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DEMUSHKIN'S THEOREM IN CODIMENSION ONE

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Abstract. Demushkin's Theorem says that any two toric structures on an affine variety X are conjugate in the automorphism group of X. We provide the following extension: Let an $(n-1)$-dimensional torus T act effectively on an n-dimensional affine toric variety X. Then T is conjugate in the automorphism group of X to a subtorus of the big torus of X.

Introduction

This paper deals with automorphism groups of toric varieties X over an algebraically closed field $\mathbb{K}$ of characteristic zero. We consider the following problem: Let $T \times X \to X$ be an effective regular torus action. When is this action conjugate in $\mathrm{Aut}(X)$ to the action of a subtorus of the big torus $T_X \subset X$? Some classical results are:

- For complete smooth X, the answer is always positive, because in this case $\mathrm{Aut}(X)$ is an affine algebraic group, see [5].
- For $X = \mathbb{K}^m$ and $\dim(T) \geq m-1$, positive answer is due to A. Białynicki-Birula, see [2] and [3].
- For X affine and $\dim(T) = \dim(X)$, positive answer is due to Demushkin [6] and Gubeladze [8].

We focus here on the case $\dim(T) = \dim(X) - 1$. As in [6] and [8], we shall assume that X has no torus factors. We do not insist on X being affine; we just require that X has no "small holes", i.e. there is no open toric embedding $X \to X'$ with $X' \setminus X$ nonempty of codimension at least two. Under these assumptions we prove, see Theorem 3.1:

Theorem. *Let $T \times X \to X$ be an effective regular action of an algebraic torus T of dimension $\dim(X) - 1$. Then T is conjugate in $\mathrm{Aut}(X)$ to a subtorus of the big torus $T_X \subset X$.*

In contrast to [6] and [8], our approach is of geometric nature. Let us outline the main ideas of the proof. According to [1], any two toric structures on X are conjugate in the automorphism group of X. Hence it suffices to extend the T-action to an almost homogeneous torus action on X. This is done in three steps:

First lift the T-action (up to a finite homomorphism $T \to T$) to Cox's quotient presentation $\mathbb{K}^m \dashrightarrow X$, see Section 1. Next extend the lifted T-action to a toric structure on $\mathbb{K}^m$. This involves linearization of a certain diagonalizable group action, see Section 2. Finally, push down the new toric structure of $\mathbb{K}^m$ to X. For this we need that X has no small holes, see Section 3.

1. Lifting torus actions

We provide here a lifting result for torus actions on a toric variety X to the quotient presentation of X introduced by Cox [4]. First we recall the latter construction. For notation and the basic facts on toric varieties, we refer to Fulton's book [7].

We shall assume that the toric variety X is *nondegenerate*, that is X admits no toric decomposition $X \cong Y \times \mathbb{K}^*$. Note that this is equivalent to requiring that every invertible $f \in \mathcal{O}(X)$ is constant.

Let X arise from a fan Δ in a lattice N. Denote the rays of Δ by $\varrho_1, \ldots, \varrho_m$. Let $Q\colon \mathbb{Z}^m \to N$ be the map sending the canonical base vector e_i to the primitive generator of ϱ_i. For a maximal cone $\tau \in \Delta$, set

$$\sigma(\tau) := \operatorname{cone}(e_i;\ \varrho_i \subset \tau).$$

Then these cones $\sigma(\tau)$ are the maximal cones of a fan Σ consisting of faces of the positive orthant in $\mathbb{Q}^m$. Moreover, $Q\colon \mathbb{Z}^m \to N$ is a map of the fans Σ and Δ. The following properties of this construction are well known:

Proposition 1.1. *Let $Z \subset \mathbb{K}^m$ be the toric variety corresponding to Σ, and let $q\colon Z \to X$ denote the toric morphism corresponding to $Q\colon \mathbb{Z}^m \to N$.*

(i) *The complement $\mathbb{K}^m \setminus Z$ is of dimension at most $m-2$.*
(ii) *The map $q\colon Z \to X$ is a good quotient for the action of $H := \ker(q)$ on Z.*
(iii) *The group H acts freely over the set of smooth points of X.*

In general, the diagonalizable group $H = \ker(q)$ may be disconnected. Hence we can at most expect liftings of a given action $T \times X \to X$ in the sense that $q\colon Z \to X$ becomes T-equivariant up to a finite epimorphism $T \to T$. But such liftings exist:

Proposition 1.2. *Notation as in 1.1. Let $T \times X \to X$ be an effective algebraic torus action. Then there exist an effective regular action $T \times Z \to Z$ and a finite epimorphism $\kappa\colon T \to T$ such that*

(i) *$t\cdot(h\cdot z) = h\cdot(t\cdot z)$ holds for all $(t,h,z) \in T \times H \times Z$,*
(ii) *$q(t\cdot z) = \kappa(t)\cdot q(z)$ holds for all $(t,z) \in T \times X$.*

Proof. First we reduce to the case that X is smooth. Suppose for the moment that the assertion is proven in the smooth case. Then we can lift the T-action over the set $U \subset X$ of smooth points. The task then is to extend the lifted action from $U' := q^{-1}(U)$ to Z.

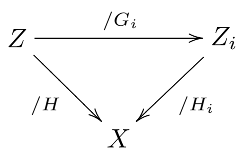
By Sumihiro's Theorem [10, Cor. 2], X is covered by T-invariant affine open subsets $V \subset X$. The inverse images $V' := q^{-1}(V)$ are affine and $V' \setminus U'$ is of codimension at least 2 in V'. This allows to extend the lifted T-action from $V' \cap U'$ to V' and hence from U' to Z.

Therefore we may assume in the remainder of this proof that X is smooth. This means that the group $H = \ker(q)$ acts freely on Z.

The most convenient way to construct the lifting is to split the procedure into simple steps. For this, let Γ denote the character group of H. Decompose Γ into a free part Γ_0 and cyclic torsion parts $\Gamma_1, \ldots, \Gamma_r$:

$$\Gamma = \Gamma_0 \oplus \Gamma_1 \oplus \ldots \oplus \Gamma_r.$$

Consider the factor groups $H_i := \mathrm{Spec}(\Gamma_0 \oplus \ldots \oplus \Gamma_i)$, and write $H_i = H/G_i$. Each of these groups defines a decomposition of the quotient presentation of X:



This observation enables us to lift in several steps. In the first one we lift with respect to the connected group H_0, and in the remaining ones we lift with respect to finite cyclic groups H_i. We shall write again Z and H instead of Z_i and H_i.

The action of H on Z defines a grading of the $\mathcal{O}_X$-algebra $\mathcal{A} := q_*(\mathcal{O}_Z)$. Namely, denoting by Γ the character group of H, we have for every open $V \subset X$ the decomposition into homogeneous functions:

$$\mathcal{O}(q^{-1}(V)) = \mathcal{A}(V) = \bigoplus_{\chi \in \Gamma} \mathcal{A}_\chi(V).$$

Since H acts freely on Z, all homogeneous components $\mathcal{A}_\chi$ are locally free $\mathcal{O}_X$-modules of rank one. We shall use this fact to make the $\mathcal{O}_X$-algebra $\mathcal{A}$ into a T-sheaf over the T-variety X. Then it is canonical to extract the desired lifting from this T-sheaf structure.

If the group H is connected, then we can prescribe T-linearizations on the $\mathcal{O}_X$-modules $\mathcal{A}_i$ corresponding the members χ_i of some lattice basis of Γ. Tensoring these linearizations gives the desired T-sheaf structure on the $\mathcal{O}_X$-algebra $\mathcal{A}$, compare also [9, Section 3].

Since X is covered by T-invariant affine open subsets, we can easily check that this T-sheaf structure of $\mathcal{A}$ arises from a regular T-action on Z that commutes with the action of H and makes the quotient map $q\colon Z \to X$ even equivariant. This settles the case of a connected H.

Assume that H is finite cyclic of order d. Let χ be a generator of Γ. Again, we choose a T-linearization of $\mathcal{A}_\chi$. But now it may happen that the induced T-linearization on $\mathcal{A}_{d\chi} = \mathcal{O}_X$ is not the canonical one. However, since $\mathcal{O}^*(X) = \mathbb{K}^*$ holds, these two linearizations only differ by a character ξ of T.

Let $\kappa\colon T \to T$ be an epimorphism such that $\xi \circ \kappa = \xi_0^d$ holds for some character ξ_0 of T. Consider the action $t{*}x := \kappa(t){\cdot}x$ on X. Then $\mathcal{A}_\chi$ is also linearized with respect to this action by setting $t{*}f := \kappa(t){\cdot}f$. Twisting with ξ_0^{-1}, we achieve that the induced linearization on $\mathcal{A}_{d\chi} = \mathcal{O}_X$ is the canonical one:

$$(t{*}f)(x) = \xi_0^{-d}(t)\xi(\kappa(t))f(t^{-1}{*}x) = f(t^{-1}{*}x).$$

The rest is similar to the preceding step: The T-sheaf structure of $\mathcal{A}$ defines a T-action $(t,z) \mapsto t{*}z$ on Z commuting with the action of H and making the quotient map $q\colon Z \to X$ equivariant with respect to $(t,x) \mapsto t{*}x$. Dividing by the kernel of ineffectivity, we can make the action on Z effective and obtain the desired lifting. □

2. Diagonalizable group actions

In this section we show that any effective regular action of an $(m-1)$-dimensional diagonalizable group G on $\mathbb{K}^m$ can be brought into diagonal form by means of an algebraic coordinate change.

Proposition 2.1. *Let $G \times \mathbb{K}^m \to \mathbb{K}^m$ be an effective algebraic action of an $(m-1)$-dimensional diagonalizable group G. Then there exist $\alpha \in \mathrm{Aut}(\mathbb{K}^m)$ and characters $\chi_i \colon G \to \mathbb{K}^*$ such that for every $g \in G$ we have*

$$\alpha(g \cdot \alpha^{-1}(z_1, \ldots, z_m)) = (\chi_1(g)z_1, \ldots, \chi_m(g)z_m).$$

Proof. Write $G = G_0 \times G_1$ with an algebraic torus G_0 and a finite abelian group G_1. According to the main result of [3], we may assume that the action of G_0 is already diagonal. The remaining task thus is to study the finite part G_1.

We consider the quotient map $p \colon \mathbb{K}^m \to \mathbb{K}^m /\!\!/ G_0$. Note that this is a toric morphism, and that $\mathbb{K}^m /\!\!/ G_0$ is either isomorphic to $\mathbb{K}$ or it is a point. We have to distinguish three cases:

Case 1. $\mathbb{K}^m /\!\!/ G_0$ is a point. Then, after a suitable permutation of coordinates, there are relatively prime integers $a, b > 0$ and a one parameter subgroup $T_0 \subset G_0$ of the form

$$t \cdot (z_1, \ldots, z_m) = (t^a z_1, \ldots, t^a z_k, t^b z_{k+1}, \ldots, t^b z_m).$$

To see this, consider the (primitive) lattice $L \subset \mathbb{Z}^m$ of the one parameter subgroups of $G_0 \subset (\mathbb{K}^*)^m$. Take a basis of the lattice L which is in Hermite normal form. Then this basis consists of vectors

$$\begin{pmatrix} c_1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} c_{m-1} \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

One can achieve this form, because $\mathbb{K}^m /\!\!/ G_0$ is a point and hence L contains a vector with only positive entries. In particular, at least one of the c_j is positive. It is then straightforward to write down a vector of L having only entries $a, b > 0$ as desired.

Now consider $g \in G_1$ and the associated translation $\eta \colon z \mapsto g \cdot z$. Then any component η_i is a T_0-homogeneous function. Moreover, since $\eta \in \mathrm{Aut}(\mathbb{K}^m)$, every component η_i contains a nontrivial linear term. This linear part is T_0-homogeneous of degree either a or b.

Since the integers a and b are relatively prime, it follows that all monomials of the coordinate functions η_i are of degree one. In other words, G acts linearly. Consequently, the G-action can be diagonalized.

Case 2. $\mathbb{K}^m /\!\!/ G_0$ is of dimension one, and p maps all coordinate hyperplanes $V(z_1), \ldots, V(z_m)$ to the point $p(0)$. Since the induced action of G_1 on $\mathbb{K}^m /\!\!/ G_0$ fixes $p(0)$, the group G_1 permutes the coordinate hyperplanes $V(z_i)$. In particular, G_1 acts by linear automorphisms. Thus the action of G can be diagonalized.

Case 3. The quotient space $\mathbb{K}^m /\!\!/ G_0$ is of dimension one, and one coordinate hyperplane, say $V(z_m)$, is not mapped to $p(0)$. For $z \in \mathbb{K}^m$ write $z = x + y$ with $x \in V(z_m)$ and $y \in \mathbb{K}e_m$. We shall show that for every $g \in G_1$ we have

$$g \cdot z = g \cdot x + \zeta_g y,$$

where ζ_g is a root of the unit. Together with cases 1 and 2 this enables us to settle case 3 by induction on m. Note that the case $m = 1$ means linearizing the action of a finite abelian group on $\mathbb{K}$.

To verify the above equation, choose a one parameter subgroup $T_0 \subset G_0$ having $V(z_m)$ as limit point set. Then T_0 commutes with the action of G_1. Moreover, taking limits in T_0 we obtain for the action of $g \in G_1$ on a point $z = x + y$:

$$\lim_{t \to 0} t \cdot (g \cdot z) = g \cdot \lim_{t \to 0} t \cdot z = g \cdot x.$$

Thus $g \cdot z$ decomposes into $g \cdot x$ and some $h(g, x, y)e_m$. Fixing $y \neq 0$, we see that h does in fact not depend on x. It follows that the second component is of the form $\zeta_g y$. This proves the desired decomposition of $g \cdot z$. □

3. Proof of the main result

We say that a toric variety X *has no small holes*, if it does not admit an open toric embedding $X \subset X'$ such that $X' \setminus X$ is nonempty of codimension at least 2 in X'. Examples are the toric varieties arising from a fan with convex support. This comprises in particular the affine ones.

Theorem 3.1. *Let X be a nondegenerate toric variety without small holes, and let $T \times X \to X$ be an effective regular action of an algebraic torus T of dimension* $\dim(X) - 1$. *Then T is conjugate in* $\mathrm{Aut}(X)$ *to a subtorus of the big torus $T_X \subset X$.*

Proof. According to [1, Theorem 4.1], any two toric structures of X are conjugate in $\mathrm{Aut}(X)$. Consequently, it suffices to show that the action of T on X extends to an effective regular action of a torus of dimension $\dim(X)$ on X.

Consider Cox's construction $q\colon Z \to X$ and its kernel $H := \ker(q)$ as defined in 1.1. Choose a lifting of the T-action to Z as provided by Proposition 1.2. This gives us an action of the $(m-1)$-dimensional diagonalizable group $G := T \times H$ on the open set $Z \subset \mathbb{K}^m$.

Since the complement $\mathbb{K}^m \setminus Z$ is of dimension at most $m - 2$, the action of G extends regularly to $\mathbb{K}^m$. Let G_0 be the (finite) kernel of ineffectivity. Applying Proposition 2.1 to the action of G/G_0, we can extend the G-action to an almost homogeneous action of a torus S on $\mathbb{K}^m$.

We claim that Z is invariant with respect to the action of S. According to [11, Corollary 2.3], the set Z is S-invariant if it is H-maximal in the following sense: If $Z' \subset \mathbb{K}^m$ is an H-invariant open subset admitting a good quotient $q'\colon Z' \to X'$ by the action of H such that Z is a q'-saturated open subset of Z', then we already have $Z' = Z$.

To verify H-maximality of Z, consider $Z' \subset \mathbb{K}^m$ and $q'\colon Z' \to X'$ as above. We may assume that Z' is H-maximal. Applying [11, Corollary 2.3] to the action of the standard torus $(\mathbb{K}^*)^m$, we obtain that Z' is invariant with respect to the action of this torus. Hence we obtain a commutative diagram of toric morphisms:

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ {\scriptstyle /\!\!/ H} \big\downarrow {\scriptstyle q} & & {\scriptstyle q'} \big\downarrow {\scriptstyle /\!\!/ H} \\ X & \longrightarrow & X' \end{array}$$

By assumption, the horizontal arrows are open toric embeddings. Moreover, the complement $X' \setminus X$ is of codimension at least two in X', because its inverse image $Z' \setminus Z$ under q' is a subset of the small set $\mathbb{K}^m \setminus Z$. By the assumption on X, we obtain $X' = X$. This verifies H-maximality of Z. Hence our claim is proved.

The rest is easy: The torus S/H acts with a dense orbit on X. Dividing S/H by the kernel of ineffectivity of this action, we obtain the desired extension of the action of T on X. □

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