Universität Konstanz

Inhomogeneous symbols, the Newton polygon, and maximal L^p-regularity

Robert Denk
Jürgen Saal
Jörg Seiler

Konstanzer Schriften in Mathematik und Informatik
Nr. 244, März 2008
ISSN 1430-3558

INHOMOGENEOUS SYMBOLS, THE NEWTON POLYGON, AND MAXIMAL L^p-REGULARITY

ROBERT DENK, JÜRGEN SAAL, AND JÖRG SEILER

Dedicated to the memory of Leonid Romanovich Volevich

Abstract. We prove a maximal regularity result for operators corresponding to rotation invariant (in space) symbols which are *inhomogeneous* in space and time. Symbols of this type frequently arise in the treatment of half-space models for (free) boundary value problems. The result is obtained by extending the Newton polygon approach to variables living in complex sectors and combining it with abstract results on $\mathcal{H}^\infty$-calculus and $\mathcal{R}$-bounded operator families. As an application we derive maximal regularity for the linearized Stefan problem with Gibbs-Thomson correction.

1. Introduction

In the theory of parabolic partial differential equations, Sobolev spaces connected to the Newton polygon appear in a natural way if the underlying symbol structure has an inherent inhomogeneity. A prominent example is the symbol $P(\xi,\lambda)=\lambda+|\xi|^2\sqrt{\lambda+|\xi|^2}$ which arises in the analysis of the Stefan problem with Gibbs-Thomson correction (cf. [11], see also Section 5 of this paper). The symbol $P(\xi,\lambda)$ is not (quasi-)homogeneous in ξ and λ which implies that standard parameter-elliptic and parabolic estimates are not available.

Typical examples of equations with inhomogeneous symbol structure are mixed-order systems ([17], [6]), free boundary value problems (see, e.g., [23], [22] for the Cahn-Hilliard equation) and boundary value problems with dynamic boundary conditions ([10], [8]). A general approach for such equations is the Newton polygon method which was developed by Gindikin and Volevich ([12], [13]). It turns out that it is possible to establish a new notion of parameter-ellipticity and parabolicity which is in fact *equivalent* to uniform a priori-estimates and maximal regularity in L^2-spaces. For results in this direction and general discussion of the Newton polygon, see also [6], [27], [9] and the references therein. The resulting class of equations were called N-elliptic with parameter and N-parabolic, respectively.

However, to our knowledge there exist no general L^p-results on N-parabolicity. For applications to nonlinear equations, as in the case of the Stefan problem, L^p-theory is necessary. The present paper establishes the first steps in this direction.

The main result states that N-parabolic scalar operators have maximal regularity in classes of L^p-Sobolev spaces anisotropic in space and time. Here maximal regularity means that the operator induces an isomorphism between the Sobolev spaces corresponding to the data and the solution of the equation. Due to the inhomogeneity of the operator, the Sobolev spaces under consideration have an inhomogeneous structure, too.

We point out that many results are known for quasi-homogeneous symbol structures and related Sobolev spaces, the simplest example being the heat equation with symbol $\lambda+|\xi|^2$ and the related solution space $W_p^1((0,T),L^p(\mathbb{R}^n))\cap L^p((0,T),W_p^2(\mathbb{R}^n))$. In contrast to this space, the Sobolev spaces considered in the present paper in general have neither homogeneity nor quasi-homogeneity with respect to time and space derivatives.

Contrary to L^2-theory, maximal regularity for L^p-Sobolev spaces does not follow directly from symbol estimates. We have to deal in a natural way with vector-valued spaces as, for instance, $W_p^s((0,T),L^p(\mathbb{R}^n))$ where Mikhlin's theorem cannot be applied. This difficulty can be overcome by the concept of $\mathcal{R}$-boundedness and $\mathcal{R}$-sectorial operators, see [18], [5]. We will briefly recall this concept in Section 2.

The operators under consideration will have rotation invariant symbols in space, i.e., they can be considered as a function of the Laplacian (more precisely, of the square root of the negative Laplacian). Observe that this holds for all examples mentioned before. To prove maximal regularity, we will essentially use $\mathcal{H}^\infty$-calculus for the negative Laplacian and the time derivative operator and apply an abstract result on joint $\mathcal{H}^\infty$-calculus due to Kalton and Weis [16]. In fact, this method works for general resolvent commuting operators admitting a bounded $\mathcal{H}^\infty$-calculus, and we will formulate our main result in this setting (Theorem 3.2 below). The results are proved simultaneously for both scales of spaces, Sobolev-Slobodeckij and Bessel potential.

In applications to boundary value problems, inhomogeneous scalar symbols often arise as the determinant of the Lopatinskii matrix related to the problem, see e.g. [22]. Therefore, the question of trace spaces of the L^p-Sobolev spaces related to the Newton polygon appears. For $p=2$, this question was answered in [7]: if the Sobolev space in the interior of the domain is defined by the Newton polygon N then the trace space is defined by a shifted version of N with shift length $\frac{1}{2}$. It turns out that a similar result holds in the L^p-case where now the shift has length $\frac{1}{p}$. The precise description of the trace space can be found in Theorem 4.1. In the proof we give an explicit construction of a right inverse to the trace operator.

The paper is organized as follows. In Section 2, we give some remarks on L^p-Sobolev spaces with exponential weight in time and summarize basic facts on $\mathcal{R}$-boundedness and $\mathcal{H}^\infty$-calculus including the properties of the Laplacian and the time derivative needed in what follows. Section 3 contains the first main result in Theorems 3.2 and 3.3 which states that an N-parabolic operator induces an isomorphism on the related inhomogeneous L^p-Sobolev spaces, that is, on its natural domain. In this context we also slightly generalize a result of [6], which gives an equivalent description of N-parabolicity. In Section 4 we deal with the trace spaces connected to the Newton polygon. The description of these spaces can be found in Theorem 4.1, the second main result of this article. In the final Section 5, we apply these results to the Stefan problem, demonstrating the usefulness of these concepts for linear and nonlinear parabolic partial differential equations.

2. Function spaces, $\mathcal{R}$-boundedness, and $\mathcal{H}^\infty$-calculus

Let us fix the notation used throughout this paper. First we introduce suitable function spaces. Let $\Omega\subseteq\mathbb{R}^m$ be open and X be an arbitrary Banach space. By $L_p(\Omega,X)$ and $H_p^k(\Omega,X)$, for $1\le p\le\infty$, $k\in\mathbb{N}$, we denote the X-valued Lebesgue and the Sobolev space of order k, respectively. We will also frequently make use of the fractional Sobolev-Slobodeckij spaces $W_p^s(\Omega, X)$, $1 \leq p < \infty$, $s \in \mathbb{R}$, which are defined by $W_p^s(\Omega, X) := B_{pp}^s(\Omega, X)$ where $B_{pp}^s(\Omega, X)$ stands for the vector-valued Besov space. For a definition and basic facts on vector-valued Besov spaces, we refer to [24]. We will only consider the cases $\Omega = \mathbb{R}^n$ and $\Omega = J$ where $J \subset \mathbb{R}$ is an interval.

For $s > 0$ an equivalent norm in $W_p^s(\Omega, X)$ is given by

$$\|g\|_{W_p^s(\Omega,X)} = \|g\|_{W^{[s],p}(\Omega,X)} + \langle\langle g\rangle\rangle_{s-[s],p,X}, \tag{2.1}$$

where

$$\langle\langle g\rangle\rangle_{s-[s],p,X} := \sum_{|\alpha|=[s]} \left(\int_\Omega \int_\Omega \frac{\|\partial^\alpha g(x) - \partial^\alpha g(y)\|_X^p}{|x-y|^{n+(s-[s])p}} \mathrm{d}x\mathrm{d}y \right)^{1/p},$$

and where $[s]$ denotes the largest integer smaller than s. Let $T \in (0, \infty]$ and $J = (0, T)$. The zero time trace version of $W_p^s(J, X)$ at $t = 0$ is defined as

$$ {}_0W_p^s(J,X) := \begin{cases} \{u \in W_p^s(J,X) : u(0) = u'(0) = \ldots = u^{(k)}(0) = 0\}, \\ \qquad\qquad \text{if} \quad k + \frac{1}{p} < s < k + 1 + \frac{1}{p},\ k \in \mathbb{N} \cup \{0\}, \\ W_p^s(J,X), \quad \text{if} \quad 0 < s < \frac{1}{p}. \end{cases} \tag{2.2}$$

Next we collect some basic facts on corresponding spaces with exponential weight $e^{-\rho p t}$. Recall that for $m \in \mathbb{N}_0$ the weighted Sobolev space is defined by

$$H_{p,\rho}^m(J,X) := \left\{ u \in \mathcal{D}'(J,X) : \Psi_\rho \left(\frac{d}{dt}\right)^k u \in L^p(J,X) \quad (0 \leq k \leq m) \right\}$$

with canonical norm

$$\|u\|_{H_{p,\rho}^m(J,X)} := \left(\sum_{k=0}^m \left\| \Psi_\rho \left(\frac{d}{dt}\right)^k u \right\|_{L^p(J,X)}^p \right)^{1/p},$$

where the operator Ψ_ρ is defined by multiplication with $e^{-\rho t}$, that is,

$$\Psi_\rho u(t) := e^{-\rho t} u(t), \quad t \in J. \tag{2.3}$$

For $s \in \mathbb{R}_+$ we define the Bessel potential and Sobolev-Slobodeckij spaces by complex and real interpolation, respectively. To be precise, for $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and integer $m > s$ we set

$$H_{p,\rho}^s(J,X) := \left[L_\rho^p(J,X), H_{p,\rho}^m(J,X)\right]_{s/m}, \tag{2.4}$$

and for $s \in \mathbb{R}_+$ and integer $m > s$ we set

$$W_{p,\rho}^s(J,X) := \left(L_\rho^p(J,X), H_{p,\rho}^m(J,X)\right)_{s/m,p}. \tag{2.5}$$

In Lemma 2.1 we will see that under suitable assumptions on X, the right-hand sides do not depend on the choice of m. Moreover, (2.4) holds also in the case $s \in \mathbb{N}$. The corresponding spaces with zero time trace at the origin ${}_0H_{p,\rho}^s(J,X)$ and ${}_0W_{p,\rho}^s(J,X)$ are defined analogously to (2.2). The results proved in this paper are obtained simultaneously for both types of spaces, Bessel potential and Sobolev-Slobodeckij. This motivates the introduction of the following notation: let $r \in \mathbb{R}$, $s \geq 0$ and

$$\mathcal{F}, \mathcal{K} \in \{H, W\}.$$

Then by $\mathcal{K}^r_{p,\rho}$ we either mean the space $H^r_{p,\rho}$ or the space W^r_p, whereas $\mathcal{F}^s_{p,\rho}(J, \mathcal{K}^r_p(\Omega))$ represents an element of the set

$$\left\{W^s_{p,\rho}\left(J, W^r_p(\Omega)\right), W^s_{p,\rho}\left(J, H^r_p(\Omega)\right), H^s_{p,\rho}\left(J, W^r_p(\Omega)\right), H^s_{p,\rho}\left(J, H^r_p(\Omega)\right)\right\}.$$

This holds for all $s > 0$ and $r \in \mathbb{R}$. In the case $s = 0$, however, we always assume $\mathcal{F} = H$, so we will not consider the case $W^0_p(J, X) = B^0_{pp}(J, X)$.

Tacitly and without any further explanations in this note we make use of the following facts.

2.1. **Lemma.** *Let $1 < p < \infty$, $s, s_1, s_2 \geq 0$, $\rho \geq 0$ such that $s_2 > s_1$, X be a UMD space (see the lines below this lemma for the definition), and $J \subseteq \mathbb{R}$ be an interval such that $J \subseteq [0, \infty)$ if $\rho > 0$. Then the space $\mathcal{F}^s_{p,\rho}(J, X)$ is well-defined and we have*

$$\begin{aligned}
\left[\mathcal{F}^{s_1}_{p,\rho}(J, X), \mathcal{F}^{s_2}_{p,\rho}(J, X)\right]_\theta &= \mathcal{F}^s_{p,\rho}(J, X), \quad s = s_1 + \theta(s_2 - s_1), \\
\left(\mathcal{F}^{s_1}_{p,\rho}(J, X), \mathcal{F}^{s_2}_{p,\rho}(J, X)\right)_{\theta,p} &= W^s_{p,\rho}(J, X), \quad s = s_1 + \theta(s_2 - s_1).
\end{aligned}$$

The assertions remain valid if $\mathcal{F}$ is replaced by ${}_0\mathcal{F}$.

Proof. First observe that for $m, \ell \in \mathbb{N}_0$ it is well-known that

$$[L^p(I, X), W^{m,p}(I, X)]_{\ell/m} = W^{\ell,p}(I, X). \tag{2.6}$$

(For $J = \mathbb{R}$ this, e.g., is a consequence of $\partial_t \in \mathcal{H}^\infty(H^k_p(J, X))$, which is shown in the first part of the proof of Proposition 2.7. The case of general J then easily follows by an extension and restriction argument.) Relation (2.6) remains true for the spaces with weight $e^{-\rho p t}$, since $\Psi_\rho : H^k_{p,\rho}(I, X) \to H^k_p(I, X)$ is an isomorphism for all $k \in \mathbb{N}_0$. Note that the UMD property of X implies the space $\mathcal{F}^s_{p,\rho}(I, X)$ to be reflexive. But then the assertion follows by the reiteration theorem for complex and real interpolation functors, respectively by the following two mixed reiteration results valid for reflexive interpolation couples E, F:

$$\begin{aligned}
[(E, F)_{\theta_0,p}, (E, F)_{\theta_1,p}]_\sigma &= (E, F)_{\theta,p}, \\
([E, F]_{\theta_0}, [E, F]_{\theta_1})_\sigma &= (E, F)_{\theta,p},
\end{aligned}$$

where $1 < p < \infty$, $0 < \theta_0 < \theta_1 < 1$, $0 < \sigma < 1$ such that $\theta = (1 - \sigma)\theta_0 + \sigma\theta_1$ (cf. [26, page 66], see also [19], [15]). □

Recall that a Banach space X is UMD, or equivalently of class $\mathcal{HT}$, if the Hilbert transform $\mathscr{F}^{-1}[i\xi/|\xi|]\mathscr{F}$ acts as a bounded operator on $L^p(\mathbb{R}, X)$ for some (and therefore all) $p \in (1, \infty)$, where $\mathscr{F}$ denotes the Fourier transform. Note that the reflexive Lebesgue, Sobolev, Sobolev-Slobodeckij, Besov, and Bessel potential spaces are known to enjoy this property. Furthermore, if X is UMD, an easy argument based on Fubini's theorem shows that also $W^{k,p}_\rho(\Omega, X)$ for $k \in \mathbb{N}$ and $1 < p < \infty$ is UMD. By an interpolation argument this property transfers to the space $\mathcal{F}^s_{p,\rho}(\Omega, X)$ for $s, \rho \geq 0$ and $1 < p < \infty$. Therefore all spaces used in this paper are UMD.

Also the next lemma is quite standard, hence we omit its proof.

2.2. **Lemma.** *Let* $1 < q < \infty$, $s, \rho, \omega \geq 0$, *and* X *be an UMD space. Further, let* $\mathcal{F} \in \{H, W\}$, $T \in (0, \infty)$, *and* $J \subseteq \mathbb{R}$ *be an interval such that* $J = (0, T)$ *if* $\rho > 0$. *We have that*

(i) $\|\cdot\|_{L^p_\rho(J,X)} \leq \|\cdot\|_{L^p_\omega(J,X)} \leq e^{(\rho-\omega)T} \|\cdot\|_{L^p_\rho(J,X)} \quad (T > 0, 0 \leq \omega \leq \rho)$,

(ii) $\Psi_\rho \in \mathrm{Isom}(\mathcal{F}^s_{p,\rho}((0,T_0),X), \mathcal{F}^s_p((0,T_0),X))$ *for each* $T_0 \in (0, \infty]$. *Furthermore, the norms*

$$\|\cdot\|_{W^s_{p,\rho}((0,T_0),X)}, \ \|\Psi_\rho \cdot\|_{W^s_p((0,T_0),X)},$$
$$\text{and } \|\cdot\|_{H^{[s]}_{p,\rho}((0,T_0),X)} + \langle\langle \Psi_\rho (d/dt)^{[s]} u \rangle\rangle_{s-[s],p,X}$$

are equivalent,

(iii) $\mathcal{F}^s_{p,\rho}(J, X) = \mathcal{F}^s_p(J, X)$ *for* $T < \infty$ *with equivalent norms,*

(iv) $\mathcal{F}^s_{p,\omega}(\mathbb{R}_+, X) \hookrightarrow \mathcal{F}^s_{p,\rho}(\mathbb{R}_+, X)$ *for* $0 \leq \omega \leq \rho$,

(v) *there exists a bounded extension operator*

$$E : \mathcal{F}^s_{p,\rho}(J, X) \to \mathcal{F}^s_{p,\omega}(\mathbb{R}_+, X)$$

simultaneously for all $1 < p < \infty$, $s, \rho, \omega \geq 0$, *and UMD spaces* X,

(vi) *statements* (i) *to* (v) *remain valid if* $\mathcal{F}$ *is replaced by* ${}_0\mathcal{F}$.

Next we clarify the notions of $\mathcal{R}$-boundedness and $\mathcal{H}^\infty$-calculus. Let X, Y be Banach spaces. By $\mathscr{L}(X, Y)$ we denote the class of all bounded operators from X to Y. The class $\mathrm{Isom}(X, Y) \subseteq \mathscr{L}(X, Y)$ denotes the subclass of isomorphisms. If $X = Y$ we write shortly $\mathscr{L}(X)$ and $\mathrm{Isom}(X)$.

2.3. **Definition.** *A family* $\mathcal{T} \subseteq \mathscr{L}(X, Y)$ *is called* $\mathcal{R}$-bounded, *if there exist a* $C > 0$ *and a* $p \in (1, \infty)$ *such that for all* $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$, *and all independent symmetric* $\{-1, 1\}$*-valued random variables* ε_j *on a probability space* $(\Omega, \mathscr{M}, \mu)$ *for* $j = 1, \dots, N$ *we have that*

$$\left\| \sum_{j=1}^N \varepsilon_j T_j x_j \right\|_{L^p(\Omega, Y)} \leq C \left\| \sum_{j=1}^N \varepsilon_j x_j \right\|_{L^p(\Omega, X)}. \tag{2.7}$$

The smallest C *such that (2.7) holds is called* $\mathcal{R}$*-bound of the family* $\mathcal{T}$ *and denoted by* $\mathcal{R}(\mathcal{T})$.

It is easy to see that $\mathcal{R}$-boundedness implies uniform boundedness. Note that the converse in general is only true in Hilbert spaces. We refer to [2] and [5] for a comprehensive introduction to the notion of $\mathcal{R}$-bounded operator families.

We denote the domain and the range of an operator A in X by $\mathscr{D}(A)$ and $\mathscr{R}(A)$ respectively. A *sectorial* operator here we define as follows:

2.4. **Definition.** *A closed operator* A *on a complex (or real) Banach space* X *is called sectorial, if it is injective,* $\overline{\mathscr{D}(A)} = \overline{\mathscr{R}(A)} = X$, $(-\infty, 0) \subset \rho(A)$, *and, if there is some* $C \geq 0$ *such that* $\|\lambda(\lambda + A)^{-1}\| \leq C$ *for all* $\lambda > 0$.

In this case (Taylor expansion) there is some $\phi \in (0, \pi)$ and a C_ϕ such that the sector

$$\Sigma_{\pi-\phi} := \{z \in \mathbb{C} \setminus \{0\} : \ |\arg z| < \pi - \phi\}$$

is contained in $\rho(-A)$, and such that $\sup\left\{\|\lambda(\lambda+A)^{-1}\| : \lambda \in \Sigma_{\pi-\phi}\right\} \leq C_\phi$. The infimum of all such ϕ is called the *spectral angle* of A and is denoted by ϕ_A. Observe that $\sigma(A) \setminus \{0\} \subset \Sigma_{\phi_A}$. Moreover, if A is sectorial and $\phi_A \leq \frac{\pi}{2}$, $-A$ generates a bounded and holomorphic C_0-semigroup on X. If additionally the above set is $\mathcal{R}$-bounded, i.e., if there is a $\varphi \in (0,\pi)$ such that

$$\mathcal{R}\left\{\lambda(\lambda+A)^{-1} : \ \lambda \in \Sigma_{\pi-\varphi}\right\} < \infty, \tag{2.8}$$

then A is called *$\mathcal{R}$-sectorial.* The infimum over all φ such that (2.8) holds is called the $\mathcal{R}$-angle of A and denoted by $\phi_A^{\mathcal{R}}$. Since the result of Weis [28], it is well-known that $\mathcal{R}$-sectoriality with $\phi_A^{\mathcal{R}} < \pi/2$ is equivalent to the important maximal regularity if the underlying Banach space X is a UMD space. In particular it implies

$$(\partial_t + A) \in \operatorname{Isom}\left(W^{1,p}((0,T),X) \cap L^p((0,T),\mathscr{D}(A)),\ L^p((0,T),X)\right),$$

for $T > 0$, $1 < p < \infty$.

A special class of sectorial operators which will frequently appear throughout this article is the set of operators admitting a *bounded $\mathcal{H}^\infty$-calculus.* In order to recall this notion, which goes back to McIntosh (see [20], [3]), we define for $\phi \in (0,\pi)$ the space

$$\mathcal{H}^\infty(\Sigma_\phi) := \{h : \Sigma_\phi \to \mathbb{C} : h \text{ is holomorphic and bounded}\}$$

equipped with the norm $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Sigma_\phi)}$ as well as its subspace $\mathcal{H}_0^\infty(\Sigma_\phi)$ given by

$$\mathcal{H}_0^\infty(\Sigma_\phi) := \left\{h \in \mathcal{H}^\infty(\Sigma_\phi) : |h(z)| \leq C\frac{|z|^s}{1+|z|^{2s}} \text{ for some } C \geq 0, s > 0\right\}. \tag{2.9}$$

Let A be a sectorial operator on X with spectral angle ϕ_A, and let $\phi \in (\phi_A, \pi)$ and $\theta \in (\phi_A, \phi)$. The path

$$\Gamma : \mathbb{R} \to \mathbb{C}, \quad \gamma(r) := \begin{cases} -r\mathrm{e}^{i\theta} & , r < 0, \\ r\mathrm{e}^{-i\theta} & , r \geq 0, \end{cases} \tag{2.10}$$

stays in the resolvent set of A with the only possible exception at $r = 0$. In view of Cauchy's integral formula, for $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$, we define $h(A)$ by the Bochner integral

$$h(A) := \frac{1}{2\pi i}\int_\Gamma h(\lambda)(\lambda - A)^{-1}\mathrm{d}\lambda, \tag{2.11}$$

which gives rise to a bounded operator on X in view of (2.9). Observe that the map

$$\Phi_A : \mathcal{H}_0^\infty(\Sigma_\phi) \to \mathscr{L}(X), \quad h \mapsto h(A),$$

is an algebra homomorphism.

2.5. **Definition.** *We say that A admits a bounded $\mathcal{H}^\infty$-calculus, if Φ_A is bounded.*

The class of all operators having this property we denote by $\mathcal{H}^\infty(X)$ and the infimum of all angles ϕ such that Φ_A is bounded is called *$\mathcal{H}^\infty$-angle* and denoted by ϕ_A^∞. Now put $g(z) := z(1+z)^{-2}$ and let $h \in \mathcal{H}^\infty(\Sigma_\phi)$. Then $g, g \cdot h \in \mathcal{H}_0^\infty(\Sigma_\phi)$ and we may set

$$h(A) = (hg)(A)g(A)^{-1},$$

initially defined on the dense subspace $\mathscr{D}(A) \cap \mathscr{R}(A)$ of X. It is easily checked that this definition coincides with the former one in case that $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$. Furthermore, the set $\mathcal{H}_0^\infty(\Sigma_\phi)$ is dense in $\mathcal{H}^\infty(\Sigma_\phi)$ with respect to the topology induced by local uniform convergence. This implies that Φ_A extends to a bounded algebra homomorphism from $\mathcal{H}^\infty(\Sigma_\phi)$ to $\mathscr{L}(X)$. Observe that it is well known that $\mathcal{H}^\infty(X)$ is contained in the class of $\mathcal{R}$-sectorial operators (see [5]).

In analogy to the definition of $\mathcal{R}$-sectoriality we say that an operator A admits an *$\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus*, if there is a $\varphi \in (0,\pi)$ such that

$$\mathcal{R}\left\{h(A):\ h \in \mathcal{H}^\infty(\Sigma_\varphi),\ \|h\|_{L^\infty(\Sigma_\varphi)} \le 1\right\} < \infty, \tag{2.12}$$

and write $A \in \mathcal{R}\mathcal{H}^\infty(X)$. The infimum of all angles such that (2.12) holds is called *$\mathcal{R}$-$\mathcal{H}^\infty$-angle* and denoted by $\phi_A^{\mathcal{R},\infty}$. The relation between the different angles which appeared for a sectorial operator A in the definitions above is

$$\phi_A \le \phi_A^{\mathcal{R}} \le \phi_A^\infty \le \phi_A^{\mathcal{R},\infty}. \tag{2.13}$$

Another notion which will appear in the next proposition is the so-called *property α*.

2.6. **Definition.** *A Banach space X is said to have property α, if there exists a $C > 0$ and a $p \in (1,\infty)$ such that for all $N \in \mathbb{N}$, $a_{jk} \in \mathbb{C}$ with $|\alpha_{jk}| \le 1$, $x_{jk} \in X$, and all independent symmetric $\{-1,1\}$-valued random variables ε_j^1 on a probability space $(\Omega_1, \mathscr{M}_1, \mu_1)$ and ε_k^2 on a probability space $(\Omega_2, \mathscr{M}_2, \mu_2)$ for $j,k = 1,\dots,N$, we have that*

$$\left\| \sum_{j,k=1}^N \varepsilon_j^1 \varepsilon_k^2 a_{jk} x_{jk} \right\|_{L^p(\Omega_1\times\Omega_2,X)} \le C \left\| \sum_{j,k=1}^N \varepsilon_j^1 \varepsilon_k^2 x_{jk} \right\|_{L^p(\Omega_1\times\Omega_2,X)}. \tag{2.14}$$

By the orthogonality of the random variables it is easy to see that Hilbert spaces enjoy this property. Moreover, Fubini's theorem implies that $H_p^k(\Omega)$ for $1 \le p < \infty$ and $k \in \mathbb{N}_0$ has property α. Furthermore, in view of the fact that property α is stable under interpolation, it can be shown that all Bessel potential and Sobolev-Slobodeckij spaces used in this note enjoy property α, too. By similar arguments as for the property UMD, it can be seen that $\mathcal{F}_{p,\rho}^s(\Omega, X)$ enjoys property α, if X does so. Therefore, all spaces appearing in this articles have this property. Note that, compared to UMD, the condition of property α is relatively weak. For instance, UMD implies reflexive, whereas the space $L^1(\Omega)$ still enjoys property α. However the two properties are completely independent, i.e., neither one implies the other.

We refer to [5] and [16] for more on $\mathcal{H}^\infty$-calculus, property α, and relations between the notions appearing above.

Two important examples admitting an $\mathcal{H}^\infty$-calculus are in order. First we consider the time derivative operator

$$Gu = \frac{d}{dt}u, \quad u \in \mathscr{D}(G) := {}_0\mathcal{F}_{p,\rho}^{s+1}(\mathbb{R}_+, X) \tag{2.15}$$

in the space ${}_0\mathcal{F}_{p,\rho}^s(\mathbb{R}_+, X)$.

2.7. **Proposition.** *Let $1 < p < \infty$, $s, \rho \ge 0$, $\mathcal{F} \in \{H, W\}$, and X be a UMD space. Then we have $G \in \mathcal{H}^\infty({}_0\mathcal{F}_{p,\rho}^s(\mathbb{R}_+, X))$ with $\mathcal{H}^\infty$-angle $\phi_G^\infty = \pi/2$.*
If X additionally has property α, then we even have $G \in \mathcal{R}\mathcal{H}^\infty({}_0\mathcal{F}_{p,\rho}^s(\mathbb{R}_+, X))$, i.e., G admits an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus on ${}_0\mathcal{F}_{p,\rho}^s(\mathbb{R}_+, X)$ with $\mathcal{R}$-$\mathcal{H}^\infty$-angle $\phi_G^{\mathcal{R},\infty} = \pi/2$.

2.8**. Remark.** *Observe that it would be sufficient to show $G \in \mathcal{H}^\infty({}_0\mathcal{F}^s_{p,\rho}(\mathbb{R}_+, X))$. This is a consequence of the fact that an $\mathcal{H}^\infty$-calculus is equivalent to an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus on Banach spaces enjoying property α, see* [16, Theorem 5.3]. *But, since the expenditure is quite the same, we prove here directly G to admit an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus.*

Proof. First we consider the operator $\widetilde{G} = d/dt$ in the space $Y := H^k_p(\mathbb{R}, X)$. Let $\phi \in (\pi/2, \pi)$ and $h \in \mathcal{H}^\infty_0(\Sigma_\phi)$. Formally we obtain

$$\begin{aligned}(\mathscr{F} h(\widetilde{G})f)(\tau) &= \frac{1}{2\pi i}\int_\Gamma h(\lambda)(\mathscr{F}(\lambda - \widetilde{G})^{-1} f)(\tau)\mathrm{d}\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma h(\lambda)(\lambda - i\tau)^{-1}\widehat{f}(\tau)\mathrm{d}\lambda \\ &= h(i\tau)\widehat{f}(\tau), \quad \tau \in \mathbb{R},\ f \in Y,\end{aligned}$$

where the path Γ is chosen as

$$\Gamma = \{re^{i\theta} :\ 0 \le r < \infty\} \cup \{re^{-i\theta} :\ 0 < r < \infty\}$$

passing from $\theta\infty$ to $-\theta\infty$ for some $\theta \in (\pi/2, \phi)$. In order to prove $\tau \mapsto h(i\tau)$ to be a multiplier on Y, we have to show that the set

$$\Big\{\tau^\ell (d/d\tau)^\ell h(i\tau) : \tau \in \mathbb{R} \setminus \{0\},\ \ell = 0, 1\Big\}$$

is $\mathcal{R}$-bounded in $\mathscr{L}(X)$. In view of the second statement in the theorem we will show that even the set

$$M := \Big\{\tau^\ell (d/d\tau)^\ell h(i\tau) : \tau \in \mathbb{R} \setminus \{0\},\ \ell = 0, 1,\ h \in \mathcal{H}^\infty_0(\Sigma_\phi),\ \|h\|_\infty \le 1\Big\}$$

is $\mathcal{R}$-bounded in $\mathscr{L}(X)$. To this end set $r(\tau) := |\tau|\sin(\phi - \pi/2)/2$, $\tau \in \mathbb{R}$. Then the ball $B_{r(\tau)}(i\tau)$ lies completely in the sector Σ_ϕ. By Cauchy's formula this implies that

$$|\frac{d}{d\tau}h(i\tau)| \le \frac{1}{r(\tau)}\max_{|z|=r(\tau)}|h(z)| \le C_\phi \frac{\|h\|_{L^\infty(\Sigma_\phi)}}{|\tau|} \le C_\phi \frac{1}{|\tau|} \qquad (\tau \in \mathbb{R} \setminus \{0\}).$$

Thus, the set M is uniformly bounded. By Kahane's contraction principle (see [5]) this implies the $\mathcal{R}$-boundedness of M. Indeed, if $N \in \mathbb{N}$, $\tau_j \in \mathbb{R}\setminus\{0\}$, $\ell_j \in \{0, 1\}$, $h_j \in \mathcal{H}^\infty_0(\Sigma_\phi)$ such that $\|h_j\|_\infty \le 1$, $x_j \in X$, and ε_j are independent symmetric $\{-1, 1\}$-valued random variables on a probability space $(\Omega, \mathscr{M}, \mu)$ for $j = 1, \dots, N$, we obtain by Kahane that

$$\begin{aligned}\Big\|\sum_{j=1}^N \varepsilon_j \tau_j^{\ell_j}(d/d\tau)^{\ell_j} h(i\tau)x_j\Big\|_{L^p(\Omega,X)} &\le 2\Big\|\sum_{j=1}^N \varepsilon_j C_\phi \|h_j\|_\infty x_j\Big\|_{L^p(\Omega,X)} \\ &\le 2C_\phi\Big\|\sum_{j=1}^N \varepsilon_j x_j\Big\|_{L^p(\Omega,X)}.\end{aligned}$$

Consequently, we deduce for the $\mathcal{R}$-bound that

$$\mathcal{R}(M) \le 2C_\phi.$$

By the operator-valued version of Mikhlin's multiplier result of Weis [28] this yields the uniform boundedness of the set

$$M_1 := \left\{ h(\widetilde{G}) : \ h \in \mathcal{H}_0^\infty(\Sigma_\phi), \ \|h\|_\infty \leq 1 \right\}$$

in $\mathscr{L}(L^p(\mathbb{R}, X))$, consequently $\widetilde{G} \in \mathcal{H}^\infty(Y)$. Since $\phi > \pi/2$ was arbitrary, we also have $\phi_{\widetilde{G}}^\infty \leq \pi/2$. On the other hand $\widetilde{G}$ is the generator of the translation group on $H_p^k(\mathbb{R}, X)$, which implies that $\phi_{\widetilde{G}} = \pi/2$. Relation (2.13) then yields $\phi_{\widetilde{G}}^\infty = \pi/2$. If X additionally admits property α then the result of Weis in the form as given in [14] even yields the set M_1 to be $\mathcal{R}$-bounded in $\mathscr{L}(L^p(\mathbb{R}, X))$. Since $\mathcal{R}$-boundedness is preserved with respect to the strong operator topology we also have $\mathcal{R}(\overline{M_1}) < \infty$ and therefore that

$$\widetilde{G} \in \mathcal{R}\mathcal{H}^\infty(Y), \quad \phi_{\widetilde{G}}^{\mathcal{R},\infty} = \pi/2.$$

Next consider G in ${}_0H_p^k(\mathbb{R}_+, X)$ for $k \in \mathbb{N}_0$. Observe that

$$(\lambda - G)^{-1} = r(\lambda - \widetilde{G})^{-1} E_0, \quad -\lambda \in \Sigma_{\varphi_0}, \ \varphi_0 \in (0, \pi/2), \tag{2.16}$$

where $r : \mathbb{R} \to \mathbb{R}_+$ denotes the restriction operator and $E_0 : \mathbb{R}_+ \to \mathbb{R}$ the extension by zero. Indeed, by the representation

$$(\lambda - \widetilde{G})^{-1} f = \int_{-\infty}^t e^{\lambda(t-s)} f(s) \mathrm{d}s$$

it can be easily seen that $r(\lambda - \widetilde{G})^{-1} E_0 f \in \mathscr{D}(G)$ for all $f \in C_c^\infty(\mathbb{R}_+, X)$. This implies

$$(\lambda - G) r (\lambda - \widetilde{G})^{-1} E_0 f = f \tag{2.17}$$

and

$$r(\lambda - \widetilde{G})^{-1} E_0 (\lambda - G) f = f \tag{2.18}$$

for all $f \in C_c^\infty(\mathbb{R}_+, X)$. The fact that $C_c^\infty(\mathbb{R}_+, X)$ lies dense in ${}_0H_p^m(\mathbb{R}_+, X)$ for all $m \in \mathbb{N}_0$ and $1 < p < \infty$ then shows that (2.17) and (2.18) remain valid for all $f \in {}_0H_p^k(\mathbb{R}_+, X)$ or $f \in \mathscr{D}(G)$ respectively. By virtue of

$$h(G) f = r h(\widetilde{G}) E_0 f \quad (f \in {}_0H_p^k(\mathbb{R}_+, X))$$

we obtain $G \in \mathcal{H}^\infty({}_0H_p^k(\mathbb{R}_+, X))$ with $\phi_G^\infty = \pi/2$, and, if X has property α, even that $G \in \mathcal{R}\mathcal{H}^\infty({}_0H_p^k(\mathbb{R}_+, X))$ with $\phi_G^{\mathcal{R},\infty} = \pi/2$.

Now, for $\rho \geq 0$ let Ψ_ρ be the operator as given in (2.3) and recall that by Lemma 2.2 (ii) $\Psi_\rho : {}_0H_{p,\rho}^k(\mathbb{R}_+, X) \to {}_0H_p^k(\mathbb{R}_+, X)$ is an isomorphism. We denote the operator G in the space ${}_0H_{p,\rho}^k(\mathbb{R}_+, X)$ by G_ρ. Observe that

$$(\lambda - G_\rho) \Psi_\rho^{-1} u = \Psi_\rho^{-1} (\lambda - p\rho - G_0) u \qquad (u \in \mathscr{D}(G_0)),$$

which implies that

$$(\lambda - G_\rho) = \Psi_\rho^{-1} (\lambda - p\rho - G_0) \Psi_\rho.$$

From $G_0 \in \mathcal{H}^\infty({}_0H_p^k(\mathbb{R}_+, X))$ it follows that $G_0 + \rho \in \mathcal{H}^\infty({}_0H_p^k(\mathbb{R}_+, X))$ and $\phi_{G_0+\rho}^\infty = \phi_{G_0}^\infty = \pi/2$. The fact that a bounded $\mathcal{H}^\infty$-calculus is invariant under conjugation with isomorphisms implies that $G_\rho \in \mathcal{H}^\infty({}_0H^k_{p,\rho}(\mathbb{R}_+, X))$ and $\phi^\infty_{G_\rho} = \pi/2$. In view of definitions (2.4) and (2.5) an interpolation argument shows that $G_\rho \in \mathcal{H}^\infty({}_0\mathcal{F}^s_{p,\rho}(\mathbb{R}_+, X))$ with $\phi^\infty_{G_\rho} = \pi/2$ for arbitrary $\rho, s \geq 0$ and $1 < p < \infty$. Finally, if X also admits property α the set

$$\{h(G_0) : \ h \in \mathcal{H}^\infty(\Sigma_\phi), \ \|h\|_\infty \leq 1\}$$

is $\mathcal{R}$-bounded in $\mathscr{L}({}_0H^k_p(\mathbb{R}_+, X)))$. Since $h \in \mathcal{H}^\infty(\Sigma_\phi)$ implies that $h_\rho \in \mathcal{H}^\infty(\Sigma_\phi)$ for $\rho \geq 0$, where $h_\rho(z) := h(z + \rho)$, we immediately see that also the set

$$\{h(G_0 + \rho) : \ h \in \mathcal{H}^\infty(\Sigma_\phi), \ \|h\|_\infty \leq 1\}$$

is $\mathcal{R}$-bounded in $\mathscr{L}({}_0H^k_p(\mathbb{R}_+, X)))$. Then, the assertion follows by an interpolation argument and in view of the facts that also the property of $\mathcal{R}$-boundedness is invariant under conjugation with isomorphisms and stable under complex and real interpolation. □

We continue with a corresponding result for the Laplacian

$$-\Delta : \mathscr{D}(-\Delta) \to \mathcal{K}^r_p(\mathbb{R}^n), \qquad \mathscr{D}(-\Delta) := \mathcal{K}^{r+2}_p(\mathbb{R}^n).$$

2.9. **Proposition.** *Let $1 < p < \infty$, $r \in \mathbb{R}$, $\rho \geq 0$, and $\mathcal{K} \in \{H, W\}$. Then $-\Delta \in \mathcal{RH}^\infty(\mathcal{K}^r_p(\mathbb{R}^n))$ and $\phi^{\mathcal{R},\infty}_{-\Delta} = 0$.*

Proof. This is analogous to the first part of the proof of Proposition 2.7. In fact, for $\phi \in (0, \pi)$ arbitrarily small and $h \in \mathcal{H}^\infty(\Sigma_\phi)$ we obtain

$$(\mathscr{F} h(-\Delta)f)(\xi) = h(|\xi|^2)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^n, \ f \in H^k_p(\mathbb{R}^n).$$

Cauchy's formula also here implies the set

$$M := \left\{\xi^\alpha D^\alpha h(|\xi|^2) : \xi \in \mathbb{R}^n \setminus \{0\}, \ \alpha \in \mathbb{N}^n_0, \ h \in \mathcal{H}^\infty(\Sigma_\phi), \ \|h\|_\infty \leq 1\right\}$$

to be uniformly bounded on the Hilbert space $\mathbb{C}$. By virtue of the fact that $\mathcal{R}$-boundedness and uniform boundedness are equivalent on Hilbert spaces, the set M in this case automatically is $\mathcal{R}$-bounded. Thus, by the n-dimensional version of the operator valued Mikhlin type multiplier result of Weis (see [14] or [5]) we have that

$$M_1 := \{h(-\Delta) : \ h \in \mathcal{H}^\infty(\Sigma_\phi), \ \|h\|_\infty \leq 1\}$$

is $\mathcal{R}$-bounded, which yields the result on the space $H^k_p(\mathbb{R}^n)$, $k \in \mathbb{N}_0$, $1 < p < \infty$. An interpolation argument implies the assertion. □

We denote by

$$D_n := (-\Delta)^{1/2}, \qquad \mathscr{D}(D_n) := {}_0\mathcal{F}^s_{p,\rho}(\mathbb{R}_+, \mathcal{K}^{r+1}_p(\mathbb{R}^n)) \tag{2.19}$$

the natural extension of $(-\Delta)^{1/2}$ to the space ${}_0\mathcal{F}^s_{p,\rho}(\mathbb{R}_+, \mathcal{K}^r_p(\mathbb{R}^n))$. The fact that for arbitrary $1 < p < \infty$ and $r \in \mathbb{R}$ the space $\mathcal{K}^r_p(\mathbb{R}^n)$ has property α, immediately implies the following result.

2.10. **Corollary.** *Let $1 < p < \infty$, $r \in \mathbb{R}$, $\rho, s \geq 0$, and $\mathcal{F}, \mathcal{K} \in \{H, W\}$. Then we have*

(i) $G \in \mathcal{RH}^\infty\left({}_0\mathcal{F}^s_{p,\rho}(\mathbb{R}_+, \mathcal{K}^r_p(\mathbb{R}^n))\right), \quad \phi^{\mathcal{R},\infty}_G = \pi/2,$

(ii) $D_n \in \mathcal{RH}^\infty\left({}_0\mathcal{F}^s_{p,\rho}(\mathbb{R}_+, \mathcal{K}^r_p(\mathbb{R}^n))\right), \quad \phi^{\mathcal{R},\infty}_G = 0,$

for the operator G as defined in (2.15) and D_n as defined in (2.19).

Proof. (i) is an immediate consequence of Proposition 2.7. For a sectorial operator A in a Banach space X it is not difficult to see that $A \in \mathcal{RH}^\infty(X)$ implies that $A^\alpha \in \mathcal{RH}^\infty(X)$ and $\phi^{\mathcal{R},\infty}_{A^\alpha} \le \phi^{\mathcal{R},\infty}_{A}$ for $\alpha \in (0,1]$. Hence Proposition 2.9 yields $D_n = (-\Delta)^{1/2} \in \mathcal{RH}^\infty(\mathcal{K}^r_p(\mathbb{R}^n))$ and $\phi^{\mathcal{R},\infty}_{D_n} = 0$. By Fubini's theorem we therefore easily deduce $D_n \in \mathcal{RH}^\infty({}_0H^k_{p,\rho}(\mathbb{R}_+, \mathcal{K}^r_p(\mathbb{R}^n)))$ for $k \in \mathbb{N}_0$ and $1 < p < \infty$. Then, an interpolation argument yields (ii). □

3. Maximal regularity for inhomogeneous symbols

Here we prove the main result, that is, the maximal regularity for inhomogeneous symbols in Bessel potential and Sobolev-Slobodeckij classes. We will restrict our considerations to rotation invariant symbols in space as they appear frequently in whole-space and half-space model problems. For fixed $\theta \in (0,\pi)$ and $\epsilon \in (0, \frac{\pi-\theta}{2})$ we will consider polynomial symbols $P\colon \overline{\Sigma}_\epsilon \times \overline{\Sigma}_\theta \to \mathbb{C}$ of the form

$$P(z,\lambda) = \sum_{m\in I} a_m z^{m_1} \lambda^{m_2} \omega(z,\lambda)^{m_3} \quad \big((z,\lambda) \in \overline{\Sigma}_\epsilon \times \overline{\Sigma}_\theta\big) \tag{3.1}$$

with $a_m \in \mathbb{C} \setminus \{0\}$, $\omega(z,\lambda) := \sqrt{\lambda + z^2}$, and $I \subset \mathbb{N}_0^3$ being a finite set of exponents. To analyze this symbol, we will follow the Newton polygon approach described in [12] and [6].

For this purpose, we define the Newton polygon $N(P) \subset [0,\infty)^2$ as the convex hull of the set

$$\{(0,0)\} \cup \bigcup_{m\in I} \big\{(m_1+m_3, m_2),\ (m_1, m_2 + \tfrac{m_3}{2}),\ (m_1+m_3, 0),\ (0, m_2 + \tfrac{m_3}{2})\big\}.$$

Denote the vertices of $N(P)$ by $v_0 := (0,0), v_1, \ldots, v_{J+1}$, numbered in counter-clockwise direction. Then for $v_j = (r_j, s_j)$ the vector $\frac{1}{\sqrt{1+\gamma_j^2}}(1,\gamma_j)$ with

$$\gamma_j := \frac{r_j - r_{j+1}}{s_{j+1} - s_j} \quad (j = 1, \ldots, J)$$

is an exterior normal to the edge $[v_j v_{j+1}]$ connecting v_j and v_{j+1}.

For simplicity, we assume that $N(P)$ has no edge parallel to the coordinate axes but not lying on the axis. More precisely, we assume

$$0 < \gamma_1 < \cdots < \gamma_J < \infty.$$

In this case, we have $N(P) = \operatorname{conv}(\tilde I)$ with

$$\tilde I := \{(0,0)\} \cup \bigcup_{m\in I} \big\{(m_1+m_3, m_2),\ (m_1, m_2 + \tfrac{m_3}{2})\big\}.$$

The main idea of the Newton polygon approach is to deal with different inhomogeneities by assigning a weight $\gamma > 0$ to the co-variable λ with respect to z, i.e., to set $|\lambda| \approx |z|^\gamma$. In a natural way, for $\gamma > 0$ the γ-degree $d_\gamma(P)$ is defined as

$$d_\gamma(P) := \max\{m_1 + \gamma m_2 + m_3 \max\{1, \gamma/2\} : m \in I\}.$$

Note that in the same way for $\omega(z,\lambda)=\sqrt{\lambda+z^2}$ the γ-degree is given by

$$d_\gamma(\omega)=\begin{cases}1, & \gamma\le 2,\\ \gamma/2, & \gamma\ge 2.\end{cases}$$

Furthermore, the γ-principal part of P is defined as

$$P_\gamma(z,\lambda):=\lim_{\rho\to\infty}\rho^{-d_\gamma(P)}P(\rho z,\rho^\gamma\lambda)\quad\big((z,\lambda)\in\Sigma_\epsilon\times\Sigma_\theta\big).$$

Obviously the "leading exponents" for weight γ are given by

$$I_\gamma:=\{m\in I: m_1+\gamma m_2+m_3\max\{1,\gamma/2\}=d_\gamma(P)\}.$$

This yields

$$P_\gamma(z,\lambda)=\sum_{m\in I_\gamma}a_m z^{m_1}\lambda^{m_2}\omega_\gamma(z,\lambda)^{m_3} \tag{3.2}$$

with

$$\omega_\gamma(z,\lambda)=\begin{cases}\sqrt{\lambda}, & \gamma>2,\\ \sqrt{\lambda+z^2}, & \gamma=2,\\ z, & \gamma<2.\end{cases}$$

Geometric observations show that I_{γ_j} consists of all $m\in I$ for which one of the points (m_1+m_3,m_2) or $(m_1,m_2+\frac{m_3}{2})$ lies on $[v_jv_{j+1}]$, so the weights γ_j correspond to the edges of the Newton polygon. Similarly, for $\gamma_{j-1}<\gamma<\gamma_j$ the set I_γ consists of all points $m\in I$ for which one of the points (m_1+m_3,m_2) or $(m_1,m_2+\frac{m_3}{2})$ are equal to v_j. These values of γ correspond to the vertices of the Newton polygon.

3.1. **Theorem.** *Let $\theta\in(0,\pi)$ and $\epsilon\in(0,\frac{\pi-\theta}{2})$. Assume that in the situation above we have*

$$P_\gamma(z,\lambda)\ne 0\quad(z\in\overline{\Sigma}_\epsilon\setminus\{0\},\ \lambda\in\overline{\Sigma}_\theta\setminus\{0\},\gamma>0). \tag{3.3}$$

Then there exist constants $\lambda_0>0$ and $C>0$ such that the inequality

$$|P(z,\lambda)|\ge C\,W(z,\lambda)\quad(z\in\overline{\Sigma}_\epsilon,\ \lambda\in\overline{\Sigma}_\theta,\ |\lambda|\ge\lambda_0) \tag{3.4}$$

holds, where the weight function W is defined by

$$W(z,\lambda):=\sum_{(n_1,n_2)\in\tilde I}|z|^{n_1}|\lambda|^{n_2}.$$

Proof. The proof follows the lines of [6], Section 2.4, and we will omit some details. Fix $\eta>0$. It was shown in [12], Section 4.2, that there exists a $\lambda_0>0$ and a partition of the form

$$\big\{(z,\lambda)\in\overline{\Sigma}_\epsilon\times\overline{\Sigma}_\theta:|\lambda|\ge\lambda_0\big\}\subset\bigcup_{j=1}^{J}G_j\cup\bigcup_{j=1}^{J+1}\tilde G_j$$

with the following properties:

(i) Let $j\in\{1,\dots,J\}$. Then for each $n=(n_1,n_2)\in\tilde I\setminus[v_jv_{j+1}]$ we have

$$|z|^{n_1}|\lambda|^{n_2}\le\eta\sum_{(n_1',n_2')\in[v_jv_{j+1}]\cap\tilde I}|z|^{n_1'}|\lambda|^{n_2'}\quad\big((z,\lambda)\in G_j\big).$$

(ii) Let $j \in \{1, \dots, J+1\}$. Then for every $(n_1, n_2) \in \tilde{I} \setminus \{v_j\}$ we have

$$|z|^{n_1}|\lambda|^{n_2} \le \eta |z|^{r_j}|\lambda|^{s_j} \quad \big((z,\lambda) \in \tilde{G}_j\big).$$

The symbol $|P(z,\lambda)|$ is estimated in each subdomain G_j, $\tilde{G}_j$ separately. We will restrict ourselves to the case $(z,\lambda) \in G_j$. The case $(z,\lambda) \in \tilde{G}_j$ can be done in a similar way.

Let $j \in \{1, \dots, J\}$. We will additionally assume $\gamma_j < 2$ and thus $\omega_\gamma(z,\lambda) = z$, the cases $\gamma_j = 2$ and $\gamma_j > 2$ can be treated analogously.

From (3.2) we obtain

$$P_{\gamma_j}(z,\lambda) = \sum_{m \in I_{\gamma_j}} a_m z^{m_1+m_3}\lambda^{m_2}. \tag{3.5}$$

Note that all exponents are integer, and $P_{\gamma_j}(z,\lambda)$ is a polynomial in (z,λ). As $v_j = (r_j, s_j)$ and $v_{j+1} = (r_{j+1}, s_{j+1})$ with $r_j > r_{j+1}$ and $s_j < s_{j+1}$, we see that all terms on the right-hand side of (3.5) have the common factor $z^{r_{j+1}}\lambda^{s_j}$, i.e., we have

$$P_{\gamma_j}(z,\lambda) = z^{r_{j+1}}\lambda^{s_j}\tilde{P}_{\gamma_j}(z,\lambda)$$

with

$$\tilde{P}_{\gamma_j} := \sum_{m \in I_{\gamma_j}} a_m z^{m_1+m_3-r_{j+1}}\lambda^{m_2-s_j}.$$

Because all exponents in P_{γ_j} lie on $[v_j v_{j+1}]$, the reduced polynomial $\tilde{P}_{\gamma_j}$ can be written as

$$\tilde{P}_{\gamma_j}(z,\lambda) = \sum_{k=0}^{r_j - r_{j+1}} c_k z^k \lambda^{s_{j+1}-s_j-k/\gamma_j}$$

with complex coefficients c_k. We have

$$c_0 z^{r_j}\lambda^{s_j} = P_\gamma(z,\lambda) \quad (\gamma_{j-1} < \gamma < \gamma_j),$$

and from (3.3) we conclude $c_0 \ne 0$. In the same way we get $c_{r_j - r_{j+1}} \ne 0$. With this and (3.3) for $\gamma = \gamma_j$ we obtain

$$\tilde{P}_{\gamma_j}(z,\lambda) \ne 0 \quad \big((z,\lambda) \in \overline{\Sigma}_\epsilon \times \overline{\Sigma}_\theta \setminus \{(0,0)\}\big).$$

As $\tilde{P}_{\gamma_j}$ is homogeneous in (z, λ^{γ_j}), we obtain an estimate of the form

$$|\tilde{P}_{\gamma_j}(z,\lambda)| \ge C_0\big(|z|^{r_j - r_{j+1}} + |\lambda|^{s_{j+1}-s_j}\big).$$

Consequently, we have

$$\begin{aligned} |P_{\gamma_j}(z,\lambda)| &\ge C_0\big(|z|^{r_j}|\lambda|^{s_j} + |z|^{r_{j+1}}|\lambda|^{s_{j+1}}\big) \\ &\ge C_1 \sum_{(n_1,n_2) \in \tilde{I} \cap [v_j v_{j+1}]} |z|^{n_1}|\lambda|^{n_2} \end{aligned} \tag{3.6}$$

with constants $C_0, C_1 > 0$.

Now we take advantage of the fact that G_j may be defined in the form

$$G_j = \{(z,\lambda) \in \overline{\Sigma}_\epsilon \times \overline{\Sigma}_\theta, C_2^{-1}|z|^{\gamma_j} \le |\lambda| \le C_2|z|^{\gamma_j}\}$$

with a constant $C_2 > 0$ (see [12] for details). Therefore

$$\lim_{|\lambda|\to\infty,\,(z,\lambda)\in G_j} \frac{\omega(z,\lambda)-\omega_{\gamma_j}(z,\lambda)}{\omega_{\gamma_j}(z,\lambda)} = \lim_{|\lambda|\to\infty,\,(z,\lambda)\in G_j} \Big(\sqrt{\frac{\lambda}{z^2}+1}-1\Big) = 0.$$

Here we used $\gamma_j < 2$. Consequently,

$$\Big|P_{\gamma_j}(z,\lambda) - \sum_{m\in I_{\gamma_j}} a_m z^{m_1}\lambda^{m_2}\omega(z,\lambda)^{m_3}\Big| \le \frac{C_1}{4} \sum_{(n_1,n_2)\in\tilde{I}\cap[v_jv_{j+1}]} |z|^{n_1}|\lambda|^{n_2}$$

for all $(z,\lambda)\in G_j$ satisfying $|\lambda|\ge\lambda_0$ for sufficiently large λ_0. Now we can estimate, using property (i),

$$\begin{aligned}
|P(z,\lambda)| &\ge |P_{\gamma_j}(z,\lambda)| - \Big|P_{\gamma_j}(z,\lambda) - \sum_{m\in I_{\gamma_j}} a_m z^{m_1}\lambda^{m_2}\omega(z,\lambda)^{m_3}\Big| \\
&\quad - \Big|\sum_{m\in I\setminus I_{\gamma_j}} a_m z^{m_1}\lambda^{m_2}\omega(z,\lambda)^{m_3}\Big| \\
&\ge \Big(C_1 - \frac{C_1}{4} - \eta C_3\Big) \sum_{(n_1,n_2)\in\tilde{I}\cap[v_jv_{j+1}]} |z|^{n_1}|\lambda|^{n_2} \\
&\ge \frac{C_1}{2} \sum_{(n_1,n_2)\in\tilde{I}\cap[v_jv_{j+1}]} |z|^{n_1}|\lambda|^{n_2} \quad \big((z,\lambda)\in G_j,\ |\lambda|\ge\lambda_0\big)
\end{aligned} \tag{3.7}$$

for λ_0 sufficiently large and η sufficiently small, where we have set

$$C_3 := \mathrm{card}(\tilde{I})\cdot\max\{|a_m| : m\in I\}.$$

For $(z,\lambda)\in G_j$ the weight function W can be estimated by

$$W(z,\lambda) = \sum_{(n_1,n_2)\in\tilde{I}} |z|^{n_1}|\lambda|^{n_2} \le \big(1+\eta\,\mathrm{card}(\tilde{I})\big) \sum_{(n_1,n_2)\in\tilde{I}\cap[v_jv_{j+1}]} |z|^{n_1}|\lambda|^{n_2}.$$

This fact and (3.7) imply the desired inequality (3.4) for $(z,\lambda)\in G_j$. □

In the next result we show how symbols satisfying condition (3.3) give rise to isomorphic operators on their natural domain arising from the vertices of the Newton polygon.

3.2. **Theorem.** *Let $1<p<\infty$, $r\in\mathbb{R}$, $\rho, s\ge 0$, and let A, B be resolvent commuting operators such that for each $\gamma\in\mathbb{R}$, $\sigma\ge 0$,*

(i) *$\mathscr{D}(A) = {}_0\mathcal{F}^{\sigma}_{p,\rho}(\mathbb{R}_+,\mathcal{K}^{\gamma+1}_p(\mathbb{R}^n))$ and $\mathscr{D}(B) = {}_0\mathcal{F}^{\sigma+1}_{p,\rho}(\mathbb{R}_+,\mathcal{K}^{\gamma}_p(\mathbb{R}^n))$,*
(ii) *$A, B\in\mathcal{H}^\infty({}_0\mathcal{F}^{\sigma}_{p,\rho}(\mathbb{R}_+,\mathcal{K}^{\gamma}_p(\mathbb{R}^n)))$ with ϕ^∞_A, ϕ^∞_B independent of γ,σ,p, and ρ.*

Furthermore, let P be a symbol as defined in (3.1) and let $v_j=(r_j,s_j)$, $j=0,\dots,J+1$ be the vertices of the Newton polygon corresponding to P. Suppose that there exist $\theta\in(\phi^\infty_B,\pi)$ and $\epsilon\in(\phi^\infty_A,\frac{\pi-\theta}{2})$ such that P satisfies condition (3.3). Then there exists a $\lambda_0>0$ such that

$$P(A,B+\lambda_0) : \mathscr{D}(P(A,B+\lambda_0)) \to {}_0\mathcal{F}^{s}_{p,\rho}(\mathbb{R}_+,\mathcal{K}^{r}_p(\mathbb{R}^n)).$$

is invertible, where

$$\mathscr{D}(P(A,B+\lambda_0)) = \bigcap_{j=1}^{J+1} {}_0\mathcal{F}^{s+s_j}_{p,\rho}(\mathbb{R}_+,\mathcal{K}^{r+r_j}_p(\mathbb{R}^n))$$

Proof. By the assumption on P and Theorem 3.1 it follows that for appropriate $\lambda_0 > 0$ the functions

$$m_j(z,\lambda) := \frac{(1+z)^{[r_j]}(1+z^{r_j-[r_j]})(1+\lambda)^{[s_j]}(1+\lambda^{s_j-[s_j]})}{P(z,\lambda+\lambda_0)}, \quad j=0,1,\ldots,J+1, \tag{3.8}$$

are uniformly bounded on $\Sigma_\epsilon \times \Sigma_\theta$, where $[s]$ denotes the largest integer smaller than $s \in \mathbb{R}$. Since $\mathcal{F}^s_{p,\rho}(\mathbb{R}_+,\mathcal{K}^r_p(\mathbb{R}^n))$ has property α, as mentioned in Remark 2.8 we know by [16, Theorem 5.3] that A even admits the stronger property of an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus. Replacing z by A in m_j, which is possible in view of $\phi^{\mathcal{R},\infty}_A < \epsilon$, we therefore obtain that

$$\mathcal{R}\left(\left\{m_j(A,\lambda), \lambda \in \Sigma_\theta\right\}\right) \le C$$

for $j = 0,\ldots,J+1$. Since A and B are resolvent commuting, $\phi^\infty_B < \theta$ and by virtue of $B \in \mathcal{H}^\infty({}_0\mathcal{F}^s_{p,\rho}(\mathbb{R}_+,\mathcal{K}^r_p(\mathbb{R}^n)))$, we may apply Theorem 4.4 in [16] to the result

$$\|m_j(A,B)\|_{\mathscr{L}({}_0\mathcal{F}^s_{p,\rho}(\mathbb{R}_+,\mathcal{K}^r_p(\mathbb{R}^n)))} \le C, \quad j=0,\ldots,J+1. \tag{3.9}$$

Note that we cannot argue directly that

$$1+B^\alpha : {}_0\mathcal{F}^{\sigma+\alpha}_{p,\rho}(\mathbb{R}_+,\mathcal{K}^\gamma_p(\mathbb{R}^n)) \to {}_0\mathcal{F}^\sigma_{p,\rho}(\mathbb{R}_+,\mathcal{K}^\gamma_p(\mathbb{R}^n))$$

is an isomorphism for arbitrary $\alpha > 0$. This is due to the fact that possibly $\phi^\infty_B > 0$ and we therefore do not have enough information on the spectrum of B^α for large $\alpha > 0$. Therefore, we split the powers in a fractional part less than 1 and an integer part. Since condition (i) is supposed to be valid for all $\gamma \in \mathbb{R}$ and $\sigma \ge 0$ we have that

$$(1+B)^k \quad : \quad {}_0\mathcal{F}^{\sigma+k}_{p,\rho}(\mathbb{R}_+,\mathcal{K}^\gamma_p(\mathbb{R}^n)) \to {}_0\mathcal{F}^\sigma_{p,\rho}(\mathbb{R}_+,\mathcal{K}^\gamma_p(\mathbb{R}^n))$$

is an isomorphism for all $k \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$, and $\sigma \ge 0$. Furthermore, condition (ii) implies that

$$\mathscr{D}(B^\alpha) \quad = \quad [{}_0\mathcal{F}^\sigma_{p,\rho}(\mathbb{R}_+,\mathcal{K}^\gamma_p(\mathbb{R}^n)), \mathscr{D}(B)]_\alpha = {}_0\mathcal{F}^{\sigma+\alpha}_{p,\rho}(\mathbb{R}_+,\mathcal{K}^\gamma_p(\mathbb{R}^n))$$

(cf. [26]) and that $\phi^\infty_{B^\alpha} \le \phi^\infty_B$ for $\alpha \in [0,1]$. Hence we have that

$$1+B^\alpha \quad : \quad {}_0\mathcal{F}^{\sigma+\alpha}_{p,\rho}(\mathbb{R}_+,\mathcal{K}^\gamma_p(\mathbb{R}^n)) \to {}_0\mathcal{F}^\sigma_{p,\rho}(\mathbb{R}_+,\mathcal{K}^\gamma_p(\mathbb{R}^n))$$

is an isomorphism for all $\alpha \in [0,1]$, $\gamma \in \mathbb{R}$, and $\sigma \ge 0$. This yields that

$$(1+B)^{[s_j]}(1+B^{s_j-[s_j]}) \ : \ {}_0\mathcal{F}^{s+s_j}_{p,\rho}(\mathbb{R}_+,\mathcal{K}^r_p(\mathbb{R}^n)) \to {}_0\mathcal{F}^s_{p,\rho}(\mathbb{R}_+,\mathcal{K}^r_p(\mathbb{R}^n))$$

is an isomorphism. An analogous argumentation for the operator A shows that also

$$(1+A^{r_j-[r_j]})(1+A)^{[r_j]} \ : \ {}_0\mathcal{F}^\sigma_{p,\rho}(\mathbb{R}_+,\mathcal{K}^{\gamma+r_j}_p(\mathbb{R}^n)) \to {}_0\mathcal{F}^\sigma_{p,\rho}(\mathbb{R}_+,\mathcal{K}^\gamma_p(\mathbb{R}^n))$$

is an isomorphism. Summarizing, we obtain that

$$\begin{aligned}(1+A^{r_j-[r_j]})(1+A)^{[r_j]}(1+B)^{[s_j]}(1+B^{s_j-[s_j]}) \quad : \quad & {}_0\mathcal{F}^{s+s_j}_{p,\rho}(\mathbb{R}_+,\mathcal{K}^{r+r_j}_p(\mathbb{R}^n)) \\ & \to {}_0\mathcal{F}^s_{p,\rho}(\mathbb{R}_+,\mathcal{K}^r_p(\mathbb{R}^n))\end{aligned}$$

is an isomorphism as well for all $j = 1, \dots, J+1$. In combination with (3.8) and (3.9) this yields the assertion. □

By employing the shift $e^{-\lambda_0 t}$ and Lemma 2.2 we immediately obtain the following result.

3.3. Theorem. *Let $r \in \mathbb{R}$, $s, \rho \geq 0$ and $1 < p < \infty$. Let G be the time derivative operator as defined in (2.15) and A an operator such that for each $\gamma \in \mathbb{R}$, $\sigma \geq 0$,*

(i) $\mathscr{D}(A) = {}_0\mathcal{F}^{\sigma}_{p,\rho}(\mathbb{R}_+, \mathcal{K}^{\gamma+1}_p(\mathbb{R}^n))$,

(ii) $A \in \mathcal{H}^\infty({}_0\mathcal{F}^{\sigma}_{p,\rho}(\mathbb{R}_+, \mathcal{K}^{\gamma}_p(\mathbb{R}^n)))$ *with ϕ^∞_A independent of γ, σ, ρ, and p.*

Furthermore, let P be a symbol satisfying the assumptions of Theorem (3.2), in particular condition (3.3) for some $\theta \in (\frac{\pi}{2}, \pi)$ and $\epsilon \in (\phi^\infty_A, \frac{\pi-\theta}{2})$. Then, if λ_0 is the constant obtained in Theorem 3.2, for $\omega \geq \lambda_0$ the operator $P(A,G) : \mathscr{D}(P(A,G)) \to {}_0\mathcal{F}^{s}_{p,\omega+\rho}(\mathbb{R}_+, \mathcal{K}^{r}_p(\mathbb{R}^n))$ is invertible, where

$$\mathscr{D}(P(A,G)) := \bigcap_{j=1}^{J+1} {}_0\mathcal{F}^{s+s_j}_{p,\omega+\rho}(\mathbb{R}_+, \mathcal{K}^{r+r_j}_p(\mathbb{R}^n)).$$

Proof. Denote by G_ρ the time derivative operator in the space ${}_0\mathcal{F}^{s}_{p,\rho}(\mathbb{R}_+, \mathcal{K}^{r}_p(\mathbb{R}^n))$. We have that

$$(\lambda - G_\omega)\Psi^{-1}_\omega u = \Psi^{-1}_\omega(\lambda - \omega - G_\rho)u \qquad (u \in \mathscr{D}(G_\rho)),$$

which implies that

$$(\lambda - G_\omega)^{-1} = \Psi^{-1}_\omega(\lambda - (G_\rho + \omega))^{-1}\Psi_\omega.$$

The Cauchy integral representation for the bounded holomorphic function $\lambda \mapsto P(A,\lambda)^{-1}$ then gives us

$$P(A, G_\omega)^{-1} = \Psi^{-1}_\omega P(A, G_\rho + \omega)^{-1}\Psi_\omega.$$

By the assumption $\omega \geq \lambda_0$, the result now follows from Theorem 3.2 with $B = G_\rho$, Theorem 2.7, and from the fact that

$$\Psi_\omega \in \mathrm{Isom}(\mathcal{F}^{s}_{p,\rho+\omega}(\mathbb{R}_+, X), \mathcal{F}^{s}_{p,\rho}(\mathbb{R}_+, X)),$$

which is an obvious consequence of Lemma 2.2(ii) for arbitrary UMD spaces X. □

4. The trace operator to the Newton polygon spaces

Theorem 3.3 is the key ingredient to obtain maximal regularity for model problems in zero time trace spaces. However, for a suitable treatment of related nonlinear problems maximal regularity of the corresponding fully inhomogeneous systems without the zero time trace assumption is required. In many applications these general systems can be reduced to zero time trace systems, if the existence of suitable extension operators for the time traces are established. In other words, it is important to have the surjectivity of the time trace operator related to the function classes determined by the Newton polygon. The purpose of this section is to derive a general result in this direction.

Before we turn our attention to the trace of the Newton polygon we need to prepare embedding results for anisotropic spaces. The first one will be obtained as a application of the mixed derivative theorem, which goes back to Sobolevskii [25] and reads as follows:

4.1. **Lemma.** *Suppose X is a UMD space which has property α. Let A and B be resolvent commuting operators satisfying $A, B \in \mathcal{H}^\infty(X)$ and $\phi_A^\infty + \phi_B^\infty < \pi$. Then, there is a constant $C > 0$ such that*

$$\|A^\sigma B^{1-\sigma} x\|_X \leq C\|Ax + Bx\|_X \quad (x \in \mathscr{D}(A) \cap \mathscr{D}(B),\ \sigma \in [0,1]).$$

Proof. Let $\theta, \epsilon \in (0, \pi)$ such that $\theta > \phi_B^\infty$, $\epsilon > \phi_A^\infty$, and $\theta + \epsilon < \pi$. We set

$$m_\sigma(z, \lambda) := \lambda^\sigma z^{1-\sigma}(\lambda + z)^{-1}.$$

By employing Youngs's inequality we easily find that

$$\begin{aligned} |m_\sigma(z,\lambda)| &\leq |\lambda|^\sigma |z|^{1-\sigma} |\lambda + z|^{-1} \\ &\leq (\sigma|\lambda| + (1-\sigma)|z|)\, |\lambda + z|^{-1} \\ &\leq C \qquad ((z,\lambda) \in \Sigma_\epsilon \times \Sigma_\theta,\ \sigma \in [0,1]). \end{aligned}$$

Since X has property α we obtain $A \in \mathcal{RH}^\infty(X)$ and $\phi_A^{\mathcal{R},\infty} = \phi_A^\infty$. Thus, the set

$$\{m_\sigma(A, \lambda) :\ \lambda \in \Sigma_\theta, \sigma \in [0,1]\}$$

is $\mathcal{R}$-bounded on X. By [16, Theorem 4.4] we conclude

$$\|m_\sigma(A, B)\|_X \leq C \quad (\sigma \in [0,1]),$$

which proves the assertion. □

4.2. **Remark.** *Note that Sobolevskii proved the above result under much less strict assumptions, see also* [11, Lemma 9.7].

As an application we obtain

4.3. **Lemma.** *Let $1 < p < \infty$, $\rho \geq 0$, $\mathcal{F}, \mathcal{K} \in \{H, W\}$, and $J \subseteq \mathbb{R}$ be an interval such that $J \subseteq \mathbb{R}_+$ if $\rho > 0$. Suppose also that $s, \alpha, \beta \geq 0$, $r \in \mathbb{R}$. Then for each $\sigma \in [0,1]$ the following embedding holds:*

$$\mathcal{F}^{s+\alpha}_{p,\rho}(J, \mathcal{K}^r_p(\mathbb{R}^n)) \cap \mathcal{F}^s_{p,\rho}(J, \mathcal{K}^{r+\beta}_p(\mathbb{R}^n)) \hookrightarrow \mathcal{F}^{s+\sigma\alpha}_{p,\rho}(J, \mathcal{K}^{r+(1-\sigma)\beta}_p(\mathbb{R}^n)).$$

Proof. W.l.o.g. we will restrict ourselves to the case $\alpha \in [0,1]$. The general case then follows by iterating. We will first assume that $J = \mathbb{R}$ (hence $\rho = 0$). We define the operators A, B in $X := \mathcal{F}^s_p(\mathbb{R}, \mathcal{K}^r_p(\mathbb{R}^n))$ by

$$\begin{aligned} Au &:= (1-\Delta)^{\beta/2} u, \qquad && u \in \mathscr{D}(A) = \mathcal{F}^s_p(\mathbb{R}, \mathcal{K}^{r+\beta}_p(\mathbb{R}^n)) \\ Bu &:= (\partial_t + 1)^\alpha u, && u \in \mathscr{D}(B) = \mathcal{F}^{s+\alpha}_p(\mathbb{R}, \mathcal{K}^r_p(\mathbb{R}^n)). \end{aligned}$$

From the proof of Proposition 2.7 we know that $B^{1/\alpha} \in \mathcal{H}^\infty(H^k_p(\mathbb{R}, \mathcal{K}^r_p(\mathbb{R}^n)))$ for $k \in \mathbb{N}_0$ and that $\phi^\infty_{B^{1/\alpha}} = \pi/2$. By an interpolation argument we see that this result still holds on the space X. Thus, we also have $B \in \mathcal{H}^\infty(X)$ and $\phi_B^\infty \leq \pi/2$. Analogously to the proof of Corollary 2.10 we obtain by virtue of Proposition 2.9 that $A \in \mathcal{H}^\infty(X)$ and $\phi_A^\infty = 0$. The Dore-Venni theorem (cf. [21, Theorem 8.4]) or once again [16, Theorem 4.4] therefore implies

$$A + B : \mathscr{D}(A) \cap \mathscr{D}(B) \to X \tag{4.1}$$

to be an isomorphism. Next we determine the domains of the operators A^σ, B^σ in X for $\sigma \in [0,1]$. Note that by the $\mathcal{H}^\infty$-calculus of B on X and the operator $(1-\Delta)^{\beta/2}$ on $\mathcal{K}_p^r(\mathbb{R}^n)$ we have that

$$\begin{aligned} \mathscr{D}(B^\sigma) &= [X, \mathscr{D}(B)]_\sigma \quad \text{and} \\ \mathscr{D}((1-\Delta)^{\beta\sigma/2}) &= [\mathcal{K}_p^r(\mathbb{R}^n), \mathcal{K}_p^{r+\beta}(\mathbb{R}^n)]_\sigma = \mathcal{K}_p^{r+\beta\sigma}(\mathbb{R}^n) \end{aligned} \tag{4.2}$$

for all $\sigma \in (0,1)$. In view of the fact that

$$\big[\mathcal{F}_p^s(\mathbb{R}, \mathcal{K}_p^r(\mathbb{R}^n)), \mathcal{F}_p^{s+\alpha}(\mathbb{R}, \mathcal{K}_p^r(\mathbb{R}^n))\big]_\sigma = \mathcal{F}_p^{s+\alpha\sigma}(\mathbb{R}, \mathcal{K}_p^r(\mathbb{R}^n)),$$

we therefore obtain

$$\mathscr{D}(B^\sigma) = \mathcal{F}_p^{s+\alpha\sigma}(\mathbb{R}, \mathcal{K}_p^r(\mathbb{R}^n)) \quad (\sigma \in [0,1]).$$

Furthermore, by the definition of fractional powers for sectorial operators given by Cauchy's formula it is clear that $A^\sigma = (1-\Delta)^{\beta\sigma/2}$. Relation (4.2) therefore yields that for $k \in \mathbb{N}_0$ the domain of A^σ in $H_p^k(\mathbb{R}, \mathcal{K}_p^r(\mathbb{R}^n))$ obviously is represented as $\mathscr{D}(A^\sigma) = H_p^k(\mathbb{R}, \mathcal{K}_p^{r+\beta\sigma}(\mathbb{R}^n))$. An interpolation argument therefore implies that in the space X we have

$$\mathscr{D}(A^\sigma) = \mathcal{F}_p^s(\mathbb{R}, \mathcal{K}_p^{r+\beta\sigma}(\mathbb{R}^n)) \quad (\sigma \in [0,1]).$$

The mixed derivative theorem, i.e. Lemma 4.1, and the invertibility of $A : \mathscr{D}(A) \to X$ and $B : \mathscr{D}(B) \to X$ now yield

$$\begin{aligned} \|u\|_{\mathcal{F}_p^{s+\alpha\sigma}(\mathbb{R}, \mathcal{K}_p^{r+(1-\sigma)\beta}(\mathbb{R}^n))} &\le C\|B^\sigma u\|_{\mathcal{F}_p^s(\mathbb{R}, \mathcal{K}_p^{r+(1-\sigma)\beta}(\mathbb{R}^n))} \\ &\le C\|A^{1-\sigma}B^\sigma u\|_X \\ &\le C\|(A+B)u\|_X \le C\|u\|_{\mathscr{D}(A)\cap\mathscr{D}(B)}. \end{aligned} \tag{4.3}$$

This proves the assertion for $J = \mathbb{R}$ and $\rho = 0$.

Suppose now that $J \subseteq \mathbb{R}$ is an arbitrary interval and let

$$E_J \in \mathscr{L}(\mathcal{F}_p^s(J, Y), \mathcal{F}_p^s(\mathbb{R}, Y))$$

be an appropriate extension operator existing simultaneously for all $p \in (1,\infty)$, $s \ge 0$, and UMD spaces Y. Note that such an extension operator can be constructed by standard methods as described in [26] or [1]. Then the result for J and $\rho = 0$ follows by first extending the functions to $\mathbb{R}$, using (4.3), and then restricting again to J. The result for $\rho \ne 0$ then is an obvious consequence of Lemma 2.2 (ii). $\square$

The next result is obtained as a consequence of a general trace result proved in [29].

4.4. **Lemma.** *Let $p, J, \rho, \mathcal{F}, \mathcal{K}$ be as in Lemma 4.3. Suppose also that $k \in \mathbb{N}_0$ and $s_1, s_2, r_1, r_2 \in \mathbb{R}$ such that*

$$\max\{0, k-1+1/p\} \le s_1 < k+1/p < s_2 < k+1+1/p.$$

Then,

$$\mathcal{F}_{p,\rho}^{s_2}(J, \mathcal{K}_p^{r_2}(\mathbb{R}^n)) \cap \mathcal{F}_{p,\rho}^{s_1}(J, \mathcal{K}_p^{r_1}(\mathbb{R}^n)) \hookrightarrow \mathrm{BUC}_\rho^k(J, W_p^{r_1-\gamma(k+1/p-s_1)}(\mathbb{R}^n)),$$

where $\gamma := (r_1 - r_2)/(s_2 - s_1)$.

Proof. Also here it is sufficient to consider the case $\rho = 0$ by virtue of Lemma 2.2 (ii). Next we show that it suffices to consider the case $k = 0$. In fact, as a consequence of Lemma 4.3 we may assume w.l.o.g. that $s_1 \geq k$. But then

$$\mathcal{F}_p^{s_2-k}(J, \mathcal{K}_p^{r_2}(\mathbb{R}^n)) \cap \mathcal{F}_p^{s_1-k}(J, \mathcal{K}_p^{r_1}(\mathbb{R}^n)) \hookrightarrow \mathrm{BUC}(J, W_p^{r_1-\gamma(k+1/p-s_1)}(\mathbb{R}^n))$$

yields that $\partial_t u \in \mathrm{BUC}(J, W_p^{r_1-\gamma(k+1/p-s_1)}(\mathbb{R}^n))$, $0 \leq \ell \leq k$, for each u in the space $\mathcal{F}_p^{s_2}(J, \mathcal{K}_p^{r_2}(\mathbb{R}^n)) \cap \mathcal{F}_p^{s_1}(J, \mathcal{K}_p^{r_1}(\mathbb{R}^n))$, and the assertion follows. Hence, it remains to prove the case $k = 0$. To this end we assume that $J = \mathbb{R}$. As in Lemma 4.3 the general result then follows by extending and restricting.

From [29] it follows that

$$H_p^{s_2}(\mathbb{R}, \mathcal{K}_p^{r_2}(\mathbb{R}^n)) \cap H_p^{s_1}(\mathbb{R}, \mathscr{D}(A^{s_2-s_1})) \hookrightarrow \mathrm{BUC}\left(\mathbb{R}, (\mathcal{K}_p^{r_2}(\mathbb{R}^n), \mathscr{D}(A))_{s_2-1/p}\right)$$

for $A = (1-\Delta)^{\gamma/2}$ considered in the space $\mathcal{K}_p^{r_2}(\mathbb{R}^n)$. Actually in [29] this result is proved for $J = \mathbb{R}_+$. But by a simple reflection argument it follows also for $J = \mathbb{R}$. In view of

$$\mathscr{D}(A^{s_2-s_1}) = \mathscr{D}((1-\Delta)^{(r_1-r_2)/2}) = \mathcal{K}_p^{r_1}(\mathbb{R}^n)$$

and

$$\begin{aligned}(\mathcal{K}_p^{r_2}(\mathbb{R}^n), \mathscr{D}(A))_{s_2-1/p} &= (\mathcal{K}_p^{r_2}(\mathbb{R}^n), \mathcal{K}_p^{r_2+\gamma}(\mathbb{R}^n))_{s_2-1/p} = W_p^{r_2+\gamma(s_2-1/p)}(\mathbb{R}^n)\\ &= W_p^{r_1-\gamma(1/p-s_1)}(\mathbb{R}^n)\end{aligned}$$

we obtain the assertion for the case $\mathcal{F} = H$. Next, let $0 < \varepsilon < \min\{s_1, 1/p - s_1\}$ and set

$$E_\pm := H_p^{s_2\pm\varepsilon}(\mathbb{R}, \mathcal{K}_p^{r_2}(\mathbb{R}^n)) \cap H_p^{s_1\pm\varepsilon}(\mathbb{R}, \mathcal{K}_p^{r_1}(\mathbb{R}^n))).$$

Analagously to the proof of Lemma 4.3 we deduce

$$A + B \in \mathrm{Isom}\left(E_\pm,\ H_p^{s_1\pm\varepsilon}(\mathbb{R}, \mathcal{K}_p^{r_2}(\mathbb{R}^n))\right)$$

for $A = (1-\Delta)^{(r_1-r_2)/2}$ and $B = (\partial_t + 1)^{s_2-s_1}$ considered in the space $H_p^{s_1\pm\varepsilon}(\mathbb{R}, \mathcal{K}_p^{r_2}(\mathbb{R}^n))$. Interpolating by the real method then implies

$$A + B \in \mathrm{Isom}\left((E_-, E_+)_{1/2,p},\ W_p^{s_1}(\mathbb{R}, \mathcal{K}_p^{r_2}(\mathbb{R}^n))\right).$$

From the fact that also

$$A + B \in \mathrm{Isom}\left(W_p^{s_2}(\mathbb{R}, \mathcal{K}_p^{r_2}(\mathbb{R}^n)) \cap W_p^{s_1}(\mathbb{R}, \mathcal{K}_p^{r_1}(\mathbb{R}^n))),\ W_p^{s_1}(\mathbb{R}, \mathcal{K}_p^{r_2}(\mathbb{R}^n))\right),$$

we conclude

$$(E_-, E_+)_{1/2,p} = W_p^{s_2}(\mathbb{R}, \mathcal{K}_p^{r_2}(\mathbb{R}^n)) \cap W_p^{s_1}(\mathbb{R}, \mathcal{K}_p^{r_1}(\mathbb{R}^n))).$$

Since the trace operator $\gamma_{t_0} u := u|_{t=t_0}$ is bounded from $E_\pm$ onto $W_p^{r_1+\gamma(1/p-s_1\pm\varepsilon)}(\mathbb{R}^n)$, again by real interpolation we obtain

$$\gamma_{t_0} \in \mathscr{L}\left(W_p^{s_2}(\mathbb{R}, \mathcal{K}_p^{r_2}(\mathbb{R}^n)) \cap W_p^{s_1}(\mathbb{R}, \mathcal{K}_p^{r_1}(\mathbb{R}^n))),\ W_p^{r_1+\gamma(1/p-s_1)}(\mathbb{R}^n)\right).$$

Strong continuity of the translation group then yields the assertion for $\mathcal{F} = W$. □

Let $N(P)$ still be the Newton polygon with vertices $v_0, v_1, \ldots, v_{J+1}$ as defined in the last section. As before, for $v_j = (r_j, s_j)$ we set

$$\gamma_j := \frac{r_j - r_{j+1}}{s_{j+1} - s_j}, \quad j = 1, \ldots, J,$$

and assume

$$0 < \gamma_1 < \gamma_2 < \cdots < \gamma_J < \infty, \tag{4.4}$$

i.e., we still avoid edges parallel to the coordinate axes, if not lying on one of the axes. For the trace result, we additionally assume that $s_j \neq k + \frac{1}{p}$ holds for all $j \in 1, \dots, J$ and $k \in \mathbb{N}_0$. For simplicity of notation, w.l.o.g. let us consider the case

$$s_1 = 0 < \frac{1}{p} < s_2 < 1 + \frac{1}{p} < s_3 < 2 + \frac{1}{p} < \cdots < s_J < J - 1 + \frac{1}{p} < s_{J+1}. \tag{4.5}$$

Furthermore, since in the sequel $1 < p < \infty$, $\rho \geq 0$, and $J = (0, T)$ will always be fixed and only the regularities r, s change, we introduce the notation

$$E(s, r) := \mathcal{F}^s_{p,\rho}(J, \mathcal{K}^r_p(\mathbb{R}^n)),$$

where still $\mathcal{F}, \mathcal{K} \in \{H, W\}$. By this notation the intersection space defined by the Newton polygon $N(P)$ is of the form

$$\mathbb{E} = \bigcap_{j=1}^{J+1} E(s_j, r_j).$$

By Lemma 4.4 we obtain for each $j = 1, \dots, J$ the sharp embedding

$$E(s_{j+1}, r_{j+1}) \cap E(s_j, r_j) \hookrightarrow \mathrm{BUC}^{j-1}_\rho \left(J, W_p^{r_j - \gamma_j(j-1+1/p-s_j)}(\mathbb{R}^n)\right).$$

This implies that the trace operators

$$T_j : \mathbb{E} \to W_p^{r_j - \gamma_j(j-1-s_j+1/p)}(\mathbb{R}^n), \quad \eta \mapsto T_j\eta := \partial_t^{j-1}\eta(0), \qquad j = 1, \dots, J,$$

are bounded. We set

$$\mathbb{F} := \Pi_{j=1}^J W_p^{r_j - \gamma_j(j-1-s_j+1/p)}(\mathbb{R}^n)$$

and define the full trace operator on $\mathbb{E}$ by

$$T : \mathbb{E} \to \mathbb{F}, \quad \eta \mapsto (T_j\eta)_{j=1}^J = \left(\partial_t^{j-1}\eta(0)\right)_{j=1}^J. \tag{4.6}$$

For the treatment of boundary value problems with fully inhomogeneous right hand sides the following result is of interest.

4.5. **Theorem.** *The trace operator T as defined in (4.6) is surjective. More precisely, there exists a bounded linear operator $R : \mathbb{F} \to \mathbb{E}$ such that $T \circ R = \mathrm{id}_\mathbb{F}$.*

Proof. Let $\sigma = (\sigma_1, \dots, \sigma_J) \in \mathbb{F}$. We set

$$\eta_j(t) := \left(\sum_{k=1}^J c_{jk} e^{-ktA^{\gamma_j}}\right) A^{-(j-1)\gamma_j}\sigma_j, \qquad t > 0,\ j = 1, \dots, J,$$

with $A := (1 - \Delta)^{-1/2}$. According to Corollary 2.10 we know that A admits a bounded $\mathcal{H}^\infty$-calculus on all spaces $E(s, r)$ and therefore the semigroups appearing in the definition of η_j are well-defined. Now determine the constants c_{jk} by requiring the η_j to satisfy

$$\partial_t^{m-1}\eta_j(0) = \delta_{j-1,m-1}\sigma_j \qquad (j, m = 1, \dots, J), \tag{4.7}$$

with the Kronecker symbol $\delta_{k,\ell}$. In view of

$$\partial_t^{m-1}\eta_j(t) = \left(\sum_{k=1}^{J}(-k)^{m-1}c_{jk}e^{-ktA^{\gamma_j}}\right)A^{-(j-m)\gamma_j}\sigma_j,$$

requirement (4.7) yields the linear systems

$$Vc_j = \mathrm{e}_j, \qquad j=1,\dots,J,$$

for $c_j := (c_{j1},\dots,c_{jJ})$. Here e_j is the j-th unit vector and V the Vandermonde matrix

$$V = \left((-k)^{m-1}\right)_{m,k=1}^{J}.$$

By virtue of $\det V = \Pi_{1\le k<\ell\le J}(\ell-k)\neq 0$ this system is uniquely solvable. This implies the existence of reals c_{jk}, $j,k=1,\dots,J$ such that (4.7) is satisfied.

Next, we claim that

$$\eta_j \in E(s_j + r_j/\gamma_j, 0)\cap E(0, r_j+\gamma_j s_j), \qquad (j=1,\dots,J). \tag{4.8}$$

Indeed, observe that

$$A^{-(j-1)\gamma_j}\sigma_j \in W_p^{r_j-\gamma_j(\frac{1}{p}-s_j)}(\mathbb{R}^n) = \left(\mathcal{K}_p^{r_j+\gamma_j(s_j-1)}(\mathbb{R}^n), D(A^{\gamma_j})\right)_{1-1/p,p}$$

where $D(A^{\gamma_j})$ denotes the domain of the operator A^{γ_j} in the space $\mathcal{K}_p^{r_j+\gamma_j(s_j-1)}(\mathbb{R}^n)$. Hence, the maximal regularity of A^{γ_j} on the space $\mathcal{K}_p^{r_j+\gamma_j(s_j-1)}(\mathbb{R}^n)$ yields

$$e^{-ktA^{\gamma_j}}A^{-(j-1)\gamma_j}\sigma_j \in E(1, r_j+\gamma_j(s_j-1))\cap E(0, r_j+\gamma_j s_j).$$

On the other hand, taking the m-th time derivative we obtain

$$\partial_t^m e^{-ktA^{\gamma_j}}A^{-(j-1)\gamma_j}\sigma_j = (-k)^m e^{-ktA^{\gamma_j}}A^{(m+1-j)\gamma_j}\sigma_j.$$

By virtue of $A^{(m+1-j)\gamma_j}\sigma_j \in W_p^{r_j-\gamma_j(m+\frac{1}{p}-s_j)}(\mathbb{R}^n)$, here the maximal regularity of A^{γ_j} on the space $\mathcal{K}_p^{r_j-\gamma_j(m+1-s_j)}(\mathbb{R}^n)$ gives us

$$\partial_t^m e^{-ktA^{\gamma_j}}A^{-(j-1)\gamma_j}\sigma_j \in E(1, r_j-\gamma_j(m+1-s_j))\cap E(0, r_j-\gamma_j(m-s_j)).$$

An application of Lemma 4.3 shows that the latter space is continuously embedded in $E(\tau, r_j-\gamma_j(m-s_j+\tau))$ for each $\tau\in[0,1]$. We can always find $m\in\mathbb{N}_0$ and $\tau\in[0,1]$ such that

$$r_j-\gamma_j(m+\tau-s_j)=0 \quad\Leftrightarrow\quad m+\tau = \frac{r_j}{\gamma_j}+s_j.$$

This implies that

$$e^{-ktA^{\gamma_j}}A^{-(j-1)\gamma_j}\sigma_j \in E(\tau+m,0) = E(r_j/\gamma_j+s_j,0).$$

Summarizing, we obtain

$$\begin{aligned}\|\eta_j\|_{E(s_j+r_j/\gamma_j,0)\cap E(0,r_j+\gamma_j s_j)} &\le C\Big(\|A^{(m+1-j)\gamma_j}\sigma_j\|_{W_p^{r_j-\gamma_j(m+1/p-s_j)}(\mathbb{R}^n)} \\ &\qquad +\|A^{-(j-1)\gamma_j}\sigma_j\|_{W_p^{r_j-\gamma_j(1/p-s_j)}(\mathbb{R}^n)}\Big) \\ &\le C\|\sigma_j\|_{W_p^{r_j-\gamma_j(j-1+1/p-s_j)}(\mathbb{R}^n)},\end{aligned} \tag{4.9}$$

which proves the claimed regularity for η_j in (4.8).

Let

$$g(r) := -\frac{r_j/\gamma_j + s_j}{r_j + \gamma_j s_j} r + \frac{r_j}{\gamma_j} + s_j = \frac{r - r_j}{r_{j+1} - r_j}(s_{j+1} - s_j) + s_j, \qquad r \in [0, r_j + \gamma_j s_j],$$

be the line connecting the points $(0, r_j/\gamma_j + s_j)$ and $(r_j + \gamma_j s_j, 0)$. By construction, it is clear that the edge $[v_j v_{j+1}]$ is a part of the graph of g. Thus, by the convexity of the Newton polygon, we see that $N(P)$ is completely contained in each of the triangles $tr(0, r_j + \gamma_j s_j, r_j/\gamma_j + s_j)$, $j = 1, \dots, J$. These geometric observations show that

$$s_k \le g(r_k) \qquad (k = 1, \dots, J+1).$$

On the other hand, Lemma 4.3 yields the embedding

$$E(s_j + r_j/\gamma_j, 0) \cap E(0, r_j + \gamma_j s_j) \hookrightarrow E(g(r), r), \qquad r \in [0, r_j + \gamma_j s_j].$$

Combining these two facts and having in mind that $r_k \in [0, r_j + \gamma_j s_j]$ for all $k = 1, \dots, J+1$ results in

$$E(s_j + r_j/\gamma_j, 0) \cap E(0, r_j + \gamma_j s_j) \hookrightarrow E(s_k, r_k) \qquad (k = 1, \dots, J+1).$$

Consequently,

$$\eta_j \in E(s_j + r_j/\gamma_j, 0) \cap E(0, r_j + \gamma_j s_j) \hookrightarrow \mathbb{E} = \bigcap_{k=1}^{J+1} E(s_k, r_k) \qquad (j = 1, \dots, J). \tag{4.10}$$

Finally, setting

$$\eta := \sum_{j=1}^{J} \eta_j,$$

we see that $\eta \in \mathbb{E}$ and that $\partial_t^{k-1}\eta(0) = \sigma_k$, $k = 1, \dots, J$. Hence, we may define the operator R by

$$R : \mathbb{F} \to \mathbb{E}, \quad \sigma \mapsto \eta.$$

Then by construction the property $T \circ R = \mathrm{id}_{\mathbb{F}}$ and the linearity of R are obvious. Relations (4.9) and (4.10) further imply that

$$\begin{aligned} \|R\sigma\|_{\mathbb{E}} &\le \sum_{j=1}^{J} \|\eta_j\|_{\mathbb{E}} \le C \sum_{j=1}^{J} \|\eta_j\|_{E(s_j + r_j/\gamma_j, 0) \cap E(0, r_j + \gamma_j s_j)} \\ &\le C \sum_{j=1}^{J} \|\sigma_j\|_{W_p^{r_j - \gamma_j(j-1+1/p-s_j)}(\mathbb{R}^n)} \\ &= C\|\sigma\|_{\mathbb{F}} \qquad (\sigma \in \mathbb{F}), \end{aligned}$$

which proves the boundedness of R. □

5. Application to a mixed order system

We demonstrate the value of the results provided above by an application to the Stefan problem with Gibbs-Thomson correction. In fact, we will be able to give a systematic and relatively short proof of maximal regularity for the linearized model. Recall that the Stefan problem is a model for phase transitions in liquid-solid systems and accounts for heat diffusion and exchange of latent heat in a homogeneous medium. The linearized one-phase model problem reads as

$$\left\{\begin{array}{rcll} (\partial_t - \Delta)v & = & f_1 & \text{in } (0,T)\times\mathbb{R}^{n+1}_+, \\ v - \Delta\sigma & = & f_2 & \text{on } (0,T)\times\mathbb{R}^n, \\ \partial_t\sigma + \partial_n v & = & f_3 & \text{on } (0,T)\times\mathbb{R}^n, \\ v|_{t=0} & = & v_0 & \text{in } \mathbb{R}^{n+1}_+, \\ \sigma|_{t=0} & = & \sigma_0 & \text{in } \mathbb{R}^n. \end{array}\right. \tag{5.1}$$

Here the unknowns are the temperature $v : (0,T)\times\mathbb{R}^{n+1}_+ \to \mathbb{R}$ and the function $\sigma : (0,T)\times\mathbb{R}^n \to \mathbb{R}$, which corresponds to the free surface given by

$$\Gamma(t) := \operatorname{graph}(\sigma(t)), \quad t \in (0,T).$$

By means of the Laplace-Fourier transform the above system is reduced to an ODE in the normal component x_n with Lopatinskii matrix

$$L(\xi,\lambda) = \begin{pmatrix} 1 & |\xi| \\ -\sqrt{\lambda+|\xi|^2} & \lambda \end{pmatrix}$$

on the boundary. Consequently, its determinant

$$\det L(\xi,\lambda) = \lambda + |\xi|^2\sqrt{\lambda+|\xi|^2}$$

is rotation invariant in ξ and inhomogeneous in λ and $|\xi|$. Replacing $|\xi|$ by z and setting

$$P(z,\lambda) := \lambda + z^2\sqrt{\lambda+z^2},$$

we easily find that the Newton polygon of P is given by

$$N(P) = \operatorname{conv}\{(0,0),(3,0),(2,1/2),(0,1)\}.$$

From this the γ-principal part of P is readily calculated to the result

$$P_\gamma(z,\lambda) = \left\{\begin{array}{rl} z^3, & 0<\gamma<2, \\ z^2\sqrt{\lambda+z^2}, & \gamma = 2, \\ z^2\sqrt{\lambda}, & 2<\gamma<4, \\ \lambda + z^2\sqrt{\lambda}, & \gamma=4, \\ \lambda, & \gamma>4. \end{array}\right.$$

For $\theta\in(0,\pi)$ and $\epsilon\in(0,(\pi-\theta)/2)$ we obviously obtain

$$P_\gamma(z,\lambda) \neq 0 \quad \left((z,\lambda)\in\overline{\Sigma}_\theta\times\overline{\Sigma}_\epsilon,\ |\lambda|>0,\ \gamma>0\right).$$

From Theorem 3.3 we conclude that there exists a $\lambda_0>0$ such that for $r\in\mathbb{R}$, $s\geq 0$, and $\rho\geq\lambda_0$ we have that

$$P \in \operatorname{Isom}\left(\mathscr{D}(P), {}_0\mathcal{F}^{s+1/2}_{p,\rho}(\mathbb{R}_+, \mathcal{K}^r_p(\mathbb{R}^n))\right) \tag{5.2}$$

where

$$\mathscr{D}(P) = {}_0\mathcal{F}^{s+1}_{p,\rho}(\mathbb{R}_+, \mathcal{K}^r_p(\mathbb{R}^n)) \cap {}_0\mathcal{F}^{s+1/2}_{p,\rho}(\mathbb{R}_+, \mathcal{K}^{r+2}_p(\mathbb{R}^n)) \cap {}_0\mathcal{F}^{s}_{p,\rho}(\mathbb{R}_+, \mathcal{K}^{r+3}_p(\mathbb{R}^n)).$$

Once (5.2) is proved, it is no longer difficult to derive maximal regularity for system (5.1) in the canonical zero time trace spaces. More precisely, if $p \in (1,\infty)$, $\rho \geq 0$, $T \in (0,\infty)$, and $J = (0,T)$, we have that for each

$$\begin{aligned} f_1 &\in {}_0\mathbb{F}_1 := L^p_\rho(J, L^p(\mathbb{R}^{n+1}_+)), \\ f_2 &\in {}_0\mathbb{F}_2 := {}_0W^{1-1/2p}_{p,\rho}(J, L^p(\mathbb{R}^n)) \cap L^p_\rho(J, W^{2-1/p}_p(\mathbb{R}^n)), \\ f_2 &\in {}_0\mathbb{F}_3 := {}_0W^{1/2-1/2p}_{p,\rho}(J, L^p(\mathbb{R}^n)) \cap L^p_\rho(J, W^{1-1/p}_p(\mathbb{R}^n)) \end{aligned}$$

there is a unique solution

$$v \in {}_0\mathbb{E}_1 := {}_0H^1_{p,\rho}(J, L^p(\mathbb{R}^{n+1}_+)) \cap L^p_\rho(J, H^2_p(\mathbb{R}^{n+1}_+)), \tag{5.3}$$

$$\sigma \in {}_0\mathbb{E}_2 := {}_0W^{3/2-1/2p}_{p,\rho}(J, L^p(\mathbb{R}^n)) \cap W^{1-1/2p}_{p,\rho}(J, H^2_p(\mathbb{R}^n)) \cap L^p_\rho(J, W^{4-1/p}_p(\mathbb{R}^n)) \tag{5.4}$$

of system (5.1). Indeed, relation (5.2) immediately implies maximal regularity for the function σ describing the free surface on the time interval $J = \mathbb{R}_+$ and for $\rho \geq \lambda_0$ and some $\lambda_0 > 0$. Lemma 2.2 (iii) then yields (5.4) for arbitrary $\rho \geq 0$ and finite time intervals $J = (0,T)$. But then the maximal regularity for the temperature v follows easily from the fact that v now can be regarded as the unique solution of the heat equation

$$\begin{cases} (\partial_t - \Delta)v = f_1 & \text{in } (0,T) \times \mathbb{R}^{n+1}_+, \\ v = f_2 + \Delta\sigma & \text{on } (0,T) \times \mathbb{R}^n, \\ v|_{t=0} = 0 & \text{in } \mathbb{R}^{n+1}_+, \end{cases}$$

and by using well-known maximal regularity results for that equation.

In order to obtain the corresponding result in general spaces without zero time trace, the result on the trace operator related to the Newton polygon Theorem 4.5 will play a crucial role. With the help of this trace result we will reduce system (5.1) with general right hand sides to a zero time trace problem. To this end, suppose $(f_1, f_2, f_3) \in \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, where $\mathbb{E}_1, \mathbb{E}_2, \mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_3$ denote corresponding general classes, and let $u_0 \in W^{2-2/p}_p(\mathbb{R}^{n+1}_+)$, $\sigma_0 \in W^{4-3/p}_p(\mathbb{R}^n)$. Without loss of generality we assume $p > 3$. Then, for compatibility reasons the data have to satisfy

$$u_0|_{\partial\mathbb{R}^{n+1}_+} - \Delta\sigma_0 = f_2|_{t=0} \quad \text{and} \tag{5.5}$$

$$f_3|_{t=0} + \partial_n u_0|_{\partial\mathbb{R}^{n+1}_+} \in W^{2-6/p}_p(\mathbb{R}^n). \tag{5.6}$$

Next, we extend u_0 to the whole space $\mathbb{R}^{n+1}$ and denote the extension by $\widetilde{u}_0$. It is well known that $\widetilde{u}_0$ can be chosen such that $\widetilde{u}_0 \in W^{2-2/p}_p(\mathbb{R}^{n+1})$. Let w be the whole space solution of the heat equation

$$\begin{cases} w_t - \Delta w = f_1 & \text{in } J \times \mathbb{R}^{n+1}, \\ w|_{t=0} = \widetilde{u}_0 & \in \mathbb{R}^{n+1}. \end{cases}$$

Then $u_1 := w|_{\mathbb{R}^{n+1}_+} \in \mathbb{E}_1$. Furthermore, let $\sigma_1 \in \mathbb{E}_2$ be an extension determined by the trace

$$\gamma\sigma_1 := (\sigma_1|_{t=0}, \partial_t\sigma_1|_{t=0}) = (\sigma_0, f_3|_{t=0} + \partial_n u_0|_{\partial_{\mathbb{R}^{n+1}_+}}) \in W^{4-3/p}_p(\mathbb{R}^n) \times W^{2-6/p}_p(\mathbb{R}^n),$$

which exists in view of the surjectivity of the trace operator

$$\gamma : \mathbb{E}_2 \to W_p^{4-3/p}(\mathbb{R}^n) \times W_p^{2-6/p}(\mathbb{R}^n)$$

given by Theorem 4.5. Now, assume (u, σ) is the desired solution of the fully inhomogeneous system (5.1). Then,

$$(u_2, \sigma_2) := (u, \sigma) - (u_1, \sigma_1)$$

satisfies the reduced system

$$\begin{cases} (\partial_t - \Delta)v &= 0 & \text{in } (0,T) \times \mathbb{R}^{n+1}_+, \\ v - \Delta\sigma &= f_2 - u_1 - \Delta\sigma_1 & \text{on } (0,T) \times \mathbb{R}^n, \\ \partial_t\sigma + \partial_n v &= f_3 - \partial_t\sigma_1 + \partial_n u_1 & \text{on } (0,T) \times \mathbb{R}^n, \\ v|_{t=0} &= 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \sigma|_{t=0} &= 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{5.7}$$

By the compatibility conditions on the data and the choice of the extension σ_1, we obviously have

$$\begin{aligned} f_2 - u_1 - \Delta\sigma_1 &\in {}_0\mathbb{F}_2, \\ f_3 - \partial_t\sigma_1 + \partial_n u_1 &\in {}_0\mathbb{F}_3. \end{aligned}$$

Thus, we may reverse the argument, i.e., we fix (u_1, σ_1) as defined above and require (u_2, σ_2) to be the unique solution of system (5.7) given by (5.3) and (5.4). Then,

$$(u, \sigma) := (u_1, \sigma_1) + (u_2, \sigma_2)$$

is the unique solution to the fully inhomogeneous system (5.1) belonging to the desired regularity classes. So, we have proved

5.1. **Theorem.** *Let $p > 3$, $\rho \geq 0$, $T \in (0, \infty)$, and $J = (0,T)$. Then for each*

$$(f_1, f_2, f_3, u_0, \sigma_0) \in \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3 \times W_p^{2-2/p}(\mathbb{R}^{n+1}_+) \times W_p^{4-3/p}(\mathbb{R}^n)$$

satisfying condition (5.5) and (5.6), there is a unique solution

$$(u, \sigma) \in \mathbb{E}_1 \times \mathbb{E}_2$$

of the linearized Stefan problem (5.1).

For $\rho = 0$ Theorem 5.1 was already obtained in [11, Theorem 1.4]. However, there the authors used a more direct approach, which seems to be difficult to generalize to symbols of more intricate structure. Our general approach also applies nicely to much more complicate symbols, as e.g. to the symbol related to the spin-coating process. This is demonstrated in [4].

References

[1] R.A. Adams. *Sobolev Spaces.* Academic Press, 1978.

[2] P. Clément, B. de Pagter, F.A. Sukochev, and H. Witvliet. Schauder decompositions and multiplier theorems. *Studia Math.*, 138:135–163, 2000.

[3] M. Cowling, I. Doust, A. McIntosh, and A. Yagi. Banach space operators with a bounded H^∞ functional calculus. *J. Austral. Math. Soc.*, 60:51–89, 1996.

[4] R. Denk, M. Geißert, M. Hieber, J. Saal, and O. Sawada. A free boundary problem associated to the spin-coating process. In preparation.

[5] R. Denk, M. Hieber, and J. Prüss. $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 166:viii+114, 2003.
[6] R. Denk, R. Mennicken, and L. Volevich. The Newton polygon and elliptic problems with parameter. *Math. Nachr.*, 192:125–157, 1998.
[7] R. Denk, R. Mennicken, and L. Volevich. Boundary value problems for a class of elliptic operator pencils. *Integral Equations Operator Theory*, 38:410–436, 2000.
[8] R. Denk, J. Prüss, and R. Zacher. Maximal L^p-regularity of parabolic problems with boundary dynamics of relaxation type. *Konstanzer Schriften in Mathematik und Informatik*, 241, 2008. Preprint, 31 pp.
[9] R. Denk and L. Volevich. Elliptic boundary value problems with large parameter for mixed order systems. *Amer. Math. Soc. Transl*, 206(2):29–64, 2002.
[10] J. Escher. Quasilinear parabolic systems with dynamical boundary conditions. *Commun. Partial Differ. Equations*, 18:1309–1364, 1993.
[11] J. Escher, J. Prüß, and G. Simonett. Analytic solutions for a Stefan problem with Gibbs-Thomson correction. *J. reine angew. Math.*, 563:1–52, 2003.
[12] S. G. Gindikin and L. R. Volevich. *The Method of Newton's Polyhedron in the Theory of Partial Differential Equations*, volume 86 of *Math. Appl. (Soviet Ser.)*. Kluwer Academic, Dordrecht, 1992.
[13] S. G. Gindikin and L. R. Volevich. Mixed problem for partial differential operators with quasihomogeneous principal part. *Translation of Mathematical Monographs*, 147, 1996. Amer. Math. Soc., Providence, Rhode Island, 233 pp.
[14] M. Girardi and L. Weis. *Criteria for $\mathcal{R}$-boundedness of operator families,* in G. Ruiz Goldstein et. al. (eds.), Evolution Equations, pp. 203-221, volume 234 of *Lecture Notes in Pure and Appl. Math.* Marcel Dekker, New York, 2003.
[15] P. Grisvard. Commutativité de deux foncteurs d'interpolation et applications. *J. Math. Pures Appl.*, 45:143–290, 1966.
[16] N. Kalton and L. Weis. The H^∞-calculus and sums of closed operators. *Math. Ann.*, 321:319–345, 2001.
[17] A. Kozhevnikov. Asymptotics of the spectrum of Douglis-Nirenberg elliptic operators on a compact manifold. *Math. Nachr.*, 182:261–294, 1996.
[18] P. C. Kunstmann and L. Weis. Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems, and H^∞-functional calculus. *Functional Analytic Methods for Evolution Equations, Lecture notes in Math.*, 1855, 2004.
[19] P. L. Lions. Une propriété de stabilité par interpolation, applications. *Compt. Rend. Acad. Sci. Paris*, 256:855–857, 1963.
[20] A. McIntosh. Operators which have an H^∞-calculus. *Proc. Centre Math. Analysis*, 14:210–231, 1986.
[21] J. Prüss. *Evolutionary Integral Equations and Applications.* Birkhäuser, 1993.
[22] J. Prüss, R. Racke, and S. Zheng. Maximal regularity and asymptotic behavior of solutions for the Cahn-Hilliard equation with dynamic boundary conditions. *Ann. Mat. Pura Appl., IV. Ser.*, 185:627–648, 2003.
[23] R. Racke and S. Zheng. The Cahn-Hilliard equations with dynamic boundary conditions. *Adv. Differential Equations*, 8:83–110, 2003.
[24] H.-J. Schmeisser: Vector-valued Sobolev and Besov spaces. *Semin. Analysis, Berlin/GDR 1985/86 Teubner-Texte Math.*, 96:4–44, 1987.
[25] P.E. Sobolevskii. Fractional powers of coercively positive sums of operators. *Soviet Math. Dokl.*, 16:1638–1641, 1975.
[26] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators.* North Holland, 1978.
[27] L. R. Volevich. Newton polygon and general parameter-elliptic (parabolic) systems. *Russian J. Math. Phys.*, 8:375–400, 2001.
[28] L. Weis. Operator-valued multiplier theorems and maximal L^p-regularity. *Math. Ann.*, 319(4):735–758, 2001.
[29] R. Zacher. Maximal regularity of type L_p for abstract parabolic volterra equations. *J. Evol. Equ.*, 5(1):79–103, 2005.