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MATHEMATICAL KNOWLEDGE IS CONTEXT DEPENDENT

Benedikt LÖWE
Universiteit van Amsterdam, Universität Hamburg &
Rheinische Friedrich-Wilhelms-Universität Bonn

Thomas MÜLLER
Rheinische Friedrich-Wilhelms-Universität Bonn

Summary

We argue that mathematical knowledge is context dependent. Our main argument is that on pain of distorting mathematical practice, one must analyse the notion of having available a proof, which supplies justification in mathematics, in a context dependent way.

> 'But a proof is sometimes a fuzzy concept, subject to whim and personality.'
> Kenneth Chang, *New York Times* (April 6, 2004)

1. *Introduction*

Mathematical knowledge appears to be of a special, privileged form. When somebody knows a mathematical fact, we say that she knows 'with mathematical certainty', and it is commonly assumed that nothing can be more firmly grounded than that. Not surprisingly, in philosophical contexts, mathematics is often used as an epistemological role model. Mathematical knowledge is assumed to be absolute and undeniably firm. The main reason for that special status lies in the fact that mathematicians *prove* their theorems: Mathematical knowledge is proven knowledge ('*more geomet*rico *demonstrata*'). What has been proven is established beyond all doubt. Thus, mathematical knowledge stands out as knowledge with a uniform witness, the notion of mathematical (deductive) proof.

This close connection between mathematical knowledge and the priviledged form of epistemic justification via mathematical proof leads to a broad consensus of how to analyse mathematical knowledge. The standard view of mathematicians and philosophers alike (which is in agreement with the common perception of the educated public) can be described the following way:

(*) *S* knows that *P* iff *S* has available a proof of *P*.

Of course, (*) is vague with respect to the two key notions of the *explanans* ("proof" and "having available"). We shall discuss the notions of proof in detail in § 2.

Assuming for a moment that we agree on what "proof" is, what does it mean to have available a proof? A literal reading in terms of having access to a material copy of the proof is inappropriate. It is too narrow, because there just aren't enough copies of proofs to back even a fraction of true mathematical knowledge claims (especially if one demands derivations, of which there are hardly any around).[1] But it is also too wide: A mathematical illiterate on the first floor of UC Berkeley's Evans Hall (the math library) has available lots and lots of proofs, but it would be odd to say that the mere location could affect any change in mathematical knowledge *(genius loci* nonwithstanding).

Thus, "having available" cannot be spelled out in terms of actual physical access; it needs to be given a modalised reading in which the epistemic subject *S* plays an active role. A reformulation of (*) that makes that modalisation explicit is the following:

(†) *S* knows that *P* iff
S could in principle generate a proof of *P*.

Of course, (†) continues to have vague terms, *viz.* "could in principle", "generate", and "proof". As mentioned before, we shall discuss "proof" in § 2; the notions of "could in principle" and "generate" will be discussed in § 3.

Following the tradition of standard (context independent) epistemology, many philosophers would like to interpret (†) by giving necessary and sufficient conditions for the right-hand side to hold, independent of the context. The general perception of the absolute nature of mathemati-

1. E.g., no living mathematician has seen a derivation of the Feit-Thompson Theorem, yet there are (many) mathematicians who know that every group of odd order is solvable. The original paper, Feit and Thompson (1963), has over 250 pages. Only specialists in finite group theory will know even an informal proof. On the other hand, the theorem is rather well known.

cal knowledge makes such a project appear more promising than in other fields of knowledge.

In such a *context independent* or *invariantist* reading, the vague notions "could in principle", "generate", and "proof" would be replaced by distinguished sharp notions "could in principle$^\sharp$", "generate$^\sharp$" and "proof$^\sharp$", leading to

(†$^\sharp$) *S* knows that *P* iff
S could in principle$^\sharp$ generate$^\sharp$ a proof$^\sharp$ of *P*.

In this paper, we shall argue that no reading of (†$^\sharp$) is adequate as an analysis of mathematical knowledge. In our argument, we shall proceed from a mildly naturalistic philosophical methodology: In philosophising about mathematics, mathematical practice must be taken seriously. If certain expressions, such as "knowledge", or "… knows that …", are used in the mathematical community, then that usage cannot be dismissed without good arguments. This is not to say that mathematical practice has the last word—but it certainly has to have the first word. Thus, we will not be satisfied with an epistemology for mathematics according to which there is no (or hardly any) mathematical knowledge in the world—mathematical practice asserts that on the contrary, there is a lot of mathematical knowledge. On the other hand, we will also not be willing to accept an epistemology that identifies all true mathematical statements as the necessary proposition 2 + 2 = 4 in disguise. Such a position consequently grants that every epistemic subject knows all mathematical truths (but may not be aware of them). There is certainly less mathematical knowledge than that!

The paper is structured as follows: In § 2, we start by briefly discussing the status of mathematics as an epistemic exception and the nature of mathematical proof. In § 3, we then move on to consider possible interpretations of "could in principle" and "generate" in connection with various notions of proof. This section contains our argument against (†$^\sharp$). Furthermore, not even the weakest notion of "proof" is necessary for mathematical knowledge: in § 4, we discuss inductive reasoning and knowledge by testimony in mathematics.

Having debunked (†$^\sharp$), we propose an alternative. In § 5, we briefly describe Lewis's contextualist analysis of knowledge and give it a mathematical reading (#′). In § 6, we tie (#′) to the Dreyfus-Dreyfus skill model to arrive at our final analysis (‡).

2. *Standard mathematical epistemology and the notion(s) of proof*

Mathematics is an *epistemic exception*[2] as compared to the other sciences. This point has been implicitly or explicitly observed by a large number of philosophers ancient and modern. Plato in the famous παις example *(Meno,* 82b–84a) shows how the slave, guided by Socrates, without any prior education or empirical data arrives at a mathematical truth. Kant, who holds that mathematical truths are synthetic *a priori,* limits the use of "knowledge" generally to proven certainties and even claims that mere belief has no place in mathematics at all *(Kritik der reinen Vernunft*, A823/B851). Frege conceives of mathematics as a branch of logic. His project of logicism makes a purely formal method the hallmark of secure knowledge *(Begriffsschrift,* (Frege, 1879, IXf.)). And to bring in a contemporary contextualist, Lewis remarks in passing that in 'the mathematics department, […] they are in confident agreement about what's true and how to tell, and they disagree only about what's fruitful and interesting' (Lewis, 2000, 187f.).

The distinguishing feature of mathematical epistemology that underlies the observed exceptional status is the robust notion of mathematical proof: in mathematics, there is deduction, in the sciences there is only induction. Moreover, as we learn early on in our education, a mathematical proof is either correct or incorrect and does not admit degrees of correctness. To use Keith Devlin's polemic words: 'Surely, any math teacher can tell in ten minutes whether a solution to a math problem is right or wrong! […] Come on folks, it's a simple enough question. Is his math right or wrong?' (Devlin, 2003)[3]

It is an empirical fact that there seem to be no lasting disagreements in mathematics. Whether someone has available a proof of P is almost never a serious matter of dispute; and thus it is natural to assume that the vague terms "has available" and "proof" in (*) can be made precise without contest. This lends intuitive support to a context independent reading *à la* ($\dagger^\sharp$).

2. This has been an important topic in the sociology of science, discussed, *e.g.*, by Mannheim, Bloor (1976) and Livingston (1986). *Cf.* also Heintz (2000, Chap. 1) and Prediger (2001, 24f.).

3. Just for the record: Of course, Devlin is playing the advocatus here, arguing that even checking proofs is not as trivial as is often believed. An observation along these lines can already be found in Locke's Essay (Locke, 1689, IV.ii.7): 'In long deductions […] the memory does not always so readily and exactly retain; therefore it comes to pass, that this is more imperfect than intuitive knowledge, and men embrace often falsehood for demonstrations.'

Let us now discuss the notion of "proof" that is so seemingly unequivocal. It is undeniably robust, but is this robustness realised in the same form everywhere? In actual mathematical practice, a wide range of texts or activities is called "proof". The guiding idea of proving something is to arrive at the result through a number of secure steps, but one needs to specify which steps may be used. Frege in his *Begriffsschrift* proposed that the steps should be so small that a mechanical procedure was available for checking each step. This led to a mathematically precise definition of formal proof which was then available for metamathematical investigations leading, *e.g.,* to Gödel's completeness and incompleteness results. We will use the term *derivation* to stand for formal proof in a mathematically well-defined system.[4] Outside meta-mathematical investigations and a few very specialised areas,[5] one will not find derivations in mathematical publications. Mathematical journals and textbooks (as well as lectures, research notes and conference talks) instead contain informal proofs.[6] The notion of informal proof does not have a mathematically precise definition—if it did, it would be just another version of derivation.

From the point of view of derivation, informal proofs contain gaps and appear to be unfinished. It is therefore tempting to see an informal proof just as an imperfect stand-in for a derivation. However, mathematical practice strongly supports the view that the important notion of proof in mathematics is not derivation, but informal proof. One reason for this is communication: 'The point of publishing a proof [...] is to communicate that proof to other mathematicians. [...] [T]he most efficient way [...] is not by laying out the entire sequence of propositions in excruciating detail' (Fallis, 2003, 55). Instead, mathematicians publish informal proofs. However, there is more to informal proof than ease of communication. It just isn't the case that mathematicians have a derivation in mind and transform it into an informal proof for publication in order to reach a wider public—the entire procedure of doing research mathematics rests on doing informal proofs. The proofs in mathematical research papers are so far removed from derivations that only a few experts could produce a derivation from them even if they wanted to,

4. There are various competing notions of derivation, but their differences do not matter for our purposes. For first-order logic, the competing formal systems are equivalent in allowing one to prove exactly the same theorems.

5. *E.g.*, the Journal of Formalized Mathematics, which focuses on derivations in the specific proof system MIZAR, *cf.* `http://www.mizar.org/JFM/`; or the publications of the Coq group discussed in § 3.

6. *Cf.*, *e.g.*, Rav (1999) for discussion of the distinction between derivations and informal proofs.

and only a minority considers that a worthwhile enterprise. We need to take seriously the fact that derivations are hardly ever used. Subscribing to the tempting image of the derivations as the real objects of mathematical study to which informal proofs are imperfect approximations would be a violation of our maxim of taking mathematical practice seriously.[7]

Informal proofs come in many flavours. One can, e.g., distinguish semi-formal textbook proofs for beginning students, graduate-level textbook proofs, journal proofs, informal research notes, and proof sketches. Each of these types is pragmatically fairly well delineated—try submitting a textbook-style proof to a mathematical journal, or presenting research note-style proofs to beginning students, and you will feel the force of the boundaries.

It is often possible to compare proofs for one and the same *P* with respect to the level of detail they exhibit. One proof may give more details than

7. Derivation is often called the gold standard of mathematical proof. That metaphor is quite telling. First, a few historical facts. Implementing a gold standard means making a fixed weight of gold the standard economic unit of account. This can, e.g., be established by using coins made of gold. More practically, gold is stored in some central reservoir, and paper money is issued as certificates entitling the holder to a fixed amount of gold. Such systems were established in the late 19th century in many Western countries, and there were earlier, similar systems in many places. A positive aspect of an international gold standard is free convertibility of currencies, which was important in boosting international trade. A negative aspect of such a system is that even though gold is nice stuff, what people actually need isn't gold (except in some cases related to dentistry), but other goods, and the scarcity or otherwise of gold dictates in effect the price of other goods. The successor of the early international gold standard, the Bretton Woods system, collapsed in the early 1970ies. Since then, many countries have sold off much of their gold. Other mechanisms of establishing trust between trading parties have proved to be more practical and more efficient.

We would like to draw a rather strict analogy between the rôle of gold for the exchange of goods and the rôle of derivation for the exchange of mathematical knowledge. Historically, of course there never was a period in the development of mathematics during which derivation was the generally accepted currency, but the logicist movement of the early 20th century surely was an attempt at establishing that currency. Just like gold vs. goods, derivation is neither the only store of value for mathematics, nor the most useful. If anything, trading in derivations is more impractical than trading in gold. (Given the scarcity of gold and the expanded international trade today, a return to an international gold standard would mean increasing the current price of gold more than tenfold. But given the scarcity of derivations, establishing derivations as the sole vehicle of mathematical justification would at present completely stop the development of mathematics.)

Other mechanisms for establishing trust in the mathematical community are well established, and they are working. Of course that does not mean that derivations are worthless. Quite on the contrary—the belief in the possibility to generate, at least in principle, a derivation corresponding to any given informal proof, may well be one of the strongest sources of mutual trust in the mathematics community. It is just that actual derivations aren't really needed—except, if you allow, for exercises in mathematical dentistry.

another, even though both are valid and complete proofs of certain types. *E.g.*, a textbook proof may contain a whole page of details for a certain inductive construction where a research note would just say "by induction". Thus it makes sense to say of a proof that it contains gaps relative to another. However, we do not subscribe to an absolute notion of "gaps in proofs", because that would presuppose an absolute standard of a "gap-less" proof.[8]

3. *Having available a proof*

In the previous section, we discussed possible readings of the vague term "proof", in this section, we shall now focus on the terms "could in principle" and "generate" used in the modalised analysis (†).

A classical example of modalisation for mathematical knowledge is Brouwer's *idealised mathematician,* the creative subject who creates his choice sequences.[9] Kitcher (1984, Chap. 6.III), in a similar vein, employs the notion of an 'ideal agent' to account for the fact that actual operations of actual agents do not suffice to establish the truths of arithmetic as he conceives it.[10] Steiner (1975, Chap. 3) explicates the modal idealisation of (†) via the following thought experiment: In order to check whether a mathematician has available a proof of *P* and thus, knows that *P,* she is asked to transform her (informal) proof into a derivation with the aid of a logician who as a Socratean 'midwife' works out the formal details, but is not otherwise mathematically creative. "If the two can bang out a formal proof, then the mathematician is said to have known the proof all along, on the basis of the informal argument" (Steiner, 1975, 100). Thus:

($\dagger^{\sharp}_{1}$) *S* knows that *P* iff with the help of a logician,
S can generate a derivation of *P.*

Brouwer, Kitcher, and Steiner give quite specific readings of modal aspects of mathematics, and Steiner gives an explicit test for 'could in principle generate a proof. This is what one needs to do if one is after an

8. Note that there is a different notion of gap in proof, which Fallis (2003) calls 'untraversed gaps' in contrast to the 'enthymematic gaps' that we just discussed: If in proving one fails to note a certain special case, the proof will be incomplete—it won't even belong to the intended class of informal proof. Here the gap terminology is appropriate in an absolute sense.

9. *Cf.* Brouwer (1929); for an historical overview of the notion, cf. Troelstra (1982).

10. *Cf.* Chihara (1990, Chap. 11.2) for decisive criticism of Kitcher's approach.

invariantist version of (†). However, mathematical practice provides counterexamples against any fixed notion—there is even knowledge without proof (*cf.* § 4 below). We will now explore the modal dimension of (†) in three steps, starting with a critique of Steiner's approach.

(i) Steiner's model ($\dagger^\sharp_1$) is open to a number of criticisms, some already voiced in the original publication.[11] The envisaged test for knowledge only replaces one form of modalisation ('has available' or 'could in principle generate') with another, not much clearer one ('can produce, with the aid of a logician, …')—and the kind of logician that is needed may well turn out not to exist. The logician's powers play a crucial role. Steiner rightly stresses that "we cannot envision a superhuman, because such a being would discover a completed proof despite the ignorance of the mathematician" (Steiner, 1975, 101f.), rendering the test useless. In practice, even if two persons cooperate in producing a derivation, the rôles will never be as clearly delineated as the test suggests. It may be fine to say that *the pair* who succeeded in writing down a derivation had available a proof (and thus, knew that *P*), but that is of course no good as a test of the *mathematician's* knowledge.

Let us now consider two variants of Steiner's modalisation. In both variants, the dubious logician is replaced by a direct appeal to the subject's capabilities. The first variant is based on derivation, the second, on informal blackboard proof.

(ii) Suppose that we want to read ($\dagger^\sharp$) by fixing "proof$^\sharp$" to mean "derivation". The task then is to try to find a good explication of 'could in principle generate'. The successes of formalised mathematics have shown that it is possible to provide derivations for many important mathematical statements, however doing so requires a long time: *e.g.,* the Coq community worked for over ten years before Geuvers, Wiedijk, and Zwanenburg were able to formalise the fundamental theorem of algebra (Geuvers et al., 2001). Now, this suggests reading 'could in principle generate' as follows:

($\dagger^\sharp_2$) *S* knows that *P* iff, given ten years,
she could write a formal derivation in the language Coq.

11. It must be said in fairness to Steiner that he does not subscribe to ($\dagger^\sharp_1$) in the end. Rather, he gives an example of mathematical knowledge without proof and then argues for a Platonist understanding of mathematical knowledge.

But compared to these ten years, the time we need to learn mathematical facts is short: many mathematicians could be in the situation that they don't know anything about *P*, but are able to learn within ten years both the mathematics needed to understand why *P* is true and then formalise it in Coq. These mathematicians would satisfy our fixed reading of ($\dagger^{\sharp}_{2}$), but by assumption do not know *P.* For particularly bright beginning students, the time of ten years might be enough to study mathematics, enter graduate school, finish a doctoral degree in mathematics, and learn Coq. Thus, the reading ($\dagger^{\sharp}_{2}$)would grant almost indefinite mathematical knowledge to everyone who has the intellectual capacity to finish a mathematics degree. Clearly, not an intended reading.

The invariantist readings ($\dagger^{\sharp}_{1}$) and ($\dagger^{\sharp}_{2}$) face another difficulty. As soon as derivations or a system like Coq play a role, we need to concede that there was no mathematical knowledge prior to a certain point in time: *e.g.,* before the *Begriffsschrift,* nobody could give a derivation of anything, because the concept of derivation had not yet been invented.[12] But mathematics is commonly taken to be the prime example of historically stable knowledge—the ancient Greeks already *knew* the Pythagorean theorem.

(iii) At the other end of the spectrum let us read ($\dagger^{\sharp}$) by fixing "proof$^{\sharp}$" to mean "informal proof on the blackboard". For many research situations in mathematics, the relevant notion of 'could in principle generate' is something like the following:

($\dagger^{\sharp}_{3}$) *S* knows that *P* iff, given a blackboard and a piece of chalk, she is able to produce an acceptable blackboard proof within an hour.

Note that we cannot restrict the timeframe for producing the proof to the time physically needed to produce the chalk markings on the blackboard, as many research mathematicians do not have all of the proofs they need for their work at their immediate cognitive disposal. They need to try one or two standard approaches to tackle the problem, remember the important details, and only after that are they able to provide an acceptable proof. If one makes this time frame too short, then one arrives at too strict a crite-

12. If you are not satisfied with taking the *Begriffsschrift* as the beginning of derivation, supply your favourite reading instead. The consequences are practically the same.

rion for knowledge. The analysis ($\dagger^{\sharp}_{3}$) is an excellent description of mathematical knowledge among researchers meeting in an office for joint work, but is inadequate for other situations. Consider a student in an oral exam asked whether *P* or non-*P* is true. Suppose that the student erroneously believes that non-*P* is true but given a blackboard, a piece of chalk and one hour of time, this particular student might be able to create a blackboard proof of *P*, first trying to prove non-*P*, failing, getting some ideas from the failed attempts, then remembering some facts and ideas from lectures, and finally proving *P*. However, in the oral exam, the examiner will not wait for an hour, the student has to rely on his belief, says 'non-*P*' and fails. Does this mean that the oral exam is not testing knowledge? In view of our methodological maxim, that would be absurd.

The analyses ($\dagger^{\sharp}_{1}$), ($\dagger^{\sharp}_{2}$), and ($\dagger^{\sharp}_{3}$) are just three possible examples for a reading of ($\dagger^{\sharp}$). For other readings, it is easy to come up with more examples or contexts of knowledge attributions that show that they are problematic.

We would like to add two relevant remarks:

First of all, our examples show that the temporal component in "could in principle" is immensely important, and that it seems hopeless to try to fix a single reading for all contexts. If one gives the subjects too much time to generate a proof, then one ends up with knowledge assertions that shouldn't be true, but if one gives them too little time, then some true knowledge assertions dissolve.

Secondly, one way to avert the move to full contextualism would be to allow the meanings of "could in principle", "generate", and "proof" to depend on *S,* but not on the general context. This would give an analysis

($\dagger^{S}$) *S* knows that *P* iff
S could in principle^{S} generate^{S} a proof^{S} of *P*,

where "could in principle^{S}", "generate^{S}", and "proof^{S}" are assignments of meanings of the vague terms to *S*. For instance, "proof^{S}" could be a formal derivation if *S* is a member of the Coq programming project, a blackboard proof if *S* is a research mathematician, and a textbook proof if *S* is a student. This obviously won't work either: Coq programmers are typically reseach mathematicians as well and may need to switch between contexts; students face different situations, *e.g.,* our student in the exam mentioned above will be assessed differently than in a tutorial session.

One step further, one could make the meanings of the vague terms dependent on S and P, leading to an analysis ($\dagger^{SP}$). The same argument shows that this cannot deal with the multitude of different contexts either.

4. *Mathematical knowledge without proof*

We have seen that proof comes in many flavours. In this section, we shall discuss examples of proper knowledge attributions in mathematics without cognitive access to any form of mathematical proof.[13]

A good historical example is reported in Pólya's study of Euler (Pólya, 1954, Chap. 2.6): It is certainly true to say that Euler *knew* that

$$1 + 1/4 + 1/9 + 1/16 + 1/25 + \ldots = \pi^2/6,$$

but Euler didn't have available (and knew that he didn't have available, nor could in principle generate) a proof of that fact—he had established it via generally shaky generalisations from finite to infinite sums, and his evidence was to a large part inductive *(i.e.*, the first 20 or so decimal places coincided). Still, it would be ahistorical to say that Euler had just guessed.[14]

Cases of knowledge without proof are not rare at all, nor are they a thing of the past. Even more important than inductive generalisations is knowledge via testimony for which proof plays hardly any rôle at all— and yet, in many mathematical contexts it is fine to base a knowledge claim on testimony. That is obvious enough for claims to mathematical knowledge in the general public: Most people haven't actually seen any mathematical proofs at all, and yet the majority of the public has mathematical knowledge of some kind, *e.g.*, elementary algorithms of arithmetic, the Rule of Three, etc.

For beginning math students, a similar observation holds: While we certainly urge them to try to learn and understand the proofs, we also concede that the students do acquire knowledge (though not a very deep kind of knowledge) by just learning theorems by heart, and that may be enough to pass a first exam. And even in the context of research mathematics, some

13. Knowledge without proof points to a vexing question in the philosophy of mathematics: Is it possible to have (a high degree of) knowledge of P by pure intuition without any formal proof in mind (the Ramanujan phenomenon)? *Cf.* Thurston (1994) for discussion of this point.

14. *Cf.* Steiner (1975, Chap. 3.IV) for a similar assessment.

knowledge is just based on trust. If one works in cooperation with others, it will not normally be possible, nor required, to learn and check all proofs.

It could be said that these examples are so vastly different from those given in § 3 that they constitute a violation of (∗) or (†). In § 5, we shall develop a context dependent reading of (†); given that the meanings of the vague terms "could in principle", "generate", and "proof" will vary according to context anyway, it will allows us to understand examples of inductive knowledge and knowledge via testimony as readings of (†), *e.g.,* by relaxing the notion "proof" (for the Euler-Ramanujam example) or "could in principle" (for cases of knowledge by testimony).

5. *Contextualism in mathematics*

Contextualism is a fairly recent attempt at answering one of the long-standing problems of epistemology, *viz*., the problem of skepticism. In spelling out contextualism, we follow David Lewis's general analysis, given in his 1996 classic, 'Elusive knowledge'. Lewis analyses the statement '*S* knows that *P*' context dependently as follows:

(#) *S* knows that *P* iff *S*'s evidence eliminates every possibility in which not-*P*—Psst!—except for those possibilities that we are properly ignoring (Lewis, 1996, 554).

The option of 'properly ignoring possibilities' allows for a spectrum of knowledge contexts from the loose standards of every-day usage (in which, *e.g.*, I know that my cat Possum is not in the study without checking the closed drawers; *cf.* Lewis (1996, 562)) to the demanding standards of epistemology (Cartesian Doubt), in which (almost) all knowledge claims are defeated. Consequently, a switch of context may destroy knowledge. According to Lewis, this both explains the force of skeptical arguments and points a way to a cure.

Lewis's paper and a number of other related works have given rise to a huge debate about details and technicalities of his version of con-textualism, dealing with important questions about the specification of 'properly ignoring possibilities' and the context changes in communicative acts. This paper is not intended to be a direct contribution to that debate.[15]

Contextualism has not been employed in the epistemology of mathematics so far. There is certainly a number of reasons why this is so. For Lewis, the main reason seems to be that he treats all true mathematical statements as the necessary proposition in disguise, thus blocking any way of distinguishing among them epistemologically. This is a consequence of Lewis's modal epistemology: A semantics for knowledge claims for Lewis must be based on possible worlds. As all mathematical statements are true in all possible worlds, modal semantics must treat all mathematical statements as the necessary proposition, modelled as the set of all worlds. As we pointed out in the introduction, given our methodology, we cannot follow Lewis here.[16] It seems obvious to us that Lewis's modal approach to epistemology can be separated from his contextualist stance, and thus we will employ a contextualist analysis of knowledge along Lewisian lines.

Our discussion in § 3 has revealed that the vague terms "could in principle", "generate", and "proof" in (†) need to be interpreted depending on the context. No fixed notion ($\dagger^{\sharp}$) of "could in principle generate a proof" yields an adequate analysis for all cases of mathematical knowledge.

Thus, contextualism wins the day. But how? Our task now is to link the general contextualist analysis of knowledge (#) to the specific case of mathematics where *S*'s evidence and the ignored possibilities must be linked to the proof or other justification that is required according to (†). As we saw, a context generally specifies a type of proof (or other justification) as appropriate. Very few contexts in mathematics demand derivations. Blackboard proofs are typical of research mathematics, and mathematical knowledge claims in the general public typically do not need to be backed by any form of proof at all. Similarly, *S*'s evidence may be interpreted as the dispositional state of mind of *S* with respect to the required form of proof of *P*. Above we gave one explication by linking that disposition to a time frame and other resources that would be required to generate a written version of the proof in question. Thus one way of writing out (#) for the case of mathematics is the following:

(#′) *S knows* that *P* iff *S*'s dispositional state of mind allows her to produce the required form of proof or justification for *P* with the resources allowed by the context.

15. Important recent contributions include MacFarlane (2005), Schaffer (2004), and DeRose (2002).

16. Incidentally, in Lewis (1993, 218), he supports something very close to our methodology, so there may be a slight tension in his position.

This analysis may be all that is needed, but it also comes with a certain problem: There does not seem to be an independent standard from which to assess the allowed resources. Thus, (#′) might be accused of being empirically void. We suggest in § 6 that the notion of mathematical skill can help to improve the analysis.

6. *Mathematical knowledge and mathematical skills*

The notion of mathematical skill links the "dispositional state of mind" of (#′) with actual performance: Skill is both a modal notion (what somebody is able to do even while not doing it) and has an empirical side (skills can be tested). Our motivation for bringing skills into the picture is that through the Dreyfus-Dreyfus model of skill acquisition there is available a semi-formal theory of skill levels that has been fruitfully applied, *e.g.,* to chess skills and nursing skills (Benner, 1984). In the Dreyfus-Dreyfus model (Dreyfus and Dreyfus, 1986), there are five levels of skill ranging from novice to expert. These levels are delineated by their different relation to explicitly formulated rules. While a novice needs to stick to explicit rules in a step-by-step fashion, experts have internalised and transgressed such rules and are able to proceed intuitively.

Certainly the link between mathematical knowledge and mathematical skills merits further investigation, which will need to be left for a future occasion. Here we merely wish to argue that a skill-based analysis is plausible.

Using the notion of skill, we can reformulate (#′), our preliminary synthesis of contextualism (#) and mathematical knowledge (†), as follows:

(‡) *S knows* that *P* iff *S*'s current mathematical skills are sufficient to produce the form of proof or justification for *P* required by the actual context.

This analysis, we claim, is adequate as a general explication of mathematical knowledge. It refers to the actual context and is thus flexible with respect to both crucial aspects of mathematical knowledge: Context determines the required form of proof or other justification, and context also sets the standard for the modal component in terms of a required skill level. Skill levels provide the link of our analysis with independent constraints that was lacking in the case of (#′)—unlike counterfactual time constraints, skill levels can be (and, more importantly: are) characterised independently of the conceptual analysis for mathematical knowledge given in this paper. Mathematical practice affirms that the concept of mathematical skill is well entrenched. It is customary to comment on students' or researchers' skills, and it is often possible to rank people with respect to their skills.[17] Skills are tested in exams and job talks, and it may well be that the aim of mathematics education is best characterised not as instilling mathematical knowledge, but as teaching mathematical skills.

7. *Conclusion*

In this paper we argued that contrary to first appearances, mathematical knowledge is not a fixed, context independent notion. Rather, we showed by appeal to mathematical practice that unless one disregards actual practice—which in our view would be just plain bad methodology—, one is forced to admit that mathematical knowledge is context dependent.

Many accounts of mathematical knowledge refer to the need to have available a proof. We concede that proof plays a crucial role in mathematics and in mathematical knowledge, but there is also mathematical knowledge without proof. Nor is proof a fixed notion: There are various forms of proof, and context determines which type of proof, if proof at all, is required. Furthermore, availability of proof is a modal notion that we suggested is best explained by reference to mathematical skills.

What then of formal derivation? The concept of derivation and its universal acceptance as a formalization of the intuitive notion of proof is important for the foundations of mathematics, but contrary to folklore, it hardly plays any rôle in determining the truth of "*S* knows that *P*"—Psst!—unless the context explicitly demands it.

17. An interesting question which again merits further investigation is the following: How finely do we need to individuate mathematical skills? Will it be enough to ascribe to persons a single "mathematical skill level", or will we need to be more topic-specific, speaking, *e.g.*, of algebraic vs. geometrical skills?

REFERENCES

Benner, Patricia (1984), *From Novice to Expert: Excellence and Power in Clinical Nursing Practice.* Addison-Wesley.

Brouwer, Luitzen E. J. (1929), "Mathematik, Wissenschaft und Sprache". *Monatshefte fur Mathematik und Physik* 36, 153–164.

Bloor, David (1976), *Knowledge and Social Imagery.* University of Chicago Press.

Chihara, Charles (1990), *Constructibility and Mathematical Existence,* Oxford University Press.

DeRose, Keith (2002), "Assertion, Knowledge and Context", *Philosophical Review* 111, 167–203.

Devlin, Keith (2003), "The shame of it, MAA Online: Devlin's Angle", May 2003. URL: http://www.maa.org/devlin/devlin_05_03.html.

Dreyfus, Hubert L. and Dreyfus, Stuart E. (1986), *Mind over Machine: The Power of Human Intuition and Expertise in the Era of the Computer,* Free Press.

Fallis, Don (2003), "Intentional gaps in mathematical proofs", *Synthese* 134, 45–69.

Feit, Walter and Thompson, John G. (1963), "Solvability of groups of odd order". *Pacific Journal of Mathematics* 13, 775–1029.

Frege, Gottlob (1879), *Begriffsschrift.* Halle.

Geuvers, Herman, Wiedijk, Freek and Zwanenburg, Jan (2001), "A Constructive Proof of the Fundamental Theorem of Algebra without using the Rationals". In: Paul Callaghan, Zhaohui Luo, James McKinna and Robert Pollack (eds.), *Types for Proofs and Programs, Proceedings of the International Workshop, TYPES 2000, Durham*, Springer LNCS 2277, 96–111.

Heintz, Bettina (2000), *Die Innenwelt der Mathematik. Zur Kultur und Praxis einer beweisenden Disziplin*. Wien: Springer.

Kitcher, Philip (1984), *The Nature of Mathematical Knowledge*. New York.

Lewis, David (1993), "Mathematics is megethology". *Philosophia Mathematica* 3, 3–23. Reprinted in his *Papers in Philosophical Logic*, Cambridge UP 1998, 203–229. Page references are to the reprint.

— (1996), "Elusive knowledge". *Australasian Journal of Philosophy* 74, 549–567. Reprinted in his *Papers in Metaphysics and Epistemology*, Cambridge UP 1999. Page references are to the original.

— (2000), "Academic appointments: Why ignore the advantage of being right?" In his *Papers in Ethics and Social Philosophy*, Cambridge University Press, 187–200.

Livingston, Eric (1986), T*he Ethnomathematical Foundations of Mathematics*. London: Routledge.

Locke, John (1689), *An Essay Concerning Human Understanding*. Ed. by P. H. Nidditch. Oxford University Press 1975.

MacFarlane, John (2005), “The assessement sensitivity of knowledge attributions”. In: T.S. Gendler and J. Hawthorne (eds.), *Oxford Studies in Epistemology*, Oxford.

Prediger, Susanne (2001), “Mathematik als kulturelles Produkt menschlicher Denktätigkeit und ihr Bezug zum Individuum”. In K. Lengnink, S. Prediger, F. Siebel (eds.): *Mathematik und Mensch. Sichtweisen der Allgemeinen Mathematik, Darmstädter Schriften zur Allgemeinen Wissenschaft* 2, Verlag Allgemeine Wissenschaft, Mühltal 2001, 21–36.

Pólya, George (1954), *Induction and Analogy in Mathematics*, Princeton University Press.

Rav, Yehuda (1999), “Why do we prove theorems?” *Philosophia Mathematica* (3) 7, 5–41.

Schaffer, Jonathan (2004), “Skepticism, contextualism, and discrimination”. *Philosophy and Phenomenological Research* 69, 138–155.

Steiner, Mark (1975), *Mathematical Knowledge*, Cornell University Press.

Troelstra, Anne S. (1982), “On the origin and development of Brouwer’s concept of choice sequence”. In: A. Troelstra and D. van Dalen (eds.), *The L. E. J. Brouwer Centenary Symposium*, Amsterdam: North Holland, 465–486.

Thurston, William P. (1994), “On proof and progress in mathematics”. *Bulletin of the American Mathematical Society* 30, 161–177.