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Embedding the prime model of real exponentiation into o-minimal exponential fields

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Abstract

Motivated by the decidability question for the theory of real exponentiation and by the Transfer Conjecture for o-minimal exponential fields, we show that, under the assumption of Schanuel's Conjecture, the prime model of real exponentiation is embeddable into any o-minimal exponential field, where the embedding is not necessarily elementary. This is a consequence of an unconditional model theoretic embeddability result that we obtain by applying Kőnig's Lemma.

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1 | INTRODUCTION

In his highly influential work [12], Tarski proved that the complete theory T_{rcf} of the real closed field $\mathbb{R}$ in the language of ordered rings $\mathcal{L}_{\mathrm{or}} = \{+, -, \cdot, 0, 1, <\}$ is decidable, by presenting an explicit quantifier-elimination algorithm for T_{rcf}. As a result, he noted that the $\mathcal{L}_{\mathrm{or}}$-structure $(\mathbb{R}, +, -, \cdot, 0, 1, <)$ is, in modern terminology, o-minimal, that is, any unary definable subset of this structure is a finite union of points and open intervals. In the same work, Tarski asked whether decidability can also be obtained for the complete theory $T_{\exp}$ of the real exponential field $\mathbb{R}_{\exp} = (\mathbb{R}, +, -, \cdot, 0, 1, <, \exp)$, where exp denotes the standard exponential function $x \mapsto \mathrm{e}^x$ on $\mathbb{R}$. Although this question is open to the date, considerable progress has been made since the 1990s: In [13], Wilkie proved that $\mathbb{R}_{\exp}$ is model complete, and thus this structure is o-minimal (see also van den Dries, Macintyre, and Marker [2]). Building on this result, Macintyre and Wilkie [9] showed that under the assumption of the real version of Schanuel's Conjecture below, $T_{\exp}$ is decidable.

Schanuel's Conjecture. *Let $n \in \mathbb{N}$ and let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ be $\mathbb{Q}$-linearly independent. Then the transcendence degree of $\mathbb{Q}(\alpha_1, \ldots, \alpha_n, \mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_n})$ over $\mathbb{Q}$ is at least n.*

In general, an ordered exponential field $\mathcal{K} = (K, +, -, \cdot, 0, 1, <, E)$ is an expansion of an ordered field $(K, +, -, \cdot, 0, 1, <)$ by an exponential E, that is, an order-preserving isomorphism from the ordered additive group $(K, +, 0, <)$ to the ordered multiplicative group $(K^{>0}, \cdot, 1, <)$. We denote the corresponding first-order language $\mathcal{L}_{\mathrm{or}} \cup \{E\}$, where E is a unary function symbol, by $\mathcal{L}_{\exp}$. Following the terminology of Krapp [6], we call an ordered exponential field $\mathcal{K}$ an EXP-field if its exponential satisfies the first-order $\mathcal{L}_{\exp}$-sentence expressing the differential equation $E' = E$ with initial condition $E(0) = 1$. Although $\mathbb{R}_{\exp}$ and, more generally, any model of $T_{\exp}$ is an o-minimal EXP-field, the following is still open:

Transfer Conjecture. *Any o-minimal* EXP-*field is elementarily equivalent to* $\mathbb{R}_{\exp}$.

Berarducci and Servi [1] showed that the Transfer Conjecture would imply that $T_{\exp}$ is decidable, motivating the study of o-minimal EXP-fields. One approach toward proving the Transfer Conjecture is to show that the (unique) prime model of real exponentiation, that is, the prime model of $T_{\exp}$, elementarily embeds into any o-minimal EXP-field, which is also the motivating question for this line of research.

Question 1.1 (Main question). Does the prime model of $T_{\exp}$ elementarily embed into any o-minimal EXP-field?

As Question 1.1 can be answered positively if and only if the Transfer Conjecture holds, it is natural to approach this question only under the assumption of Schanuel's Conjecture. While this note does not provide a complete answer to Question 1.1, we show in Theorem 4.3 that under the assumption of Schanuel's Conjecture, the prime model of $T_{\exp}$ embeds into any o-minimal EXP-field (however, this embedding might not necessarily be elementary). Theorem 4.3 is deduced from the model theoretic main result of this paper (Theorem 2.1) that establishes sufficient conditions on two structures $\mathcal{A}$ and $\mathcal{B}$ and a substructure $\mathcal{A}'$ of $\mathcal{A}$ in order that $\mathcal{A}'$ embeds into $\mathcal{B}$. By application of Theorem 2.1, we also obtain an embeddability result (Theorem 3.6) for exponential algebraic closures of $\mathbb{Z}$ within one o-minimal EXP-field into another, without assuming Schanuel's Conjecture.

1.1 | Notation and terminology

More background on the model theoretic notation and terminology we use can be found in Marker [10]. If it is clear from the context, then the $\mathcal{L}_{\mathrm{or}}$-structure of an ordered field $(K, +, -, \cdot, 0, 1, <)$ is simply denoted by K and the $\mathcal{L}_{\exp}$-structure of an ordered exponential field $(K, +, -, \cdot, 0, 1, <, E)$ simply by (K, E). Given an ordered exponential field (K, E), we say that a subfield $F \subseteq K$ is "exponentially closed" in (K, E) if the restriction $E|_F$ is an exponential on F. In this case we also denote $E|_F$ simply by E. For any structure $\mathcal{M} = (M, \ldots)$, its complete theory is denoted by $\mathrm{Th}(\mathcal{M})$ and its existential theory by $\mathrm{Th}_\exists(\mathcal{M})$. For instance, $T_{\exp} = \mathrm{Th}(\mathbb{R}_{\exp})$ denotes the theory of real exponentiation and $\mathrm{Th}_\exists(\mathbb{R}_{\exp})$ denotes the existential theory of real exponentiation. The definable closure of a set C in $\mathcal{M}$ is denoted by $\mathrm{dcl}(C; \mathcal{M})$. We say that a set is definable if it is definable *with* parameters. For a formula $\varphi(x_1, \ldots, x_n)$, we denote the subset of M^n it defines by $\varphi(\mathcal{M}) = \{(a_1, \dots, a_n) \in M^n \mid \mathcal{M} \vDash \varphi(a_1, \dots, a_n)\}$. Variable tuples are denoted by $\underline{x}$, and we only specify their length if it is of importance. If $\underline{x} = (x_1, \dots, x_n)$ and f is a unary map, we write $f(\underline{x})$ for the tuple $(f(x_1), \dots, f(x_n))$. We denote by $\mathbb{N}$ the set of natural numbers *without* 0. "Throughout the rest of this note, let (K, E) denote an o-minimal EXP-field".

2 | EMBEDDING SUBSTRUCTURES

We start by proving our model theoretic main result relying on Kőnig's Lemma, which we briefly recall in the following (see Marker [10, p. 320 f.]). A "finite branching tree" is a partial order $(T, <)$ that contains some $r \in T$ such that for any $t \in T$ we have $r \leqslant t$, $\{s \in T \mid s < t\}$ is linearly ordered by $<$ and t only has finitely many immediate successors in T (i.e., finitely many $s > t$ such that T contains no element strictly between t and s). A "path" through T is a function $f : \omega \to T$ such that $f(n) < f(n+1)$ for any $n < \omega$.

Kőnig's Lemma. *Let $(T, <)$ be an infinite finite branching tree. Then there is a path through T.*

Theorem 2.1. *Let $\mathcal{L}$ be a countable language, let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{L}$-structures and let $\mathcal{A}'$ be a countably infinite substructure of $\mathcal{A}$. Suppose that the following hold.*

- (i) $\mathcal{B} \vDash \mathrm{Th}_\exists(\mathcal{A})$.
- (ii) *For any $a \in A'$, there is an existential $\mathcal{L}$-formula $\theta(x)$ such that $\mathcal{A} \vDash \theta(a)$ and both $\theta(\mathcal{A})$ and $\theta(\mathcal{B})$ are finite (not necessarily of equal cardinality).*

Then $\mathcal{A}'$ can be embedded into $\mathcal{B}$ as an $\mathcal{L}$-substructure.

Proof. Let $(a_n)_{n<\omega}$ be an enumeration of A' and, for any $n < \omega$, let $\theta_n(x)$ be an existential $\mathcal{L}$-formula such that $\mathcal{A} \vDash \theta_n(a_n)$ and $|\theta_n(\mathcal{A})|, |\theta_n(\mathcal{B})| < \infty$. Note that $\theta_n(\mathcal{B}) \neq \emptyset$, as $\mathcal{B} \vDash \mathrm{Th}_\exists(\mathcal{A})$. Let $(\exists \underline{x}\ \psi_n(\underline{x}))_{n<\omega}$ be an enumeration of $\mathrm{Th}_\exists(\mathcal{A}')$, where each $\psi_n(\underline{x})$ is quantifier-free. For our later argument, we may assume that the free variables of each $\psi_n(\underline{x})$ are contained in $\{x_0, \dots, x_n\}$ and that $\mathcal{A} \vDash \psi_n(a_0, \dots, a_n)$, as otherwise we may reorder the enumeration and relabel the free variables in each formula.

For each $n < \omega$, let $\psi_n'(\underline{x})$ be the conjunction of $\psi_n(\underline{x})$ with $\theta_0(x_0) \wedge \dots \wedge \theta_n(x_n)$. Then $\mathcal{A} \vDash \psi_n'(\underline{a})$ and $\mathcal{B} \vDash \exists \underline{x}\ \psi_n'(\underline{x})$, as $\exists \underline{x}\ \psi_n'(\underline{x})$ is logically equivalent to a sentence in $\mathrm{Th}_\exists(\mathcal{A})$. Thus, both $\psi_n'(\mathcal{A})$ and $\psi_n'(\mathcal{B})$ are nonempty and finite. Moreover, let $T_n \subseteq B^{n+1}$ consist of all tuples $\underline{b} = (b_0, \dots, b_n) \in B^{n+1}$ with $\mathcal{B} \vDash \psi_0'(\underline{b}) \wedge \dots \wedge \psi_n'(\underline{b})$. Again, T_n is nonempty, as $\mathcal{B} \vDash \mathrm{Th}_\exists(\mathcal{A})$, as well as finite.

Let T be the disjoint union $T = \bigcup_{n=-1}^{\infty} T_n$, where $T_{-1} := \{r\}$ with $r = ()$ denoting the null tuple. For any $i, j \in \{-1, 0, 1, \dots\}$ with $i \leqslant j$ and any $s \in T_i$ and $t \in T_j$ we set $s \leqslant t$ if s is the projection of t onto its first $i+1$ components. Then $(T, <)$ is a finite branching tree with minimal element r. By Kőnig's Lemma, there is some path $f : \omega \to T$ through T. For any $n < \omega$, we let b_n be the $(n+1)$th entry of the tuple $f(n+1)$. Then $\mathcal{B} \vDash \psi_0'(\underline{b}) \wedge \dots \wedge \psi_n'(\underline{b})$.

Finally, we let $\iota : A' \to B$ map each a_n to b_n. Note that for any $n < \omega$ we have $\mathcal{B} \vDash \psi_n(\underline{b})$. Due to the enumeration we chose for $\mathrm{Th}_\exists(\mathcal{A}')$, we thus obtain that ι preserves existential formulae, that is, for any existential $\mathcal{L}$-formula $\varphi(\underline{x})$ with $\mathcal{A}' \vDash \varphi(\underline{a})$ also $\mathcal{B} \vDash \varphi(\underline{b})$. Hence, ι preserves quantifier-free formulae in both directions, showing that ι is, indeed, an $\mathcal{L}$-embedding. □

3 | EXPONENTIAL ALGEBRAIC CLOSURES

In the following, we mostly follow the terminology of Macintyre [8] and Kirby [5] adjusted to our context. For a subring $A \subseteq K$, we say that A is an "E-ring" if it is closed under E, that is, for any $a \in A$ also $E(a) \in A$. The smallest E-ring in K is denoted by $\mathbb{Z}^E$. If B is a ring, then we denote by $B[\underline{x}]_E = B[x_1, \dots, x_n]_E$ the ring of "exponential polynomials" in n variables, which consists of all functions of the form $p(\underline{x}, E(\underline{x}))$ for some $p(\underline{x}, \underline{y}) \in B[x_1, \dots, x_n, y_1, \dots, y_n]$.[1]

Definition 3.1.

(i) Let $n \in \mathbb{N}$ and let $A \subseteq K$ be an E-ring. Moreover, let $f_1, \dots, f_n \in A[x_1, \dots, x_n]_E$ and let $|J_{f_1,\dots,f_n}(\underline{x})|$ denote the determinant of the Jacobian matrix

$$J_{f_1,\dots,f_n}(\underline{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\underline{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\underline{x}) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(\underline{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\underline{x}) \end{pmatrix}.$$

Then the "Khovanskii system $S(\underline{x})$" of $f_1, \dots, f_n$ over A is the system of equations and inequations

$$f_1(\underline{x}) = \dots = f_n(\underline{x}) = 0 \text{ and} |J_{f_1,\dots,f_n}(\underline{x})| \neq 0.$$

(ii) Let B be a subset of K and let $A \subseteq K$ be the smallest E-ring containing B. An element $a_1 \in K$ is said to be "exponentially algebraic" over B if for some $n \in \mathbb{N}$ there exist a Khovanskii system $S(x_1, \dots, x_n)$ over A and $a_2, \dots, a_n \in K$ such that $(a_1, \dots, a_n)$ solves this system. The set of all elements of K that are exponentially algebraic over B is called the "exponential algebraic closure" of B in (K, E) and is denoted by $\mathrm{CL}_K^E(B)$.

The following can be obtained by simple Jacobian calculations (see also Kirby [5, Lemma 3.3] and Macintyre [8, Lemma 21]).

Lemma 3.2. *Let B be a subset of K. Then $\mathrm{CL}_K^E(B)$ is a subfield of K that is exponentially closed in (K, E).*

Note that the condition on the determinant of the Jacobian to be nonzero implies that Khovanskii systems only have isolated solutions. This is a consequence of the Implicit Function Theorem in o-minimal structures (see van den Dries [3, p. 113]).

Lemma 3.3. *Let $A \subseteq K$ be an E-ring and let $S(\underline{x})$ be a Khovanskii system over A. Then S only has finitely many solutions in K.*

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[1] Note that [8] and [5] use free E-rings in several variables, which allow multiple iterations of exponentiation. However, for the purpose of this note it suffices to work with the ring of exponential polynomials (cf. Servi [11, Proposition 4.5.4]).

Proof. It only remains to note that $S(\underline{x}) = S(x_1, \ldots, x_n)$ defines finitely many connected components in K^n (see [3, section 3.2]). □

We now investigate the connection between the definable closure and the exponential algebraic closure in o-minimal EXP-fields. First, we make an observation following from Lemma 3.3.

Observation 3.4. Let B be a subset of K. Then $\mathrm{CL}_K^E(B) \subseteq \mathrm{dcl}(B;(K,E))$.

Recall that due to definable Skolem functions in o-minimal expansions of ordered groups, for any subset B of K we have that $(\mathrm{dcl}(B;(K,E)),E) \preceq (K,E)$, that is, $\mathrm{dcl}(B;(K,E))$ is the domain of an elementary substructure of (K,E). Hence, $(\mathrm{dcl}(\emptyset;(K,E)),E)$ is the unique prime model of $\mathrm{Th}(K,E)$ (see Krapp [7, Proposition 4.75] for further details). If (K,E) is not only assumed to be an o-minimal EXP-field but already a model of real exponentiation, then we can strengthen the conclusion of Observation 3.4 to $\mathrm{CL}_K^E(B) = \mathrm{dcl}(B;(K,E))$. This result is mentioned by Macintyre in [8, Theorem 22], who attributes it to Wilkie [13]. We point out that it is also an immediate consequence of Jones and Wilkie [4, Theorem 4.2] applied to the locally polynomially bounded structure $\mathcal{M} = (K,\mathcal{F})$ with $\mathcal{F} = \{E\}$.

Proposition 3.5. *Let $(K,E) \models T_{\exp}$ and let B be a subset of K. Then*

$$\mathrm{dcl}(B;(K,E)) = \mathrm{CL}_K^E(B).$$

In particular, $(\mathrm{CL}_K^E(B),E) \preceq (K,E)$ and $(\mathrm{CL}_K^E(\emptyset),E)$ is the prime model of $T_{\exp}$.

The aim of this note is to show that, under the assumption of Schanuel's Conjecture, the prime model of $T_{\exp}$ is embeddable into any o-minimal EXP-field (see Theorem 4.3). This result will be deduced from the following.

Theorem 3.6. *Let (K_1,E_1) and (K_2,E_2) be o-minimal* EXP*-fields. Suppose that $(K_2,E_2) \models \mathrm{Th}_\exists(K_1,E_1)$. Then there exists an $\mathcal{L}_{\exp}$-embedding of the ordered exponential field $(\mathrm{CL}_{K_1}^{E_1}(\emptyset),E_1)$ into (K_2,E_2).*

Proof. We apply Theorem 2.1 to $\mathcal{A} = (K_1,E_1)$, $\mathcal{B} = (K_2,E_2)$ and $\mathcal{A}' = (\mathrm{CL}_{K_1}^{E_1}(\emptyset),E_1)$, which is countably infinite due to Lemma 3.3. To do so, we let $a \in \mathrm{CL}_{K_1}^{E_1}(\emptyset)$. Then there is a Khovanskii system $S(x,\underline{y})$ over $\mathbb{Z}^E$ such that $\mathcal{A} \models \exists \underline{y}\; S(a,\underline{y})$. Setting $\theta(x)$ to be $\exists \underline{y}\; S(x,\underline{y})$ we are done by Lemma 3.3, as $\mathcal{B} \models \mathrm{Th}_\exists(\mathcal{A})$. □

4 | PRIME MODEL OF REAL EXPONENTIATION

"In this section, we denote by $(P,\exp)$ the prime model of $T_{\exp}$". As $(P,\exp) \models T_{\exp}$, any structure containing $(P,\exp)$ as a substructure already satisfies the existential theory $\mathrm{Th}_\exists(\mathbb{R}_{\exp})$. Thus, by Proposition 3.5 and Theorem 3.6, we obtain the following.

Corollary 4.1. *There is an embedding of $(P,\exp)$ into (K,E) if and only if $(K,E) \models \mathrm{Th}_\exists(\mathbb{R}_{\exp})$.*

Corollary 4.1 shows that o-minimal EXP-fields satisfying the existential theory of real exponentiation contain the prime model of real exponentiation as a substructure. Under the assumption of Schanuel's Conjecture, any o-minimal EXP-field satisfies $\mathrm{Th}_{\exists}(\mathbb{R}_{\exp})$. This fact is basically due to Servi [11, p. 104] (see also Krapp [7, Proposition 4.57]).

Fact 4.2. *Assume Schanuel's Conjecture. Then any o-minimal* EXP*-field satisfies* $\mathrm{Th}_{\exists}(\mathbb{R}_{\exp})$.

As a result of Corollary 4.1 and Fact 4.2, we obtain the desired main result of this note.

Theorem 4.3. *Assume Schanuel's Conjecture. Then* $(P, \exp)$ *embeds into any o-minimal* EXP*-field.*

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