$\mathcal{S}_N(\cdot) : \Omega \to \mathcal{G}^{d_N}_{d_N^s}(\mathbb{C}),\ \kappa \mapsto \mathcal{S}_N(\kappa),$$
$$\mathcal{U}_N(\cdot) : \Omega \to \mathcal{G}^{d_N}_{d_N^u}(\mathbb{C}),\ \kappa \mapsto \mathcal{U}_N(\kappa),$$

satisfying the characterization: For $\operatorname{Re}\kappa > 0$ and any solution $w : \mathbb{R} \to \mathbb{C}^{d_N}$ of (3.13) we have

$$w(0) \in \mathcal{S}_N(\kappa) \iff w(+\infty) = 0 \quad \textit{and} \quad w(0) \in \mathcal{U}_N(\kappa) \iff w(-\infty) = 0.$$

(ii) *There is an analytic function $D_N(\cdot) : \Omega \to \mathbb{C}$, called* Evans function*, with the property: $D_N(\kappa) = 0$ iff there is a solution of* (3.13) *decaying as $\xi \to \infty$ and $\xi \to -\infty$.*

According to the articles [22, 3, 27, 54] the two assumptions (1) exponential decay of the coefficient matrix $\mathcal{A}(\xi;\kappa)$ to some constant asymptotic matrix $\mathcal{A}^\infty(\kappa)$ as $\xi \to \pm\infty$ and (2) the property of consistent splitting for the asymptotic matrices $\mathcal{A}^\infty(\kappa)$ in the right half-plane $\mathbb{C}_+$ are sufficient to provide a suitable setting for defining an Evans function on $\mathbb{C}_+$. Regarding an extension of the Evans function to $\overline{\mathbb{C}_+}$, one needs a further assumption (3) allowing to continue the unstable and stable spaces of the matrix $\mathcal{A}^\infty(\kappa)$ which clearly exist for $\operatorname{Re}\kappa > 0$.

Based on the following three lemmas, which state that the assumptions (1-3) are indeed met, we will give a proof of Thm. II. The lemmas will be proved in the following subsections.

Lemma 3.7. *There are constants $C_E, C > 0$ such that*

$$|\mathcal{A}(\xi;\kappa) - \mathcal{A}^\infty(\kappa)| \le C\mathrm{e}^{-C_E|\xi|}$$

holds for all $\xi \in \mathbb{R}$.

Lemma 3.8. *For any $N \in \mathbb{N}$ and any $\kappa \in \mathbb{C}_+$, the matrix*

$$\mathcal{A}^\infty(\kappa) \in \mathbb{C}^{d_N \times d_N}$$

possesses $d_N^s := N + 1$ eigenvalues,

$$\mu_k^s(\kappa) \quad \text{with } 0 \le k \le N,$$

with negative real part and $d_N^u := 3N + 3$ eigenvalues

$$\mu_k^{u_1,u_2,u_3}(\kappa) \quad \text{with } 0 \le k \le N,$$

with positive real part.

With these two lemmas at hand, an Evans function $D_N : \mathbb{C}_+ \to \mathbb{C}$ can be defined on the *open* right half-plane. To construct a continuation to some open set comprising the *closed* right half-plane we have to know how the eigenvalues of $\mathcal{A}^\infty(\kappa)$ behave when κ crosses the imaginary axis. This is the subject of the following lemma.

Lemma 3.9. *There are precisely* $2N+2$ *eigenvalues* $\mu_k^{u_1}(\kappa), \mu_k^{u_2}(\kappa)$, $k \in \{0, \cdots, N\}$, *of* $\mathcal{A}^\infty(\kappa)$ *which satisfy*

$$\mu_k^{u_1,u_2}(\kappa) \in i\mathbb{R} \quad \textit{if } \kappa \in i\mathbb{R}$$

and which cross the imaginary axis with non-zero speed when κ *moves across the imaginary axis.*

Now, we turn to the proof of Theorem II.

Proof of Theorem II. We begin with the proof of (i). For $\operatorname{Re}\kappa > 0$ Lemma 3.8 implies that the matrix $\mathcal{A}^\infty(\kappa)$ has $N+1$ eigenvalues $\mu_k^s(\kappa)$ for $k = 0, \ldots, N$ with negative real part and $3N+3$ eigenvalues $\mu_k^{u_1,u_2,u_3}(\kappa)$ for $k = 0, \ldots, N$ with positve real part, thus $\mathbb{C}^{d_N}$ splits into a direct sum,

$$\mathbb{C}^{d_N} = \mathcal{S}_N^\infty(\kappa) \oplus \mathcal{U}_N^\infty(\kappa),$$

where $\mathcal{S}_N^\infty(\kappa)$ and $\mathcal{U}_N^\infty(\kappa)$ denote the span of all (possibly generalized) eigenvectors associated with eigenvalues of negative, resp. positive, real part of the matrix $\mathcal{A}^\infty(\kappa)$; Lemma 3.8 additionally implies

$$\dim \mathcal{S}_N^\infty(\kappa) = d_N^s \quad \text{and} \quad \dim \mathcal{U}_N^\infty(\kappa) = d_N^u.$$

From Lemma 3.9 and the continuity of $\kappa \mapsto \mu(\kappa)$ we infer that this splitting can be extended to some domain Ω which is an open neighbourhood of the closed right half-plane $\overline{\mathbb{C}_+}$; however, if $\operatorname{Re}\kappa \leq 0$ the space $\mathcal{U}_N^\infty(\kappa)$ also contains (possibly generalized) eigenvectors associated with eigenvalues of non-positive real part.

This splitting is transported to $\xi = 0$ by the flow of (3.13) yielding the spaces $\mathcal{S}_N(\kappa)$ and $\mathcal{U}_N(\kappa)$.

For a more detailed argument (like in [3]), we couple (3.13) with the equation

$$\tau' = \gamma(1 - \tau^2), \tag{3.14}$$

with some suitable γ (more precisely, $0 < \gamma < \frac{1}{2}C_E$ with C_E from Lemma 3.7 as in [3]), to obtain the autonomous system (3.13), (3.14) on $\mathbb{C}^{d_N} \times [-1; +1]$. Obviously, $\xi = \pm\infty$ corresponds to $\tau = \pm 1$. We notice that the two restpoints $(w, \tau) = (0, \pm 1)$ are hyperbolic, since the linearization is given by some block matrix of the form

$$\begin{pmatrix} \mathcal{A}^\infty(\kappa) & * \\ 0 & \mp 2 \end{pmatrix},$$

hence, there are $N+1$ stable and $(3N+3)+1$ unstable eigenvalues at $(0,-1)$ and there are $(N+1)+1$ stable and $3N+3$ unstable eigenvalues at $(0,+1)$. This implies the point $(0,-1)$ has an unstable manifold $W^u(0,-1)$ of real dimension $2\cdot(3N+3)+1$ and the point $(0,+1)$ has a stable manifold $W^s(0,+1)$ of real dimension $2\cdot(N+1)+1$ each of which intersects the $\tau(0)$-section to yield the $(3N+3)$-dimensional complex space $\mathcal{U}_N(\kappa)$ and the $(N+1)$-dimensional complex space $\mathcal{S}_N(\kappa)$, respectively.

For a slightly different argument illustrating the use of the Grassmannian formulation, consider the non-linear equation

$$X' = \left(\Gamma^k \mathcal{A}(\xi;\kappa)\right)(X),$$

which is induced by the linear equation (3.13) on the Grassmannian $\mathcal{G}_k^{d_N}(\mathbb{C})$, for $k \in \{d_N^s, d_N^u\}$, or rather its autonomous version

$$X' = \left(\Gamma^k \mathcal{A}_{\kappa,c}[\tau]\right)(X), \tag{3.15a}$$

$$\tau' = \gamma(1-\tau^2), \tag{3.15b}$$

with $\mathcal{A}_{\kappa,c}[\tau(\xi)] = \mathcal{A}(\xi;\kappa)$.

For $k = d_N^u$ the restpoint $(\mathcal{U}_N^\infty(\kappa), -1)$ is hyperbolic (due to [25, Lemma 3]) and has precisely one unstable direction. Consequently, there is a one-dimensional unstable manifold $(X(\xi),\tau(\xi))$ and the unique value $X(0)$ is the sought space $\mathcal{U}_N(\kappa)$. Similarly, for $k = d_N^s$ the restpoint $(\mathcal{S}_N^\infty(\kappa), +1)$ is hyperbolic and has precisely one stable direction. Consequently, there is a one-dimensional stable manifold $(X(\xi),\tau(\xi))$ and the unique value $X(0)$ is the sought space $\mathcal{S}_N(\kappa)$.

Finally, the analyticity of the mappings $\kappa \mapsto \mathcal{S}_N(\kappa)$ and $\kappa \mapsto \mathcal{U}_N(\kappa)$ follows from the analyticity of $\kappa \mapsto \mathcal{A}(\xi;\kappa)$ and the fact that solutions of an ordinary differential equation depending analytically on some parameter are themselves analytic with respect to this parameter; the uniqueness of these mappings follows from the uniqueness of analytic continuations.

For the proof of (ii), we choose bases

$$\{\zeta_1(\kappa), \ldots, \zeta_{d_N^s}(\kappa)\} \quad \text{of } \mathcal{S}_N(\kappa)$$

and

$$\{\eta_1(\kappa), \ldots, \eta_{d_N^u}(\kappa)\} \quad \text{of } \mathcal{U}_N(\kappa)$$

which depend analytically on κ for $\kappa \in \Omega$; the existence of such bases is guaranteed by a construction due to Kato, see [37, Ch. II. §4.2]. Then, we define $D_N : \Omega \to \mathbb{C}$ as the determinant

$$D_N(\kappa) := \det(\zeta_1(\kappa), \dots, \zeta_{d_N^s}(\kappa), \eta_1(\kappa), \dots, \eta_{d_N^u}(\kappa)).$$

Clearly, $D_N(\kappa) = 0$ if and only if $\mathcal{S}_N(\kappa) \cap \mathcal{U}_N(\kappa) \neq 0$ and this is equivalent, for $\operatorname{Re}\kappa > 0$, to the existence of a decaying solution w of (3.13), i.e. $w(\pm\infty) = 0$.

The analyticity of D_N follows directly from the analyticity of the basis vectors $\zeta_i(\kappa)$ and $\eta_j(\kappa)$. □

Remark 3.10. (i) *As there is more than one choice of bases, the Evans function is unique only up to a non-vanishing factor. This non-uniqueness causes no trouble, since it does not affect the location of the zeros.*

(ii) *A closer look at the proof shows that an Evans function can be defined on the* $\mathbb{C}_+$ *for any* $\lambda \in (0, \infty)$*, equivalently for any* $c \in (0, \infty)$*, but the continuation across* $\overline{\mathbb{C}_+}$ *has only been proved for* $\lambda < \lambda_0$*, equivalently for* $c > c_0$*.*

(iii) *We emphasize that, since a spectral gap between the spaces* $\mathcal{U}^\infty(\kappa)$ *and* $\mathcal{S}^\infty(\kappa)$ *is maintained in our situation, it is very much easier to obtain a continuation of the Evans function than it is in the situation of shock waves in viscous conservation laws where a gap lemma is needed (see [27]).*

We finally show that the Evans function has a double zero at $\kappa = 0$ due to translational invariance and due to the presence of a family of waves parametrized by the speed c. This is well known for other equations (e.g. generalized KdV equation or Boussinesq equation, see [51]).

Lemma 3.11. *The Evans function constructed above satisfies*

$$D_N(0) = \partial_\kappa D_N(0) = 0.$$

Proof. By Lemma 3.2 the operator $\mathcal{L}^c$ has a Jordan chain of length two, i.e. there are solutions U_1 and U_2 satisfying

$$\mathcal{L}^c U_1 = 0 \quad \text{and} \quad \mathcal{L}^c U_2 = U_1.$$

Theorem I implies immediately there is a solution W_1 of (EVP) for $\kappa = 0$ which corresponds to U_1, i.e.

$$W_1'(\xi) = \mathbb{A}(\xi; 0) W_1(\xi).$$

Moreover, by going through the proof of Theorem I again, one finds that there exists a W_2 which corresponds to U_2 and which satisfies

$$W_2'(\xi) = \mathbb{A}(\xi;0)W_2(\xi) + \mathbb{A}_1(\xi;c)W_1(\xi)$$

with $\mathbb{A}_1$ chosen such that $\mathbb{A}(\xi;\kappa) = \mathbb{A}_0(\xi;c) + \kappa\mathbb{A}_1(\xi;c)$ holds; $\mathbb{A}_0$ and $\mathbb{A}_1$ exist, since $\mathbb{A}(\xi;\kappa)$ is linear with respect to κ.

By applying the truncation procedure from Section 3.3 we obtain functions w_1 and w_2 which satisfy

$$w_1'(\xi) = \mathcal{A}(\xi;0)w_1(\xi) \quad \text{and} \quad w_2'(\xi) = \mathcal{A}(\xi;0)w_2(\xi) + \mathcal{A}_1(\xi;c)w_1(\xi) \tag{3.16}$$

where $\mathcal{A}_1$ is induced from the above splitting of $\mathbb{A}$, i.e. $\mathcal{A}(\xi;\kappa) = \mathcal{A}_0(\xi) + \kappa\mathcal{A}_1(\xi)$.

According to [54, Section 3.3, 4.1] the property (3.16) precisely means that the Evans function has a double root in $\kappa = 0$, i.e. $D_N(0) = \partial_\kappa D_N(0) = 0$. □

Remark 3.12. (i) *The solutions W_1 and W_2 can be easily expressed in terms of (ρ^c, σ^c) by letting ψ^c denote the corresponding stream function. Then we have (as a consequence of Lemma 3.2):*

$$W_1 = (\partial_\xi \rho^c, \partial_\xi \psi^c, \partial_{\xi\xi}\psi^c, \partial_{\xi\xi\xi}\psi^c)^\mathsf{T}$$

and

$$W_2 = (\partial_c \rho^c, \partial_c \psi^c, \partial_\xi\partial_c\psi^c, \partial_{\xi\xi}\partial_c\psi^c)^\mathsf{T}.$$

(ii) *In the small-amplitude case we can calculate the leading orders of W_1 and W_2 by using the expressions from Section A.1. In this way, we obtain for W_1*

$$W_1(\xi) = \varepsilon^2 \begin{pmatrix} -\frac{1}{c_0}A_\varepsilon'(\xi)\bar{\rho}'\varphi_0 \\ A_\varepsilon'(\xi)\varphi_0 \\ A_\varepsilon''(\xi)\varphi_0 \\ A_\varepsilon'''(\xi)\varphi_0 \end{pmatrix} + \textit{higher order terms}$$

and for W_2

$$W_2(\xi) = \begin{pmatrix} -\frac{1}{c_0}(A_\varepsilon(\xi) + \frac{\xi}{2}A'_\varepsilon(\xi))\bar{\rho}'\varphi_0 \\ (A_\varepsilon(\xi) + \frac{\xi}{2}A'_\varepsilon(\xi))'\varphi_0 \\ (A_\varepsilon(\xi) + \frac{\xi}{2}A'_\varepsilon(\xi))''\varphi_0 \\ (A_\varepsilon(\xi) + \frac{\xi}{2}A'_\varepsilon(\xi))'''\varphi_0 \end{pmatrix} + \textit{higher order terms};$$

here we have used that $c = c_0 + \varepsilon^2$ *implies*

$$\frac{\partial}{\partial c} = \frac{1}{2\varepsilon}\frac{\partial}{\partial \varepsilon}$$

and

$$\frac{\partial}{\partial \varepsilon}(\varepsilon^2 A_\varepsilon(\xi)) = \frac{\partial}{\partial \varepsilon}(\varepsilon^2 A_*(\varepsilon\xi)) = 2\varepsilon\left(A_\varepsilon(\xi) + \frac{\xi}{2}A'_\varepsilon(\xi)\right)$$

neglecting higher order terms. With $V_1 := \dot{A}_*(\Xi)$ *and* $\tilde{V}_1 := A_* + \frac{\Xi}{2}\dot{A}_*(\Xi)$ *the corresponding eigenvectors for KdV are given by*

$$V = (V_1, \dot{V}_1, \ddot{V}_1) \quad \textit{and} \quad \tilde{V} = (\tilde{V}_1, \dot{\tilde{V}}_1, \ddot{\tilde{V}}_1),$$

so the relationship to KdV is also recovered here; see B.2 for the KdV eigenvalue problem.

3.4.2. Proof of Lemma 3.7 (Exponential decay)

This lemma follows from the assumption that $\psi^c(\xi, y)$ is a *regular* ISW (see Def. 2.2), which implies exponential decay of the solution and of sufficiently high derivatives. By inspection of the expressions R_1, R_2, R_3 and S_1, S_2, S_3, S_4 one finds that all terms which decay for $|\xi| \to \infty$ do this with an exponential rate.

3.4.3. Proof of Lemma 3.8 (Consistent splitting)

Proof of Lemma 3.8. We have to show that, for any κ with $\operatorname{Re}\kappa > 0$, the matrix $\mathcal{A}^\infty(\kappa)$ has precisely $3(N+1)$ eigenvalues with positive real part and $N+1$ eigenvalues with negative real part.

Step 1: Absence of imaginary eigenvalues for $\operatorname{Re}\kappa > 0$. To begin with we show that the spectrum of $\mathcal{A}^\infty(\kappa)$ does not intersect the imaginary axis, or rather that the existence of an imaginary eigenvalue of $\mathcal{A}^\infty(\kappa)$ implies $\kappa \in i\mathbb{R}$. As noted in Section 3.3 the matrix

$\mathcal{A}^\infty(\kappa)$ is block-diagonal, it thus suffices to show this separately for each block

$$\mathcal{A}_M(\kappa) = \begin{pmatrix} \frac{\kappa}{c} & 0 & -\frac{1}{c} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \delta\lambda\kappa & -\frac{1}{c}\kappa\lambda_M\delta & \delta(\lambda_M - \lambda) & \frac{\kappa}{c} \end{pmatrix}.$$

The characteristic polynomial of $\mathcal{A}_M$ is given by

$$\pi^{(M)}(\mu;\kappa) := \mu^4 - \frac{2\kappa}{c}\mu^3 + \left(\frac{\kappa^2}{c^2} + \delta(\lambda - \lambda_M)\right)\mu^2 + 2\lambda_M\delta\frac{\kappa}{c}\mu - \frac{\kappa^2}{c^2}\lambda_M\delta, \tag{3.17}$$

and setting $\mu = i\beta$ leads to the equation

$$A\hat{\kappa}^2 + iB\hat{\kappa} - C = 0$$

for $\hat{\kappa} = -\frac{\kappa}{c}$ with

$$A = \beta^2 + \lambda_M\delta,\ B = 2\beta^3 + 2\lambda_M\delta\beta,\ C = \beta^4 + \delta(\lambda_M - \lambda)\beta^2.$$

Note that

$$\begin{aligned} -(B^2 - 4AC) &= -4\beta^2(\beta^2 + \lambda_M\delta)^2 + 4(\beta^2 + \lambda_M\delta)\cdot\beta^2(\beta^2 + (\lambda_M - \lambda)\delta) \\ &= -4\beta^2(\beta^2 + \lambda_M\delta)\delta\lambda \le 0 \end{aligned}$$

and therefore any root of $AX^2 - BX + C$ is real, in particular $i\hat{\kappa} \in \mathbb{R}$, i.e. $\hat{\kappa} \in i\mathbb{R}$, hence $\kappa \in i\mathbb{R}$.

Step 2: The first part of the proof implies that the dimension of the stable resp. unstable space is the same for all κ with $\operatorname{Re}\kappa > 0$. For determining their exact dimensions, it thus suffices to consider $\kappa \in \mathbb{R}$ with $\kappa > 0$ sufficiently large. To handle this precisely, we introduce $T := \mu^{-1}$ and $K := \left(\frac{\kappa}{c}\right)^{-1}$ and investigate the equation

$$\lambda_M\delta T^4 - 2\lambda_M\delta K T^3 - \left(1 + \delta(\lambda - \lambda_M)K^2\right)T^2 + 2KT - K^2 = 0, \tag{3.18}$$

obtained from (3.17), for sufficiently small $K > 0$. We employ the Newton polygon method (see Section B.1 for a detailed description) to deduce approximate expressions of the four roots $T_j = T_j(K)$, with $j \in \{1,2,3,4\}$, for sufficiently small K. This method provides a way of finding suitable exponents γ such that inserting $T = K^\gamma t$ in (3.18) leads to a regularly perturbed equation for sufficiently small K. In order to find the exponents one has to investigate the Newton polygon associated with this equation; here one finds the two exponents $\gamma \in \{0, 1\}$.

In the first case, $\gamma = 0$, we set $K = 0$ in (3.18) to obtain the equation

$$\lambda_M \delta T^4 - T^2 = 0 \tag{3.19}$$

which possesses the roots

$$T_{1,2}(0) = 0 \quad \text{and} \quad T_{3,4}(0) = \pm \frac{1}{\sqrt{\lambda_M \delta}}. \tag{3.20}$$

Since the roots $T_{3,4}(0)$ are simple, they persist under small perturbations, hence there are two zeros $T_{3,4}(K)$ of (3.18) with approximate expressions

$$T_{3,4}(K) = \pm \frac{1}{\sqrt{\lambda_M \delta}} + \mathcal{O}(K). \tag{3.21}$$

We cannot conclude in the same way for $T_{1,2}(K)$, since $T_{1,2}(0) = 0$; the resolution of these two roots is performed next.

In the second case, $\gamma = 1$, we plug the ansatz $T = Kt$ into (3.18) and find, after dividing by K^2 and subsequently setting $K = 0$,

$$-(t-1)^2 = 0.$$

The unperturbed roots $t_{1,2}(0) = 1$ have positive real part, and so, due to continuity with respect to K, the perturbed roots $t_{1,2}(K) = 1 + o(1)$ have positive real part as well.

For the original variable T this means (3.18) has two roots $T_{1,2}(K)$ close to the origin with approximate expressions

$$T_{1,2}(K) = K + o(K). \tag{3.22}$$

Since $K > 0$ is supposed to be small, the formulas (3.21) and (3.22) imply that, for each $M \in \{0, \cdots, N\}$ there are three roots with positive real part and one root with negative real part, as claimed. □

Remark 3.13. *This proof implies that the system exhibits consistent splitting also in the open left half-plane* $\{\operatorname{Re} \kappa < 0\}$ *because it is clear, from the definition of* $\pi^{(M)}$ *in* (3.17)*, that the identity*

$$\pi^{(M)}(-\mu; -\kappa) = \pi^{(M)}(\mu; \kappa)$$

holds. We thus have the following statement, which is analogous to the assertion in

B	$p_K(B)$	$p'_K(B)$	$p''_K(B)$	$p'''_K(B)$	$p^{(iv)}_K(B)$	no. of changes
0	$+$	$+$	$+$	$+$	$+$	0
$-K/2-\tau$	$+$	$+$	$\pm$	$-$	$+$	2
$-K$	$-$	$+$	$+$	$-$	$+$	3
$-B^*$	$+$	$-$	$+$	$-$	$+$	4

Table 3.1.: Number of sign changes at some chosen points (with any sufficiently small $\tau > 0$ and some sufficiently large $B^* > 0$)

Lemma 3.8: For any κ with $\operatorname{Re}\kappa < 0$ the matrix $\mathcal{A}^\infty(\kappa)$ possesses $3N+3$ eigenvalues with negative real part and $N+1$ eigenvalues with positive real part.

3.4.4. Proof of Lemma 3.9 (Continuation across imaginary axis)

The proof of Lemma 3.9 is based on the following lemma.

Lemma 3.14. *For any $M \in \mathbb{N}$ the polynomial $p_K(B) := \pi^{(M)}(-iB; iKc)$ has precisely two real roots for any $K \in \mathbb{R}$, which coincide for $K = 0$ only.*

Proof. We have to look for roots of the real polynomial

$$p_K(B) = B^4 + 2KB^3 + (K^2 + \delta(\lambda_M - \lambda))B^2 + 2\lambda_M \delta K B + \delta\lambda_M K^2.$$

Step 1: The transformation $(B, K) \mapsto (-B, -K)$ leaves the polynomial invariant. Thus, it suffices to consider $K \geq 0$. For $K = 0$ the polynomial $p_0(B) = B^4 + \delta(\lambda_M - \lambda)B^2$ has precisely two real roots, namely $B_{1,2} = 0$ since $\lambda < \lambda_M$ for all M. Thus, it remains to treat the case $K > 0$ in the rest of the proof.

Step 2: We show that the polynomial has two roots. Since all the coefficients of $p_K(B)$ are positive, any root has to be negative. Let us consider the signs of $p_K(B)$ and its derivatives at special values of B in order to apply the Fourier-Budan theorem (see [6] for details). According to Table 3.1 the difference in the number of sign changes between the points $-B^*$ and $-K$ and between the points $-K$ and $-K/2-\tau$, where $\tau > 0$ is an arbitrary sufficiently small number, is one in each case. This implies the existence of two simple roots $B_1 \in (-B^*, -K)$ and $B_2 \in (-K, -K/2-\tau)$ for any $K > 0$. Since $\tau > 0$ is arbitrary, we conclude that B_2 is the only root in $(-K, -K/2)$, too.

Step 3: In this final step, we exclude further roots of $p_K(B)$. For sufficiently small $K > 0$, the second derivative

$$p''_K(B) = 12B^2 + 12KB + 2\left(K^2 + \delta(\lambda_M - \lambda)\right)$$

does not have a real zero, thus the polynomial $p_K(B)$ has no turning point, hence $p_K(B)$ has at most two real roots, and we are done in this case. For larger $K > 0$ and any fixed $B \in [-K/2, 0]$, we will show that $p_K(B) > 0$. To this end, we consider the derivative of $p_K(B)$, now viewed as a function $\tilde{p}(B, K)$ of K and B, along rays $B = \gamma K$ with $\gamma \in \left[-\frac{1}{2}, 0\right]$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}K} p_K(\gamma K) &= \frac{\mathrm{d}}{\mathrm{d}K} \tilde{p}(\gamma K, K) \\ &= \gamma \frac{\partial \tilde{p}(\gamma K, K)}{\partial B} + \frac{\partial \tilde{p}(\gamma K, K)}{\partial K} \\ &= 4\gamma^2(1+\gamma)^2 K^3 + 2\delta \left((2\gamma + 1)\lambda_M + (\lambda_M - \lambda)\gamma^2 \right) K > 0. \end{aligned}$$

Let $K_0 \in (0, K)$ be sufficiently small in the sense that $p_{K_0}(B_0)$ possesses exactly two real roots and $p_{K_0}(B_0) > 0$ for any $B_0 \in [-K_0/2, 0]$. Consequently,

$$p_K(B) = p_K\left(\frac{B}{K} K\right) > p_{K_0}\left(\frac{B}{K} K_0\right) > 0,$$

hence p_K has no zero in the interval $[-K/2, 0]$. Together with the result from step 2, we have thus shown that p_K possesses exactly two real roots, which are in fact simple in the case of $K \neq 0$. □

We turn now to the proof of Lemma 3.9.

Proof of Lemma 3.9. By the preceding lemma we obtain the existence of $2N + 2$ eigenvalues

$$\mu_k^{u_1,u_2}(\kappa) \text{ for } k \in \{0, \cdots, N\}$$

where

$$\mu_M^{u_1}(iKc) = -iB_1 \quad \text{and} \quad \mu_M^{u_2}(iKc) = -iB_2$$

in the notation of the preceding proof. Letting $\mu(\kappa)$ denote one of these eigenvalues, we are going to show that

$$\operatorname{Re} \mu(iKc - \varepsilon) < 0;$$

this means the $2N + 2$ eigenvalues from above move to the open left half-plane when κ crosses the imaginary axis from right to left. In order to capture this behaviour, we consider the directional derivative

$$\alpha(K) := \operatorname{Re} \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} \mu(iKc - \varepsilon) = -\operatorname{Re} \mu'(iKc)$$

of μ along the path $(-\varepsilon, iKc)$, with $\varepsilon \in \mathbb{R}$ and $|\varepsilon|$ small, that is perpendicular to the imaginary axis; note that the differentiability for $K \neq 0$ follows from the simplicity of μ as roots of some $\pi^{(M)}$, whereas case $K = 0$ needs some special treatment. By Taylor's theorem, it suffices to show

$$\alpha(K) < 0,$$

since this relation implies

$$\operatorname{Re}\mu(iKc - \varepsilon) = \operatorname{Re}\mu(iKc) + \alpha(K)\varepsilon + o(|\varepsilon|) < \operatorname{Re}\mu(iKc) = 0$$

for small positive $\varepsilon \in \mathbb{R}$.

Suppose $K \neq 0$ first. Differentiating the characteristic polynomial yields

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \pi(\mu(iKc - \varepsilon), iKc - \varepsilon) \\ &= -\partial_\mu \pi(\mu(iKc), iKc)\mu'(iKc) - \partial_\kappa \pi(\mu(iKc), iKc). \end{aligned}$$

Recalling

$$p_K(B) = \pi(-iB, iKc)$$

from the previous lemma, we immediately obtain

$$\begin{aligned} \partial_\mu \pi(-iB, iKc) &= ip_K'(B), \\ \partial_\kappa \pi(-iB, iKc) &= -\frac{i}{c}\frac{\partial p_K(B)}{\partial K}, \end{aligned}$$

and thus, for $K \neq 0$,

$$\mu'(iKc) = \frac{\frac{\partial p_K(B)}{\partial K}}{cp_K'(B)}.$$

Moreover, it was shown there that for each $M \in \{0, \cdots, N\}$ we have

$$p_K'(B_1) < 0 < p_K'(B_2) \quad \text{and} \quad \frac{\partial p_K(B_1)}{\partial K} < 0 < \frac{\partial p_K(B_2)}{\partial K}$$

implying that

$$-\alpha(K) = \operatorname{Re}\mu'(iKc) > 0 \quad \text{and} \quad \operatorname{Im}\mu'(iKc) = 0$$

for all $\mu(\kappa)$ we are considering here, so we are done.

We turn to the remaining case $K = 0$. Note first that the matrix $\mathcal{A}_M(\kappa, c)$ is diagonalizable for all $\kappa \in i\mathbb{R}$ and depends linearly, hence smoothly, on κ. Therefore, $\mu(\kappa)$ is also differentiable at $\kappa = 0$ and one may argue in a similar way as above, but since

$p_0'(0) = \frac{\partial p_0(0)}{\partial K} = 0$ we need to derive the characteristic polynomial twice to obtain an expression for $\mu'(0)$. In doing so, we find the equation

$$\pi_{\kappa\kappa}(-\mu(0),0)\mu'(0)^2 + 2\pi_{\kappa\mu}(-\mu(0),0)\mu'(0) + \pi_{\mu\mu}(-\mu(0),0) = 0$$

and, by virtue of the relation $\pi(-iB, iKc) = p_K(B)$, we compute

$$\begin{aligned} -\pi_{\mu\mu}(-\mu(0),0) &= 2\delta(\lambda_M - \lambda), \\ c\pi_{\kappa\mu}(-\mu(0),0) &= 2\delta\lambda_M, \\ -c^2\pi_{\kappa\kappa}(-\mu(0),0) &= 2\delta\lambda_M. \end{aligned}$$

Thus, we have

$$\mu'(0) = \frac{1}{c}\frac{\sqrt{\lambda_M}}{\sqrt{\lambda_M} \pm \sqrt{\lambda}},$$

which implies as above $-\alpha(0) = \operatorname{Re}\mu'(0) > 0$ and $\operatorname{Im}\mu'(0) = 0$, and the proof is complete.

□

4. Towards spectral stability of small-amplitude waves in an exponentially stratified fluid

In this chapter we consider small-amplitude waves in the Evans function framework established previously. These waves can be described approximately by Korteweg-deVries solitons and this relationship is also present on the level of spectral stability. By exploiting this fact, we show absence of unstable eigenvalues for the truncated problems in a neighbourhood of the origin. In the proof we make use of geometric singular perturbation theory.

4.1. Truncated eigenvalue problem and statement of the result

We restrict to the exponential stratification. According to Section A.1 the eigenvalue associated with a small-amplitude ISW is given in the form

$$w'(\xi) = \mathcal{A}(\xi;\kappa,\varepsilon)w(\xi) \tag{4.1}$$

with the matrix $\mathcal{A}$ having block structure

$$\mathcal{A}(\xi;\kappa,\varepsilon) = \begin{pmatrix} \mathcal{A}_0 + \mathcal{B}_{0,0} & \mathcal{B}_{0,1} & \cdots & \mathcal{B}_{0,N} \\ \mathcal{B}_{1,0} & \mathcal{A}_1 + \mathcal{B}_{1,1} & \cdots & \mathcal{B}_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}_{N,0} & \mathcal{B}_{N,1} & \cdots & \mathcal{A}_N + \mathcal{B}_{N,N} \end{pmatrix}$$

where the 4×4-blocks are given by (see Section A.1)

$$\mathcal{A}_0 = \begin{pmatrix} \frac{1}{c_0}\kappa + O(\kappa\varepsilon^2) & 0 & -\frac{1}{c_0} + O(\varepsilon^2) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_0\delta\kappa + O(\kappa\varepsilon^2) & -\frac{\lambda_0\delta}{c_0}\kappa + O(\kappa\varepsilon^2) & \frac{2\lambda_0\delta}{c_0}\varepsilon^2 + O(\varepsilon^4) & \frac{1}{c_0}\kappa + O(\kappa\varepsilon^2) \end{pmatrix},$$

$$\mathcal{A}_n = \begin{pmatrix} \frac{1}{c_0}\kappa + O(\kappa\varepsilon^2) & 0 & -\frac{1}{c_0} + O(\varepsilon^2) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_0\delta\kappa + O(\kappa\varepsilon^2) & -\frac{\lambda_n\delta}{c_0}\kappa + O(\kappa\varepsilon^2) & \delta(\lambda_n - \lambda_0) + O(\varepsilon^2) & \frac{1}{c_0}\kappa + O(\kappa\varepsilon^2) \end{pmatrix}$$

for $n \geq 1$, and

$$\mathcal{B}_{n,m} = \begin{pmatrix} \varepsilon^3 B_\varepsilon(\xi) D_{nm}^{11} & \varepsilon^3 B_\varepsilon(\xi) D_{nm}^{12} & \varepsilon^2 A_\varepsilon(\xi) D_{nm}^{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon^3 B_\varepsilon(\xi) D_{nm}^{41} & \varepsilon^3 B_\varepsilon(\xi) D_{nm}^{42} & \varepsilon^2 A_\varepsilon(\xi) D_{nm}^{43} & \varepsilon^3 B_\varepsilon(\xi) D_{nm}^{44} \end{pmatrix},$$

up to terms of higher order (i.e. $O(\varepsilon^4 + \kappa\varepsilon^4)$), where D_{nm}^{ij} denote constants independent of κ and ε and $(A_\varepsilon(\xi), B_\varepsilon(\xi))$ is the profile which satisfies

$$\begin{aligned} A_\varepsilon'(\xi) &= \varepsilon B_\varepsilon, \\ B_\varepsilon'(\xi) &= \varepsilon\left(-\frac{1}{s}A_\varepsilon - \frac{r}{s}A_\varepsilon^2\right) + O(\varepsilon^3); \end{aligned} \tag{4.2}$$

note that this equation still has the rest point $(A_\varepsilon, B_\varepsilon) = (0, 0)$ since the right hand side of the second equation still vanishes for $A_\varepsilon = 0$ (see Lemma 2.8).

By coupling Equations (4.1) and (4.2), we obtain the autonomous system

$$w'(\xi) = \mathcal{A}_{\kappa,\varepsilon}[A_\varepsilon, B_\varepsilon]w(\xi), \tag{4.3a}$$

$$\begin{pmatrix} A_\varepsilon \\ B_\varepsilon \end{pmatrix}' = \begin{pmatrix} \varepsilon B_\varepsilon \\ \varepsilon\left(-\frac{1}{s}A_\varepsilon - \frac{r}{s}A_\varepsilon^2\right) + O(\varepsilon^3) \end{pmatrix} \tag{4.3b}$$

we consider in the rest of the section; here we have used the notation $\mathcal{A}_{\kappa,\varepsilon}[A_\varepsilon(\xi), B_\varepsilon(\xi)] = \mathcal{A}(\xi; \kappa, c_0 + \varepsilon^2)$. We will give a partial answer to the question whether this system has bounded solutions for κ with $\operatorname{Re}\kappa > 0$ in the following theorem.

Theorem III. *For all $N \in \mathbb{N}$ there exist $R_0 > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ the Evans function $D_N(\kappa, c_0 + \varepsilon^2)$ associated with an ISW of speed $c_0 + \varepsilon^2$ satisfies*

$$\begin{aligned} &D_N(0, c_0 + \varepsilon^2) = D_N'(0, c_0 + \varepsilon^2) = 0 \quad \textit{and} \\ &D_N(\kappa, c_0 + \varepsilon^2) \neq 0 \textit{ for all } \kappa \textit{ with } \operatorname{Re}\kappa > 0 \textit{ and } 0 < |\kappa| \leq R_0. \end{aligned}$$

We split this theorem in two parts which treat different overlapping regimes, namely

$$0 \leq |\kappa| \leq R_1\varepsilon^3 \quad \text{and} \quad r_2\varepsilon^3 \leq |\kappa| \leq R_2$$

with an arbitrary $R_1 > 0$ and some $r_2, R_2 > 0$. The following lemma is devoted to the treatment of the inner part.

Lemma 4.1. *Let $R_1 > 0$ be arbitrary. There exists $\varepsilon_1 > 0$ such that the statement in Theorem III holds for all $\varepsilon \in (0, \varepsilon_1)$ and for all κ with*

$$0 \leq |\kappa| \leq R_1\varepsilon^3 \quad \textit{and} \quad \operatorname{Re}\kappa \geq 0.$$

The outer part is treated in the following lemma.

Lemma 4.2. *There exist $r_2, R_2, \varepsilon_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_2)$ and for all κ with*

$$r_2\varepsilon^3 \leq |\kappa| \leq R_2 \quad \textit{and} \quad \operatorname{Re}\kappa > 0$$

the system given by Equation (4.3) *has no bounded solution.*

These two lemmas jointly prove the theorem: Lemma 4.2 ensures the existence of r_2, R_2 and $\varepsilon_2 > 0$. Choosing R_1, which is arbitrary, bigger than r_2 and setting $R_0 := R_2$, $\varepsilon_0 := \min\{\varepsilon_1, \varepsilon_2\}$ finishes the proof of Theorem III.

For a concise notation it is useful to define for some $N \in \mathbb{N}$ the complementary index sets

$$I_h = \{4n + k : 1 \leq n \leq N, 3 \leq k \leq 4\}$$

and

$$I_c = \{4n + k : 0 \leq n \leq N, 1 \leq k \leq 4\} \setminus I_h.$$

We will show that all the variables carrying an index that belongs to I_h represent hyperbolic directions, which can be eliminated by performing a centre manifold reduction; afterwards only the variables carrying an index that belongs to I_c will be present. In the following we fix some $N \in \mathbb{N}$.

4.2. Proof of Lemma 4.1 (KdV regime)

Before beginning the proof we reduce to a centre manifold. By introducing $\Lambda := \kappa\varepsilon^{-3}$, we are in the regime

$$0 \leq |\Lambda| \leq R_1$$

for an arbitrary fixed $R_1 > 0$. By scaling the dependent variables as follows

$$\begin{aligned} w_1(\xi) &= \tilde{W}_1(\xi), & w_{4n+1}(\xi) &= \varepsilon W_{4n+1}(\xi),\\ w_2(\xi) &= \tilde{W}_2(\xi), & w_{4n+2}(\xi) &= \varepsilon W_{4n+2}(\xi),\\ w_3(\xi) &= \varepsilon W_3(\xi), & w_{4n+3}(\xi) &= \varepsilon^2 W_{4n+3}(\xi),\\ w_4(\xi) &= \varepsilon^2 W_4(\xi), & w_{4n+4}(\xi) &= \varepsilon^2 W_{4n+4}(\xi), \end{aligned}$$

for all $n \in \{1, \cdots, N\}$, the problem (4.3) takes to leading order the form

$$\begin{aligned} A_\varepsilon' &= \varepsilon B_\varepsilon,\\ B_\varepsilon' &= \varepsilon\left(-\frac{1}{s}A_\varepsilon - \frac{r}{s}A_\varepsilon^2\right) + O(\varepsilon^3),\\ W_1' &= O(\varepsilon),\\ W_2' &= O(\varepsilon),\\ W_3' &= O(\varepsilon),\\ W_4' &= O(\varepsilon),\\ W_{4n+1}' &= O(\varepsilon),\\ W_{4n+2}' &= O(\varepsilon),\\ W_{4n+3}' &= W_{4n+4},\\ W_{4n+4}' &= \delta(\lambda_n - \lambda_0)W_{4n+3} + O(\varepsilon^2), \end{aligned}$$

for all n with $1 \le n \le N$. We infer that, for $\varepsilon = 0$, the set

$$M_0 := \{W_j = 0 : j \in I_h\}$$

is a centre manifold, which is normally hyperbolic since the partial Jacobian matrix of this system with respect to the variables W_{4n+3}, W_{4n+4} for $n = 1, \ldots, N$ has a block structure of the form

$$\begin{pmatrix} Y_1 & * & \cdots & * \\ 0 & Y_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & Y_N \end{pmatrix} \tag{4.4}$$

where “$*$” denotes some unimportant entries and Y_n is given by

$$Y_n = \begin{pmatrix} 0 & 1 \\ \delta(\lambda_n - \lambda_0) & 0 \end{pmatrix}$$

so the eigenvalues, given by $\pm\sqrt{\delta(\lambda_n - \lambda_0)}$ for $n = 1, \dots, N$, are real and different from zero.

By virtue of Fenichel's theorem on the persistence of normally hyperbolic invariant manifolds (see [23, 24, 35]), we conclude that an invariant manifold M_ε exists for all sufficiently small ε, say $0 < \varepsilon < \delta_1$, M_ε is a graph over M_0, and, in particular, $W_j = O(\varepsilon)$ for $j \in I_h$.

In this way, we obtain a reduced system for the variables $\{W_j : j \in I_c\}$. After changing to the slow scale

$$\Xi := \varepsilon\xi,$$

setting

$$A_\varepsilon(\xi) = \tilde{A}_\varepsilon(\Xi), B_\varepsilon(\xi) = \tilde{B}_\varepsilon(\Xi),$$

and

$$W_{1,2}(\xi) = \hat{\tilde{W}}_{1,2}(\Xi), W_j(\xi) = \hat{W}_j(\Xi)$$

for all $j \in I_c \setminus \{1, 2\}$ and omitting the hat, this reduced system is of the form (with $\dot{}$ denoting the derivative with respect to Ξ)

$$\begin{aligned}
\dot{\tilde{A}}_\varepsilon &= \tilde{B}_\varepsilon, \\
\dot{\tilde{B}}_\varepsilon &= \Big(-\frac{1}{s}\tilde{A}_\varepsilon - \frac{r}{s}\tilde{A}_\varepsilon^2\Big) + O(\varepsilon^3), \\
\dot{\tilde{W}}_1 &= -\frac{1}{c_0}W_3 + O(\varepsilon^2), \\
\dot{\tilde{W}}_2 &= W_3, \\
\dot{W}_3 &= W_4, \\
\dot{W}_4 &= \tilde{\Gamma}_1\tilde{W}_1 + \tilde{\Gamma}_2\tilde{W}_2 + \Gamma_3 W_3 + O(\varepsilon), \\
\dot{W}_{4n+1} &= O(\varepsilon), \quad \text{for } 1 \le n \le N, \\
\dot{W}_{4n+2} &= O(\varepsilon), \quad \text{for } 1 \le n \le N,
\end{aligned}$$

where the coefficients are given by (see Section A.2)

$$\begin{aligned}
\tilde{\Gamma}_1(\Xi) &= \varepsilon^{-3}\mathcal{A}_0^{41} + A_*'(\Xi)D_{00}^{41} = \Lambda\frac{\int_0^1 \bar{\rho}\varphi_0'^2 \mathrm{d}y}{\int_0^1 \bar{\rho}\varphi_0^2 \mathrm{d}y} + A_*'(\Xi)\frac{\int_0^1 \bar{\rho}\varphi_0'^3 \mathrm{d}y}{\int_0^1 \bar{\rho}\varphi_0^2 \mathrm{d}y}, \\
\tilde{\Gamma}_2(\Xi) &= \varepsilon^{-3}\mathcal{A}_0^{42} + A_*'(\Xi)D_{00}^{42} = -\frac{\Lambda}{c_0}\frac{\int_0^1 \bar{\rho}\varphi_0'^2 \mathrm{d}y}{\int_0^1 \bar{\rho}\varphi_0^2 \mathrm{d}y} - \frac{2}{c_0}A_*'(\Xi)\frac{\int_0^1 \bar{\rho}\varphi_0'^3 \mathrm{d}y}{\int_0^1 \bar{\rho}\varphi_0^2 \mathrm{d}y}, \\
\Gamma_3(\Xi) &= \varepsilon^{-2}\mathcal{A}_0^{43} + A_*(\Xi)D_{00}^{43} = \frac{2}{c_0}\frac{\int_0^1 \bar{\rho}\varphi_0'^2 \mathrm{d}y}{\int_0^1 \bar{\rho}\varphi_0^2 \mathrm{d}y} - \frac{3}{c_0}A_*(\Xi)\frac{\int_0^1 \bar{\rho}\varphi_0'^3 \mathrm{d}y}{\int_0^1 \bar{\rho}\varphi_0^2 \mathrm{d}y}
\end{aligned}$$

We recall from Section 2.3 that $\tilde{A}_0$ coincides with the (ε-free) KdV soliton A_*. By the linear change of variables given by

$$W_1 = \tilde{W}_1 + \frac{1}{c_0}\tilde{W}_2, \quad W_2 = -\frac{c_0}{2}\tilde{W}_1 + \frac{1}{2}\tilde{W}_2,$$

we obtain the system

$$\begin{aligned} \dot{W}_1 &= O(\varepsilon^2), \\ \dot{W}_2 &= W_3 + O(\varepsilon^2), \\ \dot{W}_3 &= W_4, \\ \dot{W}_4 &= \Gamma_1 W_1 + \Gamma_2 W_2 + \Gamma_3 W_3 + O(\varepsilon), \\ \dot{W}_{4n+1} &= O(\varepsilon), \\ \dot{W}_{4n+2} &= O(\varepsilon), \end{aligned} \tag{EVP$_\varepsilon$}$$

with

$$\begin{aligned} \Gamma_1(\Xi) &= \frac{1}{2}\tilde{\Gamma}_1 + \frac{c_0}{2}\tilde{\Gamma}_2 = -\frac{1}{2}\dot{A}_*(\Xi)\int_0^1 \bar{\rho}\varphi_0'^3 \mathrm{d}y = -\frac{c_0 r}{3s}\dot{A}_*(\Xi), \\ \Gamma_2(\Xi) &= -\frac{1}{c_0}\tilde{\Gamma}_1 + \tilde{\Gamma}_2 = \frac{\Lambda}{s} - \frac{2r}{s}\dot{A}_*(\Xi), \\ \Gamma_3(\Xi) &= -\frac{1}{s} - \frac{2r}{s}A_*(\Xi). \end{aligned}$$

In this system the KdV eigenvalue problem becomes visible (as in Section A.2): For $\varepsilon = 0$ we obtain the system

$$\begin{aligned} \dot{W}_1 &= 0, \\ \dot{W}_2 &= W_3, \\ \dot{W}_3 &= W_4, \\ \dot{W}_4 &= \Gamma_1 W_1 + \Gamma_2 W_2 + \Gamma_3 W_3, \\ \dot{W}_{4n+1} &= 0, \\ \dot{W}_{4n+2} &= 0; \end{aligned} \tag{EVP$_0$}$$

the set $\{W_1 = 0\}$ is invariant for this flow and on this set the only non-trivial equations are the ones for W_2, W_3, W_4:

$$\begin{pmatrix} W_2 \\ W_3 \\ W_4 \end{pmatrix}_\Xi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\Lambda}{s} - \frac{2r}{s}\dot{A}_*(\Xi) & -\frac{1}{s} - \frac{2r}{s}A_*(\Xi) & 0 \end{pmatrix} \begin{pmatrix} W_2 \\ W_3 \\ W_4 \end{pmatrix}. \tag{EVP$_{\mathrm{KdV}}$}$$

This is the eigenvalue problem of KdV equation associated with a solitary wave $A_*(\Xi)$ satisfying $\ddot{A}_*(\Xi) = -\frac{1}{s}A_* - \frac{r}{s}A_*^2$.

We will show that the system (EVP$_\varepsilon$) is contained in the class of eigenvalue problems treated by Pego and Weinstein in [51], see Section B.2 for a brief overview. For this purpose we have to check the hypotheses (H1)-(H4) (in Section B.2). As it is easy to see that (H1), (H2), (H4) are satisfied, we concentrate on (H3) stating that the asymptotic matrix has a unique simple eigenvalue of smallest real part. To be more precise, let $\chi_\varepsilon(\mu;\Lambda)$ denote the characteristic polynomial of the asymptotic matrix associated with system (EVP$_\varepsilon$) for $\varepsilon > 0$; we have the following lemma.

Lemma 4.3. *There exists some $\delta_2 > 0$ such that $\chi_\varepsilon(\mu;\Lambda)$ has a unique simple eigenvalue of smallest real part for all $0 \leq |\Lambda| \leq R_1$ and for all $0 \leq \varepsilon \leq \delta_2$.*

Proof. Let $\chi_0(\mu;\Lambda)$ and $\chi_{\mathrm{KdV}}(\mu;\Lambda)$ denote the characteristic polynomials of the asymptotic matrices associated with the systems (EVP$_0$) and (EVP$_{\mathrm{KdV}}$), respectively. By inspection of these matrices, one finds

$$\chi_0(\mu;\Lambda) = (-\mu)^{2N+1}\chi_{\mathrm{KdV}}(\mu;\Lambda).$$

By Lemma B.3 the polynomial $\chi_{\mathrm{KdV}}(\mu;\Lambda)$ has the following property: There exists some $\nu > 0$ such that for all Λ with $\operatorname{Re}\Lambda \geq -\nu$ there is a unique simple root $\mu_{\mathrm{KdV}}(\Lambda)$ with smallest real part (which is negative). Consequently, the polynomial $\chi_0(\mu;\Lambda)$ has this property as well, and we have $\mu_0(\Lambda) = \mu_{\mathrm{KdV}}(\Lambda)$.

In order to show that $\chi_\varepsilon(\mu;\Lambda)$ also possesses this property for sufficiently small ε, we argue as follows with K denoting the compact set $K := \{\operatorname{Re}\Lambda \geq -\nu\} \cap \{|\Lambda| \leq R_1\}$. The equation

$$0 = \chi(\mu;\varepsilon,\Lambda) := \chi_\varepsilon(\mu;\Lambda) \tag{4.5}$$

has the solution $\mu_0(\Lambda)$ for $\varepsilon = 0$, i.e. $\chi(\mu_0(\Lambda);0,\Lambda) = 0$. Since $\mu_0(\Lambda)$ is a simple root of $\chi_0(\mu;\Lambda)$, we know

$$\frac{\partial}{\partial\mu}\chi(\mu_0(\Lambda);0,\Lambda) \neq 0,$$

thus the implicit function theorem implies that Equation (4.5) can be solved for μ in a neighbourhood of $(\varepsilon,\Lambda) = (0,\Lambda_0)$ for any $\Lambda_0 \in K$. More precisely, for any $\Lambda_0 \in K$ there exists some $\delta_2 > 0$ such that there exists a smooth function

$$\mu : (-\delta_2,\delta_2) \times \{\Lambda : |\Lambda - \Lambda_0| < \delta_2\} \to \mathbb{C}$$

with $\chi(\mu(\varepsilon,\Lambda);\varepsilon,\Lambda) = 0$. In this way, we obtain an open cover of K and its compactness allows to pass to a finite subcover. Hence, we find some $\delta_2 > 0$ and some function

$$\mu : [0, \delta_2) \times K \to \mathbb{C},$$

which is the unique simple zero $\mu_\varepsilon(\Lambda)$ of smallest real part of $\chi_\varepsilon(\mu; \Lambda)$ for all $0 < \varepsilon < \delta_2$ and all $\Lambda \in K$. □

As we have shown, hypothesis (H3) holds on the domain

$$\Omega := \{\Lambda \in \mathbb{C} : \operatorname{Re} \Lambda > -\nu\} \cap \{|\Lambda| < R_1\}$$

and, thus, the theory due to Pego and Weinstein is applicable to the system (EVP$_\varepsilon$); this is the idea underlying the proof of Lemma 4.1.

Recall that for a linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathcal{A}(x)y,$$

where $y(x)$ is a column vector, the adjoint system is given by

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -z\mathcal{A}(x),$$

where $z(x)$ is a row vector. In the following we denote by (EVP$^*_{\mathrm{KdV}}$), (EVP$^*_0$) and (EVP$^*_\varepsilon$) the adjoint systems of (EVP$_{\mathrm{KdV}}$), (EVP$_0$) and (EVP$_\varepsilon$), respectively.

Let $Z^+_{\mathrm{KdV}}(\Lambda)$, $Z^+_0(\Lambda)$, $Z^+_\varepsilon(\Lambda)$ and $Y^-_{\mathrm{KdV}}(\Lambda)$, $Y^-_0(\Lambda)$, $Y^-_\varepsilon(\Lambda)$ denote associated right, resp. left, eigenvectors of the asymptotic matrices normalized in such a way that $Y^- \cdot Z^+ = 1$ holds. Then, we obtain the following lemma which states the existence of special functions spanning the stable space and the complement of the unstable space respectively, directly by applying Lemma B.1 to each of the systems (EVP$_{\mathrm{KdV}}$), (EVP$_0$) and (EVP$_\varepsilon$).

Lemma 4.4. (i) *For $0 < \varepsilon < \delta_2$ there are differentiable functions*

$$\zeta^+_{\mathrm{KdV}}(\xi; \Lambda), \zeta^+_0(\xi; \Lambda), \zeta^+_\varepsilon(\xi; \Lambda),$$

analytic with respect to $\Lambda \in \Omega$, with the following properties:
$\zeta^+_{\mathrm{KdV}}(\xi; \Lambda)$ *solves* (EVP$_{\mathrm{KdV}}$) *and satisfies* $\mathrm{e}^{\mu_{\mathrm{KdV}}(\Lambda)\xi}\zeta^+_{\mathrm{KdV}}(\xi; \Lambda) \to Z^+_{\mathrm{KdV}}(\Lambda)$ *as* $\xi \to \infty$,
$\zeta^+_0(\xi; \Lambda)$ *solves* (EVP$_0$) *and satisfies* $\mathrm{e}^{\mu_0(\Lambda)\xi}\zeta^+_0(\xi; \Lambda) \to Z^+_0(\Lambda)$ *as* $\xi \to \infty$,
$\zeta^+_\varepsilon(\xi; \Lambda)$ *solves* (EVP$_\varepsilon$) *and satisfies* $\mathrm{e}^{\mu_\varepsilon(\Lambda)\xi}\zeta^+_\varepsilon(\xi; \Lambda) \to Z^+_\varepsilon(\Lambda)$ *as* $\xi \to \infty$.
These conditions characterize the functions uniquely up to a constant factor.

(ii) *For* $0 < \varepsilon < \delta_2$ *there are differentiable functions*

$$\eta_{\mathrm{KdV}}^{-}(\xi;\Lambda), \eta_0^{-}(\xi;\Lambda), \eta_\varepsilon^{-}(\xi;\Lambda),$$

analytic with respect to Λ*, with the following properties:*
$\eta_{\mathrm{KdV}}^{-}(\xi;\Lambda)$ *solves* $(\mathrm{EVP}_{\mathrm{KdV}}^{*})$ *and satisfies* $\mathrm{e}^{\mu_{\mathrm{KdV}}(\Lambda)\xi}\eta_{\mathrm{KdV}}^{-}(\xi;\Lambda) \to Y_{\mathrm{KdV}}^{-}(\Lambda)$ *as* $\xi \to -\infty$*,*
$\eta_0^{-}(\xi;\Lambda)$ *solves* (EVP_0^{*}) *and satisfies* $\mathrm{e}^{\mu_0(\Lambda)\xi}\eta_0^{-}(\xi;\Lambda) \to Y_0^{-}(\Lambda)$ *as* $\xi \to -\infty$*,*
$\eta_\varepsilon^{-}(\xi;\Lambda)$ *solves* $(\mathrm{EVP}_\varepsilon^{*})$ *and satisfies* $\mathrm{e}^{\mu_\varepsilon(\Lambda)\xi}\eta_\varepsilon^{-}(\xi;\Lambda) \to Y_\varepsilon^{-}(\Lambda)$ *as* $\xi \to -\infty$*.*
These conditions characterize the functions uniquely up to a constant factor.

With these functions at hand, we can define the Evans functions

$$\begin{aligned} E_{\mathrm{KdV}}(\Lambda) &= \eta_{\mathrm{KdV}}^{-}(\xi;\Lambda)\cdot\zeta_{\mathrm{KdV}}^{+}(\xi;\Lambda),\\ E_0(\Lambda) &= \eta_0^{-}(\xi;\Lambda)\cdot\zeta_0^{+}(\xi;\Lambda),\\ E_\varepsilon(\Lambda) &= \eta_\varepsilon^{-}(\xi;\Lambda)\cdot\zeta_\varepsilon^{+}(\xi;\Lambda)\end{aligned}$$

for each of the systems $(\mathrm{EVP}_{\mathrm{KdV}})$, (EVP_0) and $(\mathrm{EVP}_\varepsilon)$ in the vein of Pego and Weinstein.

So far, we have gathered all the ingredients necessary for the proof of Lemma 4.1.

Proof of Lemma 4.1. *Step 1:* For $\varepsilon = 0$, we find special solutions $\tilde{\zeta}_0^{+}$, $\tilde{\eta}_0^{-}$ to (EVP_0) and its adjoint system, namely

$$\begin{aligned}\tilde{\zeta}_0^{+} &= (0, \zeta_{\mathrm{KdV}}^{+}, 0, \dots, 0)^{\mathsf{T}},\\ \tilde{\eta}_0^{-} &= (*, \eta_{\mathrm{KdV}}^{-}, 0, \dots, 0),\end{aligned}$$

(with $*$ appropriately chosen) exhibiting the correct decay rate $\mu_0 = \mu_{\mathrm{KdV}}$ for $\xi \to \pm\infty$, respectively.

Therefore, Lemma 4.4 implies that there are complex constants $\gamma_1, \gamma_2 \in \mathbb{C}$ such that

$$\zeta_0^{+} = \gamma_1\tilde{\zeta}_0^{+} \quad \text{and} \quad \eta_0^{-} = \gamma_2\tilde{\eta}_0^{-},$$

hence

$$E_0(\Lambda) = \eta_0^{-}\cdot\zeta_0^{+} = \gamma_1\gamma_2\tilde{\eta}_0^{-}\cdot\tilde{\zeta}_0^{+} = \gamma\eta_{\mathrm{KdV}}^{-}\cdot\zeta_{\mathrm{KdV}}^{+} = \gamma E_{\mathrm{KdV}}(\Lambda)$$

with $\gamma := \gamma_1\gamma_2$ being constant. Thus, Lemma B.4 implies that $E_0(\Lambda)$ does not vanish in Ω except for $\Lambda = 0$ where a double zero is present.

Step 2: By Lemma 4.4 the functions $\zeta_\varepsilon^{+}, \eta_\varepsilon^{-}$, hence $E_\varepsilon(\Lambda)$, also exist for $0 \le \varepsilon < \delta_2$.

The Evans function $E_\varepsilon(\Lambda)$ is analytic in Λ (and continuous in ε), thus we can compare the numbers of zeros of E_ε and E_0 in a certain domain by invoking Rouchés theorem.

First, consider a small open ball U_0 centred at the the origin. By choosing ε sufficiently small, say $0 \leq \varepsilon < \delta_3$ we ensure that $|E_\varepsilon(\Lambda) - E_0(\Lambda)| < |E_0(\Lambda)|$ holds on the boundary ∂U; this is possible since E_ε is continuous in ε, coincides with E_0 for $\varepsilon = 0$, and $E_0(\Lambda)$ does not vanish on ∂U. So, the number of zeros of E_ε in U coincides with the number of zeros of E_0 in U as E_0, which is two by the previous step.

Second, for any open ball $U \subset \Omega \setminus U_0$ we find, by a similar argument, that E_ε does not vanish on U provided ε is sufficiently small. Since $\Omega \setminus U_0$ is compact, we may pass to a finite subcover to conclude that there exists some $\delta_4 > 0$ such that E_ε does not vanish on $\Omega \setminus U_0$ for all $0 \leq \varepsilon < \delta_4$.

On the other hand, E_ε has a double zero in $\Lambda = 0$ due to the generalized eigenvectors associated with horizontal shifts and changes in speed. As we have shown that there are at most two zeros, we see that $E_\varepsilon(\Lambda) \neq 0$ for all $\Lambda \in \Omega \setminus \{0\}$ with $\operatorname{Re}\Lambda \geq 0$ and for all $0 \leq \varepsilon < \varepsilon_1 := \min\{\delta_k : k = 1, \cdots, 4\}$. □

4.3. Proof of Lemma 4.2 (First outer regime)

We first introduce an appropriate scaling and second perform a centre-manifold reduction similar to the preceding section. Following the idea of Haragus and Scheel in [31], we let

$$\nu := \frac{\varepsilon}{|\kappa|^{1/3}}.$$

If we show the absence of bounded solutions for sufficiently small ν and $|\kappa|$, i.e. $0 \leq \nu \leq \nu_2$ and $0 \leq |\kappa| \leq \kappa_2$ for some $\nu_2 > 0$ and $\kappa_2 > 0$, then we have actually covered a region as asserted in the lemma, since by defining $r_2 := \frac{1}{\nu_2^3}$ and $R_2 := \kappa_2$ we find

$$|\kappa| = \delta = \frac{\varepsilon^3}{\nu^3} \geq \frac{\varepsilon^3}{\nu_2^3} \equiv r_2 \varepsilon^3.$$

Using the scaling

$$\begin{aligned}
w_1(\xi) &= W_1(\xi), & w_{4n+1}(\xi) &= |\kappa|^{\frac{1}{3}}\, W_{4n+1}(\xi),\\
w_2(\xi) &= W_2(\xi), & w_{4n+2}(\xi) &= |\kappa|^{\frac{1}{3}}\, W_{4n+2}(\xi),\\
w_3(\xi) &= |\kappa|^{\frac{1}{3}}\, W_3(\xi), & w_{4n+3}(\xi) &= |\kappa|^{\frac{2}{3}}\, W_{4n+3}(\xi),\\
w_4(\xi) &= |\kappa|^{\frac{2}{3}}\, W_4(\xi), & w_{4n+4}(\xi) &= |\kappa|^{\frac{2}{3}}\, W_{4n+4}(\xi),
\end{aligned}$$

for all n with $1 \leq n \leq N$, the problem (4.3) takes to leading order the form

$$\begin{aligned}
A_\varepsilon' &= \nu\,|\kappa|^{\frac{1}{3}}\,B_\varepsilon,\\
B_\varepsilon' &= \nu\,|\kappa|^{\frac{1}{3}}\left(-\frac{1}{s}A_\varepsilon - \frac{r}{s}A_\varepsilon^2\right) + O(\nu^3\,|\kappa|),\\
W_1' &= O(|\kappa|^{\frac{1}{3}}),\\
W_2' &= O(|\kappa|^{\frac{1}{3}}),\\
W_3' &= O(|\kappa|^{\frac{1}{3}}),\\
W_4' &= O(|\kappa|^{\frac{1}{3}}),\\
W_{4n+1}' &= O(|\kappa|^{\frac{1}{3}}),\\
W_{4n+2}' &= O(|\kappa|^{\frac{1}{3}}),\\
W_{4n+3}' &= W_{4n+4},\\
W_{4n+4}' &= \delta(\lambda_n - \lambda_0)W_{4n+3} + O(\nu^2\,|\kappa|^{\frac{2}{3}}),
\end{aligned}$$

for any $1 \leq n \leq N$. We infer that, for $|\kappa| = 0$, the set

$$M_0 := \{W_j = 0 : j \in I_h\}$$

is a centre manifold, which is normally hyperbolic since the partial Jacobian matrix of this system with respect to the variables W_{4n+3}, W_{4n+4} for $n = 1, \dots, N$ has a block structure of the form

$$\begin{pmatrix} Y_1 & * & \cdots & * \\ 0 & Y_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & Y_N \end{pmatrix} \tag{4.6}$$

with

$$Y_n = \begin{pmatrix} 0 & 1 \\ \delta(\lambda_n - \lambda_0) & 0 \end{pmatrix}$$

so the eigenvalues, given by $\pm\sqrt{\delta(\lambda_n - \lambda_0)}$ for $n = 1, \dots, N$, are real and different from zero.

By virtue of Fenichel's theorem on the persistence of normally hyperbolic invariant manifolds (see [23, 24]), we conclude that an invariant manifold M_κ exists for all sufficiently small $|\kappa| > 0$, which is a graph over M_0, and, in particular, $W_j = O(|\kappa|^{\frac{1}{3}})$ for all $j \in I_h$.

In this way, we obtain a reduced system for the variables $\{W_j : j \in I_c\}$. After changing to the slow scale

$$\Xi := |\kappa|^{\frac{1}{3}}\,\xi,$$

setting

$$W_j(\xi) = \hat{W}_j(\Xi)$$

for all $j \in I_c$ and omitting the hat, this reduced system is of the form

$$\begin{aligned}
\dot{\tilde{A}}_\varepsilon &= \nu \tilde{B}_\varepsilon,\\
\dot{\tilde{B}}_\varepsilon &= \nu\left(-\frac{1}{s}\tilde{A}_\varepsilon - \frac{r}{s}\tilde{A}_\varepsilon^2\right) + O(\nu^3\,|\kappa|^{\frac{2}{3}}),\\
\dot{W}_1 &= -\frac{1}{c_0}W_3 + O(|\kappa|^{\frac{2}{3}} + \nu^2\,|\kappa|^{\frac{2}{3}}),\\
\dot{W}_2 &= W_3,\\
\dot{W}_3 &= W_4,\\
\dot{W}_4 &= \lambda_0\delta \mathrm{e}^{i\arg(\kappa)}W_1 - \frac{\lambda_0\delta}{c_0}\mathrm{e}^{i\arg(\kappa)}W_2 + O(|\kappa|^{\frac{2}{3}} + \nu^2),\\
\dot{W}_{4n+1} &= O(|\kappa|^{\frac{1}{3}} + \nu^2\,|\kappa|^{\frac{1}{3}}),\\
\dot{W}_{4n+2} &= O(|\kappa|^{\frac{1}{3}}),
\end{aligned}$$

for all n with $1 \le n \le N$ and with $\dot{}$ denoting the derivative with respect to Ξ.

In the limit $|\kappa| = \nu = 0$ we obtain a linear system with constant coefficients, the eigenvalues of which can be easily determined: First, the three complex roots of

$$-\frac{2\lambda_0\delta}{c_0}\mathrm{e}^{i\arg\kappa}$$

lead to three hyperbolic eigenvalues (two with negative real part, one with positive real part). Second, $\mu = 0$ is a $(2N+1)$-fold eigenvalue. Hence the central space $\mathcal{M}_{0,0}$, i.e. the span of all (possibly generalized) eigenvectors associated with the zero eigenvalue, forms an invariant manifold which is normally hyperbolic.

By virtue of Fenichel's theorem on the persistence of normally hyperbolic invariant manifolds, we conclude that $\mathcal{M}_{0,0}$ is perturbed to an invariant manifold $\mathcal{M}_{\kappa,\nu}$ for all $0 \le \nu \le \nu_0$ and $0 \le |\kappa| \le \kappa_0$ with some sufficiently small ν_0, κ_0.

We recall that in the original problem for $\operatorname{Re}\kappa > 0$ the asymptotic matrix has $N+1$ eigenvalues of negative real part and $3N+3$ eigenvalues of positive real part; in the first reduction (to M_κ) N eigenvalues of negative real part and N eigenvalues of positive real part have been eliminated and in the second reduction (to $\mathcal{M}_{\kappa,\nu}$) 1 eigenvalue of negative real part and 2 eigenvalues of positive real part have been eliminated. Therefore, the reduced problem on $\mathcal{M}_{\kappa,\nu}$ has the property that all of the remaining

$$4N + 4 - ((N + N) + (1 + 2)) = 2N + 1$$

eigenvalues of the hyperbolic asymptotic matrices for $\xi \to \pm\infty$ have positive real part provided $\operatorname{Re}\kappa > 0$. This implies that all solutions to this reduced problem grow at some exponential rate as $\xi \to \infty$. Consequently, there are no bounded solutions in the present regime for $\operatorname{Re}\kappa > 0$. This finishes the proof of Lemma 4.2.

4.4. Outlook on the remaining regimes

It is beyond the scope of this thesis to fully treat all regimes of the truncated problems. However, this is the subject of on-going and future work (see [40]) and we present a first step in this direction. By application of the Newton polygon method (see B.1), we can approximately locate the spatial eigenvalues μ_k^* of $\mathcal{A}^\infty(\kappa,\varepsilon)$ in further regimes which are defined by the ratio between ε and $|\kappa|$. It seems that the following results are a promising starting point to investigate the truncated problems in the Grassmannian framework as e.g. in [25, 26].

The purpose of the following lemma is to give approximate expressions for the roots of the polynomials χ_0 and χ_n. We begin with recalling χ_0 (see Equation (3.17)):

$$\chi_0(\mu;\kappa,\varepsilon) = \mu^4 - \frac{2}{c_0}\kappa\mu^3 + \left(\frac{\kappa^2}{c_0^2} - \frac{2\delta\lambda_0}{c_0}\varepsilon^2\right)\mu^2 + \frac{2\delta\lambda_0}{c_0}\kappa\mu - \frac{\lambda_0\delta}{c_0^2}\kappa^2.$$

Lemma 4.5. (i) *There exists $r_1 > 0$ such that for $\kappa \in \mathbb{C}$ with $0 \le |\kappa|\,\varepsilon^{-3} \le r_1$:*

$$\begin{aligned}
\mu_0^s(\kappa,\varepsilon) &= -\varepsilon\sqrt{\frac{2\delta\lambda_0}{c_0}} + O(\zeta\varepsilon + \varepsilon^3),\\
\mu_0^{u_1}(\kappa,\varepsilon) &= \frac{\zeta\varepsilon^3}{2c_0} + O(\zeta^2\varepsilon^3 + \varepsilon^5),\\
\mu_0^{u_2}(\kappa,\varepsilon) &= \zeta\varepsilon + O(\zeta^2\varepsilon + \varepsilon^3),\\
\mu_0^{u_3}(\kappa,\varepsilon) &= \varepsilon\sqrt{\frac{2\delta\lambda_0}{c_0}} + O(\zeta\varepsilon + \varepsilon^3).
\end{aligned}$$

(ii) *For all $R_1 > 0$ and $\kappa \in \mathbb{C}$ with $r_1\varepsilon^3 \le |\kappa| \le R_1\varepsilon^3$:*

$$\begin{aligned}
\mu_0^s(\kappa,\varepsilon) &= \varepsilon\tau_1(\zeta) + O(\varepsilon^3),\\
\mu_0^{u_1}(\kappa,\varepsilon) &= \frac{1}{2c_0}\zeta\varepsilon^3 + O(\varepsilon^5),
\end{aligned}$$

$$\mu_0^{u_2}(\kappa,\varepsilon) = \varepsilon\tau_2(\zeta) + O(\varepsilon^3),$$
$$\mu_0^{u_3}(\kappa,\varepsilon) = \varepsilon\tau_3(\zeta) + O(\varepsilon^3),$$

with $\tau_{1,2,3}(\zeta) \in \mathbb{C}$ *denoting the roots of* $\tau^3 + \frac{1}{s}\tau - \frac{\zeta}{s} = 0$ *in such a way that* $\operatorname{Re}\tau_1(\zeta) < 0$.

(iii) *There exist* $r_2, R_2 > 0$ *such that for* $\kappa \in \mathbb{C}$ *with* $r_2\varepsilon^3 \le |\kappa| \le R_2$*:*

$$\begin{aligned}
\mu_0^s(\kappa,\varepsilon) &= \kappa^{1/3}\omega_1 + O(\rho^{1/3}\nu^2 + \rho^{2/3}),\\
\mu_0^{u_1}(\kappa,\varepsilon) &= \kappa^{1/3}\omega_2 + O(\rho^{1/3}\nu^2 + \rho^{2/3}),\\
\mu_0^{u_2}(\kappa,\varepsilon) &= \kappa^{1/3}\omega_3 + O(\rho^{1/3}\nu^2 + \rho^{2/3}),\\
\mu_0^{u_3}(\kappa,\varepsilon) &= \frac{\kappa}{2c_0} + O(\rho\nu^2 + \rho^2),
\end{aligned}$$

with $\omega_{1,2,3} \in \mathbb{C}$ *denoting the three zeros of* $\omega^3 = \frac{1}{s}$ *in such a way that* $\operatorname{Re}\omega_1 < \min\{\operatorname{Re}\omega_2, \operatorname{Re}\omega_3\}$.

(iv) *For all* $0 < r_3 < R_3$ *there exists some* $C > 0$ *such that for all* $\kappa \in \mathbb{C}$ *with* $r_3 \le |\kappa| \le R_3$ *the inequality*

$$\operatorname{Re}\mu_0^s(\kappa,\varepsilon) < -C < \min\{\operatorname{Re}\mu_0^{u_1}(\kappa,\varepsilon), \operatorname{Re}\mu_0^{u_2}(\kappa,\varepsilon), \operatorname{Re}\mu_0^{u_3}(\kappa,\varepsilon)\}$$

holds.

(v) *There exists some* $r_4 > 0$ *such that for* $\kappa \in \mathbb{C}$ *with* $r_4 \le |\kappa|$*:*

$$\begin{aligned}
\mu_0^s(\kappa,\varepsilon) &= -\sqrt{\lambda_0\delta} + O(\varepsilon^2 + \frac{1}{|\kappa|}),\\
\mu_0^{u_1}(\kappa,\varepsilon) &= \kappa\frac{1}{c_0} + O(1+\varepsilon),\\
\mu_0^{u_2}(\kappa,\varepsilon) &= \kappa\frac{1}{c_0} + O(1+\varepsilon),\\
\mu_0^{u_3}(\kappa,\varepsilon) &= \sqrt{\lambda_0\delta} + O(\varepsilon^2 + \frac{1}{|\kappa|}).
\end{aligned}$$

Proof. (i) By setting $\kappa = \zeta\varepsilon^3$ in the equation and looking at the leading order terms, the Newton polygon method yields the two exponents 1 and 3.

First, let $\mu = \varepsilon T$. Then, we find the equation

$$T^4 - \frac{2\delta\lambda_0}{c_0}T^2 + \frac{2\delta\lambda_0}{c_0}\zeta T + O(\varepsilon^2\zeta^2) = 0,$$

which is analytic with respect to ε^2 and ζ. For $\varepsilon = 0$ we immediately have $T_1 = 0$. For the remaining third-order polynomial, we can search for solutions for small ζ yielding

$$\begin{aligned} T_2(\zeta, 0) &= \zeta + O(\zeta^2), \\ T_{3,4}(\zeta, 0) &= \pm\sqrt{\frac{2\delta\lambda_0}{c_0}} + O(\zeta). \end{aligned}$$

Consequently,

$$\begin{aligned} T_2(\zeta, \varepsilon) &= \zeta + O(\zeta^2 + \varepsilon^2), \\ T_{3,4}(\zeta, \varepsilon) &= \pm\sqrt{\frac{2\delta\lambda_0}{c_0}} + O(\zeta + \varepsilon^2). \end{aligned}$$

Second, let $\mu = \varepsilon^3 T$. Then, T solves

$$\frac{2\delta\lambda_0}{c_0}\zeta T - \frac{\delta\lambda_0}{c_0^2}\zeta^2 + O(\zeta^2 + \varepsilon^2),$$

which is analytic with respect to ζ and ε^2. Thus, we find

$$T_1(\varepsilon, \zeta) = \frac{\zeta}{2c_0} + O(\zeta^2 + \varepsilon^2).$$

Returning to the original variable μ, we obtain

$$\begin{aligned} \mu_0^s(\kappa, \varepsilon) &= -\varepsilon\sqrt{\frac{2\delta\lambda_0}{c_0}} + O(\zeta\varepsilon + \varepsilon^3), \\ \mu_0^{u_1}(\kappa, \varepsilon) &= \frac{\zeta\varepsilon^3}{2c_0} + O(\zeta^2\varepsilon^3 + \varepsilon^5), \\ \mu_0^{u_2}(\kappa, \varepsilon) &= \zeta\varepsilon + O(\zeta^2\varepsilon + \varepsilon^3), \\ \mu_0^{u_3}(\kappa, \varepsilon) &= \varepsilon\sqrt{\frac{2\delta\lambda_0}{c_0}} + O(\zeta\varepsilon + \varepsilon^3). \end{aligned}$$

(ii) Setting again $\kappa = \varepsilon^3\zeta$, the Newton polygon yields the exponents 1 and 3 (as in (i)).
First, let $\mu = \varepsilon T$. For $\varepsilon = 0$, we find then $T_4 = 0$ and $T_{1,2,3}$ as the roots of the polynomial

$$T^3 - \frac{2\delta\lambda_0}{c_0}T + \frac{2\delta\lambda_0}{c_0}\zeta = 0,$$

which is the KdV polynomial χ_{KdV} from Lemma B.3. If we let $\tau_{1,2,3}(\zeta)$ denote the zeros of this polynomial such that $\operatorname{Re}\tau_1(\zeta) < 0 < \operatorname{Re}\tau_{2,3}(\zeta)$ for $\operatorname{Re}\zeta > 0$, then we find

$$T_{1,2,3}(\kappa,\varepsilon) = \tau_{1,2,3}(\zeta) + O(\varepsilon^2).$$

Second, let $\mu = \varepsilon^3 T$. Then, we find (as in (i))

$$T_4(\kappa,\varepsilon) = \frac{1}{2c_0}\zeta + O(\varepsilon^2).$$

In terms of the original variables this means

$$\begin{aligned}
\mu_0^s(\kappa,\varepsilon) &= \varepsilon\tau_1(\zeta) + O(\varepsilon^3),\\
\mu_0^{u_1}(\kappa,\varepsilon) &= \frac{1}{2c_0}\zeta\varepsilon^3 + O(\varepsilon^5),\\
\mu_0^{u_2}(\kappa,\varepsilon) &= \varepsilon\tau_2(\zeta) + O(\varepsilon^3),\\
\mu_0^{u_3}(\kappa,\varepsilon) &= \varepsilon\tau_3(\zeta) + O(\varepsilon^3).
\end{aligned}$$

(iii) In this regime, we set $\kappa = \rho\cdot \mathrm{e}^{i\varphi}$ and $\nu = \frac{\varepsilon}{\rho^{1/3}}$ and consider small ν and small ρ. Concerning small ρ the Newton polygon provides the exponents 1 and $\frac{1}{3}$.
First, we set $\mu = \rho^{1/3}T$ and find that T solves

$$T^4 - \frac{2\delta\lambda_0}{c_0}\nu^2T^2 + \frac{2\delta\lambda_0}{c_0}\mathrm{e}^{i\varphi}T + O(\rho^{2/3}) = 0,$$

which is analytic in $\rho^{1/3}$ and ν^2. For $\rho = 0$ we find $T_4 = 0$. In the remaining equation of degree three we find the solutions

$$T_{1,2,3} = \omega_{1,2,3}\mathrm{e}^{i\varphi/3} + O(\nu^2 + \rho^{1/3})$$

with $\omega_{1,2,3}$ denoting the third roots of $-\frac{2\lambda_0\delta}{c_0}$, numbered such that $\operatorname{Re}\omega_1 < 0 < \operatorname{Re}\omega_{2,3}$.
Second, we set $\mu = \rho T$ and find that T_4 is given by

$$T_4(\kappa,\varepsilon) = \frac{\mathrm{e}^{i\varphi}}{2c_0} + O(\rho + \nu^2).$$

Translating back to the original variables we find

$$\mu_0^s(\kappa,\varepsilon) = \kappa^{1/3}\omega_1 + O(\rho^{1/3}\nu^2 + \rho^{2/3}),$$

$$\mu_0^{u_1}(\kappa,\varepsilon) = \kappa^{1/3}\omega_2 + O(\rho^{1/3}\nu^2 + \rho^{2/3}),$$
$$\mu_0^{u_2}(\kappa,\varepsilon) = \kappa^{1/3}\omega_3 + O(\rho^{1/3}\nu^2 + \rho^{2/3}),$$
$$\mu_0^{u_3}(\kappa,\varepsilon) = \frac{\kappa}{2c_0} + O(\rho\nu^2 + \rho^2).$$

(iv) The existence of such a $C > 0$ follows directly from Lemma 3.8 on the consistent splitting and from the compactness of the considered domain.

(v) In this regime, we define $K := \kappa^{-1}$ and consider small K. The equation thus changes to

$$K^2\mu^4 - \frac{2}{c_0}K\mu^3 + \left(\frac{1}{c_0^2} - \frac{2\delta\lambda_0}{c_0}\varepsilon^2 K^2\right) + \frac{2\delta\lambda_0}{c_0}K\mu - \frac{\lambda_0\delta}{c_0^2} = 0.$$

Considering the Newton polygon provides the exponents -1 and 0.
First, setting $K = 0$ we find

$$\mu_0^{s}(\kappa,\varepsilon) = -\sqrt{\lambda_0\delta} + O(\varepsilon^2 + \frac{1}{|\kappa|}),$$
$$\mu_0^{u_3}(\kappa,\varepsilon) = \sqrt{\lambda_0\delta} + O(\varepsilon^2 + \frac{1}{|\kappa|}).$$

Second, after rescaling $\mu = \frac{T}{K}$ we find that T must solve the equation

$$T^2\left(T - \frac{1}{c_0}\right) + O(K^2 + \varepsilon^2 K^2) = 0,$$

which is analytic in K^2 and ε^2. The two remaining roots are thus given by

$$\mu_0^{u_1}(\kappa,\varepsilon) = \kappa\frac{1}{c_0} + O(1+\varepsilon),$$
$$\mu_0^{u_2}(\kappa,\varepsilon) = \kappa\frac{1}{c_0} + O(1+\varepsilon).$$

□

Remark 4.6. *The parts (ii) and (iii) justify a posteriori the scalings used in the proofs of Lemma 4.1 and Lemma 4.2.*

Next, we discuss $\chi_n(\mu;\kappa,\varepsilon)$ for $1 \le n \le N$, the $\varepsilon = 0$ part of which is given by (see Equation (3.17))

$$\chi_n(\mu;\kappa,\varepsilon=0) = \mu^4 - \frac{2\kappa}{c_0}\mu^3 + \left(\frac{\kappa^2}{c_0^2} - \delta(\lambda_n - \lambda_0)\right)\mu^2 + \frac{2\lambda_n\delta}{c_0}\kappa\mu - \frac{\lambda_n\delta}{c_0^2}\kappa^2.$$

Lemma 4.7. (i) *There exists some $R_1' > 0$ such that for $\kappa \in \mathbb{C}$ with $0 \leq |\kappa| \leq R_1'$:*

$$\begin{aligned}
\mu_n^s(\kappa,\varepsilon) &= -\sqrt{\delta(\lambda_n - \lambda_0)} + O(|\kappa| + \varepsilon^2),\\
\mu_n^{u_1}(\kappa,\varepsilon) &= \frac{\kappa}{c_0}\frac{\sqrt{\lambda_n}}{\sqrt{\lambda_n} - \sqrt{\lambda_0}} + O(|\kappa|^2 + \varepsilon^2),\\
\mu_n^{u_2}(\kappa,\varepsilon) &= \frac{\kappa}{c_0}\frac{\sqrt{\lambda_n}}{\sqrt{\lambda_n} + \sqrt{\lambda_0}} + O(|\kappa|^2 + \varepsilon^2),\\
\mu_n^{u_3}(\kappa,\varepsilon) &= +\sqrt{\delta(\lambda_n - \lambda_0)} + O(|\kappa| + \varepsilon^2).
\end{aligned}$$

(ii) *For all $0 < r_2' < R_2'$ there exists some $C' > 0$ such that for all $\kappa \in \mathbb{C}$ with $r_2' \leq |\kappa| \leq R_2'$ the inequality*

$$\operatorname{Re}\mu_n^s(\kappa,\varepsilon) < -C' < \min\{\operatorname{Re}\mu_n^{u_1}(\kappa,\varepsilon), \operatorname{Re}\mu_n^{u_2}(\kappa,\varepsilon), \operatorname{Re}\mu_n^{u_3}(\kappa,\varepsilon)\}$$

holds.

(iii) *There exists $r_3' > 0$ such that for $\kappa \in \mathbb{C}$ with $r_3' \leq |\kappa|$:*

$$\begin{aligned}
\mu_n^s(\kappa,\varepsilon) &= -\sqrt{\lambda_n\delta} + O(\varepsilon^2 + \frac{1}{|\kappa|}),\\
\mu_n^{u_1}(\kappa,\varepsilon) &= \kappa\frac{1}{c_0} + O(1+\varepsilon),\\
\mu_n^{u_2}(\kappa,\varepsilon) &= \kappa\frac{1}{c_0} + O(1+\varepsilon),\\
\mu_n^{u_3}(\kappa,\varepsilon) &= \sqrt{\lambda_n\delta} + O(\varepsilon^2 + \frac{1}{|\kappa|}).
\end{aligned}$$

Proof. Here the dependence of $\chi_n(\mu;\kappa,\varepsilon)$ on ε is regular, since none of the coefficients is $O(\varepsilon)$ here. Moreover, as the function $\chi_n(\mu;\kappa,\varepsilon)$ is analytic with respect to ε^2, it follows that simple roots of $\chi_n(\mu;\kappa,\varepsilon=0)=0$ are $O(\varepsilon^2)$-close to simple roots of $\chi_n(\mu;\kappa,\varepsilon)=0$ and double roots of the first equation are $O(\varepsilon)$-close to roots of the latter.

(i) The Newton polygon yields the exponents 1 and 0.
First, setting $\kappa = 0$ yields the roots

$$\begin{aligned}
\mu_n^s(\kappa,\varepsilon) &= -\sqrt{\delta(\lambda_n - \lambda_0)} + O(|\kappa| + \varepsilon^2),\\
\mu_n^{u_3}(\kappa,\varepsilon) &= +\sqrt{\delta(\lambda_n - \lambda_0)} + O(|\kappa| + \varepsilon^2),
\end{aligned}$$

Second, let $\mu = \kappa T$. Then, for $\kappa = 0$, T solves

$$\delta(\lambda_0 - \lambda_n)T^2 + \frac{2\lambda_n\delta}{c_0}T - \frac{\lambda_n\delta}{c_0^2} = 0.$$

We obtain

$$\mu_n^{u_1}(\kappa,\varepsilon) = \frac{\kappa}{c_0}\frac{\sqrt{\lambda_n}}{\sqrt{\lambda_n} - \sqrt{\lambda_0}} + O(|\kappa|^2 + \varepsilon^2),$$
$$\mu_n^{u_2}(\kappa,\varepsilon) = \frac{\kappa}{c_0}\frac{\sqrt{\lambda_n}}{\sqrt{\lambda_n} + \sqrt{\lambda_0}} + O(|\kappa|^2 + \varepsilon^2).$$

(ii) As in (iv) of the previous lemma, the existence of such a $C' > 0$ follows directly from Lemma 3.8 on the consistent splitting and from the compactness of the considered domain.

(iii) In this regime (as in (v) of the previous lemma), we define $K := \kappa^{-1}$ and consider small K. The equation thus changes to

$$K^2\mu^4 - \frac{2}{c_0}K\mu^3 + \left(\frac{1}{c_0^2} + \delta(\lambda_0 - \lambda_n)K^2\right) + \frac{2\delta\lambda_n}{c_0}K\mu - \frac{\lambda_n\delta}{c_0^2} = 0.$$

Considering the Newton polygon provides the exponents -1 and 0.
First, setting $K = 0$ we find

$$\mu_n^{s}(\kappa,\varepsilon) = -\sqrt{\lambda_n\delta} + O(\varepsilon^2 + \frac{1}{|\kappa|}),$$
$$\mu_n^{u_3}(\kappa,\varepsilon) = \sqrt{\lambda_n\delta} + O(\varepsilon^2 + \frac{1}{|\kappa|}).$$

Second, after rescaling $\mu = \frac{T}{K}$ we find that T has to solve the equation

$$T^2\left(T - \frac{1}{c_0}\right)^2 + O(K^2 + \varepsilon^2 K^2) = 0,$$

which is analytic in K^2 and ε^2. The two remaining roots are thus given by

$$\mu_n^{u_1}(\kappa,\varepsilon) = \kappa\frac{1}{c_0} + O(1 + \varepsilon),$$
$$\mu_n^{u_2}(\kappa,\varepsilon) = \kappa\frac{1}{c_0} + O(1 + \varepsilon).$$

□

5. Conclusion

In this final chapter, we make a few comments on where our method for investigating the spectral stability of internal solitary waves could lead.

We briefly summarize the principal idea of this thesis. Starting from a regular internal solitary wave as an exact solution to the Euler equations, we have obtained the associated time-independent eigenvalue problem (EVP-Euler) of the Euler equations. To decide on the spectral stability of the wave of interest, one has to determine whether this problem has a bounded solution for any $\kappa \in \mathbb{C}_+$. We have approached this question by proposing a five-step method consisting of: (i) an infinite-dimensional spatial-dynamics formulation (EVP) of the eigenvalue problem (EVP-Euler), (ii) a procedure to obtain finite-dimensional truncations (EVP_N) for each $N \in \mathbb{N}$, (iii) a definition of an Evans function $D_N(\kappa)$ for these finite-dimensional problems EVP_N, (iv) investigation of D_N for zeros κ with $\mathrm{Re}\,\kappa > 0$, (v) identification or preclusion of eigenvalues κ with $\mathrm{Re}\,\kappa > 0$ of the infinite-dimensional system EVP. From this comprehensive programme, only the steps (i)-(iv) have been carried out within this thesis. In the following we indicate directions of on-going and future work.

Full study of the small-amplitude case. The treatment of the finite-dimensional problems (EVP_N), i.e. step (iv), in the case of small-amplitude waves is only partial, as it stands, and it is the subject of on-going research with the article [40] being under preparation where we plan to consider the remaining regimes in detail. For a promising first step in this direction, see Lemma 4.5 and 4.7.

General stratifications. The rigorous results in this thesis have been obtained assuming the prototypical exponential stratification. At the moment, it is not clear how the results can be extended to general stratifications. Notably, as the matrix $\mathcal{A}^\infty(\kappa, c)$ is no longer block-diagonal then, the proof of consistent splitting, which is a crucial part in performing step (iii), needs a different treatment.

Numerical studies. Recent advances in the field of numerical Evans function calculations raise hope that the problems (EVP_N) can be studied numerically and that, in this way, a computational approach to step (iv) can be established. A stimulating question in this regard is as follows: The numerical studies performed by Lamb and collaborators (see [41, 55]) show, among many other things, how computed internal solitary waves change when their potential energy is increased and indicate that in certain cases, depending on the actual stratification, instability (wave breaking) occurs. Is it possible to capture this reported instability by investigating the associated truncated problems?

Approximation of EVP. In step (ii) we have formally applied a Galerkin-type procedure to obtain the finite-dimensional problems (EVP_N) from (EVP). It is a natural question whether the problems (EVP_N) in some strict sense *approximate* the problem (EVP). However, before this question can be posed properly, equation (EVP) has to be put on a sound functional-analytic ground. These questions are related to the step (v) which has been left to future work. It is a desired future objective to show that the unstable spectrum associated with (EVP_N) in some sense approximates the unstable spectrum associated with (EVP) (or rather the unstable spectrum of (EVP-Euler)).

Infinite-dimensional Evans function. The reformulated eigenvalue problem (EVP) is an infinite-dimensional system and it seems to be an open question whether an Evans function with the usual properties can be defined for this system. One step in this direction could consist in investigating whether the sequence of Evans functions D_N associated with (EVP_N) does converge in some sense, possibly after a suitable rescaling. In case it does, the alleged limit D would be a good candidate for such an Evans function. We also keep studying how this goal is related exactly to the work of Latushkin and others (see [44] and references therein) on infinite-dimensional Evans functions.

A. Truncated problems for small-amplitude waves

A.1. Derivation of approximate expressions for small-amplitude waves

In the case of small-amplitude waves precise formulas for the leading order terms of a travelling wave are available and yield explicit expressions of the truncated eigenvalue problem, i.e. of the matrix $\mathcal{A}(\xi;\kappa,\varepsilon)$. We derive these expressions for a general stratification and show how some of the expressions simplify in the case of an exponential stratification. This section serves as a preparation for the next section where we show how the KdV eigenvalue problem emerges as an essential part in the eigenvalue problem of small-amplitude waves.

We briefly recall the facts about small-amplitude waves from Section 2.3 that we need in the further derivation. Let $\varepsilon > 0$ and assume that the leading order of an exact ISW ψ^c of speed $c = c_m + \varepsilon^2$ is given by

$$\psi^c(\xi, y) = \varepsilon^2 A_\varepsilon(\xi)\varphi_m(y) + O(\varepsilon^4)$$

with

$$A_\varepsilon''(\xi) = \varepsilon^2 \left(-\frac{1}{s}A_\varepsilon - \frac{r}{s}A_\varepsilon^2\right) + O(\varepsilon^4)$$

(see Lemma 2.8 and Remark 2.9). The existence of such waves for $m = 0$ follows from Lemma 2.8; concerning the case $m \geq 1$ see the remarks and references at the end of Section 2.3.

From the expression for ψ^c, we immediately derive:

$$\begin{aligned}
\lambda &= \lambda_m - \frac{2\lambda_m}{c_m}\varepsilon^2 + O(\varepsilon^4),\\
\rho^c(\xi, y) &= \bar{\rho}(y) - \frac{1}{c_m}\varepsilon^2 A_\varepsilon(\xi)\bar{\rho}'(y)\varphi_m(y) + O(\varepsilon^4),\\
\sigma^c(\xi, y) &= -\varepsilon^2 A_\varepsilon(\bar{\rho}\varphi_m')' + O(\varepsilon^4).
\end{aligned}$$

A little more calculation shows

$$
\begin{aligned}
\frac{1}{\psi_y^c - c} &= -\frac{1}{c_m} + \frac{\varepsilon^2}{c_m^2}(1 - A_\varepsilon \varphi_m') + O(\varepsilon^4), \\
\frac{1}{\rho^c} &= \frac{1}{\bar\rho} + \varepsilon^2 A_\varepsilon \frac{1}{c_m}\frac{\bar\rho'}{\bar\rho^2}\varphi_m + O(\varepsilon^4), \\
\frac{1}{\psi_y^c - c}\frac{1}{\rho^c} &= -\frac{1}{c_m\bar\rho}\left(1 - \frac{\varepsilon^2}{c_m}\right) - \frac{\varepsilon^2 A_\varepsilon}{c_m^2}\left(\frac{\bar\rho'\varphi_m}{\bar\rho^2} + \frac{\varphi_m}{\bar\rho}\right) + O(\varepsilon^4).
\end{aligned}
$$

By using these identities, we will obtain sufficiently good approximations of $\left\langle U_M^l, \mathbb{A}U_N^k\right\rangle$ with $M, N \in \mathbb{N}$ and $k, l \in \{1,2,3,4\}$. More precisely, we have the following approximations neglecting errors of the orders ε^4 and $\kappa\varepsilon^4$:

$$
\begin{aligned}
\langle \bar\rho'\chi_M, R_1(\bar\rho'\chi_N)\rangle &= \frac{\kappa}{c_m}\left(1 - \frac{\varepsilon^2}{c_m}\right)\delta_{MN} + \varepsilon^2 A_\varepsilon'(-\frac{1}{c_m})\int_0^1 \bar\rho\varphi_m\chi_M(\chi_N' + \frac{\bar\rho''}{\bar\rho'}\chi_N)\mathrm{d}y \\
&\quad + \kappa\varepsilon^2 A_\varepsilon \frac{1}{c_m^2}\int_0^1 \bar\rho\varphi_m'\chi_M\chi_N\mathrm{d}y \\
\langle \bar\rho'\chi_M, R_2(\chi_N)\rangle &= -\varepsilon^2 A_\varepsilon' \frac{1}{c_m^2}\int_0^1 \bar\rho\varphi_m\chi_M\chi_N'\mathrm{d}y \\
\langle \bar\rho'\chi_M, R_3(\chi_N)\rangle &= -\frac{1}{c_m}\left(1 - \frac{\varepsilon^2}{c_m}\right)\delta_{MN} + \varepsilon^2 A_\varepsilon \frac{1}{c_m^2}\int_0^1 \frac{\bar\rho\bar\rho''}{\bar\rho'}\varphi_m\chi_M\chi_N\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
\langle \chi_M, S_1(\bar\rho'\chi_N)\rangle &= -\kappa\lambda_m\left(1 - \frac{2\varepsilon^2}{c_m}\right)\int_0^1 \bar\rho'\chi_M\chi_N\mathrm{d}y \\
&\quad + \varepsilon^2 A_\varepsilon' \int_0^1 -\bar\rho'\varphi_m''\chi_N\chi_M + (\bar\rho'\chi_N)'\chi_M(\lambda_m\varphi_m - \varphi_m')\mathrm{d}y \\
&\quad + \kappa\varepsilon^2 A_\varepsilon(-\frac{\lambda_m}{c_m})\int_0^1 2\bar\rho'\varphi_m'\chi_M\chi_N + \frac{\bar\rho'^2}{\bar\rho}\varphi_m\chi_M\chi_N\mathrm{d}y \\
\langle \chi_M, S_2(\chi_N)\rangle &= \frac{\kappa}{c_m}\left(1 - \frac{\varepsilon^2}{c_m}\right)\left(\lambda_m\int_0^1 \bar\rho'\chi_M\chi_N\mathrm{d}y + \mu_M\delta_{MN}\right) \\
&\quad + \varepsilon^2 A_\varepsilon' \frac{1}{c_m}\int_0^1 \bar\rho\Big[\varphi_m\chi_N''\chi_M' - \varphi_m'\chi_N''\chi_M - 2\varphi_m'\chi_N'\chi_M'\Big]\mathrm{d}y \\
&\quad + \varepsilon^2\kappa A_\varepsilon \frac{1}{c_m^2}\int_0^1 -\bar\rho''\varphi_m\chi_N'\chi_M + \frac{\bar\rho'^2}{\bar\rho}\varphi_m\chi_N'\chi_M + \bar\rho\varphi_m'\chi_N''\chi_M\mathrm{d}y
\end{aligned}
$$

$$
\langle \chi_M, S_3(\chi_N)\rangle = \mu_N\delta_{MN} - \frac{2\varepsilon^2\lambda_m}{c_m}\int_0^1 \bar\rho'\chi_M\chi_N\mathrm{d}y
$$

$$
\begin{aligned}
&+ \varepsilon^2 A_\varepsilon \frac{1}{c_m} \int_0^1 \left((\bar\rho' \varphi_m')' - 2\lambda_m \bar\rho'' \varphi_m + \lambda_m \frac{\bar\rho'^2}{\bar\rho} \varphi_m \right) \chi_M \chi_N \\
&\quad + \left((\bar\rho' \varphi_m)' - \frac{\bar\rho'^2}{\bar\rho} \varphi_m \right) \chi_M \chi_N' \mathrm{d}y \\
&+ \kappa \varepsilon^2 A_\varepsilon' \frac{1}{c_m} \int_0^1 \bar\rho' \varphi_m \chi_M \chi_N \mathrm{d}y \\
\langle \chi_M, S_4(\chi_N) \rangle = &\frac{\kappa}{c_m} \left(1 - \frac{\varepsilon^2}{c_m} \right) \delta_{MN} \\
&+ \varepsilon^2 A_\varepsilon' \frac{1}{c_m} \int_0^1 \bar\rho' \varphi_m \chi_M \chi_N - \bar\rho \varphi_m \chi_N' \chi_M \mathrm{d}y \\
&+ \kappa \varepsilon^2 A_\varepsilon \frac{1}{c_m^2} \int_0^1 \bar\rho \varphi_m' \chi_M \chi_N \mathrm{d}y
\end{aligned}
$$

In a preparatory step we give formal approximations of the operators R_1, R_2, R_3 and S_1, S_2, S_3, S_4 which are obtained by plugging in the expressions from above into the definitions of the operators according to Theorem I.

$$
\begin{aligned}
R_1 &= \frac{\kappa}{c_m} \left(1 - \frac{\varepsilon^2}{c_m} \right) + \varepsilon^2 \kappa \frac{1}{c_m^2} A_\varepsilon \varphi_m' + \varepsilon^2 (-\frac{1}{c_m}) A_\varepsilon' \varphi_m \partial_y + O(\varepsilon^4) \\
R_2 &= \varepsilon^2 (-\frac{1}{c_m^2}) A_\varepsilon' \bar\rho' \varphi_m \partial_y + O(\varepsilon^4) \\
R_3 &= \left(-\frac{1}{c_m} + \frac{\varepsilon^2}{c_m^2} \right) \bar\rho' + \varepsilon^2 \frac{1}{c_m^2} A_\varepsilon \bar\rho'' \varphi_m + O(\varepsilon^4)
\end{aligned}
$$

$$
\begin{aligned}
S_1^0 &= -\frac{\kappa \lambda_m}{\bar\rho} \left(1 - \frac{2\varepsilon^2}{c_m} \right) - \varepsilon^2 A_\varepsilon' \frac{\varphi_m''}{\bar\rho} - \varepsilon^2 \kappa \frac{\lambda_m}{c_m} A_\varepsilon \left(\frac{\varphi_m'}{\bar\rho} + \frac{(\bar\rho \varphi_m)'}{\bar\rho^2} \right) + O(\varepsilon^4) \\
S_1^1 &= \varepsilon^2 A_\varepsilon' \left(\lambda_m \frac{\varphi_m}{\bar\rho} - \frac{\varphi_m'}{\bar\rho} \right)
\end{aligned}
$$

$$
\begin{aligned}
S_2^1 &= \kappa \left(\frac{1}{c_m} - \frac{\varepsilon^2}{c_m^2} \right) \frac{\bar\rho'}{\bar\rho} + \varepsilon^2 A_\varepsilon' \frac{1}{c_m \bar\rho} \left(2\lambda_m \bar\rho' \varphi_m \right) \\
&\quad + \kappa \varepsilon^2 A_\varepsilon \frac{1}{c_m^2 \bar\rho} \left(\frac{\bar\rho'}{\bar\rho^2} (\bar\rho \varphi_m)' - (\bar\rho' \varphi_m)' \right) + O(\varepsilon^4) \\
S_2^2 &= \kappa \left(\frac{1}{c_m} - \frac{\varepsilon^2}{c_m^2} \right) - \varepsilon^2 A_\varepsilon' \frac{\bar\rho' \varphi_m}{c_m \bar\rho} + \kappa \varepsilon^2 \frac{1}{c_m^2} A_\varepsilon \varphi_m' + O(\varepsilon^4) \\
S_2^3 &= -\frac{1}{c_m} \varepsilon^2 A_\varepsilon' \varphi_m + O(\varepsilon^4)
\end{aligned}
$$

$$
\begin{aligned}
S_3^0 &= \lambda_m \frac{\bar\rho'}{\bar\rho}\left(1 - \frac{2\varepsilon^2}{c_m}\right) + \varepsilon^2 A_\varepsilon \frac{1}{c_m\bar\rho}\left(\lambda_m \bar\rho' \varphi_m' - \lambda_m \bar\rho'' \varphi_m + \lambda_m \frac{\bar\rho'^2}{\bar\rho}\varphi_m + \bar\rho'\varphi_m'' + \bar\rho''\varphi_m' - (\bar\rho\varphi_m')''\right) \\
&\quad + \kappa\varepsilon^2 A_\varepsilon' \frac{\bar\rho'\varphi_m}{c_m\bar\rho} + O(\varepsilon^4) \\
S_3^1 &= -\frac{\bar\rho'}{\bar\rho} + \varepsilon^2 A_\varepsilon \frac{1}{c_m\bar\rho}\left((\bar\rho'\varphi_m)' - \frac{\bar\rho'^2}{\bar\rho}\varphi_m\right) \\
S_3^2 &= -1 \\
S_4^0 &= \frac{\kappa}{c_m}\left(1 - \frac{\varepsilon^2}{c_m}\right) + \varepsilon^2 A_\varepsilon' \frac{\bar\rho'\varphi_m}{c_m\bar\rho} + \kappa\varepsilon^2 \frac{1}{c_m^2} A_\varepsilon \varphi_m' \\
S_4^1 &= -\frac{1}{c_m}\varepsilon^2 A_\varepsilon' \varphi_m
\end{aligned}
$$

From these approximations we derive the expressions mentioned above by calculating the scalar products explicitly; integration by parts then helps to simplify the terms.

Consequently, we can write down the leading orders of the matrices $\mathcal{A}_{MN}$ and $\mathcal{B}_{MN}$ explicitly. We define $\gamma_{MN} := \int_0^1 \bar\rho' \chi_M \chi_N \mathrm{d}y$. Then $\mathcal{A}_{MN}(\kappa, c)$ is, up to higher order errors, given by:

$$
\begin{pmatrix}
\frac{\kappa}{c_m}\left(1 - \frac{\varepsilon^2}{c_m}\right)\delta_{\mathrm{MN}} & 0 & -\frac{1}{c_m}\left(1 - \frac{\varepsilon^2}{c_m}\right)\delta_{\mathrm{MN}} & 0 \\
0 & 0 & \delta_{\mathrm{MN}} & 0 \\
0 & 0 & 0 & \delta_{\mathrm{MN}} \\
-\kappa\lambda_m\left(1 - \frac{2\varepsilon^2}{c_m}\right)\gamma_{\mathrm{MN}} & \frac{\kappa}{c_m}\left(1 - \frac{\varepsilon^2}{c_m}\right)(\lambda_m\gamma_{\mathrm{MN}} + \mu_N\delta_{\mathrm{MN}}) & \mu_N\delta_{\mathrm{MN}} - 2\varepsilon^2\frac{\lambda_m}{c_m}\gamma_{\mathrm{MN}} & \frac{\kappa}{c_m}\left(1 - \frac{\varepsilon^2}{c_m}\right)\delta_{\mathrm{MN}}
\end{pmatrix}.
$$

The matrix $\mathcal{B}_{\mathrm{MN}}(\xi; \kappa, c)$ is, up to higher order errors, given by:

$$
\mathcal{B}_{\mathrm{MN}} = \begin{pmatrix}
\varepsilon^2 A_\varepsilon'(\xi) D_{\mathrm{MN}}^{11} & \varepsilon^2 A_\varepsilon'(\xi) D_{\mathrm{MN}}^{12} & \varepsilon^2 A_\varepsilon(\xi) D_{\mathrm{MN}}^{13} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\varepsilon^2 A_\varepsilon'(\xi) D_{\mathrm{MN}}^{41} & \varepsilon^2 A_\varepsilon'(\xi) D_{\mathrm{MN}}^{42} & \varepsilon^2 A_\varepsilon(\xi) D_{\mathrm{MN}}^{43} & \varepsilon^2 A_\varepsilon'(\xi) D_{\mathrm{MN}}^{44}
\end{pmatrix}
$$

with the following constants, which are independent of ε and κ:

$$
\begin{aligned}
D_{\mathrm{MN}}^{11} &= -\frac{1}{c_m}\int_0^1 \bar\rho\varphi_m \chi_M(\chi_N' + \frac{\bar\rho''}{\bar\rho'}\chi_N)\mathrm{d}y, \\
D_{\mathrm{MN}}^{12} &= -\frac{1}{c_m^2}\int_0^1 \bar\rho\varphi_m \chi_M \chi_N' \mathrm{d}y, \\
D_{\mathrm{MN}}^{13} &= \frac{1}{c_m^2}\int_0^1 \frac{\bar\rho\bar\rho''}{\bar\rho'}\varphi_m \chi_M \chi_N \mathrm{d}y, \\
D_{\mathrm{MN}}^{41} &= \int_0^1 -\bar\rho'\varphi_m''\chi_N\chi_M + (\bar\rho'\chi_N)'\chi_M(\lambda_m\varphi_m - \varphi_m')\mathrm{d}y,
\end{aligned}
$$

$$D^{42}_{\mathrm{MN}} = \frac{1}{c_m}\int_0^1 \bar\rho\Big[\varphi_m\chi''_N\chi'_M - \varphi'_m\chi''_N\chi_M - 2\varphi'_m\chi'_N\chi'_M\Big]\mathrm{d}y,$$

$$D^{43}_{\mathrm{MN}} = \frac{1}{c_m}\int_0^1 \left((\bar\rho'\varphi'_m)' - 2\lambda_m\bar\rho''\varphi_m + \lambda_m\frac{\bar\rho'^2}{\bar\rho}\varphi_m\right)\chi_M\chi_N, + \left((\bar\rho'\varphi_m)' - \frac{\bar\rho'^2}{\bar\rho}\varphi_m\right)\chi_M\chi'_N\mathrm{d}y,$$

$$D^{44}_{\mathrm{MN}} = \frac{1}{c_m}\int_0^1 \bar\rho'\varphi_m\chi_M\chi_N - \bar\rho\varphi_m\chi'_N\chi_M\mathrm{d}y.$$

For the exponential density stratification, given by $\bar\rho(y) = \mathrm{e}^{-\delta y}$ with some $\delta > 0$, the identity

$$\bar\rho''(y) = -\delta\bar\rho'(y) = \delta^2\bar\rho(y)$$

serves to simplify the entries of $\mathcal{A}_{\mathrm{MN}}$ and $\mathcal{B}_{\mathrm{MN}}$. We obtain again $\mathcal{A}_{\mathrm{MN}} = 0$ for $M \neq N$ due to

$$\gamma_{\mathrm{MN}} = \int_0^1 \bar\rho'\chi_M\chi_N\mathrm{d}y = -\delta\int_0^1 \bar\rho\chi_M\chi_N\mathrm{d}y = -\delta\delta_{\mathrm{MN}} = 0 \quad \text{for } M \neq N.$$

Moreover, we find

$$\mathcal{A}_{00} = \begin{pmatrix} \frac{1}{c_0}\kappa + O(\kappa\varepsilon^2) & 0 & -\frac{1}{c_0} + O(\varepsilon^2) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_0\delta\kappa + O(\kappa\varepsilon^2) & -\frac{\lambda_0\delta}{c_0}\kappa + O(\kappa\varepsilon^2) & \frac{2\lambda_0\delta}{c_0}\varepsilon^2 + O(\varepsilon^4) & \frac{1}{c_0}\kappa + O(\kappa\varepsilon^2) \end{pmatrix},$$

and

$$\mathcal{A}_{\mathrm{NN}} = \begin{pmatrix} \frac{1}{c_0}\kappa + O(\kappa\varepsilon^2) & 0 & -\frac{1}{c_0} + O(\varepsilon^2) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_0\delta\kappa + O(\kappa\varepsilon^2) & -\frac{\lambda_n\delta}{c_0}\kappa + O(\kappa\varepsilon^2) & \delta(\lambda_n - \lambda_0) + O(\varepsilon^2) & \frac{1}{c_0}\kappa + O(\kappa\varepsilon^2) \end{pmatrix}$$

for $N \geq 1$. The concrete form of the constants D^{41}_{mm}, D^{42}_{mm} and D^{43}_{mm} will be important, so we write them down (this follows from a direct calculation):

$$D^{41}_{\mathrm{mm}} = \frac{\int_0^1 \bar\rho(\varphi'_m)^3\mathrm{d}y}{\int_0^1 \bar\rho\varphi_m^2\mathrm{d}y}, \tag{A.1a}$$

$$D^{42}_{\mathrm{mm}} = -\frac{2}{c_m}\frac{\int_0^1 \bar\rho(\varphi'_m)^3\mathrm{d}y}{\int_0^1 \bar\rho\varphi_m^2\mathrm{d}y}, \tag{A.1b}$$

$$D^{43}_{\mathrm{mm}} = -\frac{3}{c_m}\frac{\int_0^1 \bar{\rho}(\varphi_m')^3 \mathrm{d}y}{\int_0^1 \bar{\rho}\varphi_m^2 \mathrm{d}y}. \tag{A.1c}$$

A.2. Relation to the eigenvalue problem associated with KdV solitons

It is the central observation for exploring the stability of small-amplitude ISWs that the eigenvalue problem of KdV equation associated with a soliton, which enters the description of small solutions, appears in the truncated problems as well and becomes visible after a suitable rescaling.

In fact, by the scaling

$$\begin{aligned}
\Xi &= \varepsilon\xi,\\
w_{4m+1}(\xi) &= \tilde{W}_{4m+1}(\Xi),\\
w_{4m+2}(\xi) &= \tilde{W}_{4m+2}(\Xi),\\
w_{4m+3}(\xi) &= \varepsilon W_{4m+3}(\Xi),\\
w_{4m+4}(\xi) &= \varepsilon^2 W_{4m+4}(\Xi),\\
w_{4n+1}(\xi) &= \varepsilon W_{4n+1}(\Xi),\\
w_{4n+2}(\xi) &= \varepsilon W_{4n+2}(\Xi),\\
w_{4n+3}(\xi) &= \varepsilon^2 W_{4n+2}(\Xi),\\
w_{4n+4}(\xi) &= \varepsilon^2 W_{4n+2}(\Xi),
\end{aligned}$$

for all $n \neq m$, one obtains a singularly perturbed system of ordinary differential equations in which the KdV eigenvalue problem appears in the block

$$\mathcal{A}_m + \mathcal{B}_{mm} = \begin{pmatrix} 0 & 0 & -\frac{1}{c_m} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \tilde{\Gamma}_1 & \tilde{\Gamma}_2 & \Gamma_3 & 0 \end{pmatrix}$$

with the entries:

$$\tilde{\Gamma}_1(\Xi) = \varepsilon^{-3}\mathcal{A}^{41}_m + \dot{A}_*(\Xi)D^{41}_{\mathrm{mm}} = \Lambda\frac{\int_0^1 \bar{\rho}\varphi_m'^2 \mathrm{d}y}{\int_0^1 \bar{\rho}\varphi_m^2 \mathrm{d}y} + \dot{A}_*(\Xi)\frac{\int_0^1 \bar{\rho}\varphi_m'^3 \mathrm{d}y}{\int_0^1 \bar{\rho}\varphi_m^2 \mathrm{d}y}$$

$$\tilde{\Gamma}_2(\Xi) = \varepsilon^{-3}\mathcal{A}_m^{42} + \dot{A}_*(\Xi)D_{\mathrm{mm}}^{42} = -\frac{\Lambda}{c_m}\frac{\int_0^1 \bar{\rho}\varphi_m'^2 \mathrm{d}y}{\int_0^1 \bar{\rho}\varphi_m^2 \mathrm{d}y} - \frac{2}{c_m}\dot{A}_*(\Xi)\frac{\int_0^1 \bar{\rho}\varphi_m'^3 \mathrm{d}y}{\int_0^1 \bar{\rho}\varphi_m^2 \mathrm{d}y}$$

$$\Gamma_3(\Xi) = \varepsilon^{-2}\mathcal{A}_m^{43} + A_*(\Xi)D_{\mathrm{mm}}^{43} = \frac{2}{c_m}\frac{\int_0^1 \bar{\rho}\varphi_m'^2 \mathrm{d}y}{\int_0^1 \bar{\rho}\varphi_m^2 \mathrm{d}y} - \frac{3}{c_m}A_*(\Xi)\frac{\int_0^1 \bar{\rho}\varphi_m'^3 \mathrm{d}y}{\int_0^1 \bar{\rho}\varphi_m^2 \mathrm{d}y}$$

Here, we have used Remark 2.6 stating that for $\lambda = \lambda_m$ we have $\nu_m(\lambda_m) = 0$ and $\chi_m = C\varphi_m$. This is the reason why we can work with φ_m instead of χ_m regardless of the stratification.

After the change of variables

$$\tilde{W}_{4m+1} = W_{4m+1} + \frac{1}{c_m}W_{4m+2}, \quad \tilde{W}_{4m+2} = -\frac{c_m}{2}W_{4m+1} + \frac{1}{2}W_{4m+2}$$

the block has the form

$$\mathcal{A}_m + \mathcal{B}_{mm} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Gamma_1 & \Gamma_2 & \Gamma_3 & 0 \end{pmatrix}$$

with

$$\Gamma_1(\Xi) = \frac{1}{2}\tilde{\Gamma}_1 + \frac{c_m}{2}\tilde{\Gamma}_2 = -\frac{1}{2}\dot{A}_*(\Xi)\int_0^1 \bar{\rho}\varphi_m'^3 \mathrm{d}y = -\frac{c_m r}{3s}\dot{A}_*$$

$$\Gamma_2(\Xi) = -\frac{1}{c_m}\tilde{\Gamma}_1 + \tilde{\Gamma}_2 = \frac{\Lambda}{s} - \frac{2r}{s}\dot{A}_*(\Xi)$$

$$\Gamma_3(\Xi) = -\frac{1}{s} - \frac{2r}{s}A_*(\Xi)$$

such that the subsystem for the variables $(W_{4m+2}, W_{4m+3}, W_{4m+4})$ is precisely the KdV eigenvalue problem as in (B.4). It is noteworthy that this derivation holds true for an arbitrary stratification for which small waves can be described by KdV, i.e. for all $\bar{\rho}$ satisfying

$$\int_0^1 \bar{\rho}(\varphi_0)^3 \mathrm{d}y \neq 0.$$

We show in Chapter 4 how this fact implies the absence of unstable eigenvalues for the truncated system (EVP_N) associated with a small mode-0 wave (i.e. $m = 0$) in the case of an exponential stratification.

B. Collection of background material

B.1. Newton polygon method

This method can be used to determine expressions for roots of polynomials containing a small parameter. In the presentation we follow closely [16, Section 2.8] where proofs and background material can be found; see also [59] for a profound account on the Newton polygon and its applications.

Suppose that we are interested in the roots of

$$f(w,\delta) := w^k + a_{k-1}(\delta)w^{k-1} + \cdots + a_0(\delta) = 0 \tag{B.1}$$

where $\delta \in \mathbb{C}$ is a small parameter. We assume that each coefficient function $a_j(\delta)$ for $j = 0, \cdots, k-1$ is analytic in a neighbourhood of $\delta = 0$ and possesses expansions

$$a_j(\delta) = a_j^{(p_j)}\delta^{p_j} + a_j^{(p_j+1)}\delta^{p_j+1} + \cdots .$$

Additionally, assume that there is a smallest number p_0 such that $a_0^{(p_0)} \neq 0$.

If w_* is a simple root of $f(w,0) = 0$ then the implicit function theorem ensures the existence of a smooth curve $w(\delta)$ of solutions to $f(w,\delta) = 0$ for w close w_* and δ close to zero, and consequently $w(\delta) = w_* + O(\delta)$. In general, the roots of $f(w,0)$ are not simple, and this argument breaks down; for example in the case that all $p_j > 0$ we have $f(w,0) = w^k$, thus there is only the root $w = 0$, which is in fact k-fold, and it is not clear at all how the k solutons of $f(w,\delta)$ depend upon δ.

The principal idea underlying the Newton polygon method is to find a scaling $w(\delta) = \delta^\alpha v(\delta)$ in such a way that $v(\delta)$ satisfies an equation which has simple roots and thus is amenable to the implicit function argument again. By inserting this scaling in (B.1) one has to determine α such that at least two summands have equal power in δ and all other summands have larger powers in δ. In general, there is more than one suitable exponent α, and the Newton polygon provides a systematic approach to find all of them.

The Newton polygon is constructed as follows. For all $i = 0, \cdots, k-1$ consider the points $P_i = (i, p_i)$ (omitting those i where $a_i \equiv 0$) and set $P_k := (k, 0)$. The convex polygonal line L with the property that each point P_i lies on or above L is called the

Newton polygon. L is the union of a finite number, say r, of line segments L_j with slopes $-\alpha_j$. For the line segment L_j, which connects the points P_{i_j} and $P_{i_{j-1}}$, the distance of the abscissas is $n_j := i_j - i_{j-1}$. Consequently, the slope of L_j is given by $-\alpha_j = \frac{p_{i_j} - p_{i_{j-1}}}{n_j}$.

For each $j = 1, \cdots, r$ there are n_j solutions of the form

$$w_{j,l}(\delta) = \delta^{\alpha_j} v_{j,l}(\delta) \quad \text{for } l = 1, 2, \cdots, n_j,$$

and this yields all solutions of $f(w, \delta) = 0$ since $n_1 + \cdots + n_r = k$. By inserting the ansatz $w = \delta^{\alpha_j} v$ into Equation (B.1) and dividing by the greatest common power of δ, one obtains an equation of the form $\tilde{f}(v, \delta) = 0$ for $v_{j,l}(\delta)$. This equation is analytic with respect to v and to some, in general, fractional power of δ.

To apply this method in practice one proceeds according to the recipe:

(1) Determine the numbers p_i, i.e. the highest power of δ in $a_i(\delta)$.

(2) Plot the points $P_i = (i, p_i)$ and draw the Newton polygon L of these points; $L = L_1 \cup \cdots \cup L_r$ is the union of r line segments. Find their slopes $-\alpha_j$.

(3) For each j there are n_j solutions

$$w_{j,l}(\delta) = \delta^{\alpha_j} v_{j,l}(\delta) \quad \text{with } l = 1, 2, \cdots, n_j$$

where $v_{j,l}(\delta)$ solves a rescaled equation.

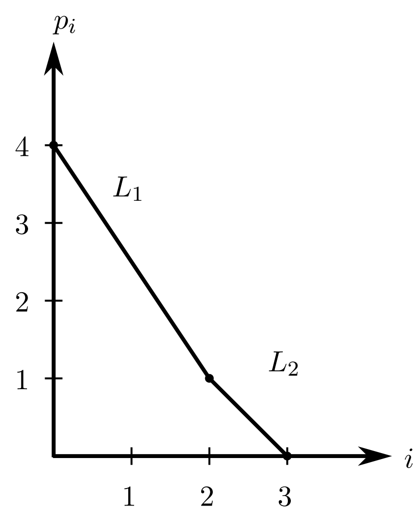

Figure B.1.: Newton polygon for the example $f(w, \delta) = w^3 - \delta w^2 + 4\delta^4$

We consider the example $f(w,\delta) = w^3 - \delta w^2 + 4\delta^4$ to illustrate the method explained above.

(1) We find $p_0 = 4$, p_1 is omitted, $p_2 = 1$, $p_3 = 0$. Hence, we have to consider the points $P_0 = (0,4)$, $P_2 = (2,1)$, $P_3 = (3,0)$.

(2) The points P_i and the Newton polygon L are plotted in Fig. B.1. We see that $r = 2$ and the slopes are $\alpha_1 = \frac{3}{2}$ and $\alpha_2 = 1$.

(3) For $j = 1$ there are $n_1 = 2$ solutions of the form

$$w_{1,l}(\delta) = \delta^{3/2} v_{1,l}(\delta) \quad \text{with } l = 1,2.$$

Rescaling and dividing by δ^4 leads to the equation

$$\delta^{1/2} v^3 - v^2 + 4 = 0,$$

which is analytic in $\delta^{1/2}$ and v. For $\delta = 0$ we find the two simple roots

$$v_{1,1}(0) = 2 \quad \text{and} \quad v_{1,2}(0) = -2,$$

this implies

$$v_{1,1}(\delta) = 2 + O(\delta^{1/2}) \quad \text{and} \quad v_{1,2}(\delta) = -2 + O(\delta^{1/2})$$

for small $\delta \neq 0$.

For $j = 2$ there is precisely $n_1 = 1$ solution of the form

$$w_2(\delta) = \delta^1 v_1(\delta).$$

Rescaling and dividing by δ^3 lead to the equation

$$v^3 - v^2 + 4\delta v = 0,$$

which is analytic in δ and v. For $\delta = 0$ we find the simple roots

$$v_2(0) = 1,$$

this implies

$$v_2(\delta) = 1 + O(\delta)$$

for small $\delta \neq 0$.

To sum up, we have found expressions for the three solutions of $f(w, \delta) = 0$ for small δ

$$\begin{aligned} w_{1,1}(\delta) &= \delta^{3/2} \left(2 + O(\delta^{1/2})\right), \\ w_{1,2}(\delta) &= \delta^{3/2} \left(-2 + O(\delta^{1/2})\right), \\ w_2(\delta) &= \delta \left(1 + O(\delta)\right), \end{aligned}$$

which coalesce in $w = 0$ for $\delta = 0$.

B.2. Spectral stability of KdV solitons following Pego and Weinstein

In their paper [51] Pego and Weinstein showed for several equations how the stability of solitons, which had been shown before by means of the moment-of-instability, is linked to properties of the Evans function. To accomplish this, they showed that the definition of the Evans function can be simplified for a quite general class of systems and they found a feasible expression for the second derivative. In Section 4.2 we show that the reduced problem obtained in the scaling regime $\kappa = O(\varepsilon^3)$ also belongs to this class.

For presenting the definition of the Evans function and the objects needed for this purpose, let us consider an abstract non-autonomous system of ODEs, given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathcal{A}(x, \Lambda) y, \tag{B.2}$$

and its adjoint problem

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -z\mathcal{A}(x, \Lambda), \tag{B.3}$$

where the coefficient matrix depends on a complex parameter $\Lambda \in \Omega$ with a simply connected domain $\Omega \subset \mathbb{C}$.

We assume that the following hypotheses are met:

(H1) The mapping $\mathcal{A} : \mathbb{R} \times \Omega \to \mathbb{C}^{m \times m}, (x, \Lambda) \mapsto \mathcal{A}(x, \Lambda)$ is continuous, and analytic in Λ for each fixed $x \in \mathbb{R}$.

(H2) For each $\Lambda \in \Omega$ the limits $\lim_{x \to \pm\infty} \mathcal{A}(x, \Lambda) = \mathcal{A}^{\pm}(\Lambda)$ exist, and are attained uniformly on compact subsets of Ω.

(H3) For $\Lambda \in \Omega$ the matrix $\mathcal{A}^{\pm}(\Lambda)$ has a unique eigenvalue $\mu^{\pm}(\Lambda)$ of smallest real part, which is simple. By $Y^{\pm}(\Lambda) \in \mathbb{C}^{1\times n}$ and $Z^{\pm}(\Lambda) \in \mathbb{C}^{n\times 1}$ we denote corresponding left, resp. right, eigenvectors normalized in such a way that $Y^{\pm}(\Lambda) \cdot Z^{\pm}(\Lambda) = 1$

(H4) Define

$$R(x,\Lambda) = \begin{cases} \mathcal{A}(x,\Lambda) - \mathcal{A}^{+\infty}, & \text{for } x > 0, \\ \mathcal{A}(x,\Lambda) - \mathcal{A}^{-\infty}, & \text{for } x < 0 \end{cases} .$$

Assume that

$$\int_{-\infty}^{+\infty} \|R(x,\Lambda)\| \,\mathrm{d}x$$

converges for all $\Lambda \in \Omega$, uniformly on compact subsets.

The hypothesis (H3) indicates that this method especially applies to situations where $\mathcal{A}^{\pm}(\Lambda)$ has precisely one eigenvalue $\mu^{\pm}(\Lambda)$ in the left half-plane and $m-1$ ones in the right half-plane for any $\Lambda \in \mathbb{C}$ with $\operatorname{Re}\Lambda > 0$ and where $\mu^{\pm}(\Lambda)$ continues being a simple eigenvalue inside a small region in the left half-plane, i.e. $\operatorname{Re}\Lambda \leq 0$. This situations actually occurs for the eigenvalue problem of KdV equation about a soliton (see below).

Since, by (H3), the eigenvalue of smallest real part is simple, the eigenspace of $\mathcal{A}^{+}(\Lambda)$ associated with $\mu^{+}(\Lambda)$ is one-dimensional as well as the eigenspace of $\mathcal{A}^{-}(\Lambda)$ associated with $\mu^{-}(\Lambda)$, which can be viewed as dual to the (possibly generalized) eigenspace associated with the $m-1$ remaining eigenvalues. That is why it is possible that the stable space and the dual of the unstable space are each spanned by one distinguished function. This is the content of the next lemma.

Lemma B.1. *[51, Proposition 1.2] There exist unique solutions $\zeta^{+}(x,\Lambda)$ of* (B.2) *and $\eta^{-}(x,\Lambda)$ of* (B.3) *which satisfy*

$$\begin{aligned} \mathrm{e}^{-\mu^{+}(\Lambda)x}\zeta^{+}(x,\Lambda) &\to Z^{+}(\Lambda) \quad \textit{as } x \to +\infty, \\ \mathrm{e}^{\mu^{-}(\Lambda)x}\eta^{-}(x,\Lambda) &\to Y^{-}(\Lambda) \quad \textit{as } x \to -\infty. \end{aligned}$$

$\zeta^{+}(x,\Lambda)$ and $\eta^{-}(x,\Lambda)$ are analytic with respect to Λ.
Any solution y of (B.2) *with $y(x) = O(\mathrm{e}^{\mu^{+}x})$ as $x \to \infty$ is a constant multiple of ζ^{+}; any solution z of* (B.3) *with $z(x) = O(\mathrm{e}^{-\mu^{-}x})$ as $x \to -\infty$ is a constant multiple of η^{-}.*

With these two solutions at hand, the Evans function can be defined on the domain Ω as the scalar product

$$E(\Lambda) := \eta^{-}(x,\Lambda) \cdot \zeta^{+}(x,\Lambda),$$

which does not depend on x (see [51, Lemma 1.3]), and it has been shown that this definition leads to the desired properties of E collected in the next proposition.

Proposition B.2. (i) *$E(\Lambda)$ is analytic for $\Lambda \in \Omega$.*

(ii) *Equation* (B.2) *has a bounded solution for some Λ iff $E(\Lambda) = 0$.*

This Evans function has notably been constructed for soliton solutions of the KdV equation. Adapted to our context, recall the soliton solution $A_*(\Xi)$ (see Section 2.3) of the equation

$$u_t = u_\Xi + r(u^2)_\Xi + s u_{\Xi\Xi\Xi}$$

with some constants $r, s \in \mathbb{R}$, $s < 0$. $A_*(\Xi)$ is the unique symmetric homoclinic solution of

$$A_{*,\Xi\Xi} = -\frac{1}{s}A_* - \frac{r}{s}(A_*)^2.$$

The associated eigenvalue problem is given by

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}_\Xi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\Lambda}{s} - \frac{2r}{s}\dot{A}_*(\Xi) & -\frac{1}{s} - \frac{2r}{s}A_*(\Xi) & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}. \tag{B.4}$$

For (a scaled variant of) this system Pego and Weinstein check the hypotheses (H1-4); notably, (H3) follows from the next lemma which we cite for reference purposes.

Lemma B.3. *There exists some $\nu > 0$ such that, for all $\Lambda \in \mathbb{C}$ with $\operatorname{Re}\Lambda \geq -\nu$, the characteristic polynomial,*

$$\chi_{\mathrm{KdV}}(\mu; \Lambda) = \mu^3 + \frac{1}{s}\mu - \frac{\Lambda}{s},$$

of the asymptotic matrix associated with (B.4) *has one unique simple root of smallest real part, denoted by $\mu_{\mathrm{KdV}}(\Lambda)$.*

Consequently, the theory due to Pego and Weinstein implies that an Evans function is available in the present context.

Lemma B.4. *There exists some $\nu > 0$ such that, for $\Omega_{\mathrm{KdV}} = \{\Lambda \in \mathbb{C} : \operatorname{Re}\Lambda \geq -\nu\}$, there is an analytic function*

$$E_{\mathrm{KdV}} : \Omega_{\mathrm{KdV}} \to \mathbb{C}$$

with the properties:

(i) *$E_{\mathrm{KdV}}(0) = E'_{\mathrm{KdV}}(0) = 0$, $E''_{\mathrm{KdV}}(0) \neq 0$, and*

(ii) *$E_{\mathrm{KdV}}(\Lambda) \neq 0$ for all $\Lambda \neq 0$ with $\operatorname{Re}\Lambda \geq 0$.*

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