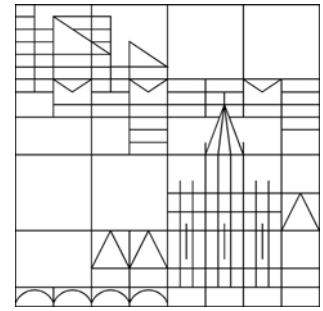


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# Compact families of piecewise constant functions in $L_p(0, T; B)$

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# COMPACT FAMILIES OF PIECEWISE CONSTANT FUNCTIONS IN $L^p(0, T; B)$

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ABSTRACT. A strong compactness result in the spirit of the Lions-Aubin-Simon lemma is proven for piecewise constant functions in time  $(u_\tau)$  with values in a Banach space. The main feature of our result is that it is sufficient to verify one uniform estimate for the time shifts  $u_\tau - u_\tau(\cdot - \tau)$  instead of all time shifts  $u_\tau - u_\tau(\cdot - h)$  for  $h > 0$ , as required in Simon's compactness theorem. This simplifies significantly the application of the Rothe method in the existence analysis of parabolic problems.

## 1. INTRODUCTION

A useful technique to prove the existence of weak solutions to nonlinear evolution equations and their systems is to semi-discretize the equations in time by the implicit Euler method (also called Rothe method [5]):

$$(1) \quad \frac{1}{\tau}(u_\tau(t) - u_\tau(t - \tau)) + A(u_\tau(t)) = f_\tau(t), \quad \tau \leq t < T, \quad u_\tau(0) \text{ given,}$$

where  $\tau > 0$  is the time step,  $A$  is an abstract (nonlinear) operator defined on a certain Banach space, and  $f_\tau$  is some (piecewise constant) function with values in a Banach space. In this way, nonlinear elliptic problems are obtained which are sometimes easier to solve. In order to pass to the limit of vanishing time steps,  $\tau \rightarrow 0$ , (relative) compactness for the sequence of piecewise constant approximate solutions  $(u_\tau)$  is needed. Since the problem is nonlinear, we need strong convergence of (a subsequence of)  $(u_\tau)$  to identify the limit. If the governing operator is monotone, the limit can be identified using Minty's trick (see, e.g., [6, Lemma 2.13]). Having suitable *a priori* estimates at hand, strong compactness can be concluded from the Aubin (or Lions-Aubin-Simon) lemma [7] which is a consequence of a compactness criterium due to Kolmogorov. However, the results of [7] are not directly applicable. Indeed, typically one can derive the uniform estimate

$$(2) \quad \|u_\tau - u_\tau(\cdot - \tau)\|_{L^1(\tau, T; Y)} = \tau \| -A(u_\tau) + f_\tau \|_{L^1(\tau, T; Y)} \leq C\tau,$$

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where  $C > 0$  does not depend on  $\tau$ , and  $Y$  is some Banach space. On the other hand, in order to apply the Aubin lemma, one needs [7, Theorem 3]

$$(3) \quad \|u_\tau - u_\tau(\cdot - h)\|_{L^1(h, T; Y)} \rightarrow 0 \quad \text{as } h \rightarrow 0, \quad \text{uniformly in } \tau > 0.$$

A possible way to avoid this problem is to construct linear interpolants of  $u_\tau$ , say  $\tilde{u}_\tau$ , for which a continuous time-derivative version of the Aubin lemma can be applied, giving  $\tilde{u} \rightarrow u$  in  $L^1(0, T; B)$  as  $\tau \rightarrow 0$  for some Banach space  $B$  [7, Corollary 4]. Since we need strong convergence of  $(u_\tau)$ , one has to show that  $u_\tau - \tilde{u}_\tau \rightarrow 0$  in  $L^1(0, T; B)$ , which might be difficult to prove (see Section 4 for a situation in which such a proof is possible).

In this note, we show that estimate (2) suffices to infer strong compactness of  $(u_\tau)$ . The main feature of our result is that it is sufficient to study the time shifts  $u_\tau - u_\tau(\cdot - \tau)$  instead of all time shifts  $u_\tau - u_\tau(\cdot - h)$  for all  $h > 0$ . This simplifies the proof of the limit  $\tau \rightarrow 0$  in (1) significantly.

For our main results, let  $T > 0$ ,  $N \in \mathbb{N}$ ,  $\tau = T/N$ , and set  $t_k = k\tau$ ,  $k = 0, \dots, N$ . Furthermore, let  $(S_h u)(x, t) = u(x, t - h)$ ,  $t \geq h > 0$ , be the shift operator. We notice that quasi-uniform time steps may be considered too [3], but they are of minor interest in the existence analysis.

**Theorem 1.** *Let  $X$ ,  $B$ , and  $Y$  be Banach spaces such that the embedding  $X \hookrightarrow B$  is compact and the embedding  $B \hookrightarrow Y$  is continuous. Furthermore, let either  $1 \leq p < \infty$ ,  $r = 1$  or  $p = \infty$ ,  $r > 1$ , and let  $(u_\tau)$  be a sequence of functions, which are constant on each subinterval  $(t_{k-1}, t_k)$ , satisfying*

$$(4) \quad \tau^{-1} \|u_\tau - S_\tau u_\tau\|_{L^r(\tau, T; Y)} + \|u_\tau\|_{L^p(0, T; X)} \leq C_0 \quad \text{for all } \tau > 0,$$

where  $C_0 > 0$  is a constant which is independent of  $\tau$ . If  $p < \infty$ , then  $(u_\tau)$  is relatively compact in  $L^p(0, T; B)$ . If  $p = \infty$ , there exists a subsequence of  $(u_\tau)$  which converges in each space  $L^q(0, T; B)$ ,  $1 \leq q < \infty$ , to a limit which belongs to  $C^0([0, T]; B)$ .

A related result in finite-dimensional spaces was recently proven by Gallouët and Latché [4, Theorem 3.4]. The same setting for degenerate elliptic-parabolic equations in  $L^1$  was considered by Andreianov [2]. In view of (3), one may conjecture that the condition  $\|u_\tau - S_\tau u_\tau\|_{L^r(\tau, T; Y)} = \mathcal{O}(\tau^\alpha)$  as  $\tau \rightarrow 0$  with  $0 < \alpha < 1$  instead of  $\mathcal{O}(\tau)$  is sufficient to obtain relative compactness. The following result shows that this is not the case (also see Theorem 5 below).

**Proposition 2.** *The factor  $\tau^{-1}$  in inequality (4) cannot be replaced by  $\tau^{-\alpha}$  for  $0 < \alpha < 1$ .*

This note is organized as follows. In Section 2, Theorem 1 is shown; the proof of Proposition 2 is presented in Section 3. Finally, we comment these results in Section 4.

## 2. PROOF OF THEOREM 1

The proof of Theorem 1 is based on a characterisation of the norm of fractional Sobolev spaces. Let  $1 \leq q < \infty$ ,  $0 < \sigma < 1$ , and let  $Y$  be a Banach space. The fractional Sobolev

space  $W^{\sigma,q}(0,T;Y)$  is the space of (equivalence classes of) functions  $u \in L^q(0,T;Y)$  with finite Slobodeckii norm

$$\|u\|_{W^{\sigma,q}(0,T;Y)} = \left( \|u\|_{L^q(0,T;Y)}^q + |u|_{\dot{W}^{\sigma,q}(0,T;Y)}^q \right)^{1/q},$$

where

$$|u|_{\dot{W}^{\sigma,q}(0,T;Y)} = \left( \int_0^T \int_0^T \frac{\|u(t) - u(s)\|_Y^q}{|t - s|^{1+\sigma q}} ds dt \right)^{1/q}$$

is the Slobodeckii semi-norm. Fractional Sobolev spaces in time have also been proven to be a useful tool in [3].

**Lemma 3.** *Let  $1 \leq q < \infty$ ,  $0 < \sigma < 1$  with  $\sigma q < 1$  and let  $u \in L^q(0,T;Y)$  be a piecewise constant function with (a finite number of) jumps of height  $[u]_k \in Y$  at points  $t_k$ ,  $k = 1, \dots, N-1$ . Then  $u \in W^{\sigma,q}(0,T;Y)$  and*

$$\|u\|_{W^{\sigma,q}(0,T;Y)} \leq \|u\|_{L^q(0,T;Y)} + C_{\sigma,q,T}^{1/q} \sum_{k=1}^{N-1} \|[u]_k\|_Y,$$

where  $C_{\sigma,q,T} = 2(2^{\sigma q} - 1)T^{1-\sigma q}/(\sigma q(1 - \sigma q))$  does not depend on  $N$ .

*Proof.* We may assume that  $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$  and that  $u(t) = u_k$  for  $t_{k-1} < t \leq t_k$  where  $k = 1, \dots, N$ . Then  $[u]_k = u_{k+1} - u_k$ ,  $k = 1, \dots, N-1$ , and

$$u(t) = u_k = u_1 + \sum_{j=1}^{k-1} (u_{j+1} - u_j) = u_1 + \sum_{j=1}^{N-1} [u]_j H_{t_j}(t)$$

for  $t_{k-1} < t \leq t_k$ , where  $H_{t_j}$  is the shifted Heaviside function

$$H_{t_j}(t) = \begin{cases} 0 & \text{for } 0 < t \leq t_j, \\ 1 & \text{for } t_j < t < T. \end{cases}$$

By definition of the  $W^{\sigma,q}(0,T;Y)$  norm and the semi-norm property of  $|\cdot|_{\dot{W}^{\sigma,q}(0,T;Y)}$ , we find that

$$\begin{aligned} \|u\|_{W^{\sigma,q}(0,T;Y)} &= \left( \|u\|_{L^q(0,T;Y)}^q + |u|_{\dot{W}^{\sigma,q}(0,T;Y)}^q \right)^{1/q} \\ &\leq \|u\|_{L^q(0,T;Y)} + |u|_{\dot{W}^{\sigma,q}(0,T;Y)} \\ &\leq \|u\|_{L^q(0,T;Y)} + |u_1|_{\dot{W}^{\sigma,q}(0,T;Y)} + \sum_{j=1}^{N-1} \|[u]_j\|_Y |H_{t_j}|_{\dot{W}^{\sigma,q}(0,T)} \\ (5) \qquad &= \|u\|_{L^q(0,T;Y)} + \sum_{j=1}^{N-1} \|[u]_j\|_Y |H_{t_j}|_{\dot{W}^{\sigma,q}(0,T)}. \end{aligned}$$

It remains to compute the seminorm of  $H_{t_j}$ :

$$\begin{aligned} |H_{t_j}|_{\dot{W}^{\sigma,q}}^q &= \int_0^T \int_0^T \frac{|H_{t_j}(t) - H_{t_j}(s)|^q}{|t-s|^{1+\sigma q}} ds dt = 2 \int_0^{t_j} \int_{t_j}^T \frac{|H_{t_j}(t) - H_{t_j}(s)|^q}{|t-s|^{1+\sigma q}} ds dt \\ &= 2 \int_0^{t_j} \int_{t_j}^T |t-s|^{-1-\sigma q} ds dt = \frac{2}{\sigma q(1-\sigma q)} ((T-t_j)^{1-\sigma q} + t_j^{1-\sigma q} - T^{1-\sigma q}). \end{aligned}$$

The right-hand side can be interpreted as a function of  $\vartheta = t_j \in [0, T]$  whose maximum is achieved at  $\vartheta = T/2$ . Therefore,

$$|H_{t_j}|_{\dot{W}^{\sigma,q}}^q \leq \frac{2}{\sigma q(1-\sigma q)} \left( 2 \left( \frac{T}{2} \right)^{1-\sigma q} - T^{1-\sigma q} \right) = \frac{2}{\sigma q(1-\sigma q)} (2^{\sigma q} - 1) T^{1-\sigma q} = C_{\sigma q, T}.$$

Inserting this estimate in (5), the result follows.  $\square$

For later use, we remark that the calculations in (5) and below show that

$$(6) \quad |u|_{\dot{W}^{\sigma,1}(0,T;Y)} \leq C_{\sigma q, T}^{1/q} \sum_{k=1}^{N-1} \|[u]_k\|_Y.$$

*Proof of Theorem 1.* The idea of the proof is to apply Corollary 5 in [7]: If  $(u_\tau)$  is bounded in  $L^p(0, T; X) \cap W^{\sigma, \varrho}(0, T; Y)$ , where  $\sigma > \max\{0, 1/\varrho - 1/p\}$ , then  $(u_\tau)$  is relatively compact in  $L^p(0, T; B)$  if  $p < \infty$ ,  $\varrho = 1$  and in  $C^0([0, T]; B)$  if  $p = \infty$ ,  $\varrho > 1$ .

First we consider the case  $p < \infty$  and  $\varrho = 1$ . Let  $\sigma \in (0, 1)$  satisfy  $1 - 1/p < \sigma < 1$  and let  $u_\tau(t) = u_k$  for  $t_{k-1} < t < t_k$ ,  $k = 1, \dots, N$ . Then

$$\begin{aligned} \sum_{k=1}^{N-1} \|[u_\tau]_k\|_Y &= \sum_{k=1}^{N-1} \|u_{k+1} - u_k\|_Y = \tau^{-1} \sum_{k=1}^{N-1} \int_{t_k}^{t_{k+1}} \|u_{k+1} - u_k\|_Y dt \\ (7) \quad &= \tau^{-1} \|u_\tau - S_\tau u_\tau\|_{L^1(\tau, T; Y)} \leq C_0. \end{aligned}$$

Since  $L^p(0, T; X) \hookrightarrow L^1(0, T; Y)$ , Lemma 3 shows that  $(u_\tau)$  is bounded in  $W^{\sigma, \varrho}(0, T; Y)$ , and the corollary applies.

It remains to discuss the case  $p = \infty$  and  $\varrho > 1$ . We define the piecewise linear interpolants

$$\tilde{u}_\tau(t) = \begin{cases} u_1 & \text{for } 0 \leq t \leq t_1, \\ u_k - \frac{t_k - t}{\tau}(u_k - u_{k-1}) & \text{for } t_{k-1} \leq t \leq t_k, \quad 2 \leq k \leq N. \end{cases}$$

Let  $(S_\tau u_\tau)(t) = u_1$  for  $0 \leq t < t_1$ . We observe that

$$(8) \quad \begin{cases} \tilde{u}_\tau'(t) = \frac{1}{\tau}(u_\tau(t) - (S_\tau u_\tau)(t)), & 0 \leq t \leq T, \quad t \neq t_k, \\ \|\tilde{u}_\tau(t)\|_X \leq \|u_\tau(t)\|_X + \|(S_\tau u_\tau)(t)\|_X, & 0 \leq t \leq T, \end{cases}$$

which implies that  $\|\tilde{u}_\tau\|_{L^p(0, T; X)} \leq 2 \|u_\tau\|_{L^p(0, T; X)}$ . Now we apply Theorem 1 to  $(u_\tau)$  with  $p = 1$  instead of  $p = \infty$ , and we apply Corollary 5 in [7] to  $(\tilde{u}_\tau)$  with  $\sigma = 1$ . We end up with a subsequence of  $(u_\tau)$  (not relabeled) such that  $u_\tau \rightarrow u^*$  in  $L^1(0, T; B)$ , and we may

assume that the associated subsequence  $(\tilde{u}_\tau)$  of piecewise linear interpolants converges to a limit  $\hat{u}$  in the topology of  $C^0([0, T]; B)$ . Next we know, for  $k = 1, \dots, N$  and  $t_{k-1} < t < t_k$ , that

$$(9) \quad \|u_\tau(t) - \tilde{u}_\tau(t)\|_Y = \frac{t_k - t}{\tau} \|u_\tau(t) - (S_\tau u_\tau)(t)\|_Y \leq \|u_\tau(t) - (S_\tau u_\tau)(t)\|_Y,$$

from which we infer that  $\|u_\tau - \tilde{u}_\tau\|_{L^1(0, T; Y)} \leq C_0 \tau$ . Notice that the embeddings  $L^1(0, T; B) \hookrightarrow L^1(0, T; Y)$  and  $C^0([0, T]; B) \hookrightarrow L^1(0, T; Y)$  are both continuous, hence  $u^* = \hat{u}$ .

Since  $(\tilde{u}_\tau)$  converges in  $C^0([0, T]; B)$  to  $\hat{u}$ , there exists a constant  $\hat{C} > 0$  such that  $\|\tilde{u}_\tau\|_{L^\infty(0, T; B)} \leq \hat{C}$  for all  $\tau$ , and then also  $\|u_\tau\|_{L^\infty(0, T; B)} \leq \hat{C}$  for all  $\tau$ . The desired convergence of  $(u_\tau)$  to  $u^*$  in any space  $L^q(0, T; B)$  for  $1 \leq q < \infty$  follows from interpolation between  $\|u_\tau - u^*\|_{L^1(0, T; B)} \rightarrow 0$  and  $\|u_\tau - u^*\|_{L^\infty(0, T; B)} \leq 2\hat{C}$ , which completes the proof.  $\square$

*Remark 4.* Estimates (6) and (7) imply that, for all piecewise constant functions  $u \in L^1(0, T; Y)$  with jumps at  $t_k = k\tau$ ,

$$|u|_{\dot{W}^{\sigma, 1}(0, T; Y)} \leq C_{\sigma q, T}^{1/q} \sum_{k=1}^{N-1} \|[u]_k\|_Y \leq \tau^{-1} C_{\sigma q, T}^{1/q} \|u - S_\tau u\|_{L^1(\tau, T; Y)}.$$

By Lemma 5 of [7], there exists an inverse inequality for all  $u \in W^{\sigma, 1}(0, T; Y)$  and all  $\sigma \in (0, 1)$ :

$$\|u - S_\tau u\|_{L^1(\tau, T; Y)} \leq C_3 \tau^\sigma |u|_{\dot{W}^{\sigma, 1}(0, T; Y)},$$

where  $C_3 > 0$  depends on  $\sigma$  and  $T$ . In this sense, the chain of inequalities

$$\tau |u|_{\dot{W}^{\sigma, 1}(0, T; Y)} \leq \tau^\sigma C_{\sigma q, T}^{1/q} C_3 |u|_{\dot{W}^{\sigma, 1}(0, T; Y)}$$

is almost sharp since we can choose  $\sigma$  as close to one as we wish.  $\square$

### 3. PROOF OF PROPOSITION 2

We construct a sequence  $(u_\tau)$  satisfying the assumptions of Theorem 1 with  $\tau^{-\alpha}$  ( $0 < \alpha < 1$ ) in (4) instead of  $\tau^{-1}$ , but not possessing a convergent subsequence in  $L^p(0, T; B)$ , where  $p < \infty$ .

Take  $X = Y = B = \mathbb{C}$  and  $(0, T) = (0, 1)$ . For  $\beta \geq 1$ , define the function

$$f_\beta(t) := (\beta p + 1)^{1/p} t^\beta, \quad 0 \leq t \leq 1.$$

Then we have  $\|f_\beta\|_{L^p(0, T)} = 1$ . For later use, we remark that

$$(10) \quad \lim_{\beta \rightarrow \infty} f_\beta(t) = 0,$$

for each fixed  $t \in [0, 1)$ , uniformly on compact sub-intervals  $[0, t_*] \subset [0, 1)$ .

Since  $\alpha < 1$ , we may choose a real number  $0 < \gamma \leq \min\{1, p(1 - \alpha)\}$ . We set  $\beta(\tau) = \tau^{-\gamma}$  and

$$u_\tau(t) := \begin{cases} f_{\beta(\tau)}(k\tau) & \text{for } k\tau \leq t < (k+1)\tau, \ k \in \{0, 1, \dots, N-1\}, \\ f_{\beta(\tau)}((N-1)\tau) & \text{for } t = 1. \end{cases}$$

The function  $u_\tau$  has jumps of height  $[u_\tau]_k$  at the values  $t = k\tau$  for  $1 \leq k \leq N - 1$ , and all jumps have the same sign. In particular,

$$\begin{aligned} \sum_{k=1}^{N-1} \|[u_\tau]_k\|_Y &= \sum_{k=1}^{N-1} [u_\tau]_k = f_{\beta(\tau)}(1 - \tau) = (\tau^{-\gamma}p + 1)^{1/p}(1 - \tau)^{\tau^{-\gamma}}, \\ 1 &\geq (1 - \tau)^{\tau^{-\gamma}} \geq (1 - \tau)^{1/\tau} \geq \frac{1}{2e}. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} \tau^{-\alpha} \|u_\tau - S_\tau u_\tau\|_{L^1(\tau, T; Y)} &= \tau^{1-\alpha} \sum_{k=1}^{N-1} \|[u_\tau]_k\|_Y = \tau^{1-\alpha} (\tau^{-\gamma}p + 1)^{1/p} (1 - \tau)^{\tau^{-\gamma}} \\ &\leq \tau^{1-\alpha} (\tau^{-\gamma}p + 1)^{1/p} \leq \left(\frac{1}{2}\right)^{1-\alpha} \left(\left(\frac{1}{2}\right)^{-\gamma} + 1\right)^{1/p}, \end{aligned}$$

for all  $\tau \in (0, 1/2)$ , since  $1 - \alpha - \gamma/p \geq 0$ . Hence, (4) holds. But the sequence  $(u_\tau) \subset L^p(0, T; B)$  does not possess a converging subsequence, which can be seen as follows. Fix  $t \in [0, 1)$ . Then  $0 \leq u_\tau(t) \leq f_{\beta(\tau)}(t)$ , and (10) implies the pointwise convergence  $\lim_{\tau \rightarrow 0} u_\tau(t) = 0$ , uniform on compact sub-intervals  $[0, t_*] \subset [0, 1)$ . Thus, the pointwise limit of the subsequence must be the zero function. However, this is impossible, because of the following uniform lower bound:

$$\int_0^1 |u_\tau(t)|^p dt \geq \int_0^{1-\tau} |f_{\beta(\tau)}(t)|^p dt = (1 - \tau)^{\tau^{-\gamma}p+1} \geq \frac{1}{2} \left((1 - \tau)^{\tau^{-\gamma}}\right)^p \geq \frac{1}{2} (2e)^{-p},$$

showing the claim.

#### 4. COMMENTS

Let  $X$ ,  $B$ , and  $Y$  be Banach spaces such that the embedding  $X \hookrightarrow B$  is dense and compact, the embedding  $B \hookrightarrow Y$  is continuous, and there exist  $0 < \theta < 1$ ,  $C_\theta > 0$  such that for all  $u \in X$ , the interpolation inequality

$$(11) \quad \|u\|_B \leq C_\theta \|u\|_X^\theta \|u\|_Y^{1-\theta}$$

holds. The setting which we have in mind relates to (1), with given  $u(0) \in B$ . In this situation, a slightly weaker version of Theorem 1 can be derived directly from the Aubin lemma.<sup>1</sup> Indeed, since  $X$  is dense in  $B$ , we may approximate  $u(0) \in B$  by  $u_0 \in X$ , and we define the piecewise linear interpolant by

$$\tilde{u}_\tau(t) = u_k - \frac{t_k - t}{\tau} (u_k - u_{k-1}), \quad t_{k-1} \leq t \leq t_k, \quad 1 \leq k \leq N.$$

We suppose that  $u_0$  and  $u_1$  satisfy

$$(12) \quad \tau \|u_0\|_X^p \leq C_1, \quad \|u_0 - u_1\|_Y \leq C_1$$

<sup>1</sup>The authors are grateful to one of the referees for this observation.



for some constant  $C_1 > 0$  independent of  $\tau$ . The first bound can always be satisfied; the second bound is a mild condition related to the construction of the sequence  $(u_k)$ . If this sequence is defined according to (1), the second bound can be replaced by the regularity assumption  $\tau\|A(u_1)\|_Y \leq C$  for some constant  $C > 0$  independent of  $\tau$  since  $\|u_1 - u_0\|_Y \leq \tau\|A(u_1)\|_Y + \tau\|f_\tau(\tau)\|_Y$ .

Now we make the agreement that  $(S_\tau u_\tau)(t) = u_0$  for  $0 \leq t < t_1 = \tau$ . Then (8) still holds. It follows from (4) that

$$\|\tilde{u}'_\tau\|_{L^1(0,T;Y)} = \|u_1 - u_0\|_Y + \tau^{-1} \|u_\tau - S_\tau u_\tau\|_{L^1(\tau,T;Y)} \leq C_1 + C_0.$$

Furthermore, using (8) and (4) again,

$$\|\tilde{u}_\tau\|_{L^p(0,T;X)} \leq \tau^{1/p} \|u_0\|_X + 2 \|u_\tau\|_{L^p(0,T;X)} \leq C_1^{1/p} + 2C_0.$$

Hence, by the Aubin lemma [7, Corollary 4], up to a subsequence,  $\tilde{u}_\tau \rightarrow u$  in  $L^p(0, T; B)$  as  $\tau \rightarrow 0$ . By the interpolation inequality (11) and by (9),

$$\begin{aligned} \|u_\tau - \tilde{u}_\tau\|_{L^1(0,T;B)} &\leq C_\theta \|u_\tau - \tilde{u}_\tau\|_{L^1(0,T;X)}^\theta \|u_\tau - \tilde{u}_\tau\|_{L^1(0,T;Y)}^{1-\theta} \\ &\leq C_\theta \left( \|u_\tau\|_{L^1(0,T;X)} + \|\tilde{u}_\tau\|_{L^1(0,T;X)} \right)^\theta \|u_\tau - S_\tau u_\tau\|_{L^1(0,T;Y)}^{1-\theta}. \end{aligned}$$

We remark that  $\|u_\tau - S_\tau u_\tau\|_{L^1(0,T;Y)} \leq \tau(\|u_1 - u_0\|_Y + C_0)$ , which implies that  $u_\tau - \tilde{u}_\tau \rightarrow 0$  in  $L^1(0, T; B)$ . Since  $\tilde{u}_\tau \rightarrow u$  in  $L^p(0, T; B)$ , we find that  $u_\tau \rightarrow u$  in  $L^q(0, T; B)$  for all  $q < p$ . Notice, however, that Theorem 1 allows us to conclude this result up to  $q = p$  without assuming (11) and (12).

Proposition 2 shows that the exponent of the factor  $\tau$  in (4) cannot be raised. However, when allowing for arbitrary time shifts  $S_h$ , the factor can be replaced by  $h^{-\alpha}$ , where  $0 < \alpha < 1$ , under some conditions. An example, adapted to our situation, can be found in [1, Theorem 1.1]:

**Theorem 5** (Amann). *Let (11) hold. Furthermore, let  $0 < s < 1$ ,  $1 \leq p < \infty$ , and  $F \subset L^p(0, T; Y)$ . Assume that there exists  $C_2 > 0$  such that each  $u \in F$  satisfies the following infinite collection of inequalities:*

$$h^{-s} \|u - S_h u\|_{L^1(\tau,T;Y)} + \|u\|_{L^p(0,T;X)} \leq C_2 \quad \text{for all } h > 0.$$

*Then  $F$  is relatively compact in  $L^q(0, T; B)$  for all  $q < p/((1 - \theta)(1 - s)p + \theta)$ .*

Notice that  $q = p$  is admissible if  $(1 - \theta)(1 - s)p + \theta < 1$  which is equivalent to  $s > 1 - 1/p$ . Thus, if we wish to allow for arbitrary large  $p \geq 1$ , we have to require the condition  $s = 1$ , which corresponds to the result of Theorem 1. On the other hand, in applications, often  $p = 2$ , and compactness follows even for  $s < 1$ , namely for any  $s > 1/2$ .

In the special situation when we have the triple  $X \hookrightarrow B \hookrightarrow X'$ , where  $Y = X'$  is the dual space of  $X$  and  $B$  is a Hilbert space, the assumptions of Amann's theorem hold with  $\theta = 1/2$ . Then  $q < 2p/((1 - s)p + 1)$ , and we see that  $2p$  is an upper bound for  $q$ . This corresponds to the result of Walkington [8, Theorem 3.1 (1)].

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