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L^p-L^q-Estimate of the Linear Equations of Thermoelasticity for Rhombic Media in $\mathbb{R}^2$

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Abstract: We determine the L^p-L^q-estimate of the linear equations of thermoelasticity for rhombic media in $\mathbb{R}^2$. For this purpose we will transform the equations to an evolution equation showing that this equation has a unique solution with the help of the semigroup theory. In order to obtain the L^p-L^q-estimate, we will first apply the Fourier transformation to the evolution equation, thus determining the spectral representation of the solution. Then we will expand the eigenvalues of the Fourier transform operator using the Newton polygon procedure. The L^p-L^q-estimate is finally obtained by applying the method of stationary phase.

1 Introduction

Why do we study L^p-L^q-estimates of the linear equations of thermoelasticity for rhombic media in $\mathbb{R}^2$? To answer this question we shall have a look back.

For differential operators the characteristic manifold of which has k vanishing main curvatures at the most we obtain the decay rate $(n-k)/2$ for the dimension n. In the case of the equations of elasticity of isotropic media, for example, $k = 1$ for $n = 3$, the decay rate thus corresponding to 1.

In the case of the equations of elasticity of cubic media, unfortunately, $k = 2$ for $n = 2$. In order to obtain a decay of the solution also for this case we use the method of stationary phase ([5]). The decay rate depends on the order of the main curvatures' vanishing. If the curvature does not vanish or if it vanishes of first or second order, respectively, we get $\frac{1}{2}$, $\frac{1}{3}$ or $\frac{1}{4}$ as decay rates. In the case of the equations of thermoelasticity of cubic media we get the decay rate $\frac{1}{2}$ ([1]) for $n = 2$ applying the method of stationary phase.

It was necessary to study another problem of this kind in order to find out whether either the isotropic case or the cubic case must be regarded as a special case. The calculation of the L^p-L^q-estimates for the linear equations of thermoelasticity of rhombic media showed that such calculations do not only become more and more sophisticated, but that the isotropic case proves to be the special case due to the decay rate $\frac{1}{2}$ found in it. We also have to assume that, applying the method of stationary phase, we cannot expect any better decay rates in the cases of the other anisotropic media.

In the following we consider the Cauchy problem for a homogeneous and initially rhombic medium in $\mathbb{R}^2$. It is given by

$$u_{tt} - \mathcal{D}' S \mathcal{D} u + \gamma \nabla \theta = 0, \tag{1.1}$$

$$\theta_t - \Delta \theta + \gamma \nabla' u_t = 0 \tag{1.2}$$

and the prescribed initial values

$$u(t=0) = u_0, \;\; u_t(t=0) = u_1, \;\; \theta(t=0) = \theta_0, \tag{1.3}$$

where u and θ are functions of $x \in \mathbb{R}^2$ and $t \geq 0$. The displacement vector of the two-dimensional medium is described by $u = u(t, x)$, the temperature difference between the absolute temperature T and the fixed reference temperature T_0 by $\theta = \theta(t, x) = T(t, x) - T_0$, and the coupling coefficient by γ. $\mathcal{D}$ is the formal generalized gradient

$$\mathcal{D} := \begin{pmatrix} \partial_1 & 0 \\ 0 & \partial_2 \\ \partial_2 & \partial_1 \end{pmatrix}$$

and

$$S := \begin{pmatrix} \nu & \kappa & 0 \\ \kappa & \omega & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

contains the moduls of elasticity $\kappa,\ \mu,\ \nu,\ \omega \in \mathbb{R}$, which describe the rhombic medium. For physical reasons, S has to be positive. This means that

$$\mu,\ \nu,\ \omega > 0 \tag{1.4}$$

and

$$\nu\omega > \kappa^2. \tag{1.5}$$

As a result,

$$\nu + \omega - 2|\kappa| > 0. \tag{1.6}$$

Using the semigroup theory we will now demonstrate that our initial value problem (1.1) to (1.3) has a unique solution. For this purpose we transform the linear equations of thermoelasticity formally to an evolution equation of first order in time and define

$$v = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} := \begin{pmatrix} S\mathcal{D}u \\ u_t \\ \theta \end{pmatrix}$$

as well as

$$A := \begin{pmatrix} 0 & -S\mathcal{D} & 0 \\ -\mathcal{D}' & 0 & \gamma\nabla \\ 0 & \gamma\nabla' & -\Delta \end{pmatrix}.$$

Thus, v satisfies the differential equation

$$v_t + Av = 0 \tag{1.7}$$

and the initial condition

$$v(t = 0) = \begin{pmatrix} S\mathcal{D}u_0 \\ u_1 \\ \theta_0 \end{pmatrix} =: v_0. \tag{1.8}$$

As in section 2.1 in [3] we take the Hilbert space

$$\mathcal{H} := L^2(\mathbb{R}^2)$$

as a basis for our initial value problem and provide it with the weighted inner product

$$(\cdot,\cdot)_{\mathcal{H}} := (\cdot, Q\cdot),$$

where

$$Q := \begin{pmatrix} S^{-1} & 0 & 0 \\ 0 & id & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbb{R}^2)$. This means that the norm of the Hilbert space $\mathcal{H}$ corresponds to the energy. If we choose

$$D(A) := \{v \in \mathcal{H} |\ v^2 \in H^1(\mathbb{R}^2),\ v^3 \in H^1(\mathbb{R}^2),\ Av \in \mathcal{H}\},$$

the following is valid:

Proposition 1.1 ([3], Theorem 2.2) *$-A$ is the generator of a contraction semigroup, and for each $v_0 \in D(A)$ there exists a unique solution*

$$v \in C^0([0,\infty), D(A)) \cap C^1([0,\infty), \mathcal{H})$$

with $v(t) = e^{-tA}v_0$.

In order to obtain the asymptotic behavior of the solution we decompose the solution implicitly as in the cubic case [1]. For this purpose we need the asymptotic expansion of the Fourier transform operator's eigenvalues, which we determine with the help of the Newton polygon procedure. In order to obtain the asymptotic behavior of the solution we split up the solution due to the varying expansion of the eigenvalues. Here it is important whether the real parts of the eigenvalues disappear. The part depending on eigenvalues the real parts of which do not disappear at all can be estimated in a similar way as the curl-free part in the isotropic case [4]. To estimate the part depending on eigenvalues the real parts of which disappear we use the method of stationary phase. All in all we shall prove the same decay rates for rhombic media as for cubic media, which correspond to the decay rates of the divergence-free part of isotropic media:

$$|v(t)|_q \leq c(1+t)^{-\frac{1}{2}(1-\frac{2}{q})}\|v_0\|_{N_p,p},$$

where $2 \leq q \leq \infty$, $\frac{1}{p}+\frac{1}{q}=1$ as well as $c>0$ and $N_p \in \mathbb{N}$ are constants which depend on q only.

For the nonlinear equations of thermoelasticity of isotropic media in $\mathbb{R}^3$ Racke proved the existence of a unique, global and classical solution, decisively considering the decay rate of the temperature in his proof. Due to the lower decay rate of the temperature with rhombic media and especially with cubic media it is not possible to prove the existence of a unique, global and classical solution for the nonlinear equations of thermoelasticity of rhombic media in $\mathbb{R}^2$, particularly not for cubic media in $\mathbb{R}^2$.

2 Spectral Representation of the Operator

The first step to derive the L^p-L^q-estimate is the spectral representation of the operator A. Applying the Fourier transformation to the equations (1.7) and (1.8), for each $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ we obtain

$$\hat{v}_t(t,\xi) + \hat{A}(\xi)\hat{v}(t,\xi) = 0, \tag{2.1}$$

$$\hat{v}(t=0,\xi) = \hat{v}_0(\xi) \tag{2.2}$$

mit

$$\hat{A}(\xi) := i \begin{pmatrix} 0 & 0 & 0 & -\nu\xi_1 & -\kappa\xi_2 & 0 \\ 0 & 0 & 0 & -\kappa\xi_1 & -\omega\xi_2 & 0 \\ 0 & 0 & 0 & -\mu\xi_2 & -\mu\xi_1 & 0 \\ -\xi_1 & 0 & -\xi_2 & 0 & 0 & \gamma\xi_1 \\ 0 & -\xi_2 & -\xi_1 & 0 & 0 & \gamma\xi_2 \\ 0 & 0 & 0 & \gamma\xi_1 & \gamma\xi_2 & -i\,|\xi|^2 \end{pmatrix}.$$

For the characteristic polynomial P_0 of $\hat{A}(\xi)$ we calculate

$$\begin{aligned} P_0(\xi,\lambda) &= \det(\hat{A}(\xi) - \lambda\, id) \\ &= \lambda\Big(\lambda^5 - |\xi|^2\lambda^4 + \alpha|\xi|^2\lambda^3 - \beta|\xi|^4\lambda^2 + \delta|\xi|^4\lambda - \eta|\xi|^6\Big) \\ &=: \lambda P(\xi,\lambda). \end{aligned}$$

The meaning of α, β, δ and η is:

$$\begin{aligned} \alpha\left(\frac{\xi_1^2}{|\xi|^2}, \frac{\xi_2^2}{|\xi|^2}\right) &:= \mu + \nu + \gamma^2 - \frac{\xi_2^2}{|\xi|^2}(\nu - \omega), \\ \beta\left(\frac{\xi_1^2}{|\xi|^2}, \frac{\xi_2^2}{|\xi|^2}\right) &:= \mu + \nu - \frac{\xi_2^2}{|\xi|^2}(\nu - \omega), \\ \delta\left(\frac{\xi_1^2}{|\xi|^2}, \frac{\xi_2^2}{|\xi|^2}\right) &:= \mu(\nu + \gamma^2) + \frac{\xi_1^2\xi_2^2}{|\xi|^4}(\nu - \kappa - 2\mu)(\kappa + \nu + 2\gamma^2) - \\ &\quad - \frac{\xi_1^2\xi_2^2}{|\xi|^4}(\nu - \omega)(\nu + \gamma^2) - \frac{\xi_2^4}{|\xi|^4}(\nu - \omega)\mu \end{aligned}$$

or

$$\begin{aligned} \eta\left(\frac{\xi_1^2}{|\xi|^2}, \frac{\xi_2^2}{|\xi|^2}\right) &:= \mu\nu + \frac{\xi_1^2\xi_2^2}{|\xi|^4}(\nu - \kappa - 2\mu)(\kappa + \nu) - \frac{\xi_1^2\xi_2^2}{|\xi|^4}(\nu - \omega)\nu - \\ &\quad - \frac{\xi_2^4}{|\xi|^4}(\nu - \omega)\mu, \end{aligned}$$

respectively. For all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ α, β, δ and η are positive functions according to [2].

The matrix $\hat{A}(\xi)$ is diagonalizable for $|\xi| \neq 0$ under certain conditions regarding the moduls of elasticity κ, μ, ν, ω and the coupling coefficient γ. A detailed representation of the facts is provided by

Lemma 2.1 *If κ, μ, ν, ω and γ satisfy the inequations*

$$\nu \geq \gamma^2, \;\; \omega \geq \gamma^2, \;\; \frac{\nu\omega - \kappa^2}{\nu + \omega - 2\kappa} \geq \gamma^2, \;\; \frac{\nu\omega - (\kappa + 2\mu)^2}{\nu + \omega - 2(\kappa + 2\mu)} \geq \gamma^2 \tag{2.3}$$

as well as

$$\begin{aligned} (\mu - \nu - \gamma^2)^2 \geq \gamma^4, \;\; (\mu - \omega - \gamma^2)^2 &\geq \gamma^4, \\ (\kappa + \mu + \gamma^2)^2\, \frac{4(\mu - \nu - \gamma^2)(\mu - \omega - \gamma^2) - 4(\kappa + \mu + \gamma^2)^2}{(2\mu - \nu - \omega - 2\gamma^2)^2 - 4(\kappa + \mu + \gamma^2)^2} &\geq \gamma^4 \end{aligned} \tag{2.4}$$

and

$$\mu + \nu \geq 6\gamma^2, \;\; \mu + \omega \geq 6\gamma^2, \tag{2.5}$$

the eigenvalues of $\hat{A}(\xi)$ are different for $|\xi| \neq 0$, $\hat{A}(\xi)$ thus being diagonalizable for $|\xi| \neq 0$.

However, before starting to reflect on the actual proof we should note that we only have to consider the conditions

$$\frac{\nu\omega - (\kappa + 2\mu)^2}{\nu + \omega - 2(\kappa + 2\mu)} \geq \gamma^2$$

and

$$(\kappa + \mu + \gamma^2)^2 \, \frac{4(\mu - \nu - \gamma^2)(\mu - \omega - \gamma^2) - 4(\kappa + \mu + \gamma^2)^2}{(2\mu - \nu - \omega - 2\gamma^2)^2 - 4(\kappa + \mu + \gamma^2)^2} \geq \gamma^4,$$

if

$$\nu + \omega - 2\kappa - 4\mu \neq 0$$

or

$$(2\mu - \nu - \omega - 2\gamma^2)^2 - 4(\kappa + \mu + \gamma^2)^2 \neq 0,$$

respectively, which is equivalent to $\nu+\omega-2\kappa-4\mu \neq 0$ due to condition (1.6). For $\nu+\omega-2\kappa-4\mu = 0$, both conditions have to be ignored completely.

We will prove Lemma 2.1 by showing that $\hat{A}(\xi)$ does neither have a double real eigenvalue nor a double nonreal eigenvalue. For this purpose, however, we need a lower bound s for the real eigenvalues and the minimum m of the function h with $h(\Theta) = \alpha^2(\Theta) - 4\delta(\Theta)$ for the nonreal eigenvalues. The following is valid according to [2]:

$$s = \min\left\{ \frac{\nu}{\nu + \gamma^2}, \frac{\omega}{\omega + \gamma^2}, \frac{\nu\omega - \kappa^2}{\nu\omega - \kappa^2 + (\nu + \omega - 2\kappa)\gamma^2}, \frac{\nu\omega - (\kappa + 2\mu)^2}{\nu\omega - (\kappa + 2\mu)^2 + \Big[\nu + \omega - 2(\kappa + 2\mu)\Big]\gamma^2} \right\} |\xi|^2$$

and

$$\begin{aligned} h(\Theta) \;\geq\; & \min\Bigg\{ (\mu - \nu - \gamma^2)^2, (\mu - \omega - \gamma^2)^2, \\ & (\kappa + \mu + \gamma^2)^2 \frac{4(\mu - \nu - \gamma^2)(\mu - \omega - \gamma^2) - 4(\kappa + \mu + \gamma^2)^2}{(2\mu - \nu - \omega - 2\gamma^2)^2 - 4(\kappa + \mu + \gamma^2)^2} \Bigg\} \\ =:\; & m \;\geq\; 0. \end{aligned}$$

$s > \frac{1}{2}|\xi|^2$ and $m \geq \gamma^4$ due to the conditions (2.3) or (2.4), respectively. We shall now prove Lemma 2.1.

Proof: Since P is a polynomial of odd degree and $-A$ generates a contraction semigroup, $\hat{A}(\xi)$ has at least one positive eigenvalue. If nonreal eigenvalues exist, these are conjugate-complex to each other as the coefficients of P are real. P is decomposed into linear factors over $\mathbb{C}$:

$$P(\xi, \lambda) = \prod_{j=1}^{5} (\lambda - \lambda_j(\xi)).$$

Thus

$$\sum_{j=1}^{5} \lambda_j(\xi) = |\xi|^2,$$

which means that the sum of the real parts of the eigenvalues of $\hat{A}(\xi)$ equals $|\xi|^2$.

To show that $\hat{A}(\xi)$ does not have a double real eigenvalue we use the lower bound s. We have chosen the conditions (2.3) such that s is always greater than $\frac{1}{2}|\xi|^2$. Suppose that $\hat{A}(\xi)$ has a double real eigenvalue. Then the sum of the real parts of all eigenvalues is greater than $|\xi|^2$. Contradiction! For $\hat{A}(\xi)$ does only have one positive real eigenvalue under the conditions (2.3).

To show that $\hat{A}(\xi)$ does not have a double nonreal eigenvalue we use the minimum m of the function h. We have already chosen the conditions (2.4) such that m is greater than or equal to γ^4. Suppose that $\lambda = x + iy$ with $y \neq 0$ is a double nonreal eigenvalue of $\hat{A}(\xi)$. As each double zero of P is generally also a zero of $\frac{\partial P}{\partial \lambda}$, λ solves the two equations

$$\begin{aligned} P(|\xi|, \Theta, \lambda) &= \lambda^5 - |\xi|^2\lambda^4 + \alpha(\Theta)|\xi|^2\lambda^3 - \beta(\Theta)|\xi|^4\lambda^2 + \\ &\quad + \delta(\Theta)|\xi|^4\lambda - \eta(\Theta)|\xi|^6 \\ &= 0, \end{aligned} \tag{2.6}$$

$$\begin{aligned} \frac{\partial P}{\partial \lambda}(|\xi|, \Theta, \lambda) &= 5\lambda^4 - 4|\xi|^2\lambda^3 + 3\alpha(\Theta)|\xi|^2\lambda^2 - 2\beta(\Theta)|\xi|^4\lambda + \\ &\quad + \delta(\Theta)|\xi|^4 \\ &= 0. \end{aligned} \tag{2.7}$$

The following is valid for the imaginary part of equation (2.6) and for the real part of equation (2.7):

$$\begin{aligned} \mathrm{Im}\Big(P(|\xi|, \Theta, x, y)\Big) &= y\Big[y^4 - 10x^2y^2 + 5x^4 - 4|\xi|^2x(x^2 - y^2) + \\ &\quad + \alpha(\Theta)|\xi|^2(3x^2 - y^2) - 2\beta(\Theta)|\xi|^4x + \\ &\quad + \delta(\Theta)|\xi|^4\Big] \\ &= 0, \end{aligned} \tag{2.8}$$

$$\begin{aligned} \mathrm{Re}\left(\frac{\partial P}{\partial \lambda}(|\xi|, \Theta, x, y)\right) &= 5(x^4 - 6x^2y^2 + y^4) - 4|\xi|^2x(x^2 - 3y^2) + \\ &\quad + 3\alpha(\Theta)|\xi|^2(x^2 - y^2) - 2\beta(\Theta)|\xi|^4x + \\ &\quad + \delta(\Theta)|\xi|^4 \\ &= 0. \end{aligned} \tag{2.9}$$

Dividing equation (2.8) by $-y$ and adding equation (2.9) to it, we obtain

$$4y^4 - 20x^2y^2 + 8|\xi|^2xy^2 - 2\alpha(\Theta)|\xi|^2y^2 = 0$$

or

$$y^2 = 5x^2 - 2|\xi|^2x + \frac{1}{2}\alpha(\Theta)|\xi|^2, \tag{2.10}$$

respectively. We transform equation (2.8):

$$\begin{aligned} &y^4 - \Big(10x^2 - 4|\xi|^2x + \alpha(\Theta)|\xi|^2\Big)y^2 + \\ &\quad + 5x^4 - 4|\xi|^2x^3 + 3\alpha(\Theta)|\xi|^2x^2 - 2\beta(\Theta)|\xi|^4x + \delta(\Theta)|\xi|^4 = 0. \end{aligned}$$

Together with equation (2.10) we get

$$y^4 = 5x^4 - 4|\xi|^2 x^3 + 3\alpha(\Theta)|\xi|^2 x^2 - 2\beta(\Theta)|\xi|^4 x + \delta(\Theta)|\xi|^4. \tag{2.11}$$

Squaring equation (2.10) and inserting the result into equation (2.11), we obtain

$$\begin{aligned} 0 &= 20x^4 - 16|\xi|^2 x^3 + (4|\xi|^4 + 2\alpha(\Theta)|\xi|^2)x^2 - 2\gamma^2|\xi|^4 x + \\ &\quad + \Big(\frac{1}{4}\alpha^2(\Theta) - \delta(\Theta)\Big)|\xi|^4 \\ &= \Big(20x^2 - 16|\xi|^2 x + 2\alpha(\Theta)|\xi|^2\Big)x^2 + \\ &\quad + \Big(4x^2 - 2\gamma^2 x + \frac{1}{4}\alpha^2(\Theta) - \delta(\Theta)\Big)|\xi|^4 \\ &=: Ax^2 + B|\xi|^4. \end{aligned}$$

If we take B as a parabola in x, the function B has its minimum in $\frac{1}{4}\gamma^2$ so that

$$B \geq -\frac{1}{4}\gamma^4 + \frac{1}{4}\Big(\alpha^2(\Theta) - 4\delta(\Theta)\Big) = -\frac{1}{4}\gamma^4 + \frac{1}{4}h(\Theta).$$

$h(\Theta) \geq \gamma^4$ and thus $B \geq 0$ due to condition (2.4). We refute our assumption by showing that $Ax^2 + B|\xi|^4 > 0$ for all $|\xi| \neq 0$. Since the real eigenvalue of P is greater than $\frac{1}{2}|\xi|^2$ and the sum of the real parts of the eigenvalues of P equals $|\xi|^2$, $0 \leq x < \frac{1}{8}|\xi|^2$. $B > 0$ for $x = 0$ and $A > 0$ for $0 < x \leq \frac{1}{8}\alpha(\Theta)$. Consequently, $Ax^2 + B|\xi|^4 > 0$, provided that $0 \leq x \leq \frac{1}{8}\alpha(\Theta)$. The following is valid for $\frac{1}{8}\alpha(\Theta) < x < \frac{1}{8}|\xi|^2$:

$$\begin{aligned} A &= 20\left(x - \frac{2}{5}|\xi|^2\right)^2 - \frac{16}{5}|\xi|^4 + 2\alpha(\Theta)|\xi|^2 \\ &> 20\left(\frac{1}{8}|\xi|^2 - \frac{2}{5}|\xi|^2\right)^2 - \frac{16}{5}|\xi|^4 + 2\alpha(\Theta)|\xi|^2 \\ &= -\frac{27}{16}|\xi|^4 + 2\alpha(\Theta)|\xi|^2. \end{aligned}$$

Thus

$$\begin{aligned} Ax^2 + B|\xi|^4 &> -\frac{27}{16}|\xi|^4 x^2 + 2\alpha(\Theta)|\xi|^2 x^2 + 4|\xi|^4 x^2 - 2\gamma^2|\xi|^4 x + \\ &\quad + \frac{1}{4}\Big(\alpha^2(\Theta) - 4\delta(\Theta)\Big)|\xi|^4 \\ &= 2\alpha(\Theta)|\xi|^2 x^2 + \frac{1}{16}|\xi|^4(37x - 32\gamma^2)x + \frac{1}{4}h(\Theta)|\xi|^4 \\ &> 2\alpha(\Theta)|\xi|^2 x^2 + \frac{1}{16}|\xi|^4\left(\frac{37}{8}\alpha(\Theta) - 32\gamma^2\right)x + \frac{1}{4}h(\Theta)|\xi|^4, \end{aligned}$$

which is, however, greater than or equal to zero if $\frac{37}{8}\alpha(\Theta) - 32\gamma^2 \geq 0$ or the conditions (2.5) are met. Contradiction! □

If the conditions of Lemma 2.1 are met and if $P_j(\xi)$ is the projector onto the eigenspace to the eigenvalue $\lambda_j(\xi)$, the spectral decomposition of $\hat{A}(\xi)$ is:

$$\hat{A}(\xi) = \sum_{j=1}^{5} \lambda_j(\xi) P_j(\xi).$$

Hence, the solution $\hat{v}$ of the transformed equations (2.1) and (2.2) can be represented as follows:

$$\hat{v}(t,\xi) = \sum_{j=1}^{5} e^{-\lambda_j(\xi)t} P_j(\xi)\hat{v}_0(\xi).$$

3 Eigenvalues of the Operator

From the proof of Lemma 2.1 we know that, under the conditions (2.3) to (2.5), $\hat{A}(\xi)$ has one real eigenvalue and two pairs of conjugate-complex eigenvalues, where the real eigenvalue is positive and the real part of the nonreal eigenvalues not negative.

In the rhombic case we find the following decompositions for the polynomial P in the points $\Theta = 0$, $\Theta = 1$ and $\Theta = \frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu}$:

$$\begin{aligned} P(|\xi|, \Theta = 0, \lambda) &= \left[\lambda^3 - |\xi|^2\lambda^2 + (\nu+\gamma^2)|\xi|^2\lambda - \nu|\xi|^4\right]\left[\lambda^2 + \mu|\xi|^2\right], \\ P(|\xi|, \Theta = 1, \lambda) &= \left[\lambda^3 - |\xi|^2\lambda^2 + (\omega+\gamma^2)|\xi|^2\lambda - \omega|\xi|^4\right]\left[\lambda^2 + \mu|\xi|^2\right] \end{aligned}$$

or

$$\begin{aligned} &P\left(|\xi|, \Theta = \frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu}, \lambda\right) = \\ &= \Bigg\{\lambda^3 - |\xi|^2\lambda^2 + \\ &\quad + \left[\mu + \frac{1}{2}(\nu+\kappa) + \gamma^2 - \frac{1}{2}\frac{(\nu-\omega)(\nu-\kappa-2\mu)}{\nu+\omega-2\kappa-4\mu}\right]|\xi|^2\lambda - \\ &\quad - \left[\mu + \frac{1}{2}(\nu+\kappa) - \frac{1}{2}\frac{(\nu-\omega)(\nu-\kappa-2\mu)}{\nu+\omega-2\kappa-4\mu}\right]|\xi|^4\Bigg\} \cdot \\ &\quad \cdot \left\{\lambda^2 + \left[\frac{1}{2}(\nu-\kappa) - \frac{1}{2}\frac{(\nu-\omega)(\nu-\kappa-2\mu)}{\nu+\omega-2\kappa-4\mu}\right]|\xi|^2\right\}, \end{aligned}$$

respectively. Hence, we obtain purely imaginary eigenvalues for $\hat{A}(\xi)$ in the points $\Theta = 0$, $\Theta = 1$ and $\Theta = \frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu}$. The next lemma will show us that these points are the only points in which $\hat{A}(\xi)$ has purely imaginary eigenvalues.

Lemma 3.1 *Let $|\xi| \neq 0$. In the case of $(\nu-\kappa-2\mu)(\omega-\kappa-2\mu) > 0$ the real part of the eigenvalues of $\hat{A}(\xi)$ disappears in the points $\Theta = 0$, $\Theta = \frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu}$ and $\Theta = 1$. In the case of $(\nu-\kappa-2\mu)(\omega-\kappa-2\mu) \leq 0$ with $\nu \neq \omega$ the real part of the eigenvalues of $\hat{A}(\xi)$ only disappears in the points $\Theta = 0$ or $\Theta = 1$.*

BEWEIS: Let $\lambda = iy$ with $y \neq 0$. Then, the following equations result from the real and imaginary part of the polynomial P for y:

$$y^4 - \beta|\xi|^2y^2 + \eta|\xi|^4 = 0, \quad (3.1)$$

$$y^4 - \alpha|\xi|^2y^2 + \delta|\xi|^4 = 0. \quad (3.2)$$

Subtracting equation (3.2) from equation (3.1), we obtain

$$(\alpha-\beta)|\xi|^2y^2 + (\eta-\delta)|\xi|^4 = 0$$

or

$$y^2 = \Big[\mu + \Theta(1-\Theta)(\nu+\omega-2\kappa-4\mu)\Big]|\xi|^2, \tag{3.3}$$

respectively. If we insert equation (3.3) into equation (3.1), then

$$-\Theta(1-\Theta)\Big[\Theta(\nu+\omega-2\kappa-4\mu)-(\nu-\kappa-2\mu)\Big]^2 = 0. \tag{3.4}$$

Equation (3.4) is now only valid if

$$\begin{aligned}\Theta &= 0,\\ \Theta &= 1\end{aligned}$$

or

$$\Theta(\nu+\omega-2\kappa-4\mu)-(\nu-\kappa-2\mu) \;=\; 0. \tag{3.5}$$

In the case of $\nu+\omega-2\kappa-4\mu=0$, equation (3.5) is equivalent to

$$\nu-\kappa-2\mu=0.$$

For this reason, $\omega-\kappa-2\mu=0$ and thus $\nu=\omega=\kappa+2\mu$. In the case of $\nu+\omega-2\kappa-4\mu\neq 0$,

$$\tilde{\Theta} = \frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu} \tag{3.6}$$

solves equation (3.5). If $\tilde{\Theta}\leq 0$ or $\tilde{\Theta}\geq 1$, the following results from equation (3.6):

$$(\nu-\kappa-2\mu)(\omega-\kappa-2\mu)\leq 0,$$

and equation (3.4) is valid for $\Theta=0$ and $\Theta=1$ only. If $0<\tilde{\Theta}<1$, the following results from equation (3.6):

$$(\nu-\kappa-2\mu)(\omega-\kappa-2\mu)> 0,$$

and equation (3.4) is valid for $\Theta=0$, $\Theta=\frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu}$ and $\Theta=1$. □

We shall now expand the eigenvalues of $\hat{A}(\xi)$ for $|\xi|\to 0$ and $|\xi|\to\infty$. Since P is a polynomial of fifth degree, an explicit solution formula for its zeros does not exist. Nevertheless it is possible to expand the eigenvalues of $\hat{A}(\xi)$ in power series with fractional powers at least for $|\xi|\to 0$ and $|\xi|\to\infty$. For this purpose we will keep Θ fixed from now on and apply a method based on Newton. Newton's approach was to conceive of $P(|\xi|,\lambda)=0$ as an implicit equation for λ and to calculate λ using an approximate procedure which gives an expansion of λ according to powers of $|\xi|$. If the conditions of the implicit function theorem are met, λ can be represented in $|\xi|$ by a convergent power series. In general, however, this is not the case, which is also demonstrated by the following example:

$$P(|\xi|,\lambda)=|\xi|^3-\lambda^2=0$$

namely has the solution

$$\lambda(|\xi|)=|\xi|^{3/2}.$$

This example already shows that, in any case, we need fractional powers of $|\xi|$ to represent λ.

Using the following proposition proved in [2] we summarize the asymptotic expansions of the eigenvalues for $|\xi|\to 0$ and $|\xi|\to\infty$, numbering the eigenvalues as follows: In $\Theta=0$, let λ_1 be real, $\operatorname{Re}\lambda_{2,3}>0$, $\operatorname{Re}\lambda_{4,5}=0$ and $\operatorname{Im}\lambda_{2,4}>0$. If κ, μ, ν, ω and γ meet the requirements of Lemma 2.1, we have to distinguish the cases

(a) $\max\{\mu-\nu, \mu-\omega\} \le 0$,

(b) $\mu-\nu \le 0 \wedge \mu-\omega \ge 2\gamma^2$,

(c) $\mu-\nu \ge 2\gamma^2 \wedge \mu-\omega \le 0$,

(d) $\min\{\mu-\nu, \mu-\omega\} \ge 2\gamma^2$

due to the conditions (2.4). These conditions yield $\kappa+\mu+\gamma^2 \ne 0$ and therefore the additional cases

(i) $\kappa+\mu+\gamma^2 < 0$,

(ii) $\kappa+\mu+\gamma^2 > 0$.

Proposition 3.2 *The eigenvalues of $\hat{A}(\xi)$ can be expanded for $|\xi| \to 0$ and $|\xi| \to \infty$ in power series with fractional powers.*

The expansions of the real eigenvalue λ_1 are

- *for* $|\xi| \to 0$: $\lambda_1(|\xi|,\Theta) = \dfrac{\eta(\Theta)}{\delta(\Theta)}|\xi|^2 + O(|\xi|^4)$,
- *for* $|\xi| \to \infty$: $\lambda_1(|\xi|,\Theta) = |\xi|^2 - \gamma^2 + O(|\xi|^{-2})$.

The following is valid for the eigenvalues $\lambda_{2,3}$ and $\lambda_{4,5}$:

1. *In the case of $(\nu-\kappa-2\mu)(\omega-\kappa-2\mu) \le 0$ with $\nu \ne \omega$ the real part of the eigenvalues disappears in $\Theta = 0$ and $\Theta = 1$.*

   (a) If $\max\{\mu-\nu, \mu-\omega\} \le 0$, the expansions of the eigenvalues are as follows:

   - *for* $|\xi| \to 0$:

     $$\lambda_{2/3}(|\xi|,\Theta) = \pm i\sqrt{\frac{\alpha(\Theta)}{2} + \frac{1}{2}\sqrt{\alpha^2(\Theta) - 4\delta(\Theta)}}|\xi| + t_1'(\Theta)|\xi|^2 + O(|\xi|^3),$$

     where $t_1'(\Theta) > 0$ for all $\Theta \in [0,1]$,

     $$\lambda_{4/5}(|\xi|,\Theta) = \pm i\sqrt{\frac{\alpha(\Theta)}{2} - \frac{1}{2}\sqrt{\alpha^2(\Theta) - 4\delta(\Theta)}}|\xi| + t_1''(\Theta)|\xi|^2 + O(|\xi|^3),$$

     where $t_1''(\Theta) = 0$ for $\Theta = 0$ and $\Theta = 1$,

   - *for* $|\xi| \to \infty$:

     $$\lambda_{2/3}(|\xi|,\Theta) = \pm i\sqrt{\frac{\beta(\Theta)}{2\eta(\Theta)} - \frac{1}{2\eta(\Theta)}\sqrt{\beta^2(\Theta) - 4\eta(\Theta)}}^{\,-1}|\xi| + \\ + t_1'(\Theta) + O(|\xi|^{-1}),$$

     where $t_1'(\Theta) > 0$ for all $\Theta \in [0,1]$,

     $$\lambda_{4/5}(|\xi|,\Theta) = \pm i\sqrt{\frac{\beta(\Theta)}{2\eta(\Theta)} + \frac{1}{2\eta(\Theta)}\sqrt{\beta^2(\Theta) - 4\eta(\Theta)}}^{\,-1}|\xi| + \\ + t_1''(\Theta) + O(|\xi|^{-1}),$$

     where $t_1''(\Theta) = 0$ for $\Theta = 0$ and $\Theta = 1$.

   (b) If $\mu-\nu \le 0$ and $\mu-\omega \ge 2\gamma^2$, the expansions of the eigenvalues are as follows:

- *for* $|\xi| \to 0:$

  $\lambda_{2/3}(|\xi|,\Theta) = \pm i\sqrt{\frac{\alpha(\Theta)}{2} + \frac{1}{2}\sqrt{\alpha^2(\Theta) - 4\delta(\Theta)}}|\xi| + t_1'(\Theta)|\xi|^2 + O(|\xi|^3),$

  where $t_1'(\Theta) = 0$ *for* $\Theta = 1,$

  $\lambda_{4/5}(|\xi|,\Theta) = \pm i\sqrt{\frac{\alpha(\Theta)}{2} - \frac{1}{2}\sqrt{\alpha^2(\Theta) - 4\delta(\Theta)}}|\xi| + t_1''(\Theta)|\xi|^2 + O(|\xi|^3),$

  where $t_1''(\Theta) = 0$ *for* $\Theta = 0,$
- *for* $|\xi| \to \infty:$

  $\lambda_{2/3}(|\xi|,\Theta) = \pm i\sqrt{\frac{\beta(\Theta)}{2\eta(\Theta)} - \frac{1}{2\eta(\Theta)}\sqrt{\beta^2(\Theta) - 4\eta(\Theta)}}^{\,-1}|\xi| +$
  $\qquad + t_1'(\Theta) + O(|\xi|^{-1}),$

  where $t_1'(\Theta) = 0$ *for* $\Theta = 1,$

  $\lambda_{4/5}(|\xi|,\Theta) = \pm i\sqrt{\frac{\beta(\Theta)}{2\eta(\Theta)} + \frac{1}{2\eta(\Theta)}\sqrt{\beta^2(\Theta) - 4\eta(\Theta)}}^{\,-1}|\xi| +$
  $\qquad + t_1''(\Theta) + O(|\xi|^{-1}),$

  where $t_1''(\Theta) = 0$ *for* $\Theta = 0.$

(c) If $\mu - \nu \geq 2\gamma^2$ *and* $\mu - \omega \leq 0$*, the expansions of the eigenvalues are as follows:*

- *for* $|\xi| \to 0:$

  $\lambda_{2/3}(|\xi|,\Theta) = \pm i\sqrt{\frac{\alpha(\Theta)}{2} - \frac{1}{2}\sqrt{\alpha^2(\Theta) - 4\delta(\Theta)}}|\xi| + t_1'(\Theta)|\xi|^2 + O(|\xi|^3),$

  where $t_1'(\Theta) = 0$ *for* $\Theta = 1,$

  $\lambda_{4/5}(|\xi|,\Theta) = \pm i\sqrt{\frac{\alpha(\Theta)}{2} + \frac{1}{2}\sqrt{\alpha^2(\Theta) - 4\delta(\Theta)}}|\xi| + t_1''(\Theta)|\xi|^2 + O(|\xi|^3),$

  where $t_1''(\Theta) = 0$ *for* $\Theta = 0,$
- *for* $|\xi| \to \infty:$

  $\lambda_{2/3}(|\xi|,\Theta) = \pm i\sqrt{\frac{\beta(\Theta)}{2\eta(\Theta)} + \frac{1}{2\eta(\Theta)}\sqrt{\beta^2(\Theta) - 4\eta(\Theta)}}^{\,-1}|\xi| +$
  $\qquad + t_1'(\Theta) + O(|\xi|^{-1}),$

  where $t_1'(\Theta) = 0$ *for* $\Theta = 1,$

  $\lambda_{4/5}(|\xi|,\Theta) = \pm i\sqrt{\frac{\beta(\Theta)}{2\eta(\Theta)} - \frac{1}{2\eta(\Theta)}\sqrt{\beta^2(\Theta) - 4\eta(\Theta)}}^{\,-1}|\xi| +$
  $\qquad + t_1''(\Theta) + O(|\xi|^{-1}),$

  where $t_1''(\Theta) = 0$ *for* $\Theta = 0.$

(d) If $\min\{\mu - \nu, \nu - \omega\} \geq 2\gamma^2$*, the expansions of the eigenvalues are as follows:*

- *for* $|\xi| \to 0:$

  $\lambda_{2/3}(|\xi|,\Theta) = \pm i\sqrt{\frac{\alpha(\Theta)}{2} - \frac{1}{2}\sqrt{\alpha^2(\Theta) - 4\delta(\Theta)}}|\xi| + t_1'(\Theta)|\xi|^2 + O(|\xi|^3),$

  where $t_1'(\Theta) > 0$ *for all* $\Theta \in [0,1],$

$$\lambda_{4/5}(|\xi|,\Theta) = \pm i\sqrt{\frac{\alpha(\Theta)}{2} + \frac{1}{2}\sqrt{\alpha^2(\Theta) - 4\delta(\Theta)}}|\xi| + t_1''(\Theta)|\xi|^2 + O(|\xi|^3),$$

where $t_1''(\Theta) = 0$ *for* $\Theta = 0$ *and* $\Theta = 1$,

- *for* $|\xi| \to \infty$:

$$\lambda_{2/3}(|\xi|,\Theta) = \pm i\sqrt{\frac{\beta(\Theta)}{2\eta(\Theta)} + \frac{1}{2\eta(\Theta)}\sqrt{\beta^2(\Theta) - 4\eta(\Theta)}}^{\,-1}|\xi| + $$
$$+ t_1'(\Theta) + O(|\xi|^{-1}),$$

where $t_1'(\Theta) > 0$ *for all* $\Theta \in [0,1]$,

$$\lambda_{4/5}(|\xi|,\Theta) = \pm i\sqrt{\frac{\beta(\Theta)}{2\eta(\Theta)} - \frac{1}{2\eta(\Theta)}\sqrt{\beta^2(\Theta) - 4\eta(\Theta)}}^{\,-1}|\xi| + $$
$$+ t_1''(\Theta) + O(|\xi|^{-1}),$$

where $t_1''(\Theta) = 0$ *for* $\Theta = 0$ *uad* $\Theta = 1$.

2. *In the case of* $(\nu - \kappa - 2\mu)(\omega - \kappa - 2\mu) > 0$ *the real part of the eigenvalues disappears in* $\Theta = 0$, $\Theta = \frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu}$ *and* $\Theta = 1$.

   (a) *If* $\max\{\mu - \nu, \mu - \omega\} \le 0$, *the expansions of the eigenvalues* $\lambda_{2,3}$ *and* $\lambda_{4,5}$ *have the same form as in 1 (a) for* $|\xi| \to 0$ *and* $|\xi| \to \infty$.

      (i) *For* $\kappa + \mu + \gamma^2 < 0$ *the coefficients satisfy:*

      $t_1'(\Theta) = 0$ *in* $\Theta = \frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu}$ *as well as*
      $t_1''(\Theta) = 0$ *in* $\Theta = 0$ *and* $\Theta = 1$.

      (ii) *For* $\kappa + \mu + \gamma^2 > 0$ *the coefficients satisfy:*

      $t_1'(\Theta) > 0$ *for all* $\Theta \in [0,1]$ *as well as*
      $t_1''(\Theta) = 0$ *in* $\Theta = 0$, $\Theta = \frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu}$ *and* $\Theta = 1$.

   (b) *If* $\mu - \nu \le 0$, $\mu - \omega \ge 2\gamma^2$, *the expansions of the eigenvalues* $\lambda_{2,3}$ *and* $\lambda_{4,5}$ *have the same form as in 1 (b) for* $|\xi| \to 0$ *and* $|\xi| \to \infty$.

      (i) *For* $\kappa + \mu + \gamma^2 < 0$ *the coefficients satisfy:*

      $t_1'(\Theta) = 0$ *in* $\Theta = \frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu}$ *and* $\Theta = 1$ *as well as*
      $t_1''(\Theta) = 0$ *in* $\Theta = 0$.

      (ii) *For* $\kappa + \mu + \gamma^2 > 0$ *the coefficients satisfy:*

      $t_1'(\Theta) = 0$ *in* $\Theta = 1$ *as well as*
      $t_1''(\Theta) = 0$ *in* $\Theta = 0$ *and* $\Theta = \frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu}$.

   (c) *If* $\mu - \nu \ge 2\gamma^2$, $\mu - \omega \le 0$, *the expansions of the eigenvalues* $\lambda_{2,3}$ *and* $\lambda_{4,5}$ *have the same form as in 1 (c) for* $|\xi| \to 0$ *and* $|\xi| \to \infty$.

      (i) *For* $\kappa + \mu + \gamma^2 < 0$ *the coefficients satisfy:*

      $t_1'(\Theta) = 0$ *in* $\Theta = 1$ *as well as*
      $t_1''(\Theta) = 0$ *in* $\Theta = 0$ *and* $\Theta = \frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu}$.

      (ii) *For* $\kappa + \mu + \gamma^2 > 0$ *the coefficients satisfy:*

      $t_1'(\Theta) = 0$ *in* $\Theta = \frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu}$ *and* $\Theta = 1$ *as well as*
      $t_1''(\Theta) = 0$ *in* $\Theta = 0$.

(d) If $\min\{\mu-\nu,\mu-\omega\}\geq 2\gamma^2$*, the expansions of the eigenvalues* $\lambda_{2,3}$ *and* $\lambda_{4,5}$ *have the same form as in 1 (d) for* $|\xi|\to 0$ *and* $|\xi|\to\infty$*.*

(i) Forr $\kappa+\mu+\gamma^2<0$ *the coefficients satisfy:*

$t_1'(\Theta)>0$ *for all* $\Theta\in[0,1]$ *as well as*
$t_1''(\Theta)=0$ *in* $\Theta=0$, $\Theta=\frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu}$ *and* $\Theta=1$.

(ii) For $\kappa+\mu+\gamma^2>0$ *the coefficients satisfy:*

$t_1'(\Theta)=0$ *in* $\Theta=\frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu}$ *as well as*
$t_1''(\Theta)=0$ *in* $\Theta=0$ *and* $\Theta=1$.

By expanding the eigenvalues it is possible to show the boundedness of the projectors

$$P_j(\xi)=\prod_{\substack{k=1\\k\neq j}}^{5}\frac{\hat{A}(\xi)-\lambda_k(\xi)id}{\lambda_j(\xi)-\lambda_k(\xi)},$$

in the zero point and at infinity, where $j\in\{1,\dots,5\}$. Obviously, the following is valid for $|\xi|\to 0$:

$$P_j(\xi)=O(1).$$

For $|\xi|\to\infty$ it can be also demonstrated in a similar way as used in [1] that

$$P_j(\xi)=O(1).$$

4 Method of Stationary Phase

As we are now familiar with the expansion of eigenvalues in the zero point and at infinity, we are able to calculate the time-asymptotic behavior of solution v of the differential equation (1.7) in relation to the initial condition (1.8) for infinite times. Recalling the representation of solution $\hat{v}$ of the transformed equations (2.1) and (2.2) and performing the Fourier backtransformation we obtain for v:

$$v(t,x)=\frac{1}{2\pi}\sum_{j=1}^{5}\int_{\mathbb{R}^2}e^{i(x\xi+i\lambda_j(\xi)t)}P_j(\xi)\hat{v}_0(\xi)\,d\xi.$$

This means that the decay behavior of v is determined by the integrals

$$I_j(t,x):=\int_{\mathbb{R}^2}e^{i(x\xi+i\lambda_j(\xi)t)}P_j(\xi)\hat{v}_0(\xi)\,d\xi,$$

where $j\in\{1,\dots,5\}$. Let the subscript j be fixed in the following.

In order to calculate the time-asymptotic behavior of I_j for $t\to\infty$ we apply the method of stationary phase, focusing on the behavior of the phase

$$\sigma_j\left(\frac{x}{t},\xi\right):=\frac{x}{t}\xi+i\lambda_j(\xi).$$

For our further reflections we parameterize $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$ by polar coordinates:

$$\begin{aligned}\xi_1&:=r\cos\varphi,\\ \xi_2&:=r\sin\varphi,\end{aligned}$$

where $r \geq 0$ and $0 \leq \varphi < 2\pi$, and define

$$U_\varepsilon := \left\{ 0 \leq \varphi < 2\pi \,\middle|\, \sin^2\varphi \leq \varepsilon \;\vee\; 1 - \sin^2\varphi \leq \varepsilon \;\vee\; \left| \frac{\nu - \kappa - 2\mu}{\nu + \omega - 2\kappa - 4\mu} - \sin^2\varphi \right| \leq \frac{\varepsilon}{2} \right\}$$

as well as

$$\mathbb{R}^2_\varepsilon := \{(\xi_1, \xi_2) \in \mathbb{R}^2 |\ \xi_1 = r\cos\varphi,\ \xi_2 = r\sin\varphi,\ \text{wobei } r \geq 0,\ \varphi \in U_\varepsilon\}.$$

Since

$$\Theta = \frac{\xi_2^2}{|\xi|^2} = \sin^2\varphi,$$

U_ε is an ε-neighborhood of $\Theta = 0$, $\Theta = 1$ and $\Theta = \frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu}$. In the following we will determine the decay behavior of the integral I_j in dependence on its phase.

If the phase σ_j satisfies one of the two conditions

(A) $\forall\ \xi \in \mathbb{R}^2 :\ \operatorname{Im}\sigma_j(\frac{x}{t}, \xi) \neq 0$,

(B) $\exists\ k_0, K_0 > 0\ \ \exists\ C_j > 0\ \ \exists\ \varepsilon_0 > 0\ \ \forall\ 0 < \varepsilon < \varepsilon_0\ \ \forall\ \xi \in \mathbb{R}^2_\varepsilon$*:*
$|\nabla_\xi \sigma_j(\frac{x}{t}, \xi)| \geq C_j$ *für* $|x| \leq \frac{1}{2}k_0 t$ *oder* $|x| \geq 2K_0 t$,

I_j decays in the L^p-L^q-estimate as $(1+t)^{-(1-\frac{2}{q})}$. A detailed representation of the facts is provided by

Proposition 4.1 *Let* $2 \leq q \leq \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

(a) If $N_p > 3(1 - \frac{2}{q})$ *(if* $q \in \{2, \infty\}$*, then* $N_p \geq 3(1 - \frac{2}{q})$*) and the condition (A) is satisfied, then* $c = c(q) > 0$ *exists. Thus, the following is valid for each* $v_0 \in W^{N_p,p}$ *and* $t \geq 0$ *gilt:*

$$|I_j(t)|_q \leq c(1+t)^{-(1-\frac{2}{q})} \|v_0\|_{N_p,p}.$$

(b) If $N_p \leq 3$ *and the condition (B) is satisfied, then* $c = c(q) > 0$ *exists. Thus, the following is valid for each* $v_0 \in W^{N_p,p}$ *and* $t \geq 0$*:*

$$|I_j(t)|_q \leq c(1+t)^{-(1-\frac{2}{q})} \|v_0\|_{N_p,p}.$$

Proof: (a) We subdivide the proof into two steps. In the first step we determine the L^2-L^2-estimate

$$|I_j(t)|_2 \leq c|v_0|_2$$

for I_j and in the second step the special L^1-L^∞-estimate

$$|I_j(t)|_\infty \leq c(1+t)^{-1} \|v_0\|_{3,1},$$

$c > 0$ being a constant in both estimates. The claim finally results from the interpolation of the two estimates.

The L^2-L^2-estimate is obvious. Since $-A$ generates a contraction semigroup, the eigenvalues have a non-negative real part. Thus,

$$|I_j(t)|_2 = 2\pi\left|e^{-\lambda_j t}P_j\hat{v}_0\right|_2 \le 2\pi|e^{-\lambda_j t}|_\infty |P_j\hat{v}_0|_2 \le c|\hat{v}_0|_2 \;=\; c|v_0|_2,$$

for the projector P_j is bounded.

We shall now prove the L^1-L^∞-estimate. Due to proposition 3.2 there are positive constants r, R, c_j and C_j so that for each $|\xi| \le r$

$$\operatorname{Im}\sigma_j\left(\frac{x}{t},\xi\right) \ge c_j|\xi|^2$$

and for each $|\xi| \ge R$

$$\operatorname{Im}\sigma_j\left(\frac{x}{t},\xi\right) \ge C_j.$$

In the case of $r < |\xi| < R$ $\operatorname{Im}\sigma_j$ is also greater than zero. Since $[r,R]$ is compact, $\tilde{C}_j > 0$ exists so that for each $|\xi| \le r$

$$\operatorname{Im}\sigma_j\left(\frac{x}{t},\xi\right) \ge c_j|\xi|^2$$

and for each $|\xi| > r$

$$\operatorname{Im}\sigma_j\left(\frac{x}{t},\xi\right) \ge \tilde{C}_j.$$

Thus we obtain

$$\begin{aligned}
|I_j(t)|_\infty &= \left|\int_{\mathbb{R}^2} e^{i\xi\cdot - \lambda_j(\xi)t}P_j(\xi)\hat{v}_0(\xi)\,d\xi\right|_\infty \\
&\le \Bigg| |e^{i\cdot}P_j\hat{v}_0|_\infty \int_{|\xi|\le r} \left|e^{-\lambda_j(\xi)t}\right| d\xi + \\
&\qquad + |e^{i\cdot}(1+|\cdot|)^3 P_j\hat{v}_0|_\infty \int_{|\xi|>r} \left|e^{-\lambda_j(\xi)t}\frac{1}{(1+|\xi|)^3}\right| d\xi \Bigg|_\infty .
\end{aligned}$$

We perform an upper estimate of the integral

$$\int_{|\xi|\le r} e^{-\lambda_j(\xi)t}\,d\xi$$

with the help of the transformation $y = \sqrt{t}\xi$:

$$\int_{|\xi|\le r} e^{-\lambda_j(\xi)t}\,d\xi \le \int_{|\xi|\le r} e^{-c_j|\xi|^2 t}\,d\xi = \frac{1}{t}\int_{|y|\le r\sqrt{t}} e^{-c_j|y|^2}\,dy.$$

This is the reason why for $t \ge 1$

$$\int_{|\xi|\le r} e^{-\lambda_j(\xi)t}\,d\xi \le \frac{1}{t}\int_{\mathbb{R}^2} e^{-c_j|y|^2}\,dy \le c(1+t)^{-1}$$

and for $0 \leq t \leq 1$

$$\int\limits_{|\xi|\leq r} e^{-\lambda_j(\xi)t}\, d\xi \leq \frac{1}{t} \int\limits_{|y|\leq r\sqrt{t}} dy \leq \pi r^2.$$

Hence, the following is valid for all t

$$\int\limits_{|\xi|\leq r} e^{-\lambda_j(\xi)t}\, d\xi \leq c(1+t)^{-1}.$$

The Integral

$$\int\limits_{|\xi|>r} e^{-\lambda_j(\xi)t} \frac{1}{(1+|\xi|)^3}\, d\xi$$

remains bounded due to the weight $(1+|\xi|)^{-3}$ and decays exponentially in time:

$$\int\limits_{|\xi|>r} e^{-\lambda_j(\xi)t} \frac{1}{(1+|\xi|)^3}\, d\xi \leq c e^{-\tilde{C}_j t}.$$

However, we have to accept higher Sobolev norms in the estimate because of the weight:

$$|e^{ix\cdot}(1+|\cdot|)^3 P_j \hat{v}_0|_\infty \leq c \sum_{k=0}^{3} \Big| |\cdot|^k P_j \hat{v}_0 \Big|_\infty \leq c \sum_{k=0}^{3} \Big| |\cdot|^k \hat{v}_0 \Big|_\infty.$$

We estimate each of the addends

$$\Big| |\cdot|^k \hat{v}_0 \Big|_\infty$$

as follows:

$$\begin{aligned}
\Big| |\cdot|^k \hat{v}_0 \Big|_\infty &= \left| \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} |\cdot|^k e^{-ix\cdot} v_0(x)\, dx \right|_\infty \\
&\leq c \sum_{l=0}^{k} \sum_{m=1}^{2} \left| \int\limits_{\mathbb{R}^2} \left[\frac{\partial^l}{\partial x_m^l} e^{-ix\cdot} \right] v_0(x)\, dx \right|_\infty \\
&\leq c \sum_{l=0}^{k} \sum_{m=1}^{2} \left| \frac{\partial^l}{\partial x_m^l} v_0 \right|_1.
\end{aligned}$$

Altogether we finally obtain the L^1-L^∞-estimate

$$|I_j(t)|_\infty \leq c(1+t)^{-1} \|v_0\|_{3,1}.$$

(b) We proceed analogously to (a). It is sufficient to show the L^∞-L^∞-estimate

$$|I_j(t)|_\infty \leq c t^{-1} \sum_{k=0}^{3} \| |\cdot|^k \hat{v}_0 \|_{1,\infty}.$$

As the real part of the eigenvalue λ_j is not negative and P_j is bounded, the following is valid for each $0 \leq t \leq 1$:

$$|I_j(t)|_\infty = \left| \int\limits_{\mathbb{R}^2} e^{i(\xi\cdot + i\lambda_j(\xi)t)} P_j(\xi)\hat{v}_0(\xi)\, d\xi \right|_\infty \leq c \int\limits_{\mathbb{R}^2} |\hat{v}_0(\xi)|\, d\xi = c|\hat{v}_0|_1.$$

In combination with Theorem 6.4 from [1] we obtain the L^1-L^∞-estimate

$$|I_j(t)|_\infty \leq c(1+t)^{-1}\|v_0\|_{3,1}$$

from the last two estimates, c being a positive constant in all estimates.

In order to prove the L^∞-L^∞-estimate, we split the integral I_j using the definition of $\mathbb{R}^2_\varepsilon$:

$$I_j(t,x) = \int\limits_{\mathbb{R}^2_\varepsilon} e^{it\sigma(\frac{x}{t},\xi)} P_j(\xi)\hat{v}_0(\xi)\, d\xi + \int\limits_{\mathbb{R}^2 \setminus \mathbb{R}^2_\varepsilon} e^{it\sigma(\frac{x}{t},\xi)} P_j(\xi)\hat{v}_0(\xi)\, d\xi.$$

The second integral can be estimated in the same way since the imaginary part of the phase σ_j differs from zero in $\mathbb{R}^2 \setminus \mathbb{R}^2_\varepsilon$. To obtain the time-asymptotic behavior of the first integral, we define

$$L := \sum_{k=1}^{2} |\nabla_\xi\, \sigma|^{-2} \frac{\partial \sigma}{\partial \xi_k} \frac{\partial}{\partial \xi_k}$$

and the operator formally adjoint to L

$$L^* u := -\sum_{k=1}^{2} \frac{\partial}{\partial \xi_k} \left(u |\nabla_\xi\, \sigma|^{-2} \frac{\partial \sigma}{\partial \xi_k} \right).$$

Since the projector P_j as well as all the derivatives of P_j are bounded and $L\, e^{it\sigma} = ite^{it\sigma}$, we obtain the following for the decay behavior in this case by partial integration:

$$\begin{aligned}
\left| \int\limits_{\mathbb{R}^2_\varepsilon} e^{it\sigma(\frac{x}{t},\xi)} P_j(\xi)\hat{v}_0(\xi)\, d\xi \right| &= \left| \frac{1}{it} \int\limits_{\mathbb{R}^2_\varepsilon} L\left[e^{it\sigma(\frac{x}{t},\xi)} \right] P_j(\xi)\hat{v}_0(\xi)\, d\xi \right| \\
&= \frac{1}{t} \left| \int\limits_{\partial\mathbb{R}^2_\varepsilon} |\nabla_\xi\, \sigma|^{-2} \frac{\partial \sigma}{\partial \nu} e^{it\sigma(\frac{x}{t},\xi)} P_j(\xi)\hat{v}_0(\xi)\, do_\xi - \right. \\
&\qquad \left. - \int\limits_{\mathbb{R}^2_\varepsilon} e^{it\sigma(\frac{x}{t},\xi)} L^* P_j(\xi)\hat{v}_0(\xi)\, d\xi \right| \\
&\leq \frac{c}{t} |\,(1+|\cdot|)^2 P_j\hat{v}_0\,|_\infty \int\limits_0^\infty \frac{1}{r^2}\, dr + \\
&\qquad + \frac{c}{t} |\,(1+|\cdot|)^3 L^* P_j\hat{v}_0\,|_\infty \int\limits_0^\infty \frac{1}{r^2}\, dr \\
&\leq \frac{c}{t} \sum_{k=0}^{3} \|\,|\cdot|^k P_j\hat{v}_0\,\|_{1,\infty} \leq \frac{c}{t} \sum_{k=0}^{3} \|\,|\cdot|^k \hat{v}_0\,\|_{1,\infty},
\end{aligned}$$

where ν is the outer unit normal at $\partial\mathbb{R}^2_\varepsilon$. □

Now we determine the L^p-L^q-estimate for I_j in the case of $\frac{1}{2}k_0 t < |x| < 2K_0 t$. For this purpose we introduce a transformation T which is similar to the transformation used for polar coordinates. Instead of integrating over circles we integrate over the curve

$$\Gamma_y(\varphi) := \Gamma(y,\varphi),$$

where $y > 0$, $\varphi \in U_\varepsilon$ and

$$\operatorname{Im}\lambda_j(\Gamma_y(\varphi)) = y.$$

We define the transformation T by

$$T\colon (0,\infty)\times U_\varepsilon \to (0,\infty)\times U_\varepsilon,\ \ (y,\varphi)\mapsto (r,\varphi).$$

In order for T to be bijective and $|\det T'| < \infty$, we choose $\varepsilon > 0$ such that

$$\frac{1}{2}\operatorname{Im}\lambda_j(r,\tilde\varphi) \le \operatorname{Im}\lambda_j(r,\varphi) \le 2\operatorname{Im}\lambda_j(r,\tilde\varphi)$$

and

$$\frac{1}{2}\frac{\partial}{\partial r}\operatorname{Im}\lambda_j(r,\tilde\varphi) \le \frac{\partial}{\partial r}\operatorname{Im}\lambda_j(r,\varphi) \le 2\frac{\partial}{\partial r}\operatorname{Im}\lambda_j(r,\tilde\varphi),$$

if

$$(\sin^2\tilde\varphi = 0 \ \wedge\ \sin^2\varphi \le \varepsilon) \quad\vee\quad (\sin^2\tilde\varphi = 1 \ \wedge\ 1-\sin^2\varphi \le \varepsilon) \quad\vee$$
$$\left(\sin^2\tilde\varphi = \frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu} \ \wedge\ \left|\frac{\nu-\kappa-2\mu}{\nu+\omega-2\kappa-4\mu} - \sin^2\varphi\right| \le \frac{\varepsilon}{2}\right).$$

As an abbreviation we define

$$s := \left|\begin{pmatrix} x\\ t\end{pmatrix}\right|,$$
$$\zeta := \frac{1}{s}\begin{pmatrix} x\\ t\end{pmatrix}$$

and

$$\Phi_j(y,\varphi,\zeta) := \zeta\begin{pmatrix} r(y,\varphi)\cos\varphi\\ r(y,\varphi)\sin\varphi\\ -y+ix_j(y,\varphi)\end{pmatrix},$$

where

$$x_j := \operatorname{Re}\lambda_j$$

and ζ is an element from the set

$$M := \left\{\zeta\in S^2 \,\middle|\, \zeta_3 > 0,\ \frac{k_0}{\sqrt{k_0^2+4}} \le |(\zeta_1,\zeta_2)| \le \frac{2K_0}{\sqrt{4K_0^2+1}}\right\}.$$

We further define

$$SP^j_\varepsilon := \left\{(y,\varphi,\zeta)\in(0,\infty)\times U_\varepsilon\times M \,\middle|\, \frac{\partial\Phi_j}{\partial\varphi}(y,\varphi,\zeta) = 0\right\}.$$

If the phase Φ_j satisfies one of the two conditions

(C) $\exists\, c_j > 0 \;\; \exists\, \varepsilon > 0 \;\; \forall\, (y, \varphi, \zeta) \in SP_\varepsilon^j : \;\; \left| \frac{\partial^2 \Phi_j}{\partial \varphi^2}(y, \varphi, \zeta) \right| \geq c_j y,$

(D) $\exists\, c_j, C_j > 0 \;\; \exists\, r > 0 \;\; \forall\, 0 \leq \varphi < 2\pi \;\; \forall\, \zeta \in M$*:*
$\left| \frac{\partial^2 \Phi_j}{\partial \varphi^2}(y, \varphi, \zeta) \right| \geq c_j y^2$ *für* $y < r$ *und* $\left| \frac{\partial^2 \Phi_j}{\partial \varphi^2}(y, \varphi, \zeta) \right| \geq C_j$ *für* $y \geq r,$

I_j decays in the L^p-L^q-estimate as $(1+t)^{-\frac{1}{2}(1-\frac{2}{q})}$. A detailed representation of the facts is provided by

Proposition 4.2 *Let* $2 \leq q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $N_p \leq 3$. *If either condition (C) or condition (D) is satisfied, then* $c = c(q) > 0$ *exists. Thus, the following is valid for each* $v_0 \in W^{N_p,p}$ *and* $t \geq 0$*:*

$$|I_j(t)|_q \leq c(1+t)^{-\frac{1}{2}(1-\frac{2}{q})} \|v_0\|_{N_p,p}.$$

Proof: We subdivide the proof again into two steps. In the first step we determine the L^2-L^2-estimate

$$|I_j(t)|_2 \leq c|v_0|_2$$

for I_j and in the second step the special L^∞-L^∞-estimate

$$|I_j(t)|_\infty \leq c t^{-\frac{1}{2}} \sum_{k=0}^{3} \| |\cdot|^k \hat{v}_0 \|_{1,\infty}.$$

As the real part of the eigenvalue λ_j is not negative and P_j is bounded, the following is valid for each $0 \leq t \leq 1$:

$$|I_j(t)|_\infty = \left| \int_{\mathbb{R}^2} e^{i(\xi \cdot + i\lambda_j(\xi)t)} P_j(\xi) \hat{v}_0(\xi)\, d\xi \right|_\infty \leq c \int_{\mathbb{R}^2} |\hat{v}_0(\xi)|\, d\xi = c|\hat{v}_0|_1.$$

In combination with Theorem 6.4 from [1] we obtain the L^1-L^∞-estimate

$$|I_j(t)|_\infty \leq c(1+t)^{-\frac{1}{2}} \|v_0\|_{3,1}$$

from the last two estimates, c being a positive constant in all estimates. The claim finally results from the interpolation of the L^2-L^2-estimate with the L^1-L^∞-estimate.

We have already demonstrated the L^2-L^2-estimate in the proof of Proposition 4.1.

In order to prove the L^∞-L^∞-estimate for condition (C), we split the integral I_j using the definition of $\mathbb{R}^2_\varepsilon$. The following is valid:

$$I_j(t,x) = \int_{\mathbb{R}^2_\varepsilon} e^{i(x\xi + i\lambda_j(\xi)t)} P_j(\xi) \hat{v}_0(\xi)\, d\xi + \int_{\mathbb{R}^2 \setminus \mathbb{R}^2_\varepsilon} e^{i(x\xi + i\lambda_j(\xi)t)} P_j(\xi) \hat{v}_0(\xi)\, d\xi.$$

The second integral can be estimated in the same way as in the proof of Proposition 4.1 since the imaginary part of the phase σ_j differs from zero in $\mathbb{R}^2 \setminus \mathbb{R}^2_\varepsilon$. To obtain the time-asymptotic behavior of the first integral, we apply the transformation T to the integral and get

$$\int_{\mathbb{R}^2_\varepsilon} e^{i(x\xi + i\lambda_j(\xi)t)} P_j(\xi)\hat{v}_0(\xi)\,d\xi \;=$$

$$= \int_0^\infty r \int_{U_\varepsilon} e^{i\left[xr\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} - ty_j(r,\varphi) + itx_j(r,\varphi)\right]} P_j(r,\varphi)\hat{v}_0(r,\varphi)\,d\varphi\,dr$$

$$= \int_0^\infty \int_{U_\varepsilon} e^{i\left[xr(y,\varphi)\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix} - ty + itx_j(y,\varphi)\right]} P_j(y,\varphi)\hat{v}_0(y,\varphi)\cdot$$

$$\cdot\, r(y,\varphi)\left|\frac{\partial r}{\partial y}(y,\varphi)\right| d\varphi\,dy$$

$$= \int_0^\infty y \int_{U_\varepsilon} e^{is\Phi_j(y,\varphi,\zeta)} P_j(y,\varphi)\hat{v}_0(y,\varphi)\frac{r(y,\varphi)}{y}\left|\frac{\partial r}{\partial y}(y,\varphi)\right| d\varphi\,dy.$$

This means that the time-asymptotic behavior of the integral

$$\int_{\mathbb{R}^2_\varepsilon} e^{i(x\xi + i\lambda_j(\xi)t)} P_j(\xi)\hat{v}_0(\xi)\,d\xi$$

is determined by the asymptotic behavior of the integral

$$J_j(s,y) := \int_{U_\varepsilon} e^{is\Phi_j(y,\varphi,\zeta)} P_j(y,\varphi)\hat{v}_0(y,\varphi)\frac{r(y,\varphi)}{y}\left|\frac{\partial r}{\partial y}(y,\varphi)\right| d\varphi$$

for $s \to \infty$. The phase of J_j is Φ_j. Due to condition (C) there is no point of stationary phase of order two in $\mathbb{R}^2_\varepsilon$. For this reason we obtain

$$|J_j(s,y)| \;\le$$

$$\le s^{-\frac{1}{2}} y^{-\frac{1}{2}} \sqrt{\frac{2\pi}{\inf\limits_{(y,\varphi,\zeta)\in SP^j_\varepsilon} y^{-1}\left|\frac{\partial^2\Phi_j}{\partial\varphi^2}(y,\varphi,\zeta)\right|}} \cdot \left\| P_j(y)\hat{v}_0(y)\frac{r(y)}{y}\left|\frac{\partial r}{\partial y}(y)\right|\right\|_{1,\infty}.$$

This estimate is uniform for all $\zeta \in M$. Thus,

$$\int_{\mathbb{R}^2_\varepsilon} e^{i(x\xi + i\lambda_j(\xi)t)} P_j(\xi)\hat{v}_0(\xi)\,d\xi \;\le$$

$$\le \int_0^\infty s^{-\frac{1}{2}} y \sqrt{\frac{2\pi}{c_0 y}}(1+y)^{-3}\sum_{k=0}^{3}\left\|(1+y)^k P_j(y)\hat{v}_0(y)\frac{r(y)}{y}\left|\frac{\partial r}{\partial y}(y)\right|\right\|_{1,\infty} dy.$$

As $\frac{r}{y}$ and $\frac{\partial r}{\partial y}$ as well as P_j and the derivatives of P_j are bounded, the following is valid:

$$\begin{aligned}\int\limits_{\mathbb{R}^2_\varepsilon} e^{i(x\xi+i\lambda_j(\xi)t)} P_j(\xi)\hat{v}_0(\xi)\,d\xi &\le cs^{-\frac{1}{2}} \int\limits_0^\infty \frac{1}{(1+y)^2}\,dy \sum_{k=0}^{3} \left\| \cdot^k P_j \hat{v}_0 \right\|_{1,\infty} \\ &\le cs^{-\frac{1}{2}} \sum_{k=0}^{3} \left\| |\cdot|^k P_j \hat{v}_0 \right\|_{1,\infty} \\ &\le ct^{-\frac{1}{2}} \sum_{k=0}^{3} \left\| |\cdot|^k \hat{v}_0 \right\|_{1,\infty}.\end{aligned}$$

In order to prove the L^∞-L^∞-estimate for condition (D), we proceed in the same way as before and obtain

$$\begin{aligned}&\int\limits_{\mathbb{R}^2_\varepsilon} e^{i(x\xi+i\lambda_j(\xi)t)} P_j(\xi)\hat{v}_0(\xi)\,d\xi \le \\ &\le \int\limits_0^r ys^{-\frac{1}{2}} \sqrt{\frac{2\pi}{c_0 y^2}} (1+y)^{-3} \cdot \\ &\quad \cdot \sum_{k=0}^{3} \left\| (1+y)^k P_j(y) \hat{v}_0(y) \frac{r(y)}{y} \left| \frac{\partial r}{\partial y}(y) \right| \right\|_{1,\infty} dy + \\ &\quad + \int\limits_r^\infty ys^{-\frac{1}{2}} \sqrt{\frac{2\pi}{C_0}} (1+y)^{-3} \cdot \\ &\quad \cdot \sum_{k=0}^{3} \left\| (1+y)^k P_j(y) \hat{v}_0(y) \frac{r(y)}{y} \left| \frac{\partial r}{\partial y}(y) \right| \right\|_{1,\infty} dy\end{aligned}$$

or

$$\int\limits_{\mathbb{R}^2_\varepsilon} e^{i(x\xi+i\lambda_j(\xi)t)} P_j(\xi)\hat{v}_0(\xi)\,d\xi \le ct^{-\frac{1}{2}} \sum_{k=0}^{3} \left\| |\cdot|^k \hat{v}_0 \right\|_{1,\infty},$$

respectively. The proof of the proposition is now complete. □

In [2] we demonstrate that one of the conditions (A) to (D) is always satisfied.

5 L^p-L^q-Estimate

From the Propositions 4.1 and 4.2 we now obtain the following theorem for the time-asymptotic decay behavior of solution v of the equations (1.7) und (1.8):

Theorem 5.1 *If $2 \le q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $N_p \le 3$ (if $q \in (2,\infty)$), then $N_p > 3(1 - \frac{2}{q})$) and the conditions of Lemma 2.1 are satisfied, then $c = c(q) > 0$ exists. Thus, the following is valid for each $v_0 \in W^{N_p,p}$ and $t \ge 0$:*

$$|v(t)|_q \le c(1+t)^{-\frac{1}{2}(1-\frac{2}{q})} \|v_0\|_{N_p,p}.$$

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