Social Networks 78 (2024) 54–64

Contents lists available at ScienceDirect

Social Networks

journal homepage: www.elsevier.com/locate/socnet

The interplay of structural features and observed dissimilarities among centrality indices

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ARTICLE INFO

Keywords:
Centrality
Correlation
Network topology
Threshold graphs

ABSTRACT

An abundance of centrality indices has been proposed which capture the importance of nodes in a network based on different structural features. While there remains a persistent belief that similarities in outcomes of indices is contingent on their technical definitions, a growing body of research shows that structural features affect observed similarities more than technicalities. We conduct a series of experiments on artificial networks to trace the influence of specific structural features on the similarity of indices which confirm previous results in the literature. Our analysis on 1163 real-world networks, however, shows that little of the observations on synthetic networks convincingly carry over to empirical settings. Our findings suggest that although it seems clear that (dis)similarities among centralities depend on structural properties of the network, using correlation type analyses do not seem to be a promising approach to uncover such connections.

1. Introduction

Over the past decades, hundreds of centrality indices have been proposed in the literature (Jalili et al., 2015; Lü et al., 2016). Each index is supposed to assess the structural importance of nodes in networks based on very different assumptions of what constitutes a central position within a network. While we have developed an understanding for what individual indices are measuring (Freeman, 1978; Borgatti, 2005; Borgatti and Everett, 2006), there is still a considerable lack of knowledge about when and why indices produce similar or different results. That is, how correlated centrality indices are.

When correlation among indices is assessed, usually one of two perspectives is adopted. First, high correlation can be a desirable outcome. Many of the traditional indices from the early years of social network analysis were not designed with large scale data in mind and computational complexity was not considered an issue. However, already quadratic runtime can prohibit the use of an index in contemporary empirical settings. Finding computationally efficient alternatives which correlate strongly with computationally expensive indices thus allows to either approximate the expensive measures or even replace them entirely (Li et al., 2015; Ficara et al., 2022).

However, high correlation with existing measures can also be regarded as undesirable from the perspective of crafting novel indices. If a new index is consistently found to correlate highly with existing ones, then it is unclear what new concept of being central the index is actually assessing. Correlation analyses are thus often used in the process of defining new indices (e.g Newman, 2005; Estrada et al., 2009; Chen et al., 2012) based on the assumption that low correlations "indicate distinctive measures likely to be associated with different outcomes" (Valente et al., 2008). Hence, if a new index does not correlate strongly with others, then it does assess centrality from a novel point of view and can be regarded as a viable new measure. Evidently, this approach is already questionable from a substantive point of view, since centrality indices are supposed to be operationalize measurement of empirical phenomena and should not evolve simply out of technical novelty.

Both perspectives highlight the assumption that correlation is contingent on conceptual and technical similarities between indices. Highly correlated indices are expected to share some technical commonalities and similar ideas of what "being central" means. Conceptually different indices, on the other hand, are expected to not be strongly correlated. Schoch et al. (2017), however, show that correlations are not necessarily indicative of such similarity. The reason being that most existing indices induce the same ranking, i.e. are perfectly correlated, on threshold graphs (Schoch and Brandes, 2016). The authors employ a measure, the majorization gap, which allows for the assessment of the distance between a network and its closest threshold graph. The lower this distance, the higher the assumed correlation among any two indices. This connection is illustrated on a set of 60 social networks using degree, betweenness, closeness, and eigenvector centrality. Oldham et al. (2019) extend this analysis and address the relation between

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https://doi.org/10.1016/j.socnet.2023.11.006

Available online 6 December 2023

correlation and topological features with a larger scale study using 17 indices and 8 topological measures on a set of 212 diverse real-world networks. They found that the connection between correlation and the majorization gap is weak at best and the modularity of a network can explain correlation more robustly. Both studies provide some evidence for the importance of network structure for observed similarities rather than technical definition of indices.

In this work, we re-examine the existing results that connect structural features with the correlation among indices using a different set of analytical tools. We start by introducing the technical background that forms the basis of our work and motivate a new measure of rank dissimilarity which we employ throughout our analyses instead of using correlation indices. Afterwards, we perform several experiments on carefully crafted networks to trace the association between specific topological features on dissimilarities in rankings between indices. We end with a large scale analysis on 1163 real-world networks from a diverse background and analyze the connection between indices and structural features using tools from statistical entropy analysis to find the best structural predictors for high (or low) rank dissimilarities among centrality indices. We end with a discussion on the effectiveness of using such an approach to uncover connections between network structure and dissimilarities of indices and propose some alternative research directions.

2. Network centrality, neighborhood-inclusion and induced rankings

We only consider simple undirected graph $G = (V, E)$, the (open) neighborhood $N(u)$ of a node $u \in V$ is defined as $N(u) = v : u, v \in E$, and the closed neighborhood is given by $N[u] = N(u) \cup u$. If the neighborhood of a node u is a subset of the closed neighborhood of a node v, $N(u) \subseteq N[v]$, we speak of neighborhood inclusion. This concept defines a preorder, a reflexive and transitive binary relation, among nodes in a network, and we say that u is dominated by v if $N(u) \subseteq N[v]$. Schoch and Brandes (2016) showed that if u is dominated by v, then u is always less central than v. That is, $c(u) \leq c(v)$ holds for appropriately defined indices $c : V \to \mathbb{R}$, and we say that the index c preserves neighborhood-inclusion. This preservation is strict if $N(u) \subset N[v]$ implies $c(u) < c(v)$. For the technical details of what constitutes an "appropriately defined'' index we refer to the original work. For our purpose, it suffices to say that all commonly used indices fulfill this property, which includes degree, betweenness (Freeman, 1977), closeness (Sabidussi, 1966; Bavelas, 1948), and eigenvector centrality (Bonacich, 1972).

One consequence of this result is that centrality indices can induce the same ranking if the neighborhood-inclusion is total. Graphs of this kind are referred to as threshold graphs (Mahadev and Peled, 1995) or nested graphs (Mariani et al., 2019). Threshold graphs also exhibit a (nested) core–periphery structure, as illustrated in Fig. 1. A core–periphery graph consists of a clique as the core and a periphery which is a set of nodes that are pairwise disconnected and are only connected to core nodes. To construct a random threshold graph, we generate a random binary sequence and iterate through the sequence as follows. For each zero we encounter, an isolated node is added to the graph. For each one, we add a node that is connected to all nodes that have already been added to the network.

To understand what we mean by "inducing the same ranking", we first note that there are five different configurations of pairs $u, v \in V$ when comparing two rankings induced by indices c_1 and c_2:

(i) *concordance* if $c_1(u) > c_1(v)$ and $c_2(u) > c_2(v)$
(ii) *discordance* if $c_1(u) > c_1(v)$ and $c_2(u) < c_2(v)$
(iii) *tie* if $c_1(u) = c_1(v)$ and $c_2(u) = c_2(v)$
(iv) *right tie* if $c_1(u) \neq c_1(v)$ and $c_2(u) = c_2(v)$
(v) *left tie* if $c_1(u) = c_1(v)$ and $c_2(u) \neq c_2(v)$

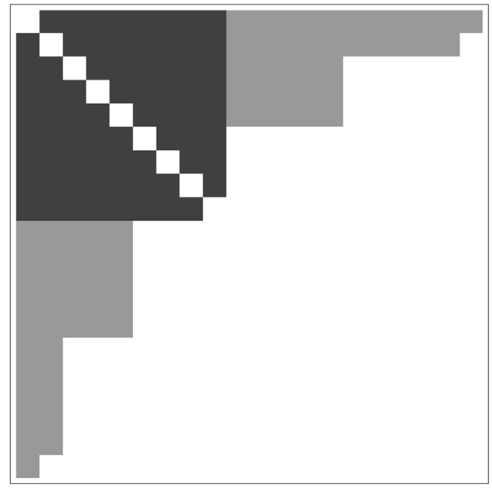

Fig. 1. Patterning of the adjacency matrix of a threshold graph. Dark gray shows the core and light gray the nested periphery.

Rank based correlation indices which are commonly used when rankings of indices are compared, e.g. Spearman's ρ or Kendall's τ, combine counts of these configurations into a value between −1 and 1.

Inducing the same ranking in the strictest sense means that we should only observe concordant or tied pairs ((i) and (iii)). It is easy to verify that this is the case if both c_1 and c_2 strictly preserve the neighborhood-inclusion preorder. Any standard rank based correlation index will yield a perfect correlation of 1 in this case.

Left or right ties, however, can still occur when comparing an index that strictly preserves neighborhood-inclusion with one that does not. A prominent index which does not fulfill strict preservation is betweenness. Hence, comparing betweenness with, say closeness, can yield many left ties. An example of this behavior is shown in Fig. 2.

Betweenness does not distinguish between nodes A, B, C, E, G, and I while closeness only assigns the same rank to A and B. Most rank correlation indices assume that this should decrease correlation. For the example graph Kendall's $\tau = 0.83$, although no discordant pairs are present. We argue that left and right ties merely distinguish between fine and coarse-grained indices and it is not necessary to count these ties negatively towards correlation. This gives rise to a weak form of the requirement of "inducing the same ranking", which is simply that there should not be any discordant pairs between the rankings. In the remainder, we will use this weak formulation and we compute the fraction of discordant pairs as a measure of dissimilarity among indices, that is,

$$\frac{n_d(c_1, c_2)}{\binom{n}{2}},$$

where $n_d(c_1, c_2)$ is the number of discordant pairs. We refer to this fraction simply as rank dissimilarity. The measure is zero if indices induce the same ranking in the weak sense and one if the rankings are exactly reversed. Hence, a low fraction of discordant pairs, i.e. low rank dissimilarity, corresponds to high correlation.

3. Network topology and its association with centrality rankings

Early work that related network topology to correlation among centrality indices largely revolved around stability of measures when networks are sampled or under different types of errors in the network (e.g. missing edges and wrong edges). The focus here is not on the correlation between indices, but rather to correlate the scores of one index on an observed network and a sub-sampled version of it. Among the first in this line of research was the study conducted by Galaskiewicz (1991) which investigated the stability of degree-like measures with

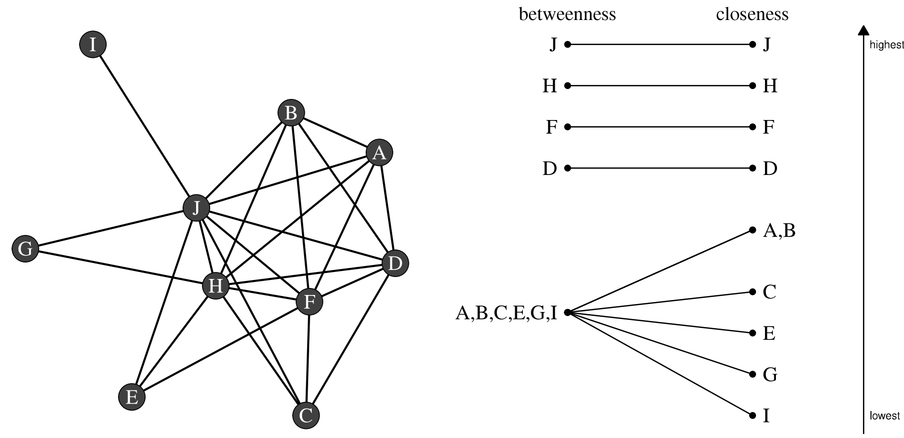

Fig. 2. (Left) Threshold graph and (Right) betweenness and closeness ranking.

different sampling procedures. One of his findings suggested that more accurate approximations of actor's centrality scores can be obtained in sparser networks and for less popular actors. The study, however, was limited in scope since it only used degree on two networks from a similar context. Costenbader and Valente (2003) conducted a similar study but used a set of 11 centrality indices and more than 60 networks of varying size. The authors found that network characteristics rather than sampling level are important for the correlation of an index applied to the actual and sampled data. The direction of association, though, varies. Density was found to be positively associated with integration and radiality, but negatively with degree and betweenness.

Instead of sampling networks, Borgatti et al. (2006) introduce various types of errors to the data and examine the stability of centrality scores. All experiments are carried out on uniform random graphs. They found that centrality indices have similar robustness patterns for varying network size and density and different error types have similar robustness profiles. Frantz et al. (2009) extended this analysis to different network topologies. Among those are small-world, core–periphery, and scale-free graphs. The authors find that the underlying network topology has a measurable effect on the robustness of indices. This effect also outweighs the effect of network size, density and various error types and levels have on the robustness of indices.

Platig et al. (2013) and Niu et al. (2015) investigate the robustness of indices on different network topologies under specific link errors and manipulation. Both find that indices are relatively robust on heterogeneous topologies, specifically for top ranked nodes.

Besides the literature on robustness under uncertainty and errors, there is also a large body of research that investigates the correlation between pairs of indices (Lee, 2006; Li et al., 2015; Batool and Niazi, 2014; Lozares et al., 2015; Shao et al., 2018; Rastogi and Jain, 2020). Structural properties of the networks are mostly ignored and the goal usually is to identify redundancies or conceptually different indices.

Comparably little research focuses exclusively on the association of network structure and correlation between indices. Valente et al. (2008) found on a set of 58 network datasets that density is positively correlated with the average correlation among 11 different centrality indices. Schoch et al. (2017) reanalyzed the data and found that the majorization gap and the spectral gap can explain the correlation between standard indices better than density on these networks. Both measures are related to the findings of Schoch and Brandes (2016) on the importance of neighborhood-inclusion and threshold graphs on centrality indices. The majorization gap approximates the topological distance of a network to the closest threshold graph via the degree sequence (Arikati and Peled, 1994). The higher the value, the larger the distance to a uniquely ranked graph and thus the more supposed degrees of freedom to rank nodes differently. The spectral gap makes use of the fact that threshold graphs have an idealized core–periphery structure. This suggests that graphs close to being a threshold graph should have structural properties of such networks (Borgatti and Everett, 2000). Intrinsically, the spectral gap is a way to a priori determine the number of clusters k in spectral clustering tasks (von Luxburg, 2007). Given the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of the adjacency matrix of a graph, the parameter k is chosen such that $|\lambda_i - \lambda_{i+1}|$ is small for $1 \leq i < k$ and $|\lambda_k - \lambda_{k+1}|$ is comparably large. For core–periphery graphs, we should expect that a gap appears at $k = 1$, since only one densely connected subgraph, the core, is present. For our purposes, we define the spectral gap as $\frac{\lambda_2}{\lambda_1}$. That is, the larger the gap, the smaller the fraction.

The work of Oldham et al. (2019) is the most comprehensive in terms of relating network structure to associations among centrality indices. Their study investigates the correlation among 17 indices on a diverse set of more than 200 real-world networks. The goal of the study was to investigate how a set of eight structural properties influences the overall correlation among indices. The chosen properties include the spectral gap and the majorization gap, as well as modularity, degree assortativity, clustering coefficient, density, global and diffusion efficiency. Modularity assesses the quality of a clustering of a network (Newman, 2006). If the value is high for a computed clustering, then the network consists of densely connected clusters with sparse connections in between groups. Degree assortativity measures the extent to which nodes connect to other nodes with comparable degree (Newman, 2003). A score close to 1 indicates that nodes largely connect to nodes with the same degree as themselves. A value close to −1 means that high degree nodes connect mostly to low degree nodes. The clustering coefficient computes the proportion of closed triangles present in the network and is used as an indicator for how nodes in a network tend to cluster together (Watts and Strogatz, 1998). Global efficiency is defined as the inverse of the average characteristic path length between all pairs of nodes in the network (Latora and Marchiori, 2001).

The results suggest that density, global efficiency, modularity, majorization gap, and spectral gap are associated with the average correlation among indices. The structural effect of the majorization gap, however, was found to be rather weak, especially on random surrogate graphs derived from the empirical network. While the majorization gap increased on the randomized instances, the correlations did on average not decrease. Overall, modularity was found to have the strongest association, even on randomized networks. The higher its value, the lower the observed correlation among indices. This observation has also been made in prior work (Guimerà and Nunes Amaral, 2005).

In the following, we will investigate some potential connections between topological features and correlation among centrality indices. Note, however, that we will use the concept of rank dissimilarities instead. In terms of indices, we employ the indices degree, betweenness, closeness, and eigenvector centrality. Limiting the choice of indices to them is motivated by the fact that they are the most widely used and best understood indices. Finding connections among them can serve as a starting point to expand the analysis to larger sets of indices.

In our analysis on real networks in Section 6, we investigate the influence of the already mentioned measures density, majorization gap, spectral gap modularity, degree assortativity, clustering coefficient, and global efficiency. We also add the diameter (i.e. the length of the longest shortest path in the network) and a measure of "core-peripheriness", which uses the discrete core–periphery model by Borgatti and Everett (2000). That is, we calculate the correlation between the adjacency matrix and an idealized core–periphery matrix. The rationale of adding a measure of core-peripheriness is that since threshold graphs have a (nested) core–periphery structure, networks with a high core-peripheriness should also exhibit less discordance.

4. Statistical entropy analysis

Associations between network structure and rank dissimilarities are commonly assessed via correlation over a set of networks. The higher the absolute correlation, the stronger the influence of the feature on discordance. For our analysis on real networks in Section 6, we move away from this simple measure and use more powerful techniques from statistical entropy analysis, which also allow us to assess the influence of combinations of structural features on dissimilarities.

Statistical entropy analysis is a general exploratory method that can be used to analyze and test complicated dependence structures in multivariate data with variables observed on a common domain and finite range spaces. The systematic use of multidimensional entropies is developed by Frank (2011) and Frank and Shafie (2016). The authors show how combinations of multivariate entropies can reveal redundancies among variables due to low variation, implying that they are almost constant, or because they are uniquely determined by other variables. For each variable that is not uniquely determined by others, it might be of interest to know if it is almost uniquely determined with high prediction power by some combinations of other variables. In the following we introduce some specific entropy quantities that are used as dependence, independence, and conditional independence measures.

The joint entropy for a pair of variables conveys the association strength between them. It is a relative measure of any patterns resulting from the frequencies of joint outcomes on the two variables; while higher values of joint entropy indicate a strong association between them, zero value signifies complete independence. That is, if we observe a high joint entropy between a topological feature and discordance between a pair of indices, then there is a strong association between them. If the entropy is zero, then the topological feature is completely independent of the discordance. To explore joint entropies among a set of variables, the association graph can be used. In such a graph, variables are the nodes and they are connected if their joint entropy exceeds a certain threshold. The further away the threshold from the maximum joint entropy, the more links we see in the graph.

Joint entropies focus on the associations between two variables, but using higher order entropies, we can assess if combinations of features can explain discordance in any meaningful way. The expected conditional entropies can be used to check whether any response variable can be explained as a function of one or several other variables. It is a measure of how many outcomes there are of the response on average when the outcomes are given for one or two predictors. Thus, it can also be called the prediction power of the response given one or two other variables. The lower the values computed on this measure, the better the prediction. In the limit case, when the expected conditional entropy equals zero, we have unambiguous prediction meaning each unique (joint) outcome of the predictor(s) yields a unique outcome of the response. In our analyses, we want to know if any pairwise combinations of structural features can predict discordance between indices well.

Table 1
Parameters used to create 7 threshold graphs as basis for the edge rewiring experiment.

Number of nodes	100	250	500	1000	2500	5000	10 000
Density	0.1	0.1	0.05	0.025	0.005	0.0025	0.001

For more technical details, including formal definitions, see Frank (2011) and Frank and Shafie (2016).

5. Rank dissimilarity on classes of random graphs

Uniform random graphs have often been employed to analyze the correlation among centrality indices (Frantz et al., 2009; Costenbader and Valente, 2003; Ronqui and Travieso, 2015; Shao et al., 2018; Ficara et al., 2022). Their regular structure, however, does not allow much room for structural variation to occur and as such, these models are not necessarily adequate to use in analyses of rank dissimilarities, specifically in conjunction with structural features. In this section, we focus on two classes of random graphs and construct a third group of graphs in order to (a) carefully trace the interplay of structural features and their association with rank dissimilarities and (b) to construct maximally dissimilar rankings. The goal is to review postulated associations between network structure and rank dissimilarities in a controlled setting and to demonstrate that high rank dissimilarity can theoretically be achieved for any pair of indices.

5.1. Rank dissimilarities on edge rewired threshold graphs

We here investigate the connection between the distance of a network to its closest threshold graph and rank dissimilarities. As a byproduct, our experiment will also show that density can be completely independent of rank dissimilarities.

We start by constructing seven threshold graphs with the parameters shown in Table 1. Note, that we vary both, the number of nodes and the density. This does make the comparison slightly inconsistent across the networks. However, more important for us are the impact of the structural changes that we introduce to the graphs in the form of edge rewirings.

For each graph, we perform the following iterative process. In step $i \in \{1, 2, \ldots, 100\}$, we rewire each edge with a probability of $p_i = \frac{i}{100}$ and calculate rank dissimilarities for the standard indices. This process is repeated 10 times for each i and the average dissimilarity is recorded. The described iteration ensures that the density is kept fix on each graph, but we slowly move away from the underlying threshold graph towards a uniform random graph. Fig. 3 shows the association of this rewiring with the rank dissimilarities.

The figure shows that the rank dissimilarities increases quickly for small rewiring probabilities but decreases again when the probability approaches one. Several individual differences in terms of baseline threshold graph and pairs of indices can be observed, yet the overall trend is similar for each configuration. An apparent outlier being closeness and eigenvector centrality, where dissimilarity is consistently low.

The result confirms the observations of Oldham et al. (2019) concerning the limits of using the distance to a threshold graph (in this case the majorization gap) to explain rank dissimilarities since we observe low dissimilarities for low and high distances. But it also directly provides an explanation. The relation between rank dissimilarities and majorization gap is not linear but roughly follows an inverse u-shape. Fig. 4 illustrates this by showing the average rank dissimilarity for all networks and all centrality pairs in relation with the majorization gap.

Using the majorization gap in a simple linear regression thus cannot yield a high predictive power for rank dissimilarities between indices, given that the relation is not monotonic.

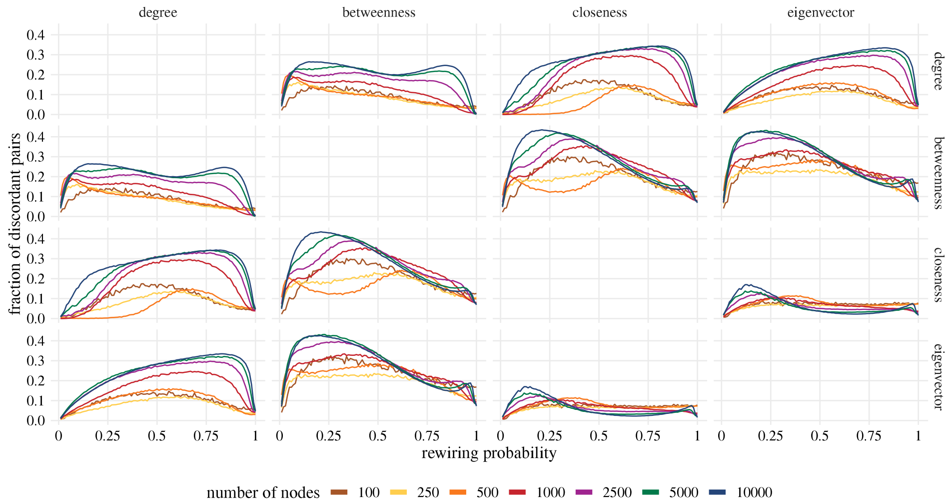

Fig. 3. Results of the rewiring experiment. Each panel shows the fraction of discordant pairs between centrality indices on a threshold graph with increasing edge rewiring probability.

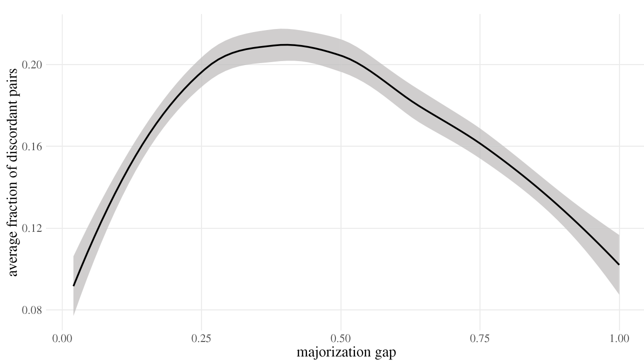

Fig. 4. Average fraction of discordant pairs for increasing majorization gap of all graphs in the rewiring experiment.

5.2. Rank dissimilarities on clustered graphs

In this section, we explore the connection between modularity and rank dissimilarities between indices to reevaluate the result of Oldham et al. (2019) who found that modularity was the strongest and most robust topological feature which correlates with rank dissimilarities.

To investigate this relationship, we generate clustered graphs with the algorithm introduced by Lancichinetti et al. (2008). The algorithm generates graphs which can account for heterogeneities in the distributions of node degrees and also of cluster sizes, referred to as LFR benchmark graphs. We use the LFR algorithm to create a set of graphs with default parameters and with $n = 1000$ nodes, average degree 15 and cluster sizes ranging from 20 to 50. The parameter μ is used to control the fraction of edges running between clusters. Values close to 0 create fragmented networks and numbers close to 1 create graphs without any clear cluster structure. For increasing values of μ between 0 and 1, we compute a clustering using the Louvain algorithm (Blondel et al., 2008). Oldham et al. (2019) additionally used a consensus clustering procedure to address algorithmic degeneracies, but we found that this is not necessary in this particular constructed case. Additionally, we record the rank dissimilarities among the standard indices. The results for one iteration of the experiment are shown in Fig. 5.[1]

[1] Repeating the experiment with different parameter setting did not have any measurable effect on the results.

At first glance, the results are in line with the observations of the literature. The rank dissimilarities increase with increasing fragmentation of the network in all cases. Note, however, that modularity is perfectly correlated with the spectral gap ($\rho = 1$). Hence, although there seems to indeed be a connection between modularity and rank dissimilarities, the perfect correlation of modularity and the spectral gap mitigates this result. We cannot disentangle exactly which might actually cause the rank dissimilarities in the first place.

This observation points to the fact that even if a seeming strong association is found between a structural feature and rank dissimilarity, there is no guarantee that there does not exist a confounding structural feature which yields the exact same explanation. We address this issue in the analysis of real-world networks using statistical entropies.

5.3. Constructing networks with high discordance

The existence of networks without any discordant pairs, and consistently low discordance on random graphs raise the question if there also exist classes of graphs where we can consistently observe high rank dissimilarities among pairs of indices. To the best of our knowledge, no such class has been identified in the literature. We here again take a more experimental approach to construct instances of graphs which are yield high rank dissimilarities between a given pair of indices. This task can be formulated as an optimization problem of the form

$$\arg\max_{G \in \mathcal{G}} n_d(c_1, c_2)$$

where $\mathcal{G}$ is the set of all simple graphs. Constraints can be added to make the problem more tangible such as fixing the number of nodes and/or the density of the network. In the following, we only fix the number of nodes of the graph and define

$$\arg\max_{G \in \mathcal{G}} n_d(c_1, c_2) \quad \text{s.t.} \quad V(G) = n \tag{1}$$

Solving this optimization problem remains practically infeasible given the large outcome space. The global optimum can be approximated using heuristics such as simulated annealing (Kirkpatrick et al., 1983) or genetic algorithms (Holland, 1992). We here employ the simulated annealing approach starting from a uniform random graph with fixed n but random density. In each optimization step, an edge is either created, deleted, or rewired and $n_d(c_1, c_2)$ is computed. The graph is accepted as a new temporary solution if the number of discordant pairs has increased. If the number has decreased, the graph is accepted only with a probability which decreases with each iteration step. Worse

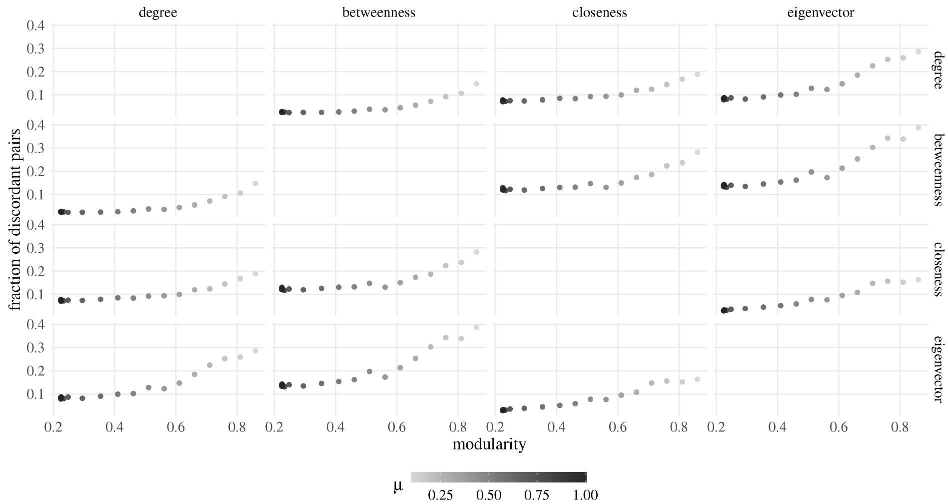

Fig. 5. Cluster graph experiment with LFR benchmark graphs ($n = 1000$). Each panel shows the rank dissimilarities between centrality indices in relation to the modularity of a clustering obtained with the Louvain method.

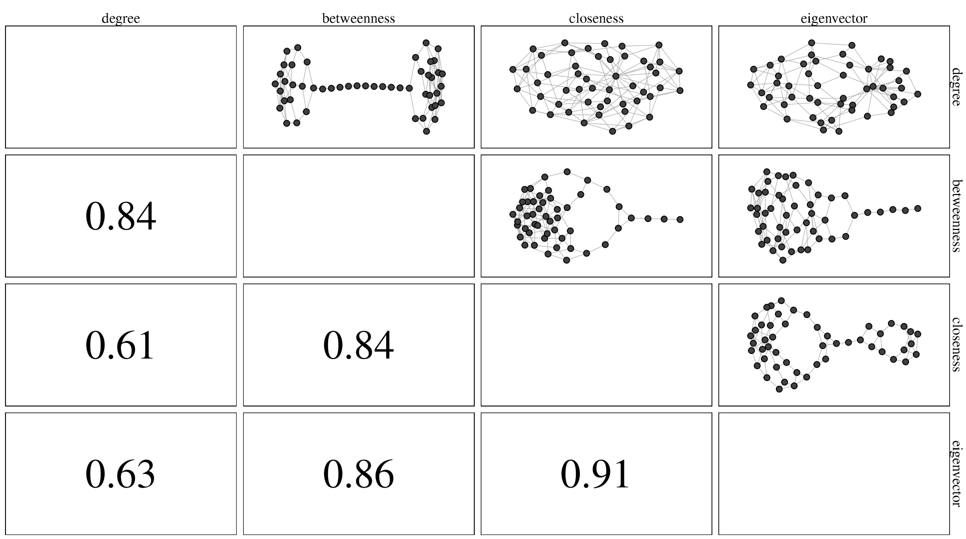

Fig. 6. Approximate solutions for Eq. (1) with $n = 50$ for the standard centrality indices. Upper diagonal shows the graphs and lower diagonal the rank dissimilarities.

Table 2
Maximum rank dissimilarity found solving Eq. (1) for different values of n.

	$n = 10$	$n = 15$	$n = 25$	$n = 50$	$n = 100$
Degree-betweenness	0.71	0.73	0.78	0.84	0.75
Degree-closeness	0.53	0.54	0.64	0.61	0.40
Degree-eigenvector	0.44	0.49	0.61	0.63	0.66
Betweenness-closeness	0.71	0.78	0.83	0.84	0.76
Betweenness-eigenvector	0.76	0.83	0.86	0.86	0.76
Closeness-eigenvector	0.53	0.78	0.84	0.91	0.79

solutions are accepted to get out of local optima. Fig. 6 shows some approximated solutions of the optimization problem with a fixed $n = 50$ and the standard indices in Eq. (1). For all pairs of indices, we find a graph where more than 60% of the node pairs are discordant. In four cases even more than 80%. Interestingly, we found the highest discordance for eigenvector centrality and closeness, which had the lowest overall dissimilarity in Section 5.1. Table 2 shows the highest achieved rank dissimilarities for networks with up to 100 nodes.

Naturally, the same restrictions as for threshold and random graphs hold for these type of networks. It is not likely to encounter such extreme cases of discordance in empirical settings. Yet, they highlight that even if two indices are found to consistently produce similar rankings, there do exist graphs, or even classes of graphs, where they do not.

6. Real-world networks

All experiments up to now have dealt with random and artificially crafted networks. In this section, we turn to empirical networks and investigate if any of the topological features density, diameter, spectral gap, majorization gap, modularity, core-peripheriness, degree assortativity, clustering coefficient, or global efficiency is associated with rank dissimilarities. For this purpose, we analyze a set of 1163 networks including a diverse set of social networks,[2] movie networks (Kaminski et al., 2018), animal social networks (Sah et al., 2019), and covert networks.[3] All networks are available in the R package networkdata (Schoch, 2022c). For all networks, we again calculate rank dissimilarities between degree, betweenness, closeness, and eigenvector centrality.

[2] Downloaded from http://moreno.ss.uci.edu/data.

[3] Downloaded from https://sites.google.com/site/ucinetsoftware/datasets/covert-networks.

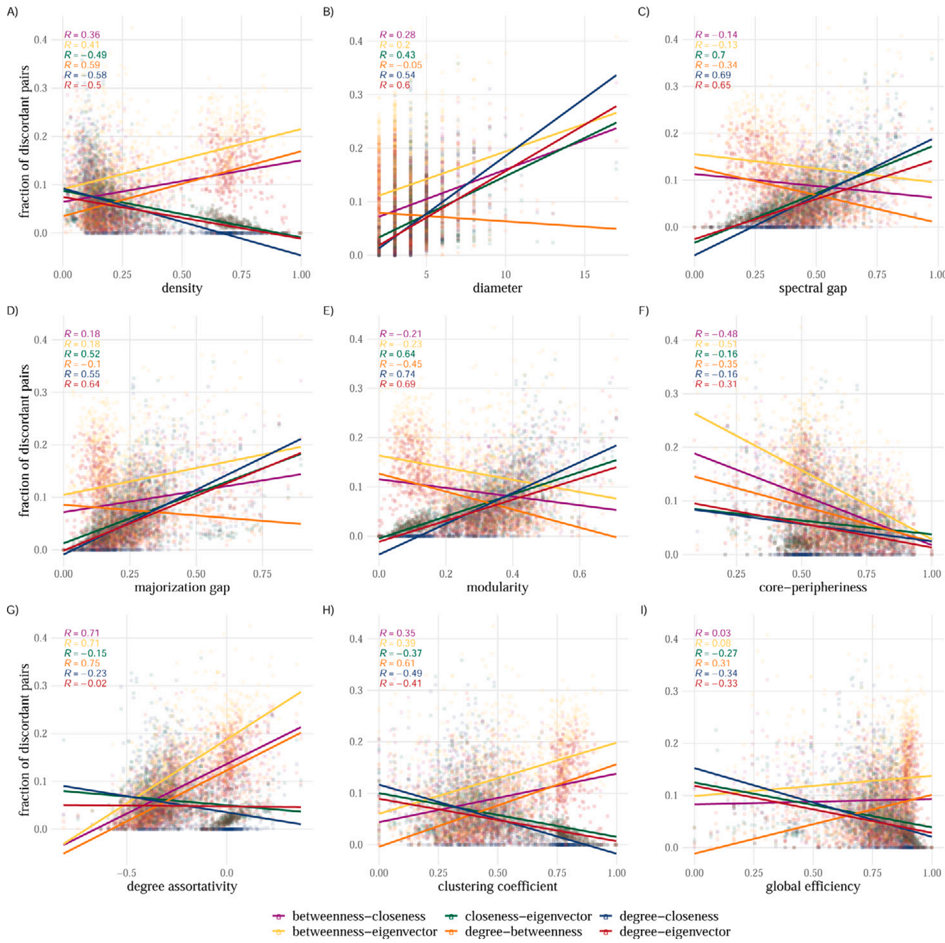

Fig. 7. Association between structural features and rank dissimilarities between centrality indices in 1163 real networks.

6.1. Correlation between rank dissimilarity and structural features

When comparing rank dissimilarities between indices with topological features, we refrain from calculating mean dissimilarities for networks, since we assume that structural properties influence the association between pairs of indices differently. Fig. 7 shows that this is indeed largely the case.

At first glance, there is no property that overall correlates strongly with rank dissimilarities. However, there are several interesting observations to be made. First, for most of the topological features, we see that the discordance between betweenness and the other indices is reversed. For instance, the higher the density, the lower the discordance between closeness, degree and eigenvector, but the higher the discordance between betweenness and those indices. The same holds for the spectral gap, modularity, degree assortativity, clustering coefficient and global efficiency. The only topological feature where all agree is the core-peripheriness with the expected outcome that an increasing core–periphery structure leads to less discordance. However, the association is only weak at best. The feature with the lowest association is global efficiency with all correlations being around zero.

6.2. Joint entropies of structural features and rank dissimilarities

To determine the significance of the results obtained from the correlation analysis above, we now turn to a more nuanced analysis using the statistical entropy approach introduced in Section 4. As a first step, we aggregate the outcome spaces of all available variables (topological features and discordance) to get reliable estimates for the entropies of their frequency distributions. This is done via a median split to ensure approximately two equally sized categories. The dichotomization of all frequency distributions are shown in Fig. 8.

Table 3 shows the computed joint entropies between the structural features and rank dissimilarities. The two strongest associations are between degree assortativity and discordance between any index with betweenness.

Besides looking for strong associations, it is also worth checking for features that are completely independent of rank dissimilarities, i.e. when the joint entropy is zero. All such cases are highlighted in Table 3. Diameter, for instance, has one of the strongest associations with discordance between degree and closeness, but it is completely independent of the discordance between betweenness and closeness, and betweenness and eigenvector. We also find that core-peripheriness is independent of discordance between degree and eigenvector centrality, and the majorization gap of betweenness and eigenvector centrality. Some relations are also close to being independent. Of particular interest is the role of density, which is almost independent of the discordance between betweenness and all other indices. Overall, though, no clear patterns do emerge. The most consistent observation is the role of degree assortativity for rank dissimilarities with betweenness.

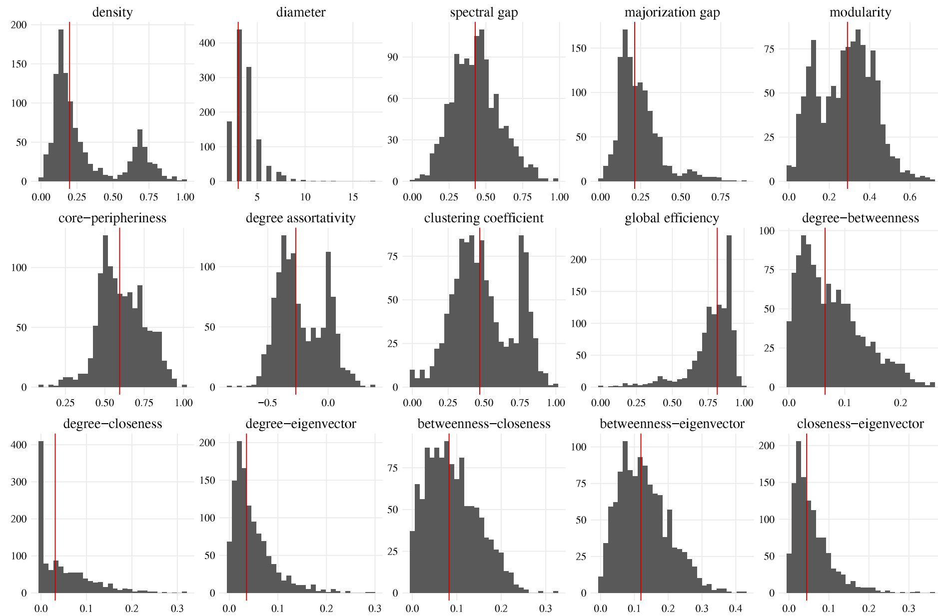

Fig. 8. Distribution of topological features of the 1163 real networks. The red lines represent approximately 50% of the cumulative frequency distributions and is used to dichotomize each variable. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

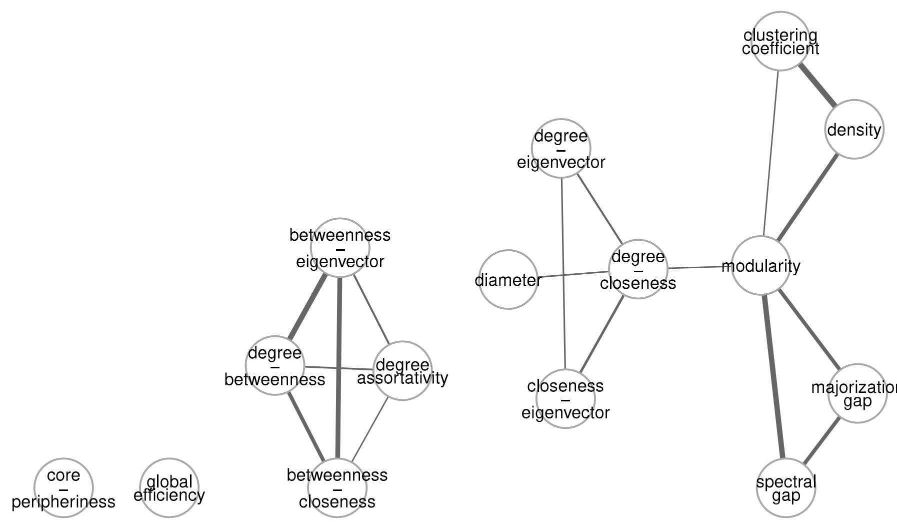

Fig. 9. Summary of the strongest associations between network properties and rank dissimilarities. Thicker edges indicate stronger associations.

Table 3
Joint entropy matrix among variables representing various network properties and rank dissimilarities. The highlighted cells indicate independence.

	Density	Diameter	Spectral gap	Majorization gap	Modularity	Core-peripheriness	Degree assortativity	Clustering coefficient	Global efficiency
Degree-betweenness	0.098	0.006	0.068	0.015	0.072	0.097	0.276	0.182	0.058
Degree-closeness	0.229	0.268	0.223	0.187	0.266	0.007	0.043	0.141	0.150
Degree-eigenvector	0.178	0.189	0.202	0.224	0.221	0.000	0.009	0.114	0.077
Betweenness-closeness	0.038	0.000	0.029	0.001	0.036	0.160	0.256	0.070	0.018
Betweenness-eigenvector	0.058	0.000	0.030	0.000	0.033	0.143	0.290	0.096	0.029
Closeness-eigenvector	0.129	0.158	0.191	0.159	0.179	0.009	0.029	0.076	0.076

6.3. Association graph of features

Thus far, we have focused on individual pairs of structural properties and rank dissimilarities. A more complete picture of the hidden dependence structure can be obtained using the association graph. Recall, that the association graph connects variables that have a joined entropy above a predefined threshold. Fig. 9 shows this graph with a threshold of 0.25. The threshold was chosen to only show the strongest variable associations since the maximum observed entropy is 0.29.

The graph confirms the observation of Fig. 7 that betweenness behaves quite different from other indices on these networks. We can see a clique consisting of variables degree assortativity and rank dissimilarities degree-betweenness, betweenness-eigenvector, and betweenness-closeness. We also note that modularity is part of two cliques and

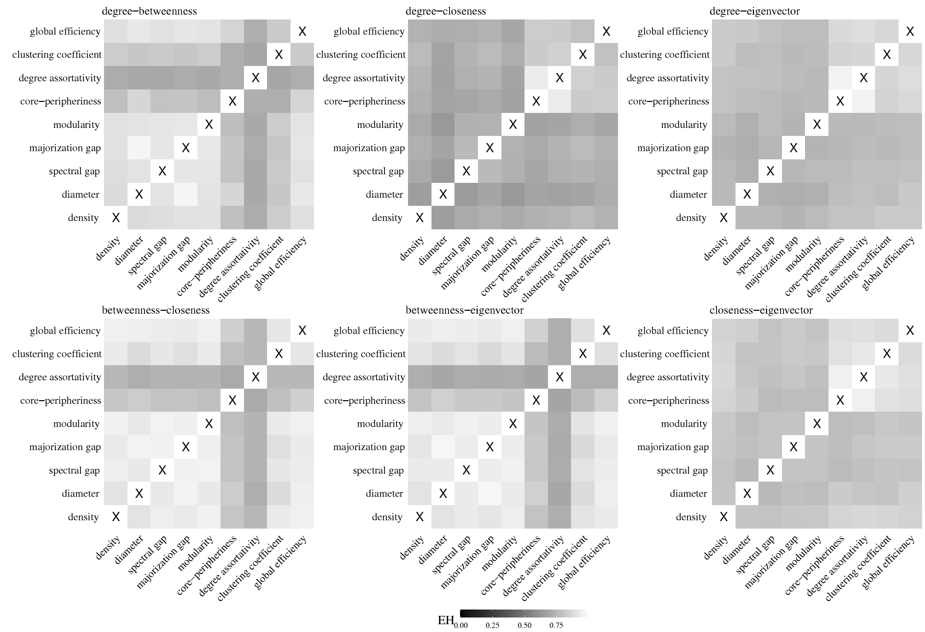

Fig. 10. Prediction strength based on expected conditional entropies (EH) when pairs of structural variables are used to predict rank dissimilarities among centrality indices.

acts as a bridge to degree-closeness. This can be interpreted as the following: conditional on modularity, degree-closeness is independent of the spectral and majorization gap, and clustering coefficient and density, respectively.

Higher order dependencies and prediction powers

In a last analysis, we move away from just associating a single structural feature with discordance using expected conditional entropies and prediction powers. Prediction powers of different variables can conveniently be compared by using prediction plots that display a color matrix with rows for one predictor and columns for another, and with colors representing the power of predicting the response as measured by expected conditional entropies, denoted EH. Fig. 10 shows such plots for predicting rank dissimilarities by using different pairs of structural variables.

As before, we can observe that betweenness follows different patterns. Degree assortativity works well with any other structural feature to predict discordance in this case. However, this is largely due to the fact that it is already a very good predictor by itself. Interestingly, core-peripheriness in conjunction with any other feature is the second-best predictor, although, by itself, it has very low association with rank dissimilarities. The overall best predictions are obtained for the discordance between degree and closeness with modularity and the diameter.

7. Discussion

We have re-examined existing results for the correlation among centrality indices and the association of structural features therewith using a different analytical set of tools. In our experiments, we can observe that clear patterns emerge on random and hand-crafted networks for modularity, spectral and majorization gap. However, only fairly weak associations remain when tested on a large set of real networks and other structural features, such as degree assortativity, are more dominant in predicting rank dissimilarities. On the LFR benchmark graphs, modularity and the spectral gap are highly correlated, and for the real networks, they form a clique in the association graph together with the majorization gap. This does not come as a surprise, since networks with high modularity do not have a core–periphery structure. Conversely, networks with a large spectral gap and small majorization gap do not have a cluster structure. Hence, disentangling which of the measures is most indicative is not possible without a deeper analysis.

We presented a statistical entropy approach that goes beyond the usually employed correlation analysis to create association graphs among all structural features and rank dissimilarities, and to uncover higher order dependencies. We consistently observe that all pairs of degree, closeness, and eigenvector centrality behave coherently but betweenness follows very different rules. The overall strongest association we observe is that of degree assortativity with discordance between betweenness and any other indices.

General differences between betweenness and other indices have been reported before (e.g. for small networks Nakao, 1990). An explanation for this result is offered by the classification of indices into radial and medial measures by Borgatti and Everett (2006). Radial indices assess walks that start from or end with a given node, while medial indices are based on the number of walks that pass through a given node. Degree, closeness, and eigenvector centrality can all be classified as radial, whereas betweenness as a medial index. The authors argue that radial and medial indices are complementary, where radial indices assess group membership and medial indices bridging between groups. Our results confirm that the distinction between radial and medial indices can in empirical settings indeed be meaningful and may thus serve as a good assistance in the process of deciding which index to choose for an analysis.

Overall, our results suggest that using correlation (or rank dissimilarities) to study associations among indices and structural features leaves too many ambiguities to make conclusive statements about the roles of specific structural features since there are still too many unknown factors at play. Additionally, our experiments show that high correlations (or low discordance) can always be achieved by comparing indices on threshold graphs and uniform random graphs, and low correlations (high discordance) can be artificially created via the introduced graph generator. The use of a correlation analysis to justify new indices is thus questionable given that low or high associations between a set of indices can be crafted at will.

This result also has some implications for empirical research in general. Centrality indices are often used as independent or explanatory variables in empirical settings. Hypotheses typically state that the level of some variable of the network is either positively or negatively associated with some index (a *centrality effect*). Ideally "some index" is chosen based on substantive arguments ("Why should the index work best?"), yet frequently the decision is based on a comparative analyses ("Which index works best?"). The fact that high (or low) rank dissimilarities can neither be attributed to technical differences nor unambiguously to any structural features, however, means that results from comparative analyses are difficult to interpret. The same holds for explaining a good (or bad) performance of an index. An alternative to comparative analyses of a set of indices are more holistic approaches such as probabilistic assessment of centrality without using indices (Schoch, 2018).

If we deem correlations or rank dissimilarities insufficient then what are viable alternatives to uncover structural similarities or differences among indices? A promising line of research is looking for analytical connections. Several have been found over the years such as closeness and betweenness (Brandes et al., 2016), degree and closeness (Evans and Chen, 2022) and walk based indices (Benzi and Klymko, 2013). While analytical connections may not directly translate into general explanations for observed high (or low) correlations, they can point to important structural features for smaller groups of indices. A large spectral gap, for instance, implies that the scores of indices based on the eigendecomposition are dominated by the eigenvector associated with the largest eigenvalue and thus correlate with eigenvector centrality (Benzi and Klymko, 2013).

Although the majorization gap is only weakly associated with discordance, constructing distance measures to artificial graph classes like threshold graphs may provide further insights into the interplay of indices. The majorization gap is only a rough estimate for the distance to the closest threshold graph. To date, however, it remains the only computationally feasible method to compute a distance to the class of threshold graphs. While the rewiring experiment shows a somewhat consistent behavior, that is the majorization gap increases monotonically while moving away from a threshold graph, there is no guarantee that this consistency caries over to real networks.

Besides the neighborhood-inclusion preorder, there may be other structural features of networks that drive all measures of centrality we miss by only focusing on the distance to threshold graphs. Uncovering such universal properties will give rise to additional uniquely ranked graphs and a more complete picture of what part of the centrality ranking is predetermined by the network structure. Section 5.3 also made a case for investigating the reverse case, that is, graphs that lead to extreme rank dissimilarities among pairs of indices. Properly defining such graphs, or classes thereof, can provide information about structural features that make indices highly variable. The optimization problem presented here is flexible enough to search for graphs with high discordance given any structural feature as a constraint. However, finding optimal solutions for any instantiation of the problem is infeasible and given that centrality values need to be recalculated in each step only manageable for small networks.

Replication material

Data and code to reproduce the results shown in the paper can be obtained from https://github.com/schochastics/centrality-correlation. We used R version 4.2.2 (R. Core Team, 2022) and the following R packages: ggpubr v. 0.5.0 (Kassambara, 2022), ggraph v. 2.1.0 (Pedersen, 2022), igraph v. 1.4.0 (Csardi and Nepusz, 2006), netrankr v. 1.2.0 (Schoch, 2022a), netropy v. 0.1.0 (Shafie, 2022), netUtils v. 0.8.1 (Schoch, 2022b), and networkdata v 0.1.14 (Schoch, 2022c)

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