Erschienen in: Journal of Non-Crystalline Solids ; 172-174 (1994), Part 1. - S. 241-247
http://dx.doi.org/10.1016/0022-3093(94)90442-1

The Kohlrausch law as a limit solution to mode coupling equations

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Abstract

The α-equations of the idealized mode coupling theory for density fluctuations in simple liquids are studied in the limit of large wavevectors, q. The Kohlrausch function $\Phi_q(t) = f_q^c\, e^{-\Gamma_q (t/\tau)^b}$ is obtained with $\Gamma_q \propto q$ for $q \to \infty$. This result reflects Levy's generalized central limit theorem.

1. Introduction

The non-exponentional time dependence of the final decay into equilibrium, i.e., of the α-process, is a characteristic feature of the dynamics of supercooled liquids. The mode coupling theory (MCT) (see Refs. [1] and [2] for recent reviews) has derived a closed set of equations for the α-dynamics of density fluctuations. The solutions are uniquely determined by the equilibrium structure. The α-equations of the idealized MCT aim at describing the slowing of the structural relaxation due to the cage effect known from liquid theory [3]. For supercooled liquids, this description is valid above a certain temperature, T_c, below which solid like, or activated, transport processes influence the α-process. In colloidal suspensions, however, the α-process is found to be frozen out below T_c indicating the applicability of the idealized MCT above and below T_c [4]. For temperatures, T, close to T_c and neglecting solidlike transport, a diverging α-timescale, τ, is found, $\tau \propto (T - T_c)^{-\gamma}$. If τ is long compared with some characteristic microscopic time, t_0, the time–temperature superposition principle is obtained for the auto correlation function of variable X [5]:

$$\Phi_X(t) = \hat{\Phi}_X(\hat{t}), \quad T \to T_c +, \quad \tau \to \infty,$$
$$\hat{t} = t/\tau = \text{const.} \tag{1}$$

The equations for the α-master functions, $\hat{\Phi}_X(\hat{t})$, in general depend on the details of the equilibrium structure of the system studied. Nevertheless, it could be shown rigorously that the initial decay of the $\hat{\Phi}_X(\hat{t})$ obeys the so-called von Schweidler law [6]:

$$\hat{\Phi}_X(\hat{t}) = f_X^c - h_X \hat{t}^b \quad \text{for} \quad \hat{t} \to 0. \tag{2}$$

The MCT α-equations have numerically been solved for two simple liquids specified by their static structure factor, S_q. The resulting correlation functions of a hard sphere system (HSS) [7] and a soft sphere binary mixture (BM) [8] were found to show non-exponentional decay which could

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approximatively be parameterized by the Kohlrausch function

$$\hat{\Phi}_X^K(\hat{t}) = f_X^K \, e^{-(\hat{t}/\hat{\tau}_X)^{\beta_X}}. \qquad (3)$$

The Kohlrausch functions, however, exhibited small but systematic deviations from the solutions of the α-equations in general and the parameters in Eq. (3), especially the stretching exponent β_X, were dependent on the variable X studied. In this paper, it is shown that the MCT α-equations for density fluctuations, $X = \varrho_q$, in simple liquids are solved by the Kohlrausch function in the limit of large wavevectors, q. As discussed below, this result can be interpreted in view of Levy's generalization of the central limit theorem [9]. The relevance of this theorem to glassy dynamics was emphasized by Shlesinger and others [10] and its connection to the MCT was discussed in Ref. [2]. Levy proved that, for appropriately normalized sums of N stochastically independent processes, each with equal probability density $f(\omega)$, the Kohlrausch law e^{-t^β} is obtained as the characteristic function of the limit distribution, if the probability density, $f(\omega)$, has a power law behaviour at large frequencies, $f(\omega) \propto \omega^{-1-\beta}$ for $\omega \gg 1$ with $0 < \beta \leqslant 2$.

2. Theory

2.1. Reformulation of the MCT α-equations

The MCT for simple liquids studies the solutions of a closed set of equations for normalized density correlation functions $\Phi_q(t) = \langle \varrho_q^*(t)\varrho_q \rangle / \langle |\varrho_q|^2 \rangle$ (see review Ref. [5] for a summary of results and proofs). The full functions, $\Phi_q(t)$, can be calculated in principle; they show regular short time behavior: $\Phi_q(t) = 1 - \frac{1}{2}(t\Omega_q)^2 + \cdots$ for $t \to 0$, with Ω_q given by sum rules [3]. Close to T_c a fraction f_q^c of $\Phi_q(t)$, $0 < f_q^c < 1$, relaxes via the α-process with the slow relaxation time, τ, as noted in Eq. (1). The normalized α-master functions will be studied:

$$\hat{\varphi}_q(\hat{t}) = \hat{\Phi}_q(\hat{t})/f_q^c. \qquad (4)$$

The carets are dropped in the following as only α-quantities are discussed. Normalizing the α-correlators to $\varphi_q(t = 0) = 1$ bears the advantage that they can be viewed as characteristic functions. The $\varphi_q(t)$ can be calculated from the α-scaling equation:

$$\varphi_q(t) = \mu_q(t) - f_q^c \left\{ \frac{d}{dt} \int_0^t dt' \, \mu_q(t - t') \varphi_q(t') \right\}, \qquad (5a)$$

where the memory functions, $\mu_q(t)$, are given by quadratic polynomials in the $\varphi_q(t)$:

$$\mu_q(t) = \int_{-q/\sqrt{2}}^{q/\sqrt{2}} d\bar{p} \int_0^\infty d\bar{k} \, \tfrac{1}{2} v_{q,\bar{k}\bar{p}}^c \, \varphi_{((q/2)+[(\bar{k}-\bar{p})/\sqrt{2}])}(t) \times \varphi_{((q/2)+[(\bar{k}+\bar{p})/\sqrt{2}])}(t). \qquad (5b)$$

In Eq. (5b), an appropriate choice of integration variables $\bar{k}$, $\bar{p}$ was made. It reflects momentum conservation and dimension $d = 3$. The scaled vertices are determined by the equilibrium static structure factor, S_q, evaluated at the critical temperature, T_c, entering into $V_{q;k,p}^c$; see Ref. [5] for the exact formula which will not be needed in this paper.

$$v_{q;\bar{k},\bar{p}}^c = \frac{1}{4\pi^2} \frac{kp}{q} \frac{1}{f_q^c} V_{q;k,p}^c f_k^c f_p^c \geqslant 0,$$

$$k = \frac{q}{2} + \frac{\bar{k} + \bar{p}}{\sqrt{2}}, \quad p = \frac{q}{2} + \frac{\bar{k} - \bar{p}}{\sqrt{2}}. \qquad (6)$$

The symmetry of the vertices $V_{q,kp}^c = V_{q,pk}^c$ leads to $v_{q;\bar{k},-\bar{p}} = v_{q;\bar{k},\bar{p}}$. For simple liquids, $V_{q;k,p}^c = \mathcal{O}(q^{-2})$ for $q \to \infty$ follows from the general property $S_q \to 1$.

The α-scaling equation (5) is evaluated at a critical point where the following two sets of equations hold. First, a non-vanishing solution f_q^c exists of the equation

$$\frac{1}{1 - f_q^c} = \mu_q(0) = \int_{-q/\sqrt{2}}^{q/\sqrt{2}} d\bar{p} \int_0^\infty d\bar{k} \, \tfrac{1}{2} v_{q;\bar{k},\bar{p}}^c. \qquad (7a)$$

The non-zero solution, f_q^c, was of course already anticipated in the definition of the scaled vertices. Second, the maximal non-degenerate eigenvalue, E, equals 1. Denoting the corresponding eigenvector by $\Gamma_q = h_q/f_q^c$, the eigenvalue equation runs:

$$E\Gamma_q = (1 - f_q^c)^2 \int_{-q/\sqrt{2}}^{q/\sqrt{2}} d\bar{p} \int_0^\infty d\bar{k} \; v_{q;\bar{k},\bar{p}}^c \times \Gamma_{((q/2)+[(\bar{k}-\bar{p})/\sqrt{2}])} \quad \text{and} \quad E \stackrel{!}{=} 1. \qquad (7b)$$

The eigenvector, Γ_q, appears as relaxation rate in the von Schweidler short time expansion:

$$\varphi_q(t) = 1 - \Gamma_q t^b \quad \text{for} \quad t \to 0, \tag{8}$$

where the exponent b (and γ entering in τ) is given by the exponent parameter λ of the MCT which in turn is determined from $S_q(T_c)$ [5].

2.2. Solutions for large wavevectors q

Except for the short time expansion (8), no further analytical results are known for the wavevector-dependent α-equations (5). For large wavevectors, however, $q \to \infty$, the equations can be simplified without taking specific properties of the static structure, S_q, into account. The α-relaxation strength, f_q^c, decays to zero in this limit due to the vanishing of the vertices $V^c_{q;kp}$ for large q:

$$f_q^c \to 0 \quad \text{for } q \to \infty. \tag{9}$$

Readily it can be seen from Eq. (5a) that this leads to the following simplification:

$$\varphi_q(t) = \mu_q(t) + \mathcal{O}(f_q^c) \quad \text{with} \quad f_q^c \to 0 \quad \text{for} \quad q \to \infty. \tag{10}$$

Scaled α-correlator and memory function become equal for large q. This emphasizes that a Markovian assumption for $\mu_q(z) \approx i\mu_q$ is invalid for the α-process. Due to the normalization of the correlators, $\varphi_q(t)$, the integral over the scaled vectors can be read off; it can be expressed in terms of a positive weight $\rho^q_{\bar{p}}$:

$$\rho^q_{\bar{p}} = \int_0^\infty \mathrm{d}\bar{k}\, \tfrac{1}{2} v^c_{q;\bar{k}\bar{p}} \geqslant 0, \tag{11a}$$

with

$$\int_{-q/\sqrt{2}}^{q/\sqrt{2}} \mathrm{d}\bar{p}\, \rho^q_{\bar{p}} = \mu_q(0) \to 1 \quad \text{for} \quad q \to \infty, \tag{11b}$$

For large q, the integral over $\rho^q_{\bar{p}}$ in Eq. (11b) becomes normalized to unity. The symmetry in the vertices, v, leads to $\rho^q_{\bar{p}}$ being symmetric in $\bar{p}$, $\rho^q_{-\bar{p}} = \rho^q_{\bar{p}}$. Without knowing the actual solutions of the equations (7a) for the non-ergodicity parameters, no more exact properties of $\rho^q_{\bar{p}}$ can be stated. Generically due to its normalization (11b), $\rho^q_{\bar{p}}$ will for fixed $\bar{p}$ decrease like $1/q$ for large q. Since $v^c_{q;\bar{k}\bar{p}} \geqslant 0$ one can use the mean value theorem in the $\bar{k}$ integrations. The eigenvalue condition (7b) for example becomes

$$\Gamma_q = (1 - f_q^c)^2 \int_{-q/\sqrt{2}}^{q/\sqrt{2}} \mathrm{d}\bar{p}\, \rho^q_{\bar{p}}\, 2\Gamma_{((q/2)+[(\bar{K}^q_{\bar{p}}+\bar{p})/\sqrt{2}])}\,. \tag{12}$$

The $\bar{K}^q_{\bar{p}}$ measure the wavevector region where the microscopic structure in the vertices $v^c_{q;\bar{k}\bar{p}}$ cannot be neglected. They will be connected to physically relevant length scales like q_p the position of the first peak in the static structure factor, S_q. For the results in this paper, it will become crucial that the $\bar{K}^q_{\bar{p}}$ which appear in Eq. (12) and in a similar use (Eq. (18)) of the mean value theorem stay finite in the limit of large wavevectors:

$$\bar{K}^q_{\bar{p}} < K < \infty \quad \text{for} \quad q \to \infty. \tag{13}$$

This property could not yet be proven rigorously. It is a consequence of Eq. (9) and could be deduced if the asymptotic behaviour of f_q^c for large q was known.

The eigenvalue condition (12) has except for an arbitrary scaling factor an unique solution Γ_q. Irrespective of the form of the vertices $V^c_{q;kp}$, as long as the weight $\rho^q_{\bar{p}}$ does not become degenerate [9], Γ_q can at most diverge linearly in q for large q. The following ansatz for Γ_q uses an arbitrary but bounded function, Δ_q, to show this:

$$\Gamma_q = (q/q_0) - \Delta_q, \quad \text{with } |\Delta_q| \leqslant \Delta < \infty. \tag{14}$$

If this ansatz is inserted into Ref. (12), the leading contribution for large q is cancelled on both sides:

$$\frac{q}{q_0} - \Delta_q \to \frac{q}{q_0} + \sqrt{2}/q_0 \int_{-q/\sqrt{2}}^{q/\sqrt{2}} \mathrm{d}\bar{p}\, \rho^q_{\bar{p}} \bar{K}^q_{\bar{p}} - 2\int_{-q/\sqrt{2}}^{q/\sqrt{2}} \mathrm{d}\bar{p}\, \rho^q_{\bar{p}} \Delta_{((q/2)+[(\bar{K}^q_{\bar{p}}+\bar{p})/\sqrt{2}])} \quad \text{for } q \to \infty. \tag{15}$$

Due to $|\Delta_q| \leqslant \Delta$ and inequality (13), at most terms constant for large q are left in Eq. (15) as the two integrals can be estimated to be smaller than K and Δ, respectively. The linear divergence of Γ_q is connected to the expression of the memory function,

$\mu_q(t)$, as a quadratic polynomial in $\varphi_q(t)$ and to the momentum conservation applied in dimension $d = 3$. Δ_q has still to be determined by an integral equation corresponding to Eq. (12). If the following limit exists,

$$\Gamma_0 q_0 = \lim_{q\to\infty} \langle \bar{K}_q \rangle_{\bar{p}} = \lim_{q\to\infty} \int_{-q/\sqrt{2}}^{q/\sqrt{2}} \mathrm{d}\bar{p}\; \rho_{\bar{p}}^q \bar{K}_{\bar{p}}^q, \tag{16}$$

then the constant terms in Eq. (15) are approximately cancelled by $\Delta_q \to \Gamma_0$ for $q \to \infty$. $\Gamma_0 q_0$ can be used to check the condition (13). The final form for the rate Γ_q shall be denoted by $\Gamma_q = (q/q_0) - \Gamma_0 - \tilde{\Delta}_q$, where $|\tilde{\Delta}_q| \leqslant \tilde{\Delta} < \infty$ is assured and $|\tilde{\Delta}_q| \to 0$ for $q \to \infty$ can be expected.

The divergence of the rate Γ_q opens up a time window where, for large q, the memory function $\mu_q(t)$ and also $\varphi_q(t)$, due to Eq. (10), can be calculated. The exact von Schweidler asymptote (8) generally is not the short time expansion of a Kohlrausch function. If one sets

$$\varphi_q(t) = \mathrm{e}^{-\Gamma_q t^b} \quad \text{for } t \ll t_q^{\mathrm{K}}, \tag{17}$$

this will be valid for times short compared to a q-dependent upper cut-off time, t_q^{K}. The following calculation shows that for large q the time t_q^{K} becomes approximately constant. Using the mean value theorem of integrations in Eq. (5b), $\varphi_q(t)$ becomes, for large wavevectors,

$$\varphi_q(t) \to \int_{-q/\sqrt{2}}^{q/\sqrt{2}} \mathrm{d}\bar{p}\; \rho_{\bar{p}}^q \varphi_{((q/2)+[(\dot{K}_{\bar{p}}^q+\bar{p})/\sqrt{2}])}(t) \times \varphi_{((q/2)+[(\dot{K}_{\bar{p}}^q-\bar{p})/\sqrt{2}])}(t) \quad \text{for } q \to \infty. \tag{18}$$

For finite values of q, the ansatz (17) will not be more valid than its short time expansion, the von Schweidler law, which rigorously is valid for $t \ll 1$ only. Ansatz (17) with expressions (14) and (16) for Γ_q now leads to

$$\varphi_q(t) \to \int_{-q/\sqrt{2}}^{q/\sqrt{2}} \mathrm{d}\bar{p}\; \rho_{\bar{p}}^q \exp\left\{ -t^b \left[\frac{q}{q_0} + \sqrt{2}\frac{\dot{K}_{\bar{p}}^q}{q_0} - 2\Gamma_0 - 2\tilde{\Delta}_{((q/2)+[(\dot{K}_{\bar{p}}^q+\bar{p})/\sqrt{2}])} \right] \right\} \quad \text{for } q \to \infty$$

$$= \mathrm{e}^{-\Gamma_q t^b} \int_{-q/\sqrt{2}}^{q/\sqrt{2}} \mathrm{d}\bar{p}\; \rho_{\bar{p}}^q \exp\left\{ -t^b \left[\sqrt{2}\frac{\dot{K}_{\bar{p}}^q}{q_0} - \Gamma_0 + \tilde{\Delta}_q - 2\tilde{\Delta}_{((q/2)+[(\dot{K}_{\bar{p}}^q+\bar{p})/\sqrt{2}])} \right] \right\}$$

$$\approx \mathrm{e}^{-\Gamma_q t^b} \mathrm{e}^{-t^b[2\Gamma_0 + 3\tilde{\Delta}]}. \tag{19}$$

This shows that self consistently for large q the Kohlrausch function (17) is obtained as a solution to the MCT α-equations for a finite time window, which is restricted by the mentioned validity of the von Schweidler law and the second factor in Eq. (19). The solution (17) is valid for

$$t \ll t_q^{\mathrm{K}} \to \min\{(2\Gamma_0 + \tilde{\Delta})^{-1/b}, 1\} \quad \text{for } q \to \infty. \tag{20}$$

Due to the divergence of the rate Γ_q, the correlator $\varphi_q(t)$ follows the Kohlrausch function (17) for an increasing part of its decay for large q. The stretching exponent, β^{K}, equals the von Schweidler exponent, b, in this limit.

3. Numerical results for simple liquids

It is interesting to study whether the large q results of the previous section can be observed in the numerical solutions of the α-MCT equations for the HSS [7] and the BM [8]. There, q was increased up to $qa = 30$ in order to calculate the integrals in Eq. (5b) without cut off effects. The length scale, a, is connected to the density n via $\frac{4}{3}\pi a^3 n = 1$; the peak of the static structure factor lies at $q_p a \approx 4.3$ in these systems.

Fig. 1 shows the scaled vertices, $v_{q;\bar{k}\bar{p}}$, as a function of k and p for the largest wave vector $qa = 29.9$ calculated in Ref. [7]. The integration variable, $\bar{p}$, runs parallel to the line $k + p = q$ and $\bar{k}$ perpendicular to it. The decrease of $v_{q;\bar{k}\bar{p}}^{\mathrm{c}}$ for $k + p = q + \sqrt{2}\bar{k}$ and $\bar{k}$ becoming large can be observed. The strongly cleft and picked structure in the normalized $v_{q;\bar{k}\bar{p}}^{\mathrm{c}}$ influences via $\bar{K}_{\bar{p}}^q$ in Eq. (12) the range of validity of the large q results. In Fig. 2 one notices that the corrections $(-\Gamma_0 + \tilde{\Delta}_q)$ are not yet negligible even for the largest wavevectors studied. The asymptotes added in Fig. 2 are fitted to the rates Γ_q for even larger q, $qa \geqslant 50$. The tagged particle dynamics, whose Γ_q^{s} are included in Fig. 2, can easily be included into the discussion of Section 2.2. Numerical solutions for $\varphi_q(t)$ in the HSS are shown for four wave vectors in Fig. 3. Two

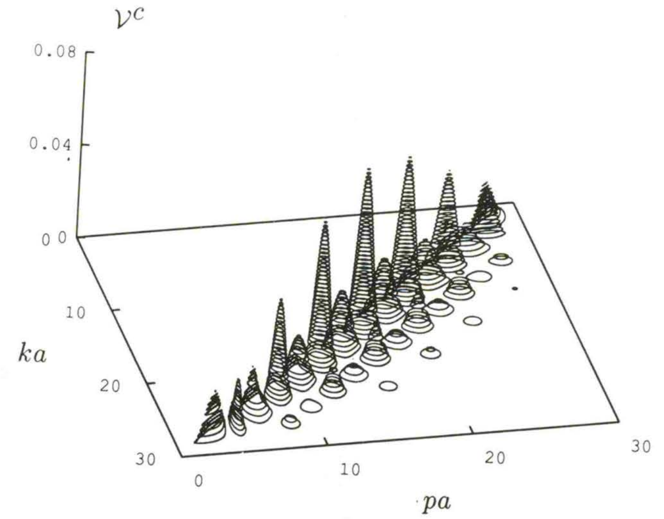

Fig. 1. Rescaled vertices, $v^c_{q,\bar{k}\bar{p}}$, of a hard sphere system (HSS) [7] for $qa = 29.9$; $\bar{k}$ and $\bar{p}$ are connected to k and p via: $k + p = q + \sqrt{2}\bar{k}$ and $k - p = \sqrt{2}\bar{p}$.

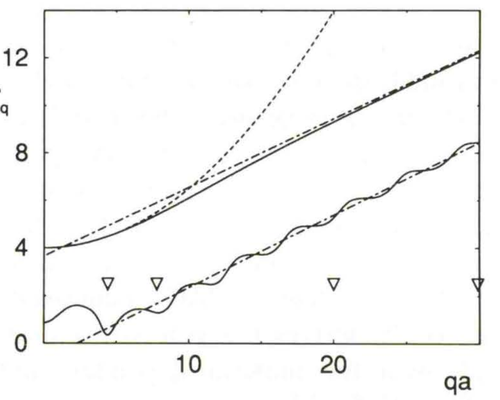

Fig. 2. Decay rates Γ_q and Γ^s_q for coherent and incoherent density fluctuations in the HSS, the latter with a vertical off set of 4. The large q asymptotes are shown as dot-dashed lines with parameters: $q_0 = 3.3$, $\Gamma_0 = 0.7$ for Γ_q and $q_0 = 3.5$, $\Gamma_0 = 0.3$ for Γ^s_q. The quadratic small q asymptote to Γ^s_q is shown as a dashed curve. Four wavevectors are marked by triangles.

wavevectors, q_p and q close to the position of the second maximum in S_q, are representative for intermediate wavevectors; the other two wavevectors, $qa = 20$ and $qa = 29.9$, illustrate the large q behaviour. The expected trend of Eq. (17) becoming a better approximation for large wavevectors can be observed; exponent b equals $b = 0.53$ in the HSS. Mainly due to the limited range of the von

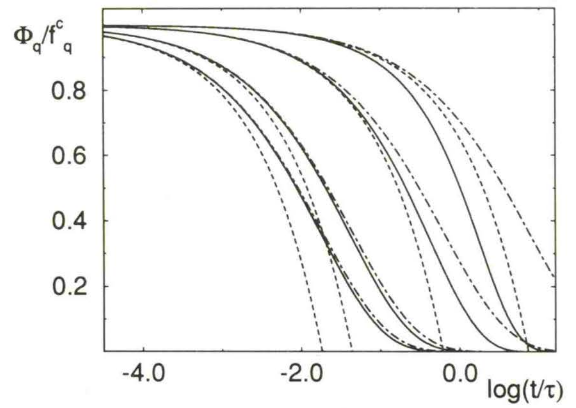

Fig. 3. For four wave vectors q (indicated by triangles in Fig. 2) numerical solutions, $\varphi_q(t)$, of the HSS (solid curves) are compared to the von Schweidler short time expansion (short dashes) and the Kohlrausch function (Eq. (17), chain curves) with Γ_q taken from Fig. 2. From right to left the q-values increase ($qa = 4.4, 7.8, 20, 29.9$).

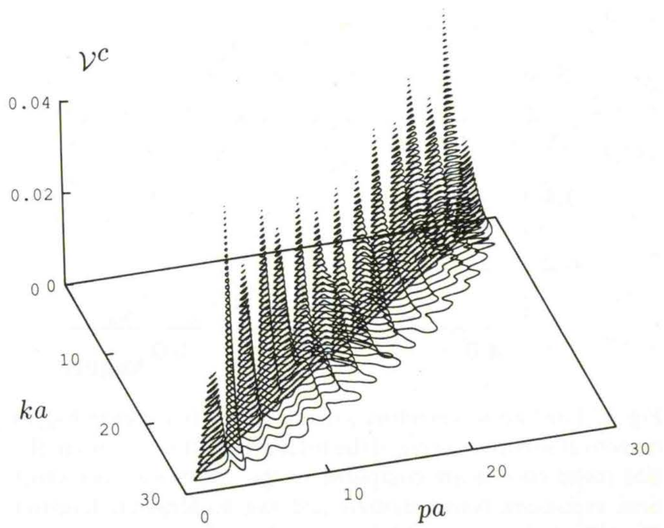

Fig. 4. Rescaled vertices, $v^c_{q,\bar{k}\bar{p}}$, of the memory function, $K^{nn}_q(t)$, determining the total density fluctuations in a binary soft sphere mixture (BM) [8] for $qa = 29.9$; $\bar{k}$ and $\bar{p}$ are connected to k and p via: $k + p = q + \sqrt{2}\bar{k}$, $k - p = \sqrt{2}\bar{p}$.

Schweidler asymptote for not too large wavevectors, the upper cut-off time, t^K_q, is rather small value $t^K_q \approx 10^{-2}$ for $q \to \infty$.

In Fig. 4, the scaled vertices, $v^c_{q;\bar{k}\bar{p}}$, for the memory function ($K^{(nn)}_q(t)$ in the notation of Eq. [8]) describing the total density fluctuations in the BM

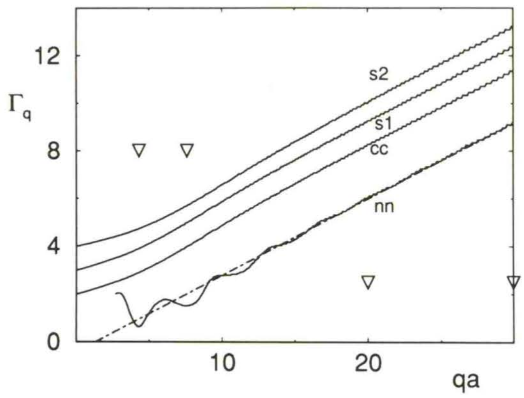

Fig. 5. Decay rates, Γ_q^r, for total density (r = nn), concentration (r = cc) and tagged particle fluctuations of species 1 (r = s1) or 2 (r = s2) in the BM, the latter with vertical off sets of 2, 3 and 4. The large q asymptote for the total density fluctuations is shown by a chain curve with $q_0 = 3.2$ and $\Gamma_0 = 0.3$. Four wavevectors are marked by triangles.

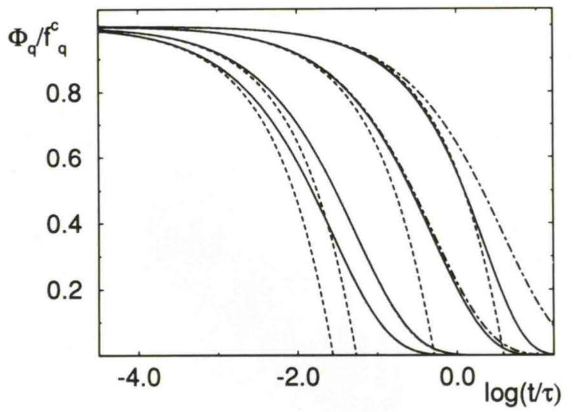

Fig. 6. For four wavevectors q (indicated by triangles in Fig. 5) numerical solutions, $\varphi_q(t)$, of the total density fluctuations in the BM (solid curves) are compared to the von Schweidler short time expansion (short dashes) and the Kohlrausch function (Eq. (17), chain curves) with Γ_q^{nn} taken from Fig. 5. From right to left, the q-values increase (qa = 4.3,7.6,20,29.9). For the two largest wavevectors, the dot-dashed Kohlrausch curves almost coincide with the numerical solutions.

for $qa = 29.9$ are shown. The smoother variation of the $v^{\mathrm{c}}_{q;\bar{k}\bar{p}}$ with increasing $\bar{k}$ leads to smaller corrections to the linear asymptote in the damping rates, Γ_q, as shown in Fig. 5. There also the Γ_q corresponding to concentration and tagged particle fluctuations are displayed. The smaller corrections to the large q asymptotes in Γ_q in this system correspond to a closer agreement of $\varphi_q(t)$ with the Kohlrausch behavior for the same four wavevectors as chosen in Fig. 3; this is demonstrated in Fig. 6 where the exponent b equals $b = 0.62$. Eq. (17) with predetermined rate Γ_q describes more than 90% of the α-decay of $\varphi_q(t)$ for the largest wavevectors considered. This is due to a rather large upper cut-off time, $t_q^{\mathrm{K}} \approx 10^{-1}$.

4. Discussion and conclusions

The above study of the large q behaviour of the MCT α-equations aims at understanding better the numerical solutions of these equations. In this context, the Kohlrausch function plays an important role since it is the characteristic function of a limit distribution. Basically, two conditions have to be fulfilled for Levy's theorem to be applicable [9].

First, the probability density, $f(\omega)$, of each of the single stochastic variables has to have a power law tail $f(\omega) \propto \omega^{-1-\beta}$ for $\omega \gg 1$. Due to the regular short time behaviour of a fluctuation function, which would lead to an overall Gaussian decay, it is clear that some prescription has to be given as to how the microscopic timescales drop out of the α-equations before Levy's theorem can be applied. Further, a common von Schweidler high frequency wing of the α-process has to be explained. Finding asymptotic scaling laws the MCT achieves this goal and rigorously derives the von Schweidler asymptote (2) with the material-dependent exponent b [6]. Crucial for this is the appearance of an intermediate dynamical process, the β-process in MCT notation, which introduces power law behaviours.

The second condition for Levy's theorem is the assumption of stochastically independent processes. Considering the $\varphi_q(t)$ as characteristic functions belonging to some stochastic variables, the α-equations (5) show that in general the variables are not independent. The straightforward application of Levy's theorem fails to incorporate the influence of the static structure, S_q, on the $\varphi_q(t)$. For large q, however, the memory functions (18) are made up of an increasing number of decreasing contributions due to $f_q^c \to 0$; this will hold for more general vertices as well. When neglecting $\dot{K}^q_{\tilde{p}}$, Eq. (18) shows that the correlators, $\varphi_q(t)$, exhibit stability [9] in the following restricted sense: $\Phi^n(t) = \Phi(t/n^\beta)$ for $n = 2$. This holds for arbitrary n if $\Phi(t)$ belongs to a stable probability distribution. Stable distributions are the only possible candidates for limit theorems like Levy's. Naturally a probabilistic reasoning applies better to the BM where more correlators are coupled than in the one-component HSS. A strong argument for the probabilistic interpretation of the large q results is the aspect that they depend on general properties of the coupling vertices only and that no detailed information on S_q is needed.

Perhaps the generic results of the α-equations of the idealized MCT can now be understood from two known limiting results. The first ansatz neglected any wavevector dependence in Eq. (5) and studied $\Phi(t)$ and $m(t) = v\Phi^2(t)$ [11]. It lead to exponentional α-decay. For large wavevectors, an increasing number of correlators contributes to the memory function $m_q(t)$ in Eq. (5b) and leads to the Kohlrausch function with $\beta = b$. The essential prerequisite for this result is the von Schweidler asymptote with $b < 1$, which is one of the central results of the MCT for the β-process [5].

The author cordially thanks A. Latz and W. Götze for valuable discussions and W. Götze and L. Sjögren for critical comments on the manuscript. Financial support from the Deutsche Forschungsgemeinschaft under contract Go 154/8-1 is also acknowledged.

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