First publ. in: Forum mathematicum 8 (1996), pp. 723-735

Valuation Bases for Extensions of Valued Vector Spaces *

Salma Kuhlmann

1. 6. 1995

Abstract

Let (V, v) be any valued vector space, and (V_0, v) a subspace. Then (V, v) admits a valuation basis over (V_0, v) if and only if it admits a nice composition series over (V_0, v). We show that this is always the case if $v(V \setminus V_0)$ is reversely well ordered. If $v(V_0)$ is reversely well ordered, we show that V_0 is nice in any extension, and that it admits a valuation basis over every subspace. Finally, we show that the property of admitting a valuation basis is preserved under countable dimensional extensions.

1 Introduction

Valued vector spaces played a historical role in the development of infinite abelian group theory in the last three decades. They were intensively used by group theorists seeking to generalize Ulm's structure theorem for countable p-groups (cf. [G] for a survey). Indeed, the socle of a p-group, endowed with the height function, may be viewed as a valued $\mathbb{F}_p$-vector space, with ordinal values (cf. [KAP] for these notions from abelian group theory, and [F1] for a comprehensive study on valued vector spaces and applications to p-groups and mixed groups). In this approach, the interest is often focused on that special case, where the valuation is the height function, and the value set an ordinal number. Our interest for valued vector spaces in the wide sense comes from a different direction; indeed they emerge quite naturally in the context of ordered and valued fields. For instance, Brown's theorem stating that a valued vector space of countable dimension admits a valuation basis (cf. [B]) was used by the author to show a structure theorem for countable exponential fields (cf. [KS] and [K-K]). Also, in the context of valuation theory of fields, an intensive study of general valued modules is now given in [KF]. In the present paper, we concentrate on the notion of valuation independence, and on the question of existence of a valuation basis. Nice subspaces play naturally an important role.

*This paper represents some results from the author's doctoral thesis.

In Section 2, we recall some definitions and results that we shall need in the subsequent sections.

We begin Section 3 with the definition of a nice composition series for a valued vector space V over a subspace V_0, and show the existence of such to be equivalent to that of a valuation basis over V_0 (Theorem 14). Note that it has been shown (cf. [H-W], Theorem 3.8) that for a valuated p-group, the existence of a nice composition series is equivalent to that of a pseudo p-basis.

We then investigate some cases where conditions on the value set imply the existence of such composition series. Recall that valued vector spaces which have a finite value set enjoy two important properties: they admit a valuation basis over every subspace, and they are nice in every extension. We show (Corollary 16) that the same holds with the assumption that the value set is inversely well ordered (instead of finite). If for the extension $V_0 \subset V$, we have that $v(V \setminus V_0)$ is reversely well ordered, we can even show that V will admit a nice composition series over V_0 (Corollary 15).

Next, we define the quotient valuation on a quotient by a nice subspace (note that niceness is necessary since we work with arbitrary totally ordered value sets, not necessarily having suprema). This permits us to relativize the previous results of the section, and thus study subextensions $V_0 \subset V_1 \subset V$. For instance, we show (Corollary 20) that if the value set of V_1/V_0 is reversely well ordered, then V_1 inherits from V_0 the property of being nice, or that of being maximally valued.

We close section 3 with the definition of "defectless extensions" and show (Theorem 22) that the extension $V_0 \subset V$ is defectless if and only if V_0 is nice in V. This notion of "defectless" comes from valuation theory: given an extension of valued fields, we may consider the larger field as a general valued vector space (in the sense of [F3]) over the smaller one. The extension is said to be defectless if every finitely generated subspace admits a valuation basis. In general, Theorem 22 will not hold, but it does in our case where the valuation is trivial on the ground field.

The last section is devoted to proving that a countable dimensional extension of a space admitting a valuation basis, admits a valuation basis as well (Corollary 24). This generalizes a result of Fuchs ([F2], Lemma 7 and Theorem 8) where this was shown to hold if the value set is an ordinal number.

2 Preliminaries on valuation independence and niceness

In the sequel, let V be a vector space over a field K, and Γ a totally ordered set with last element ∞. A surjective map

$$v : V \longrightarrow \Gamma$$

is a *valuation on* V (and (V, v) is a *valued vector space*) if for all $x, y \in V$ and $k \in K$, the following holds:

(V1) $v(x) = \infty$ if and only if $x = 0$,

(V2) $v(kx) = v(x)$ if $k \neq 0$,

(V3) $v(x+y) \geq \min\{v(x), v(y)\}$.

Condition (V3) is also called the triangle inequality. The following is a consequence of the above axioms:

$$v(x) \neq v(y) \Longrightarrow v(x+y) = \min\{v(x), v(y)\}\ .$$

The restriction of v to a subspace V_0 of V is a valuation on V_0 (that we denote also by v). Now let (V_0, v_0) and (V, v) be valued vector spaces. Suppose that V_0 is a subspace of V and that $v_0(V_0) \subset v(V)$. We will say that (V, v) is an *extension* of (V_0, v_0), and write $(V_0, v_0) \subset (V, v)$ if $v(x) = v_0(x)$ for all $x \in V_0$ (that is, v_0 is just the restriction of v to V_0). We say that the extension $(V_0, v) \subset (V, v)$ is *immediate* if and only if for all nonzero $x \in V$ there exists $y \in V_0$ such that $v(x-y) > v(x)$. Note that if $(V_0, v) \subset (V, v)$ is a proper immediate extension, then $v(V \setminus V_0)$ contains an infinite increasing sequence. If (V_0, v_0) admits no proper immediate extensions, we shall say that V_0 is *maximally valued*.

We say that $V_0 \subset V$ is a *finite extension* (or V is *finitely generated* over V_0) if and only if V/V_0 is of finite K–dimension.

If $\{x_i\,;\ i \in I\} \subset V$, then

$$\langle \{x_i\,;\ i \in I\} \rangle$$

will denote the K–subspace of V generated by the elements x_i, that is the subspace consisting of all sums $\sum_{i \in I} k_i x_i$ where $k_i = 0$ except for a finite number of $i \in I$ (by convention, $\langle \emptyset \rangle = 0$). Let $\{V_i\,;\ i \in I\}$ be a family of subspaces of V such that $V = \bigoplus_{i \in I} V_i$ (i.e. V is the direct sum of the V_i's). We then say that V is the *coproduct* of $\{V_i\,;\ i \in I\}$ (or: $\{V_i\,;\ i \in I\}$ is an *orthogonal family of subspaces*) if for every sum $\sum_{i \in I} x_i \in V$ such that $x_i \in V_i$ for all $i \in I$ and $x_i = 0$ except for a finite number of $i \in I$, then

$$v\left(\sum_{i \in I} x_i\right) = \min_{i \in I} v(x_i)\ .$$

We denote the coproduct of $\{V_i\,;\ i \in I\}$ by $\perp_{i \in I} V_i$, and if V_i is orthogonal to V_j, we write $V_i \perp V_j$.

Now let V_0 be a subspace of V: The subset $\{x_i\,; i \in I\} \subset V$ is *independent over* V_0 if and only if $\{x_i + V_0\,; i \in I\}$ is K–linearly independent in V/V_0, and $\{x_i\,; i \in I\}$ is a *basis of* V *over* V_0 if and only if $\{x_i + V_0\,; i \in I\}$ is a K–basis of V/V_0 (if and only if $V = \bigoplus_{i \in I} K x_i \oplus V_0$).

Note that if $\{x_i\,;\ i \in I\}$ is independent over V_0, then in particular $\{x_i\,;\ i \in I\}$ is K– linearly independent.

We say that $\{x_i\,;\ i \in I\} \subset V$ is *valuation independent over* V_0 if and only if $0 \notin \{x_i\,;\ i \in I\}$, and for all $z_0 \in V_0$, $k_i \in K$ such that $k_i = 0$ except for a finite number of $i \in I$, we have

$$v\left(\sum_{i \in I} k_i x_i + z_0\right) = \min_{\{i \in I;\, k_i \neq 0\}} \{v(x_i), v(z_0)\}\ .$$

If moreover $\{x_i\,;\ i \in I\} \subset V$ is a K–basis of V over V_0 then $\{x_i\,;\ i \in I\}$ is a *valuation basis of* V *over* V_0 (if and only if $V = \perp_{i \in I} K x_i \perp V_0$).

Note that by definition, the element 0 never belongs to a valuation independent set. Also, by convention, $\min \emptyset = \infty$, and $\emptyset$ is valuation independent over V_0. The following lemma is easy to show:

Lemma 1 *Let $\mathcal{B}$, $\mathcal{B}'$ be two subsets of V. Then $\mathcal{B} \cup \mathcal{B}'$ is valuation independent over V_0 if and only if $\mathcal{B}$ is valuation independent over V_0 and $\mathcal{B}'$ is valuation independent over $\langle \mathcal{B}, V_0 \rangle$. Moreover, the increasing union of a chain of valuation independent sets is again valuation independent.*

Proposition 2 *The subset $\{x_i\,;\, i \in I\} \subset V \setminus \{0\}$ is valuation independent over V_0 if and only if for all $y \in \langle \{x_i\,;\, i \in I\} \rangle$, if $y = \sum_{i \in I} k_i x_i$ then*

$$\max_{z \in V_0} v(y - z) = \min_{\{i \in I\,;\, k_i \neq 0\}} v(x_i) \ .$$

Proof: Clearly, $\{x_i\,;\, i \in I\} \subset V$ is valuation independent over V_0 if and only if for every such sum $y \in \langle \{x_i\,;\, i \in I\} \rangle$, and for all $z \in V_0$ we have

$$v(y) = \min_{\{i \in I\,;\, k_i \neq 0\}} v(x_i) \ ,$$

and

$$v(y + z) = \min\{v(y), v(z)\} \ .$$

Now assume these last equations hold. Then certainly $v(y) \geq v(y + z)$, for all $z \in V_0$ whence $v(y) = \max_{z \in V_0} v(y - z)$. Conversely, if

$$\max_{z \in V_0} v(y - z) = \min_{\{i \in I\,;\, k_i \neq 0\}} v(x_i) \ ,$$

then since on the other hand $v(y) \geq \min_{\{i \in I\,;\, k_i \neq 0\}} v(x_i)$, equality holds. Moreover, we then have for all $z \in V_0$

$$v(y) \geq v(y + z) \geq \min\{v(y), v(z)\}$$

which shows that the last part of this inequality cannot be strict. $\square$

Remark 3 If $\{x_i\,;\, i \in I\}$ is valuation independent over V_0 then $\{x_i\,;\, i \in I\}$ is independent over V_0: for if not we would have k_i, not all zero, $k_i = 0$ except for a finite number of $i \in I$, such that $z_0 = \sum_{i \in I} k_i x_i \in V_0$ and

$$v\left(\sum_{i \in I} k_i x_i - z_0\right) = \infty \neq \min_{\{i \in I\,;\, k_i \neq 0\}} \{v(x_i), v(z_0)\} \ .$$

It follows that a valuation basis of V over V_0 is always a maximal valuation independent set in V over V_0 .

Note that by Zorn's Lemma, there exists always in V a (possibly empty) maximal valuation independent set $\mathcal{B}$ over V_0, and we have the following characterization of immediate extensions:

Lemma 4 *Let $\mathcal{B}$ be maximal valuation independent in V over V_0. Then $\mathcal{B} = \emptyset$ if and only if $V_0 \subset V$ is an immediate extension.*

Proof: First note that by Proposition 2, if $0 \neq x \in V$ then $\{x\}$ is valuation independent over V_0 if and only if

$$\max_{z \in V_0} v(x - z) = v(x)$$

Now clearly, this last condition holds for some $x \in V$ if and only if the extension $V_0 \subset V$ is not immediate, whence the result. □

Theorem 5 *For every $V_0 \subset V$, there exists V_1 such that $V_0 \subset V_1 \subset V$ and*
1) V_1 admits a valuation basis over V_0,
2) $V_1 \subset V$ is an immediate extension.

Proof: Let $\mathcal{B} \subset V$ a maximal valuation independent set over V_0 and set

$$V_1 = \langle \mathcal{B} \cup V_0 \rangle .$$

Obviously, V_1 satisfies 1). Moreover $\emptyset$ is maximal valuation independent in V over V_1: if not, there exists $\mathcal{B}' \neq \emptyset$, $\mathcal{B}' \subset V$, $\mathcal{B}'$ valuation independent over V_1; it follows by Lemma 1 that $\mathcal{B} \cup \mathcal{B}'$ is valuation independent over V_0, this contradicts the maximality of $\mathcal{B}$. Now by Lemma 4, the extension $V_1 \subset V$ is immediate. □

Let us now recall the definition and basic properties of nice subspaces. We say that the subspace V_0 of V is *nice in* V if for all $x \in V$, the subset $v(x + V_0) = \{v(y);\, y - x \in V_0\}$ of $v(V)$ admits a maximum. Now Let $x \in V$, and assume that the coset $x + V_0$ admits a representative $x_0 \in x + V_0$ satisfying $v(x_0) = \max v(x + V_0)$ ($x_0 = 0$ if and only if $x \in V_0$). Such a representative is called a *proper representative*. Hence, V_0 is nice in V if for all $x \in V$, the coset $x + V_0$ admits a proper representative.

Using Proposition 2, it is straightforward to show the following

Lemma 6 *Let $x \in V \setminus V_0$ and $x_0 \in x + V_0$. Then x_0 is a proper representative if and only if $\{x_0\}$ is a valuation basis of $\langle V_0 \cup \{x\} \rangle$ over V_0. Consequently, V_0 is nice in V if and only if for all $x \in V \setminus V_0$, the subspace $\langle V_0 \cup \{x\} \rangle$ admits a valuation basis over V_0.*

Note that if V_0 is nice in V, then V_0 is nice in every subextension as well. Also, V_0 is nice in V if and only if for all $x \in V$, V_0 is nice in $\langle V_0 \cup \{x\} \rangle$. Moreover, Lemma 4 together with the last lemma show that V_0 is nice in $\langle V_0 \cup \{x\} \rangle$ if and only if the extension $V_0 \subset \langle V_0 \cup \{x\} \rangle$ is not immediate.

Corollary 7 *Every maximal immediate extension W of V_0 in V is nice in V. In particular, if V_0 is maximally valued, then V_0 is nice in every extension V.*

Proof: Let $V_0 \subset W$ be a maximal immediate extension of V_0 in V. Suppose that $x \in V \setminus W$, then clearly the extension $W \subset \langle W \cup \{x\} \rangle$ cannot be immediate (otherwise, the extension $V_0 \subset \langle W \cup \{x\} \rangle$ would be immediate as well, which contradicts the maximality of W). So by the note following Lemma 6, W is nice in V. □

Corollary 8 *If V admits a valuation basis over V_0, then V_0 is nice in V.*

Proof: Let $\mathcal{B}$ the valuation basis of V over V_0 and let $x \in V \setminus V_0$. Write $x = \sum_{i \in I} k_i b_i + z_0$ with $b_i \in \mathcal{B}$ and $z_0 \in V_0$. Then by Proposition 2, $v(x + V_0)$ admits $v(x - z_0)$ as its maximum. Hence V_0 is nice in V. $\square$

Note that the converse is in general not true.

Corollary 9 *There exists V_1 such that $V_0 \subset V_1 \subset V$ and*
1) V_0 is nice in V_1,
2) $V_1 \subset V$ is an immediate extension.

Proof: Follows from Theorem 5 and Corollary 8. $\square$

In studying extensions $V_0 \subset V$, with V_0 nice in V, it is often quite useful to consider the quotient space V/V_0. In this case, one can endow it with a canonical valuation $\overline{v}$ (see Lemma 10 below).

For the rest of this section, let us assume always that V_0 is nice in V.

Recall that then, every class $x + V_0$ has a proper representative. Define for all $x \in V$:

$$\overline{v}(x + V_0) = \max_{z \in V_0} v(x + z) \; ;$$

this maximum exists for all $x \in V$. It is clear that $\overline{v}$ is well defined. Note that $\overline{v}(V/V_0) \subset v(V)$ and that $\overline{v}(x+V_0) = v(x)$ if and only if x is a proper representative.

Lemma 10 *$\overline{v}$ is a valuation on V/V_0.*

Proof: We have $\overline{v}(0+V_0) = \max_{z \in V_0} v(0+z) = \infty$, but if $x \notin V_0$, then $0 \notin x + V_0$ and $\overline{v}(x + V_0) = \max_{z \in V_0} v(x + z) \neq \infty$. Clearly, condition (V2) holds. Now we check the triangle inequality: Without loss of generality, let x (respectively y) be a proper representative of $x + V_0$ (respectively of $y + V_0$). We then have

$$\begin{aligned} \overline{v}(x + y + V_0) &= \max_{z \in V_0} v(x + y + z) \geq v(x + y) \\ &\geq \min\{v(x), v(y)\} = \min\{\overline{v}(x + V_0), \overline{v}(y + V_0)\} \; . \end{aligned}$$

$\square$

Note that in general, $x + y$ is not a proper representative of $x + y + V_0 = (x + V_0) + (y + V_0)$. But we have:

Lemma 11 *Let $\{x_i + V_0 \, ; \, i \in I\} \subset V/V_0$. Then $\{x_i \, ; \, i \in I\}$ is independent for the valuation v over V_0 if and only if $\{x_i + V_0 \, ; \, i \in I\}$ is independent for the valuation $\overline{v}$ and the x_i are proper representatives. In this case, every finite sum $\sum_{i \in I} k_i x_i$ is a proper representative of $\sum_{i \in I} k_i(x_i + V_0)$.*

Proof: By Proposition 2, $\{x_i\,;\, i \in I\} \subset V$ is independent for the valuation over V_0 if and only if for all $k_i \in K$ such that $k_i = 0$ except for a finite number of $i \in I$,

$$\max_{z \in V_0} v(\sum_{i \in I} k_i x_i + -z) = \min_{\{i \in I\,;\, k_i \neq 0\}} v(x_i)\ .$$

Take $k_j = 0$ for $j \neq i$, this equation implies that x_i is a proper representative. Since

$$\max_{z \in V_0} v(\sum_{i \in I} k_i x_i - z) = \overline{v}\left(\sum_{i \in I} k_i x_i + V_0\right) = \overline{v}\left(\sum_{i \in I} k_i (x_i + V_0)\right)$$

and

$$\min_{\{i \in I\,;\, k_i \neq 0\}} v(x_i) = \min_{\{i \in I\,;\, k_i \neq 0\}} \overline{v}(x_i + V_0)$$

for proper representatives x_i, the assertion of Proposition 2 is equivalent to the assertion of the lemma. □

Lemma 12 *Suppose $V_0 \subset V_1 \subset V$. Then $V_1 \subset V$ is immediate if and only if $(V_1/V_0, \overline{v}) \subset (V/V_0, \overline{v})$ is immediate.*

Proof: $\Rightarrow$: Show that for all $x \in V$ there exists $y \in V_1$ such that $\overline{v}(x - y + V_0) > \overline{v}(x + V_0)$. Without loss of generality, x is a proper representative, so $\overline{v}(x + V_0) = v(x)$. By hypothesis there exists $y \in V_1$ such that $v(x-y) > v(x)$, this implies

$$\overline{v}(x - y + V_0) = \max_{z \in V_0} v(x - y + z) \geq v(x - y) > v(x) = \overline{v}(x + V_0)\ .$$

$\Leftarrow$: Suppose $(V_1/V_0, \overline{v}) \subset (V/V_0, \overline{v})$ is immediate, and let $x \in V$. There exists $x_1 \in V_1$ such that $\overline{v}(x - x_1 + V_0) > \overline{v}(x + V_0)$, i.e.

$$\max_{z \in V_0} v(x - x_1 + z) > \max_{z \in V_0} v(x + z) \geq v(x)\ .$$

Then there exists $z_0 \in V_0$ such that $v(x - x_1 + z_0) > v(x)$. Hence $y = x_1 - z_0 \in V_1$ satisfies $v(x-y) > v(x)$. □

We close this section with the following well known result, which can be proved by standard arguments using the definition of niceness (see e.g. [KF], where this result is even proved for valued modules):

Corollary 13 *Suppose that $V_0 \subset V_1 \subset V$. Then V_1 is nice in V if and only if V_1/V_0 is nice in V/V_0.*

3 Nice subspaces and defectless extensions

Let V_0 be a subspace of V. A *nice composition series* of V over V_0 is a sequence $\{V_\mu\,;\, 0 \leq \mu < \nu\}$ of subspaces, indexed by some ordinal ν, such that
V_μ is nice in $V_{\mu+1}$ and $\dim(V_{\mu+1}/V_\mu) \leq 1$ if $\mu + 1 < \nu$,
$V_\lambda = \bigcup_{0 \leq \mu < \lambda} V_\mu$ for every limit ordinal $\lambda < \nu$ and
$V = \bigcup_{0 \leq \mu < \nu} V_\mu$.
Obviously, if V admits a nice composition series $\{V_\mu\,;\, 0 \leq \mu < \nu\}$ over V_0, then it admits one over every V_μ as well.

Theorem 14 *V admits a nice composition series over V_0 if and only if V admits a valuation basis over V_0.*

Proof: Assume V admits a nice composition series $\{V_\mu\,;\, 0 \le \mu < \nu\}$ over V_0. Hence, every $V_{\mu+1}$ admits a valuation basis $\{y_\mu\}$ over V_μ. By induction and Lemma 1, $\{y_\mu, \ldots, y_{\mu+n}\}$ is a valuation basis of $V_{\mu+n+1}$ over V_μ. Similarly, it follows from Lemma 1 that $\{y_\mu\,;\, 0 \le \mu < \lambda\}$ is a valuation basis of V_λ over V_0 for every limit ordinal $\lambda < \nu$. By induction, we obtain the assertion.
Conversely, let $\mathcal{B} = \{y_\mu\,;\, 0 \le \mu < \nu\}$ be a valuation basis of V over V_0. For $0 \le \mu < \nu$, set

$$V_\mu = \langle V_0, \{y_\alpha\,;\, \alpha < \mu\}\rangle.$$

It follows again by Lemma 1 that $\{V_\mu\,;\, 0 \le \mu < \nu\}$ is a nice composition series of V over V_0. □

Let α be an ordinal. In the sequel, we shall denote by α^* the order type of the reversed ordering on α.

Corollary 15 *If $v(V \setminus V_0) = \alpha^*$, then V admits a valuation basis over V_0 (in particular, V_0 is nice in V).*

Proof: Without loss of generality, we may assume that V is a proper extension of V_0. Let $\mathcal{B} = \{b_\mu\,;\, 0 \le \mu < \nu\}$ be a basis of V over V_0. For $0 \le \mu < \nu$, set

$$V_\mu = \langle V_0, \{b_\alpha\,;\, \alpha < \mu\}\rangle.$$

We show that $\{V_\mu\,;\, 0 \le \mu < \nu\}$ is a nice composition series of V over V_0. We only have to show that V_μ is nice in $V_{\mu+1}$. But $v(V_{\mu+1} \setminus V_\mu) \subset v(V \setminus V_0)$. Thus $v(V_{\mu+1} \setminus V_\mu)$ is the reverse of an ordinal as well and hence contains no infinite increasing sequences. It follows that the extension $V_\mu \subset V_{\mu+1}$ is not immediate, hence the assertion. □

Corollary 16 *If $v(V_0) = \alpha^*$, then V_0 is nice in V. Further, V_0 admits a valuation basis over every subspace.*

Proof: The hypothesis implies that for all $x \in V$, $v(V_0 + Kx)$ is also the reverse of an ordinal, and so is $v(V_0 + Kx \setminus V_0)$. Then by Corollary 15, V_0 is nice in $V_0 + Kx$. Now let W be a subspace of V_0. Then $v(V_0 \setminus W) \subset v(V_0)$, hence $v(V_0 \setminus W)$ is also the reverse of an ordinal. The second assertion follows now by Corollary 15. □

Note that in the last corollary, V will not necessarily admit a valuation basis over V_0.

Corollary 17 *If $\dim V \le \aleph_0$, then V admits a valuation basis over every subspace V_0 of finite dimension.*

Proof: Write $V = \bigcup_{1 \leq \mu < \aleph_0} V_\mu$ where the V_μ are subspaces of finite dimension and $\dim V_{\mu+1}/V_\mu \leq 1$. The dimension of V_μ being finite, $v(V_\mu)$ is finite; then by Corollary 16, V_μ is nice in $V_{\mu+1}$. Now, the assertion follows from Theorem 14. □

We have seen in Theorem 14 that a space V admits a valuation basis over a given nice subspace V_0 if and only if V has a nice composition series over V_0. But of course, such a composition series does not always exist. For instance, the 0 space is nice in every space V, but not every V admits a valuation basis. However, we shall now describe some more cases where niceness implies the existence of nice composition series. To this end, we shall apply the results so far obtained in this section to the valued vector space $(V/V_0, \overline{v})$, and this in turn will give us information about the extension $V_0 \subset V$.

Proposition 18 *Let V_0 be nice in V. Then $(V/V_0, \overline{v})$ admits a valuation basis if and only if V admits a valuation basis over V_0.*

Proof: Follows by Lemma 11. □

From this proposition and Corollary 17, we obtain the following strengthening of Corollary 17.

Corollary 19 *Let V_0 be nice in V. If $\dim V/V_0 \leq \aleph_0$, then V admits a valuation basis over V_0.*

Note that this last corollary gives the converse to Corollary 8 in the case of countable dimensional extensions.

Corollary 20 *Let V_0 be nice in V. Let $V_0 \subset V_1 \subset V$ and $\overline{v}(V_1/V_0) = \alpha^*$. Then V_1 admits a valuation basis over V_0 and V_1 is again nice in V. If moreover V_0 is maximally valued, then V_1 is maximally valued as well.*

Proof: Since $\overline{v}(V_1/V_0) = \alpha^*$, then V_1/V_0 admits a valuation basis over $\{0\}$ (by Corollary 16). Hence by Proposition 18, V_1 admits a valuation basis over V_0. By Corollary 16, V_1/V_0 is nice in V/V_0, and by Corollary 13 it follows that V_1 is nice in V.

If moreover V_0 is maximally valued, then V_0 is nice in every extension V' of V_1 and hence V_1 is nice in every extension V'. This implies by the note following Lemma 6 that V_1 admits no proper immediate extension. □

As a corollary, we get yet another generalization of Corollary 17:

Corollary 21 *Suppose that V admits a valuation basis over V_0, and let $V_0 \subset V_1 \subset V$ such that $V_0 \subset V_1$ is a finite extension. Then V admits a valuation basis over V_1.*

Proof: Let $\mathcal{B}$ a valuation basis over V_0. Since $V_0 \subset V_1$ is finite, there exists a partition $\mathcal{B} = \mathcal{B}' \dot\cup \mathcal{B}''$ such that $\mathcal{B}'$ is finite and $V_1 \subset \langle \mathcal{B}', V_0 \rangle =: W$. By Corollary 8, V_0 is nice in V. So by Corollary 20, V_1 is nice in V and hence in W. Then by Corollary 19, W admits a valuation basis $\mathcal{B}^*$ over V_1. By Lemma 1, $\mathcal{B}''$ is a valuation basis of V over W and $\mathcal{B}'' \cup \mathcal{B}^*$ is a valuation basis of V over V_1. □

We say that $V_0 \subset V$ is a *defectless extension* if and only if for every finite subextension $V_0 \subset V_1$, there exists a valuation basis of V_1 over V_0. Note that if $V_0 \subset W \subset V$ and if $V_0 \subset V$ is a defectless extension, then $V_0 \subset W$ is a defectless extension.

Theorem 22 *V_0 is nice in V if and only if $V_0 \subset V$ is a defectless extension.*

Proof: $\Leftarrow$: follows from the definition and Lemma 2.6.
$\Rightarrow$: If V_0 is nice in V, then by Corollary 20, every finite extension of V_0 admits a valuation basis over V_0, hence $V_0 \subset V$ is a defectless extension. □

4 Extensions of spaces admitting a valuation basis

Theorem 23 *Suppose that V_0 admits a valuation basis and that $V_0 \subset V$ is immediate. Let $W \subset V$ such that*

$$V = V_0 \oplus W\ .$$

then there exist $V_1, V_2 \subset V_0$ such that
i) $V_2 \subset V_2 \oplus W$ is immediate and $\dim V_2 \leq \max\{\aleph_0, \dim W\}$
ii) V_1 and V_2 admit a valuation basis
iii) $V_0 = V_1 \perp V_2$, and hence

$$V = V_1 \perp (V_2 \oplus W)\ .$$

Proof: Let $\mathcal{B}$ a valuation basis of V_0. We first show the following fact: if U is a subspace of V, then there exists $\mathcal{B}_U \subset \mathcal{B}$ such that

$$\operatorname{card} \mathcal{B}_U \leq \max\{\aleph_0, \dim U\}$$

and

$$\forall 0 \neq u \in U\ \exists y \in \langle \mathcal{B}_U \rangle :\ v(u-y) > v(u)\ . \tag{1}$$

For let $\{u_i\,; i \in I\}$ be a maximal valuation independent set in U. Then the extension $\langle u_i\,;\ i \in I \rangle \subset U$ is immediate (cf. Lemma 4), and $\operatorname{card}\{u_i\,;\ i \in I\} \leq \dim U$ (cf. Remark 3). Further, since $V_0 \subset V$ is immediate, for all $i \in I$ there exists a finite subset $\mathcal{B}_i \subset \mathcal{B}$ and $y_i \in \langle \mathcal{B}_i \rangle$ such that

$$v(u_i - y_i) > v(u_i).$$

Set $\mathcal{B}_U = \bigcup_{i\in I} \mathcal{B}_i$, then if $\sum k_i u_i \in \langle u_i\,;\ i \in I\rangle$, we have $\sum k_i y_i \in \langle \mathcal{B}_U \rangle$ and

$$\begin{aligned} v\left(\sum k_i u_i - \sum k_i y_i\right) &= v\left(\sum k_i (u_i - y_i)\right) \geq \min_{i\in I\,;\,k_i\neq 0} v(u_i - y_i) \\ &> \min_{i\in I\,;\,k_i\neq 0} v(u_i) = v\left(\sum k_i u_i\right)\ , \end{aligned}$$

hence $\mathcal{B}_U$ satisfies assertion (1) for $0 \neq u \in \langle u_i\,;\ i \in I\rangle$. We claim now that $\mathcal{B}_U$ satisfies assertion (1) for all $0 \neq u \in U$. Indeed let $0 \neq u \in U$. Since the extension $\langle u_i\,;\ i \in I\rangle \subset U$ is immediate, let $u' \in \langle u_i\,;\ i \in I\rangle$ such that $v(u - u') > v(u)$. Now let $y \in \langle \mathcal{B}_U \rangle$ such that $v(u' - y) > v(u')$. A straightforward argument, using the triangle inequality, shows then that $v(u - y) > v(u)$.

Let us define by induction on $n \in \omega$ an increasing sequence $\mathcal{B}_n$ of subsets of $\mathcal{B}$ as follows:

$$\mathcal{B}_1 = \mathcal{B}_W \ \text{ and } \ \mathcal{B}_{n+1} = \mathcal{B}_n \cup \mathcal{B}_{\langle \mathcal{B}_n, W\rangle}\ .$$

Set

$$\mathcal{B}_\omega = \bigcup_{n\in\omega} \mathcal{B}_n \ \text{ and } \ V_2 = \langle \mathcal{B}_\omega \rangle\ .$$

Hence $\dim V_2 = \operatorname{card} \mathcal{B}_\omega \leq \max\{\aleph_0, \dim W\}$. If $0 \neq x \in V_2 \oplus W$, then there exists $n \in \omega$ such that $x \in \langle \mathcal{B}_n, W\rangle$. Consequently, there exists $y \in \langle \mathcal{B}_{n+1}\rangle$ such that

$$v(x - y) > v(x)\ ,$$

which implies assertion i).

Set $V_1 = \langle \mathcal{B} \setminus \mathcal{B}_\omega \rangle$. Assertion ii) is clear, and also that $V_0 = V_1 \perp V_2$. To prove assertion iii) that $V = V_1 \perp V_2 \oplus W$, it suffices to verify the following general fact: if $V_i \perp V_j$ and $V_j \subset V_{j'}$ is immediate, then $V_i \perp V_{j'}$. The proof is straightforward. □

Corollary 24 *Suppose that V_0 admits a valuation basis and that $\dim(V/V_0) \leq \aleph_0$. Then V admits a valuation basis.*

Proof: By Theorem 5 and Lemma 1, it is clear that it suffices to consider the case where $V_0 \subset V$ is immediate. By hypothesis, there exists $W \subset V$ with $\dim W \leq \aleph_0$ and

$$V = V_0 \oplus W\ .$$

Let

$$V = V_1 \perp (V_2 \oplus W)$$

be the corresponding decomposition of V, as described in Theorem 23 above. Since $\dim(V_2 \oplus W) \leq \aleph_0$, it follows by Corollary 17 that $V_2 \oplus W$ admits a valuation basis. To see that V admits a valuation basis, it suffices to verify the following general fact: if V_i admits a valuation basis $\mathcal{B}_i$ for all $i \in I$, then $V = \perp_{i\in I} V_i$ admits $\bigcup_{i\in I} \mathcal{B}_i$ as a valuation basis. The proof is straightforward. □

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Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
D-69120 Heidelberg, Germany