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FORMATION OF SINGULARITIES FOR ONE-DIMENSIONAL RELAXED COMPRESSIBLE NAVIER-STOKES EQUATIONS

YUXI HU, REINHARD RACKE AND NA WANG

ABSTRACT. We investigate the formation of singularities in one-dimensional hyperbolic compressible Navier-Stokes equations, a model proposing a relaxation leading to a hyperbolization through a nonlinear Cattaneo law for heat conduction as well as through the constitutive Maxwell type relations for the stress tensor. By using the entropy dissipation inequality, which gives the lower energy estimates of the local solutions without any smallness condition on initial data, and by constructing some useful averaged quantities we show that there are in general no global C^1 solutions for the studied system with some large initial data.

This appears as a remarkable contrast to the situation without relaxation, i.e. for the classical compressible Navier-Stokes equations, where global large solutions exist. It also contrasts the fact that for the linearized system associated to the classical resp. relaxed compressible Navier-Stokes equations, the qualitative behavior is exactly the same: exponential stability in bounded domains and polynomial decay without loss of regularity for the Cauchy problem.

KEYWORDS: singularities; compressible Navier-Stokes equations; large data
AMS CLASSIFICATION CODE: 35 L 60, 35 B 44, 76 N 10

1. INTRODUCTION

In this paper, we consider the system of one-dimensional non-isentropic compressible Navier-Stokes equations,

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \rho u_t + \rho u u_x + p_x = S_x, \\ \rho e_t + \rho u e_x + p u_x + q_x = S u_x, \end{cases} \tag{1.1}$$

where ρ, u, e, p, S, q denote the fluid density, velocity, specific internal energy per unit mass, pressure, stress tensor, heat flux, respectively. To make the above system complete, we need to impose some constitutive equations for both q and S. Instead of using the classical relations

$$q = -\kappa\theta_x, \qquad S = \mu u_x,$$

with positive constants κ, μ and θ denoting the temperature, we shall consider the relaxed versions in form of the (nonlinear) Cattaneo law of heat conduction

$$\tau_1(q_t + u \cdot q_x) + q + \kappa\theta_x = 0, \tag{1.2}$$

and the Maxwell type constitutive relations for the stress tensor

$$\tau_2(S_t + u \cdot S_x) + S = \mu u_x. \tag{1.3}$$

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Here $\tau_1, \tau_2 > 0$ are constant relaxation parameters, turning the classical system (corresponding to $\tau_1 = \tau_2 = 0$) of essentially *parabolic* type into a mainly *hyperbolic* one. The constitutive relations (1.2) and (1.3) respect the Galilean invariance, resulting in the nonlinear terms $u \cdot q_x$ and $u \cdot S_x$, respectively, cp. [3] for the flux relation (1.3). The linearized version of (1.2) is usually called Cattaneo's law.

In the constitutive relation (1.3), in its linearized form: $\tau_2 S_t + S = \mu u_x$, the positive parameter τ_2 is the relaxation time describing the time lag in the response of the stress tensor to the velocity gradient. In fact, even in simple fluid, water for example, the"time lag" exists, but it is very small ranging from 1 ps to 1 ns, see [23, 36]. However, Pelton et al. [26] showed that such a"time lag" cannot be neglected, even for simple fluids, in the experiments of high-frequency (20GHZ) vibration of nano-scale mechanical devices immersed in water-glycerol mixtures. It turned out that, cp. also [1], equation (1.3) provides a general formalism to characterize the fluid-structure interaction of nano-scale mechanical devices vibrating in simple fluids. A similar relaxed constitutive relation was already proposed by Maxwell in [24], in order to describe the relation of stress tensor and velocity gradient for a non-simple fluid.

Moreover, we assume that the internal energy e and the pressure p satisfy the following constitutive relations,

$$e = C_v\theta + \frac{\tau_1}{\kappa\theta\rho}q^2 + \frac{\tau_2}{\mu\rho}S^2, \tag{1.4}$$

$$p = R\rho\theta - \frac{\tau_1}{2\kappa\theta}q^2 - \frac{\tau_2}{2\mu}S^2, \tag{1.5}$$

with positive constants C_v, R denoting the heat capacity at constant volume and the gas constant, respectively, such that they satisfy the usual thermodynamic equation

$$\rho^2 e_\rho = p - \theta p_\theta.$$

The dependence on q^2 term of the internal energy is indicated in paper [4], where they rigorously prove that such constitutive equations are consistent with the second law of thermodynamics if and only if one use the relaxation equation (1.2), see also [2, 5, 38]. Since we also consider a relaxation for the stress tensor S, it is motivated, naturally, by [4] that the internal energy should also depend on S in a quadratic form. Indeed, the authors [10] show that, under the above constitutive laws, the relaxed system (1.1)-(1.3) has a dissipative entropy which implies the compatibility with the second law of thermodynamics.

We shall consider the Cauchy problem for the functions

$$(\rho, u, \theta, S, q) : \mathbb{R} \times [0, +\infty) \to \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$$

with initial condition

$$(\rho(x, 0), u(x, 0), \theta(x, 0), S(x, 0), q(x, 0)) = (\rho_0, u_0, \theta_0, S_0, q_0). \tag{1.6}$$

The local existence for (1.1)-(1.6) has been established by the authors in [10], as well as the global existence of solutions with *small* initial data. So, it is a natural question that whether the smooth solutions exist for any large initial data. Note that, when $\tau_1 = \tau_2 = 0$, the above system is reduced to the classical compressible Navier-Stokes equations for which smooth solutions exist globally for arbitrary initial data away from vacuum, see [21] and the reference cited therein. On the other hand, the authors have proved that when the relaxation parameters go to zero, smooth solutions of system (1.1)-(1.6) converge to that of classical system. This indicates that the relaxed system exhibits a similar qualitative behavior as the classical system. However, and surprisingly, we show that there are in general no C^1 solutions for system (1.1)-(1.6) with some large initial data. That is, we have another and more complex result in comparison to [12, 13] of a *nonlinear* system where the relaxation process turns a (globally) well-posed system into a not (globally) well-posed one. This sheds light on the difficulties in proving some global existence results in fluid dynamics, and in finding the "correct" model.

We remark that a qualitative change was observed before for certain thermoelastic systems in *bounded domains*, where the non-relaxed system is exponentially stable, while the relaxed one is not, see Quintanilla and Racke resp. Fernández Sare and Muñoz Rivera [28, 6] for plates, and Fernández Sare and Racke [7] for Timoshenko beams. For the corresponding *Cauchy problem* a relaxation leads to a loss of regularity (for the notion of *loss of regularity* see Section 4), see Racke and Ueda [33] for plates, and Said-Houari and Kasimov [34] for Timoshenko beams.

These observations were made for, and these results were proved for *linear* systems. Here, we have the new interesting and somehow surprising effect, that the linearized system while introducing a relaxation remains exponentially stable and the Cauchy problem keeps the decay rates without loss of regularity, while the nonlinear one changes the behavior essentially (global existence to blow-up) when a relaxation is introduced.

The method we use to prove the blow-up result is mainly motivated by Sideris' paper [37] where he showed that any C^1 solutions of compressible Euler equations must blow up in finite time. As was shown in [10], the system (1.1)-(1.5) is a strictly hyperbolic system which indicates an important property of finite propagation speed. The finite propagation speed property allows us to define some averaged quantities as in [37] and finally show a blow-up of solutions in finite time by establishing a Riccati-type inequality.

The linearized system with relaxation ($\tau_1, \tau_2 > 0$) behaves qualitatively the same as the classical non-relaxed one ($\tau_1 = \tau_2 = 0$). That is, linear similarity up to similarity of nonlinear systems for small data does not imply similar behavior for nonlinear systems with large data. Here we remind of the case of *incompressible* Navier-Stokes equations, for which the relaxed case was studied in Racke and Saal [31, 32] and in Schöwe [35] – the question of blow-up remains yet as open as for the classical Navier-Stokes equations in 3-d. We also remember the case of semi-linear heat resp. damped wave equation with the same critical exponent, see Section 4 for details.

Hu and Wang [12, 13] showed blow-up results for both one-dimensional and multi-dimensional *isentropic* Navier-Stokes equations. However, they only considered the isentropic case and linearized constitutive relations. Here, the nonlinearities appearing in (1.2) and (1.3), i.e., $u \cdot q_x$ and $u \cdot S_x$, will cause a lot of technical problems in the proof of the main result. We shall use some delicate bootstrap skills to overcome these difficulties.

The paper is organized as follows. In Section 2 we recall the local existence theorem and the finite propagation speed property, and then present the main theorem on the blow-up of solutions in finite time. The proof of this main theorem is given in Section 3. In Section 4 we demonstrate that the two linearized systems, associated to the relaxed resp. non-relaxed (classical) system, have the same qualitative behavior: exponential stability in bounded domains and no regularity loss for the Cauchy problem.

Finally, we introduce some notation. $W^{m,p} = W^{m,p}(\mathbb{R})$, $0 \le m \le \infty$, $1 \le p \le \infty$, denotes the usual Sobolev space with norm $\|\cdot\|_{W^{m,p}}$, H^m and L^p stand for $W^{m,2}$ resp. $W^{0,p}$.

2. Assumptions and statement of the main result

First, we choose $\delta > 0$ small enough such that $p_\rho, p_\theta, e_\theta$ are positive and bounded away from zero and $|p_S|, |p_q|$ are sufficiently small as functions of (ρ, θ, q, S) on

$$\Omega := (1-\delta, 1+\delta) \times (1-\delta, 1+\delta) \times (-\delta, \delta) \times (-\delta, \delta).$$

Now, we present a local existence theorem for the problem (1.1)-(1.6), see [10].

Lemma 2.1. *Let* $(\rho_0, u_0, \theta_0, q_0, S_0) : \mathbb{R} \to \mathbb{R}$ *be given with*

$$(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0) \in H^2,$$
$$\forall x \in \mathbb{R}, \quad (\rho_0, \theta_0, q_0, S_0) \in \Omega.$$

Then, the initial value problem (1.1)-(1.6) *has a unique solution* (ρ, u, θ, q, S) *on a maximal time interval* $[0, T_0)$*, for some* $T_0 > 0$*, with*

$$(\rho - 1, u, \theta - 1, q, S) \in C^0([0, T_0), H^2) \cap C^1([0, T_0), H^1)$$

and

$$\forall x \in \mathbb{R}, \quad \forall t \in [0, T_0), \quad (\rho(x,t), \theta(x,t), q(x,t), S(x,t)) \in \Omega.$$

The following lemma states the finite propagation speed property which is guaranteed by the strict hyperbolicity of the system (1.1)-(1.6) given for q small enough, cp. [10].

Lemma 2.2. *Let* (ρ, u, θ, q, S) *be a local solutions to* (1.1)-(1.6) *on* $[0, T_0)$*. Let* $M > 0$*. Assume the initial data* $(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0)$ *are compactly supported in* $(-M, M)$ *and* $(\rho_0, \theta_0, q_0, S_0) \in \Omega$*. Then, there exists a constant* σ *such that*

$$(\rho(\cdot,t), u(\cdot,t), \theta(\cdot,t), q(\cdot,t), S(\cdot,t) = (1, 0, 1, 0, 0) := (\bar{\rho}, \bar{u}, \bar{\theta}, \bar{q}, \bar{S})$$

on $D(t) = \{x \in \mathbb{R} \big| |x| \geq M + \sigma t\}, \quad 0 \leq t < T_0.$

In the sequel we will assume

$$\delta < \frac{\bar{\theta}}{2} = \frac{1}{2}. \tag{2.1}$$

One may observe that this does not put a restriction on u. Indeed, essentially it will be u for which a blow-up is shown. Let us define some useful averaged quantities:

$$F(t) := \int_{\mathbb{R}} x\rho(x,t)u(x,t)\mathrm{d}x,$$
$$G(t) := \int_{\mathbb{R}} (E(x,t) - \bar{E})\mathrm{d}x,$$

where

$$E(x,t) := \rho(e + \frac{1}{2}u^2)$$

is the total energy and

$$\bar{E} := \bar{\rho}(\bar{e} + \frac{1}{2}\bar{u}^2) = C_v.$$

We mention that the functional defined above exists since the solution $(\rho - 1, u, \theta - 1, q, S)$ is zero on the set $D(t)$ defined in Lemma 2.2.

Now, we are ready to show our main result.

Theorem 2.3. *We assume that the initial data satisfy the assumption in Lemma 2.1 and 2.2. Moreover, we assume that*

$$G(0) > 0. \tag{2.2}$$

Then, there exists u_0 *satisfying*

$$F(0) > \max\left\{\frac{32\sigma \max \rho_0}{3 - \gamma}, \frac{4\sqrt{\max \rho_0}}{\sqrt{3 - \gamma}}\right\} M^2, \quad 1 < \gamma := 1 + \frac{R}{C_v} < 3 \tag{2.3}$$

such that the length T_0 *of the maximal interval of existence of a smooth solution* (ρ, u, θ, q, S) *of* (1.1)-(1.6) *is finite, provided the compact support of the initial data is sufficiently large.*

The assumption $1 < \gamma < 3$ holds for the elementary kinetic theory of gases, cp. [37, p. 478]. We also remark that the original system can be made dimensionless by the following change of variables.

$$\tilde{x} = \frac{x}{L_r}, \tilde{t} = \frac{t}{T_r}, \tilde{u} = \frac{uT_r}{L_r}, \tilde{\rho} = \frac{L_r\rho}{A_r}, \tilde{\theta} = \frac{\theta}{K_r}, \tilde{p} = \frac{T_r^2 p}{L_r A_r}, \tilde{S} = \frac{T_r^2 S}{L_r A_r}, \tilde{q} = \frac{T_r^3 q}{L_r^2 A_r}, \tilde{e} = \frac{T_r^2 e}{L_r^2},$$

$$\tilde{\tau}_1 = \frac{\tau_1}{T_r},\ \tilde{\tau}_2 = \frac{\tau_2}{T_r},\ \tilde{\kappa} = \frac{K_r T_r^3 \kappa}{L_r^3 A_r},\ \tilde{\mu} = \frac{T_r \mu}{L_r A_r},$$

where L_r, T_r, A_r, K_r are characteristic reference length, time, mass and temperature, respectively.

3. Proof of Theorem 2.3

From equations $(1.1)_2$ and $(1.1)_3$, we can get the equation for E:

$$E_t + (uE + up - uS + q)_x = 0,$$

which implies that $G(t)$ is a constant and

$$G(t) = G(0) > 0. \tag{3.1}$$

On the other hand, we have

$$\begin{aligned} F'(t) &= \int_{\mathbb{R}} (\rho u)_t \cdot x \mathrm{d}x \\ &= \int_{\mathbb{R}} \{(-\rho u^2)_x - p_x + S_x\} \cdot x \mathrm{d}x \\ &= \int_{\mathbb{R}} \rho u^2 \mathrm{d}x + \int_{\mathbb{R}} (p - \bar{p}) \mathrm{d}x - \int_{\mathbb{R}} S \mathrm{d}x. \end{aligned}$$

By the constitutive equation (1.4) and (1.5), we know

$$\int_{\mathbb{R}} (p - \bar{p}) \mathrm{d}x = \int_{\mathbb{R}} (R\rho\theta - \frac{\tau_1}{2\kappa\theta} q^2 - \frac{\tau_2}{2\mu} S^2 - R\bar{\rho}\bar{\theta}) \mathrm{d}x$$

and

$$R\rho\theta = \frac{R}{C_v}\rho e - \frac{\tau_1 R}{C_v \kappa\theta} q^2 - \frac{\tau_2 R}{C_v \mu} S^2.$$

So, using (3.1), we derive that

$$\begin{aligned} \int_{\mathbb{R}} (p - \bar{p}) \mathrm{d}x &= \int_{\mathbb{R}} \left\{ \frac{R}{C_v}(\rho e - \bar{\rho}\bar{e}) - \frac{\tau_1(2\gamma - 1)}{2\kappa\theta} q^2 - \frac{\tau_2(2\gamma - 1)}{2\mu} S^2 \right\} \mathrm{d}x \\ &= \int_{\mathbb{R}} \frac{R}{C_v} \{(E - \frac{1}{2}\rho u^2) - \bar{E}\} \mathrm{d}x - \int_{\mathbb{R}} (\frac{\tau_1(2\gamma - 1)}{2\kappa\theta} q^2 + \frac{\tau_2(2\gamma - 1)}{2\mu} S^2) \mathrm{d}x \\ &\geq -\frac{\gamma - 1}{2} \int_{\mathbb{R}} \rho u^2 \mathrm{d}x - \int_{\mathbb{R}} (\frac{\tau_1(2\gamma - 1)}{2\kappa\theta} q^2 + \frac{\tau_2(2\gamma - 1)}{2\mu} S^2) \mathrm{d}x, \end{aligned}$$

where $\gamma = \frac{R}{C_v} + 1$.

So, using the Hölder inequality, we derive that

$$F'(t) \geq \frac{3 - \gamma}{2} \int_{\mathbb{R}} \rho u^2 \mathrm{d}x - \int_{\mathbb{R}} \frac{\tau_1(2\gamma - 1)}{2\kappa\theta} q^2 \mathrm{d}x - \int_{\mathbb{R}} (\frac{\tau_2(2\gamma - 1)}{2\mu} + \frac{1}{2}) S^2 \mathrm{d}x - (M + \sigma t). \tag{3.2}$$

By definition of $F(t)$, we know

$$\begin{aligned} F^2(t) &= \left(\int_{\mathbb{R}} x\rho(x,t)u(x,t)\mathrm{d}x\right)^2 \\ &\leq \int_{B_t} x^2\rho\mathrm{d}x \cdot \int_{B_t} \rho u^2\mathrm{d}x \\ &\leq (M+\tilde{\sigma}t)^2 \int_{B_t} \rho\mathrm{d}x \cdot \int_{B_t} \rho u^2\mathrm{d}x \\ &= (M+\tilde{\sigma}t)^2 \int_{B_t} \rho_0\mathrm{d}x \cdot \int_{B_t} \rho u^2\mathrm{d}x \\ &\leq 2\max\rho_0(M+\tilde{\sigma}t)^3 \int_{\mathbb{R}} \rho u^2\mathrm{d}x, \end{aligned}$$

where $B_t = \{x \in \mathbb{R} | |x - \tilde{\sigma}t| \leq M\}$ and $\tilde{\sigma} \geq \sigma$ can be chosen arbitrary. For simplicity, we still denote $\tilde{\sigma}$ by σ in the following calculations. Therefore, we have

$$F'(t) \geq \frac{3-\gamma}{4\max\rho_0(M+\sigma t)^3}F^2 - \int_{\mathbb{R}} \frac{\tau_1(2\gamma-1)}{2\kappa\theta}q^2\mathrm{d}x - \int_{\mathbb{R}} \frac{\tau_2(2\gamma-1)+\mu}{2\mu}S^2\mathrm{d}x - (M+\sigma t). \tag{3.3}$$

Let

$$c_2 := \frac{\sigma}{M}, \qquad c_3 := \frac{3-\gamma}{4\max\rho_0 M^3}.$$

Assume for the moment

$$F(t) \geq c_1 > 0 \tag{3.4}$$

and

$$M + \sigma t = M(1+c_2t) \leq \frac{c_3}{2(1+c_2t)^3}F^2, \tag{3.5}$$

where c_1 is to be determined later. Under the given assumption, and in particular using (2.1) inequality (3.3) reduces to

$$F'(t) \geq \frac{c_3}{2(1+c_2t)^3}F^2 - \frac{\tau_1(2\gamma-1)}{\kappa\bar{\theta}}\int_{\mathbb{R}} q^2\mathrm{d}x - \frac{\tau_2(2\gamma-1)+\mu}{2\mu}\int_{\mathbb{R}} S^2\mathrm{d}x \tag{3.6}$$

which implies

$$\frac{F'(t)}{F^2} \geq \frac{c_3}{2(1+c_2t)^3} - \frac{\tau_1(2\gamma-1)}{c_1^2\kappa\bar{\theta}}\int_{\mathbb{R}} q^2\mathrm{d}x - \frac{\tau_2(2\gamma-1)+\mu}{c_1^2 2\mu}\int_{\mathbb{R}} S^2\mathrm{d}x. \tag{3.7}$$

Now, we use the following entropy dissipation equation derived in paper [10] as follows:

$$\begin{aligned} &\left[C_v\rho(\theta - \ln\theta - 1) + R(\rho\ln\rho - \rho + 1) + (1-\frac{1}{2\theta})\frac{\tau_1}{\kappa\theta}q^2 + \frac{1}{2}\rho u^2 + \frac{\tau_2}{2\mu}S^2\right]_t \\ &+[\rho u C_v(\theta - \ln\theta - 1) + u(1-\frac{1}{2\theta})\frac{\tau_1}{\kappa\theta}q^2 + \frac{\tau_2}{2\mu}uS^2 + R\rho u\ln\rho - R\rho u - \frac{q}{\theta} + \frac{1}{2}\rho u^3 + pu + q - Su]_x \\ &+\frac{q^2}{\kappa\theta^2} + \frac{S^2}{\theta\mu} = 0. \end{aligned} \tag{3.8}$$

Let

$$H_0 := \int_{\mathbb{R}} \left(C_v\rho_0(\theta_0 - \ln\theta_0 - 1) + R(\rho_0\ln\rho_0 - \rho_0 + 1) + (1-\frac{1}{2\theta_0})\frac{\tau_1}{\kappa\theta}q_0^2 + \frac{\tau_2}{2\mu}S_0^2\right)\mathrm{d}x,$$

then (3.8) implies

$$\int_0^t\int_{\mathbb{R}}\frac{q^2}{\kappa\theta^2}\mathrm{d}x\mathrm{d}t+\int_0^t\int_{\mathbb{R}}\frac{S^2}{\theta\mu}\mathrm{d}x\mathrm{d}t\le H_0+\frac{\max\rho_0}{2}\|u_0\|_{L^2}^2.$$

Therefore, we have

$$\frac{\tau_1(2\gamma-1)}{c_1^2\kappa\bar{\theta}}\int_0^t\int_{\mathbb{R}}q^2\mathrm{d}x\mathrm{d}t+\frac{\tau_2(2\gamma-1)+\mu}{c_1^2 2\mu}\int_0^t\int_{\mathbb{R}}S^2\mathrm{d}x\mathrm{d}t\le c_4+c_5\|u_0\|_{L^2}^2, \tag{3.9}$$

where

$$c_4:=\frac{1}{c_1^2}\left[\bar{\theta}(4\tau_1(2\gamma-1)+\tau_2(2\gamma-1)+\mu)H_0\right],\quad c_5:=\frac{1}{c_1^2}\left[\bar{\theta}(4\tau_1(2\gamma-1)+\tau_2(2\gamma-1)+\mu)\frac{\max\rho_0}{2}\right].$$

Using the above estimates and integrating the inequality (3.7) over $(0,t)$, we have

$$\frac{1}{F_0}-\frac{1}{F}\ge-\frac{c_3}{4c_2(1+c_2t)^2}+\frac{c_3}{4c_2}-c_4-c_5\|u_0\|_{L^2}^2. \tag{3.10}$$

Now we assume

$$F_0>\frac{8c_2}{c_3} \tag{3.11}$$

and

$$c_4+c_5\|u_0\|_{L^2}^2\le\frac{c_3}{8c_2}. \tag{3.12}$$

Then, we have

$$\frac{1}{F_0}\ge\frac{1}{F_0}-\frac{1}{F}\ge-\frac{c_3}{4c_2(1+c_2t)^2}+\frac{c_3}{8c_2}, \tag{3.13}$$

which mean T_0 cannot be arbitrary large without contradicting (3.11).

Now, define $c_1:=\frac{2c_2}{c_3}$. We first show the a priori estimate (3.4) hold. From (3.13), we have

$$\frac{1}{F}\le\frac{1}{F_0}+\frac{c_3}{4c_2(1+c_2t)^2}-\frac{c_3}{8c_2}\le\frac{c_3}{4c_2(1+c_2t)^2} \tag{3.14}$$

which means

$$F\ge\frac{4c_2}{c_3}(1+c_2t)^2\ge 2c_1. \tag{3.15}$$

This close the a priori assumption (3.4) by noting that $F_0\ge 2c_1$.

To show the a priori estimate (3.5) hold, we only need to show the following inequality:

$$M(1+c_2t)\le\frac{c_3}{4(1+c_2t)^3}F^2. \tag{3.16}$$

As a first step, we need (3.16) hold for $t=0$, that is,

$$F_0^2\ge\frac{4M}{c_3}=\frac{16M^4\max\rho_0}{3-\gamma}. \tag{3.17}$$

Using (3.15) and definition of c_2 and c_3, the inequality (3.16) is equivalent to

$$\sigma^2\ge\frac{3-\gamma}{16\max\rho_0}, \tag{3.18}$$

which is satisfied naturally since σ can be chosen arbitrarily large.

Thus, the proof will be finished if we can show there exists u_0 such that (3.11), (3.12), and (3.17) hold and the assumption (2.2) is satisfied. As in [11], we choose $u_0 \in H^2(\mathbb{R}) \cap C^1(\mathbb{R})$ as follows:

$$u_0(x) := \begin{cases} 0, & x \in (-\infty, -M], \\ \frac{L}{2}\cos(\pi(x+M)) - \frac{L}{2} & x \in (-M, -M+1], \\ -L, & x \in (-M+1, -1], \\ L\cos(\frac{\pi}{2}(x-1)), & x \in (-1, 1], \\ L, & x \in (1, M-1], \\ \frac{L}{2}\cos(\pi(x-M+1)) + \frac{L}{2} & x \in (M-1, M], \\ 0, & x \in (M, \infty), \end{cases} \tag{3.19}$$

where L is a positive constant to be determined later. We assume $M \geq 4$. Assumption (2.2) can easily be satisfied since it is equivalent to requiring

$$\int_{\mathbb{R}} \left(\rho_0 e_0 - \bar{\rho}\bar{e} + \frac{1}{2}u_0^2 \right) dx > 0,$$

which is satisfied by choosing $\rho_0\theta_0 > \bar{\rho}\bar{\theta} = 1$. Since

$$F_0 = \int_{\mathbb{R}} x\rho_0(x)u_0(x)\mathrm{d}x \geq \frac{L}{2}\min\rho_0 M^2,$$

we can choose L large enough, and independent of M, such that

$$\frac{L}{2}\min\rho_0 > \max\left\{ \frac{32\sigma \max\rho_0}{3-\gamma}, \frac{4\sqrt{\max\rho_0}}{\sqrt{3-\gamma}} \right\}$$

Therefore, (3.11) and (3.17) hold. On the other hand, since $\|u_0\|_{L^2}^2 \leq 2L^2M$, we can choose M sufficiently large such that

$$\bar{\theta}(8\gamma\tau_1 + 2\gamma\tau_2 + \mu)(H_0 + \max\rho_0 ML^2) \leq \frac{2\sigma\max\rho_0}{(3-\gamma)}M^2.$$

Therefore, (3.12) holds and the proof is finished.

4. Linear stability

The linearized system associated to (1.1)-(1.3) has the form

$$\begin{cases} \rho_t + u_x = 0, \\ u_t - S_x + R\theta_x + R\rho_x = 0, \\ C_v\theta_t + Ru_x + q_x = 0, \end{cases} \tag{4.1}$$

$$\tau_1 q_t + q + \kappa\theta_x = 0, \tag{4.2}$$

$$\tau_2 S_t + S - \mu u_x = 0, \tag{4.3}$$

with initial conditions

$$(\rho(x,0), u(x,0), \theta(x,0), S(x,0), q(x,0)) = (\rho_0, u_0, \theta_0, S_0, q_0). \tag{4.4}$$

Case 1: Bounded domain, $x \in (0,1)$.
Here we consider the boundary conditions

$$u(t,0) = u(t,1) = 0, \quad q(t,0) = q(t,1) = 0. \tag{4.5}$$

Without loss of generality, we assume

$$\int_0^1 \rho_0(x)\mathrm{d}x = \int_0^1 \theta_0(x)\mathrm{d}x = 0, \tag{4.6}$$

which implies by the equations $(4.1)_1$ and $(4.1)_3$

$$\int_0^1 \rho(t,x)\mathrm{d}x = \int_0^1 \theta(t,x)\mathrm{d}x = 0. \tag{4.7}$$

Defining the energy terms

$$E_1(t) := \int_0^1 \left(\frac{R}{2}\rho^2 + \frac{1}{2}u^2 + \frac{C_v}{2}\theta^2 + \frac{\tau_1}{2\kappa}q^2 + \frac{\tau_2}{2\mu}S^2\right)\mathrm{d}x,$$

$$E_2(t) := \int_0^1 \left(\frac{R}{2}\rho_t^2 + \frac{1}{2}u_t^2 + \frac{C_v}{2}\theta_t^2 + \frac{\tau_1}{2\kappa}q_t^2 + \frac{\tau_2}{2\mu}S_t^2\right)\mathrm{d}x,$$

and

$$E(t) := E_1(t) + E_2(t),$$

we will prove the following result in exponential stability:

Theorem 4.1. *There are constants $C, d > 0$ such that for all $t \geq 0$ we have*

$$E(t) \leq CE(0)\mathrm{e}^{-dt}.$$

Proof. We have the basic energy estimates:

$$\frac{\mathrm{d}E_1}{\mathrm{d}t} + \int_0^1 \left(\frac{1}{\kappa}q^2 + \frac{1}{\mu}S^2\right)\mathrm{d}x = 0, \tag{4.8}$$

and

$$\frac{\mathrm{d}E_2}{\mathrm{d}t} + \int_0^1 \left(\frac{1}{\kappa}q_t^2 + \frac{1}{\mu}S_t^2\right)\mathrm{d}x = 0. \tag{4.9}$$

By equations $(4.1)_1$, (4.2), (4.3), we have

$$\int_0^1 \theta_x^2\mathrm{d}x \leq C\int_0^1 (q_t^2 + q^2)\mathrm{d}x \tag{4.10}$$

and

$$\int_0^1 \rho_t^2\mathrm{d}x = \int_0^1 u_x^2\mathrm{d}x \leq C\int_0^1 (S_t^2 + S^2)\mathrm{d}x. \tag{4.11}$$

Then, using the boundary condition for u and (4.7), we derive, using the Poincáre inequality,

$$\int_0^1 (\theta^2 + u^2)\mathrm{d}x \leq C\int_0^1 (q_t^2 + q^2 + S_t^2 + S^2)\mathrm{d}x. \tag{4.12}$$

Now, multiplying $(4.1)_2$ by u_t and integrating over $(0,1)$, using (4.10) and (4.11), we get

$$\begin{aligned}\int_0^1 u_t^2\mathrm{d}x &= -\int_0^1 R\rho_x u_t\mathrm{d}x - \int_0^1 R\theta_x u_t\mathrm{d}x + \int_0^1 S_x u_t\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 R\rho u_x\mathrm{d}x - \int_0^1 R\rho_t u_x\mathrm{d}x - \int_0^1 R\theta_x u_t\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 Su_x\mathrm{d}x + \int_0^1 S_t u_x\mathrm{d}x \\ &\leq \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (R\rho u_x - Su_x)\mathrm{d}x + \frac{1}{2}\int_0^1 u_t^2\mathrm{d}x + C\int_0^1 (q^2 + q_t^2 + S^2 + S_t^2)\mathrm{d}x \end{aligned} \tag{4.13}$$

Similarly, multiplying $(4.1)_3$ by θ_t and integrating over $(0,1)$, we get

$$\begin{aligned}C_v\int_0^1\theta_t^2\mathrm{d}x&=-\int_0^1 Ru_x\theta_t\mathrm{d}x-\int_0^1 q_x\theta_t\mathrm{d}x\\&\le\frac{C_v}{2}\int_0^1\theta_t^2\mathrm{d}x+\frac{1}{2C_v}\int_0^1 R^2u_x^2\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 q\theta_x\mathrm{d}x+\int_0^1 q_t\theta_x\mathrm{d}x\\&\le\frac{C_v}{2}\int_0^1\theta_t^2\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 q\theta_x\mathrm{d}x+C\int_0^1(q^2+q_t^2+S^2+S_t^2)\mathrm{d}x\end{aligned}\tag{4.14}$$

Let $\psi(t,x):=\int_0^x\rho(t,x)\mathrm{d}x$, then $\psi(0)=\psi(1)=0$. Multiplying $(4.1)_2$ by ψ and integrating over $(0,1)$ yields

$$\begin{aligned}R\int_0^1\rho^2\mathrm{d}x&=-\int_0^1 R\rho_x\psi\mathrm{d}x=\int_0^1 u_t\psi\mathrm{d}x+\int_0^1 R\theta_x\psi\mathrm{d}x-\int_0^1 S_x\psi\mathrm{d}x\\&\le\frac{R\pi^2}{8}\int_0^1\psi^2\mathrm{d}x+\frac{2}{R\pi^2}\int_0^1 u_t^2\mathrm{d}x+\frac{3R}{8}\int_0^1\rho^2\mathrm{d}x+\int_0^1(R\theta^2+\frac{2}{R}S^2)\mathrm{d}x\\&\le\frac{1}{2}R\int_0^1\rho^2\mathrm{d}x+\frac{2}{R\pi^2}\int_0^1 u_t^2\mathrm{d}x+\int_0^1(R\theta^2+\frac{2}{R}S^2)\mathrm{d}x,\end{aligned}\tag{4.15}$$

where we have used

$$\int_0^1\psi^2\mathrm{d}x\le\frac{1}{\pi^2}\int_0^1\psi_x^2\mathrm{d}x=\frac{1}{\pi^2}\int_0^1\rho^2\mathrm{d}x.$$

Hence, using (4.13), we get

$$\frac{R}{2}\int_0^1\rho^2\mathrm{d}x\le\frac{4}{R\pi^2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1(R\rho u_x-Su_x)\mathrm{d}x+C\int_0^1(q_t^2+q^2+S_t^2+S^2)\mathrm{d}x\tag{4.16}$$

Let the Lyapunov function F be given by

$$F:=E_1+E_2+\varepsilon(R\rho\rho_t-S\rho_t)-\varepsilon q\theta_x.$$

Combining the above estimates, by choosing sufficiently small ε, there exists $\delta>0$ such that

$$\frac{\mathrm{d}F}{\mathrm{d}t}+\delta E\le 0\tag{4.17}$$

and positive constants C_1 and C_2 such that

$$C_1E(t)\le F(t)\le C_2E(t).\tag{4.18}$$

Thus, the exponentially stability follows as usual from (4.17) and (4.18).

□

Case 2: Cauchy problem, $x\in\mathbb{R}$.
We follow Jiang and Racke [17, section 3.2.1] which is based on the work of Kawashima [20].

We rewrite the system $(4.1)--(4.3)$ as symmetric-hyperbolic system,

$$A^0V_t+A^1V_x+BV=0,\tag{4.19}$$

where

$$A^0=\begin{pmatrix}R&0&0&0&0\\0&1&0&0&0\\0&0&C_v&0&0\\0&0&0&\frac{\tau_1}{\kappa}&0\\0&0&0&0&\frac{\tau_2}{\mu}\end{pmatrix},A^1=\begin{pmatrix}0&R&0&0&0\\R&0&R&0&-1\\0&R&0&1&0\\0&0&1&0&0\\0&-1&0&0&0\end{pmatrix},B=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&\frac{1}{\kappa}&0\\0&0&0&0&\frac{1}{\mu}\end{pmatrix}.$$

Applying the Fourier transform, we obtain

$$A^0\hat{V}_t + i|\xi|A^1(\omega)\hat{V} + B\hat{V} = 0, \tag{4.20}$$

where $A^1(\omega) = A^1\omega$ and $\omega = \frac{\xi}{|\xi|}, \xi \in \mathbb{R}$.

Note that $A^0, A^1(\omega), B$ are all real and symmetric and B is positive semi-definite. Then we take the inner product of (4.20) (in $\mathbb{C}^5$) with $\hat{V}$ and take the real part of both sides of the resulting equation to deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} < A^0\hat{V}, \hat{V} > + < B\hat{V}, \hat{V} >= 0, \tag{4.21}$$

where $< \cdot, \cdot >$ denotes the standard inner product in $\mathbb{C}^5$. Let

$$K := \begin{pmatrix} 0 & R & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ 0 & -2C_v & 0 & \frac{N\kappa}{\tau_1} & 0 \\ 0 & 0 & -\frac{N}{C_v} & 0 & \frac{\mu}{\tau_2} \\ 0 & 0 & 0 & -\frac{\kappa}{\tau_1} & 0 \end{pmatrix}. \tag{4.22}$$

where $N > 0$ is a number to be chosen large enough later. Then, simple calculations imply

$$KA^0 = \begin{pmatrix} 0 & R & 0 & 0 & 0 \\ -R & 0 & 2C_v & 0 & 0 \\ 0 & -2C_v & 0 & N & 0 \\ 0 & 0 & -N & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

which is an anti-symmetric matrix, and, for $\beta > 0$,

$$\beta KA^1 + B = \begin{pmatrix} \beta R^2 & 0 & \beta R^2 & 0 & -\beta R \\ 0 & \beta R & 0 & 2\beta & 0 \\ -2\beta RC_v & 0 & -2\beta RC_v + \beta\frac{N\kappa}{\tau_1} & 0 & 2\beta C_v \\ 0 & -\beta(\frac{R}{C_v}N + \frac{\mu}{\tau_2}) & 0 & \frac{1}{\kappa} - \beta\frac{N}{C_v} & 0 \\ 0 & 0 & -\beta\frac{\kappa}{\tau_1} & 0 & \frac{1}{\mu} \end{pmatrix}.$$

Now, multiplying (4.20) by $-i|\xi|K(\omega)$, with $K(\omega) := K\omega$, and then taking the inner product with $\hat{V}$, noting that $iK(\omega)A^0$ is hermitean and B is positive semi-definite, we obtain, after taking the real part of the resulting equality,

$$\begin{aligned} -\frac{|\xi|}{2}\frac{\mathrm{d}}{\mathrm{d}t} < iK(\omega)A^0\hat{V}, \hat{V} > +\xi^2 < \mathrm{sym}(K(\omega)A^1(\omega))\hat{V}, \hat{V} > &= \mathrm{Re}\left\{i|\xi| < K(\omega)B\hat{V}, \hat{V} >\right\} \\ &\le \varepsilon|\xi|^2|\hat{V}|^2 + C(\varepsilon) < B\hat{V}, \hat{V} >, \end{aligned} \tag{4.23}$$

where $\mathrm{sym}[K(\omega)A^1(\omega)]$ denotes the symmetric part of $K(\omega)A^1(\omega)$ and $0 < \varepsilon < 1$ is to be determined below.

Define

$$E^\beta(t) = \frac{1}{2} < A^0\hat{V}, \hat{V} > -\frac{\beta}{2}\frac{|\xi|}{1+|\xi|^2} < iK(\omega)A^0\hat{V}, \hat{V} >, \tag{4.24}$$

where β is now a small positive constant to be determined later on. Then (4.21) $\times (1 + |\xi|^2) +$ (4.23) $\times \beta$ yields

$$\begin{aligned} (1+|\xi|^2)\frac{\mathrm{d}}{\mathrm{d}t}E^\beta + |\xi|^2 < \{\mathrm{sym}[\beta K(\omega)A(\omega)] + B\}\,\hat{V}, \hat{V} > + < B\hat{V}, \hat{V} > \\ \le \beta\varepsilon|\xi|^2|\hat{V}|^2 + \beta C(\varepsilon) < B\hat{V}, \hat{V} > . \end{aligned} \tag{4.25}$$

It can easily be seen that there is a small constant $\beta_0 > 0$ such that E^β is equivalent to $|\hat{V}|^2$. Moreover, the matrix

$$\mathrm{sym}[\beta K(\omega)A^1(\omega)] + B = \mathrm{sym}[\beta KA^1] + B = \mathrm{sym}[\beta KA^1 + B]$$

is positive definite for any $\beta \in (0, \beta_0]$, if N is large enough and β_0 is small enough. This can be seen as follows.

$$sym(\beta KA^1 + B) =$$
$$\begin{pmatrix} \beta R^2 & 0 & \frac{\beta}{2}(R^2 - 2RC_v) & 0 & -\frac{\beta}{2}R \\ 0 & \beta R & 0 & \frac{\beta}{2}(2 - \frac{R}{C_v}N - \frac{\mu}{\tau_2}) & 0 \\ \frac{\beta}{2}(R^2 - 2RC_v) & 0 & \beta(\frac{\kappa}{\tau_1}N - 2RC_v) & 0 & \frac{\beta}{2}(2C_v - \frac{\kappa}{\tau_1}) \\ 0 & \frac{\beta}{2}(2 - \frac{R}{C_v}N - \frac{\mu}{\tau_2}) & 0 & \frac{1}{\kappa} - \beta\frac{N}{C_v} & 0 \\ -\frac{\beta}{2}R & 0 & \frac{\beta}{2}(2C_v - \frac{\kappa}{\tau_1}) & 0 & \frac{1}{\mu} \end{pmatrix}.$$

Let d_j denote the j-th principle minor of the matrix $sym(\beta KA^1 + B)$, $j = 1, \ldots, 5$.

$$d_1 = \beta R^2 > 0, \quad d_2 = \beta R^3 > 0.$$

Let

$$A_3 := \begin{pmatrix} R^2 & 0 & \frac{1}{2}(R^2 - 2RC_v) \\ 0 & R & 0 \\ \frac{1}{2}(R^2 - 2RC_v) & 0 & \frac{\kappa}{\tau_1}N - 2RC_v \end{pmatrix}.$$

Then, $d_3 = \beta^3 \det(A_3)$ and

$$\det(A_3) = R^3(\frac{\kappa}{\tau_1}N - 2RC_v - \frac{1}{4}(R - 2C_v)^2).$$

We can choose N independent of β such that

$$\frac{\kappa}{\tau_1}N > 2RC_v + \frac{1}{4}(R - 2C_v)^2 = \frac{1}{4}(R^2 + 4C_v^2).$$

implying $d_3 > 0$. Now N is fixed. For small β, we observe that

$$d_4 = \frac{1}{\kappa}\det(A_3)\beta^3 + O(\beta^4), \quad \text{as}\, \beta \to 0,$$
$$d_5 = \frac{1}{\kappa\mu}\det(A_3)\beta^3 + O(\beta^4), \quad \text{as}\, \beta \to 0,$$

which gives $d_3, d_4 > 0$ by choosing β_0 sufficiently small. Thus, the second term on the left-hand-side of (4.25) is bounded from below by $C(\beta_0)|\xi|^2|\hat{V}|^2$. Now, choose ε and β such that $\varepsilon = \frac{C(\beta_0)}{2\beta_0}$ and $\beta = \min\{\beta_0, \frac{1}{C(\varepsilon)}\}$. Then, the estimate (4.25) implies

$$E_t^\beta + C_1 h(|\xi|)E^\beta(t) \le 0, \qquad \text{with} \quad h(r) := \frac{r^2}{1 + r^2}. \tag{4.26}$$

Thus, we have

Lemma 4.2. *There are positive constants C and C_1 such that the solutions of* (4.20) *satisfy*

$$|\hat{V}(t,\xi)|^2 \le Ce^{-C_1 h(|\xi|)t}|\hat{V}(0,\xi)|^2, \qquad \text{for} \quad (t,\xi) \in \mathbb{R}^+ \times \mathbb{R}, \tag{4.27}$$

where $h(r) = \frac{r^2}{1+r^2}$.

As a consequence, we obtain in a standard manner (see [17]) the decay rates of solutions to the Cauchy problem,

Theorem 4.3. *Let $l \geq 0$, and $0 \leq k \leq l$ be integers, and let $p \in [1,2]$. Assume that $V(0) \in H^l(\mathbb{R}) \cap L^p(\mathbb{R})$. Then we have*

$$\|\partial_x^l V(t)\|^2 \leq C\left\{e^{-C_1 t}\|\partial_x^l V(0)\|^2 + (1+t)^{-(2\lambda+l-k)}\|\partial_x^k V(0)\|_{L^p}^2\right\} \tag{4.28}$$

where $\lambda = \frac{1}{2p} - \frac{1}{4}$ and C_1 is the same constant as in Lemma 4.2.

For $p=1$, $k=l$, we get the L^1-L^2 decay of order $-1/4 = -n/4$ (space dimension $n=1$),and for the decay of the l-th derivative of V one needs at most l derivatives of the data. That is, there is no so-called *loss of regularity*, which is typical for systems not experiencing a loss of exponential stability in bounded domains.

As mentioned in the Introduction, there is a loss of regularity for example for the thermoelastic plate equation, where one has (cp. [33] for the meaning of the dependent variables)

$$\begin{aligned}\|\partial_x^k(u_t, \Delta u, \theta, \tau q)(t)\|_{L^2} \leq & C(1+t)^{-n/4-k/2}\|(u_1, \Delta u_0, \theta_0, \tau q_0)\|_{L^1} \\ & + C(1+t)^{-\ell/2}\|\partial_x^{k+\ell}(u_1, \Delta u_0, \theta_0, \tau q_0)\|_{L^2},\end{aligned} \tag{4.29}$$

where, for $k, \ell \geq 0$, the *loss of regularity* is visible in the last term of (4.29) requiring $k+\ell$ derivatives of the data to obtain a decay for k derivatives of the solution at time t. For further examples see [9, 14, 40].

Here we have the same situation as for the non-relaxed case $\tau_1 = \tau_2 = 0$ of the classical compressible Navier-Stokes equations, where the exponential stability in bounded domains and the decay is known, cp. Jiang [15, 16] and Li and Liang [22] for the nonlinear situation; for the linearized system, the exponential decay in bounded domains and the decay without loss of regularity for the Cauchy problem can be proved as for the case $\tau_1, \tau_2 > 0$ above.

That is, the linearized system with relaxation ($\tau_1, \tau_2 > 0$) behaves qualitatively the same as the classical non-relaxed one ($\tau_1 = \tau_2 = 0$). This is also know for the system of thermoelasticity, where one also has a similar behavior with respect to global existence for small data and for the blow-up for large data, see [17, 29].

Here we have shown that the latter does no longer hold, i.e., although the linearized systems behave the same, and although for small data the behavior is comparable, we have for large data a blow-up for the relaxed system while there are global large solutions for the classical compressible Navier-Stokes system.

We remark that the equations of thermoelasticity, where one does not "loose" anything when relaxing the equations, neither in the linearized nor in the nonlinear framework, seems to build an exceptional case which is pointed out for linear systems in Racke [30].

Therefore, we now experienced that linear similarity up to similarity of nonlinear systems does not imply similar behavior for nonlinear systems with large data. Here we remind of the case of *incompressible* Navier-Stokes equations, for which the relaxed case was studied in Racke and Saal [31, 32] and in Schöwe [35] – the question of blow-up remains yet as open as for the classical Navier-Stokes equations in 3-d.

One might also compare the situation with the semi-/linear heat equation resp. the damped wave equation. Here we have the situation that for

$$u_t - \Delta u = u^p$$

resp.

$$u_{tt} - \Delta u + u_t = u^p$$

we have for the linearized systems that they behave similar for bounded domains in $\mathbb{R}^n$ (exponential stability), they have the same (e.g.) L^1-L^∞-decay rates $-n/2$ for the Cauchy problem, with improvements for derivatives, and they have exactly the same *critical exponent* $p_c = 1 + 2/n$ with the property that global small solutions exist for $p > p_c$, while solutions blow up even for small data if $1 < p \leq p_c$, see the work of Todorova and Yordanov [39] and Zhang [41] for the damped wave equation, and the references there as well as the survey by Galaktinov and Vázquez [8] for the heat equation.

Finally, we recall the *isentropic* case, which was discussed by Hu and Wang in [12],

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \rho u_t + \rho u u_x + p_x = S_x, \\ \tau S_t + S = \mu u_x. \end{cases} \tag{4.30}$$

While for $\tau = 0$ global large solutions exist, they show a blow-up for large data if $\tau > 0$, i.e. the relaxation has an effect in the nonlinear case for large data. On the other hand, we again have the similarity of the linearized systems,

$$\begin{cases} \rho_t + u_x = 0, \\ u_t + R_1 \rho_x - S_x = 0, \\ \tau S_t + S - \mu u_x = 0, \end{cases} \tag{4.31}$$

for some $R_1 > 0$. For $\tau = 0$ we derive that u satisfies a wave equation with Kelvin-Voigt damping,

$$u_{tt} - R_1 u_{xx} - \mu u_{txx} = 0.$$

For this equation, with initial conditions, appropriate normalizations and, for bounded domains, associated boundary conditions, it is well known that in bounded domains $I = (a, b)$ we have exponential stability, and for the Cauchy problem, there is no loss of regularity: see Ponce [27] for the latter for the former one may simply use the Lyapunov functional

$$L(t) := \int_a^b \left(u_t^2 + (R_1 + \varepsilon)u_x^2 + \varepsilon u_t u\right) dx$$

to conclude, for sufficiently small $\varepsilon > 0$, that the energy

$$E_{is}(t) := \int_a^b \left(u_t^2 + R_1 u_x^2\right) dx$$

tends to zero exponentially.

For $\tau > 0$ we can derive the third-order equation

$$\tau u_{tt} + u_{tt} - R_1 u_{xx} - (\tau R_1 + \mu) u_{txx} = 0.$$

This equation if of Jordan-Moore-Gibson-Thompson type, and the exponential stability in bounded domains is known as well as the non-loss of regularity, see Kaltenbacher, Lasiecka, Marchand [18] and Kaltenbacher, Lasiecka and Popieszalska [19] for bounded domains, and Pellicer and Said-Houari [25] for the Cauchy problem. The there needed stability condition here turns into the satisfied condition $\mu > 0$.

Therefore, for the isentropic case, we have the same phenomenon as for the non-isentropic case presented here.

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